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An introduction to TWG1:
Argumentation and proof

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Introduction

Argumentation and proof is a research area that continues to attract the wide interest of the mathematics education research community, which was also evident in CERME13. CERME13 was the first in-person conference after a period of online meetings, which was reflected on the great interest of the participants who were actively engaged in practicing and promoting communication, cooperation, and collaboration. In Thematic Working Group 1 (TWG1; "Argumentation and proof"), 48 participants from 20 different countries discussed and reflected upon the 34 papers and 6 posters presented in TWG1. The papers of TWG1 were organized in four topics: a) Argumentation and proof in teacher training and education, b) Argumentation and proof in school and university mathematics, c) Theoretical and epistemological perspectives about argumentation and proof, and d) Argumentation and proof in mathematical activity, modelling, and problem solving. The work of TWG1 was organized in whole-group sessions and (parallel) split-group sessions to maximize active participation and to ensure the coherence and unity of the TWG1 spirit in the split-groups. The main points of interest of each split-group session were reported in a whole-group session, so that an overview of all TWG1 papers would be obtained. In this introductory chapter, drawing upon the discussions of our group, the TWG1 papers are organized in three themes: a) Theoretical and epistemological perspectives about argumentation and proof, b) Argumentation and proof in teacher education and training, and c) Argumentation and proof in school and university.

Theoretical and epistemological perspectives about argumentation and proof

This theme appears to transcend our CERME meetings with various theoretical perspectives being employed to expand the answer space of already posed questions and to pose new questions and ideas pertaining to argumentation and proof. Considering argumentation, Sheena Tan discussed the relationship between the dialectic and the dialogic aspects of argumentation, drawing upon Toulmin and Bakhtin, respectively. On the other hand, Amedeo Matteucci and Francesco Saverio Tortoriello drew upon scientific debate theory (classic and modern) to propose the employment of debate in mathematics teaching, while Marta Saccoletto and Pier Luigi Ferrari discussed the interactional and

linguistic aspects of argumentation. Moreover, Miglena Asenova introduced novel theoretical lenses to analyse both the epistemic and logical aspects of students' argumentation.

Considering proof, Thibaut Trouvé discussed the notion of genericity in mathematics education differentiating proof on a generic instance from the generic character of a proof to discuss the transition from pragmatic to conceptual proofs. Paolo Boero and Nadia Azrou employed Habermas's rationality and introduced the construct rationality into proving to investigate the interplay between content knowledge and meta-knowledge in prospective teachers' proving behaviour. Joachim Frans and Karen Francois re-visited the notion of explanatory proof by proposing a research approach that employs the lenses of philosophy of mathematics and mathematics education research in mathematical research papers and mathematics textbooks. Further, Leander Kempen and Eva Müller-Hill discussed a conceptualization of the enculturation function of mathematical proof and its pedagogical implications in the teaching and learning of mathematics, linking the general education discourse about science with teacher competence. Finally, Katalin Gosztonyi and Simon Modeste attempted to establish a theoretical model for describing networks of problems, their interrelations, and their relations with mathematical content, for didactical purposes.

Argumentation and proof in teacher education and training

In the past, at CERME12, several contributions investigated the teaching argumentation and proof, to address the need for such research projects that was noted in the TWG1 Introduction section of the CERME11 proceedings. In CERME13, the teaching of argumentation and proof continued to gather the interest of several researchers, spanning across the educational levels. A significant number of contributions explicitly concentrated on this theme by considering teacher education and training, including pre-service, novice, and experienced teachers.

Considering the elementary level, pre-service primary school teachers were the population of interest of four contributions: Lara Gayer investigated the different understandings of pre-service primary school teachers while they were engaged in example-based visual proofs; Marit Buset Langfeldt, Anita Valenta, and Torkel Haugan Hansen explored the ways that pre-service teachers' use of the word explain when evaluating the students' arguments; Jakub Michal investigated the pre-service teachers' conceptions and beliefs of reasoning and proof in primary school mathematics; Lisa Eggerichs focused on the mathematical justification strategies that pre-service teachers employ when they attempt to produce a valid proof in the field of arithmetic. Furthermore, Dimitrios Deslis, Andreas Stylianides and Mateja Jamnik considered both pre-service and in-service Greek primary school teachers to investigate their mathematical knowledge and views about Lakatos-style proving activity with the purpose of identifying qualitatively different profiles (by utilizing latent profile analysis).

Considering the secondary education level, Peter Vankúš and Michaela Vargová concentrated on pre-service teachers to analyze their errors and misconceptions with respect to five proving tasks. Orly Buchbinder discussed a dual case study that explored the ways that novice teachers recontextualize the teaching of mathematics via reasoning proving by following for two years two beginning secondary mathematics teachers as students in a capstone course and as full interns in secondary schools. Maria Alessandra Mariotti, Eszter Kónya, and Zoltán Kovács explored pre-service

mathematics teachers' justifications and refutations for construction of a regular pentagon by paper folding, while engaged in collaborative argumentation. Lakatos-style proving activity was also at the crux of a study by Mei Yang, Andreas Stylianides, and Mateja Jamnik who focused on Chinese pre-service and in-service secondary mathematics teachers' orientations of noticing and on the ways that these orientations are linked with the teachers' framings of their professional obligations. Furthermore, Fiene Bredow and Christine Knipping investigated in-service teachers' actions that support or hinder students' mathematical argumentation processes in eight-grade classrooms.

Researchers attempted to consider multiple perspectives in their investigations about teaching argumentation and proof. At the university level, Nadia Azrou focused on the university teaching of proof by considering the views of both university students and their instructors about the teaching and the learning of proof and proving and the related difficulties. Horacio Solar, Andrés Ortiz, Victoria Arriagada, and Marco Catalán investigated the development of argumentation in the different phases of the modelling cycle, including in-service teachers at the elementary and secondary education levels. Following an interdisciplinary perspective, Andreas Moutsios-Rentzos, in a theoretical paper, employed a systemic communicational approach to link argumentation in mathematics and physics (in textbooks and task design) with the purpose of supporting teachers who teach the two courses to gain deeper understanding of what constitutes institutionally acceptable scientific evidence and scientific inference in mathematics and physics.

Argumentation and proof: from school to university

Another central theme of TWG1 in CERME13, which was also present in CERME12, concentrated on the different aspects of the teaching and learning of argumentation and proof at the different educational levels. Regarding the school level, Kayo Miura and Yusuke Shinno employed Toulmin's scheme and Balacheff's levels of validation of proof to identify the structural characteristics of generic arguments using examples and counterexamples as produced by primary school students. Furthermore, Mickaël Da Ronch, Marie-Line Gardes, and Ismaïl Mili discussed a study that explored the potential of problems to practice a research activity in mathematics at elementary school, whilst Da Zhou presented a study that focussed on evaluating the sixth graders' cognitive and categorical processes of argumentation. Moreover, Trond Stølen Gustavsen and Andrea Hofmann presented two papers deriving from a project examining proof-based teaching in grade 8: in the first paper they investigated the possibility of employing proof-based teaching of fraction multiplication, whilst in their second paper they discussed the opportunities that six eighth graders had for proving the rule for fraction multiplication with two proper fractions.

Considering the university level, Valentina Postelnicu, Mario Gonzalez, and Florin Postelnicu discussed a study investigating the didactical engineering involved in designing tasks that may facilitate the undergraduate students to understand the principle of mathematical induction by using analogies. Ieva Kilienė and Rimas Norvaiša discussed a study with first year university mathematics students, which focused on their construction of a deductive argument when solving word problems.

Several papers in our group focussed on the socio-cultural and other contextual aspects of argumentation and proof. Karolína Mottlová and Ieva Kilienė considered the linguistic specificity of Czech and Lithuanian languages to compare students' problem-solving strategies and approaches

when solving "What numbers make sense?" problems. Kinga Szűcs considered upper-secondary school textbooks in different German federal states and in Hungary to investigate the significance of proof-related reasoning related to the law of sines. Furthermore, Jarmila Novotná and Mária Slavíčková analyzed two textbooks (one from Slovakia and one from the Czech Republic) to investigate their different approaches to reasoning and proving in topics about symmetry. Yusuke Shinno and Takeshi Miyakawa considered another topic in geometry, the Pythagorean theorem, to investigate the cultural aspects of what is valued as proof and proving in a Japanese ninth grade classroom. Finally, Cécile Ouvrier-Buffet discussed students' conceptions of proof at the transition between secondary and tertiary level, with a particular interest to propose an instrument that may enable collaborations in Europe about this topic.

Conclusions and future directions for TWG01

We posit that CERME13 offered the TWG1 participants the opportunity to be engaged in rich, broad, and deep, discussions about a variety of issues and perspectives that spanned across different educational levels and contexts; some enriched ideas discussed in previous CERMEs, whereas others provided novel perspectives and ideas. Our fruitful discussions offered a series of questions that paint potential paths of future research projects.

The role of the broad notion of *context* in argumentation and proof research appeared to be an increasingly complex factor, as it may involve: a) *intra-mathematical* aspects (including the specific mathematical content that a study may entail or the specificities of different educational levels), b) *interdisciplinary* aspects and links with other disciplines, and c) *socio-cultural* aspects. For example, we pondered about: What is the role of mathematical content in argumentation and proof research? What is the role of sociocultural aspects and language in argumentation and proof research? What are the interactions of research in mathematical argumentation and proof and other disciplines? Taking also into consideration the fact that there is an increased adoption of *blended learning environments*, theoretical and methodological tools need to be adopted, adapted, and/or developed to address this contextual complexity.

Considering the teaching and learning of argumentation and proof, we noticed a growing number of studies, which also appear to address the complexity of the mathematics classroom (and sometimes beyond the mathematics classroom), to consider the *multiple, potentially diverging perspectives of the interacting roles and intentions within a complex educational system*. Such approaches complement analytical approaches by allowing gaining deeper understanding about the complex phenomena of argumentation and proof in a given educational setting, thus allowing a more effective design for supporting the teachers and the learners.

These areas of interest echo the ideas discussed in our TWG1 meetings, and do not imply that specific lines of research are favoured over others. Argumentation and proof are approached from diverse perspectives (and in other TWGs) and in TWG1 we are committed to voicing this variety.