The Mathematics Enthusiast

Manuscript 1626

# The spirit of mathematical modeling – a philosophical study on the occasion of 50 years of mathematical modeling education

Peter Frejd

Pauline Vos

**The spirit of mathematical modeling –**

**a philosophical study on the occasion of 50 years of mathematical modeling education**

Peter Frejd[1]
Linköping University, Sweden

Pauline Vos
University of Agder, Norway

Abstract: We mark the 50th anniversary of mathematical modeling education by reviving the term *the spirit of mathematical modeling* (SoMM), which idealistically reflects core aspects of mathematical modeling. The basis of our analysis is the notion of *bildung*, which is an educational philosophy that strives for harmonizing heart, mind, social life and culture. We built SoMM on five descriptions of mathematical modeling: two research studies from the 1970s, two studies about the work of professional modelers, and one about an environmental school project. We captured SoMM as a collection of aspects at the micro, meso and macro level: at the micro level, we found aspects such as agency, anticipating, scrutiny and critique as part of SoMM; at the meso level, we found collaborating, consulting and navigating social norms; and at the macro level, interdisciplinarity, relevance and social justice. Through the lens of bildung-based educational philosophies, we see that instruction and assessment traditions have transposed mathematical modeling into 'teachable' practices that drift away from SoMM. We recommend focusing more on *fostering* mathematical modeling and to assess students through alternative formats (e.g., group projects).

*Keywords*: Agency; bildung; collaboration; evidence-based education; fostering; social justice; mathematical modelling education

## 1 Introduction

Mathematical modeling, that is, using mathematics to solve real world problems, has gained importance in curricula around the globe (Common Core State Standards Initiative, 2010; Niss & Blum, 2020). This development means that we have witnessed a substantial change: around 1970, modeling was still marginal in mathematics curricula, but from 1990 onwards, mathematical modeling started to make its appearance into mainstream mathematics education in a growing number of countries (Blum & Niss, 1991). The ongoing development has led to new manifestations within mathematics education, such as new lesson materials, in which mathematical modeling activities for middle schools, high schools, and teacher education are bundled (e.g., Erbaş et al. 2016; Gould et al., 2012; LEMA, 2009; Maaß, 2008; Maaß et al., 2018). Also, international comparative assessment studies now include modeling tasks (OECD,

[1] peter.frejd@liu.se

2016; Turner, 2007; Stacey, 2015), and in research, we see an increasing number of academic journals publishing 'special issues' on mathematical modeling education, such as in ZDM in 2006 and 2018 (Kaiser, Blomhøj, & Sriraman, 2006; Schukajlow, Kaiser, & Stillman, 2018), MERJ in 2010 (Stillman, Brown, & Galbraith, 2010), AEIM in 2021 (Barquero, 2020), and ESM, MTL and Quadrante in 2022 (Carreira, & Blum, 2021; Cevikbas, Kaiser, & Schukajlow, 2022; Frejd & Vos, 2022; Schukajlow, Kaiser, & Stillman, 2021).

The movement that advocated mathematical modeling as an integral part of mathematics education started sometime in the 1970s. In their historical account, Houston, Galbraith and Kaiser (2009) mark the start in the year 1973, when a survey about mathematical graduates in the industries revealed that many of them did not have the competences that the industries expected from them (McLone, 1973, as cited in Houston, Galbraith, & Kaiser, 2009). At that time, most western universities and schools still offered a type of academic mathematics, known as *New Math*, which was shaped and framed by pure mathematicians and was related to the 'Sputnik shock', when Russia launched a satellite showing their technological supremacy (Kilpatrick, 2012). However, from 1970 onwards, employers and governments as 'clients of education' started increasingly to demand that schools and universities should not just focus on the transmission of scholarly knowledge but be effective in preparing school leavers for the job market (Teddlie & Stringfield, 2007). As a result, for example, language education was reformed from plain word reading to reading comprehension (Pearson & Cervetti, 2015). Regarding mathematics education, there came a call for problem solving and mathematical modeling (Kilpatrick, 2012). Some of the reasons given to advance mathematical modeling pointed at the position of future citizens and employees. One argument was the *utility* argument, which entailed that mathematics education should provide students with tools "to utilize mathematics for solving problems in or describing aspects of specific extra-mathematical areas and situations, whether referring to other subjects or occupational contexts ("mathematics as a service subject") or to the actual or future everyday lives of students" (Blum & Niss, 1991, p. 43). Another argument (Blum & Niss, 1991, p. 43) was the *critical competence argument*, which focuses on preparing students in all grades to:

> … live and act with integrity as private and social citizens, possessing a critical competence in a society the shape and functioning of which are being increasingly influenced by the utilization of mathematics through applications and modelling. The aim of such a critical competence is to enable students to "see and judge" independently, to recognize, understand, analyse and assess representative examples of actual uses of mathematics, including (suggested) solutions to socially significant problems.

Taking 1973 as starting point, we mark with this paper 50 years of advancing mathematical modeling education and aim to look backward and forward. In particular, we want to revive discussions from the early days on what is important in mathematical modeling education, that is, what comprises the 'soul' of mathematical modeling.

In looking backward and forward, our analysis cannot be detached from broader discussions on the goals of education across the globe (Biesta, 2010). One side in the discussion aims for more evidence-based education that favors accountability, whereby stakeholders in an educational system are informed about qualities, improvements and effectivity through 'evidence', which is based on tests that measure the qualities of students, teachers, textbooks, schools, curricula, and so forth. The tests needed for this quality management are based on sets of listable and testable criteria. The tests are called 'standardized' tests to express their quality. Students' competences are expressed as scores on these. The other side in the discussion aims for a *bildung-based* education (Biesta, 2010) and draws on arguments from many ranges, among which are writings by important historical figures. For instance, in the US, the writings by the *Founding Fathers* such as George Washington or Benjamin Franklin are cited to explain that the goals of education are political and moral, for students to develop honesty, integrity, and compassion and learn to make good judgements, although these are assets that cannot be validly and reliably tested (Rothstein & Jacobsen, 2005). In some European countries, the argumentation builds on the respective terms *Bildung* (in Germany), *bildning* (Swedish), *dannelse* (in Norwegian and Danish), or *vorming* (in Dutch), which express that education should be a life-long process of both personal and cultural maturation, with an important role for creativity, responsibility and values, whereby the individual's mind and heart harmonize with broader social life and culture (Bildung, n.d.). The weakness of such idealistic and vague goals, however, is that they cannot be objectively evaluated.

In this paper, we do not aim at resolving the two sides of the discussion but consider the distinction useful for discussing important aspects in mathematical modeling education. Over the past 50 years, the conceptualizations of *mathematical modeling* have evolved. This happened not because of changing times, but adaptations always occur when certain knowledge and skills from outside school are adapted to become 'teachable' to students. This process of *didactic transposition* is affected by constraints within the target institution and by actors and interests influencing the process (Chevallard, 1991; Barquero, Bosch, & Romo, 2018). Within this process, educational philosophies, whether aiming for more evidence-based or bildung-based education, are at play in conceptualizations and implementations.

Several metaphors have been used over the years to define and describe central aspects of mathematical modeling such as, “the core of modeling” (Greer & Verschaffel, 2007, p. 219) or “at the very heart of mathematical modeling” (Naylor, 1989, p. 324). Another metaphor, used in particular within the International Community of Teachers of Mathematical Modeling and Applications (ICTMA), is *the spirit of mathematical modeling* (Dyke, 1987; Hersee, 1987; Kaiser & Brand, 2015; Lamon, 2003). This metaphor has, for example, been used to clarify two contrasting goals of modeling education (Dyke, 1987, p. 43, our italics).

> In designing a modelling course within a BTEC scheme, it is necessary to be very prescriptive in order to satisfy the criteria laid down by BTEC. This might be thought to be against *the spirit of true mathematical modelling*, which should be as open-ended as possible.

This quote highlights that a teacher or educator should, on the one hand, design modeling activities that fit criteria defined by some authority (evidence-based). On the other hand, the designed activities should offer openness to students (bildung-based). The second side of mathematical modeling is, apparently, described by the metaphor *the spirit of true mathematical modeling*. This quote by Dyke (1987) suggests that the two sides of the more general educational debate, evidence based and bildung- based, are also present in discussions about mathematical modeling education. Also, it suggests that the bildung-based side connects to something called *the spirit of mathematical modeling*.

In this paper, we aim to discuss the two sides of modeling education and, in particular, study:

> *What aspects are in the spirit of mathematical modeling?*

To answer this question, we selected and analyzed literature published in the last 50 years. Before presenting the methodology and the actual analysis, we start by describing some recent developments in research on mathematical modeling education.

## 2 Background

### 2.1 Strands in recent research on mathematical modeling education

Over the past 50 years, the body of research on mathematical modeling education has steadily grown. In these studies, researchers emphasize different aspects within mathematical modeling activities. According to reviews by Cevikbas et al. (2021), Geiger and Frejd (2015), Kaiser and Brand (2015), Schukajlow et al. (2018) and Stillman (2019), the largest strand of research focuses on modeling competencies. These are an individual student’s cognitive abilities to solve modeling problems. A considerable number of researchers define modeling competencies by building on the so-called modeling cycle (Blum, 2015), which incorporates the phases a modeler undertakes: translating a problem into mathematics and constructing a model of the situation, as well as interpreting and evaluating the mathematical outcomes. Additionally, some definitions include willingness (Maaβ, 2006), readiness to act (Kaiser, 2017) and metacognition (Stillman, 1998; Vorhölter, 2018). Much research on mathematical modeling competencies is qualitative and aims to describe how students, often working in groups, cope with tasks (Cevikbas et al., 2022). Approximately one third of the research on competencies is quantitative and based on measuring competencies. Here, researchers slice the modeling competencies according to the phases in the modeling cycle, and sometimes further refine the competencies into sub-competencies. They then map students' answers on written tests to score the (sub-)competencies. Typical research here has a pretest-intervention-posttest design, whereby a control group will not experience the intervention (Cevikbas et al., 2022) and the experimental group will show improved performances, as shown by a significant increase in their scores.

Geiger and Frejd (2016) already observed that the research on mathematical modeling competencies focuses primarily on the cognition of the individual student rather than on task design, students' collaboration, their creativity, their experiences of the relevance of mathematics, and so forth. We add that, in particular, the quantitative research on modeling competencies clearly reflects the evidence-based educational philosophy, whereby the main focus is on finding 'evidence' through a measurement of 'what works', with the spotlight on students working individually, large samples, average scores, reliability measures of instruments, and statistical analyses (Frejd & Vos, 2022).

Among the researchers that engage with foci other than competencies, we mention three strands: the Models and Modeling perspective (Lesh & Doerr, 2003), the Ethnomodeling approach (Rosa & Orey, 2010), and the research based on the Anthropological Theory of Didactics (ATD) (Barquero, Bosch, & Gascon, 2019). These three approaches analyze the teaching and learning of mathematical modeling in different ways. The Models and Modeling perspective looks at students' mathematical thinking in relation to how students explore and develop models based on scenarios, which can be realistic or mathematical (Lesh & Doerr, 2003). In this perspective, all mathematical thinking is based on manipulating models, whereby they define models as "conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation system, and that are used to construct, describe, or explain the behaviors of other system(s) – perhaps so that the other system can be manipulated or predicted intelligently" (Lesh & Doerr, 2003, p. 10).

In the Ethnomodeling approach, modeling is viewed as a methodology for connecting academic and cultural aspects of mathematics (Rosa & Orey, 2010). The goal of Ethnomodeling is to make "critical analysis of the generation and production of knowledge (creativity), and forms an intellectual process for its production, the social mechanisms of institutionalization of knowledge (academics), and its transmission (education)" (Rosa & Orey, 2010, p.18).

The research based on ATD considers "that in a modeling activity both system and model have a praxeological structure and that the modeling activity is a process of reconstruction and articulation of mathematical praxeologies which become progressively broader and more complex." (Barquero, Bosch, & Gascon, 2019, p. 2051). This means that activities, tasks, and models are all considered as constructs with theoretical (logos) and practical (praxis) sides, which are interconnected, transformed, reformulated, constrained by institutions, and so forth. Within the ATD-based research, questions lead to new questions, articulations to new articulations, tasks to new tasks, models to new models, and so forth.

The three 'minority' approaches within the research on mathematical modeling education are very different from each other, but they have in common that they take a wider perspective than the cognitivist focus on students' competencies. These research strands build on research in general mathematics education concerning respectively, problem-solving as developed by George Polya and others (the Models and Modeling perspective), Ethnomathematics as developed by Ubiratan d'Ambrozio (the Ethnomodeling perspective), and the Anthropological Theory of Didactics (ATD) as developed by Yves Chevallard. They thereby perceive students within socio-cultural environments with certain mathematical and socio-cultural conventions, rules and constraints, as having a history and a future, and as moving between institutions (home, school), just like the teachers, the educational designers, and even the researchers themselves. Although unarticulated, these strands may align more with bildung-based educational philosophies; they clearly do not align with evidence-based education.

Thus, while an evidence-based philosophy is quite obvious in quantitative research on modeling competencies, it remains unclear what would be the cornerstones of a bildung-based philosophy for mathematical modeling education. As the first exploration showed, openness in tasks, students' collaboration, their creativity, their experiences of the relevance of mathematics and a holistic view on heart, mind, social life and culture seem in the spirit of mathematical modeling. We therefore turn to conceptualize the *spirit of mathematical modeling* with reference to the last 50 years of the advancement of the research field of mathematical modeling; with this, we aim to identify characteristics that fit a bildung-based philosophy on mathematical modeling.

**2.2 The spirit of mathematical modeling – a conceptualization**

The spirit of mathematical modeling is a term used in the literature to idealistically capture important aspects of mathematical modeling (Dyke, 1987; Hersee, 1987; Kaiser & Brand, 2015; Lamon, 2003). More generally, "*the spirit of something*" is an idiom for "the feeling, quality, or disposition characterizing something" (Merriam-Webster, n.d.-b). The wording "*X is in the spirit of Y*" means that X fits qualities, intentions and norms of Y, which often are unstated, implicit, and possibly romanticized. For example, one may say "*it is in the spirit of teamwork to respect one another*" to suggest that teamwork entails a certain 'spirit' regarding mutual trust among members of the team.

The notion of the spirit of mathematical modeling was used in the early debates on mathematical modeling education. In a historical overview, Kaiser and Brand (2015, our italics) write that, initially, "the emphasis (..) lay on the development of courses and their evaluation, the creation of new examples and the widening of mathematical themes to be tackled with *a modelling spirit*" (p. 130). At the 5$^{th}$ conference of the International Community of Teachers of Mathematics held in 1993, there were elaborate discussions among the participants regarding "the question, what and why to assess, how to assess modeling without destroying *the spirit of modelling*" (Kaiser & Brand, 2015, p. 132; see also Schukajlow et al., 2018, p. 7, our italics). The first quote places an emphasis on the fostering of mathematical modeling within the bildung-based philosophy, whereas the second quote expresses tensions between the assessment of modeling according to a curriculum-driven philosophy and something termed as *the spirit of mathematical modeling*, which even can be destroyed.

We already gave the example quote from Dyke (1987) above to illustrate the tensions between the two educational philosophies, evidence-based and bildung-based, within mathematical modeling education. Also, Lamon (2003) and Hersee (1987) expressed these concerns. Lamon (2003) reports on a tension with teaching goals, namely, that students do not recognize the value in the mathematics they learn. However, "by not assuming too much about their [students] preparation, and by taking the time to guide their entry *into [the] spirit of mathematical modeling*, dramatic progress can be achieved in changing these attitudes in a relatively short period of time" (Lamon, 2003, p. 91, our italics). This quote indicates that it is possible to change students' beliefs about the usefulness of mathematics if teachers do not consider students' pre-understanding, assumingly in accordance with some curriculum standards, to achieve and provide students with ample of time to introduce themselves to the spirit of mathematical modeling. Hersee (1987) also identifies a tension in mathematics teaching when it comes to introducing new applications to students, due to time pressure and teachers' insecurity in new teaching approaches. According to Hersee (1987, p. 231, our italics), the goal of including realistic application in teaching "has *a modelling spirit*; to show the usefulness and applicability of mathematics and encourage students to use it". The tension described in this quote is that the applied teaching approach is not in line with the modeling spirit, since it requires some other type of teaching approach.

The spirit of mathematical modeling is not clearly defined by the authors quoted above. We draw on their descriptions, and define that:

> *The spirit of mathematical modeling* is a metaphor for important principles of mathematical modeling that build on a bildung-based educational philosophy.

We will further on refer to it as ***SoMM***. A perspective on SoMM raises the question: what aspects are 'in the spirit of mathematical modeling'? Or in short: what aspects are in SoMM? In the remaining text, we aim to answer this question. We start by explaining the methodology applied in this study.

## 3 Methodology

The mathematics education research literature is frequently centered around how results are obtained derived by some method, whereas less literature reports on and explains the rationale for the design of the research, namely the *methodology* (Burton, 2005). Ernest (1998, p. 35) defines that "[e]ducational-research *methods* are specific and concrete approaches. In contrast, educational-research *methodology* is a *theory* of methods – the underlying theoretical framework and the set of epistemological (and ontological) assumptions that determine a way of viewing the world and, hence, that underpin the choice of research methods". Thus, making transparent and replicable research requires some type of methodological approach that explicitly declares and specifies an applicable theoretical framework together with arguments concerning its potential and deficits for exploring some phenomena. For most research this is an accurate approach, but in this research that aims to capture aspects that are in SoMM, we do not adopt a specific theoretical framework. Adopting a theoretical perspective affects one's views; some things seem obvious and other things become obscured. As outlined in the introduction, the diverse descriptions of mathematical modeling display features of bohemianism (Bohemianism, 2022). It means that "various artistic or academic communities…. lives and acts with no regard for conventional rules of behavior" (Bohemianism, 2022). In educational research, the core conventional rule of behavior is to build on other researchers' work from different areas of interest, describing the richness of the field while at the same time displaying a field of converging understanding concerning the phenomena of interest. However, for the field of mathematical modeling within educational research, that is not the case; the researchers in academic communities mainly refer to the research within that particular community, which leads to the development of isolated islands of research.

This paper aims to break away from the core conventional rule of behavior and rather look beyond the isolated islands. First, we adopt the epistemological assumptions of Biesta (2010) that educational researchers may have different goals (evidence-based, bildung-based). Second, we assume that mathematical modeling, metaphorically, has a certain spirit, which is a socio-cultural construct. Both determine our way of viewing the world, and for researching it, we will apply a thematic analysis of some literature (Braun & Clarke, 2006). A thematic analysis is "a method for identifying, analyzing and reporting patterns (themes) within data" (Braun & Clarke, 2006, p. 78) that provides both flexibility and theoretical freedom by not assuming a constructionist or essentialist approach. Braun and Clarke (2006) describe two types of approaches for thematic analysis: *inductive* and *theoretical*. An inductive analysis is data driven, which Braun and Clarke (2006, p. 83) define as "a process of coding the data *without* trying to fit it into a pre-existing coding frame, or the researcher's analytic preconceptions", whereas a theoretical analysis is "driven by the researcher's theoretical or analytic interest in the area driven" (p. 84). Since we aim to identify semantic themes reflecting the main patterns in our data from both analytic and data-driven interests, we firstly performed a theoretical thematic analysis and secondly an inductive thematic analysis. We hoped that the combination of thematic analyses might strengthen the aspects that are in the spirit of mathematical modeling without losing the rich description of the data or a more detailed analysis of a particular set of data, compared to selecting only one type of thematic analysis (Braun & Clarke, 2006).

Mathematical modeling, metaphorically as a *spirit*, is a socio-cultural construct that requires an analysis at least three different levels of the socio-cultural world, in which the construct can be 'constructed'. We chose the following interacting levels that assist in connecting individuals and their social and cultural environment: the level of the individual (*micro*), the level of groups of individuals (*meso*) and the level of broader social and cultural structures (*macro*) (Jaspal, Carriere & Moghaddam, 2016). The first analysis (the theoretical thematical analysis) with the purpose of synthesizing our data was therefore to code our data in the analytic themes of micro, meso and macro levels. According to Bolíbar (2016), using different methods of analysis of socio-cultural phenomena articulated at the three levels enriches the research results.

In the second analysis (the inductive theoretical analysis), we conducted an overall review of the data in each coding theme (micro, meso and macro) to identify a set of final subthemes linked to the three coding levels. Our guiding principle was that data within the subthemes should "cohere together meaningfully, while there should be clear and identifiable distinctions between themes" (Braun and Clarke 2006, p. 91). Based on these subthemes, we constructed our final thematic map (Braun and Clarke 2006, pp. 89-91), which is depicted in Figure 1. Before we provide our analysis, however, we present our sample of research literature.

### 3.1 Our sample of literature

As a point of departure for capturing aspects of the spirit of mathematical modeling (SoMM), we selected a sample of literature, which includes five papers. These five papers are summarized in Table 1. We selected them to cover both historical and modern descriptions, and to cover both features of school context and professional practices. Case 1 and 2 situate SoMM in a historical perspective on teaching and learning modeling (Pollack, 1969) and on modeling as a professional practice (McLone, 1973). Case 3 and 4 present more recent insights into the work of professional modelers; one is a multiple case study of what model constructors do prior to, during and after the constructing of a mathematical model (Frejd & Bergsten 2016), whereas the second is a news report that discusses how pandemic models assisted in understanding the COVID-19 pandemic, and where they fell short (The Public Health Agency of Sweden, 26 May 2021). Finally, the recent newspaper item "8th grade students doing research on waste recycling" (Nødland Skogedal, 2019) also concerns students' spontaneous modeling within a school context.

Table 1: Five cases paving the way for identifying aspects in SoMM

| Case | Description |
|---|---|
| **1. History:** ***A new approach for teaching and learning mathematics***<br>Pollack (1969) | Pollak (1969) argued why application problems should have a place in mathematics classrooms and how this could be done. He discarded word problems as 'false' and 'whimsical' and pleaded for including realistic and genuine applications "for a complete and honest presentation of mathematics in our schools" (p. 403). He characterized mathematical modeling education as different from standard mathematics education, for instance, in students learning to formulate problems rather than just solve them, working collaboratively in 'consultation groups', working on a task for an hour, and expecting students to be creative. Also, teachers needed to be flexible. |
| **2. History:** ***A new approach for research in mathematics education with focus on modeling***<br>McLone (1973) | An early empirical study about mathematical modeling education was McLone (1973, as cited in Thwaites, 1973 and in Houston, Galbraith, & Kaiser, 2009). McLone reported on a large-scale survey of modelers' work in industrial workplaces. He piloted a questionnaire and sent it to more than 500 employers, asking them about the strengths and weaknesses of the work done by recently graduated mathematicians. They easily engaged in solving problems by applying mathematical techniques, but were less skilled in formulating problems, planning, and making a critical evaluation of the completion of their work. Moreover, they had difficulties determining how to communicate it to others. |
| **3.** ***Modeling as a professional practice***<br>(Frejd & Bergsten, 2016) | Frejd and Bergsten (2016) conducted a multiple case study of nine professional modelers working in industries (e.g., aircraft, finance), public services (traffic, forest measurement) and at universities (research on climate, spread of diseases between trees, structure of materials). The aim was to capture characteristics of the work by the *constructors* of mathematical models, as opposed to the *operators* (those using the mathematical models as black boxes) and the *consumers* (citizens who acquire the outcomes of the work by constructors and operators). In all cases, the professional modelers received their task from a *client*, after which they entered into *a pre-construction phase*, in which they took much time to study the problem environment and consult with others. For instance, they maintained a dialogue with the client to check and re-check for goals, accuracy, and effectiveness of the mathematical model. Also, they consulted with external experts to improve the quality of their work by verifying alternative solutions and better models. In the construction phase, they made use of computer programs and other tools, and again consulted with experts. They accepted that their models would always have limitations and intrinsic insecurities, for instance, because they were based on uncontrollable sources. Therefore, they strived to construct models that yielded sufficiently acceptable and reasonable solutions, again in dialogue with the client. Regarding the *post-construction phase*, they were aware of the risks that their models could yield inaccurate, or even harmful results (e.g., yielding losses for a company), and expressed ethical considerations (e.g., ethnicity as a variable, minimal costs at the expense of animal health). |

| | |
|---|---|
| **4. News:** ***Pandemic models assisted us in understanding the development of the pandemic*** (Jöud et al., 2021) | When Covid-19 spread across the world, mathematical modelers developed mathematical models to predict how the pandemic would develop. However, how reliable were these pandemic models? News media in Sweden reported on a meta-analysis (Jöud et al., 2021), which examined and evaluated 20 pandemic models (compartment models, statistical models and one agent-based model) developed by researchers. The results show that these models assisted policy makers, researchers, and citizens in getting a better grip on how the pandemic developed by providing predictions that the virus would differ across regions and that less physical interaction in social situations would reduce the infection rate. The study also identified constraints and limitations in many of the models scrutinized. In several models, it was unclear which data had been used to create the models, and what methods had been used in calibrating them. The study ended with the recommendation: "Researchers, authorities and others who publish pandemic models must be clear in their communication with regard to intended recipients (other researchers, authorities, the public, etc.), the purpose of the model (scenario or prediction), data and assumptions used, and how the reliability of the outcome can be validated" (Jöud et al., 2021, p.41, our translation). |
| **5. News:** ***8$^{th}$ grade students doing research on waste recycling*** (Nødland Skogedal, 2019) | In a school in Norway, 8$^{th}$ grade students received a problem from the local recycling enterprise concerning how to improve waste recycling. Although this question did not look mathematical, the students used mathematics to analyze their own waste sorting by exploring, "How much waste in the black bins (non-recyclable waste) should be in the other ones?" The students decided collaboratively to start with the black garbage bins in the school yard. They sorted that waste into paper, plastics, food, glass, metal, and the remainder and then measured everything in kg. The students constructed a mathematical model, visualized their findings in their presentation to the leaders from the recycling enterprise. They found that approximately 90% of the waste in the black garbage bins was recyclable and should have been placed into recycling bins that were not present in the school yard. To address this dilemma, the students decided to send a delegation to the school authorities and request more recycling bins in the school yard. To convince the school authorities, they used the mathematical model developed as a basis for their arguments. The students succeeded and improved the waste recycling at their school (see also Frejd et al., in press; Vos & Frejd, 2020; 2022). |

### 3.2 Analysis

As indicated in the methodology section, the thematic analysis was carried out in two phases: firstly, as a theoretical thematical analysis and secondly, as an inductive thematical analysis. The reliability of the coding in our analysis builds on discussions and final agreement between the two authors. In this section, we provide a brief example of how the theoretical thematic analysis was developed followed by a more in-depth description of how the aspects that are in

SoMM were developed during the inductive thematical analysis. Thereafter, we will present our results.

### 3.2.1 *Theoretical thematical analysis*

All papers were closely read by the two authors, whereby we identified the three levels (micro, meso and macro) in the five cases in relation to aspects in mathematical modeling. A rubric with the three levels (see Table 2) was used to classify segments in the text and to facilitate our discussions. Both authors individually made notes (codes) in separated rubrics, and the rubrics were then compared and discussed. Below in Table 2, we illustrate an example on our methodological procedure for case 2 (McLone, 1973).

Table 2: An example of the coding procedure in the theoretical thematical analysis

| **Levels of an individual's relationship with the socio-cultural world** | **Paper 2**<br>**New research in mathematics education with focus on modeling (McLone, 1973)** |
|---|---|
| ***Micro***<br>(Individual attitudes, personality traits, attributes and tendencies) | engagement, applying mathematics, posing problems, ingenuity, planning and critiquing one's own work, knowing when to finish |
| ***Meso***<br>(Social group communities of individuals, such as families, neighborhood, co-workers, etc.) | communicating results to other workers, industrial aspects within the work of modeling, receiving critique and critiquing other's work |
| ***Macro***<br>(Societal ideologies, cultural aspects, and social representations) | communicating results to an external audience such as clients or citizens, industrial value of mathematical modeling |

Table 2 illustrates the thematic analysis for Case 2, the paper by McLone (1973), showing how we collected aspects in modeling mainly at the micro level with an emphasis on individual skills, such as being interested and motivated to solve modeling problems by applying mathematics (engagement and applying mathematics). Other personal traits needed to become productive include being able to plan and evaluate (planning and critiquing) and knowing when the work is done. We consider these as aspects of SoMM. Communication is identified at both the meso and macro level. Here, the difference is about interacting with different social groups of people, that is, when closer to the modeler, such as co-workers (meso) or interacting with clients outside the industry or the general public (macro). The use of workplace mathematics per se (meso) and the value of using it (macro) are also emphasized as aspects of SoMM.

### 3.2.2 *Inductive thematical analysis*

The five cases in Table 1 capture aspects found in SoMM. The five rubrics identified in the theoretical thematic analysis were used as a starting point for the second part, the thematic analysis. The goal of this process was to identify aspects of SoMM as subthemes at the micro, meso and macro levels.

The first cluster of aspects in SoMM related to **the micro level** is connected to *metacognition*. Maaβ (2006), Stillman (1998), Vorhölter (2018) and others already pointed at the importance of metacognition for regulating and coordinating the many processes in modeling. There are different conceptualizations of metacognition, most of which focus on individual and group processes, but to our knowledge none has extended it to the world outside the modelers' team. When the students in the waste sorting project, Case 5, mentioned that they had to report their results back to their client, the local recycling enterprise, they decided to also share their results with the school authorities. Thus, during the modeling work, aims and processes need to be coordinated and regulated considering goals external to the group of modelers. These can be goals from the client, but also pertain to the didactical contract from the teacher. We capture this under *scrutiny* and *critique* as second cluster of aspects at the micro level connected to the pre-constructing phase in Case 3. Scrutiny and critique entail how a modeler conceptualizes a fuzzy, ill-articulated problem from a client, poses questions (to each other, to experts and the client) and reads intentions into these, asking critically whether the work will lead to a satisfying answer, how other modelers have tackled similar problems, how to communicate the results back to the client, and so forth.

A third aspect that reflects SoMM at the micro level is *flexibility-creativity*, which we combine because they relate to creatively dealing with unexpected situations (see Case 1) and reflectively changing initial plans. Also, teachers in mathematical modeling classes need flexibility and creativity, since they can expect their students to come with unexpected turns. A fourth aspect in this same cluster at the micro level is indicated as *anticipation*, which captures the foresight concerning what to do later (Niss, 2010). This is done continuously throughout all modeling activities. For example, in Case 5, when the students in the waste project had decided they would send a delegation to the school leadership, and thus changed their aim, they went back to their model and made it 'neater'. Going back and forth, both in activities and in reflecting about these, also reflects SoMM. We name this *iterativity,* which entails a disposition towards making enriched repetitions.

An aspect that is undeniably common in the five cases in Table 1, and connected to **the meso level**, is collaboration. Blum (2002) named collaboration as a core component of mathematical modeling. Therefore, we say that collaboration is in SoMM, and it can be between students, between students and experts, between teachers designing modeling tasks, between teachers and researchers, and so forth. Collaboration does not just entail 'groupwork' but also mutual responsibility for understanding each other, responsibility for the collective progress, and collective agency in decision making. In Case 1, Pollak (1969) wrote about students working in 'consultation groups', exemplified in Case 5, when 8th grade students were asked to assist the local recycling enterprise. Consultation goes beyond the modelers' in-group. Consultation can be with experts for improving approaches, and with the client as an 'intended recipient' of the solution. This is the practice of professional modelers, Case 3 and Case 4, who start from a dialogue with a client, consult with each other, and thereafter repeatedly consult with the client and each other during the modeling process. Thus, *collaboration* and *consultation* are in SoMM.

Another cluster of aspects in SoMM, at the meso level, suggests that modelers are encouraged, if not expected, to make their own choices, which is related to independence (Burkhardt, 2020). Thus, in modeling classrooms, students' independence does not entail 'doing their own thing' but making autonomous yet negotiated decisions. This is described in Case 5 when the students in the waste project changed the initial problem into a researchable question and took the initiative to send a delegation to the school authorities. Such responsible and reflective activities can be captured by the term *agency*. This aspect means that teachers are not supposed to provide students with 'correct' answers, but rather offer independence-preserving support (Blum & Borromeo Ferri, 2009; Buchholtz & Mesrogli, 2013; Doerr & Ärlebäck, 2015). Thus, *agency* is in SoMM.

Also, students engaging in extended investigations for a stretch of time, at least for one lesson but possibly for several weeks, reflects SoMM (Niss, 1993). Mathematical modeling cannot be a quick job applying factual knowledge and standard algorithms as argued in Case 1. It requires time to deal with many intertwined processes, which are (meta-)cognitive, social, and otherwise complex. Thus, the social norms that allow for *ample time* are in SoMM.

In mathematical modeling, the engagement increases when the problem cannot easily be solved yet seems to be within reach. Case 2 reports on mathematical modelers immersing themselves into the modeling work and finding it hard to conclude it. Similarly, English (2019) and Gjesteland and Vos (2019) write that an important aspect of mathematical modeling is flow, which is the phenomenon that one forgets time because the work is so challenging and interesting. Flow is observed when the school bell rings and students still continue the modeling work. Thus, social norms that enable *flow* are in SoMM.

Another aspect in SoMM, connected to meso, refers to *accountability*: modelers are responsible for their advice to their clients in light of the problem situation (Jablonka, 1997) during the post-construction phase (see Case 3). In the pandemic modeling of Case 4, incorrect assumptions may cause queues of ambulances at hospitals. Thus, modeling is not a fun game but has an accountability agenda, for example, regarding medical or environmental health. In pure mathematics, a calculation error by a factor of 10 might be considered "silly… [but in modeling]… it shows no judgment at all" (Pollak, 2015, p. 268). Within SoMM, better to make a close, rough estimate than a neat calculation with a huge error.

In summarizing, the six aspects named above – collaboration; consultation; agency; ample time; flow and accountability – can be clustered under meso in mathematical modeling, combining dialogical processes within and between social groups with the autonomy and persistence of the modelers.

A cluster of aspects was identified at **the macro level**; it connected to societal values (Lindgreen et al., 2020). According to Skovsmose (2021), mathematical models provide a substitute representation of reality, thus forming the dynamics of a situation, and/or imposing actions to undertake. In Case 5 and 4, mathematical models offered a description or representation of reality, which facilitated a certain understanding of the pandemic and the waste sorting, respectively. Also, the models became an essential part within the different discourses and were used to decide what actions to undertake to convince politicians or school authorities. We name this aspect *design,* aiming at reflectively intervening, shaping and accomplishing something (Merriam-Webster, n.d.-a). Design was clearly present in Case 5, where students took care of the presentation of their results, so these had an impact and in Case 4 the COVID-19 models were used to discuss how to flatten the curve to decrease infection rates. We can also observe this aspect in teachers' work when they create learning environments and (re-)design new modeling tasks. In many ways, design reflects SoMM.

A second cluster of aspects at the macro level is related to the quality of the described problem situations. None of the five cases in Table 1 includes artificially constructed situations to make students apply some recently learnt concepts or algorithms. Rather, the context situations display *relevance*, namely that there is a certain urge from a client regarding the solution (Hernandez-Martinez & Vos, 2018). A third aspect that reflects SoMM is *situatedness*. Mathematical modeling exists by virtue of real-life problems, embedding mathematics in contexts, whereby variables used have a dimension (kilos, Euros, etc.) and a meaning (waste, healthcare costs, etc.). This situatedness implies that knowledge of such contexts assists the modelers, for example, in better making assumptions. Simultaneously, the modeling activity advances one's understanding of the problem context, and potentially the discipline that studies such problems. For example, the 'waste problem' in Case 5 can be presented as an isolated, local problem; however, the project can also teach students about the discipline of environmental science regarding costs, human behavior, toxic waste reduction, and so forth. Integrating mathematics with medical, environmental, natural and other sciences is therefore an aspect in SoMM, addressed as *interdisciplinarity* (Borromeo Ferri & Mousoulides, 2017). Within this cluster, macro, knowledge about the value of mathematics in society is another aspect, named *usefulness*. In fact, one does not need to know the exact mathematics to understand that it plays a role in scientific discourses in predicting pandemics (see Case 4), building safety, assessing financial investments, and so forth (Frejd, 2020; Vos, 2020). In relation to this aspect, we add the critical knowledge about *justice* on how mathematical models are also used in selection and evaluation processes, thereby sometimes willingly increasing gender, ethnic and other discrimination (O'Neil, 2016).

## 4 Results of analysis

The captured aspects in SoMM identified above are depicted in Figure 1 as a dreamcatcher, our final thematic map. A dreamcatcher, originally *asabikeshiinh* in the Ojibwe language, is a handmade willow hoop made by the first Americans. It has a 'spiritual' use to trap nightmares, while allowing the good dreams to pass through the center and filter them down through the feathers to the child sleeping. In other words, it is used to 'protect' deeper feelings, qualities and dispositions of modeling. Central to Figure 1 stands *SoMM*, and the aspects we identified as being in SoMM hang as feathers, without hierarchy, with some of them grouped. The idea of a dreamcatcher is that other feathers can be added, so anyone is invited to change, add or remove aspects. A dreamcatcher description is intuitive, fuzzy and able to be adjusted and modified for new conditions connected to the ongoing process of harmonizing heart, mind, social life and culture as a part of bildung-based education.

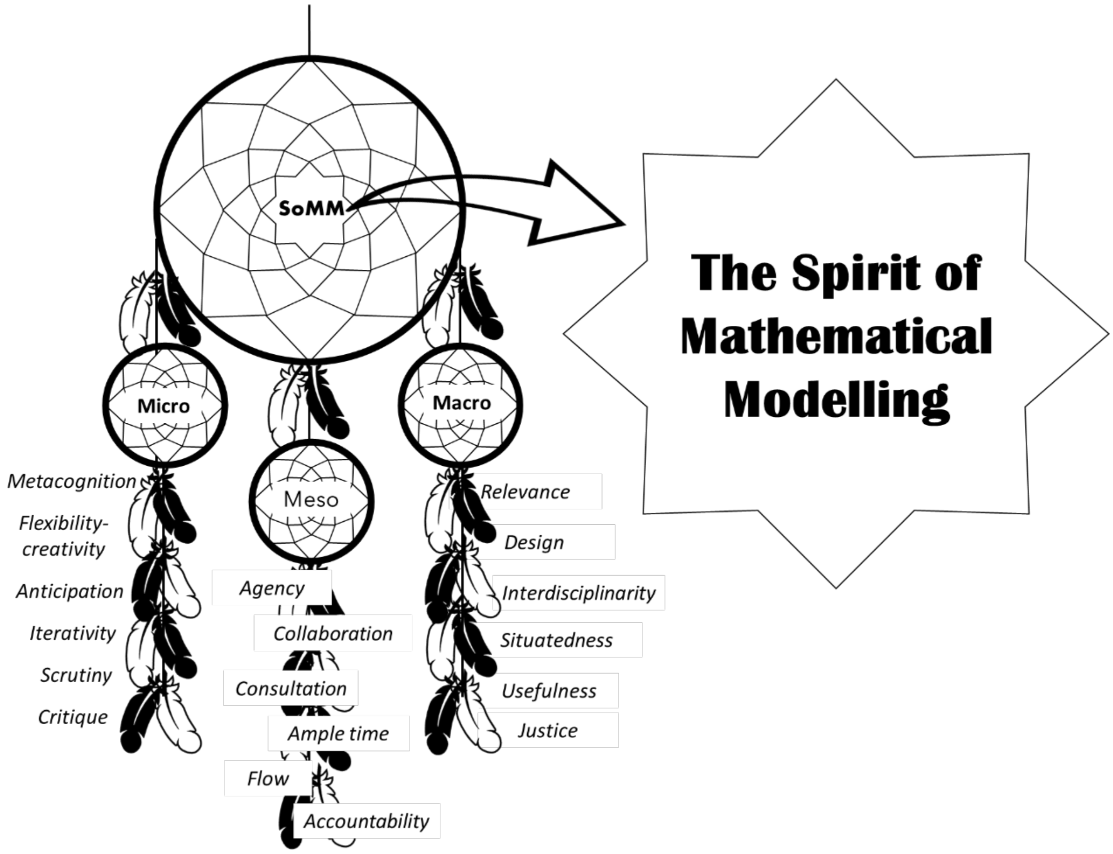


Figure 1: A dreamcatcher capturing SoMM.

.

## 5 Discussion and Conclusion

In this paper, we looked back at 50 years of mathematical modeling education. We noticed that in the beginning years, some authors described a dilemma in mathematical modeling education. The dilemma involved one side, captured as aspects that were in SoMM, and another with aspects that were related to objectively measuring students' performances. In this paper, we related these two sides to the wider educational discussion between proponents of evidence-based education and proponents of bildung-based education. Within the 50 years of advancing and developing mathematical modeling research, we see a clear move towards more cognitivist research focusing on individual students' modeling competencies, where a considerable amount of research focuses on the measurement of it. This type of research clearly connects to a philosophy of evidence-based education.

In this paper, we tried to gain better insight into aspects that could be part of a bildung-based mathematical modeling education. We argued that there is more to modeling than competencies, conceptualized SoMM from a bildung-based perspective, and asked the question: *what aspects are in SoMM*?

Five cases from the literature prompted us to characterize aspects that are in SoMM, whereby we perceive SoMM as an idealistic, complex and open-ended collection of notions caught in a dreamcatcher. SoMM captures, for example, socio-cultural norms regarding interactions across educational and professional practices, which remain obscured when the focus is on modeling competencies. Also, the cluster of societal value offers differences. A teacher explaining about mathematical models used in pandemic times may discuss their role in predicting a pandemic and policy making. Hence, (s)he may foster SoMM without fostering modeling competencies since the students will not need to perform the modeling of the pandemic themselves. To understand the value of models in the real world, one does not need to fully understand such models, although some modeling competencies may support this understanding, but not necessarily. In addition, the cluster of metacognition tends go beyond the modeling competencies. For instance, when researchers adopt a perspective of the modeling cycle, planning and anticipating is expected to occur at the start of the modeling processes. However, when adopting a SoMM perspective, these activities also may occur when preparing a presentation of results and adjusting the language of communication thereof.

In the above paragraph emerged a term, *fostering*, that describes an activity carried out by a teacher. Fostering is a certain type of teaching, aiming at promoting growth and development through encouragement, nurturing, and offering time and facilities. Fostering is more student centered, informal, experiential, and inspiring than direct instruction (Frejd & Vos, 2021). Kokkinos (2009) argues that "[t]he difference between teaching and fostering is that teaching requires direct instruction and fostering indicates an indirect, exploratory approach" (p.4). Clearly, fostering is in SoMM yet we did not include it into the dreamcatcher in Figure 1, because it is a teacher activity and not a modeler's activity. Teacher activities that are in SoMM can yield a new string of feathers in the SoMM dream catcher, see Figure 1.

A focus on mathematical modeling competencies is often connected to empirical research based on relatively short activities within traditional school environments, whereby students individually work on tasks containing a written description of a problem during a limited time. To move away from an evidence-based focus, we framed SoMM from professional modelers' activities (Cases 1, 3, and 4), and by choosing a special school regime, as in Case 5, whereby students started from a fuzzy, unsharp question from a client (how to improve waste sorting), rephrased it (how much waste is wrongly binned?), consulted with experts, engaged in an inquiry to which even the teacher did not know the answer, and used their results to change waste sorting facilities at their school. SoMM revives the focus on students' project-based work within technology-rich environments, modeling weeks, excursions to workplaces, and students creating products (posters, presentations, research reports). Also, SoMM opens the door of the classroom and invites the out-of-school world of professional mathematical modelers. In this way, the fostering of SoMM can also entail that teachers and students discuss historical end current instances where mathematical models play a role in safety, health, police investigations and other societal areas. Under the regime of fostering modeling competencies, such a discussion may remain under the radar, since students do not need to engage in a modeling activity in order to understand its role in society.

Returning to the differences between an evidence-based focus and a bildung-based focus on mathematical modeling, we can ask the question: what consequences will there be if the evidence-based philosophy outweighs the bildung-based philosophy? This question addresses the what, how and why of measuring to obtain evidence. Niss (1993) warned that "what you assess is what you get", meaning that only those components that are assessed will be taken seriously by teachers, researchers and other stakeholders. Nevertheless, he argued, "if modelling and application work is to be taken seriously in and by such an educational system it has to be subject to assessment in some form or another" (p. 43). We want to rephrase this warning by saying that if only competencies are measured, then only these will be taken seriously in future mathematical modeling education, and the many aspects that we identified as being in SoMM may remain obscured. This implies that researchers and teachers overemphasizing cognitive activities neglect important aspects of modelling. However, we do not plead for a full core bildung-based education. If the fostering of SoMM is to be taken seriously (Niss,1993), it must be evaluated somehow, albeit beyond modeling competencies. It is therefore necessary to include aspects, such as *designing*, *consulting*, and providing *ample time* to make the modes of assessment more valid. Frejd (2013) suggested that there should be a more prominent role in mathematics education for assessment modes such as projects, reports, hands-on tests, portfolios, and oral presentations. We contend that there are interesting assessment formats that allow for *collaboration* and *consultation*, *accountability* (justifying choices), *critique*, *iteratively* and *flexibility-creativity*, all of which are at the heart of SoMM.

We finally discuss the difference between assessing and measuring. While measuring refers to a quantitative description of a phenomenon, such as achievement, assessing and evaluation concern the judgement of how this measurement fits a set of agreed standards (Stolz & Webster, 2015). The inherent nature of SoMM is the lack of hard criteria. However, the ultimate way of evaluating the quality of mathematical modeling work is determining its societal value. In the cases of the professional modelers of the pandemic and the 8th grade students in the waste project, the modeling work was assessed in the public arena and had to prove its value in the real world, thus outside the classroom. It thus remains a major aspect to strive for fostering mathematical modeling and assessing students in such a way that they reflects the many aspects in the spirit of mathematical modeling.

**References**

Barquero, B. (2020). Introducción a 'Investigaciones sobre enseñanza y aprendizaje de la modelización matemática: Aproximaciones a la problemática de su diseño, implementación y análisis'. *Avances de Investigación en Educación Matemática*, 17, 1–4. https://doi.org/10.35763/aiem.v0i17.326

Barquero, B., Bosch, M., & Gascón, J. (2019). The unit of analysis in the formulation of research problems: The case of mathematical modelling at university level. *Research in Mathematics Education, 21*(3), 314–330. https://doi.org/10.1080/14794802.2019.1624602

Barquero, B., Bosch, M., & Romo, A. (2018). Mathematical modelling in teacher education: dealing with institutional constraints. *ZDM Mathematics Education*, *50*(1), 31–43. https://doi.org/10.1007/s11858-017-0907-z

Biesta, G. (2010). Why 'what works' still won't work. From evidence-based education to value-based education. *Studies in Philosophy and Education*, *29*(5), 491–503. https://doi.org/10.1007/s11217-010-9191-x

Bildung (n.d.). https://en.wikipedia.org/wiki/Bildung

Blum, W. (2002). ICMI Study 14: Applications and modelling in mathematics education–Discussion document. *Educational Studies in Mathematics*, *51*(1), 149–171. https://doi.org/10.1023/A:1022435827400

Blum, W. (2015). Quality teaching of mathematical modelling: What do we know, what can we do? In S. J. Cho (Ed.), *The Proceedings of the 12th International Congress on Mathematical Education* (pp. 73–96). Springer.

Blum, W. & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects: state, trends and issues in mathematics instruction. *Educational Studies in Mathematics, 22*(1), 37–68. https://doi.org/10.1023/A:1022435827400

Blum, W., & Borromeo Ferri, R. (2009). Mathematical modelling: Can it be taught and learnt?. *Journal of mathematical modelling and application, 1*(1), 45–58.

Bolíbar, M. (2016). Meso, micro, macro: broadening the 'social' of social network analysis with a mixed methods approach. *Qual Quant*, 50, 2217–2236. https://doi.org/10.1007/s11135-015-0259-0

Bohemianism (2022, September 1). In *Wikipedia*. https://en.wikipedia.org/wiki/Bohemianism

Borromeo Ferri, R., & Mousoulides, N. (2017). Mathematical modelling as a prototype for interdisciplinary mathematics education? – Theoretical reflections. In T. Dooley & G. Gueudet (Eds.), *Proceedings of CERME 10* (pp. 900–907). ERME.

Buchholtz, N., & Mesrogli, S. (2013). A whole week of modelling–examples and experiences of modelling for students in mathematics education. In G. Stillman, G. Kaiser, W. Blum & J. Brown (Eds.), *Teaching mathematical modelling: connecting to research and practice* (pp. 307–316). Springer. https:// doi.org/10.1007/978-94-007-6540-5_25

Burkhardt, H. (2020). Modelling in school mathematics: Past achievements—current challenges. In F. K. S. Leung (Ed.), *Mathematical Modelling Education in East and West* (pp. 341–361). Springer. https://doi.org/10.1007/978-3-030-66996-6_29

Burton, L. (2005). Methodology and methods in mathematics education research: where is the "why"? In S. Goodchild & L. D. English (Eds.), *Researching mathematics classrooms: a critical examination of methodology* (pp. 1–10). Information Age Pub.

Carreira, S., & Blum, W. (2021). Mathematical modelling in the teaching and learning of mathematics: Part 1. *Quadrante, 30*(1), 1–7. https://doi.org/10.48489/quadrante.24926

Cevikbas, M., Kaiser, G., & Schukajlow, S. (2021). A systematic literature review of the current discussion on mathematical modelling competencies: State-of-the-art developments in conceptualizing, measuring, and fostering. *Educational Studies in Mathematics*, 109, 205–236. https://doi.org/10.1007/s10649-021-10104-6

Chevallard, Y. (1991). *La transposition didactique—Du savoir savant au savoir enseigné*. La Pensée Sauvage.

Common Core State Standards Initiative. (2010). *Common Core State Standards for Mathematics.* Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Doerr, H., & Ärlebäck, J. B. (2015). Fostering students' independence in modelling activities. In *Proceedings of the 9th Congress of the European Society for Research in Mathematics Education* (pp. 855–861). ERME.

Dyke, P. P. G. (1987). Mathematical modelling in Naval architecture. In J. S. Berry, D. N. Burghes, I. D. Huntley, D. J. G. James, & A. O. Moscardini (Eds.), *Mathematical modelling courses* (pp. 40–50). Ellis Horwood.

English, L. D. (2019). Commentary: Flow and mathematical modelling: Issues of balance. In S. Chamberlin & B. Sriraman (Eds.), *Affect in Mathematical Modeling* (pp. 211–217). Springer. https://doi.org/10.1007/978-3-030-04432-9_13

Ernest, P. (1998). The epistemological basis of qualitative research in mathematics education: A postmodern perspective. In D. A. Grouws (Ed.), *Qualitative research methods in mathematics education* (pp. 22–39). National Council of Teachers of Mathematics.

Erbaş, A. K., Çetinkaya, B., Alacacı, C., Çakıroğlu, E., Aydoğan Yenmez, A., Şen Zeytun, A., Korkmaz, H., Kertil, M., Didiş, M. G., Baş, S., & Şahin, Z. (2016). *Lise matematik konuları için günlük hayattan modelleme soruları* [Everyday Life Modeling Questions for High School Math Subjects]. TÜBA, Turkish Academy of Sciences.

Frejd, P. (2013). *Modes of modelling assessment – A literature review. Educational Studies in Mathematics*, *84*(3), 413–438. https://doi.org/10.1007/s10649-013-9491-5

Frejd, P. (2020). Teachers' possibilities to generate science capital for modelling. In G. A. Stillman, G. Kaiser & E. Lampen (Eds*.), Mathematical modelling education and sense-making* (pp. 39–49). Springer. https://d,oi.org/10.1007/978-3-030-37673-4_4

Frejd, P., Ärlebäck, J. B., & Vos, P. (in press). Fostering student's mathematical literacy through mathematical modelling tasks. In O. H. Bolstad, S. Goodchild, & M. Goos (Eds.), *International Perspectives on Teaching and Learning for Mathematical Literacy*. Brill.

Frejd, P., & Bergsten, C. (2016). Mathematical modelling as a professional task. *Educational Studies in Mathematics*, *91*(1), 11–35. https://doi.org/10.1007/s10649-015-9654-7

Frejd, P., & Vos, P. (2022). A commentary on the Special Issue "Innovations in measuring and fostering mathematical modelling competencies". *Educational Studies in Mathematics*, *109*(2), 455-468. https://doi.org/10.1007/s10649-021-10122-4

Geiger, V., & Frejd, P. (2015). A reflection on mathematical modelling and applications as a field of research: Theoretical orientation and diversity. In G. A. Stillman, W. Blum & M. Salett Biembengut (Eds.), *Mathematical modelling in education research and* *practice: Cultural, social and cognitive influences* (pp. 161–171). Springer. https://doi.org/10.1007/978-3-319-18272-8_12

Gjesteland, T., & Vos, P. (2019). Affect and mathematical modeling assessment – A case study on students' experience of challenge and flow during a compulsory mathematical modeling task by engineering students. In S. Chamberlain & B. Sriraman (Eds), *Affect in Mathematical Modeling* (pp. 257–272). Springer. https://doi.org/10.1007/978-3-030-04432-9_16

Gould, H., Murray, D. R., & Sanfratello, A. (2012). *Mathematical Modeling Handbook*. COMAP.

Greer, B., & Verschaffel, L. (2007). Modelling Competencies — Overview. In W. Blum, P. Galbraith, H. Henn & M, Niss. (Eds.), *Modelling and Applications in Mathematics Education. 14th* (pp. 219–224). Springer. https://doi.org/10.1007/978-0-387-29822-1_22

Hernandez-Martinez, P., & Vos, P. (2018). "Why do I have to learn this?" – A study from mathematical modelling education about the relevance of mathematics. *ZDM Mathematics Education*, *50*(1-2), 245–257. https://doi.org/10.1007/s11858-017-0904-2

Hersee, J. (1987). Modelling versus applications. In J. Berry, D. Burghes, I. Huntley, D. James & A. Moscardini (Eds.), *Mathematical modelling courses* (pp. 227–239). Ellis Horwood.

Houston, K., Galbraith, P., & Kaiser, G. (2009). *ICTMA: The first twenty-five years*. History of ICMI. https://www.icmihistory.unito.it/ictma.phpJ

Jablonka, E. (1997). What makes a model effective and useful (or not)?. In S. K. Houston, W. Blum, I. D. Huntley & N. T. Neill (Eds.), *Teaching and learning mathematical modelling* (pp. 39–50). Albion Publishing. https://doi.org/10.1007/978-0-387-29822-1_19

Jaspal, R., Carriere, K. R., & Moghaddam, F. M. (2016). Bridging Micro, Meso, and Macro Processes in Social Psychology. In J. Valsiner, G. Marsico, N. Chaudhary, T. Sato, & V. Dazzani (Eds.), *Psychology as the Science of Human Being. Annals of Theoretical Psychology* (pp. 265–276). Springer. https://doi.org/10.1007/978-3-319-21094-0_15

Jöud, A., Gerlee, P., Spreco, A., & Timpka, T. (2021). *Sammanställning och utvärdering av modeller för pandemiprediktion I Sverige under 2020* [Compilation and assessment of models for forecasting the progress of the COVID-19 pandemic in Sweden during 2020]. Retrieved from Linköping University Electronic Press website: http://urn.kb.se/resolve?urn=urn:nbn:se:liu:diva-175787

Kaiser, G. (2017). The teaching and learning of mathematical modeling. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 267–291). National Council of Teachers of Mathematics.

Kaiser, G., Blomhøj, M., & Sriraman, B. (2006). Towards a didactical theory for mathematical modelling. *ZDM Mathematics Education*, 38, 82–85. https://doi.org/10.1007/BF02655882

Kaiser, G., & Brand, S. (2015). Modelling competencies: Past development and further perspectives. In G. A. Stillman, W. Blum & M. Salett Biembengut (Eds*.), Mathematical Modelling in Education Research and Practice* (pp. 129–147). Springer. https://doi.org/10.1007/978-3-319-18272-8_10

Kilpatrick, J. (2012). The new math as an international phenomenon. *ZDM Mathematics Education*, *44*(4), 563–571. https://doi.org/10.1007/s11858-012-0393-2

Lamon, S. J. (2003). Powerful modelling tools for high school algebra. In S. J. Lamon, W. A. Parker, & K. Houston (Eds.), *Mathematical modelling: A way of life. ICTMA11* (pp. 81–94). Horwood Pub. https://doi.org/10.1533/9780857099549.2.81

Lesh, R., & Doerr, H. M. (2003). Foundations of models and a modeling perspective on mathematics teaching, learning, and problem solving. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 3–33). Erlbaum.

LEMA (2009). *Reality-based tasks for school*. International Centre for STEM Education. https://www.matematikksenteret.no/sites/default/files/attachments/Booklet%20LEMA-prosjekt.pdf.

Lindgreen, A., Di Benedetto, C. A., Clarke, A. H., Evald, M. R., Bjørn-Andersen, N., & Lambert, D. M. (2020). How to define, identify, and measure societal value. *Industrial Marketing Management*. https://doi.org/10.1016/j.indmarman.2020.05.013

Maaβ, K. (2006). What are modelling competencies? *ZDM Mathematics Education*, *38*(2), 113–142. https://doi.org/10.1007/BF02655885

Maaß, K. (2008). *Mathematisches Modellieren – Aufgaben für die Sekundarstufe I*. Cornelsen Scriptor.

Maaß, J., O'Meara, N., Johnson, P., & O'Donoghue, J. (2018). *Mathematical Modelling for Teachers*. Springer. https://doi.org/10.1007/978-3-030-00431-6

Merriam-Webster. (n.d.-a). Design. In *Merriam-Webster.com dictionary*. Retrieved July 8, 2021, from https://www.merriam-webster.com/dictionary/design

Merriam-Webster. (n.d.-b). Spirit. In *Merriam-Webster.com dictionary.* Retrieved July 8, 2021, from https://www.merriam-webster.com/dictionary/spirit

Naylor, T. (1989). The unifying effects of mathematical modelling in a college of higher education. In Blum et al. (Eds.). *Applications and modelling in learning and teaching mathematics* (pp. 323–329). Ellis Horwood Limited

Niss, M. (1993). Assessment of mathematical applications and modelling in mathematics teaching. In J. de Lange (Ed.), *ICTMA 5–- Innovation in mathematics education by modelling and applications* (pp. 41–51). Ellis Horwood.

Niss, M. (2010). Modeling a crucial aspect of students' mathematical modeling. In R. Lesh, et al. (Eds.), *Modelling students' mathematical competencies* (pp. 43–59). Springer. https://doi.org/10.1007/978-1-4419-0561-1_4

Niss, M., & Blum, W. (2020). *The learning and teaching of mathematical modelling (1st ed.)*. Routledge. https://doi.org/10.4324/9781315189314

Nødland Skogedal, M. (April 2019). Åttendeklassinger har forsket på søppelsortering [8th grade students doing research on waste recycling]. *Agder- Næring*, https://issuu.com/gunnarskarpodde/docs/agdern_ring_01_2019_-_nett

OECD. (2016*). PISA 2015 Assessment and analytical framework: Science, reading, mathematics and financial literacy*. OECD publishing.

O'Neil, C. (2016). *Weapons of math destruction: How big data increases inequality and threatens democracy*. Allen Lane.

Pearson, P. D., & Cervetti, G. N. (2015). Fifty years of reading comprehension theory and practice. In P. D. Pearson & E. H. Hiebert (Eds.), *Research-based practices for teaching Common Core literacy* (pp. 1–24). Teachers College Press.

Pollak, H. O. (1969). How can we teach applications of mathematics? *Educational Studies in Mathematics*, 2, 393–404. https://doi.org/10.1007/BF00303471

Pollak, H. O. (2015). The place of mathematical modelling in the system of mathematics education: Perspective and prospect. In G. A. Stillman, W. Blum & M. S. Biembengut (Eds.), *Mathematical Modelling in Education Research and Practice* (pp. 265–276). Springer. https://doi.org/10.1007/978-3-319-18272-8_21

Rosa, M., & Orey, D. (2010). Etnomodeling as a Pedagogical Tool for the Ethnomathematics Program. *Revista Latinoamericana de Etnomatemática, 3*(2). 14– 23

Rothstein, R., & Jacobsen, R. (2005). The Goals of Education. *Phi Delta Kappan, 88*(4), 264–272. https://doi.org/10.1177/003172170608800405

Schukajlow, S., Kaiser, G., & Stillman, G. (2018). Empirical research on teaching and learning of mathematical modelling: a survey on the current state-of-the-art. *ZDM Mathematics Education, 50*(1-2), 5–18. https://doi.org/10.1007/s11858-018-0933-5

Schukajlow, S., Kaiser, G., & Stillman, G. (2021). Modeling from a cognitive perspective: theoretical considerations and empirical contributions. *Mathematical Thinking and Learning*, 1–11. https://doi.org/10.1080/10986065.2021.2012631

Skovsmose, O. (2021). Mathematics and crises. *Educational Studies in Mathematics*. 108, 369–383. https://doi.org/10.1007/s10649-021-10037-0

Sriraman, B. (2013). (Why) Yet another issue on Problem Solving?. *The Mathematics Enthusiast*, *10*(1), 1–2. https://doi.org/10.54870/1551-3440.1256

Stacey, K. (2015). The real world and the mathematical world. In K. Stacey & R. Turner (Eds.), *Assessing mathematical literacy* (pp. 57–84). Springer. https://doi.org/10.1007/978-3-319-10121-7_3

Stillman, G. A. (2019). State of the art on modelling in mathematics education—Lines of inquiry. In G. A. Stillman & J. P. Brown (Eds.), *Lines of Inquiry in Mathematical Modelling Research in Education, ICME-13 Monographs* (pp. 1–20). Springer. https://doi.org/10.1007/978-3-030-14931-4_1

Stillman, G. (1998). The emperor's new clothes? Teaching and assessment of mathematical applications at the senior secondary level. In P. Galbraith, W. Blum, G. Booker & I. D. Huntley (Eds.), *Mathematical modelling: Teaching and assessment in a technology-rich world* (pp. 243–253). Horwood.

Stillman, G., Brown, J., & Galbraith, P. (2010). Researching applications and mathematical modelling in mathematics learning and teaching. *Mathematics Education Research Journal, 22*(2), 1–6. https://doi.org/10.1007/BF03217561

Teddlie, C., & Stringfield, S. (2007). A history of school effectiveness and improvement research in the USA focusing on the past quarter century. In T. Townsend (Ed.), *International handbook of school effectiveness and improvement* (pp. 131–166). Springer.

The Public Health Agency of Sweden. (26 May 2021). Modeller för smittspridning bidrog till att förstå pandemins utveckling [Pandemic models helped us to understand the development of the pandemic]. https://www.folkhalsomyndigheten.se/nyheter-och-press/nyhetsarkiv/2021/maj/modeller-for-smittspridning-bidrog-till-att-forsta-pandemins-utveckling/

Thwaites, B. (1973). Review of the training of mathematicians: A research report by R. R. McLone. *The Mathematical Gazette*, *57*(402), 348–350.

Turner, R. (2007). Modelling and applications in PISA. In W. Blum, P. Galbraith, H.-W. Henn & M. Niss (Eds.), *ICMI Study 14, Modelling and applications in mathematics education* (pp. 433-440). Springer. https://doi: 10.1007/978-0-387-29822-1_48

Vorhölter, K. (2018). Conceptualization and measuring of metacognitive modelling competencies: Empirical verification of theoretical assumptions. *ZDM Mathematics Education*, *50*(1), 343–354. https://doi.org/10.1007/s11858-017-0909-x

Vos, P. (2020). On science museums, science capital and the public understanding of mathematical modelling. In G. A. Stillman, et al. (Eds.), *Mathematical Modelling Education and Sense-making* (pp. 63–73). Springer. https://doi.org/10.1007/978-3-030-37673-4_6

Vos, P., & Frejd, P. (2020). The object-tool duality in mathematical modelling - a framework to analyze students' appropriation of Sankey diagrams to model dynamic processes. *Avances de Investigación en Educación Matemática*, 17, 52–66. https://doi.org/10.35763/aiem.v0i17.305

Vos, P., & Frejd, P. (2022). Grade 8 students appropriating Sankey diagrams: The first cycle in an educational design research. *Journal on Mathematics Education*, *13*(2), 289–306. http://doi.org/10.22342/jme.v13i2.pp289-306