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The Black-Litterman model

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Abstract

The Markowitz model has two problematic tendencies; unintuitive portfolios and portfolios with high transaction costs. The Black-Litterman model was made as an improvement of the Markowitz model. It uses a Bayesian approach to combine the views of the investor with the equilibrium portfolio. The main purpose of the model is to create intuitive portfolios and limit the transaction costs. With the computer power available today, the implementation of mathematical models are an important issue. Using the programming language R, I will in this thesis look at possible ways of implementing the Black-Litterman model. Todays investment firms have different ways of expressing their views on future asset performance. A common method is to use a scorecard. It is interesting to see how the scorecards properties make simplifying conditions to the Black-Litterman model. By using both existing R packages and self-made R code, the Black and Litterman model is applied to the R language to find the best approach.

Contents

1 Introduction

1.1 Qantitative portfolio optimization

In 1952, Harry Markowitz [1] published an article named *Portfolio Selection* in the *Journal of Finance*. This article is today known as the "*cornerstone*" [2] of modern quantitative portfolio theory. Markowitz represented through this article a simple but brilliant mathematical model for use in portfolio optimization. But the model was not flawless. Richard O. Michaud discussed in his article *The Markowitz Optimization Enigma: Is "Optimized" Optimal?* in 1989 different reasons for why the Markowitz model was not such a good model, he also revealed the fact that few investors had used the model since its discovery in 1952. Although Michaud mentioned several valid reasons for why the Markowitz model wasn't such a good model, two of them where crucial; its tendency to create unintuitive portfolios and portfolios with high transaction costs. In 1992 Fisher Black and Robert Litterman stated in their article *Global Portfolio Optimization* that "*...asset allocation models have not played the important role they should in global portfolio management*". They made in this article an effort to solve the problems related to the Markowitz model. They did this, not by changing the model itself, but by changing the model input. The mean variance model builds on a "null portfolio", while the Black-Litterman(B-L) model builds on the equilibrium portfolio. The B-L model then take into consideration the views of the investor, this means that the investors makes "bets" on expected returns. Dependent on the investors faith in each view, he gives them a confidence, where the confidence determines the impact of the view on the equilibrium portfolio. The B-L model uses a Bayesian approach to combine these "bets", or views with the equilibrium portfolio.

The B-L model is today used by some of the largest investment companies in the world. This proves that Black and Litterman have reached their goal of making the quantitative models more applied in portfolio investment decisions. The investment companies often use what is called *scorecards* as a tool when making investment decisions. These scorecards serves as a result list, with the winner on the top as the hottest investment subject. Each of the possible assets are graded after different factors, such as expected return, dividend yield and liquidity. The scorecard therefore contains the information necessary to implement in the B-L model. If the investor picks five assets from the scorecard, the Black-Litterman model can then make a good framework for calculating suggestions to how much the investor should invest in each of the assets. There are few articles to be found that discusses the appliance of the B-L model to a scorecard, I have therefore spent time researching this topic, and as I reveal later in the thesis, uncovered interesting results.

Another goal in my research was to look at the implementation of the B-L model in the free programming language R. One of the packages already developed in the R environment, is the BLCOP package. But the package proves insufficient under the assumptions of a scorecard. One of the goals with research is therefore to uncover useful methods of applying R to the B-L model with scorecard assumptions.

1.2 The R programming language

R is a free to use software programming language based on the programming language S that provides a variety of statistical and graphical opportunities. It was developed by the two statisticians Robert Gentleman and Ross Ihaka [3] and first represented trough the internet portal Statlib in 1993. It immediately gained feedback, and especially Martin Mächler was showing interest in their work. Martin persuaded them to publish their work as free software, and it was made available under the terms of the Free Software Foundation's GNU general license in 1995. R has since then become one of the main tools for statisticians developing statistical software.

I have chosen to use R as a statistical tool when working with the Markowitz and Black-Litterman model. There already exist packages developed for portfolio optimization purposes. One of the more extensive packages covering the Markowitz framework is the fPortfolio package. This is a relatively easy to use package, with intelligent solutions for implementing a wide range of constraints to the portfolio. Eric Zivot [4] has created a set of functions for portfolio optimization which he calls *portfolio.r* (from here on I will use portfolior instead of portfolio.r). The functions does not have the extensive framework of the fPortfolio package, but the simpler format makes it easier to implement your own estimates, as the posterior expected excess returns and covariance matrix of the B-L model.

For tests in the Markowitz framework I am going to use the portfolior functions. I will in the Black-Litterman framework use my own functions BL() and BLvar(), and the BLCOP package. The BL() function is for the case when we only have absolute views, and the investor has no view on the confidence. The BLvar() function is essentially the same function as the BL() function, but for the case where the investor has confidences about his views. The reason why I created two different functions is that the BL() functions requires less inputs. I use the BLCOP package for the case of relative views.

1.3 Data

In my research I will use a portfolio of 5 stocks from the OBX stock index. The OBX stock index consists of the 25 most traded stocks of the Norwegian main stock index OSEBX. When applying the B-L model, a market portfolio is needed to calculate equilibrium expected excess returns. I will use the OSEBX as the market portfolio, with its respective market return and market variance. My portfolio will consist of the following 5 stocks (I will from here on describe the shares by their ticker):

- *Det norske oljeselskap*(DETNOR)
- *Norwegian Air Shuttle*(NAS)
- *Subsea 7*(SUBC)
- *Telenor Group*(TEL)
- *TGS-NOPEC*(TGS)

With respect to diversification within industries, this is not a very well build portfolio. The companies *DETNOR*, *SUBC* and *TGS* are part of the Norwegian oil industry, all engaged in the exploration and recovering of oil and gas. *NAS* is a low cost airline with a seemingly bright future, and therefore possibly one of the better investment opportunities in todays stock market. *TEL* is the largest telecommunications company in Norway. As represented, the portfolio has a overweight of firms in the Norwegian oil industry. As the petroleum industries represent one third of the state income in Norway, investments linked to the petroleum industry are hard to avoid. Anyway, diversification is unimportant in the context of my research, as there are other aspects that will be of interest for me. For my purposes, this is a good portfolio.

2 The Markowitz Model

2.1 Introduction

With his publication of the article *Portfolio Selection* in 1952, Harry Markowitz gave life to one of the foundations of modern portfolio theory. He was a pioneer that traveled into the new world of quadratic portfolio optimization. There were previously made little research on the mathematical relations within portfolios of assets. In his article, Markowitz claims that each investor wants to maximize expected return and at the same time minimize the variance, or the risk of the portfolio. As he explained mathematically it's not enough to look at each single asset when creating a portfolio; you also have to take in to consideration the correlation between the assets. The computation of portfolio variance depends on the variance of each single asset and the covariance between the different assets. Markowitz argues that the investor, by taking into account the correlation between the assets, will generate portfolios with higher returns for the same or lower risk.

In his article, Markowitz stated that *"...the process of selecting a portfolio may be divided into two stages"* [1]. The first stage refers to the collection of relevant data and model input. Markowitz focuses on the second stage, the computation of the portfolio. The Markowitz mean variance model is the foundation of many other developments in modern portfolio theory, one of these is the Black-Litterman model. The B-L model builds on the Markowitz model, and it is hence important to understand the Markowitz model when applying the B-L model. The models focus is to enhance the Markowitz model by interfering with what Markowitz called the first part of the portfolio optimization process.

Richard O. Michauds [5] discusses in his article *The Markowitz Optimization Enigma: Is Optimized Optimal?* reasons for why the Markowitz model is seldom used by the investor. Michaud reveals in his discussion both negative and positive arguments for the Markowitz model. The positive arguments mentioned by Michaud are some of the reasons for why the model today stands as one of the pillars on modern portfolio theory.

2.2 The Model

In the Markowitz mean variance model, the inputs needed are the various assets expected return and variance-covariance matrix. It is assumed that the investor prefers to maximize the expected return given a certain risk or minimize the risk given a certain expected return.

The investor can do so by solving the following optimization problems:

$$\begin{aligned} &\min_{\boldsymbol{w}} \boldsymbol{\sigma}_p^2 = \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w} \\ \text{subject to} & \\ &R_p = \boldsymbol{w}^T \boldsymbol{R} \\ &\boldsymbol{w}^T \mathbf{1} = 1 \end{aligned}$$

or

$$\underset{w}{\text{Max}}\ R_p = \boldsymbol{w}^T \boldsymbol{R}$$

subject to

$$\sigma_p^2 = \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w}$$

$$\boldsymbol{w}^T \boldsymbol{1} = 1 \ ,$$

where

$\boldsymbol{w}$ = the vector of portfolio weights.

$\boldsymbol{w}^{opt}$ = the Markowitz optimized portfolio weights.

σ_p^2 = the portfolio variance.

$\boldsymbol{\Sigma}$ = the variance-covariance matrix between the different portfolio assets.

$\boldsymbol{R}$ = the vector of portfolio asset expected returns.

R_p = the portfolio expected return.

Every weight w_i tells how much of the investors initial wealth is invested in asset i. Because the total value of the portfolio cannot be larger than our initial wealth, the constraint $\boldsymbol{w}\boldsymbol{1} = 1$ is added. The initial wealth is what we have available for investing in the portfolio. Since the optimization problems are dual problems, solving them gives a single optimal solution [4] (the proof can be found under Appendix A):

$$\boldsymbol{x}_w = \boldsymbol{B}_w^{-1} \boldsymbol{a} \ .$$

In a case where we have a n asset portfolio the $\boldsymbol{w}^{opt}$ will be the n first rows of the $\boldsymbol{x}_w$ vector.

2.3 The tangency portfolio

If we take into account the first optimization problem of Markowitz, where we minimize the variance subject to a given return, we can find the efficient frontier. The efficient frontier represents all those portfolios that subject to a expected return, minimize the risk. This means we have to solve the following optimization problem for a set of expected returns:

$$\operatorname*{Max}_{\boldsymbol{w}} R_p = \boldsymbol{w}^T \boldsymbol{R}$$

subject to

$$\sigma_p^2 = \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w}$$

$$\boldsymbol{w}^T \mathbf{1} = 1 \,.$$

I will now use an example to show one way of finding the efficient frontier using the portfolio I represented in the introduction. By using the efficient.frontier() function in *portfolior*, I can find a set of portfolios on the efficient frontier. As input to the Markowitz model, I calculate from the five assets historical prices the historical average return and the variance-covariance matrix. The portfolio is created on the first of March 2013 based on historical data from the past year. To calculate the historical average and the variance-covariance matrix, I have made a simple function in R named Mark() which output can be used directly in the portfolior functions. The variance-covariance matrix and the expected returns vector are calculated by Mark()$CM and Mark()$ER respectively.

```
> ER<-Mark()$ER
> ER
    DETNOR        NAS        SUBC        TEL        TGS
0.04664701 1.10013264 0.09968745 0.19925025 0.35341081
```

```
> CM<-Mark()$CM
> CM
           DETNOR        NAS       SUBC        TEL        TGS
DETNOR 0.12971546 0.05180166 0.06174439 0.02632477 0.06563738
NAS    0.05180166 0.16825281 0.04508897 0.02307640 0.05065959
SUBC   0.06174439 0.04508897 0.10334572 0.02983291 0.07574867
TEL    0.02632477 0.02307640 0.02983291 0.04439916 0.02486181
TGS    0.06563738 0.05065959 0.07574867 0.02486181 0.11092227
```

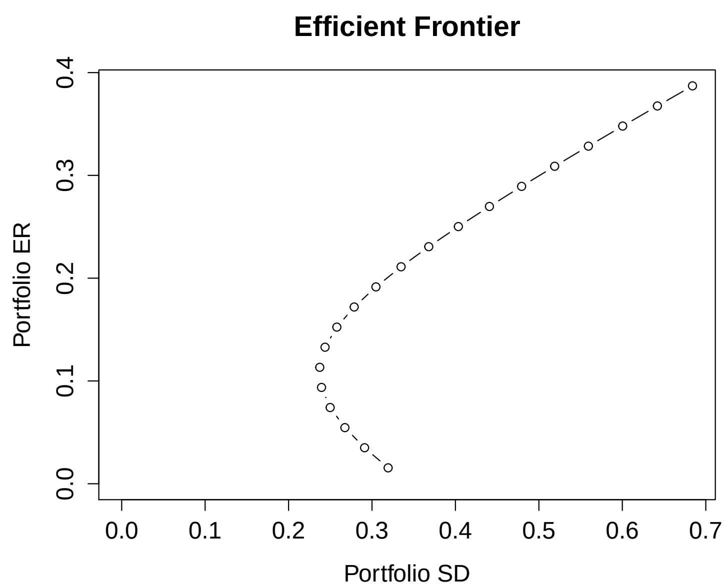
```
> efficient<-efficient.frontier()
> efficient
```

```
Call:
efficient.frontier()

Frontier portfolios' expected returns and standard deviations
   port 1 port 2 port 3 port 4 port 5 port 6 port 7 port 8 port 9
ER 1.5136 1.4266 1.3395 1.2525 1.1654 1.0784 0.9913 0.9043 0.8172
SD 0.5024 0.4728 0.4436 0.4149 0.3867 0.3593 0.3328 0.3074 0.2834
   port 10 port 11 port 12 port 13 port 14 port 15 port 16 port 17
ER 0.7302  0.6431  0.5561  0.4691  0.3820  0.2950  0.2079  0.1209
SD 0.2612  0.2414  0.2245  0.2113  0.2024  0.1985  0.1998  0.2063
   port 18 port 19 port 20
ER  0.0338 -0.0532 -0.1403
SD  0.2174  0.2326  0.2510
```

Represented here are 20 portfolios on the efficient frontier. The plot of these frontier portfolios makes a "bullet" formed shape and is therefore usually referred to as the *"Markowitz bullet"*.

```
> plot.Markowitz(efficient)
```



From the efficient frontier, or the Markowitz bullet, we can see that the approximate possible minimum standard deviation is 20 %. To find the actual minimum variance portfolio weights, we have to solve the following optimization problem:

$$\underset{\boldsymbol{w}}{\text{Min}}\ \sigma_p^2 = \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w}$$

subject to

$$\boldsymbol{w}^T \mathbf{1} = 1 \ .$$

This leads to the optimal solution

$$\boldsymbol{z}_m = \boldsymbol{A}_w^{-1} \boldsymbol{b} \ .$$

The proof for this solution is available in appendix A under the Markowitz section. This optimization problem is implemented in the R package portfolior and fPortfolio. In fPortfolio, finding the optimal portfolio subject to the posterior expected excess returns and the posterior variance-covariance matrix, can be done by using the optimalPortfolios.fPort() function. Although the fPortfolio package is a much more extensive package than portfolior, portfolior is easier to use for my purposes. My focus will not be the implementation of advanced constraints, those are available and carefully explained in "*Portfolio Optimization with R/Rmetrics"* [6], but on the implementation of the B-L model to R and the benefits of using the B-L model compared to the Markowitz model. I will now find the global minimum variance portfolio for the first of March 2013.

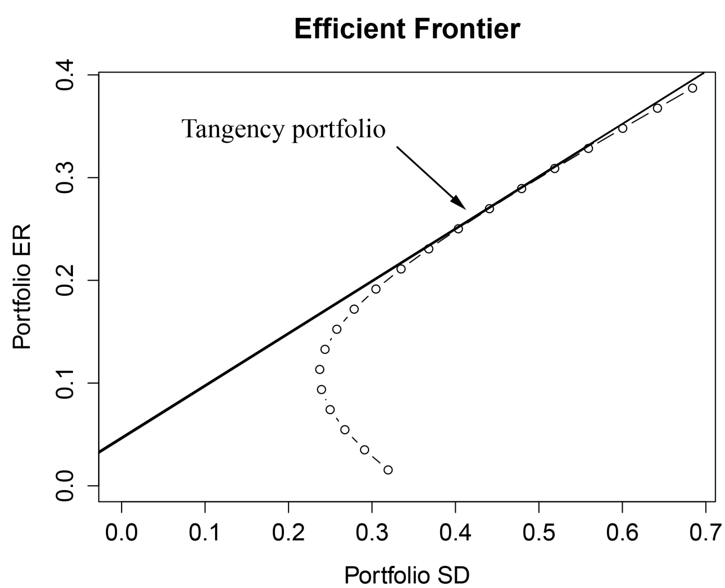
```
> globalMin.portfolio()
Call:
globalMin.portfolio()

Portfolio expected return:     0.273206
Portfolio standard deviation:  0.1983128
Portfolio weights:
DETNOR    NAS   SUBC    TEL    TGS
0.0597 0.0771 0.0228 0.7372 0.1032
```

We see that the optimization problem produces a portfolio with a standard deviation of 19.83 % and an expected return of 27.32 %. I have now shown how to find the efficient portfolios of assets. But not all the portfolios make an equally good investment, so which one is the optimal portfolio to choose? The answer was given by William Sharp [7] as the "*reward to volatility ratio*" in the article *Mulutal Fund Performance* in 1966. The reward to volatility ratio was later named the *"Sharp ratio"* which is the name mostly used today. The Sharp ratio rates a portfolio after its amount of excess return($r_p - r_f$) per percentage of portfolio volatility(σ_p):

$$SR = \frac{r_p - r_f}{\sigma_p}.$$

The portfolio on the efficient frontier with the highest Sharp ratio is then the "*risk neutral*" investors portfolio of choice. With the efficient frontier plot present, we can draw a line from the point $(0, r_f)$ that is tangent with the efficient frontier. The portfolio on the tangency point will then be the *tangency portfolio*. Unfortunately, the portfolior package does not contain a function for producing the tangency line. The plot with the tangency portfolio show an approximation of the tangency line. The tangency point is the optimal efficient portfolio with the highest Sharp ratio, and thereby named the tangency portfolio.



The tangency portfolio

Even though the tangency portfolio is the optimal solution on the tangency line, it is not the only optimal solution. One can by forming a portfolio that consist of the risk free asset and the tangency portfolio achieve a portfolio with the same Sharp ratio. The line drawn from the risk free position through the tangency portfolio, is called the "*Capital Allocation Line*", and represents all the possible optimal solutions. Where an investor chooses to place himself on the Capital Allocation line, is dependent on the investors risk aversion. The Capital Allocation line(CAL) can be presented by the following equation(The notation is found under appendix A in the Markowitz section):

$$r_p = r_f + \frac{r_T - r_f}{\sigma_T}\sigma_p \, .$$

We see here that the Sharp ratio is the slope of the CAL. The following equation is a interesting rewriting of the CAL(Proof in appendix A under the Markowitz section):

$$r_p = (1 - \frac{\sigma_p}{\sigma_T})r_f + \frac{\sigma_p}{\sigma_T}r_T \, .$$

We here see the portfolio expected return represented as a weighted average between the risk free asset(r_f) and the tangency portfolio(r_T). The weights are decided by how much risk the investor is willing to take. A risk averse investor would like a low portfolio volatility σ_p, which would again lead to a lower expected return, as my following calculations prove. To interpret the function I find the partial derivative of r_p with respect to σ_p:

$$\frac{\partial r_p}{\partial \sigma_p} = \frac{r_T}{\sigma_T} - \frac{r_f}{\sigma_T} \, .$$

Because the following have to be true:

$$r_f < r_T \, ,$$

we have that

$$\frac{\partial r_p}{\partial \sigma_p} > 0 \, .$$

This tells us that there is a positive relationship between portfolio volatility and portfolio expected return, which is intuitive. In the rest of my thesis I will assume that the investor wants to take on the tangency portfolio, and hence us this in my tests. To find the tangency portfolio, the following optimization problem needs to be solved:

$$\underset{\boldsymbol{w}}{\text{Max}} \text{ Sharpe's ratio} = \frac{r_p - r_f}{\sigma_p}$$

subject to

$$\boldsymbol{w}^T \mathbf{1} = 1 \text{ .}$$

Since we have that $\sigma_p = (\boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w})^{\frac{1}{2}}$ and $r_p = \boldsymbol{w}^T \boldsymbol{R}$, the Sharp ratio can be written in matrix form:

$$SR = \frac{\boldsymbol{w}^T \boldsymbol{R} - r_f}{(\boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w})^{\frac{1}{2}}} \text{ .}$$

We get the Lagrangian problem:

$$\mathcal{L}(w, \lambda) = (\boldsymbol{w}^T \boldsymbol{R} - r_f)(\boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w})^{-\frac{1}{2}} + \lambda(\boldsymbol{w}^T \mathbf{1} - 1) \text{ .}$$

Writing the first order conditions:

$$\begin{aligned} \frac{\partial \mathcal{L}(\boldsymbol{w}, \lambda)}{\partial \boldsymbol{w}} &= \boldsymbol{R}(\boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w})^{-\frac{1}{2}} - (\boldsymbol{w}^T \boldsymbol{R} - r_f)(\boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w})^{-\frac{3}{2}} \boldsymbol{\Sigma} \boldsymbol{w} + \lambda \mathbf{1} = \mathbf{0} \\ \frac{\partial \mathcal{L}(\boldsymbol{w}, \lambda)}{\partial \lambda} &= \boldsymbol{w}^T \mathbf{1} - 1 = 0 \text{ .} \end{aligned}$$

It can be shown that the solution for $\boldsymbol{w}$, where $\boldsymbol{w}$ is the tangency portfolio, is the following:

$$\boldsymbol{w} = \frac{\Sigma^{-1}(\boldsymbol{R} - r_f \mathbf{1})}{\mathbf{1}^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{R} - r_f \mathbf{1})} \text{ .}$$

In R, I can by using tangency.portfolio() function in the portfolior package find the tangency portfolio:

```
> tangency.portfolio()
Call:
tangency.portfolio()

Portfolio expected return:     1.491078
Portfolio standard deviation:  0.4946824
Portfolio weights:
 DETNOR      NAS    SUBC      TEL      TGS
-0.5399  1.1613 -0.6676  0.4185  0.6277
```

We see here one of the problems present when using the Markowtz model, there are ”extreme” positions in the portfolio shares, with large short positions in DETNOR and SUBC and a very large long position in NAS. The B-L model limits these problems, but it is nevertheless necessary with constraints to limit the exposure to extreme positions. As I have explained here, the tangency portfolio, or a combination of the tangency portfolio and the risk free investment, is the optimal investment. Throughout the paper I will therefore assume the investor to take the tangency portfolio, and all my model testing will be with the tangency portfolio.

2.4 Problems with the Markowitz model

In 1989 Richard O. Michaud published the article ”The Markowitz Otimization Enigma: Is 'Optimized' Optimal?” in the *FINANCIAL ANALYSTS JOURNAL*. Michaud here highlights the fact that even though the Markowitz model is one of the main models in the academic world, it's practical use has been almost non-existent. But even though the model has problematic shortcomings, there are several practical elements with the Markowitz model. Michaud [5] lists the following elements:

- *Satisfaction of client objectives and constraints*: The Markowitz model environment makes it easy to include constraints concerning the investors limitations and objectives.
- *control of portfolio risk exposure*: The Markowitz model makes it easy to control the portfolios exposure to risk components.
- *Implementation of style objectives and market outlook*: The investor can by choice in the

various risk factors reflect his/her investment style, philosophy and market outlook in the Markowitz model framework.

- *Efficient use of investment information*: The Markowitz model provides a optimal solution given the available information.
- *Timely portfolio changes*: In 1989 Michaud pointed out the Markowitz models ability to process large amounts of information. Because of a large increase in available computer power since 1989, this statement is even more accurate today.

I will later on discuss the Black-Litterman model as an improvement of the Markowitz model. We will see that the B-L model manages to get rid of or decrease some of the elements that makes the Markowitz model of little use for investors. The positive elements of the B-L model will pretty much be the same as the Markowitz model.

Michaud states that the Markowitz hasn't had a very large impact on the investment strategies since it's publication in 1952. Because of the development of new models since 1989, such as the B-L model in 1992, this is today a truth with modifications. It's a fact that the Markowitz model itself isn't used by many investors, but the B-L model, which is an improvement of the Markowitz model, is another story. As an example; in the article *The Intuition Behind Black-Litterman Model Portfolios* written by Robert Litterman and Guangliang He [8] in 1999, it is said that the B-L model is used by the Quantitative Strategies Group at Goldman Sachs. This is not only the case for Goldman Sachs, for large investment firms it is today common with sections working with quantitative modeling.

Michaud mention several good reasons for why the Markowitz model may not be such a good model, but not all of them are still valid. Michaud argued that there were some "less robust" reasons for not using the Markowitz model. One of these was the political reason; he claimed that there where political reasons within the firm that prevented the use of the Markowitz model in portfolio optimization. The implementation of the quantitative models in the investment strategy would mean that some of the current managers specialized in qualitative portfolio optimization would have to be replaced by new experts on quantitative portfolio optimization. As Michaud states, no one easily gives up a position of power. Today, this argument is no longer valid. As I mentioned, this is because the investment bureaus have made use of the quantitative investment strategies in their investment decisions. However, Michaud presented limitations and problems with the model that are true also today:

- It is anecdotally known that the Markowitz model produces portfolios that are unintuitive, this because of badness in the variance-covariance matrix.
- The Markowitz model produces portfolios with high transaction costs. This can be traced to the fact that the Markowitz model produces what Michaud calls "*estimation error maximizes*". The Markowitz model is known to create "unique solutions", but this is only true under the assumptions that the expected return and variance estimates are without estimation error, which isn't true; all expected return and variance estimators are subject to estimation error. Because of this the Markowitz model "*...significantly overweighs (underweights) those securities that have large (small) estimated expected returns, negative (positive) correlations and small (large) variances*".
- It is common for users of the Markowitz model to apply historical returns, which isn't optimal because there exists better methods for finding expected return and variance-covariance estimates.
- The Markowitz model often ignores factors that are important in investment management decisions. One of these factors is the liquidity factor. It has happened that investors blindly using quantitative models have taken considerable long positions in low liquidity firms, causing large problems when trying to leave positions. Investing in low liquidity firms may also affect the stock price, leading the stock price to rise (fall) when investing(selling).
- The Markowitz model can in some cases produce unstable optimal solutions. Small deviations in input create very different optimal portfolios, making the portfolio both unintuitive and expensive to hold because of transaction costs. According to Michaud this is dependent

on a ill conditioned variance-covariance matrix caused by insufficient historical data.

- The Markowitz model tends to rely to much on portfolios with a low estimated volatility, creating a less diversified portfolio.

Some of these problems can be discussed, such as the liquidity problem. The liquidity problem can be taken care of by a proper portfolio constraint. But as this has been a trap entered by investors in the past, it's still a valid argument, mainly due to an investors unawareness. This reveals another flaw with the Markowitz model, if for any reason an investor does not implement an important constraint to the model, he could end up with an unfortunate portfolio.

3 The Black-Litterman model

3.1 Introduction

The Markowitz model has different problems, such as unintuitivity, ill-behaved portfolios and error maximization. Although the model has been an academic success, these problems have rendered the model of little use for investors. In 1992 Robert Litterman and Fischer Black [9] proposed a improvement to this model that where suppose to make quantitative optimization tools of more use to investors. They claimed that by taking the CAPM equilibrium into consideration, they would significantly improve the model. With the equilibrium portfolio as a starting point, the investor would then make adjustments depending on the investors views on the market. One of the benefits of the B-L model is that there is no need for the investor to have views on all the portfolio assets. The investor only adds his/her views when they have one, and else use the equilibrium expected excess returns. Although the B-L model is a large improvement from the Markowitz model, it isn't actually changing the model. In the article "Portfolio Selection" Harry Markowitz [1] write about the two stages of portfolio optimization. The first stage being the collection and final beliefs about future returns, and the second stage the use of these beliefs to create a portfolio. As Markowitz states, his model is part of the last stage, concerning the use of available information and the creation of an efficient portfolio. The B-L model only tries to improve the Markowitz model by interfering with the first stage. So, the B-L model, rather than being a whole new optimization model, is just an add-on to the Markowitz model, interfering with the mean variance input. But this doesn't make it any less useful, the model generates portfolios with considerable differences from those created by the Markowitz model.

3.2 The model

The model uses a weighted average of the implied equilibrium excess returns and the investors views. The weights are decided by the confidence on each of the factors, and I will later on explain how to find the confidence of the implied equilibrium excess returns and the investors views. In the derivation of the model I will use the following notation:

$\boldsymbol{R}$ =The vector of expected asset returns

$\boldsymbol{w}$ =The vector of portfolio weights

$\boldsymbol{A}$ =The risk aversion factor

A_{mkt} =The market risk aversion factor

$\boldsymbol{\Sigma}$ =The variance covariance matrix

r_f =The risk free rate of return

$E(r_m)$ =The expected market return

$\boldsymbol{E(r)}$ =The vector of the investors expected returns

σ_m^2 =The variance of the expected market return

$\boldsymbol{\pi}$ =The vector of implied market equilibrium expected excess returns

3.2.1 The implied equilibrium expected excess returns

Let's say we have the investors utility function:

$$U = \boldsymbol{w}^T \boldsymbol{R} - \frac{1}{2} A \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w} \,.$$

We maximize the investor utility function to find the utility maximizing weights:

$$\frac{\partial U}{\partial \boldsymbol{w}} = \boldsymbol{R} - A \boldsymbol{\Sigma} \boldsymbol{w} = 0 \,.$$

We then solve for R

$$\boldsymbol{R} = A \boldsymbol{\Sigma} \boldsymbol{w} \,.$$

By substituting the weights($\boldsymbol{w}$) with the market capitalization weights($\boldsymbol{w}_{mkt}$), and use the market risk aversion factor(A), we get the implied equilibrium expected excess returns($\boldsymbol{\pi}$). The market risk aversion factor is found by using the following formula:

$$A_m = \frac{E(r_m) - r_f}{\sigma_m^2} \,.$$

The market capitalization weights are found by dividing the total market value of each asset on the total market value of the portfolio:

$$\boldsymbol{w}_{mkt} = \begin{pmatrix} w_1 \\ w_2 \\ . \\ . \\ . \\ w_n \end{pmatrix} = \begin{pmatrix} \frac{\text{Asset 1 market kap. value}}{\text{Portfolio market kap. value}} \\ \frac{\text{Asset 2 market kap. value}}{\text{Portfolio market kap. value}} \\ . \\ . \\ . \\ \frac{\text{Asset } n \text{ market kap. value}}{\text{Portfolio market kap. value}} \end{pmatrix} .$$

We get the implied equilibrium expected excess return vector:

$$\boldsymbol{\pi} = A_m \boldsymbol{\Sigma} \boldsymbol{w}_{mkt} \, .$$

Where $\boldsymbol{\Sigma}$ is the variance-covariance matrix of the implied equilibrium excess returns. We have that

$$\boldsymbol{R} \sim \mathcal{N}(\boldsymbol{\pi}, \boldsymbol{\Sigma}) \, .$$

The confidences of the implied equilibrium expected excess returns are found by finding the inverse of the variance-covariance matrix:

$$\text{The confidence of the implied equilibrium expected excess returns} = \boldsymbol{\Sigma}^{-1} \, .$$

3.2.2 The investors views

The investor views are implemented into the model to adjust the equilibrium expected excess returns for the investor's views on the future returns. I denote the number of investor views by k. To implement the views we first make a $k * 1$ matrix and name it $\boldsymbol{K}$. The $\boldsymbol{K}$ matrix represents all the investors views on the expected return:

$$\boldsymbol{K} = \begin{pmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_k \end{pmatrix} .$$

We need to "connect" the view matrix to the assets to apply the investors views, to do this we use a "link matrix". The link matrix is a $k * n$ matrix where the values is either positive, negative or zero depending on the views effect on the linked asset.

An example of a link matrix with five assets and three views:

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} .$$

Because there is an uncertainty about the views we have to add an error term to the views:

$$\boldsymbol{K} + \boldsymbol{\varepsilon}$$

where

$$\varepsilon_i \sim \mathcal{N}(0, \boldsymbol{\Omega}) .$$

We are going to assume that the error terms are normally distributed with zero mean and $\boldsymbol{\Omega}$ variance. $\boldsymbol{\Omega}$ represents a variance-covariance matrix. The example link matrix $\boldsymbol{P}$ reveals two kinds of views, the relative views and the absolute views. The first row of the P matrix represents a *absolute view*. In the absolute view the investor believes the asset will achieve a certain return independent from the other assets in the portfolio. In the *relative views*, represented in row 2 and 3 of the matrix P, the investors have beliefs on how two or more assets are going to perform relative to each other. I will later on discuss how to apply investor scorecards to the B-L model; a property of the scorecards is that they only contain absolute views.

We now have the uncertainty of the views, represented by the $\boldsymbol{\Omega}$ matrix. The confidence is found by taking the inverse of the $\boldsymbol{\Omega}$ matrix.

$$\text{The confidence of the investors views} = \boldsymbol{\Omega}^{-1} .$$

Black and Litterman [10] suggest that the views uncertainty could be computed as

$$\boldsymbol{\Omega} = \tau \boldsymbol{P} \boldsymbol{\Sigma} \boldsymbol{P}^T$$

where τ is a scalar. Which value to choose as the scalar has been subject to some disagreement. Black and Litterman operated with 0.025, or a value close to zero, while other practitioners use 1. The scalar can be used to "tune" the model. In the absence of investor opinions on the views uncertainty, this formula can be used to compute a variance-covariance matrix.

3.2.3 Derivation of the model

We now have all the elements necessary to derive the Black-Litterman model. I will derive the model as represented by George A. Christodoulakis [11] with a few changes. The B-L model represents the expectation and the variance of the multivariate normal distribution of the expected returns given the equilibrium expected excess returns. To derive the model we need Bayes theorem. Bayes theorem states that the probability distribution of an event A given another event B, can be written:

$$P(A|B) = \frac{P(A|B)P(A)}{P(B)} \text{ .}$$

It is usefull to keep in mind the multivariate normal distribution function

$$f_x(x_1, ..., x_k) = \frac{1}{\sqrt{(2\alpha^k)|\boldsymbol{\Sigma}|}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})} \text{ ,}$$

where $\boldsymbol{\mu}$ is the mean, and $\boldsymbol{\Sigma}$ the variance-covariance matrix in the multivariate normal distribution.

The events A and B will now be the expected return and the equilibrium expected excess return. By substituting A and B we get the problem we need to solve to find the B-L model.

$$P(\boldsymbol{P(r)}|\boldsymbol{\pi}) = \frac{P(\boldsymbol{\pi}|\boldsymbol{E(r)})P(\boldsymbol{E(r)})}{P(\boldsymbol{\pi})} \text{ .}$$

Where E(r) are the prior beliefs of the investor. We have assumed that

$$\varepsilon_i \sim \mathcal{N}(0, \boldsymbol{\Omega}) \text{ ,}$$

which means that

$$\boldsymbol{E(r)} \sim \mathcal{N}(\boldsymbol{K}, \boldsymbol{\Omega}) \text{ .}$$

We also need the probability density function(pdf) to the equilibrium expected excess returns given the prior expected returns. We make the following assumption

$$\boldsymbol{\pi}|\boldsymbol{E(r)} \sim \mathcal{N}(\boldsymbol{E(r)}, \boldsymbol{\Sigma}) \text{ .}$$

From the assumed distributions, we can create the pdf's of the prior expected returns and the equilibrium expected excess returns given the prior expected returns:

$$pdf(\boldsymbol{E(r)}) = \frac{1}{\sqrt{(2\alpha^k)|\boldsymbol{\Omega}|}} exp\Big(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{K})^T \boldsymbol{\Omega}^{-1}(\boldsymbol{x} - \boldsymbol{K})\Big)$$
$$pdf(\boldsymbol{\pi}|\boldsymbol{E(r)}) = \frac{1}{\sqrt{(2\alpha^k)|\tau\boldsymbol{\Sigma}|}} exp\Big(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{E(r)})^T (\tau\boldsymbol{\Sigma})^{-1}(\boldsymbol{x} - \boldsymbol{E(r)})\Big) \text{.}$$

Christodoulakis [11] states that since the pfd of the equilibrium expected excess returns are a constant, it will be absorbed in to the integrating constant of the pdf of the prior expected returns given the equilibrium expected excess returns. Because of this, we can find the mean and variance of the posterior pdf by taking the product between the pdf of the prior returns and the pdf of the equilibrium expected excess returns given the prior expected returns:

$$exp\Big(-\frac{1}{2}(\boldsymbol{\pi} - \boldsymbol{E(r)})^T (\tau\boldsymbol{\Sigma})^{-1}(\boldsymbol{\pi} - \boldsymbol{E(r)}) - \frac{1}{2}(\boldsymbol{E(r)} - \boldsymbol{K})^T \boldsymbol{\Omega}^{-1}(\boldsymbol{E(r)} - \boldsymbol{K})\Big) \text{.}$$

We need to rearrange the exponent such that it represents the exponent in the multivariate normal distribution function.:

$$= exp\Big(- \frac{1}{2}\Big((\boldsymbol{\pi} - \boldsymbol{E}(\boldsymbol{r}))^T(\tau\boldsymbol{\Sigma})^{-1}(\boldsymbol{\pi} - \boldsymbol{E}(\boldsymbol{r})) + (\boldsymbol{E}(\boldsymbol{r}) - \boldsymbol{K})^T\boldsymbol{\Omega}^{-1}(\boldsymbol{E}(\boldsymbol{r}) - \boldsymbol{K})\Big)\Big)$$

$$= exp\Big(- \frac{1}{2}\Big((\boldsymbol{\pi} - \boldsymbol{E}(\boldsymbol{r}))^T(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} - (\boldsymbol{\pi} - \boldsymbol{E}(\boldsymbol{r}))^T(\tau\boldsymbol{\sigma})^{-1}\boldsymbol{E}(\boldsymbol{r}) + (\boldsymbol{PE}(\boldsymbol{r}) - \boldsymbol{K})^T\boldsymbol{\Omega}^{-1}\boldsymbol{PE}(\boldsymbol{r})$$

$$- (\boldsymbol{PE}(\boldsymbol{r}) - \boldsymbol{K})^T\boldsymbol{\Omega}^{-1}\boldsymbol{q}\Big)\Big)$$

$$= exp\Big(- \frac{1}{2}\Big(\boldsymbol{\pi}^T(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} - \boldsymbol{E}(\boldsymbol{r})^T(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} - \boldsymbol{\pi}^T(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{E}(\boldsymbol{r})^T(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{E}(\boldsymbol{r})$$

$$+ (\boldsymbol{PE}(\boldsymbol{r}))^T\boldsymbol{\Omega}^{-1}\boldsymbol{PE}(\boldsymbol{r}) - \boldsymbol{K}^T\boldsymbol{\Omega}^{-1}\boldsymbol{PE}(\boldsymbol{r}) - (\boldsymbol{PE}(\boldsymbol{r}))^T\boldsymbol{\Omega}^{-1}\boldsymbol{K} + \boldsymbol{K}^T\boldsymbol{\Omega}^{-1}\boldsymbol{K}\Big)\Big)$$

$$= exp\Big(- \frac{1}{2}\Big(\boldsymbol{\pi}^T(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} - \boldsymbol{E}(\boldsymbol{r})^T(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} - \boldsymbol{\pi}^T(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{E}(\boldsymbol{r})^T(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{E}(\boldsymbol{r})$$

$$+ \boldsymbol{E}(\boldsymbol{r})^T\boldsymbol{P}^T\boldsymbol{\Omega}^{-1}\boldsymbol{PE}(\boldsymbol{r}) - \boldsymbol{K}^T\boldsymbol{\Omega}^{-1}\boldsymbol{PE}(\boldsymbol{r}) - \boldsymbol{E}(\boldsymbol{r})^T\boldsymbol{P}^T\boldsymbol{\Omega}^{-1}\boldsymbol{K} + \boldsymbol{K}^T\boldsymbol{\Omega}^{-1}\boldsymbol{K}\Big)\Big)$$

$$= exp\Big(- \frac{1}{2}\Big(\boldsymbol{E}(\boldsymbol{r})^T((\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}^T\boldsymbol{\Omega}^{-1}\boldsymbol{P})\boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{\pi}^T(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} - \boldsymbol{E}(\boldsymbol{r})^T(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi}$$

$$- \boldsymbol{\pi}^T(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{E}(\boldsymbol{r}) - \boldsymbol{K}^T\boldsymbol{\Omega}^{-1}\boldsymbol{PE}(\boldsymbol{r}) - \boldsymbol{E}(\boldsymbol{r})^T\boldsymbol{P}^T\boldsymbol{\Omega}^{-1}\boldsymbol{K} + \boldsymbol{K}^T\boldsymbol{\Omega}^{-1}\boldsymbol{K}\Big)\Big)$$

$$= exp\Big(- \frac{1}{2}\Big(\boldsymbol{E}(\boldsymbol{r})^T\left((\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}^T\boldsymbol{\Omega}^{-1}\boldsymbol{P}\right)\boldsymbol{E}(\boldsymbol{r}) - (\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi}\boldsymbol{E}(\boldsymbol{r}) - (\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi}\boldsymbol{E}(\boldsymbol{r}) - \boldsymbol{P}^T\boldsymbol{\Omega}^{-1}\boldsymbol{KE}(\boldsymbol{r})$$

$$+ \boldsymbol{\pi}^T(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} + \boldsymbol{K}^T\boldsymbol{\Omega}^{-1}\boldsymbol{K}\Big)\Big)$$

$$= exp\Big(- \frac{1}{2}\Big(\boldsymbol{E}(\boldsymbol{r})^T\left((\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}^T\boldsymbol{\Omega}^{-1}\boldsymbol{P}\right)\boldsymbol{E}(\boldsymbol{r}) - 2\left((\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} + \boldsymbol{P}^T\boldsymbol{\Omega}^{-1}\boldsymbol{K}\right) + \boldsymbol{\pi}^T(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} + \boldsymbol{K}^T\boldsymbol{\Omega}^{-1}\boldsymbol{K}\Big)\Big).$$

If we have that:

$$\boldsymbol{H} = (\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}^T\boldsymbol{\Omega}^{-1}\boldsymbol{P}$$

$$\boldsymbol{C} = (\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} + \boldsymbol{P}^T\boldsymbol{\Omega}^{-1}\boldsymbol{K}$$

$$\boldsymbol{A} = \boldsymbol{\pi}^T(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} + K^T\boldsymbol{\Omega}^{-1}\boldsymbol{K}\,,$$

we can write:

$$= exp\Big(- \frac{1}{2}\Big(\boldsymbol{E}(\boldsymbol{r})^T\boldsymbol{HE}(\boldsymbol{r}) - 2\boldsymbol{C}^T\boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{A}\Big)\Big)$$

$$= exp\Big(- \frac{1}{2}\Big(\boldsymbol{E}(\boldsymbol{r})^T\boldsymbol{H}^T\boldsymbol{H}\boldsymbol{H}^{-1}\boldsymbol{E}(\boldsymbol{r}) - 2\boldsymbol{C}^T\boldsymbol{H}^{-1}\boldsymbol{HE}(\boldsymbol{r}) + \boldsymbol{A}\Big)\Big)$$

$$= exp\Big(- \frac{1}{2}\Big((\boldsymbol{HE}(\boldsymbol{r}) - \boldsymbol{C})^T\boldsymbol{H}^{-1}(\boldsymbol{HE}(\boldsymbol{r}) - \boldsymbol{C}) - \boldsymbol{C}^T\boldsymbol{H}^{-1}\boldsymbol{C} + \boldsymbol{A}\Big)\Big)$$

$$= exp\Big(- \frac{1}{2}(\boldsymbol{A} - \boldsymbol{C}^T\boldsymbol{H}^{-1}\boldsymbol{C})\Big)exp\Big(- \frac{1}{2}(\boldsymbol{HE}(\boldsymbol{r}) - \boldsymbol{C})^T\boldsymbol{H}^{-1}(\boldsymbol{HE}(\boldsymbol{r}) - \boldsymbol{C})\Big).$$

We have that

$$= exp\Big(- \frac{1}{2}(\boldsymbol{A} - \boldsymbol{C}^T \boldsymbol{H}^{-1}\boldsymbol{C})\Big)$$

is part of the integrating constant. To find the posterior expected excess returns and variance-covariance matrix, I have to do one final operation:

$$\begin{aligned}
&exp\Big(- \frac{1}{2}(\boldsymbol{H}\boldsymbol{E}(\boldsymbol{r}) - \boldsymbol{C})^T \boldsymbol{H}^{-1}(\boldsymbol{H}\boldsymbol{E}(\boldsymbol{r}) - \boldsymbol{C})\Big)\\
&exp\Big(- \frac{1}{2}(\boldsymbol{H}\boldsymbol{E}(\boldsymbol{r}) - \boldsymbol{C})^T \boldsymbol{H}^{-1}\boldsymbol{H}\boldsymbol{H}^{-1}\boldsymbol{H}\boldsymbol{H}^{-1}(\boldsymbol{H}\boldsymbol{E}(\boldsymbol{r}) - \boldsymbol{C})\Big)\\
&exp\Big(- \frac{1}{2}(\boldsymbol{H}^{-1}\boldsymbol{H}\boldsymbol{E}(\boldsymbol{r}) - \boldsymbol{H}^{-1}\boldsymbol{C})^T \boldsymbol{H}\boldsymbol{H}^{-1}\boldsymbol{H}(\boldsymbol{H}^{-1}\boldsymbol{H}\boldsymbol{E}(\boldsymbol{r}) - \boldsymbol{H}^{-1}\boldsymbol{C})\Big)\\
&exp\Big(- \frac{1}{2}(\boldsymbol{E}(\boldsymbol{r}) - \boldsymbol{H}^{-1}\boldsymbol{C})^T \boldsymbol{H}(\boldsymbol{E}(\boldsymbol{r}) - \boldsymbol{H}^{-1}\boldsymbol{C})\Big).
\end{aligned}$$

This is the exponent of the multivariate normal distribution. The mean and variance is represented by $\boldsymbol{H}^{-1}\boldsymbol{C}$ and $\boldsymbol{H}^{-1}$ respectively, which is the posterior expected excess return and variance-covariance matrix of the B-L model:

$$\begin{aligned}
&\boldsymbol{E}(\boldsymbol{r}) - r_f = \boldsymbol{H}^{-1}\boldsymbol{C} = [(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}^T\boldsymbol{\Omega}^{-1}\boldsymbol{P}]^{-1}[(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} + \boldsymbol{P}^T\boldsymbol{\Omega}^{-1}\boldsymbol{K}]\\
&Var(r) = \boldsymbol{H}^{-1} = [(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}^T\boldsymbol{\Omega}^{-1}\boldsymbol{P}]^{-1}\ .
\end{aligned}$$

3.3 Implementation in R

3.3.1 Relative views

As explained earlier, an investor has relative views when he thinks of the expected return of one asset relative to another. To implement the relative views in R, I will use the BLCOP package. The creator of the BLCOP package, Francisco Gochez [12], has written a useful paper called *Notes on the BLCOP Package*, where he gives an introduction to the package. To illustrate how you can implement relative views in R by using the BLCOP package, I am going to use my portfolio of 5 assets, and construct random views. I will implement the following two random views to the respective assets on the first of March 2013:

- *View 1*: The investor expects DETNOR to outperform NAS by 5% in expected return with a confidence of 20%.
- *View 2*: The investor expects SUBC to outperform DETNOR by 4% in expected return with a confidence of 25%.

First I have to create the pick matrix($\boldsymbol{P}$). If we combine the views, the pick matrix will have the following form:

$$\boldsymbol{P} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

I am going to separate the views such that we get two pick-matrices. The pick-matrix of view 1 will be the first row of $\boldsymbol{P}$, and the pick matrix of view 2 will be the second row of $\boldsymbol{P}$. We need to create the pick-matrices in R to implement them in the BLCOP package. This is done by the following R commands:

```
#View 1:
pickMatrix<-matrix(c(1,-1,0,0,0),nrow=1,ncol=5)
pickMatrix
     [,1] [,2] [,3] [,4] [,5]
[1,]    1   -1    0    0    0

#View 2:
pickMatrix2<-matrix(c(-1,0,1,0,0),nrow=1,ncol=5)
pickMatrix2
     [,1] [,2] [,3] [,4] [,5]
[1,]   -1    0    1    0    0
```

The next step is to create the views in R such that the BLCOP package can use them in the B-L model. We will now have to tell R the expected return and confidence of our view:

```
- View 1:
> views<-BLViews(P=pickMatrix, q=0.05, confidences=0.2,
                   assetNames=colnames(obx1))
> views
1 : 1*DETNOR+-1*NAS=0.05  + eps. Confidence: 0.2
```

I have now implemented my first view in R, before I add my second view, I will look at the portfolio created by the current posterior expected excess return and variance-covariance matrix. To find the posterior expected excess return and variance-covariance matrix, I need a market benchmark. Because I have a portfolio of Norwegian stocks at hand, I use the OSEBX index as a benchmark. By implementing the historical prices for my portfolio and the OSEBX index, the views and τ in the BLCOP package, the BLPosterior() function will create the posterior expected excess returns and variance-covariance matrix needed as input in the Markowitz model. It will later on be interesting to see how the second view will change the optimal portfolio. Black and Litterman argued that τ should be close to zero, and often set the value to 0.025, I will do the same. I will also need the historical risk free returns to create the posterior expected excess returns and variance-covariance matrix, for this I will use the 10 year Norwegian T-bills.

```
> marketPosterior<-BLPosterior(as.matrix(monthobxret1),
           views,tau=0.025,marketIndex=as.matrix(monthlyOSEBXrets1)
                   ,riskFree=as.matrix(monthobligrate))
> marketPosterior
Prior means:
     DETNOR          NAS         SUBC          TEL          TGS
-0.01028499  0.04220088  0.00814294  0.01302360  0.02080217
Posterior means:
      DETNOR           NAS          SUBC           TEL           TGS
-0.010284734  0.042199824  0.008142725  0.013023507  0.020802273
Posterior covariance:
             DETO          NAS          SUBC          TEL          TGS
DETO 0.0056065199 0.0005203223 0.003905025 0.002752687 0.0027401353
NAS  0.0005203223 0.0216844611 0.008201606 0.004523309 0.0005997117
SUBC 0.0039050246 0.0082016063 0.009358051 0.005001387 0.0026262989
TEL  0.0027526867 0.0045233089 0.005001387 0.005033596 0.0035071181
TGS  0.0027401353 0.0005997117 0.002626299 0.003507118 0.0051040303
```

While testing the BLCOP package, I found that the easiest way of implementing the historical values, was to implement them as returns, rather than historical prices in R. It is also important to remember that all the historical data need to have the exact same period of time for the package to work. After we have found the posterior expected excess returns, the results can be used by the fPortfolio package. The following function from fPortfolio calculates the optimal portfolio given a ”long only” constraint. This means that the investor is unable to short stocks.

```
> optimalPortfolios.fPort(marketPosterior,optimizer="tangencyPortfolio")
$priorOptimPortfolio
```

```
Title:
 MV Tangency Portfolio
 Estimator:         .priorEstim
 Solver:            solveRquadprog
 Optimize:          minRisk
 Constraints:       LongOnly

Portfolio Weights:
DETNOR    NAS   SUBC    TEL    TGS
0.0000 0.3229 0.0000 0.0000 0.6771

Covariance Risk Budgets:
DETNOR    NAS   SUBC    TEL    TGS


Target Return and Risks:
  mean     mu    Cov  Sigma   CVaR    VaR
0.0000 0.0277        0.0689 0.0000 0.0000

Description:
 Wed May 29 21:18:31 2013 by user: Christopher

$posteriorOptimPortfolio

Title:
 MV Tangency Portfolio
 Estimator:         .posteriorEstim
 Solver:            solveRquadprog
 Optimize:          minRisk
 Constraints:       LongOnly

Portfolio Weights:
DETNOR    NAS   SUBC    TEL    TGS
0.0000 0.3228 0.0000 0.0000 0.6772

Covariance Risk Budgets:
DETNOR    NAS   SUBC    TEL    TGS


Target Return and Risks:
  mean     mu    Cov  Sigma   CVaR    VaR
```

```
0.0000 0.0277       0.0697 0.0000 0.0000

Description:
 Wed May 29 21:18:31 2013 by user: Christopher

attr(,"class")
[1] "BLOptimPortfolios"
```

This portfolio is subject to the view that DETNOR outperforms NAS with 5%. The posterior means show that the NAS and TEL shares are the one with the lowest expected return, which makes the mean variance optimized portfolio intuitive, by only investing in the three stocks that has a considerable expected return. Now, lets see what happens when we implement the last view to our portfolio. After creating the pick-matrix, I can add views by using the addBLviews() function in BLCOP.

```
> finViews<-matrix(ncol=5,nrow=1,dimnames=list(NULL,colnames(obxreturns)))
> finViews[,1:5]<-c(-1,0,1,0,0)
> finViews
     DETNOR NAS SUBC TEL TGS
[1,]     -1   0    1   0   0
> views<-addBLViews(finViews,0.04,0.25,views)
> views
1 : 1*DETNOR+-1*NAS=0.05  + eps. Confidence: 0.2
2 : -1*DETNOR+1*SUBC=0.04  + eps. Confidence: 0.25
```

The next step is to create the new posterior expected excess returns and variance-covariance matrix.

```
> marketPosterior<-BLPosterior(as.matrix(monthobxret1),
            views,tau=0.025,marketIndex=as.matrix(monthlyOSEBXrets1)
                  ,riskFree=as.matrix(monthobligrate))
> marketPosterior
Prior means:
     DETNOR         NAS        SUBC         TEL         TGS
-0.01028499  0.04220088  0.00814294  0.01302360  0.02080217
Posterior means:
     DETNOR         NAS        SUBC         TEL         TGS
-0.01028267  0.04219131  0.00814151  0.01302301  0.02080322
Posterior covariance:
             DETO           NAS         SUBC          TEL           TGS
```

```
DETO 0.0056065167 0.0005203358 0.003905028 0.002752688 0.0027401341
NAS  0.0005203358 0.0216844044 0.008201590 0.004523302 0.0005997167
SUBC 0.0039050283 0.0082015903 0.009358045 0.005001384 0.0026263000
TEL  0.0027526882 0.0045233023 0.005001384 0.005033595 0.0035071186
TGS  0.0027401341 0.0005997167 0.002626300 0.003507119 0.0051040298
```

Then the posterior expected excess returns are implemented in the fPortfolio package where we find the mean variance optimized portfolio.

```
> optimalPortfolios.fPort(marketPosterior,optimizer="tangencyPortfolio")
$priorOptimPortfolio

Title:
 MV Tangency Portfolio
 Estimator:         .priorEstim
 Solver:            solveRquadprog
 Optimize:          minRisk
 Constraints:       LongOnly

Portfolio Weights:
DETNOR    NAS   SUBC    TEL    TGS
0.0000 0.3229 0.0000 0.0000 0.6771

Covariance Risk Budgets:
DETNOR    NAS   SUBC    TEL    TGS


Target Return and Risks:
  mean     mu    Cov  Sigma   CVaR    VaR
0.0000 0.0277        0.0689 0.0000 0.0000

Description:
 Wed May 29 21:22:58 2013 by user: Christopher

$posteriorOptimPortfolio

Title:
 MV Tangency Portfolio
 Estimator:         .posteriorEstim
 Solver:            solveRquadprog
 Optimize:          minRisk
 Constraints:       LongOnly
```

```
Portfolio Weights:
DETNOR    NAS   SUBC    TEL    TGS
0.0000 0.3228 0.0000 0.0000 0.6772

Covariance Risk Budgets:
DETNOR    NAS   SUBC    TEL    TGS


Target Return and Risks:
  mean     mu    Cov  Sigma   CVaR    VaR
0.0000 0.0277        0.0697 0.0000 0.0000

Description:
 Wed May 29 21:22:58 2013 by user: Christopher

attr(,"class")
[1] "BLOptimPortfolios"
```

Notice that the changes in the view is almost non-existent. To find the reason for this, we need to look at the variance-covariance matrix of the historical returns:

```
> var(monthobxret1)
             DETO          NAS         SUBC          TEL          TGS
DETO 0.0054697758 0.0005076303 0.003809780 0.002685548 0.0026733028
NAS  0.0005076303 0.0211555770 0.008001568 0.004412985 0.0005850841
SUBC 0.0038097798 0.0080015682 0.009129806 0.004879402 0.0025622428
TEL  0.0026855479 0.0044129847 0.004879402 0.004910825 0.0034215786
TGS  0.0026733028 0.0005850841 0.002562243 0.003421579 0.0049795418
```

The variance-covariance matrix reveals that there is a high certainty of the equilibrium expected excess returns. Recall the B-L model

$$\boldsymbol{E}(\boldsymbol{r}) - r_f = [(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}^T\boldsymbol{\Omega}^{-1}\boldsymbol{P}]^{-1}[(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} + \boldsymbol{P}^T\boldsymbol{\Omega}^{-1}\boldsymbol{K}]$$

$$Var(r) = [(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}^T\boldsymbol{\Omega}^{-1}\boldsymbol{P}]^{-1}\ \textbf{.}$$

A low variances on the equilibrium expected excess returns will lead to a larger weight on the equilibrium portfolio proportionate to the investor's views, and a change in the investor's views will have less impact on the optimal portfolio.

3.3.2 Scorecards

Scorecards are used as a tool by investors to get an overview of their possible investments and the qualities of these investments. The different investments get scores after how well they are expected to perform in the future. You could compare the scorecard to the results of a sports game, where the athlete with the best performance gets place number 1, and the athlete with the next best performance place number 2 on the results lists. The scorecard would be the ”result list”, except here the athletes (assets) are evaluated on their expected future performance. The assets ”place” on the scorecards would normally not be the only information the investor implements in the scorecard, the scorecards normally also containt information on the assets operational performance, momentum and valuation. I will now look at how to implement the views of the scorecards into the B-L model.

I will assess two different types of scorecards:

- Scorecards with only the investors expected return.
- Scorecards with both the investors expected return and the certainty of these views.

I begin with the appliance of the B-L model to the scorecards without uncertainty-estimates. I will here assume that the investor choose to calculate the uncertainty estimates with the function

$$\boldsymbol{\Omega} = \tau \boldsymbol{P}^T \boldsymbol{\Sigma} \boldsymbol{P} \, .$$

Assuming we have a 5 asset portfolio, the scorecard will give us one absolute view on each of the portfolio assets. This means that the link matrix will be the following:

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} .$$

This gives us two simplifying conditions for the B-L model:

1. $\boldsymbol{P} = \boldsymbol{I}$.

2. $\boldsymbol{\Omega} = \tau \boldsymbol{P}^T \boldsymbol{\Sigma} \boldsymbol{P}$.

The next step is to implement the simplifying conditions to the B-L model:

$$E(r) - r_f = [(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}^T \boldsymbol{\Omega}^{-1} \boldsymbol{P}]^{-1} [(\tau\boldsymbol{\Sigma})^{-1} \boldsymbol{\pi} + \boldsymbol{P}^T \boldsymbol{\Omega}^{-1} \boldsymbol{K}] \; .$$

Substituting $\boldsymbol{\Omega} = \tau \boldsymbol{P}^T \boldsymbol{\Sigma} \boldsymbol{P}$:

$$E(r) - r_f = [(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}^T (\tau \boldsymbol{P}^T \boldsymbol{\Sigma} \boldsymbol{P})^{-1} \boldsymbol{P}]^{-1} [(\tau\boldsymbol{\Sigma})^{-1} \boldsymbol{\pi} + \boldsymbol{P}^T (\tau \boldsymbol{P}^T \boldsymbol{\Sigma} \boldsymbol{P})^{-1} \boldsymbol{K}] \; .$$

Given the fact that $\boldsymbol{P}$ is the identity matrix, we get

$$\begin{aligned}
E(r) - r_f &= [(\tau\boldsymbol{\Sigma})^{-1} + (\tau\boldsymbol{\Sigma})^{-1}]^{-1} [(\tau\boldsymbol{\Sigma})^{-1} \boldsymbol{\pi} + (\tau\boldsymbol{\Sigma})^{-1} \boldsymbol{K}] \\
E(r) - r_f &= [(2(\tau\boldsymbol{\Sigma}))^{-1}]^{-1} [(\tau\boldsymbol{\Sigma})^{-1} \boldsymbol{\pi} + (\tau\boldsymbol{\Sigma})^{-1} \boldsymbol{K}] \\
E(r) - r_f &= [(2(\tau\boldsymbol{\Sigma}))^{-1}]^{-1} (\tau\boldsymbol{\Sigma})^{-1} [\boldsymbol{\pi} + \boldsymbol{K}] \\
E(r) - r_f &= \frac{1}{2} [\boldsymbol{\pi} + \boldsymbol{K}] \; .
\end{aligned}$$

Rewriting the posterior variance-covariance matrix:

$$\begin{aligned}
Var(r) &= [(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}^T \boldsymbol{\Omega}^{-1} \boldsymbol{P}]^{-1} \\
Var(r) &= [(\tau\boldsymbol{\Sigma})^{-1} + (\tau\boldsymbol{\Sigma})^{-1}]^{-1} \\
Var(r) &= 2\tau\boldsymbol{\Sigma} \; .
\end{aligned}$$

We here see that the appliance of scorecards with only the expected returns, leaves the B-L model as the arithmetic mean between the investors expected returns and the equilibrium expected excess returns.

In the case where $\boldsymbol{\Omega}$ is given by the investor, we get the following rewriting of the Black-Litterman model:

$$E(r) - r_f = [(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}^T\boldsymbol{\Omega}^{-1}\boldsymbol{P}]^{-1}[(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} + \boldsymbol{P}^T\boldsymbol{\Omega}^{-1}\boldsymbol{K}]$$
$$E(r) - r_f = [(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{\Omega}^{-1}]^{-1}[(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} + \boldsymbol{\Omega}^{-1}\boldsymbol{K}]$$

and

$$Var(r) = [(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{\Omega}^{-1}]^{-1} .$$

The only difference from the original B-L model is that, due to the fact that $\boldsymbol{P}$ is the identity matrix, it can be removed from the equation. We here see that the scorecards leave the Black-Litterman model with one or more simplifying conditions. Scorecards commonly contain a large number of investment opportunities, but because it consists of absolute views, it is possible to implement the views in an efficient manner. In such a matter, the BLCOP package proves insufficient. It is possible to use the BLCOP model, but it would require a great amount of time. This is why I have created my own functions in R that takes advantage of the simplifying conditions, and thus makes the implementation of the scorecard much less time consuming. I have made two functions, one for each of the mentioned scorecard types. An advantage of these functions compared to the BLCOP package is that they don't require an input pick matrix. The function for the appliance of scorecards with or without views on the certainty are respectively (the code can be found in Appendix D):

- BLvar()
- BL()

I will now represent two different scorecards, one with certainty estimates, and one without certainty estimates on the views. I will by using the R functions BL() and BLvar() find the optimal portfolio subject to these scorecards. Have in mind that the scorecards are simplifications of scorecards used in the real world, as they are far more complex, and contain much more information regarding each of the assets. My goal with this illustration is not to make the portfolio with the highest return and lowest volatility, but to illustrate how the scorecards could work in a portfolio optimization setting. I will therefore use random numbers in the estimation of both the expected return and the certainty estimates of the expected returns.

This is an example of a scorecard without certainty estimates on the views

Scorecard 1			
	Market capitalization	Expected return	Score
DETNOR	12586,27	0,27449	4,9
NAS	7964,22	0,414532	4,8
SUBC	47738,43	0,471416	4,6
TEL	193121,4	0,118059	4
TGS	22335	0,019781	3,5

This is an example of a scorecard with certainty estimates on the views

Scorecard 2				
	Market capitalization	Expected return	Certanties of the expected returns	Score
DETNOR	12586,27	0,274490353	0,249376245	4,9
NAS	7964,22	0,414531649	0,289580421	4,8
SUBC	47738,43	0,471415579	0,387926034	4,6
TEL	193121,4	0,118058596	0,307287056	4
TGS	22335	0,019781038	0,389492562	3,5

To use scorecard 1 to find the optimal portfolio, I have to use the BL() function. Keep in mind that I also need the variance-covariance matrix of the portfolio, the expected market return, and the variance of the expected market return as input in the B-L model. To calculate these inputs I use historical data of the OSEBX and the different shares from the first of March 2013 and one year back in time.

```
> postReturns<-BL()$posteriorReturns
```

```
> postSigma<-BL()$posteriorSigma
> postReturns
             [,1]
DETNOR 0.22376782
NAS    0.28649261
SUBC   0.33021222
TEL    0.14364289
TGS    0.09875831
> postSigma
            DETNOR         NAS        SUBC         TEL         TGS
DETNOR 0.006460036 0.002579805 0.003074969 0.001311016 0.003268846
NAS    0.002579805 0.008379257 0.002245502 0.001149242 0.002522928
SUBC   0.003074969 0.002245502 0.005146781 0.001485726 0.003772404
TEL    0.001311016 0.001149242 0.001485726 0.002211149 0.001238158
TGS    0.003268846 0.002522928 0.003772404 0.001238158 0.005524105
```

To create the tangency portfolio subject to the posterior expected excess returns and variance-covariance matrix, I use the tangency.portfolio() function of the portfolior package:

```
> tangency.portfolio()
Call:
tangency.portfolio()

Portfolio expected return:     0.5264594
Portfolio standard deviation:  0.08946058
Portfolio weights:
[1]  0.1704  0.3596  1.4382  0.1341 -1.1022
```

We see that the unconstrained portfolio produces extreme positions. There is a long position of 143.8% in SUBC and a short position of -110.2% in TGS. Both are extreme positions, the only way to get rid of these extreme positions would be to set constraints, such as an investment percentage constraint with the absolute value of 0.3. I will for now look at the no constraint portfolio to get a clearer image of what happens under the different scenarios. The next step is to look at the scorecard with certainty estimates on the expected returns. The posterior expected excess returns are now the product of a weighted average between the equilibrium expected excess returns, and the investors expected returns.

```
> postReturns<-BLvar()$posteriorReturns
> postSigma<-BLvar()$posteriorSigma
> postReturns
            [,1]
DETNOR 0.1551332
NAS    0.1478968
SUBC   0.1692492
TEL    0.1373691
TGS    0.1548497
> postSigma
            DETNOR         NAS        SUBC         TEL         TGS
DETNOR 0.014676918 0.005404878 0.006743488 0.002868699 0.007158613
NAS    0.005404878 0.018853406 0.004775647 0.002547644 0.005349036
SUBC   0.006743488 0.004775647 0.011976808 0.003366618 0.008560980
TEL    0.002868699 0.002547644 0.003366618 0.005345440 0.002741602
TGS    0.007158613 0.005349036 0.008560980 0.002741602 0.012777969
```

we see here that the posterior expected excess returns produces quite different results than previously, to get a better picture of what happens, we can take a look at the equilibrium expected excess returns:

```
> equilibrium<-BL()$eq
> equilibrium
             [,1]
DETNOR 0.09304529
NAS    0.07845357
SUBC   0.10900887
TEL    0.08922718
TGS    0.09773558
```

We know that without estimates on the certainties of the expected returns, the posterior expected excess returns are the arithmetic average between the investors expected return, and the equilibrium expected excess returns.

4 Testing the models

4.1 Model test

Earlier in the thesis, when I discussed Michauds article, there were revealed different reasons for why the Markowitz model may not be such a good model. Two of the reasons are the models tendency to produce ”unintuitive portfolios” and portfolios with ”high transaction costs”. According to Fischer Black and Robert Litterman, the Black-Litterman model will take care of these issues. I will therefore, when testing the model, focus on these two elements. When testing portfolio optimization tools, you could say that the most important test subject should be the portfolio return. But when using optimization tools, such as the Markowitz model and the B-L model, the return on the optimized portfolio is dependent on the investors ability to predict asset returns. One of the main reasons for using these models, are their ability to process large amounts of information in a short time, and it will thus be appropriate to test how well the models process information into optimized portfolios. The test will be done using scorecards without uncertainty estimates on the views. This means that the portfolio optimization will be done under the simplifying conditions of the scorecard as represented earlier in the thesis:

$$E(r) - r_f = \frac{1}{2}[\boldsymbol{\pi} + \boldsymbol{K}]$$

$$Var(r) = 2\tau\boldsymbol{\Sigma} \text{ .}$$

When testing the models ability to process information, I have to look at two aspects; the sudden changes in the portfolio input, and the over time changes in the portfolio input. I have chosen to test the two models against each other by making a set of four hypotheses, all in favour of the B-L model:

- *Hypothesis 1*: Changes in the portfolio over time will be more intuitive in the B-L model than in the Markowitz model.

- *Hypothesis 2*: There will incur less transaction costs in a portfolio over time with the use of the B-L model than the Markowitz model.
- *Hypothesis 3*: Sudden changes in the portfolio input parameters will produce more intuitive changes to the portfolio with the B-L model than with the Markowitz model.
- *Hypothesis 4*: There will incur less transaction costs in a portfolio with sudden changes in the input parameters with the use of the B-L model than with the Markowitz model.

To test these hypotheses I am going to look at the portfolio development of a five stock portfolio over a year with daily prices. The portfolio will be as represented under the data section. I will extract the expected returns from self-made scorecards consisting of random expected returns and the market capitalization value.

My scorecard for 01.03.2012

	Expected returns	Market capitalization	Score
TGS	0.294201858	16568.57	5
TEL	0.173969609	166769.6	4.6
SUBC	0.22233094	47245.84	4.5
NAS	0.075746237	2860.01	4.1
DENTOR	0.064300456	11640.34	4

The portfolio will have a monthly restructuring at the first of every month. To calculate the variance-covariance matrices I will use the historical prices of the portfolio assets.

Hypothesis 1

In an intuitive portfolio you can expect the change in portfolio weights to be positively correlated with the change in the expected returns. One way to test for intuition can therefore be to find the correlation between the change in the portfolio weights and expected returns. But by doing so, a problem occurs. Let's say we have a portfolio of two assets:

- Asset A.
- Asset B.

I use R_i as the previous expected returns, and r_i as notation of the present expected return. If we have that

$$R_A < r_A \ ,$$

$$R_B < r_B$$

and

$$r_A < r_B \ .$$

When calculating the new portfolio, the intuitive change to the portfolio would then be that w_A decreases as w_B increases. We here see one problem by using the correlation parameter directly; both the negative and the positive correlation could be intuitive answers. For the correlation parameter to be a useful tool for testing we need to somehow transform the data. If we assume that the weights are stationary parameters they can be normalized. The correlation between the normalized views and the weights will create a better parameter for testing the intuition of the portfolio. I normalize the view using the following method:

$$W_i = w_i - \bar{w} + \frac{1}{n}$$

where:

w_i = view number i.

W_i = the normalized view number i.

n = the number of views.

Since the hypothesis states that a portfolio created by the B-L model is more intuitive in the long run, it has to be rejected. The test parameters reveal that both the models create intuitive portfolios. As long as the portfolios average intuition parameter is higher than 0.4, the portfolio is regarded as intuitive.

The correlation between the normalized views and the weights in the Markowitz portfolio

DENTOR	NAS	SUBC	TEL	TGS
0,818826	0,729893	0,829435	0,724637	0,856544
The average correlation:				
0,791867				

The correlation between the normalized views and the weights in the B-L portfolio

DENTOR	NAS	SUBC	TEL	TGS
0,654316	0,416414	0,294953	0,552231	0,707178
The average correlation:				
0,525018				

Hypothesis 2

The large transaction costs was mentioned by Michaud and other as one of the reasons for why the Markowitz model did such a poor job as a optimizing tool in real life. To test for transaction costs, I set a standard price of one unit per change in the portfolio percentage weight. This leaves the percentage change in the portfolio weight the cost of keeping the portfolio.

The percentage changes in the portfolios are calculated by using the following formula:

$$c_{n,i} = w_{n,t-1} - w_{n,t}$$

$$C_n = \sum_{i=1}^{T} |c_{n,i}|$$

$$P_m = \sum_{n=1}^{N} C_n \,.$$

Where:

$w_{n,t-1}$ = the weight in period $t-1$ for asset n.

$w_{n,t}$ = the weight in period t for asset n.

$c_{n,i}$ = the i'th transaction cost between the weigt in time $t-1$ and t for asset n.

C_n = the total transaction cost for asset n.

P_m = the total portfolio transaction cost using method m.

My research revealed the following results from my test portfolio:

$$P_{Markowitz} = 35,1833$$

$$P_{Black-Litterman} = 43,9169 \text{.}$$

This means that the transaction costs occurring with the Black-Litterman model is higher than the Markowitz model. The hypothesis will then have to be rejected as it is not true that the B-L model produces portfolios with lower transaction costs than the Markowitz model.

Hypothesis 3

To test how sudden changes in the model input affect the optimized portfolio, I will give five sudden changes to each of the monthly portfolios. The changes will be the following:

- *Change 1*: DETNOR +5%.
- *Change 2*: NAS +5%.
- *Change 3*: SUBC -5%.
- *Change 4*: TEL -5%.
- *Change 5*: TGS -5%.

To measure the intuition I use the same test as under *hypothesis 1*. The test reveals the following intuition parameters: The test reveals one month where the Markowitz model produces an unintuitive portfolio, the first of September. The B-L model seems in this case to take care of the problem with unintuitive portfolio changes, which indicates that the hypothesis is true. However, the other monthly portfolio changes seem to reveal no further evidence for unintuitive portfolios, both models appear to produce intuitive portfolios. Since the average portfolios of both models are intuitive, you can expect the both models to produce intuitive portfolios, and the hypothesis is rejected.

The portfolio intuition parameter

Dates	The Markowitz model	The B-L model
01.03.2012	0,745232481	0,56264415
01.04.2012	0,810208658	0,742769953
01.05.2012	0,83899436	0,847451572
01.06.2012	0,846139213	0,844754385
01.07.2012	0,872455415	0,873392506
01.08.2012	0,80619912	0,833416488
01.09.2012	**0,289956993**	**0,869699291**
01.10.2012	0,546326374	0,851722539
01.11.2012	0,882596766	0,926512276
01.12.2012	0,90892217	0,940196195
01.01.2013	0,786484463	0,872584467
01.02.2013	0,786209903	0,86175033
01.03.2013	0,604018266	0,75030708
Average	0,747980322	0,829015479

Hypothesis 4

The portfolio transaction costs will be calculated as with the tests over time.

The transaction costs		
Dates	The Markowitz model	The B-L model
01.03.2012	2,9899	2,7967
01.04.2012	0,7902	0,8076
01.05.2012	3,384	3,9534
01.06.2012	0,8989	1,2636
01.07.2012	0,9001	0,9082
01.08.2012	1,3062	0,9981
01.09.2012	**5,9811**	**1,9599**
01.10.2012	**7,3445**	**0,7059**
01.11.2012	1,0342	0,555
01.12.2012	0,6696	0,4429
01.01.2013	2,119	0,9459
01.02.2013	0,7138	0,7742
01.03.2013	**12,9495**	**1,9625**
Total	**41,081**	18,0739

The bolded rows in the table represent the dates where the sudden changes results in a high transaction cost for the Markowitz portfolio. Common for all these dates are that the B-L model produces a portfolio with a considerably lower transaction cost. This leaves evidence that the B-L portfolio produces portfolios with lower transaction costs. Since the total transaction cost is over twice as high for the Markowitz model than the B-L model, this leaves no doubt that the B-L model produces portfolios with lower transaction costs than the Markowitz model. I do not reject the hypothesis, and we can assume the B-L model to produce portfolios with lower transaction costs than the Markowitz model.

4.2 Conclusion

As I have mentioned, the B-L model was made to take care of two problems concerning the use of the Markowitz model; high transaction costs and unintuitive portfolios. For the long run changes in my portfolio, I rejected both the hypothesis that the B-L model produced more intuitive portfolios, and portfolios with lower transaction costs than the Markowitz model. What one has to remember when interpreting these results, is that the test was done for a special case of the B-L model. It may actually be true that the scorecard assumptions to the B-L model limit some of the effects of the original B-L model. It's also important to remember that this is a portfolio with few assets, which makes it less likely for unintuitive changes in the portfolio weights to happen. Michaud mentioned that unintuitive changes to the portfolio occurs due to badness in the variance-covariance matrix. Badness in the variance-covariance matrix is due to insufficient or poor data. I used in my test data with daily changes in the asset prices; it is therefore not surprising that the Markowitz model delivers intuitive portfolios. My point here is that the tests I have done don't prove that the B-L model fails to improve the Markowitz model. I draw my conclusion that in the long run, under scorecard assumptions, with daily data; the B-L model doesn't appear to improve the Markowitz model.

The sudden changes in the portfolio however, reveal other results. With respect to the intuitive portfolio, the Markowitz portfolio show just small signs of problems, but not enough to draw any conclusions. It is however interesting to see that for the one case where the Markowitz portfolio appear unintuitive, the B-L portfolio appear intuitive. This raises the question; what if the B-L model manages to limit the problems with the Markowitz model when they appear? Before we draw any conclusions, we need to look at the transaction costs. My tests leaves no doubt that, in my case, and for sudden changes, the B-L model produces portfolios with lower transaction costs than the Markowitz model.

To answer my question, I can only make suggestions. To draw any certain conclusions on the B-L model, it would be necessary to test the model under various cases; with no scorecard assumptions, weekly data, monthly data, etc. My tests suggest that the B-L model manages to limit the problems of the Markowitz model, when they appear.

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5 Appendix A

5.1 Markowitz optimization problems

I use the method of lagrange multipliers to find maximum or minimum of the optimization problems.

Problem 1: Find the portfolio $\boldsymbol{w}$ that has the highest expected return for a given level of risk as measured by portfolio variance.

$$\underset{\boldsymbol{w}}{\text{Min}}\ \sigma_p^2 = \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w}$$

subject to

$$R_p = \boldsymbol{w}^T \boldsymbol{R}$$
$$\boldsymbol{w}^T \boldsymbol{1} = 1 \,.$$

Lagrangian:

$$\mathcal{L}(w, \lambda_1, \lambda_2) = \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w} + \lambda_1 (\boldsymbol{w}^T \boldsymbol{R} - R_p) + \lambda_2 (\boldsymbol{w}^T \boldsymbol{1} - 1) \,.$$

FOC:

$$\frac{\partial \mathcal{L}(\boldsymbol{w}, \lambda_1, \lambda_2)}{\partial \boldsymbol{w}} = 2\boldsymbol{\Sigma} \boldsymbol{w} + \lambda_1 \boldsymbol{R} + \lambda_2 \boldsymbol{1} = \boldsymbol{0}$$
$$\frac{\partial \mathcal{L}(\boldsymbol{w}, \lambda_1, \lambda_2)}{\partial \lambda_1} = \boldsymbol{w}^T \boldsymbol{R} - R_p = 0$$
$$\frac{\partial \mathcal{L}(\boldsymbol{w}, \lambda_1, \lambda_2)}{\partial \lambda_2} = \boldsymbol{w}^T \boldsymbol{1} - 1 = 0 \,.$$

Writing the FOCs in matrix form

$$\begin{pmatrix} 2\boldsymbol{\Sigma} & \boldsymbol{R} & \mathbf{1} \\ \boldsymbol{R}^T & 0 & 0 \\ \mathbf{1}^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{w} \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ R_p \\ 1 \end{pmatrix}$$

or

$$\boldsymbol{B}_w \boldsymbol{x}_w = \boldsymbol{a} \text{ ,}$$

where

$$\boldsymbol{B}_w = \begin{pmatrix} 2\boldsymbol{\Sigma} & \boldsymbol{R} & \mathbf{1} \\ \boldsymbol{R}^T & 0 & 0 \\ \mathbf{1}^T & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{x}_w = \begin{pmatrix} \boldsymbol{w} \\ \lambda_1 \\ \lambda_2 \end{pmatrix} \qquad \text{and} \qquad a = \begin{pmatrix} \mathbf{0} \\ R_p \\ 1 \end{pmatrix} .$$

Solve for $\boldsymbol{x}_w$ to find the optimal portfolio weights

$$\boldsymbol{x}_w = \boldsymbol{B}_w^{-1} \boldsymbol{a} \text{ .}$$

Problem 2: Find the portfolio $\boldsymbol{w}$ that has the lowest portfolio variance target to a expected return.

$$\operatorname*{Max}_{\boldsymbol{w}} R_p = \boldsymbol{w}^T \boldsymbol{R}$$

subject to

$$\sigma_p^2 = \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w}$$

$$\boldsymbol{w}^T \mathbf{1} = 1 \text{ .}$$

Because optimization problem 1 and 2 are dual problems, solving both gives the same optimal solution:

$$\boldsymbol{x}_w = \boldsymbol{B}_w^{-1} \boldsymbol{a} \text{ .}$$

Problem 3: Find the global minimum variance portfolio.

$$\underset{\boldsymbol{w}}{\text{Min}}\ \sigma_p^2 = \boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w}$$

subject to

$$\boldsymbol{w}^T\mathbf{1} = 1\ .$$

Lagrangian:

$$\mathcal{L}(\boldsymbol{w},\lambda) = \boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w} + \lambda(\boldsymbol{w}^T\mathbf{1} - 1)\ .$$

FOC:

$$\begin{aligned}
\frac{\partial\mathcal{L}(\boldsymbol{w},\lambda)}{\partial\boldsymbol{w}} &= \frac{\partial\boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w}}{\partial\boldsymbol{w}} + \frac{\partial}{\partial\boldsymbol{w}}\lambda(\boldsymbol{w}^T\mathbf{1} - 1) = 0\\
\frac{\partial\mathcal{L}(\boldsymbol{w},\lambda)}{\partial\boldsymbol{w}} &= 2\boldsymbol{\Sigma}\boldsymbol{w} + \lambda\mathbf{1} = \mathbf{0}\\
\frac{\partial\mathcal{L}(\boldsymbol{w},\lambda)}{\partial\lambda} &= \frac{\partial\boldsymbol{w}^T\boldsymbol{\Sigma}\boldsymbol{w}}{\partial\lambda} + \frac{\partial}{\partial\lambda}\lambda(\boldsymbol{w}^T\mathbf{1} - 1) = 0\\
\frac{\partial\mathcal{L}(\boldsymbol{w},\lambda)}{\partial\lambda} &= \boldsymbol{w}^T\mathbf{1} = 1\ .
\end{aligned}$$

Writing the FOC's in matrix form

$$\begin{pmatrix} 2\boldsymbol{\Sigma} & \mathbf{1} \\ \mathbf{1}^T & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{w} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}.$$

The FOCs are the linear system

$$\boldsymbol{A}_w\boldsymbol{z}_w = \boldsymbol{b}$$

where

$$\boldsymbol{A}_w = \begin{pmatrix} 2\boldsymbol{\Sigma} & \mathbf{1} \\ \mathbf{1}^T & 0 \end{pmatrix}, \qquad \boldsymbol{z}_w = \begin{pmatrix} \boldsymbol{w} \\ \lambda \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}.$$

Solve for $\boldsymbol{z}_w$ to find the optimal portfolio weights:

$$\boldsymbol{z}_w = \boldsymbol{A}_w^{-1}\boldsymbol{b}\ .$$

5.1.1 The tangency portfolio

The Capital Allocation Line(CAL):

$$r_p = r_f + \frac{r_T - r_f}{\sigma_T}\sigma_p \, .$$

Notation:

r_p = the portfolio return.

r_f = the risk free return.

r_T = the return of the tangency portfolio

σ_T = the volatility of the tangency portfolio.

σ_p = the portfolio variance.

Rewriting the CAL:

$$r_p = r_f + \frac{\sigma_p r_T - \sigma_p r_f}{\sigma_T}$$

$$r_p = r_f + \frac{\sigma_p r_T}{\sigma_T} - \frac{\sigma_p r_f}{\sigma_T}$$

$$r_p = (1 - \frac{\sigma_p}{\sigma_T})r_f + \frac{\sigma_p}{\sigma_T}r_T \, .$$

Finding the partial derrivative with respect to σ_p:

$$\frac{\partial r_p}{\partial \sigma_p} = \frac{r_T}{\sigma_T} - \frac{r_f}{\sigma_T} \, .$$

5.2 The Black Litterman model

5.2.1 Bayes Theorem

We have two possible events:

- A.
- B.

By using Bayes Law we can decompose the joint likelihood of event A and B

$$
\begin{aligned}
P(A|B) &= P(A|B)P(B) \\
&= P(B|A)P(A) \text{ ,}
\end{aligned}
$$

we get Bayes theorem

$$
P(A|B) = \frac{P(A|B)P(A)}{P(B)} \text{ .}
$$

6 Appendix B

6.1 R code assumptions

The self-made R code

- The risk free rate is constant for each investment period

6.2 Model assumptions

Assumptions of the Markowitz model

- The expected returns are normally distributed.

- The market consist of rational investors.
- Investors are risk averse.
- Increased expected return is regarded as positive.
- There is a risk-return tradeoff.
- There is absence of arbitrage.
- Markets are efficient, which means all information are available and markets adjust accordingly.

The Black-Litterman model assumptions

- Investors have their own views they believe can lead to a better portfolio.
- Given the fact that the investor makes changes to the market portfolio, one have to assume that the market isn't totally efficient.
- Each of the views has its own risk expectation, either from the investor or the formula $\boldsymbol{\Omega} = \tau \boldsymbol{P}^T \boldsymbol{\Sigma} \boldsymbol{P}$.
- The expected excess returns are unobservable.
- The probability distribution of the expected excess return can be written as a product of two multivariate normal distributions.

7 Appendix C

7.1 Test data

For the testing of the Markowitz model and the Black-Litterman model, the following data was needed:

- The market capitalization weights for the assets at the first of each month between 01.03.2012 and 01.03.2013.
- The investors views at the first of each month between 01.03.2012 and 01.03.2013.
- The daily price development of the portfolio assets between 01.03.2011 and 01.03.2013.
- The daily prices from the OSEBX index between 01.03.2011 and 01.03.2013.

The investors views

The invesors views

Date	DETNOR	NAS	SUBC	TEL	TGS
01.03.2012	0,0643	0,075746	0,222331	0,174202	0,294202
01.04.2012	0,033763	0,438211	0,435143	0,3568	0,135733
01.05.2012	0,143603	0,126774	0,2156	0,14389	0,228039
01.06.2012	0,254689	0,456215	0,406547	0,365422	0,479592
01.07.2012	0,465336	0,369096	0,437654	0,33587	0,343094
01.08.2012	0,450289	0,303307	0,458879	0,271445	0,163289
01.09.2012	0,160811	0,478413	0,076198	0,019924	0,09659
01.10.2012	0,107101	0,305369	0,330923	0,075028	0,393956
01.11.2012	0,359436	0,107729	0,47799	0,433096	0,092359
01.12.2012	0,027573	0,421024	0,473115	0,495174	0,485084
01.01.2013	0,452109	0,427613	0,429491	0,259121	0,072531
01.02.2013	0,472171	0,41934	0,417846	0,371036	0,275983
01.03.2013	0,033763	0,438211	0,435143	0,3568	0,135733

The market capitalization values

The asset capitalization value

Date	DETNOR	NAS	SUBC	TEL	TGS
01.03.2012	12586,27	7964,22	47738,43	193121,4	22335
01.04.2012	12213,39	6009,2	47140,37	190469,4	21451,68
01.05.2012	11608,35	5059,83	46471,95	175026	18772,8
01.06.2012	10297,22	4905,11	45873,89	180485,8	18431,48
01.07.2012	11224,61	4290,02	44396,38	175805,9	20022,17
01.08.2012	12292,71	3749,41	47738,43	176897,9	19815,44
01.09.2012	11908,96	3906,36	47738,43	165822,3	17810,12
01.10.2012	10425,13	3819,16	44642,63	160674,4	18305,53
01.11.2012	9977,43	3888,92	42567,06	160095,5	16763,68
01.12.2012	9561,7	3060,56	41054,31	141762,2	15272,84
01.01.2013	10521,07	3374,47	52170,95	169181,9	17114,69
01.02.2013	11288,57	3906,36	54176,19	171111,7	16514,54
01.03.2013	11640,34	2860,01	47245,84	166769,6	16568,57

8 Appendix D

8.1 Guide to use the self-made R functions

To use the BL() and BLvar() functions, you need to the following R packages:

- ”timeDate”.
- ”timeSeries”.
- ”OpenMx”.

To install the packages, copy and paste the following R code in to the console:

```
install.packages("timeDate")
install.packages("timeSeries")
source('http://openmx.psyc.virginia.edu/getOpenMx.R')
```

To activate the pacakges, use the following commands:

```
library("OpenMx", lib.loc="location")
library("timeDate", lib.loc="location")
library("timeSeries", lib.loc="location")
```

For the code to work, the ”location” of the ”library” folder has to be entered into the code. Beware that the OpenMx package only works with a 32-bit version of R. These packages have to be active before the code is copied into the console and activated.

Data import

First of all; the data need to be on a one year basis ranging from old to new prices for the functions to work propperly. Also; to use the functions, the data imported to R need to have a certain form and format. A practical solution of importing data to R, is to produce the data in Excel and save it as cvs files. To import the csv files the following function can be used:

```
mktc(market capitalization values) <- read.csv("file path",
                                                sep=";", dec=",")
```

The various data need to be imported separately into R in the following format (Note: The self-made functions will take care of conversions to the matrix format):

- ***The investor views :*** the investors views need to be on a row vector form.
- ***The certainty estimates in the investors views :*** The certainties on the investors views needs to be imported on row vector form.
- ***The asset capitalization values :*** The market capitalization values need to be imported on row vector form.
- ***The market index price development :*** The market index needs to be imported on column vector form ranging from old to new prices.
- ***The portfolio asset price development :*** The portfolio asset price development needs to be imported in a n by m matrix $\boldsymbol{A}_{i,j}$ form, start where

  i = the number of working days during a year

  j = the number of portfolio assets.

  Other model inputs:

- ***The risk free rate of return :*** the risk free rate of return is implemented as a constant.
- ***τ:*** A scalar, usually set between 0 and 1.

Examples of data in Excel:

	A	B	C	D	E	F	G
1	DETNOR	NAS	SUBC	TEL	TGS		
2	11640,34	2860,01	47245,84	166769,6	16568,57		
3							
4							
5							
6							
7							
8							
9							
10							
11							

The capitalization values

	A	B	C	D	E	F	G
1	DETNOR	NAS	SUBC	TEL	TGS		
2	0,0143	0,075746	0,222331	0,17397	0,294202		
3							
4							
5							
6							
7							
8							
9							
10							
11							

The investors views

	A	B	C	D	E	F	G
1	DETNOR	NAS	SUBC	TEL	TGS		
2	27,7	113	139,1	91,75	145,4		
3	27	110,5	137,18	91,25	145,2		
4	27,1	112	137,56	91,45	145,3		
5	27,5	113	140,54	91,9	147,7		
6	27,3	109	139,87	91,1	144,6		
7	27	113	139,58	92,8	146,2		
8	27	116,5	135,54	93	145,5		
9	27,3	115,5	132,65	92,35	139		
10	27	111,5	128,51	90,95	139,6		
11	26	109	130,15	90,25	136,8		

The portfolio asset price development

	A	B	C	D	E	F	G
1	444,23						
2	442,39						
3	445,91			Old prices			
4	448,58						
5	446,33						
6	447,09						
7	444,26						
8	435,72			New prices			
9	431,91						
10	428,1						
11	420,23						

The market index

BL()

To use the BL() function, you need the following input:

- ***The investors views***
- ***The asset capitalization values***
- ***The market index price development***
- ***The portfolio asset price development***
- ***The risk free rate of return***
- τ

The BL() function with arguments(inputs):

```
BL(x=the market index, y=the risk free rate,
   z=the asset price development, v=the market capitalization value,
   q=the investors views, tau=a scalar)
```

The function can be used to find:

  - The posterior expected returns = BL()$posteriorReturns.
  - The posterior sigma = BL()$posteriorSigma.
  - The equilibrium expected excess returns = BL()$eq.
  - The equilibrium weights = BL()$EQw.

BLvar()

To use the BLvar() function you need one additional model input:

- ***The certainty estimates of the investors views***

The BLvar() function with arguments:

```
BLvar(x=The market index,y=the risk free rate,
      z=the asset price development, v=the market capitalization value,
      q=the investors views,o=the investors certainty estimates,
      tau=a scalar)
```

The function can be used to find:

- The posterior expected returns = BLvar()$posteriorReturns.
- The posterior sigma = BLvar()$posteriorSigma.

Mark()

I used this function to test the Markowitz model. It's a simple function, but it converts the data to a format such that it can be used by the portfolior functions. To use the Mark() function you need the following input:

- ***The investors views***
- ***The portfolio asset price development***

The Mark() function with arguments:

```
Mark(x=the asset price development, y=the investors views)
```

The function can be used to find the following:

- The expected portfolio returns = Mark()$views
- The variance-covariance matrix = Mark()$CM
- The historical returns = Mark()$ER

8.2 R Code

To use the code, copy and paste it into an R script and activate by pressing ”control-shift-enter”. To avoid unwanted numbers in the code, I have removed the page numbering from here on.

```
#############BLvar() function###########

#x<-the historical market returns
#y<-the risk free rate
#z<-the matrix of the different portfolio asset price time seires.
#v<-the different market capitalization values
#q<-the investors views in the form as a row vector.
#tau<-a scalar
#o<- this is the investors uncertainty about the views.

BLvar<-function(x,y,z,v,q,o,tau){

#####Finding the equilibrium returns and the variance/covariance matrix


    erm<-((sum(returns(as.timeSeries(x),method="simple"))/
             length(as.matrix(x)))
          *length(as.matrix(x)))
    #The returns() function is part of the timeSeries package
    A<-(erm-y)/(var(returns(as.timeSeries(x)))*length(as.matrix(x)))

    mktw<-(1/sum(v))*v

    pi<-(as.numeric(A)*cov(returns(as.timeSeries(z),method="simple"))
        %*%t(as.matrix(mktw))
        *length(as.matrix(x)))
  sigma <- cov(returns(as.timeSeries(z)))*length(as.matrix(z))

  ##Where pi is the vector of implied equilibrium excess returns
  #and sigma is the variance/covariance matrix


#####Creating the view matrix, the omega matrix and the link matrix:

  omega <- vec2diag(as.matrix(o))
  ##The package OpenMX is needed. OpenMx only run
  # on a 32-bit version of R.
  #To download the package, you need to copy and paste
```

```
  #the following link into the console:
  #source('http://openmx.psyc.virginia.edu/getOpenMx.R')


  K <- as.matrix(q)


  ##Where omega is the unceratinty matrix of the views,
  #K is the vector of investors views and P is the  link matrix.

#####Finding the posterior returns

  rpost <- (solve((solve(tau*sigma))+solve(omega))%*%
    (solve(tau*sigma)%*%pi+solve(omega)%*%t(K))+y)
  ##Because I work with returns in the markowitz framework,
  #and not excess returns, I need to add the risk free
  #rate of return to the posterior excess returns.

  sigmapost <- (solve((solve(tau*sigma))+solve(omega)))

  list(posteriorReturns=rpost,posteriorSigma=sigmapost
       ,sigma=sigma)
}
################################################################

#############BL() function###############

#Finding the equilibrium returns
#x<-the historical market returns
#y<-the risk free rate
#z<-the matrix of the different portfolio asset price time seires.
#v<-the different market capitalization values
#q<-the investors views in the form of a row vector.
#tau<-a scalar


BL<-function(x,y,z,v,q,tau){

####Finding the equilibrium returns

    erm<-((sum(returns(as.timeSeries(x),method="simple"))))
    #The returns() function is part of the timeSeries package
```

```
    A<-(erm-y)/(var(returns(as.timeSeries(x)))*length(as.matrix(x)))

    mktw<-(1/sum(v))*v

    R<-(as.numeric(A)*cov(returns(as.timeSeries(z),method="simple"))
        %*%t(as.matrix(mktw))
        *length(as.matrix(x)))
  ##Where R is the vector of implied equilibrium excess returns and
  #sigma is the variance-covariance matrix

####Finding the posterior returns
  postR<-0.5*(R+t(as.matrix(q)))+y

####Finding the posterior sigma
  postSigma<-2*tau*cov(returns(as.timeSeries(z),
                              method="simple"))*length(as.matrix(x))

  list(posteriorReturns=postR,posteriorSigma=postSigma,eq=R,EQw=mktw)

}

#####################################################################

##########Mark()###########

Mark<-function(x,y){

  returns <- returns(as.timeSeries(x),method="simple")

  er <- colSums(returns)
  cov <- cov(returns)*252

mktv<-as.matrix(y)

list(ER=er,CM=cov,views=mktv)

}

####################################################################
#Functions for portfolio analysis
# to be used in Introduction to Computational Finance
# & Financial Econometrics
```

```
# last updated: August 8, 2012 by Hezky Varon
#               November 7, 2000 by Eric Zivot
#               Oct 15, 2003 by Tim Hesterberg
#               November 18, 2003 by Eric Zivot
#               November 9, 2004 by Eric Zivot
#               November 9, 2008 by Eric Zivot
#               August 11, 2011 by Eric Zivot
#
# Functions:
# 1. efficient.portfolio compute minimum variance portfolio
#         subject to target return
# 2. globalMin.portfolio compute global minimum variance portfolio
# 3. tangency.portfolio   compute tangency portfolio
# 4. efficient.frontier  compute Markowitz bullet
# 5. getPortfolio     create portfolio object

stopifnot("package:quadprog" %in% search()  ||  require("quadprog",
                                                               quietly = TRUE) )

getPortfolio <-
  function(er, cov.mat, weights)
  {
    # contruct portfolio object
    #
    # inputs:
    # er    N x 1 vector of expected returns
    # cov.mat    N x N covariance matrix of returns
    # weights  N x 1 vector of portfolio weights
    #
    # output is portfolio object with the following elements
    # call original function call
    # er   portfolio expected return
    # sd   portfolio standard deviation
    # weights N x 1 vector of portfolio weights
    #
    call <- match.call()

    #
    # check for valid inputs
    #
    asset.names <- names(er)
    weights <- as.vector(weights)
```

```
    names(weights) = names(er)
    er <- as.vector(er) # assign names if none exist
    if(length(er) != length(weights))
      stop("dimensions of er and weights do not match")
    cov.mat <- as.matrix(cov.mat)
    if(length(er) != nrow(cov.mat))
      stop("dimensions of er and cov.mat do not match")
    if(any(diag(chol(cov.mat)) <= 0))
      stop("Covariance matrix not positive definite")

    #
    # create portfolio
    #
    er.port <- crossprod(er,weights)
    sd.port <- sqrt(weights %*% cov.mat %*% weights)
    ans <- list("call" = call,
                "er" = as.vector(er.port),
                "sd" = as.vector(sd.port),
                "weights" = weights)
    class(ans) <- "portfolio"
    ans
  }

efficient.portfolio <-
  function(er, cov.mat, target.return=0.1,shorts=T)
  {
    # compute minimum variance portfolio subject to target return
    #
    # inputs:
    # er     N x 1 vector of expected returns
    # cov.mat     N x N covariance matrix of returns
    # target.return   scalar, target expected return
    # shorts          logical, allow shorts is TRUE
    #
    # output is portfolio object with the following elements
    # call     original function call
    # er     portfolio expected return
    # sd     portfolio standard deviation
    # weights     N x 1 vector of portfolio weights
    #
    call <- match.call()
```

```
#
# check for valid inputs
#
asset.names <- names(er)
er <- as.vector(er) # assign names if none exist
N <- length(er)
cov.mat <- as.matrix(cov.mat)
if(N != nrow(cov.mat))
  stop("invalid inputs")
if(any(diag(chol(cov.mat)) <= 0))
  stop("Covariance matrix not positive definite")
  # remark: could use generalized inverse if cov.mat
  # is positive semidefinite.

#
# compute efficient portfolio
#
if(shorts==TRUE){
  ones <- rep(1, N)
  top <- cbind(2*cov.mat, er, ones)
  bot <- cbind(rbind(er, ones), matrix(0,2,2))
  A <- rbind(top, bot)
  b.target <- as.matrix(c(rep(0, N), target.return, 1))
  x <- solve(A, b.target)
  w <- x[1:N]
} else if(shorts==FALSE){
  Dmat <- 2*cov.mat
  dvec <- rep.int(0, N)
  Amat <- cbind(rep(1,N), er, diag(1,N))
  bvec <- c(1, target.return, rep(0,N))
  result <- solve.QP(Dmat=Dmat,dvec=dvec,Amat=Amat,bvec=bvec,meq=2)
  w <- round(result$solution, 6)
} else {
  stop("shorts needs to be logical. For no-shorts, shorts=FALSE.")
}

#
# compute portfolio expected returns and variance
#
names(w) <- asset.names
er.port <- crossprod(er,w)
sd.port <- sqrt(w %*% cov.mat %*% w)
```

```
    ans <- list("call" = call,
                "er" = as.vector(er.port),
                "sd" = as.vector(sd.port),
                "weights" = w)
    class(ans) <- "portfolio"
    ans
  }

globalMin.portfolio <-
  function(er, cov.mat, shorts=TRUE)
  {
    # Compute global minimum variance portfolio
    #
    # inputs:
    # er N x 1 vector of expected returns
    # cov.mat N x N return covariance matrix
    # shorts          logical, allow shorts is TRUE
    #
    # output is portfolio object with the following elements
    # call original function call
    # er portfolio expected return
    # sd portfolio standard deviation
    # weights N x 1 vector of portfolio weights
    call <- match.call()

    #
    # check for valid inputs
    #
    asset.names <- names(er)
    er <- as.vector(er) # assign names if none exist
    cov.mat <- as.matrix(cov.mat)
    N <- length(er)
    if(N != nrow(cov.mat))
      stop("invalid inputs")
    if(any(diag(chol(cov.mat)) <= 0))
      stop("Covariance matrix not positive definite")
    # remark: could use generalized inverse if cov.mat is positive
    # semi-definite

    #
    # compute global minimum portfolio
    #
```

```
    if(shorts==TRUE){
      cov.mat.inv <- solve(cov.mat)
      one.vec <- rep(1,N)
      w.gmin <- rowSums(cov.mat.inv) / sum(cov.mat.inv)
      w.gmin <- as.vector(w.gmin)
    } else if(shorts==FALSE){
      Dmat <- 2*cov.mat
      dvec <- rep.int(0, N)
      Amat <- cbind(rep(1,N), diag(1,N))
      bvec <- c(1, rep(0,N))
      result <- solve.QP(Dmat=Dmat,dvec=dvec,Amat=Amat,bvec=bvec,meq=1)
      w.gmin <- round(result$solution, 6)
    } else {
      stop("shorts needs to be logical. For no-shorts, shorts=FALSE.")
    }

    names(w.gmin) <- asset.names
    er.gmin <- crossprod(w.gmin,er)
    sd.gmin <- sqrt(t(w.gmin) %*% cov.mat %*% w.gmin)
    gmin.port <- list("call" = call,
                      "er" = as.vector(er.gmin),
                      "sd" = as.vector(sd.gmin),
                      "weights" = w.gmin)
    class(gmin.port) <- "portfolio"
    gmin.port
  }


tangency.portfolio <-
  function(er,cov.mat,risk.free, shorts=T)
  {
    # compute tangency portfolio
    #
    # inputs:
    # er    N x 1 vector of expected returns
    # cov.mat    N x N return covariance matrix
    # risk.free  scalar, risk-free rate
    # shorts          logical, allow shorts is TRUE
    #
    # output is portfolio object with the following elements
    # call   captures function call
    # er   portfolio expected return
```

```
# sd   portfolio standard deviation
# weights  N x 1 vector of portfolio weights
call <- match.call()

#
# check for valid inputs
#
asset.names <- names(er)
if(risk.free < 0)
  stop("Risk-free rate must be positive")
er <- as.vector(er)
cov.mat <- as.matrix(cov.mat)
N <- length(er)
if(N != nrow(cov.mat))
  stop("invalid inputs")
if(any(diag(chol(cov.mat)) <= 0))
  stop("Covariance matrix not positive definite")
# remark: could use generalized inverse if cov.mat is
# positive semi-definite

#
# compute global minimum variance portfolio
#
gmin.port <- globalMin.portfolio(er, cov.mat, shorts=shorts)
if(gmin.port$er < risk.free)
  stop("Risk-free rate greater than avg return on
       global minimum variance portfolio")

#
# compute tangency portfolio
#
if(shorts==TRUE){
  cov.mat.inv <- solve(cov.mat)
  w.t <- cov.mat.inv %*% (er - risk.free) # tangency portfolio
  w.t <- as.vector(w.t/sum(w.t)) # normalize weights
} else if(shorts==FALSE){
  Dmat <- 2*cov.mat
  dvec <- rep.int(0, N)
  er.excess <- er - risk.free
  Amat <- cbind(er.excess, diag(1,N))
  bvec <- c(1, rep(0,N))
  result <- solve.QP(Dmat=Dmat,dvec=dvec,Amat=Amat,bvec=bvec,meq=1)
```

```
    w.t <- round(result$solution/sum(result$solution), 6)
  } else {
    stop("Shorts needs to be logical. For no-shorts, shorts=FALSE.")
  }

  names(w.t) <- asset.names
  er.t <- crossprod(w.t,er)
  sd.t <- sqrt(t(w.t) %*% cov.mat %*% w.t)
  tan.port <- list("call" = call,
                   "er" = as.vector(er.t),
                   "sd" = as.vector(sd.t),
                   "weights" = w.t)
  class(tan.port) <- "portfolio"
  tan.port
}

efficient.frontier <-
  function(er, cov.mat, nport=20, alpha.min=-0.5, alpha.max=1.5,
           shorts=TRUE)
  {
    # Compute efficient frontier with no short-sales constraints
    #
    # inputs:
    # er   N x 1 vector of expected returns
    # cov.mat   N x N return covariance matrix
    # nport   scalar, number of efficient portfolios to compute
    # shorts          logical, allow shorts is TRUE
    #
    # output is a Markowitz object with the following elements
    # call   captures function call
    # er   nport x 1 vector of expected returns on efficient porfolios
    # sd   nport x 1 vector of std deviations on efficient portfolios
    # weights  nport x N matrix of weights on efficient portfolios
    call <- match.call()

    #
    # check for valid inputs
    #
    asset.names <- names(er)
    er <- as.vector(er)
    N <- length(er)
    cov.mat <- as.matrix(cov.mat)
```

```
if(N != nrow(cov.mat))
  stop("invalid inputs")
if(any(diag(chol(cov.mat)) <= 0))
  stop("Covariance matrix not positive definite")

#
# create portfolio names
#
port.names <- rep("port",nport)
ns <- seq(1,nport)
port.names <- paste(port.names,ns)

#
# compute global minimum variance portfolio
#
cov.mat.inv <- solve(cov.mat)
one.vec <- rep(1, N)
port.gmin <- globalMin.portfolio(er, cov.mat, shorts)
w.gmin <- port.gmin$weights

if(shorts==TRUE){
  # compute efficient frontier as convex combinations of two
  # efficient portfolios
  # 1st efficient port: global min var portfolio
  # 2nd efficient port: min var port with ER = max of ER
  # for all assets
  er.max <- max(er)
  port.max <- efficient.portfolio(er,cov.mat,er.max)
  w.max <- port.max$weights
  a <- seq(from=alpha.min,to=alpha.max,length=nport)
   # convex combinations
  we.mat <- a %o% w.gmin + (1-a) %o% w.max
  # rows are efficient portfolios
  er.e <- we.mat %*% er
  # expected returns of efficient portfolios
  er.e <- as.vector(er.e)
} else if(shorts==FALSE){
  we.mat <- matrix(0, nrow=nport, ncol=N)
  we.mat[1,] <- w.gmin
  we.mat[nport, which.max(er)] <- 1
  er.e <- as.vector(seq(from=port.gmin$er, to=max(er),
                      length=nport))
```

```
      for(i in 2:(nport-1))
        we.mat[i,] <- efficient.portfolio(er, cov.mat, er.e[i],
         shorts)$weights
    } else {
      stop("shorts needs to be logical. For no-shorts,
      shorts=FALSE.")
    }

    names(er.e) <- port.names
    cov.e <- we.mat %*% cov.mat %*% t(we.mat)
    # cov mat of efficient portfolios
    sd.e <- sqrt(diag(cov.e))
    # std devs of efficient portfolios
    sd.e <- as.vector(sd.e)
    names(sd.e) <- port.names
    dimnames(we.mat) <- list(port.names,asset.names)

    #
    # summarize results
    #
    ans <- list("call" = call,
                "er" = er.e,
                "sd" = sd.e,
                "weights" = we.mat)
    class(ans) <- "Markowitz"
    ans
  }

#
# print method for portfolio object
print.portfolio <- function(x, ...)
{
  cat("Call:\n")
  print(x$call, ...)
  cat("\nPortfolio expected return:    ", format(x$er, ...), "\n")
  cat("Portfolio standard deviation: ", format(x$sd, ...), "\n")
  cat("Portfolio weights:\n")
  print(round(x$weights,4), ...)
  invisible(x)
}

#
```

```
# summary method for portfolio object
summary.portfolio <- function(object, risk.free=NULL, ...)
  # risk.free risk-free rate. If not null then
  # compute and print Sharpe ratio
  #
{
  cat("Call:\n")
  print(object$call)
  cat("\nPortfolio expected return:    ", format(object$er, ...), "\n")
  cat(  "Portfolio standard deviation: ", format(object$sd, ...), "\n")
  if(!is.null(risk.free)) {
    SharpeRatio <- (object$er - risk.free)/object$sd
    cat("Portfolio Sharpe Ratio:       ", format(SharpeRatio), "\n")
  }
  cat("Portfolio weights:\n")
  print(round(object$weights,4), ...)
  invisible(object)
}
# hard-coded 4 digits; prefer to let user control,
# via ... or other argument

#
# plot method for portfolio object
plot.portfolio <- function(object, ...)
{
  asset.names <- names(object$weights)
  barplot(object$weights, names=asset.names,
          xlab="Assets", ylab="Weight",
          main="Portfolio Weights", ...)
  invisible()
}

#
# print method for Markowitz object
print.Markowitz <- function(x, ...)
{
  cat("Call:\n")
  print(x$call)
  xx <- rbind(x$er,x$sd)
  dimnames(xx)[[1]] <- c("ER","SD")
  cat("\nFrontier portfolios' expected
      returns and standard deviations\n")
```

```
  print(round(xx,4), ...)
  invisible(x)
}
# hard-coded 4, should let user control

#
# summary method for Markowitz object
summary.Markowitz <- function(object, risk.free=NULL)
{
  call <- object$call
  asset.names <- colnames(object$weights)
  port.names <- rownames(object$weights)
  if(!is.null(risk.free)) {
    # compute efficient portfolios with a risk-free asset
    nport <- length(object$er)
    sd.max <- object$sd[1]
    sd.e <- seq(from=0,to=sd.max,length=nport)
    names(sd.e) <- port.names

    #
    # get original er and cov.mat data from call
    er <- eval(object$call$er)
    cov.mat <- eval(object$call$cov.mat)

    #
    # compute tangency portfolio
    tan.port <- tangency.portfolio(er,cov.mat,risk.free)
    x.t <- sd.e/tan.port$sd # weights in tangency port
    rf <- 1 - x.t # weights in t-bills
    er.e <- risk.free + x.t*(tan.port$er - risk.free)
    names(er.e) <- port.names
    we.mat <- x.t %o% tan.port$weights
     # rows are efficient portfolios
    dimnames(we.mat) <- list(port.names, asset.names)
    we.mat <- cbind(rf,we.mat)
  }
  else {
    er.e <- object$er
    sd.e <- object$sd
    we.mat <- object$weights
  }
  ans <- list("call" = call,
```

```
              "er"=er.e,
              "sd"=sd.e,
              "weights"=we.mat)
  class(ans) <- "summary.Markowitz"
  ans
}

print.summary.Markowitz <- function(x, ...)
{
  xx <- rbind(x$er,x$sd)
  port.names <- names(x$er)
  asset.names <- colnames(x$weights)
  dimnames(xx)[[1]] <- c("ER","SD")
  cat("Frontier portfolios' expected returns
       and standard deviations\n")
  print(round(xx,4), ...)
  cat("\nPortfolio weights:\n")
  print(round(x$weights,4), ...)
  invisible(x)
}
# hard-coded 4, should let user control

#
# plot efficient frontier
#
# things to add: plot original assets with names
# tangency portfolio
# global min portfolio
# risk free asset and line connecting rf
# to tangency portfolio
#
plot.Markowitz <- function(object, plot.assets=FALSE, ...)
  # plot.assets logical. If true then plot asset sd and er
{
  if (!plot.assets) {
    y.lim=c(0,max(object$er))
    x.lim=c(0,max(object$sd))
    plot(object$sd,object$er,type="b",xlim=x.lim, ylim=y.lim,
         xlab="Portfolio SD", ylab="Portfolio ER",
         main="Efficient Frontier", ...)
  }
  else {
```

```
    call = object$call
    mu.vals = eval(call$er)
    sd.vals = sqrt( diag( eval(call$cov.mat) ) )
    y.lim = range(c(0,mu.vals,object$er))
    x.lim = range(c(0,sd.vals,object$sd))
    plot(object$sd,object$er,type="b", xlim=x.lim, ylim=y.lim,
         xlab="Portfolio SD", ylab="Portfolio ER",
         main="Efficient Frontier", ...)
    text(sd.vals, mu.vals, labels=names(mu.vals))
  }
  invisible()
}
```