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Advancing engineering students' conceptual understanding through puzzle-based learning: a case study with exact differential equations

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Current views on the teaching of differential equations (DEs) are shifting towards the use of graphical and numerical methods. Motivated by recent research suggesting that puzzle-based learning (PzBL) can improve the teaching and learning of STEM subjects and by the lack of relevant studies for DEs, we designed two tasks—sophism and paradox—to explore undergraduate engineering students' conceptual understanding of a classical topic—exact DEs—and to analyse the process of meaning-making during collaborative learning in small groups. One hundred and thirty-five undergraduate engineering students from a public university in Iran participated. In response to recent research signalling the tendency of the students to procedural learning of DEs, we analyse how the students in our study engaged in small group work on puzzle tasks, gaining a more conceptual understanding of exact DEs and acknowledging the efficiency of PzBL in their responses to a questionnaire and in interviews.

> Just as scientific theory remains unchallenged until conceptual or empirical anomalies become apparent [. . .], students operating at the frontiers of their conceptual knowledge have no reason to build new conceptual structures unless their current knowledge results in obstacles, contradictions or surprises. (Cobb, 1988, p. 82)

1. Introduction

Engineering plays a key role in social and economic development, contributing to an improvement in the quality of people's lives (Grigg, 2018). Mathematics is vital for the education of future engineers; it provides 'a training in rational thinking', serves as 'one of the principal tools that engineers use for the derivation of quantitative information about natural systems' and offers 'a training for adaptation to the future' (Mustoe & Walker, 1970, p. 390). The world of engineering is changing, and education must adapt rapidly to new growing demands (Writh *et al.*, 2017) because traditional study curricula no longer provide the foundations for ensuring engineering success in the 21st century (Marshall, 2003).

One of the main reasons for why first-year engineering students often drop out of university is related to difficulties with mathematical disciplines and the struggle to relate mathematics to their engineering studies (Venable *et al.*, 1995; Bernold *et al.*, 2007; Godfrey *et al.*, 2010; Paura & Arhipova, 2014). Many future engineers complain about the disparity between their courses in mathematics and engineering and would prefer teaching that emphasises the value of mathematics through contextualised problems and examples (Harris *et al.*, 2015). In fact, Sazhin (1998, p. 142) quoted his student as saying, 'We are mostly not academics but practical engineers; we forget what we are told but never forget what we see or discover for ourselves!' and suggested that 'engineering students need to have it explained to them why knowledge of mathematics is essential for their future practical work' (p. 151).

Flegg *et al.* (2012) reported that in response to a questionnaire, a majority of first-year engineering students agreed or strongly agreed that mathematics is relevant to their future careers, giving the highest ratings for the ways of thinking (82%), ideas (79%) and mathematical skills (76%) and the lowest agreement for those skills related to formulating and solving engineering problems (59%). On the other hand, research has indicated that a number of students graduate with a wealth of knowledge in engineering and computer science but are not capable of applying what they have learned for solving real-world problems; they lack critical and creative thinking, good communication skills and a teamwork mindset (Mills & Treagust, 2003; Kamsah, 2004; Falkner *et al.*, 2012). These problems may be rooted in learning practices, which are often reduced to finding solutions for drill exercises (Michalewicz & Michalewicz, 2008; Klymchuk, 2017). University mathematics courses do not appeal to some first-year engineering students (Klymchuk, 2017), and this is particularly disappointing because mathematical disciplines contribute significantly to the development of logical thinking and students' ability to properly analyse real-world phenomena. As Devlin (2001, p. 22) stressed, 'The main benefit they [software engineers] got from the mathematics they learned in academia was the experience of rigorous reasoning with purely abstract objects and structures.' In fact, 68% of the engineering students surveyed responded that 'the rigorous aspects of mathematics would be important for them in the future' (Flegg *et al.*, 2012, p. 729).

Widely acknowledged for its rigour and accuracy, mathematics is also notorious for fallacies and mistakes. Not surprisingly, both the words paradox (παράδοξο—combining the two words 'contrary to' and 'opinion') and sophism (i.e., originating from σόφισμα—'I am wise') emerge from the Greek school of philosophy because it is exactly 'in the Greek philosophical tradition the cognitive role of ignorance and errors, and their relationships with fallacies, true belief, or even knowledge were investigated thoroughly' (Acerbi, 2008, p. 544). One of the important works from the Greek philosophers, *Pseudaria*, is ascribed to Euclid; it is one of the lost and little-known ancient sources, for which 'we have only their probable title and a handful of outlines, a few lines long, of their contents' (Acerbi, 2008, p. 511); we also know that it was mentioned by Proclus of Athens (410–485), the most authoritative Greek Neoplatonist philosopher. *Pseudaria* was a collection of false proofs in antiquity, 'a mathematical work particularly open to philosophical connections' (Acerbi, 2008, p. 523), where 'false proofs were expounded and criticised, pointing out where the error lay and probably making appeal to true and basic results' (Acerbi, 2008, p. 513). For centuries, mathematicians and philosophers have disputed the role of ambiguity in mathematics and its impact on the development of creativity and new mathematical ideas; indeed, bringing this debate to the mathematics classroom for pedagogic purposes is beneficial for students.

> Logic abhors the ambiguous, the paradoxical, and especially the contradictory, but the creative mathematician welcomes such problematic situations because they raise the question, 'What is going on here?' Thus, the problematic signals a situation that is worth investigating. The problem is a potential source of new mathematics. (Byers, 2007, p. 6)

No wonder, puzzle-based learning (PzBL) is now being recognised as a new efficient method for developing students' thinking skills and problem-solving strategies, especially when it comes to real-world and unstructured problems (Badger *et al.*, 2012; Falkner *et al.*, 2012; Kawash, 2012; Thomas *et al.*, 2013). PzBL engages students with puzzle tasks to advance the development of high-order thinking (e.g., critical, creative and lateral thinking) and problem-solving skills (Michalewicz & Michalewicz, 2008). For the purposes of the current study, a puzzle task is a 'non-standard, non-routine, unstructured question presented in an entertaining way' (Klymchuk, 2017, p. 1106). These tasks differ from routine problems that can be solved with long, mechanical, boring and complex calculations as well as from procedural problems that can be solved by rote learning. Puzzles are known as a type of task that make students think (Klymchuk, 2017). In the present paper, we focus on two types of puzzle tasks: sophism and paradox. Here, sophism is conceptualised as an 'intentionally invalid reasoning that looks formally correct, but in fact contains a subtle mistake or flaw' (Klymchuk, 2017, p. 1106) and paradox as a 'surprising, unexpected, counter-intuitive statement that looks invalid but in fact is true' (Klymchuk, 2017, p. 1106). This conceptualization is described in more detail in the section on PzBL.

PzBL makes the classroom environment enjoyable, increases students' motivation to learn mathematics, helps students learn mathematical concepts in a meaningfully way and stimulates curiosity and participation in classroom discussions (Michalewicz & Michalewicz, 2008; Klymchuk, 2017). The purpose of puzzle tasks is to encourage engineering and mathematics students to reflect on strategies that can be used for solving problems not familiar to students (Falkner *et al.*, 2010), contribute to the development of professional skills (Klymchuk, 2017) and inform students about the historical development of mathematics (Klymchuk & Staples, 2013).

In this paper, we explore the effect of PzBL on the development of students' conceptual understanding of one important class of differential equations (DEs): exact equations. A literature search revealed no studies addressing the impact of PzBL on the quality of teaching and learning of DEs; however, several papers explored its use in the teaching of calculus (Badger *et al.*, 2012; Klymchuk & Staples, 2013). In fact,

> Research in the teaching and learning of undergraduate mathematics is a relatively recent phenomenon, and the topic of differential equations in this research context is still newer. At the start of many new areas of investigation, a firm grounding in students' conceptions can be an important component for future curricular and instructional design ... We need to explore the variety of ways in which content, instruction and technology can be profitably coordinated to promote student learning. If mathematics educators and mathematicians work together to address these issues, then reform in differential equations can be paradigmatic for effective change in university-level mathematics. (Rasmussen, 2001, p. 84)

Seeking to fill the knowledge gap and investigate the opportunities offered by PzBL, we address the following research question: *How do challenging tasks in the format of paradoxes and sophisms enhance students' engagement in understanding the concept of exact DEs and promote advanced mathematical thinking?*

2. Literature review

We start with a review of the relevant research on the teaching and learning of DEs and exact DEs before proceeding to the results on the development of advanced mathematical thinking and PzBL.

2.1. *Teaching and learning of differential equations*

DEs are used for modelling various real-world phenomena in many disciplines, including engineering (Rowland, 2006; Arslan, 2010a; Camacho-Machín & Guerrero-Ortiz, 2015; Khotimah & Masduki, 2016). In the engineering curriculum, standard compulsory courses in DEs are usually offered after the calculus sequence. Future engineers should master how to analyse different DEs analytically and numerically and use this knowledge in applied problems (Kwon, 2002), but they are often inclined to learn only the methods for finding solutions to DEs, even if they are pertinent only to particular classes of equations (Keene, 2007). Yap (2010) highlighted that several students perceived DEs courses only as a means to help them learn a variety of techniques to solve standard classes of DEs, including, for instance, separable, homogeneous and exact equations. Consequently, over the past two decades, the teaching of DEs has been changing towards a more extensive use of graphical and numerical methods and digital technology (Rasmussen, 2001; Stephan & Rasmussen, 2002; Camacho-Machín & Guerrero-Ortiz, 2015).

Traditionally, ‘differential equations, even relatively simple ones, seem to be a stumbling block for many students’ (Sazhin, 1998, p. 147). Students struggle to choose suitable methods for solving DEs and find the evaluation of integrals they encounter in the solving process to be demanding (Camacho-Machín *et al.*, 2012). The conversion of algebraic solutions to graphical format is often strenuous (Camacho-Machín *et al.*, 2012) as, in general, is the understanding of the geometrical approach to DEs (Habre, 2003). Students face difficulties with the conceptual understanding of material, even though they may have good computational skills; ‘algebraic solutions of DEs can be found even without a deep understanding and conceptualisation of DEs’ (Arslan, 2010b, p. 887). However, a procedural understanding of DEs does not necessarily provide the basis for a conceptual understanding although ‘an individual who has learned conceptually will also be able to answer procedural questions’ (Arslan, 2010a, p. 104). Many problems with sense-making in DEs stem from the fact that students usually relate DEs and their solutions by relying on algebraic techniques (Raychaudhuri, 2008; Arslan, 2010a). By sense-making, we mean making sense of the external world while meaning-making relates it to our inner world and is crucial for the development of advanced mathematical thinking. Noticeably, even students who learn DEs procedurally still find it difficult to recall appropriate methods for solving problems (Kwon *et al.*, 2005). It has been suggested that the use of numerical and graphical approaches can facilitate students’ conceptual understanding of DEs (Rasmussen, 2001), even though several authors have reported students’ difficulties with the use of digital technology (Rowland, 2006; Camacho-Machín & Guerrero-Ortiz, 2015).

A number of studies have explored students’ understanding of the concept of a DE (Rasmussen, 2001; Habre, 2003; Rowland & Jovanoski, 2004; Raychaudhuri, 2008; Arslan, 2010a, 2010b; Treffert-Thomas *et al.*, 2018), and ‘research has pointed to the various challenges that students face with this concept’ (Rasmussen & Wawro, 2017, p. 555). Rasmussen (2001) reported students’ difficulties and confusion arising from the interpretation of solutions as functions rather than numbers. Raychaudhuri (2008) analysed the structure of the definition of a solution to a DE, suggesting a ‘context-entity-process-object’ frame. Introducing a solution to a DE using dual processes, one upholding the required property and another generating the object, she explored students’ reasoning about the concept.

Mallet & McCue (2009) investigated students' understanding, asking them to suggest possible solutions to first- and second-order linear DEs prior to introducing analytical solution methods. Recently, Treffert-Thomas *et al.* (2018) brought the problem back into focus, acknowledging students' unexpected struggle with the concepts of particular and general solutions in an assignment on existence and uniqueness theorems. Pointing out that the teaching and learning of DEs demand a high level of conceptual understanding and require additional effort for solving problems, Bibi *et al.* (2019) suggested that the use of nonroutine problems can enhance teaching and learning. This agrees with the work of Stephan & Rasmussen (2002), who argued that 'students create meaningful mathematical ideas as they engage in challenging tasks' (p. 461), see also Treffert-Thomas *et al.* (2018).

2.2. *Exact differential equations*

Exact DEs are taught in all introductory DE courses and in many calculus courses. This is one of a few important classes of nonlinear DEs for which closed-form solutions can be found by using a well-defined procedure (see Appendix 1). Exactness is an important property, not only for the study of conservative vector fields in physics. It is also useful for solving first-order linear DEs and many nonlinear higher-order DEs whose left-hand sides can be written as total differentials. Exact DEs have numerous applications; they are used, for instance, for finding electric potential at a point in a two-dimensional static electric field in physics and for finding the streamlines of two-dimensional incompressible planar flows in hydrodynamics. Exact DEs find applications in Complex Analysis for sketching a planar force field by finding and sketching the family of curves tangent to the field (see Zill & Shanahan, 2003).

Although exact DEs are important for applications, students do not always understand the solution technique correctly, partly because 'exactness is a fragile condition in the sense that seemingly minor alterations in an exact equation can destroy its exactness' (Larson *et al.*, 1997, p. 1093). Farlina *et al.* (2018) reported that 65% of mathematics education students at a college in Indonesia could not solve the initial value problem for an exact DE. Only 10% of the students solved the problem correctly, whereas 35% of students did not even know what solution method should be applied. In another study, 86.9% of Turkish teacher candidates correctly solved an exact DE on a test but failed to demonstrate a conceptual understanding of the problem (Arslan, 2010b). Interestingly enough, 96% of students correctly solved a DE that was both exact and homogeneous, while the number of correct solutions dropped to 77% for the exact equation because several students could not identify the type of equation and solution method (Arslan, 2010a). González-Gaxiola & Hernández Linares (2010) argued that students in regular DE courses usually experience difficulties with their conceptual understanding of exact DEs, memorise traditional solution procedure and apply it to rote-memory tasks designed by lecturers. To facilitate a better understanding of the concept and solution technique, González-Gaxiola & Hernández Linares (2010) suggested an 'alternative direct method' for solving exact DEs based on the concept of the essential part of a twice continuously differentiable function that contains only mixed derivatives.

Realising engineering students' need for basic knowledge of DEs and understanding the difficulties they experience, one may reasonably argue the following:

> What can we expect students to get out of an elementary course in differential equations? I reject the 'bag of tricks' answer to this question. A course taught as a bag of tricks is devoid of educational value. One year later, the students will forget the tricks, most of which are useless anyway. (Rota, 1997, p. 9)

We respectfully disagree with Rota's labelling of solution techniques for DEs as 'tricks' and view a trick as 'an intellectual move which is key to solving a task, but which is unique to that task, or to very few disparate tasks' (Badger *et al.*, 2012, p. 5). In fact, every diligent student should have a variety of such smart tricks in their mathematics toolbox. 'Tricks' do not trivially extend to a variety of other cases; their use brings a flavour of originality, elegance and creativity to one's mathematical argument. However, we fully support Rota's criticism of the confusing practice of teaching the integration factor techniques via exact DEs without a careful explanation of the relevant details and pose (in a different format) his question to the students in our study: 'At this point, some bright students will ask the following question: Are the DEs $M\,dx + N\,dy = 0$ and $qM\,dx + q\,N\,dy = 0$ 'the same' or are they 'different'?' (Rota, 1997, p. 6).

2.3. Development of advanced thinking

A course in DEs is difficult, and conceptual understanding is important yet not easily achieved. Previous studies have suggested that engagement with higher-order thinking (HOT) tasks contributes to students' development of a conceptual understanding of mathematics (e.g., Smith & Stein, 1998; Radmehr & Vos, 2020). HOT tasks 'are often less structured, more complex, and longer than tasks to which students are typically exposed' (Stein *et al.*, 1996, p. 462) and have been found to be associated with student self-monitoring and making conceptual connections. High-level tasks—also termed 'doing mathematics' by Smith & Stein (1998)—are closely related to the 'powerful tasks' introduced by Krainer (1993) as those tasks that 'provide opportunities for action and reflection and for making ideas of pupils 'public' and for relating them to other conceptions' (p. 69).

High-level tasks contribute to the development of students' advanced mathematical thinking, which can be understood as 'thinking that requires deductive and rigorous reasoning about mathematical notions that are not entirely accessible to us through our five senses' (Edwards *et al.*, 2005, pp. 17–18). The complex process of developing advanced mathematical thinking often creates cognitive conflicts that could impede students' learning (Tall, 1992). On the other hand, 'oversimplified environments designed to protect students from confusion may only serve to provide implicit regularities [. . .] causing serious conflict at a later stage' (Tall, 1992, p. 508).

Therefore, the selection of tasks and organisation of students' work are very important in our study. Our choice of problems was guided by the following criteria: (i) relevance to students' classroom experience; (ii) 'challenging enough to require, and elicit, metacognitive behaviour, while simultaneously being within the capacity of the subjects to solve with existing knowledge'; and (iii) 'a blend of genuine 'problems' and routine exercises' (Goos & Galbraith, 1996, p. 238). Cohen (1994) conjectured that 'given an ill-structured problem and a group task, productivity will depend on interaction' (p. 8), suggesting that efficient interaction should involve sharing ideas, hypotheses and possible solution strategies. Previous studies (e.g., Goos *et al.*, 2002; Goos, 2004; Fani & Ghaemi, 2011) have indicated that peer interactions can generate a collaborative Vygotsky's zone of proximal development (ZPD) in which the opportunities for students to construct their ideas and personal mathematical insights are created. However, the processes of meaning-making are not merely reduced to reaching a group consensus on an answer or solution; they are much more complex (Kruger, 1993). Because 'the potential for small group work to develop students' mathematical thinking and problem-solving abilities has remained largely unexplored' (Goos *et al.*, 2002, p. 194), we organised students' work as collaborative learning where 'participants are making a coordinated, continuing attempt to solve a problem or in some other way construct common knowledge' (Mercer & Howe, 2012, p. 11). Following the suggestion that students are more likely to engage in open, extended discussions and take more independent ownership of cocreated knowledge in the absence of the teacher (Barnes & Todd, 1997), we granted them such an opportunity to eliminate the possibilities for the teacher to ease the cognitive demand of the tasks by premature content-related interventions. Motivated by the claim that 'if more time were spent in classrooms with students engaged in working on cognitively demanding nonroutine tasks, as opposed to exercises in which a known procedure is practiced, students' opportunities for thinking and learning would likely be enhanced' (Simon & Tzur, 2004, p. 92), we view sophism and paradox tasks as 'validation tasks' where 'validation can include asking and answering questions, assenting to claims, constructing subproofs, remembering or finding and interpreting other theorems and definitions, complying with instructions (e.g., to consider or name something), and conscious (but probably nonverbal) feelings of rightness or wrongness' (Selden & Selden, 2003, p. 5).

2.4. *Puzzle-based learning*

Being known for centuries as a subject of recreational mathematics (O'Beirne, 1965), nowadays, puzzles are more often brought to the mathematics classroom. Thomas *et al.* (2013) compared PzBL with the six characteristics of problem-based learning (PBL) identified by Barrows (1996), concluding the following:

> PBL is (i) student centred and (ii) occurs in small groups, (iii) teachers are facilitators, (iv) the problems stimulate learning and (v) are a vehicle for development of problem-solving skills, and (vi) new information is acquired through self-directed learning. It is the contention of the authors that all of these apply or can apply to PzBL, given the right choice of puzzles. (p. 126)

Pointing at the missing focus on problem-solving skills in most engineering and computer science curricula, Falkner *et al.* (2010) claimed that 'students often have difficulty applying independent thinking or problem-solving skills regardless of the nature of a problem. At the same time, educators are interested in teaching 'thinking skills' rather than 'teaching information and content' (p. 20). This agrees with the observation that 'in a typical engineering curriculum, there is no space for a course on generic thinking skills. Lecturers hope that their students develop and enhance their generic thinking skills solving specific problems from the course' (Klymchuk, 2017, p. 1107). Arguing in favour of educational puzzles that support problem-solving skills and creative thinking, Falkner *et al.* (2010) suggested that puzzles should satisfy most of the following criteria: *independence*, *generality*, *simplicity*, *Eureka factor*, and *entertainment factor*; although Thomas *et al.* (2013) argued that generality is a characteristic of all problems and that not all puzzles meet the simplicity requirement. As highlighted in the introduction, in the current study; we focus on two types of puzzles known as sophism and paradox.

A sophism is a mathematical statement containing an invalid argument that seems correct (Klymchuk, 2017). Engaging with sophisms, students analyse tasks in an attempt to find invalid reasoning(s). This process often requires reflection on the properties postulated in definitions, assumptions in theorems and conditions for applying solution techniques, hence contributing to students' deeper understanding of the concept(s) incorporated in the tasks (Klymchuk & Staples, 2013). The second type of the puzzle problem, a paradox, is a valid mathematical argument that is counterintuitive (Klymchuk, 2017). Paradoxes help students realise that sometimes, their intuition about a mathematical concept or problem may be incorrect (Klymchuk & Staples, 2013).

We focus on paradoxes and sophisms because of their significance for advancing mathematical thinking (Klymchuk & Staples, 2013). Working with sophisms, students look for errors or inconsistencies in logical constructions and reasoning, gaining a deeper understanding of relevant concepts and ideas. The discovery of skilfully set logical traps in seemingly correct arguments brings an unforgettable feeling known as a *Eureka* moment or Martin Gardner's *Aha!* (Michalewicz & Michalewicz, 2008). Paradoxes alert students to the fact that their intuition about a mathematical concept, result or solution may at times be misleading; paradoxes 'can generate curiosity, increase motivation, create an effective environment for debate, encourage the examination of underlying assumptions, and show that faulty logic and erroneous arguments are not an uncommon feature of the mathematical enterprise' (Kleiner & Movshovitz-Hadar, 1994, p. 973). In this project, an important motivational factor for us was the fact that both paradoxes and sophisms are known to be strongly related to cognitive conflicts. Sophisms have enormous pedagogic value for two reasons: they demonstrate, on the one hand, that 'in mathematics, as in other areas, to err is human. The freedom to err is at the heart of developing mathematical knowledge' (Movshovitz-Hadar & Hadass, 1990, p. 266). On the other hand, 'it is insightful to find the invalid logic underlying a mistaken proof. The outcome of this process is further development, refinement and purification of mathematical concepts and theorems' (Movshovitz-Hadar & Hadass, 1990, p. 266). We were also encouraged by the work of Klymchuk (2017), who argued that 'interesting puzzles, paradoxes and sophisms can engage students' emotions, creativity and curiosity and also enhance their conceptual understanding, critical thinking skills, problem-solving strategies and lateral thinking "outside the box"' (p. 1106) and the findings of Badger *et al.* (2012), whose teaching experience 'strongly suggests that embedding puzzles in the curriculum enhances students' learning by developing their general problem-solving and independent learning skills' (p. 1).

3. Methods

A sequential mixed methods study (Creswell, 2014) was designed to explore the development of undergraduate students' meaning-making in the process of solving puzzle problems with exact DEs. The tasks were designed, piloted with 11 students majoring in mathematics and refined with the help of three instructors teaching DE courses. Then, volunteer undergraduate engineering students who were enrolled in DE courses solved four tasks, analyses of two of which are reported in the current paper. Right after solving the puzzle tasks, the students completed a questionnaire on their attitudes towards PzBL. Using stratified random sampling (Teddlie & Yu, 2007) based on the performance in the test and responses to the questionnaire, 13 students were selected for a semistructured interview to further explore their attitudes towards PzBL; these findings will be published elsewhere.

3.1. Participants and procedure

One hundred and thirty-five undergraduate engineering students (55 female and 80 male) from a public university in eastern Iran enrolled in several DE courses and volunteered to participate. The students came from different engineering majors—computer (N = 35, 25.9%), metallurgical (N = 24, 17.8%), chemical (N = 16, 11.9%), civil (N = 15, 11.1%), industrial (N = 14, 10.4%), electrical (N = 12, 8.9%), mechanical (N = 11, 8.1%) and other majors (N = 8, 5.9%)—and worked on the tasks for 60 to 90 minutes in 47 small groups (41 groups of three and 6 groups of two) that were formed according to their own preferences. Having the students work in small groups allowed us to gain better insights both into individual student contributions and peer collaboration; it also helped in more broadly interpreting students' actions and utterances (Dreyfus *et al.*, 2018). This format fits well with the nature of puzzle problems, which stimulate students' active engagement with the tasks and classroom discussions (Thomas *et al.*, 2013); indeed, puzzle problems prepare students for teamwork, as practising engineers do in large-scale projects (Graaff, 2012). The students were asked to solve the tasks in groups and audio-recorded their discussions; a similar approach has been used to explore students' processes of constructing abstract mathematical knowledge (e.g., Dreyfus *et al.*, 2015). Thirteen students were invited to participate in semi-structured interviews to further explore their engagement with the tasks. To more accurately represent the characteristics of a larger student cohort, we included students exhibiting different levels of success in the problem-solving session.

3.2. *Definition of exact differential equation in the context of the study*

In the context of the study, the exact DE is defined as follows:

Definition. The DE $M(x,y)dx + N(x,y)dy = 0$ (1) is exact if there exists a two-variable function, $f(x,y)$, such that $df = M(x,y)dx + N(x,y)dy$. Therefore, if (1) is exact, then $df = 0$ (Kerayechian, 2020).

After discussing this definition and solving a few examples, the following theorem is presented; however, it might not be proven for engineering students in the lecture.

Theorem (Kerayechian, 2020). Assume that in (1) the functions $M(x,y)$ and $N(x,y)$ are continuous in a region from the *xy*-plane, called *R*, and have continuous first-order partial derivatives in this region. Then, the necessary and sufficient condition for (1) to be exact is

$$M_y\,(x,y) = N_x\,(x,y)\,, \forall\,(x,y) \in R.$$

In light of this theorem, condition $M_y(x,y) = N_x(x,y)$ is usually used by students for checking the exactness of DEs.

3.3. *Puzzle problems*

The first task was a *sophism* with a seemingly correct argument containing a hidden error designed to explore students' conceptual understanding of exact DEs and their critical thinking skills. We expected that students would recall that multiplication by an integrating factor converts a non-exact DE into an exact one, recognise that a common factor in a DE is not necessarily an integrating factor and reason that the elimination of a common factor may impact the exactness of the given DE. This task could also promote the development of students' general reasoning skills by highlighting that concrete examples do not prove general statements. This could lead to a change in students' solution strategy—rather than construct more examples in support of a statement whose validity cannot be established by a rigorous proof, they must come up with a counterexample to the statement under consideration. Engaging with the tasks presented in the form of sophisms or paradoxes, students still need to check the exactness criterion $M_y = N_x$ for a number of DEs to find relevant examples or counterexamples; the use of this routine helps them develop the procedural skills needed for solving DEs analytically. Furthermore, after finding a counterexample, students may become curious about how the elimination of a common factor impacts the exactness of a DE. Because in many problems integration factors are used to solve DEs, this inquiry also leads to a better conceptual understanding of the impact of integrating factors on the exactness of DEs.

Task 2 was a *paradox*; it included a statement that might intuitively seem incorrect but is in fact correct. We believed that those students who understood the concept of an exact DE and associated solution method would compare two responses, concluding that both forms for the general solution are correct. We also anticipated that some students may mistakenly think that a DE can have only one integrating factor or that the general solution of a DE is unique and because two different general solutions are found for the same DE, one of them should be wrong.

1. Verify the following statement. Please explain your reasons.

'Factoring out a common factor and its elimination from a differential equation (DE) does not impact the exactness of a DE'.

For example, a DE

$$-\frac{1}{y}\sin\frac{x}{y}\,dx+\frac{x}{y^2}\,\sin\frac{x}{y}\,dy=0 \tag{A}$$

is exact because $M_y=\frac{1}{y^2}\,\sin\frac{x}{y}+\frac{x}{y^3}\,\cos\frac{x}{y}=N_x$. If we factor out $\sin\frac{x}{y}$ in (A) and then eliminate it, we have, respectively,

$$\sin\frac{x}{y}\,\left(-\frac{1}{y}\,dx+\frac{x}{y^2}\,dy\right)=0$$

and

$$-\,\frac{1}{y}\,dx+\frac{x}{y^2}\,dy=0. \tag{B}$$

DE (B) is still exact because $M_y=\frac{1}{y^2}=N_x$. Consider now a DE

$$e^{x+y}\left(x^2+y^2\right)dx+e^{x+y}\,\left(x^2+y^3\right)dy=0. \tag{C}$$

Equation (C) is not exact because
$M_y=e^{x+y}\,\left(x^2+y^2\right)+2\,y\,e^{x+y}\neq e^{x+y}\,\left(x^2+y^3\right)+2\,x\,e^{x+y}=N_x$. If we factor out e^{x+y} in (C) and eliminate it, we have, respectively,

$$e^{x+y}\left(\left(x^2+y^2\right)dx+\left(x^2+y^3\right)dy\right)=0$$

and

$$\left(x^2+y^2\right)dx+\left(x^2+y^3\right)dy=0. \tag{D}$$

This new DE (D) is also not exact because

$$M_y=2\,y\neq 2\,x=N_x.$$

Thus, factoring out and eliminating a common factor does not impact the exactness of a DE.

2. Reza, Ali and Ehsan decided to study together for a DE exam. Ehsan asked his friends how a DE

$$2ydx + xdy = 0, (x, y > 0) \tag{E}$$

can be solved with an integrating factor. Reza and Ali solved equation (E) separately for him. Based on their responses, Ehsan concluded that this DE has two integrating factors and that both functions defined implicitly by equations $yx^2 = c_2$ and $2\,x\,\sqrt{y} = c_1$ are general solutions. Is this possible? Justify your answer.

Reza's solution:

$$\frac{N_x - M_y}{M} = -\frac{1}{2\,y} \implies \mu(y) = e^{-\int \frac{1}{2\,y}\,dy} = \frac{1}{\sqrt{y}}.$$

Now, we multiply the DE by the integrating factor and the new DE

$$2\,\sqrt{y}\,dx + \frac{x}{\sqrt{y}}\,dy = 0 \tag{F}$$

is exact because $M_y = \frac{1}{\sqrt{y}} = N_x$. We can solve (F) using the standard method:

$$\int 2\,\sqrt{y}\,dx = 2\,x\sqrt{y} + Q(y).$$

Differentiation with respect to y yields

$$\frac{2\,x}{2\,\sqrt{y}} + Q'(y) = \frac{x}{\sqrt{y}} \implies Q'(y) = 0$$

and we set $Q(y) = 0$. Therefore, $F(x, y) = 2\,x\sqrt{y}$ and $2x\sqrt{y} = c_1$ would be the general solution of the given DE.

Ali's solution:

$$\frac{M_y - N_x}{N} = \frac{1}{x} \implies \mu(x) = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x.$$

Multiply (E) by the integrating factor, then the new DE

$$2yx\,dx + x^2\,dy = 0 \tag{G}$$

is exact because $M_y = 2\,x = N_x$. We solve (G) using the standard method:

$$\int 2yx\,dx = yx^2 + Q(y).$$

Differentiate the result with respect to y:

$$x^2 + Q'(y) = x^2 \implies Q'(y) = 0$$

and we set $Q(y) = 0$. Therefore, $F(x, y) = yx^2$ and $yx^2 = c_2$ would be the general solution for the given DE.

4.1. Data analysis

We transcribed the audio-recorded files of the group discussions and analysed them along with the written solutions. To achieve this, we used inductive content analysis (Vaismoradi *et al.*, 2013), which proved to be a useful approach for constructing the descriptions and interpretations of phenomena in a conceptual form (Elo & Kyngäs, 2008). Inductive content analysis can be used when no previous studies related to the phenomena under research are known (Hsieh & Shannon, 2005). In this approach, categories are derived from the data, and the researcher(s) should avoid using preconceived categories (Moretti *et al.*, 2011). Because the literature search revealed no examples of studies on the development of students' conceptual understanding and advanced mathematical thinking in DEs through small group work with puzzle problems, inductive content analysis is a suitable approach for the present study.

First, we scrutinised how each group solved the tasks and what type(s) of arguments were discussed in the groups. We also explored students' mathematical understanding and misunderstandings when engaging with these tasks. In the current study, open coding was used to create categories directly from the data and to achieve a general description of students' competencies in solving the sophism and paradox tasks. For each task, the findings are reported in relation to three aspects: (1) students' written responses and their discussion in groups; (2) students' understanding of the underlying concepts related to first-order DE; and (3) students' errors and misconceptions during their work with these tasks. Similar solutions, reasoning demonstrating understanding or a lack there of and typical errors and argumentation were first grouped and then summarised. These are presented below, with some sample responses from the students' work and the arguments used.

5. Results

In this section, we analyse the data collected in the study.

5.1. Task 1: Sophism

Forty groups (85.1%) solved Task 1 correctly, while seven groups (14.9%) failed to do so. Analysis of the audio records revealed that the solution strategy adopted by these 40 groups was to check the exactness criterion (2) for equations (A) and (B) and then for (C) and (D). Four groups (8.5%) solved DEs (A) and (C) and modified equations (B) and (D), noticing that the answers differed and concluding that this was influenced by the elimination of common factors. The following excerpt from one of the group discussions illustrates typical reasoning:

Sadra [1]: Factoring out the common factor and eliminating it from (A) has no impact on the exactness of this DE. However, we should examine whether the general solutions of both DEs (A) and (B) are identical. I will solve the first one. Arman, please solve the second one!

Arman: We solve (A) using the method:

$$\int \frac{-1}{y}\sin\frac{x}{y}\,dx = \cos\frac{x}{y} + P(y).$$

Now, differentiation with respect to y leads to

$$\frac{x}{y^2}\sin\frac{x}{y} + P'(y) = \frac{x}{y^2}\sin\frac{x}{y} \Longrightarrow P'(y) = 0 \Longrightarrow P(y) = c,$$

Therefore, the general solution is $f(x, y) = \cos\frac{x}{y} = c$.

Sadra: I solved it using the six-step method as well:

$$\int \frac{-1}{y}\,dx = \frac{-x}{y} + P(y).$$

By differentiating with respect to y, we have

$$\frac{x}{y^2} + P'(y) = \frac{x}{y^2} \quad\Longrightarrow\quad P'(y) = 0 \Longrightarrow P(y) = c.$$

Thus, the general solution is $f(x, y) = \frac{-x}{y} = c$.

Arman: The general solutions here are not identical. The general solutions found for the DE should only differ by a constant, but these two are different.

Sadra: Therefore, we cannot conclude that eliminating a common factor has no impact on the DE.

Twenty-six out of the 40 groups (55.3%) that solved this task correctly explained that if a DE is non-exact, it can be converted to an exact form by multiplication with an integrating factor. They realised that if a common factor is an integrating factor or a part of it, its elimination influences the exactness of the DE; otherwise, it has no effect on the exactness of a DE. A typical response was, 'This statement is not true. By multiplying with an integrating factor, a non-exact equation becomes exact. Consequently, if an integrating factor is factored out and eliminated from an exact equation, the equation would become non-exact'. Remarkably, 17 groups (36.2%) acknowledged that a mathematical statement cannot be proved by verification using just a few examples; this indicates that these students were aware that the type of reasoning, which is known in the literature as empirical argument (Stylianides, 2008) or naive empiricism (Balacheff, 1988), is not considered a valid argument in mathematics. Furthermore, these students highlighted that counter examples can be used for refuting a mathematical proposition. An excerpt from one of the group discussions illustrates this argument.

Golnar: These two examples confirm the claim, but we cannot conclude that the statement is true. To prove a mathematical statement, we should use valid mathematical reasoning.

Fatemeh: One way to show that a mathematical statement is false is by using a counterexample. We should find a counterexample to this statement, or we should prove it.

1 Pseudonyms are used for the students' names.

This statement is not correct; if we consider the following DE, we see that it is exact, but if we factor out x and eliminate it from the DE, it becomes non-exact.

$$4\,x^3dy + 12x^2y\,dx = 0 \quad \Longrightarrow \quad M_y = 12\,x^2 = N_x\,,$$

and equation is exact. Now, factor out x and eliminate it:

$$4x^2dy + 12\,xydy = 0 \quad \Longrightarrow \quad M_y = 8x \neq 12x = N_x\,,$$

and the new DE is not exact.

Fig. 1. A counterexample provided for Task 1.

All 17 groups provided counterexamples to show that the statement is not correct; one counterexample is presented in Fig. 1.

Seven groups (14.9%) concluded that the elimination of expression in a DE impacts its solution. An excerpt from a relevant group discussion is provided below.

Zohreh: The statement seems incorrect. In Calculus [Zohreh misspoke here, her comment should have referred to Algebra], when solving equations, we cannot eliminate a common factor that includes a variable(s) from both sides of the equation because it is possible that some of the equation's solutions would be eliminated. For instance, if the common factor x were eliminated from both sides of the equation $x\,(x-2) = x$, *we would only have* $x = 2$ *as the solution, while* $x = 0$ *is another solution for this equation.*

Mozhgan: I agree. It seems that factoring a common factor out in a DE and eliminating it could impact its solutions.

The analysis of Task 1 suggests that a substantial number of students advanced their conceptual understanding of the exact DE and solution procedure, made meaningful connections with related material from Calculus and provided grounded reasons for responses. We make this conclusion based on the careful inspection of students' written solutions and transcripts of small group discussions. Only four groups (8.5%) concluded that the elimination of a common factor in a DE has no effect on its exactness. A sample response was as follows: 'Solving a DE, we can cancel the common factor on both sides of the equation, and this does not impact the solution. Because multiplication with an integrating factor makes a DE exact, this gives us an independent solution that has no effect on the main DE's solution.' Three groups (6.4%) did not have clear ideas about how a mathematical proposition could be justified, mistakenly believing that one can prove mathematical statements by providing an example satisfying the claim. For instance, one of the groups responded, 'we always do it [factor out and eliminate a common factor] in mathematics. This statement is confirmed by two examples. One of the ways to prove mathematical propositions is by using examples; therefore, the conclusion is correct'. Two students in different groups could not recall the exactness test. One of them said, 'I do not remember how to distinguish an exact DE from a non-exact because this topic was included in the midterm exam, which we had a month ago.' However, other group members explained to these two students how to check a DE for exactness.

5.2. *Task 2: Paradox*

Twenty-five groups (53.2%) concluded that both solutions in Task 2 are correct, while the attempts of 22 groups (46.8%) were not successful. Analysis of the group discussions revealed that 41 groups (87.2%)

Yes, it is possible. For example, if we square both sides of the Reza's general solution, we can find Ali's general solution:

$$F(x, y) = \sqrt{y}\, 2x = c_1$$

$$\xrightarrow{\text{Square both sides}} \quad 4x^2y = c_1^2 \quad \Longrightarrow \quad x^2y = \frac{c_1^2}{4} \quad \Longrightarrow \quad x^2y = c_2.$$

Fig. 2. Sample response for Task 2.

were unsure about the correctness of Ali's and Reza's solutions and re-examined their arguments. The students in three groups (6.4%) initially believed that a DE cannot have more than one integrating factor because this would then lead to multiple solutions, but after group discussions, they came to the conclusion that this is possible. An excerpt from one of the group discussions is as follows:

Raha: These two solutions seem to be correct, but I am confused about why they solved the DE using two different integrating factors. Is this possible?

Jana: I do not think so. I have not seen non-exact DEs that have more than one integrating factor. I think a DE could have more than one integrating factor only if the integrating factors are similar, like x and 2x, that differ only by a constant multiple.

Sofia: I think it is possible. We always attempt to find one integrating factor, and when we find it, we multiply a non-exact DE through this factor and solve it without paying attention to whether other integrating factors exist.

Jana: Can you give an example of a DE that has two integrating factors?

Sofia: Give me a moment. Let me check on the internet. I found it! For instance, a DE

$$3\,xy + y^2 + \left(x^2 + xy\right)\frac{dy}{dx} = 0$$

has two integrating factors, $h(x) = x$ and $h\,(x, y) = \frac{1}{xy\,(2x+y)}$.

Twenty-two groups (46.8%) demonstrated a good understanding of the concept of exact DE and the solution method by following the reasoning evoked by Ali and Reza. A typical response was, 'Let's first examine the solutions. The DE (E) is not exact. The steps taken to obtain the integrating factor are correct. There is also nothing wrong with the differentiation and integration on the way to a solution.' Sixteen groups (34%) realised that the general solution of a DE can be represented differently. Of these 16 groups, 13 groups (27.7%) came to this conclusion, showing that two solutions are equivalent (Fig. 2). Eleven groups (23.4%) mentioned that an integrating factor for a non-exact DE is not unique: 'Yes, a DE may have several integrating factors.' These findings demonstrate that many students were not only capable of explaining and justifying the six-step method for solving exact DEs (see the Appendix), but, contrary to the lack of conceptual understanding documented in previous studies (e.g., Rasmussen, 2001; Raychaudhuri, 2008; Arslan, 2010b; Treffert-Thomas *et al.*, 2018), we are inclined to conclude that the students in our teaching experiment meaningfully used the concepts of a DE and its general solution.

The data indicated that the students in the two groups did not remember how to make a non-exact DE exact; the team members in nine groups had forgotten how to solve exact DEs, so they asked their peers for help. An excerpt from a group discussion is provided below.

Mahdi: In my opinion, in Ali's solution, x^2 on the right-hand side of equation (G) should be replaced with $2x$.

Mahsa: No, you are wrong. I think you forgot the method for solving exact DEs. His solution is correct. After integrating $2yx$ with respect to x, we differentiate the outcome with respect to y and equate the result to x^2, the coefficient of dy in the exact DE (G) obtained by using the integrating factor.

Mahdi: Oh, I recall this now!

Six groups (12.8%) erroneously believed that a non-exact DE cannot have more than one integrating factor. A sample response was as follows: 'It is impossible for a DE to have more than one integrating factor. I have not seen any DE that could be solved using two different integrating factors'. Five groups (10.6%) believed that the general solution for a DE is unique and did not realise that it could be represented differently by algebraically equivalent expressions. For instance, one of the students argued that 'a DE may have more than one integrating factor, but it cannot have two different general solutions.' All five groups solved the DE (E) using the six-step method and compared Ali's or Reza's solution with their own. Three groups decided that Ali's solution was correct, and two groups favoured Reza's solution. Four groups (8.5%) solved (E) as a separable DE and found solutions different from Ali's and Reza's, concluding that both solutions in the problem are incorrect because if a DE can be solved as a separable equation, it cannot be solved using the method for exact DEs. The following excerpt from the discussion of one of these four groups illustrates the false argument used:

Arash: Look, (E) can be solved as a separable DE!

Parsa: Yes, I think we should use the method we learned for solving first-order DEs, checking their type in a certain order. First, we explore whether the DE is separable, then homogeneous and then exact.

Shahab: So let's solve it by considering (E) as a separable DE and then check Ali's and Reza's solutions.

Note: The group's written solution is as follows:

$$2y\,dx + x\,dy = 0 \Longrightarrow -\frac{2y}{dy} = \frac{x}{dx} \Longrightarrow -\frac{1}{2}\int \frac{1}{y}dy = \int \frac{1}{x}dx$$
$$\Longrightarrow -\frac{1}{2}\ln y = \ln x \Longrightarrow \frac{1}{\sqrt{y}} = x \Longrightarrow y = \frac{1}{x^2}.$$

Parsa: So this equation is separable; therefore, we do not need to solve it by the method for exact DEs.

Two groups (4.3%) encountered difficulties with the verification of Ali's and Reza's solutions because of committing a computational error in the differentiation with respect to y and concluding that 'Reza made a mistake in differentiation. The derivative of $2\sqrt{y}\;x + Q(y)$ with respect to y is $\frac{2}{2\sqrt{y}} + Q'(y)$, so his solution is not correct.' Two groups (4.3%) believed that an integrating factor for (E) should be a function of both x and y because both variables are present in a DE. One of the groups claimed, 'No, it is unacceptable. Both Ali's and Reza's solutions are incorrect. For equations that include x and y, the integrating factor should be $z = f(x, y)$, not just a function of a single variable x or y'.

6. Discussion and conclusions

The purpose of the current research was to explore how PzBL can be used to enhance the teaching and learning of one of the classical topics in DEs. We focused our attention on the development of advanced mathematical thinking among engineering students engaged in solving two puzzle problems (sophism and paradox) for exact DEs. The literature review did not reveal studies analysing the effect of PzBL on students' sense-making in DEs, though several authors addressed the use of PzBL in mathematics teaching (Falkner *et al.*, 2010, 2012; Badger *et al.*, 2012; Thomas *et al.*, 2013; Klymchuk, 2017). Our contribution adds new knowledge to the research on students' collaborative learning by providing an example of productive student work on two types of non-traditional tasks designed for promoting the conceptual understanding of exact DEs. The main difficulties for students with such non-standard tasks were related to the need for verification of two chains of logical reasoning; this experience is similar to the procedure of the proof validation (Selden & Selden, 2003). The complexity of comprehensive tasks with high cognitive demand (Doyle, 1983; Krainer, 1993; Smith & Stein, 1998) breaks the usual study routines of engineering students; this effect was additionally amplified by the lack of similar previous learning experience and the unusual features of puzzle-like problems.

Knowledge of DEs is important for engineering students, but it does not come in a straightforward manner, especially the conceptual understanding of DEs (González-Gaxiola & Hernández Linares, 2010; Arslan, 2010a, 2010b; Farlina *et al.*, 2018). The use of PzBL in our study brings new teaching ideas into a DE class, inviting students to experience ambiguity as a powerful source for creativity in mathematics and to share the excitement of independent mathematical discovery. Being well aware of current educational tendencies favouring geometrical approaches, numerical methods and digital tools, we deliberately selected one traditional topic in DEs to demonstrate new opportunities for student learning. Our research makes an important step towards ambitious goals in teaching DEs for understanding the concepts rather than learning the procedures pointed out by Rota (1997):

> What matters is their [students'] getting a feeling for the importance of the subject, their coming out of the course with the conviction of the inevitability of differential equations, and with enhanced faith in the power of mathematics. These objectives are better achieved by stretching the students' minds to the utmost limits of cultural breadth of which they are capable and by pitching the material at a level that is just a little higher than they can reach. (p. 9)

Analysis of the data indicates that learners were engaged in *explorative talk,* which is defined as a 'discussion in which partners engage critically but constructively with each other's ideas . . . Proposals may be challenged and counter-challenged ... Knowledge is made publicly accountable, and reasoning is visible in the talk' (Mercer, 2000, p. 98).

The account of students' collaborative learning illustrates progressive development of the shared understanding of the material, confirming the positive impact of small group work on the co-creation and expansion of common knowledge within a collaborative ZPD. Our findings lend additional support to the conclusion of Mercer & Howe (2012) that 'collective, goal-directed, curriculum-based activity amongst students without a teacher present can offer distinctive and valuable benefits for students' learning and the development of their understanding' (p. 17). To this end, we designed challenging questions to explore students' conceptual understanding of exact DEs. We allowed sufficient time for discussions and encouraged students to put their knowledge in their own words, take turns in discussions, question others and justify their own views, as recommended by Mercer & Howe (2012). Such learning environments contribute to the development of students' 'metacognitive awareness of the learning functions of talk and an appreciation of its potential value as a cultural and psychological tool' (Mercer & Howe, 2012, p. 18).

Our findings demonstrate a considerable difference in students' success with the tasks: 85.1% for sophism and 55.3% for paradox. We relate this to the need for understanding the concept of the general solution in the paradox problem, in addition to the conceptual understanding of exact DEs and solution methods required for both tasks. In fact, several groups had difficulties establishing the equivalence of solutions in Task 2 and struggled with the interpretation of solutions, which is consistent with previous studies (e.g., Rasmussen, 2001; Raychaudhuri, 2008; Arslan, 2010a, 2010b). We also observed that several students who adopted a procedural approach towards learning could not recall the test for exactness and solution method, which agrees with the findings reported in the literature (Arslan, 2010a, 2010b; González-Gaxiola & Hernández Linares, 2010; Farlina *et al.*, 2018).

Thomas *et al.* (2013) discussed the pedagogic value of PzBL from the standpoint of the Index of Learning Styles® introduced by Felder & Brent (2005) to assess learning preferences of students in various disciplines including engineering. They concluded that 'it might seem that PzBL is not a teaching method that would be favoured by many engineering students' (Thomas *et al.*, 2013, pp. 127–128) and provided four reasons explaining why 'puzzles are traditionally presented as stand-alone entities with little attempt to put them in some broader context' (ibid, p. 128). Nevertheless, students' responses to our questionnaire show that 67.9% enjoyed the sophism task, 85.9% agreed that paradoxes are entertaining, and 71.6% concurred that solution of sophisms and paradoxes improves students' mathematical understanding and enhances problem-solving skills. Furthermore, 63.4% and 67.2% of students responded that solving sophisms and paradoxes helps to improve thinking skills. Students also confirmed their positive opinion about the usefulness of PzBL in the interviews: Behzad spoke about critical thinking and the need for a thorough analysis of the problem: 'To solve paradoxes and sophisms correctly, students need to critique them. They need to consider all possibilities, and different aspects of the given problem.' Fatemeh emphasised the development of understanding and problem-solving skills: 'Sophisms and paradoxes help students to become better problem solvers. They promote deep mathematical understanding.' Erfan acknowledged challenges posed by non-standard tasks and their contribution to the understanding of mathematics: 'Paradoxes and sophisms challenge students' mathematical knowledge and encourage them to gain a better mathematical understanding.' Parham mentioned entertaining nature of puzzle problems: 'Sophisms break the monotony of the classwork and might increase students' interest in solving problems.'

As expected, not all students (although a significantly smaller part) were particularly excited about PzBL. Kamran complained that 'Finding the starting point for solving puzzles and paradoxes takes too much time' whereas Nazli argued that 'Sophisms are not appropriate for engineering students, because problems we encounter in engineering students can be solved with routine algorithms. I prefer to solve routine problems because I do not like challenging questions.' Overall positive feedback from students in the questionnaire and interviews agrees with the findings of Falkner *et al.* (2012) and Klymchuk (2017) that PzBL could improve students' problem solving and thinking skills making the class environment more entertaining and enjoyable for students. Promoting the use of PzBL in teaching mathematics, Thomas *et al.* (2013), argued that 'puzzle-based learning is under-used in the teaching of mathematics to engineers. Embedding puzzles in other teaching enhances students' learning by developing their problem-solving and independent-learning skills, whilst increasing their motivation to learn mathematics' (p. 131).

Having tried PzBL with engineering students, we wholeheartedly support Thomas *et al.*'s (2013) recommendation and acknowledge the positive impact of PzBL on students' conceptual understanding and the dynamics of collaborative learning. In agreement with other researchers, we confirm that it is a useful approach contributing to a quality improvement of teaching and learning in DEs; it helps students gain a deeper conceptual understanding of the material, preparing them for future engineering careers. Our study demonstrates that PzBL not only brings entertainment and creativity to the mathematics classroom but that it also meets the important requirements for engineering education set over 50 years ago:

However, in the end, it is not the number of tricks that the student has learned, but rather the understanding of the concepts and the awareness of the relevance of the techniques, which is important—and, finally, his approach to the new learning which an encounter with a new problem may demand. (Bickley, 1964, p. 383)

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Appendix 1: Exact Differential Equations: Exactness Test, Properties and Solution Method

For a DE

$$M(x,y)\,dx + N(x,y)\,dy = 0, \tag{1}$$

the '*exactness test*' (cf. Theorem 2.5 in Abell & Braselton, 2018, p. 56) requires that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{2}$$

Verification of the exactness test tacitly assumes the existence of continuous partial derivatives in the domain of interest (see, e.g., Abell & Braselton, 2018, p. 56). The necessary part of the exactness test is an immediate consequence of the classical result in Analysis, establishing the equality of mixed partial derivatives (Clairaut, 1740). However, the proof of sufficiency part was revisited by Hellman (1964), who argued that it is 'not quite simple and is not handled too well' in some textbooks. Condition (2) is equivalent to the fact that the vector field $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ is the gradient of some scalar function $f(x,y)$ called *potential*. Then, the general solution $f(x,y) = C$ to the exact DE (1) can be also obtained by finding a potential function for a conservative vector field $\mathbf{F}(x,y)$. The equation $f(x,y) = C$ defines level curves of the potential function. Alternatively, given a function $F = F(x,y)$ with continuous partial derivatives F_x and F_y, then $F(x,y) = C$ is a solution to the DE

$$F_x\,(x,y)\,dx + F_y\,(x,y)\,dy = 0, \tag{3}$$

where the expression on the left-hand side of equation (3) is called the *exact (total) differential* of the function F. This interpretation suggests a powerful technique for solving the DE (1) by rewriting its left-hand side as an exact differential; with the use of integrating factors, the technique expands further to DEs which can be made exact, and to linear DEs. However, transformation of the left-hand side to the exact differential is difficult for students, and most textbooks recommend, with some variations, the following practical six-step procedure (cf. Abell & Braselton, 2018, p. 59):

Step 1. Check that equation (1) satisfies the exactness condition (2).

Step 2. Integrate the equation $\frac{\partial F(x,y)}{\partial x} = M(x,y)$ that relates like terms for dx in (1) and (3) with respect to x to obtain

$$F\,(x,y) = \Phi\,(x,y) + \varphi(y), \tag{4}$$

where Φ is an antiderivative of $M(x,y)$ with respect to x and φ is an unknown function of y.

Step 3. Differentiate equation (4) with respect to y to obtain

$$\frac{\partial F\,(x,y)}{\partial y} = \frac{\partial \Phi\,(x,y)}{\partial y} + \varphi'(y). \tag{5}$$

Step 4. Equating the right-hand side of equation (5) to $N\,(x,y)$, find the equation for $\varphi'(y)$:

$$\varphi'(y) = N\,(x,y) - \frac{\partial \Phi\,(x,y)}{\partial y}. \tag{6}$$

Step 5. Integrate (6) with respect to y, set the constant of integration to zero, and substitute the result of integration, $\varphi(y)$, in (4) obtaining the expression for $F(x,y)$.

Step 6. Set

$$F\,(x,y) = C \tag{7}$$

to obtain an implicit form for the solution of the exact DE (1). If possible, solve equation (7) for y. If needed, apply the initial conditions to find a particular solution of the given DE.

Remark 1. One may repeat the above procedure by starting with the integration with respect to y in Step 2 (i.e., swapping the variables x and y) because for exact equations, it does not matter whether they are solved for x or y.

Remark 2. An elegant method for the integration of exact DEs using the line integral can be found, for instance, in a classical text by Ritger & Rose (1968) but is no longer used in most modern texts on DEs. Precisely for the use 'in an elementary course in differential equations', Firey (1961) did not employ Gauss's theorem for proving that the line integral of a differential form satisfying the exactness condition (2) in a simply connected plane region has the same value along any continuously differentiable arc joining two given points. This theorem furnishes a general formula for constructing the solutions for exact equations.

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