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A Uniform Quantificational Logic for Algebraic Notions of Context

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Summary

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The equations of the algebra partitions context terms into equivalence classes. A formal semantics is defined, containing models that map equivalence classes of certain context terms to sets of first order structures.

The corresponding Hilbert system incorporates the algebraic equations as axioms asserted in context. In this way a uniform logic for arbitrary algebras of context is obtained. Soundness and completeness are proved.

In semigroups of contexts, where combination of contexts is associative, finite ground algebraic equations correspond to contingent equivalence between certain logical formulas.

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Algebras of contexts
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Abstract

A quantificational framework of formal reasoning is proposed, which emphasises the pattern of entering and exiting context. Contexts are modelled by an algebraic structure which reflects the order and manner in which context is entered into and exited from.

The equations of the algebra partitions context terms into equivalence classes. A formal semantics is defined, containing models that map equivalence classes of certain context terms to sets of first order structures. The corresponding Hilbert system incorporates the algebraic equations as axioms asserted in context. In this way a uniform logic for arbitrary algebras of context is obtained. Soundness and completeness are proved.

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Some contextual reasoning systems in the literature are inherently associative, and we present those as special cases.

Keywords: Formalization of context, logic of contextual assertions, algebras of context

1 Introduction

It is commonly held that all reasoning takes place within context of some sort or another. The seminal papers of McCarthy et.al. [12, 11], which call for a treatment of contexts as explicit mathematical objects, have spurred quite a bit of effort towards devising formal systems of reasoning that reflect this adequately.

Context tends to change in the course of reasoning, for instance if a reasoner is concerned with several independent tasks in quasi-parallel, or if she(/he/it) encounters a very general subtask that can be solved in a very large class of contexts, or a very special subtask that can only proceed within some particular and highly specific context. Thus entering and exiting contexts are fundamental operations in contextual reasoning.

In logical terms, assuming a language of contexts and a language of formulas, the pattern of entering and exiting context can be modelled by deduction rules specifying the context before and after transition, and what formulas are taken to hold in the old and the new context.

Following [9, 12, 5, 4] we adopt the notation

$$x : \lambda \tag{1.1}$$

for asserting a formula λ in context x, and the formula syntax

$$ist(c, \lambda) \tag{1.2}$$

for truth of λ in context c. Here, c is an atomic context name, while x denotes an accumulated context composed, in general, of various atomic constituents. The syntax of the special *ist* predicate is restricted to atomic contexts in its first coordinate. Assertions of the form (1.1) are used to keep track of the surrounding context in a chain of reasoning which might take into account several axioms or premises of the form (1.2). Below, we shall formalize the accumulation of several contexts $c, d, \ldots$ into a composite context x by a generic operator $\oplus$, and specify the manner in which context accumulates by giving algebraic equations on $\oplus$ terms.

Now we proceed to discuss rules for moving in and out of context. Consider these two labelled deduction rules for entering and exiting context, which bear a vague resemblance to the rule of necessitation and the inverse rule of necessitation in modal logic:

$$\text{Enter: } \frac{\vdash u : ist(v, \lambda)}{\vdash w : \lambda} \qquad\qquad \text{Exit: } \frac{\vdash x : \lambda}{\vdash y : ist(z, \lambda)} \tag{1.3}$$

Our investigation concerns the interplay between u, v, w, resp. x, y, z, in (1.3). It will take us beyond the mainland of modal logic, as suggested by such formulas as

$$\forall v . p(v) \rightarrow ist(v, \lambda) \tag{1.4}$$

which are usually not well treated there.

By the Enter rule, one may pass from $ist(v, \lambda)$ asserted in u to λ asserted in w. In the premise of this transition, the context of reasoning is u, and the asserted formula expresses that λ is true in context v. The result of the transition is that the new context of reasoning is w, and the asserted formula is λ. In a sense, contextual information passes from the asserted formula into the surrounding context of reasoning. The new context of reasoning w expresses the combined context after, in context u, having additionally entered context v. In what follows, w will be modelled as an algebraic combination of u and v.

Vice versa, the premise of the Exit rule is that the context of reasoning is x, and that λ is asserted there. The result is a new context of reasoning y, where it is asserted that λ is true in context z. Information is taken from the surrounding context of reasoning and put into the asserted formula. The context x has, so to speak, been split in two: a part z has been chipped off and used to express that λ is true there, and the residual part y is the new context of further reasoning. In our models, x will be an algebraic combination of y and z.

Taking this one step further, we consider an arbitrary series of interleaved Enter and Exit transitions. Upon entering a context it is desirable to leave a trace which shows the accumulated context after entering. A trace can have information about the context that is entered into, and/or of the surrounding context at the time.

Successive Enters augment the current context of reasoning with additional contextual information $b, c, \ldots$ by removing contextual items from formulas of the form $ist(b, ist(c, \ldots))$.

Conversely, on exiting from a context, the trace is picked up and modified for further reasoning purposes. Successive Exits construct nested formulas of the form $ist(d, ist(e, \ldots))$ by extracting parts $\ldots, e, d$ from the accumulated context of reasoning.

On this view then, the current context of reasoning can be compared to a data structure, which accumulates items of context during Enter transitions, and which releases items of context during Exit transitions.

We have purposely made no assumptions about the order and manner in which items of context are stored in the accumulated context of reasoning, and we are going to develop the theory in general for any equationally specified discipline, i.e. any algebra of context combination.

Let us compare this view with some contextual reasoning systems in the literature. In [4], we find a quantified logic of context where entering a context leaves a fairly uninformative trace, including only the last context that was entered:

$$\text{Enter: } \frac{\vdash x : ist(c, \lambda)}{\vdash c : \lambda} \qquad\qquad \text{Exit: } \frac{\vdash c : \lambda}{\vdash x : ist(c, \lambda)} \tag{1.5}$$

As another example, the accumulated context of reasoning could be the sequence of contexts entered into and not exited from, suggesting a stack discipline of reasoning. This is the approach taken in the propositional logic of context in [5] and the quantificational logic of context in [14], and corresponds to these rules:

$$\text{Enter: } \frac{\vdash \vec{c} : ist(c, \lambda)}{\vdash \vec{c}c : \lambda} \qquad\qquad \text{Exit: } \frac{\vdash \vec{c}c : \lambda}{\vdash \vec{c} : ist(c, \lambda)} \tag{1.6}$$

In the system we are about to develop, it is by no means prescribed that the contextual items taken from *ist* formulas in an Enter transition, will occur in any particular order during later Exits, or indeed that the same contextual items will occur at all. A particular algebra might change the contribution of one particular item of context if it occurs among certain others in a particular $\oplus$ term, for instance. We shall cater for arbitrary equationally specified disciplines of context combination.

We therefore allow any context constructor $\oplus$ which combines a previously accumulated context x with the context c being entered into:

$$\text{Enter: } \frac{\vdash x : ist(c, \lambda)}{\vdash x \oplus c : \lambda} \qquad\qquad \text{Exit: } \frac{\vdash x \oplus c : \lambda}{\vdash x : ist(c, \lambda)} \tag{1.7}$$

Along with these deduction rules the algebraic properties of the context constructor $\oplus$ are specified by equations. For example, the case of (1.6) corresponds to a purely associative constructor:

$$(u \oplus v) \oplus w = u \oplus (v \oplus w) \tag{1.8}$$

whereas (1.5) corresponds to a stronger condition on $\oplus$:

$$u \oplus v = v \tag{1.9}$$

2 Overview

We now proceed to develop the theory generally for arbitrary algebras of contexts. The paper is organized as follows: In the next section, we define algebras of contexts. Then, the language of contextual formulas is defined, and some notational conventions established. Some options for semantical interpretation are discussed, arriving at a framework which defines truth of asserted formulas. Further, axioms schemata and deduction rules are put forward and discussed, and their soundness and completeness with respect to the semantical framework proved. A rule for exchanging algebraically equal contexts is defined and demonstrated to be sound, and a connection with substructural logic is pointed out. Then, the class of associative finite ground algebras is defined, and a simplified presentation of their axiomatics is given. In this way, we obtain as special cases systems where the algebras are sets and bags (multisets), as well as the free semigroup (corresponding to (1.6)) and the flat semigroup (corresponding to (1.5)). The latter two coincide with the systems in [14] resp [4]. In the concluding section, we point out some directions of further research.

3 Algebras of context

Let a countable set C of contexts be given a priori, and take it as the carrier of an algebra with one binary operator $\oplus$ and equations

$$y_i = z_i, \quad 1 \leq i \leq N \tag{3.1}$$

for some $N > 0$, where y_i and z_i are terms on $\oplus$.

As an example consider bags (multisets) over C, generated by the $\oplus$ operation. In a bag, as opposed to a sequence, the order of elements is immaterial. The relevant equations express associativity and commutativity of $\oplus$:

$$(u \oplus v) \oplus w \quad = \quad u \oplus (v \oplus w) \tag{3.2}$$

$$u \oplus v \quad = \quad v \oplus u \tag{3.3}$$

In a proper set over C, where repeated context entries don't count, the $\oplus$ operation is also idempotent:

$$u \oplus u \quad = \quad u \tag{3.4}$$

We denote the set of all $\oplus$-terms over C by $C^{\oplus}$, and the set of equivalence classes imposed by the algebraic equations we denote by $C^{\oplus}_{\equiv}$.

It is sometimes convenient to include a special context ϵ, such that

$$\epsilon \oplus u = u = u \oplus \epsilon \tag{3.5}$$

For example, in applications where there is an outermost supercontext, enclosing all other contexts, that could be ϵ.

Any desired properties of $\oplus$ are to be specified as algebraic equations in the axiomatics, including any or all of those mentioned here. Note that accociativity is not a priori assumed.

In assertions of the form

$$x : \lambda$$

with $x \in C^{\oplus}, \lambda \in \mathcal{L}$, both sides will mention contexts, but in very different syntactic regimes. To the left, there is an arbitrary term from $C^{\oplus}$, whereas the right hand side only mentions contexts from the carrier C, and only in subformulas of the form $ist(c, \chi)$. We observed above how context is exchanged across the dividing colon during Enter and Exit transitions, and we now proceed to identify a subset of $C^{\oplus}$ which is going to be of frequent use later:

Definition 3.1 (x-continuants)
For $x \in C^{\oplus}$, an x-continuant is any term $((x \oplus c_1) \oplus c_2) \ldots \oplus c_m$ where $m \geq 0$ and $c_i \in C, 1 \leq i \leq m$.

Note that x itself is an x-continuant, with $m = 0$. The parenthetical structure is a part of the definition, so the following are examples of x-continuants

$$x \oplus c, \qquad (x \oplus c) \oplus d$$

but

$$x \oplus (c \oplus d)$$

is not an x-continuant.

It may well be that different x-continuants are equal by force of the algebraic equations, for example

$$(((x \oplus c_1) \oplus c_2) \ldots \oplus c_m) = (((x \oplus d_1) \oplus d_2) \ldots \oplus d_n)$$

but this does not in general imply that

$$((c_1 \oplus c_2) \ldots \oplus c_m) = ((d_1 \oplus d_2) \ldots \oplus d_n)$$

unless additional information about $\oplus$ is available.

The model structure below focuses on equivalence classes of x-continuants, so it's worth coining a term:

Definition 3.2 (x-bundles)
For $x \in C^{\oplus}$, the set of x-bundles is the quotient of the set of x-continuants under the equivalence relation imposed by the algebraic equations.

The word 'bundle' was suggested by the thought of different linear strands bound together at both ends; all starting at x and all finally being equal.

4 Formula language

Based on the discussion so far, we will use these language components:

- disjoint sorts C for contexts, and T for other objects of discourse
- names (constants) for each of the elements of C and T
- the $\oplus$ function symbol
- predicates $p(t_1, \ldots, t_m)$ on sorted coordinates, including:
- the identity predicate $t_1 = t_2$ for each sort
- truth-functional connectives $\neg, \rightarrow, \vee, \wedge, \leftrightarrow$
- the *ist* modality

For simplicity, our language will have no other (non-constant) function symbols than $\oplus$. We elect to work in a quantified predicate logic, so we'll incorporate these additional elements:

- (infinitely many) variables of each sort

- quantifiers $\forall, \exists$

Let therefore $\mathcal{L}$ be a sorted first order predicate language with identity, augmented by the special modality $ist(c, \lambda)$. The sorts will be C for contexts and T for other objects of discourse. More precisely, the language $\mathcal{L}$ of well-formed formulas can be defined in this way:

DEFINITION 4.1 ($\mathcal{L}$, the set of well-formed formulas)

$$\mathcal{L} ::= \; P \mid \neg\mathcal{L} \mid \mathcal{L} \to \mathcal{L} \mid \forall V.\mathcal{L} \mid ist(C, \mathcal{L})$$

where P is a set of atomic predicates on sorted terms, including the identity predicate for each sort, V is a set of sorted variables, and C is a set of context names.

Let us establish the following notational conventions:

- $\exists$, $\vee$, $\wedge$, and $\leftrightarrow$ are the usual abbreviations.
- lower-case b, c, d, e denote constants or variables of sort C
- lower-case u, v, w denote (sorted) variables
- lower-case x, y, z denote context terms, i.e. elements of $C^{\oplus}$
- lower-case greek letters $\lambda, \chi, \mu, \ldots$ denote formulas from $\mathcal{L}$
- upper-case greek letters $\Omega, \Gamma, \ldots$ denote sets of formulas
- $\perp$ denotes an arbitrary propositional contradiction, e.g. $\lambda \wedge \neg\lambda$.
- the result of replacing all free occurrences of v in λ with free u is denoted λ^v_u.
- $ist(c_1, c_2, \ldots, c_m, \lambda)$ is shorthand for $ist(c_1, ist(c_2, \ldots, ist(c_m, \lambda) \ldots))$. When $m \leq 0$, this is just λ.

5 Semantical structure and truth conditions

Formulas $\lambda \in \mathcal{L}$ are always asserted in context given by terms $x \in C^{\oplus}$:

$$x : \lambda$$

and we now present a semantical framework for interpreting such pairs as true or false.

As already discussed, x denotes an accumulated context composed by the $\oplus$ operator, and λ is a quantificational formula possibly containing $ist(\ldots, \ldots)$ subformulas. We are on familiar ground as far as the propositional connectives are concerned, and even the quantifiers can be dealt with by techniques familiar from modal logic. The critical aspect is the case where λ is $ist(c, \chi)$ for some c and λ, and we refer back to our discussion of Enter/Exit transitions: a model should assign the same truth value to $x : ist(c, \chi)$ as to $x \oplus c : \chi$. Now χ might in turn be $ist(d, \mu)$, leading to $(x \oplus c) \oplus d : \mu$, and so on.

We see that a model for asserted formulas of the form $x : \lambda$ must also be prepared to deal with $y : \mu$ whenever y is an x-continuant. But there is a complication: For x-continuants y, z that are equal by force of the algebraic equations, a model needs to assign the same truth value to

$$y : \mu \qquad \text{and} \qquad z : \mu$$

A model is therefore going to take x-bundles rather than x-continuants as arguments.

We take the view that a context can be vague, in the sense of encompassing a set of different possible circumstances. Any particular circumstance will be modelled as a first-order interpretation of the symbols and predicates of the formula language.

Thus, our models for assertions $x : \lambda$ will be maps from x-bundles to sets of first-order interpretations. Intuitively, truth in a model of a formula asserted in a context requires truth at each first-order interpretation bundled to that context.

Moreover, we do not commit to consistence of contexts, thus admitting the possibility of modelling contradictory contexts. Such a commitment can be made, if desired, by restricting the model structure so that the image of an x-bundle is always nonempty, and enriching the axiomatics correspondingly.

A point frequently made is that distinct contexts may well have different formula languages. This feature could be built into our system by mapping contexts to subsets of $\mathcal{L}$, and letting the semantical interpretation be three-valued. This technique was used in [5]. We believe this issue can be dealt with most naturally in a multi-language system in the style of [15], and it remains our ambition to endow multi-language systems with an algebraic superstructure reminiscent of the one we are discussing here. That will be in a future paper, however.

We are now ready to define the class of models for interpretation of asserted formulas. Let the set C of contexts and the set T of objects of discourse be given a priori. These are nonempty and no more than countable, and shall stay fixed throughout. Let the constants (names) of each sort of the language be rigid designators, i.e. let them correspond 1-1 to elements of C resp T, so that we may identify the sets of constants of sort C with C itself, and correspondingly with T.

Definition 5.1 (Rigid interpretations)
A rigid interpretation is a first-order interpretation of the language, in which:

- the domain for objects of discourse is T
- the domains for contexts is $C^{\oplus}_{\equiv}$, the qoutient of the set of $\oplus$ terms under the equivalence relation imposed by the algebraic equations
- each constant of sort T is interpreted according to the 1-1 correspondence mentioned above
- each constant of sort C is interpreted as its own equivalence class modulo the algebraic equations
- the $\oplus$ symbol is interpreted homomorphically, i.e. $x \oplus y$ is interpreted as the set of terms $\widehat{x} \oplus \widehat{y}$ such that $\widehat{x}$ is in the set interpreting x and $\widehat{y}$ is in the set interpreting y.
- the identity predicate for each sort is interpreted as the corresponding identity relation

Definition 5.2 (x-models)
An x-model is a function from x-bundles to sets of rigid interpretations.

Given an x-model M and an x-continuant y we shall frequently (in fact, usually) stretch the syntax a little and write $M(y)$ to denote the set of rigid interpretations that M associates with the x-bundle containing y.

We can now define truth and falsity of formulas asserted in context. Let $x \in C^{\oplus}, \lambda \in \mathcal{L}$, and let M be an x-model:

$$M \models x : \lambda \text{ iff } M, I \models x : \lambda \text{ for all } I \in M(x)$$

where for $I \in M(x)$

$$\begin{array}{lcl} M, I \models & x : p & \text{iff } I \text{ interprets } p \text{ as true} \\ M, I \models & x : \neg\lambda & \text{iff } M, I \not\models x : \lambda \\ M, I \models & x : \lambda \to \gamma & \text{iff } M, I \models x : \lambda \ \text{ implies } M, I \models x : \gamma \\ M, I \models & x : \forall v.\lambda & \text{iff } M, I \models x : \lambda^v_t \ \text{ for all } t \text{ of correct sort} \\ M, I \models & x : ist(c, \lambda) & \text{iff } M, J \models x \oplus c : \lambda \ \text{ for all } J \in M(x \oplus c) \end{array} \tag{5.1}$$

Validity of an asserted formula is defined as truth in all models of the relevant context:

Definition 5.3 (Validity)
We say that a formula λ is valid in context x, or synonymously that $x : \lambda$ is valid, in symbols

$$\models x : \lambda$$

iff $M \models x : \lambda$ for all x-models M.

6 Axiomatic presentation

The following table gives axiom schemata and rules of deduction which are sound and complete with respect to the semantical framework.

Most of the axioms are asserted in a general context x. This x is a variable which ranges over all contexts.

The left and right sides, y_i resp. z_i, of each algebraic equation $x : y_i = z_i$, are terms from $C^{\oplus}$.

Reflexivity and congruence together imply symmetry and transitivity of the $=$ predicate, so we get all the familiar properties of equality.

PL and MP govern the the classical connectives.

K and Exit are the foundation of a normal multi-modal system, but the *ist* predicate has greater expressivity than standard modalities, since we can quantify over the first coordinate of an *ist*.

The $\mathrm{G}^{2,0,1,1}$ axiom and the axiom of nesting both have to do with x-continuant contexts. $\mathrm{G}^{2,0,1,1}$ forces truth 'twice removed', so to speak, and later we shall see that $\mathrm{G}^{2,0,1,1}$ generalizes to more deeply nested *ist* formulas, cfr (6.2).

The axiom of nesting relates bundles of x-continuants to equivalence of the correspondingly nested *ist* formulas. Note that x occurs on both sides of the colon in this schema, providing a connection between the formula being asserted and the context in which the assertion is made. In a later section, we'll look at a class of algebras which make the axiom of nesting redundant.

UI and UG govern the classical properties of quantification, and BF is the axiom which corresponds to the condition of invariant domains of interpretation in the model structure.

TABLE 1. **Axiom schemata and deduction rules**

Rules for changing context:

$$\text{Enter: } \frac{\vdash x : ist(c, \lambda)}{\vdash x \oplus c : \lambda} \qquad \text{Exit: } \frac{\vdash x \oplus c : \lambda}{\vdash x : ist(c, \lambda)}$$

Equational properties:

Reflexivity: $\vdash x : y = y$

Congruence: $\vdash x : y = z \rightarrow (\lambda^v_y \rightarrow \lambda^v_z)$

Algebraic equations: $\vdash x : y_i = z_i \qquad y_i, z_i \in C^{\oplus} \qquad 1 \leq i \leq N$

Propositional properties:

PL: $\vdash x : \lambda$ whenever λ is an instance of a propositional tautology

$$\text{MP: } \frac{\vdash x : \lambda \quad \vdash x : \lambda \rightarrow \chi}{\vdash x : \chi}$$

Modal properties:

K: $\vdash x : ist(c, \lambda \rightarrow \chi) \rightarrow (ist(c, \lambda) \rightarrow ist(c, \chi))$

$G^{2,0,1,1}$: $\vdash x : \neg ist(c, ist(d, \lambda)) \rightarrow ist(c, \neg ist(d, \lambda))$

Nesting: $\vdash x : ((x \oplus c_1) \ldots \oplus c_m) = ((x \oplus d_1) \ldots \oplus d_n) \rightarrow$
$(ist(c_1, \ldots, c_m, \lambda) \rightarrow ist(d_1, \ldots, d_n, \lambda))$

Quantificational properties:

UI: $\vdash x : \forall v.\lambda \rightarrow \lambda^v_t$ UG: $\dfrac{\vdash x : \xi \rightarrow \lambda^v_t}{\vdash x : \xi \rightarrow \forall v.\lambda}$ where v is not free in ξ

BF: $\vdash x : \forall v.ist(c, \lambda) \rightarrow ist(c, \forall v.\lambda)$ where c is not the variable v

DEFINITION 6.1 (Theoremhood)
We say that a formula λ is derivable in context x, or synonymously that $x : \lambda$ is a theorem, in symbols

$$\vdash x : \lambda$$

iff $x : \lambda$ is an instance of an axiom schema, or follows from other theorems by applications of the deduction rules.

Derived schemata:
Let us note that the following useful schemata are derivable. For $n > 0$:

$$\begin{aligned}
\mathrm{K}^n: \quad & \vdash x : ist(c_1, \ldots, c_n, \lambda \to \chi) \to \\
& \qquad (ist(c_1, \ldots, c_n, \lambda) \to ist(c_1, \ldots, c_n, \chi)) && (6.1)\\
\mathrm{G}^{n+1,0,n,1}: \quad & \vdash x : \neg ist(c_1, \ldots, c_{n+1}, \lambda) \to \\
& \qquad ist(c_1, \ldots, c_n, \neg ist(c_{n+1}, \lambda)) && (6.2)\\
\mathrm{BF}^n: \quad & \vdash x : \forall v. ist(c_1, \ldots, c_n, \lambda) \to ist(c_1, \ldots, c_n, \forall v.\lambda) \\
& \qquad \text{where } v \text{ does not occur in } c_1, \ldots, c_n && (6.3)\\
\mathrm{AND}^n: \quad & \vdash x : ist(c_1, \ldots, c_n, \lambda \wedge \chi) \leftrightarrow \\
& \qquad ist(c_1, \ldots, c_n, \lambda) \wedge ist(c_1, \ldots, c_n, \chi) && (6.4)\\
\mathrm{OR}^n: \quad & \vdash x : ist(c_1, \ldots, c_n, \lambda) \vee ist(c_1, \ldots, c_n, \chi) \to \\
& \qquad ist(c_1, \ldots, c_n, \lambda \vee \chi) && (6.5)
\end{aligned}$$

The proofs are included in an appendix. Note that $\mathrm{G}^{1,0,0,1}$ is not a theorem of this system:

$$\nvdash x : \neg ist(c, \lambda) \to ist(c, \neg\lambda)$$

7 Soundness

We verify that the axiom schemata are semantically valid and that the deduction rules preserve semantic validity, formally in the case of $G^{2,0,1,1}$ and by brief comments and remarks for the rest.

The Enter/Exit rules are easily seen to be valid from the *ist* clause of the model conditions. Reflexivity and congruence are valid by the rigidity requirements. As for nesting, the premise places the two x-continuants in the same x-bundle, and validity follows.

In the algebraic equations, the terms y_i resp. z_i are interpreted as the same equivalence class of terms by the rigidity requirements, and validity follows by rigidity of identity interpretation.

Any instance of PL is valid because the model conditions for x-models respect the truth-functional connectives, and MP preserves validity for the same reason. Also, K is valid since for any x-model M and any context c, the M also respects the truth-functional connectives when considered as an $x \oplus c$-model.

Validity of UI follows directly from the $\forall$ clause of the model conditions. UG is seen to preserve validity by realizing that t in the schema of the premise of the rule is arbitrary, and comparing the model conditions for the premise and the conclusion. Barcan's axiom BF is valid because every interpretation has the same domains for each sort.

Concerning $G^{2,0,1,1}$, take any x-model M and an interpretation $I \in M(x)$ and assume that $M, I \models x : \neg ist(c, ist(d, \lambda))$. It then follows that $M, J \not\models x \oplus c : ist(d, \lambda)$ for some $J \in M(x \oplus c)$, and that $M, H \not\models (x \oplus c) \oplus d : \lambda$ for some $H \in M((x \oplus c) \oplus d)$.

For validity of $G^{2,0,1,1}$, we now require $M, I \models x : ist(c, \neg ist(d, \lambda))$, which is equivalent to $M, Q \models (x \oplus c) : \neg ist(d, \lambda)$ for every $Q \in M(x \oplus c)$, which is equivalent to $M, R \not\models (x \oplus c) \oplus d : \lambda$ for some $R \in M((x \oplus c) \oplus d)$ for every such $Q \in M(x \oplus c)$. By the argument in the preceding paragraph, H fills the requirement for such an R.

8 Completeness

We begin by setting out definitions of consistency and maximality indexed by contexts.

DEFINITION 8.1 (x-consistency)
Let $x \in C$.

- a formula λ is x-consistent iff $\nvdash x : \neg\lambda$.
- a finite set of formulas is x-consistent iff their conjunction is x-consistent
- an infinite set of formulas is x-consistent iff every finite subset is x-consistent.

DEFINITION 8.2 (x-maximality)
Let $x \in C$.

A set Λ of formulas is x-maximal iff it is x-consistent and for every formula λ, $\Lambda \cup \{\lambda\}$ is x-consistent only if $\lambda \in \Lambda$.

The properties of x-maximal sets are the familiar ones, cfr. e.g. [6, 10]. We shall construct x-maximal sets in a way that reflects our axiomatics:

LEMMA 8.3 (cfr. Lindenbaum's lemma)
Every x-consistent set of formulas can be extended to a x-maximal set.

PROOF. take a set Γ_0 of x-consistent formulas and an enumeration of all formulas $\mathcal{L} = \{\gamma_1, \gamma_2, \ldots\}$, and construct Γ_i from Γ_{i-1}, $i > 0$, as follows: If $\Gamma_{i-1} \cup \{\gamma_i\}$ is x-inconsistent, let

$$\Gamma_i = \Gamma_{i-1}$$

otherwise construct Γ_i from Γ_{i-1} by adding γ_i in any case, and:

- if γ_i is $\neg ist(c_1, \ldots c_m, \neg\forall v.\lambda)$ for some $c_1, \ldots c_m \in C$ with $m \geq 0$, then also adding $\neg ist(c_1, \ldots c_m, \neg\lambda^v_u)$ where u is an unused variable of correct sort
- if γ_i is $\neg ist(c_1, \ldots c_m, \forall v.\lambda)$ for some $c_1, \ldots c_m \in C$ with $m \geq 0$, then also adding $\neg ist(c_1, \ldots c_m, \lambda^v_u)$ where u is an unused variable of correct sort

The construction of Γ_i from Γ_{i-1} can be formalised as follows:

$$\begin{aligned}\Gamma_i = \Gamma_{i-1} &\cup \{\gamma_i\}\\ &\cup\{\neg ist(c_1, \ldots c_m, \neg\lambda^v_u) \mid \gamma_i \text{ is } \neg ist(c_1, \ldots c_m, \neg\forall v.\lambda)\}\\ &\cup\{\neg ist(c_1, \ldots c_m, \lambda^v_u) \mid \gamma_i \text{ is } \neg ist(c_1, \ldots c_m, \forall v.\lambda)\}\end{aligned}$$

The so-called 'witness'-formulas, that that are added contingently if γ_i has one of the prescribed forms, provide the proper trace of existential formulas. Let us see that they can be added x-consistently. The set Γ_{i-1} is finite, so let ξ be the formula $\bigwedge_{\gamma \in \Gamma_{i-1}} \gamma$ and assume that $\xi \wedge \gamma_i$ is x-consistent, i.e.

$$\nvdash x : \xi \to \neg\gamma_i$$

or equivalently

$$\nvdash x : \xi \wedge \gamma_i \to \neg\gamma_i \tag{8.1}$$

- when γ_i is $\neg ist(c_1, \ldots c_m, \neg\forall v.\lambda)$, suppose for the sake of argument that $\xi \wedge \gamma_i \wedge \neg ist(c_1, \ldots c_m, \neg\lambda^v_u)$ is x-inconsistent, i.e.

  $$\vdash x : \xi \wedge \gamma_i \rightarrow ist(c_1, \ldots c_m, \neg\lambda^v_u)$$

  Then by UG

  $$\vdash x : \xi \wedge \gamma_i \rightarrow \forall v.ist(c_1, \ldots c_m, \neg\lambda)$$

  and by BF^m and PL

  $$\vdash x : \xi \wedge \gamma_i \rightarrow ist(c_1, \ldots c_m, \forall v.\neg\lambda)$$

  and by UI, m times Exit, K^m, and PL

  $$\vdash x : \xi \wedge \gamma_i \rightarrow ist(c_1, \ldots c_m, \neg\forall v.\lambda)$$

  which contradicts (8.1).
- when γ_i is $\neg ist(c_1, \ldots c_m, \forall v.\lambda)$, suppose that $\xi \wedge \gamma_i \wedge \neg ist(c_1, \ldots c_m, \lambda^v_u)$ is x-inconsistent, i.e.

  $$\vdash x : \xi \wedge \gamma_i \rightarrow ist(c_1, \ldots c_m, \lambda^v_u)$$

  Then by UG

  $$\vdash x : \xi \wedge \gamma_i \rightarrow \forall v.ist(c_1, \ldots c_m, \lambda)$$

  and by BF^m and PL

  $$\vdash x : \xi \wedge \gamma_i \rightarrow ist(c_1, \ldots c_m, \forall v.\lambda)$$

  which again contradicts (8.1).

Because every addition of a formula is made x-consistently, the union

$$\Gamma = \bigcup_{i=0}^{\infty} \Gamma_i$$

is x-consistent.

Γ is also x-maximal since $\Gamma \cup \{\gamma_i\}$ is x-consistent iff $\gamma_i \in \Gamma_i \subset \Gamma$. ∎

Now we fix some context x and a x-consistent formula δ, and embark on the construction of an x-model for the asserted formula $x : \delta$. First we need an x-maximal set of formulas containing δ:

Definition 8.4 (Ω, an x-maximal extension of $\{\delta\}$)
Let Ω be a set of formulas which is x-maximal, extending the set $\{\delta\}$, and constructed as in the proof of lemma 8.3.

The set Ω, being x-maximal, can be construed as a complete account of truth in context x. For every theorem $\vdash x : \lambda$, $\lambda \in \Omega$, and for every $\lambda \in \mathcal{L}$, either $\lambda \in \Omega$ or $\neg\lambda \in \Omega$. This is but one out of several possible accounts, since the construction in lemma 8.3 is not uniquely specified.

Ω also gives partial accounts of truth in x-continuant contexts, by virtue of nested $ist(\ldots, \ldots)$ formulas contained in it. By retracting the nesting of certain ist formulas in Ω, we define some formula sets which are going to be useful in defining an x-model for δ:

Definition 8.5 (Ω retracts)
For contexts $c_1, \dots, c_n \in C$, let

$$\Omega_{c_1,\dots,c_n} = \{ \quad \phi \quad | \quad ist(c_1, \dots, c_n, \phi) \in \Omega \text{ and } \phi \in P \cup \neg P\}$$

Retracts of Ω are not x-maximal, since they only contain atomic formulas and negations of atomic formulas, and they may even be x-inconsistent. Never the less, if an Ω retract is x-consistent, it gives a partial account of truth in the corresponding x-continuant context, and if so it gives rise to a set of rigid interpretations which agree with this partial account. Those interpretations will define the behaviour of our x-model of δ on the corresponding x-continuant.

We are now in a position to define an x-model M such that $M \models x : \delta$, which in turn will prove completeness.

Definition 8.6 (M, a model of $x : \delta$)
Let the mapping M from x-bundles to sets of rigid interpretations be defined as follows:

Whenever $\widehat{x}$ is an x-bundle containing $((x \oplus c_1) \dots \oplus c_m)$ let

$$M(\widehat{x}) = \{ \quad I \quad | \quad I \text{ is rigid and validates every formula in } \Omega_{c_1,\dots,c_m}\}$$

M is well-defined because of the following lemma:

Lemma 8.7 (Congruence of retracts)
If $\vdash x : ((x \oplus c_1) \dots \oplus c_m) = ((x \oplus d_1) \dots \oplus d_n)$, then $\Omega_{c_1 \dots c_m} = \Omega_{d_1 \dots d_n}$

Proof. Suppose the premise is true, and let $\lambda \in \Omega_{c_1 \dots c_m}$. Then by construction of retracts $ist(c_1, \dots, c_m, \lambda) \in \Omega$, and by x-maximality of Ω and the axiom of nesting it follows that $ist(d_1, \dots, d_n, \lambda) \in \Omega$, and hence by definition of retracts $\lambda \in \Omega_{d_1 \dots d_n}$. The converse is symmetric. ∎

The proof of completeness hinges on the fact that the x-model M models a formula asserted in a certain x-continuant context if and only if that formula is a member of the corresponding Ω retract. To help with the proof of this, let us see that there is a bonus to be had if an Ω retract is x-consistent. Recall that $\perp$ means any propositional contradiction, so no x-consistent set contains $\perp$:

Lemma 8.8 (D approximation)
For $m \geq 0$ and $c_1, \dots, c_{m+1} \in C$ and $\lambda \in \mathcal{L}$, if $ist(c_1, \dots, c_m, \perp) \notin \Omega$, then

$$\neg ist(c_1, \dots, c_{m+1}, \lambda) \in \Omega \quad \text{iff} \quad ist(c_1, \dots c_m, \neg ist(c_{m+1}, \lambda)) \in \Omega$$

Proof. If $m = 0$ the conclusion holds vacuously, so let $m > 0$. From the premise

$$ist(c_1, \dots, c_m, \perp) \notin \Omega$$

it follows that

$$ist(c_1, \dots, c_m, ist(c_{m+1}, \lambda)) \notin \Omega \text{ or } ist(c_1, \dots, c_m, \neg ist(c_{m+1}, \lambda)) \notin \Omega$$

in other words

$$\neg ist(c_1, \dots, c_{m+1}, \lambda) \notin \Omega \text{ implies } ist(c_1, \dots, c_m, \neg ist(c_{m+1}, \lambda)) \notin \Omega$$

On the other hand, suppose $\neg ist(c_1, \ldots, c_{m+1}, \lambda) \in \Omega$. Now $\mathrm{G}^{m+1,0,m,1}$ applies since $m > 0$, and we have

$$\vdash x : \neg ist(c_1, \ldots, c_{m+1}, \lambda) \rightarrow ist(c_1, \ldots, c_m, \neg ist(c_{m+1}, \lambda))$$

But then it follows that $ist(c_1, \ldots, c_m, \neg ist(c_{m+1}, \lambda)) \in \Omega$ by x-maximality of Ω. ∎

We can now prove that M is an x-model of the members of Ω, from which completeness will follow. In fact, it is easier to prove the following stronger lemma:

Lemma 8.9 (Autovalidation)
for $m \geq 0$ and $\chi \in \mathcal{L}$

$$ist(c_1, \ldots, c_m, \chi) \in \Omega \text{ iff } M \models ((x \oplus c_1) \ldots \oplus c_m) : \chi$$

The proof is included in an appendix.

Theorem 8.10 (Completeness)
$\models x : \phi$ iff $\vdash x : \phi$

Proof. Every x-consistent formula has an x-model, by lemma 8.9 with $m = 0$ and $\chi = \delta$. ∎

9 Exchanging equal contexts of reasoning

Let us see that it is safe to replace the surrounding context of reasoning with one that is equal according to the algebra, by proving that the following inference rule is semantically sound:

$$\text{RC:} \quad \frac{\vdash x : y = z \quad \vdash y : \lambda}{\vdash z : \lambda} \tag{9.1}$$

It expresses that wherever reasoning is taking place and the current context is algebraically equal to another context, then the current context can be exchanged for the other, equal context. In general, y and z can be syntactically different terms on $\oplus$, but belonging to the same equivalence class. In a way, the algebraic equations can be viewed as specifying what constitutes fair exchange of contextual currency.

In view of completeness, (9.1) can be rephrased as:

$$\text{If } \models x : y = z \text{ and } \models y : \lambda, \text{ then } \models z : \lambda \tag{9.2}$$

To verify this, we need a lemma to the effect that algebraically equal contexts have the same bundles of continuants:

Lemma 9.1 (Congruence of bundles)
If $\vdash x : y = z$
then $\vdash x : ((y \oplus c_1) \ldots \oplus c_m) = ((z \oplus c_1) \ldots \oplus c_m)$.

Proof. By induction on m:

$m = 0$: the claim is vacuously true
$m > 0$: 1 $\vdash x : ((y \oplus c_1) \ldots \oplus c_{m-1}) = ((z \oplus c_1) \ldots \oplus c_{m-1})$ ind.hyp.
2 $\vdash x : ((y \oplus c_1) \ldots \oplus c_{m-1}) = ((z \oplus c_1) \ldots \oplus c_{m-1}) \rightarrow ((y \oplus c_1) \ldots \oplus c_m) = ((z \oplus c_1) \ldots \oplus c_m)$ axiom of congruence

$3 \vdash x : ((y \oplus c_1) \ldots \oplus c_m) = ((z \oplus c_1) \ldots \oplus c_m)$ 1, 2, MP

∎

Now we are ready to exchange algebraically equal contexts of reasoning. Instead of (9.2) we shall prove the following stronger statement:

THEOREM 9.2
If $\models x : y = z$, then for every y-model M (which is then also a z-model), and every $I \in M(y)$ (hence also $I \in M(z)$), and every formula λ, $M, I \models y : \lambda$ iff $M, I \models z : \lambda$

PROOF. by induction on the structure of λ

$\lambda \in P$: I validates λ iff I validates λ, trivially.

λ is $\neg\mu$: $M, I \not\models y : \mu$ iff $M, I \not\models z : \mu$ by the inductive hypothesis.

λ is $\mu_1 \to \mu_2$: Suppose $M, I \models y : \mu_1 \to \mu_2$ and $M, I \models z : \mu_1$. By the inductive hypothesis the latter is equivalent to $M, I \models y : \mu_1$, and by the model conditions it follows that $M, I \models y : \mu_2$, which by the inductive hypothesis is equivalent to $M, I \models z : \mu_2$. The argument in the other direction is symmetric.

λ is $\forall v.\mu$: $M, I \models y : \mu^v_t$ iff $M, I \models z : \mu^v_t$ by the inductive hypothesis.

λ is $ist(c, \mu)$: We have $J \in M(y \oplus c)$ iff $J \in M(z \oplus c)$ by lemma 9.1, so we get $M, J \models y \oplus c : \mu$ iff $M, J \models z \oplus c : \mu$ by the inductive hypothesis.

∎

10 Simulating substructural implication

It turns out that the semantical framework we have given admits a natural definition of substructural implication. Let us extend $\mathcal{L}$ by adding a new binary connective $\hookrightarrow$, and let its semantics be defined as follows:

$$\models x : \lambda \hookrightarrow \chi \text{ iff for all } y, \models y : \lambda \text{ implies } \models x \oplus y : \chi \tag{10.1}$$

We can now govern the properties of $\hookrightarrow$ by varying the algebraic equations on $\oplus$, obtaining a spectrum of substructural implication systems. These are some of the possibilities (compare table 10.4 of [7]):

Associative $u \oplus (v \oplus w) = (u \oplus v) \oplus w$	Commutative $u \oplus v = v \oplus u$	Idempotent $u \oplus u = u$	Which algebra	Which substructural implication
yes	no	no	sequences	Lambek
yes	yes	no	multisets	linear
yes	yes	yes	sets	mingle

Some algebras and corresponding substructural implication systems

In the next section, we'll see that these same algebraic properties characterize useful special logics of context.

11 Associative Finite Ground Algebras

We proceed to treat a class of algebraic equations that provides a nice algebraic-logical correspondence. As an indication of what is in store here, compare the flatness schema of Buvač's quantificational logic of context [4]

$$\text{(Flat) } k : ist(k1, ist(k2, \phi)) \leftrightarrow ist(k2, \phi) \tag{11.1}$$

with the algebraic equation

$$x : u \oplus v = v \tag{11.2}$$

It is tempting to ask whether all algebraic equations have an alternative presentation as axiom schemata expressing equivalence of certain *ist*-formulas. In this section, we establish criteria on the algebra which allow us to translate the algebraic equations into such schemata.

The contexts

$$k : k1 \ldots k2 \ldots k2$$

of schema (11.1) fit the pattern

$$x : u \ldots v \ldots v$$

of the algebraic equation (11.2), but there are a couple of caveats:

- nesting of the *ist* predicate entails association of contexts into a sequence, so although $k1 \oplus k2$ suggests $ist(k1, ist(k2, \ldots))$, it is not straightforward to translate $k1 \oplus (k2 \oplus k3)$ into a nested *ist*-formula unless the algebra is associative
- the *ist* predicate has only atomic contexts $k1, k2 \in C$ in its first coordinate, whereas u and v in the algebraic equation stand in for arbitrary context terms from $C^{\oplus}$

We can steer clear of these obstacles by restricting the algebra, and this leads to a particularly neat axiomatic presentation of the corresponding systems.

Definition 11.1 (AFG algebras)
An associative finite ground algebra, abbreviated AFG algebra, is one that satisfies these criteria:

- it is associative, i.e. contains the equation

$$\vdash x : (u \oplus v) \oplus w = u \oplus (v \oplus w)$$

- every equation apart from associativity is restricted so that the variables in it can only be instantiated by constants, not by terms containing $\oplus$.
- the number of equations is finite

A logic of context where the algebra is an AFG algebra, is called an AFG logic.

In AFG algebras, every equation other than associativity can be written

$$\vdash x : c_1 c_2 \ldots c_m = d_1 d_2 \ldots d_n \tag{11.3}$$

with $m, n > 0$, since parentheses are redundant because of associativity, and the variables range over individual contexts. Terms from $C^{\oplus}$ are written as sequences of contexts, disregarding the explicit operator $\oplus$.

We are going to let each equation of the form (11.3) give rise to a corresponding axiom schema:

$$\vdash x : ist(c_1, c_2, \ldots, c_m, \lambda) \leftrightarrow ist(d_1, d_2, \ldots, d_n, \lambda) \tag{11.4}$$

which follows by associativity, congruence, nesting and MP.

We also note that in AFG algebras, the axiom of nesting is subsumed by the axioms of the form (11.4), in the sense that for any ground instance of the axiom of nesting, whenever the equality in the premise of the axiom of nesting is true, the conclusion of same instance of the axiom is provable.

To see this, we first notice that the following deduction rule is admissible:

$$\text{RRI:} \quad \frac{\vdash x : ist(c, \lambda) \rightarrow ist(c, \chi)}{\vdash x \oplus c : \lambda \rightarrow \chi} \tag{11.5}$$

In an AFG algebra, any instance of the premise of the axiom of nesting:

$$((x \oplus b_1) \ldots \oplus b_m) = ((x \oplus e_1) \ldots \oplus e_m) \tag{11.6}$$

can be written

$$a_1 \ldots a_k b_1 \ldots b_m = a_1 \ldots a_k e_1 \ldots e_n \tag{11.7}$$

where x is $a_1 \ldots a_k$ for context names $a_j, 1 \leq j \leq N$.

Because of rigidity of the $=$ predicate, if such a premise is true anywhere, it is true everywhere, and by completeness it is then also a theorem asserted in any context. So, given a theorem corresponding to the premise of the axiom of nesting:

$$\vdash x : a_1 \ldots a_k b_1 \ldots b_m = a_1 \ldots a_k e_1 \ldots e_n \tag{11.8}$$

we can prove

$$\vdash x : ist(b_1, \ldots, b_m, \lambda) \rightarrow ist(e_1, \ldots, e_n, \lambda) \tag{11.9}$$

from the axioms of the form (11.4), without using the axiom of nesting. There will then be a chain of equal terms $y_0 = \ldots = y_h$ such that

$$y_0 = a_1 \ldots a_k b_1 \ldots b_m$$

$$y_h = a_1 \ldots a_k e_1 \ldots e_n$$

and

$$\vdash x : y_{i-1} = y_i, 1 \leq i \leq h$$

because of an AFG equation of the form (11.3) and congruence. A proof of (11.9) is then obtained by a corresponding chain of inferences from this tautology

$$\vdash \epsilon : ist(a_1, \ldots, a_k, b_1, \ldots, b_m, \lambda) \rightarrow ist(a_1, \ldots, a_k, b_1, \ldots, b_m, \lambda)$$

using the corresponding AFG axioms of the form (11.4) and congruence, resulting in

$$\vdash \epsilon : ist(a_1, \ldots, a_k, b_1, \ldots, b_m, \lambda) \rightarrow ist(a_1, \ldots, a_k, e_1, \ldots, e_n, \lambda)$$

followed by k applications of RRI, ending in

$$\vdash x : ist(b_1, \ldots, b_m, \lambda) \rightarrow ist(e_1, \ldots, e_n, \lambda)$$

The derivation of the converse implication is symmetric.

On the strength of this, an AFG logic can be presented by taking associativity, as well as the algebraic equations and the induced equivalences on context terms, for granted, presenting the $\oplus$ operation, rigid interpretations, x-bundles and x-models in the terminology of the chosen algebra, giving axiom schemata of the form (11.4) in place of equations of the form (11.3) and deleting the axiom of nesting.

The following axioms and deduction rules are therefore common to all AFG systems:

Enter: $\dfrac{\vdash x : ist(c, \lambda)}{\vdash x \oplus c : \lambda}$ Exit: $\dfrac{\vdash x \oplus c : \lambda}{\vdash x : ist(c, \lambda)}$

Reflexivity: $\vdash x : y = y$

Congruence: $\vdash x : y = z \rightarrow (\lambda^v_y \rightarrow \lambda^v_z)$

PL: $\vdash x : \lambda$ whenever λ is an instance of a propositional tautology

MP: $\dfrac{\vdash x : \lambda \quad \vdash x : \lambda \rightarrow \chi}{\vdash x : \chi}$

K: $\vdash x : ist(c, \lambda \rightarrow \chi) \rightarrow (ist(c, \lambda) \rightarrow ist(c, \chi))$

$G^{2,0,1,1}$: $\vdash x : \neg ist(c, ist(d, \lambda)) \rightarrow ist(c, \neg ist(d, \lambda))$

UI: $\vdash x : \forall v.\lambda \rightarrow \lambda^v_t$ UG: $\dfrac{\vdash x : \xi \rightarrow \lambda^v_t}{\vdash x : \xi \rightarrow \forall v.\lambda}$ where v is not free in ξ

BF: $\vdash x : \forall v.ist(c, \lambda) \rightarrow ist(c, \forall v.\lambda)$ where c is not the variable v

In addition, there will be axioms of the form (11.4) corresponding to the AFG equations. In the subsections that follow, we show how this yields some context systems from the literature, as well as some new ones, by choosing different AFG algebras. For each one, we display the Enter/Exit rules, and the special axioms of the form (11.4).

11.1 Context sequences

The simplest option is to leave the algebra as purely associative, with no additional equations of the form (11.3). This expresses the intuition that changes to the surrounding context follow a LIFO discipline. This option is implicit in the propositional logic of context of Buvač, Buvač, and Mason [5], and in the quantificational logic of context of Serafini [14].

The $\oplus$ operation is then concatenation of finite sequences, denoted by juxtaposition. Each context name is identified with the singleton sequence containing itself. The equation expressing associativity is taken as implicit. An x-continuant is simply a sequence starting with x, and the concepts of x-bundles and x-continuants coincide. Identity of sequences is pairwise identity of elements.

The model structure is formulated as a mapping from finite sequences of contexts, denoted C^*, to sets of rigid first-order interpretations, which corresponds exactly to taking for granted the equivalence classes imposed by associativity alone.

This is the form of the Enter/Exit rules in purely associative AFG algebras:

$$\text{Enter: } \frac{\vdash x : ist(c, \lambda)}{\vdash xc : \lambda} \qquad \text{Exit: } \frac{\vdash xc : \lambda}{\vdash x : ist(c, \lambda)} \qquad \text{where } x \in C^*, c \in C$$

There are no explicit AFG axioms in this case, only associativity of $\oplus$ which is taken for granted.

In [5], a propositional logic of contexts is presented where the semantics is based on sequences. Its axioms and rules are identical to our K, $G^{2,0,1,1}$, PL, MP, and Exit. The proof of completeness in [5] is considerably more complex than ours, although that logic is without quantifiers and without identity predicates. and caters only for one particular algebra of contexts.

In [14], a similar system is presented on the basis of sequences of contexts. There, the logic has quantifiers but no identity predicates, and the axioms and rules coincide with our K, $G^{2,0,1,1}$, PL, MP, BF for the modal fragment. Completeness is proved by reduction to normal multi-modal $KG^{2,0,1,1}$. This seems to dispense with some of the expressivity of the special *ist* predicate, in as much as quantification over modalities is not a feature of normal multi-modal logic.

11.2 Context multisets

When dealing with independent contexts, it is reasonable to think that entering one context and then another should amount to the same as entering them in the opposite order. This corresponds to thinking about the surrounding context as a multiset, i.e. a collection where the order of the constituents is immaterial. This is expressed algebraically by commutativity of $\oplus$:

$$\vdash x : u \oplus v = v \oplus u$$

We observe that multiset equality can be expressed by associativity in conjunction with commutativity restricted to context constants:

$$\vdash x : c \oplus d = d \oplus c \tag{11.10}$$

because with associativity we can get any permutation of

$$c_1 \dots c_m$$

by a series of applications of (11.10) on adjacent elements.

Thus, we have an AFG presentation of multisets, and we get a logic of multisets of contexts by taking models to be mappings from multisets of contexts to sets of rigid first-order interpretations, taking $\oplus$ to be multiset union $\sqcup$ and identifying each context name c with the multiset $[c]$ containing only a single occurrence of itself, and taking context collections to be identical if they are equal as multisets.

This is therefore the form of the Enter/Exit rules for multisets (bags):

$$\text{Enter: } \frac{\vdash x : ist(c, \lambda)}{\vdash x \sqcup [c] : \lambda} \qquad \text{Exit: } \frac{\vdash x \sqcup [c] : \lambda}{\vdash x : ist(c, \lambda)} \qquad \text{where } x \in \text{bag}(C), c \in C$$

and the following AFG axiom schema expresses the necessary additional property of multiset continuants, namely commutativity:

$$\text{COMM:} \quad \vdash x : ist(c, d, \lambda) \leftrightarrow ist(d, c, \lambda)$$

The Hilbert system for context multisets is obtained from the common AFG rules and axioms by revising the Enter/Exit rules as shown here, and adding axiom COMM.

11.3 Context sets

If one feels that a particular item of context makes the same contribution to the accumulated outer context whether it is entered into once or more than once, then it is reasonable to say that context composition is idempotent:

$$\vdash x : u \oplus u = u \tag{11.11}$$

If taken in conjunction with commutativity as in the preceding subsection, the resulting context algebra is that of sets. Idempotence can then be adequately taken as an AFG equation

$$\vdash x : c \oplus c = c \tag{11.12}$$

because with associativity and commutativity it is possible to collect equal elements of

$$c_1, \ldots c_m$$

so that they are adjacent, and then repeatedly deleting an element where adjacent ones are equal by applying (11.12).

Thus, we get a logic of context sets by letting models be maps from sets of contexts to sets of rigid first-order interpretations, taking accumulated contexts to be identical if they are equal as sets, taking $\oplus$ to be set union $\cup$, identifying a context c with the singleton set $\{c\}$, taking this form of Enter/Exit:

$$\text{Enter: } \frac{\vdash x : ist(c, \lambda)}{\vdash x \cup \{c\} : \lambda} \qquad \text{Exit: } \frac{\vdash x \cup \{c\} : \lambda}{\vdash x : ist(c, \lambda)} \qquad \text{where } x \in 2^C, c \in C$$

and adding the following axioms to those common to all AFG systems:

$$\begin{aligned} &\text{COMM:} \quad \vdash x : ist(c, d, \lambda) \leftrightarrow ist(d, c, \lambda) \\ &\text{IDEM:} \quad \vdash x : ist(c, c, \lambda) \leftrightarrow ist(c, \lambda) \end{aligned}$$

11.4 Flat contexts

For applications where the importance of each item of context is considered to be independent of where it is examined from, the authors of [5] propose a model structure for propositional formulas asserted in context, where only the last context entered into has any bearing on validity. In [4] the same motivation and a similar constraint is applied to a quantificational logic of context. This type of context model has come to be known as 'flat'.

The following algebraic equation corresponds to flatness:

$$\vdash x : u \oplus v = v \tag{11.13}$$

Let us reflect on some of its consequences. First, we note that (11.13) subsumes associativity:

$$(u \oplus v) \oplus w = v \oplus w = u \oplus (v \oplus w)$$

With associativity, equation (11.13) can adequately be taken as an AFG equation

$$\vdash x : c \oplus d = d \tag{11.14}$$

because given a sequence of contexts containing two adjacent but otherwise arbitrary subsequences $c_1 \ldots c_m$ and $d_1 \ldots d_n$:

$$b_1 \ldots b_i c_1 \ldots c_m d_1 \ldots d_n e_1 \ldots e_j$$

the subsequence $c_1 \ldots c_m$ can be removed by repeatedly removing the left neighbour of d_1 by application of (11.14).

The equivalence classes imposed by equation (11.13) are singleton sets, since associativity of $\oplus$ is implied, and each finite but nonempty sequence is equal to the singleton sequence containing its rightmost element. The equivalence classes can therefore be identified with individual contexts.

Equation (11.13) also implies that every context c is a continuation of every other context d, so any flat c-model is also a flat d-model. We can therefore talk about flat models per se, without rooting them at context terms.

This has the interesting consequence that

$$\models c : \neg ist(d, \bot)$$

in the class of all flat models. By completeness, it follows that the logic of flat contexts has the theorem

$$\vdash c : \neg ist(d, \bot)$$

which is comparable to axiom schema (D) $\neg\Box\bot$ of modal logic. Suppose in fact that a flat model M has $M(c_1) = \emptyset$ for some context c_1. Then for every rigid interpretation $I \in M(c_1)$ we have, vacuously, $M, I \models c_1 : ist(c_2, \bot)$ for arbitrary c_2, which by flatness and the model conditions is equivalent to $M, J \models c_2 : \bot$ for every $J \in M(c_2)$. This can only be the case if also $M(c_2) = \emptyset$, and then $M(c) = \emptyset$ for every c. Hence the only flat model which verifies $c : ist(d, \bot)$ is the inconsistent model, and that verifies everything, also $c : \neg ist(d, \bot)$.

The axiom schema corresponding to (11.13) is $\vdash x : ist(c, d, \lambda) \leftrightarrow ist(d, \lambda)$, but considering that x in this case will reduce to an atomic context, the following adheres to our notational conventions:

$$\text{FLAT:} \quad \vdash b : ist(c, d, \lambda) \leftrightarrow ist(d, \lambda) \tag{11.15}$$

Thus, we get a quantificational logic of flat contexts by letting models be functions from contexts to sets of rigid first-order interpretations, considering sequences of contexts to be identical according to equality of their rightmost elements, taking this form of the Enter/Exit rules:

$$\text{Enter: } \frac{\vdash b : ist(c, \lambda)}{\vdash c : \lambda} \qquad \text{Exit: } \frac{\vdash c : \lambda}{\vdash b : ist(c, \lambda)} \qquad \text{where } b, c \in C$$

and adding axiom (11.15) to the common AFG system.

This model structure and Hilbert system for flat contexts match those in [4], but here we have arrived at them by specialization within a more general logical framework. Flat contexts were also obtained as a special case of a fibred logic of context in [8].

12 Concluding remarks

We have developed a quantificational logic of contexts which allows arbitrary algebras of context combination, and displayed several special cases within the class of Associative Finite Ground algebras, including novel systems as well as systems coinciding with those of other workers in the field.

Ground equations in semigroups were first studied by Axel Thue [17] early in the 20^{th} century, and this became one of the roots of general term rewriting systems. In the literature, there is little information about non-ground algebras of context with efficiently computable normal forms. It seems that labelled deductive systems in the style of [7] would be a natural setting in which to study this topic.

The *ist* modality is by no means the only formalization of context, see [13, 2, 15, 16] for some alternatives, and e.g. [1, 3] for surveys. Some of the alternatives cater more naturally for variations in language among contexts, and multilanguage systems for context [15] seem particularly well suited to this. Multilanguage systems with hiearchical structure have been described, but apparently there are no studies so far of multilanguage systems of context with an algebraic structure of the same scope as the one we have treated here. This therefore remains an interesting topic for future study.

Appendix: proofs of derived axiom schemata (cfr. page 16)

The proofs of K^n, $\mathrm{G}^{n+1,0,n,1}$, BF^n, and AND^n are by induction on n. The bases, where $n = 1$, are just K, $\mathrm{G}^{2,0,1,1}$, and BF in the first three cases, and for AND^n it is this:

$$\begin{array}{rll}
\mathrm{AND}: & 1 \vdash x \oplus c : \lambda \wedge \chi \to \lambda & \text{PL} \\
& 2 \vdash x : ist(c, \lambda \wedge \chi) \to ist(c, \lambda) & \text{1, K, MP} \\
& 3 \vdash x : ist(c, \lambda \wedge \chi) \to ist(c, \chi) & \text{symmetric} \\
& 4 \vdash x : ist(c, \lambda \wedge \chi) \to ist(c, \lambda) \wedge ist(c, \chi) & \text{2, 3, PL} \\
& 5 \vdash x \oplus c : \lambda \to (\chi \to \lambda \wedge \chi) & \text{PL} \\
& 6 \vdash x : ist(c, \lambda) \to (ist(c, \chi) \to ist(c, \lambda \wedge \chi)) & \text{5, K twice, PL} \\
& 7 \vdash x : ist(c, \lambda) \wedge ist(c, \chi) \to ist(c, \lambda \wedge \chi) & \text{6, PL} \\
& 8 \vdash x : ist(c, \lambda \wedge \chi) \leftrightarrow ist(c, \lambda) \wedge ist(c, \chi) & \text{4,7, PL}
\end{array}$$

For the induction steps, fix an $n > 0$ and assume K^n, $\mathrm{G}^{n+1,0,n,1}$, BF^n, and AND^n as inductive hypotheses.

$$\begin{array}{rll}
\mathrm{K}^{n+1}: & 1 \vdash x \oplus c_1 : ist(c_2, \ldots, c_{n+1}, \lambda \to \chi) \to (ist(c_2, \ldots, c_{n+1}, \lambda) \to ist(c_2, \ldots, c_{n+1}, \chi)) & \text{ind. hyp.} \\
& 2 \vdash x : ist(c_1, ist(c_2, \ldots, c_{n+1}, \lambda \to \chi) \to (ist(c_2, \ldots, c_{n+1}, \lambda) \to ist(c_2, \ldots, c_{n+1}, \chi))) & \text{1, Exit} \\
& 3 \vdash x : ist(c_1, \ldots, c_{n+1}, \lambda \to \chi) \to (ist(c_1, \ldots, c_{n+1}, \lambda) \to ist(c_1, \ldots, c_{n+1}, \chi)) & \text{2, K twice}
\end{array}$$

$$\begin{array}{rll}
\mathrm{G}^{n+2,0,n+1,1}: & 1 \vdash x \oplus c_1 : \neg ist(c_2, \ldots, c_{n+2}, \lambda) \to ist(c_2, \ldots, c_{n+1}, \neg ist(c_{n+2}, \lambda)) & \text{ind. hyp.} \\
& 2 \vdash x : ist(c_1, \neg ist(c_2, \ldots, c_{n+2}, \lambda) \to ist(c_2, \ldots, c_{n+1}, \neg ist(c_{n+2}, \lambda))) & \text{1, Exit}
\end{array}$$

$3 \vdash x : \neg ist(c_1, \ldots, c_{n+2}, \lambda) \rightarrow ist(c_1, \ldots, c_{n+1}, \neg ist(c_{n+2}, \lambda))$ 2, $G^{2,0,1,1}$, PL

BF^{n+1} : $1 \vdash x \oplus c_1 : \forall v.ist(c_2, \ldots, c_{n+1}, \lambda) \rightarrow ist(c_2, \ldots, c_{n+1}, \forall v.\lambda)$ ind. hyp.

$2 \vdash x : ist(c_1, \forall v.ist(c_2, \ldots, c_{n+1}, \lambda) \rightarrow ist(c_2, \ldots, c_{n+1}, \forall v.\lambda))$ 1, Exit

$3 \vdash x : \forall v.ist(c_1, \ldots, c_{n+1}, \lambda) \rightarrow ist(c_1, \ldots, c_{n+1}, \forall v.\lambda)$ 2, BF, PL

AND^{n+1} : $1 \vdash x \oplus c_1 : ist(c_2, \ldots, c_{n+1}, \lambda \wedge \chi) \leftrightarrow ist(c_2, \ldots, c_{n+1}, \lambda) \wedge ist(c_2, \ldots, c_{n+1}, \chi)$ ind. hyp.

$2 \vdash x : ist(c_1, ist(c_2, \ldots, c_{n+1}, \lambda \wedge \chi) \leftrightarrow ist(c_2, \ldots, c_{n+1}, \lambda) \wedge ist(c_2, \ldots, c_{n+1}, \chi))$ 1, Exit

$3 \vdash x : ist(c_1, \ldots, c_{n+1}, \lambda \wedge \chi) \leftrightarrow ist(c_1, ist(c_2, \ldots, c_{n+1}, \lambda) \wedge ist(c_2, \ldots, c_{n+1}, \chi)))$ 2, K twice

$4 \vdash x : ist(c_1, \ldots, c_{n+1}, \lambda \wedge \chi) \leftrightarrow ist(c_1, \ldots, c_{n+1}, \lambda) \wedge ist(c_1, \ldots, c_{n+1}, \chi)$ 3, AND, PL

This completes the inductive proofs of K^n, $G^{n+1,0,n,1}$, BF^n, and AND^n. Finally, the schema OR^n is an easy consequence of K^n :

OR^n : $1 \vdash x : \lambda \rightarrow \lambda \vee \chi$ PL

$2 \vdash x : ist(c_1, \ldots, c_n, \lambda) \rightarrow ist(c_1, \ldots, c_n, \lambda \vee \chi)$ 1, K^n

$3 \vdash x : ist(c_1, \ldots, c_n, \chi) \rightarrow ist(c_1, \ldots, c_n, \lambda \vee \chi)$ symmetric

$4 \vdash x : ist(c_1, \ldots, c_n, \lambda) \vee ist(c_1, \ldots, c_n, \chi) \rightarrow ist(c_1, \ldots, c_n, \lambda \vee \chi)$ 2, 3, PL

∎

Appendix: proof of Lemma 8.9 (Autovalidation)

If $ist(c_1, \ldots, c_m, \perp) \in \Omega$ then $\Omega_{c_1,\ldots,c_m}$ is x-inconsistent and then $M((x \oplus c_1) \ldots \oplus c_m) = \emptyset$ by definition, so the claim is vacuously true. Therefore we proceed with $ist(c_1, \ldots, c_m, \perp) \notin \Omega$. The proof is by structural induction on χ:

$\chi \in P$ The base case follows directly from the definition of M.

χ is $\neg \ldots$ This case splits into subcases as follows:

χ is $\neg p$ with $p \in P$: again the claim follows directly from the definition of M.

χ is $\neg\neg\mu$: we may simplify χ by deleting the double negation, and the result follows immediately by the inductive hypothesis.

χ is $\neg(\mu_1 \wedge \mu_2)$: we may rewrite χ as $\neg\mu_1 \vee \neg\mu_2$, and this case is treated below.

χ is $\neg(\mu_1 \vee \mu_2)$: we may rewrite χ as $\neg\mu_1 \wedge \neg\mu_2$, and this case is treated below.

χ is $\neg\forall v.\mu$: By the construction in lemma 8.3, we have $\neg ist(c_1, \ldots, c_m, \neg\forall v.\mu) \in \Omega$ iff also $\neg ist(c_1, \ldots, c_m, \neg\mu^v_u) \in \Omega$ for some unused u of correct sort. Therefore, by x-maximality of Ω, $ist(c_1, \ldots, c_m, \neg\forall v.\mu) \in \Omega$ iff $ist(c_1, \ldots, c_m, \neg\mu^v_u) \in \Omega$ for every u of correct sort. By inductive hypothesis the latter is true iff $M \models ((x \oplus c_1) \ldots \oplus c_m) : \neg\mu^v_u$ for every u of correct sort, in other words $M \not\models ((x \oplus c_1) \ldots \oplus c_m) : \mu^v_u$ for every u of correct sort, which is true iff, by the model conditions, $M \models ((x \oplus c_1) \ldots \oplus c_m) : \neg\forall v.\mu$

χ is $\neg ist(c_{m+1}, \mu)$: we have assumed that $ist(c_1, \ldots, c_m, \perp) \notin \Omega$, so lemma 8.8 applies, and we have this chain of equivalent statements:
$ist(c_1, \ldots, c_m, \neg ist(c_{m+1}, \mu)) \in \Omega$ iff $\neg ist(c_1, \ldots, c_{m+1}, \mu) \in \Omega$
iff $ist(c_1, \ldots, c_{m+1}, \mu) \notin \Omega$ iff $ist(c_1, \ldots, c_m, ist(c_{m+1}, \mu)) \notin \Omega$
iff (by the induction hypothesis) $M \not\models ((x \oplus c_1) \ldots \oplus c_m) : ist(c_{m+1}, \mu)$
iff $M \models ((x \oplus c_1) \ldots \oplus c_m) : \neg ist(c_{m+1}, \mu)$

χ is $\mu_1 \wedge \mu_2$ By schema (6.4), $ist(c_1, \ldots, c_m, \mu_1 \wedge \mu_2) \in \Omega$ iff $ist(c_1, \ldots, c_m, \mu_1) \wedge ist(c_1, \ldots, c_m, \mu_2) \in \Omega$ iff (by x-maximality of Ω) $ist(c_1, \ldots, c_m, \mu_1) \in \Omega$ and $ist(c_1, \ldots, c_m, \mu_2) \in \Omega$, iff, by induction, $M \models ((x \oplus c_1) \ldots \oplus c_m) : \mu_1$ and $M \models ((x \oplus c_1) \ldots \oplus c_m) : \mu_2$, iff, by the model conditions, $M \models ((x \oplus c_1) \ldots \oplus c_m) : \mu_1 \wedge \mu_2$

χ is $\mu_1 \vee \mu_2$ First we prove that $ist(c_1, \ldots, c_m, \mu_1 \vee \mu_2) \in \Omega$ implies $M \models ((x \oplus c_1) \ldots \oplus c_m) : \mu_1 \vee \mu_2$. Since we have $ist(c_1, \ldots, c_m, \perp) \notin \Omega$ it follows that either $ist(c_1, \ldots, c_m, \mu_1 \vee \mu_2) \notin \Omega$ or $ist(c_1, \ldots, c_m, \neg\mu_1 \wedge \neg\mu_2) \notin \Omega$. So if $ist(c_1, \ldots, c_m, \mu_1 \vee \mu_2) \in \Omega$ then $ist(c_1, \ldots, c_m, \neg\mu_1 \wedge \neg\mu_2) \notin \Omega$, which is to say that either $ist(c_1, \ldots, c_m, \neg\mu_1) \notin \Omega$ or $ist(c_1, \ldots, c_m, \neg\mu_2) \notin \Omega$. Now, by the inductive hypothesis, this is so iff $M \not\models ((x \oplus c_1) \ldots \oplus c_m) : \neg\mu_1$ or $M \not\models ((x \oplus c_1) \ldots \oplus c_m) : \neg\mu_2$, if and only if, by the model conditions, $M \models ((x \oplus c_1) \ldots \oplus c_m) : \mu_1 \vee \mu_2$

Now let us prove that $M \models ((x \oplus c_1) \ldots \oplus c_m) : \mu_1 \vee \mu_2$ implies $ist(c_1, \ldots, c_m, \mu_1 \vee \mu_2) \in \Omega$. The premise is true iff $M \models ((x \oplus c_1) \ldots \oplus c_m) : \mu_1$ or $M \models ((x \oplus c_1) \ldots \oplus c_m) : \mu_2$, which by induction is equivalent to $ist(c_1, \ldots, c_m, \mu_1) \in \Omega$ or $ist(c_1, \ldots, c_m, \mu_2) \in \Omega$. By schema (6.5) and x-maximality of Ω this implies that $ist(c_1, \ldots, c_m, \mu_1 \vee \mu_2) \in \Omega$.

χ is $\forall v.\mu$ by the construction in lemma 8.3, we have $\neg ist(c_1, \ldots, c_m, \forall v.\mu) \in \Omega$ iff $\neg ist(c_1, \ldots, c_m, \mu^v_u) \in \Omega$ for some u of correct sort. So, by x-maximality, $ist(c_1, \ldots, c_m, \forall v.\mu) \in \Omega$ iff $ist(c_1, \ldots, c_m, \mu^v_u) \in \Omega$ for all u of correct sort, iff, by the inductive hypothesis, $M \models ((x \oplus c_1) \ldots \oplus c_m) : \mu^v_u$ for all u of correct sort, iff, by the model conditions, $M \models ((x \oplus c_1) \ldots \oplus c_m) : \forall v.\mu$

χ is $ist(c_{m+1}, \mu)$ $ist(c_1, \ldots, c_m, ist(c_{m+1}, \mu)) \in \Omega$ is just $ist(c_1, \ldots, c_{m+1}, \mu) \in \Omega$, which, by inductive hypothesis, is true iff $M \models ((x \oplus c_1) \ldots \oplus c_{m+1}) : \mu$, in other words $M \models ((x \oplus c_1) \ldots \oplus c_m) : ist(c_{m+1}, \mu)$.

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