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Research Article

Indefinite integrals from Wronskians and related linear second-order differential equations

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ABSTRACT
Many indefinite integrals are derived for Bessel functions and associated Legendre functions from particular transformations of their differential equations which are closely linked to Wronskians. A large portion of the results for Bessel functions is known, but all the results for associated Legendre functions appear to be new. The method can be applied to many other special functions. All results have been checked by differentiation using Mathematica.

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1. Introduction

Many special functions $\bar{y}(x)$ obey a second-order linear homogeneous differential equation of the form

$$\bar{y}''(x) + \bar{p}(x)\bar{y}'(x) + \bar{q}(x)\bar{y}(x) = 0, \tag{1.1}$$

and in [1,2], the integration formula

$$\int \bar{f}(x)\left(h''(x) + \bar{p}(x)h'(x) + \bar{q}(x)h(x)\right)\bar{y}(x)\,\mathrm{d}x = \bar{f}(x)\left(h'(x)\bar{y}(x) - h(x)\bar{y}'(x)\right) \tag{1.2}$$

was derived, where $\bar{y}(x)$ is any solution of Equation (1.1) and $h(x)$ is an arbitrary twice differentiable complex-valued function of x. The function $\bar{f}(x)$ in Equation (1.2) obeys the differential equation

$$\left(\bar{f}(x)\right)' = \bar{p}(x)\bar{f}(x) \tag{1.3}$$

and hence

$$\bar{f}(x) = \exp\left(\int \bar{p}(x)\,\mathrm{d}x\right). \tag{1.4}$$

The multiplicative constant in the definition of $\bar{f}(x)$ can be arbitrarily chosen, as any choice cancels in Equation (1.2). The general solution of Equation (1.1) is of course given in terms

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of any two independent solutions of the equation as

$$\bar{y}(x) = C_1\bar{y}_1(x) + C_2\bar{y}_2 x \tag{1.5}$$

and choosing $h(x) = \bar{y}_2(x)$ and $y(x) = \bar{y}_1(x)$ in Equation (1.2) gives immediately

$$\bar{y}_1(x)\,y_2'(x) - \bar{y}_1'(x)\,\bar{y}_2 = \frac{A}{\bar{f}(x)}, \tag{1.6}$$

where A is a constant, which is Abel's identity [3,4], with the constant A determined by the chosen normalizations for $\bar{y}_1(x)$, $\bar{y}_2(x)$ and $\bar{f}(x)$.

In [1,2], various transformations of the form $\bar{y}(x) = g(x)y(x)$ were applied to Equation (1.1) and the method of fragments was applied to the resulting equations to derive many integrals of $\bar{y}_1(x)$ and $\bar{y}_2(x)$ for specific special functions. The equation obeyed by $y(x)$ is [1,2]

$$y''(x) + \left(2\frac{g'(x)}{g(x)} + \bar{p}(x)\right)y'(x) + \left(\frac{g''(x)}{g(x)} + \bar{p}(x)\frac{g'(x)}{g(x)}\right)y(x) = 0 \tag{1.7}$$

and defining

$$p(x) = 2\frac{g'(x)}{g(x)} + \bar{p}(x), \tag{1.8}$$

then

$$g(x) = \sqrt{\frac{f(x)}{\bar{f}(x)}} \tag{1.9}$$

$$y(x) = \sqrt{\frac{\bar{f}(x)}{f(x)}}\bar{y}(x) \tag{1.10}$$

and Equation (1.7) can be alternatively expressed as [1]

$$y''(x) + p(x)y'(x) + \left[\frac{1}{2}\left(p(x) - \bar{p}(x)\right)' + \frac{1}{4}\left(p^2(x) - \bar{p}^2(x)\right) + \bar{q}(x)\right]y(x) = 0. \tag{1.11}$$

One particular transformed equation which was not examined in [1,2] is

$$y''(x) + p(x)y'(x) = 0, \tag{1.12}$$

where

$$\frac{1}{2}\left(p(x) - \bar{p}(x)\right)' + \frac{1}{4}\left(p^2(x) - \bar{p}^2(x)\right) + \bar{q}(x) = 0. \tag{1.13}$$

The explicit form of $p(x)$ in Equation (1.12) could be constructed by solving Equation (1.13) as a Riccati equation using Euler's method [5], but this is not necessary.

Equation (1.12) has a constant C_1 as a solution, so its general solution is necessarily of the form

$$y(x) = C_1 + C_2 r(x). \tag{1.14}$$

The general solution of Equation (1.12) is also given by Equations (1.5) and (1.10) as

$$y(x) = \sqrt{\frac{\bar{f}(x)}{f(x)}} \left(C_1 \bar{y}_1(x) + C_2 \bar{y}_2(x)\right), \tag{1.15}$$

and there are only two ways Equation (1.15) can match the form of Equation (1.14). These are the choices

$$\sqrt{\frac{\bar{f}(x)}{f(x)}} = \frac{1}{\bar{y}_1(x)} \Rightarrow f(x) = \bar{f}(x) \bar{y}_1^2(x) \tag{1.16}$$

so that Equation (1.15) becomes

$$y(x) = C_1 + C_2 \frac{\bar{y}_2(x)}{\bar{y}_1(x)} \tag{1.17}$$

and the choice

$$\sqrt{\frac{\bar{f}(x)}{f(x)}} = \frac{1}{\bar{y}_2(x)} \Rightarrow f(x) = \bar{f}(x) \bar{y}_2^2(x) \tag{1.18}$$

which gives

$$y(x) = C_1 \frac{\bar{y}_1(x)}{\bar{y}_2(x)} + C_2. \tag{1.19}$$

The two choices for $p(x)$ in Equation (1.12) are given directly by Equation (1.3) and Equations (1.16) and (1.18) as

$$p(x) = \bar{p}(x) + 2\frac{\bar{y}_1'(x)}{\bar{y}_1(x)}, \tag{1.20}$$

$$p(x) = \bar{p}(x) + 2\frac{\bar{y}_2'(x)}{\bar{y}_2(x)} \tag{1.21}$$

resulting in the transformed differential equations

$$y''(x) + \left(\bar{p}(x) + 2\frac{\bar{y}_1'(x)}{\bar{y}_1(x)}\right) y'(x) = 0, \tag{1.22}$$

$$y''(x) + \left(\bar{p}(x) + 2\frac{\bar{y}_2'(x)}{\bar{y}_2(x)}\right) y'(x) = 0 \tag{1.23}$$

with the respective solutions

$$y_1(x) = C_1 + C_2 \frac{\bar{y}_2(x)}{\bar{y}_1(x)}, \tag{1.24}$$

$$y_2(x) = C_1 \frac{\bar{y}_1(x)}{\bar{y}_2(x)} + C_2. \tag{1.25}$$

The results above can also be proven very simply using Abel's identity (1.6) for Wronskians.

Theorem 1.1: *The differential equation*

$$y''(x) + \left(\bar{p}(x) + 2\frac{\bar{y}_1'(x)}{\bar{y}_1(x)}\right) y'(x) = 0 \tag{1.26}$$

has the general solution

$$y(x) = C_1 + C_2 \frac{\bar{y}_2(x),}{\bar{y}_1(x)} \tag{1.27}$$

where $\bar{y}_1(x)$ *and* $\bar{y}_2(x)$ *are any two solutions of Equation* (1.1).

Proof: From elementary principles $y'(x)$ is given by

$$y'(x) = C_2 \frac{\bar{y}_1(x)\bar{y}_2' - \bar{y}_1'(x)\bar{y}_2}{\bar{y}_1^2(x)} \tag{1.28}$$

and from Abel's identity equation (1.28) becomes

$$y'(x) = \frac{C_2 A}{\bar{f}(x)\bar{y}_1^2(x)}. \tag{1.29}$$

Differentiating Equation (1.29) gives

$$y''(x) = -\frac{C_2 A\left(\bar{f}'(x)\bar{y}_1^2(x) + 2\bar{f}(x)\bar{y}_1'(x)\bar{y}_1'(x)\right)}{\left(\bar{f}(x)\bar{y}_1^2(x)\right)^2} \tag{1.30}$$

and substituting Equation (1.3) into Equation (1.30) gives

$$y''(x) = -\left(\bar{p}(x) + 2\frac{\bar{y}_1'(x)}{\bar{y}_1(x)}\right) y'(x) \tag{1.31}$$

and hence $y(x)$ is a solution of Equation (1.26). As Equation (1.27) contains two independent functions, it is the general solution of Equation (1.26) and the theorem is proven. ■

The results presented above involving Wronskians can be applied to many special functions. In addition, the method of fragments [1,2] can be applied to Equations (1.22)– (1.23) to obtain new integrals. Many useful fragments can be obtained by employing recurrence relations for $\bar{y}_1'(x)$ and $\bar{y}_2'(x)$ in these equations. These include conventional recurrence relations and the recurrences introduced in [6], which give more exotic integrals. Section 2 provides integrals for Bessel functions obtained with the equations above. Some of these integrals are given in the literature and can be obtained with Mathematica [7]. However, Bessel functions seem to provide the only example where this is the case. Section 3 gives analogous results for associated Legendre functions and these results seem to be completely new, with Mathematica being unable to obtain them directly. All results presented have been checked by differentiation using Mathematica.

2. Bessel functions

The baseline Bessel equation is

$$\bar{y}''(x) + \frac{1}{x}\bar{y}'(x) + \left(1 - \frac{\nu^2}{x^2}\right)\bar{y}(x) = 0 \tag{2.1}$$

for which $\bar{f}(x) = x$. The usual two independent solutions selected for this equation are the Bessel function of first kind $J_\nu(x)$ and the Neumann function $Y_\nu(x)$, with a Wronskian given by [8]

$$W(J_\nu(x),\ Y_\nu(x)) \equiv J_\nu(x)\,Y'_\nu(x) - J'_\nu(x)\,Y_\nu(x) = \frac{2}{\pi x}. \tag{2.2}$$

However, as Equation (2.1) contains ν^2, the functions $J_{-\nu}(x)$ and $Y_{-\nu}(x)$ are also solutions of Equation (2.1). The four Wronskians linking $J_\nu(x)$ and $Y_\nu(x)$ with $J_{-\nu}(x)$ and $Y_{-\nu}(x)$ are [8]

$$W(J_\nu(x), J_{-\nu}(x)) = -\frac{2\sin(\nu\pi)}{\pi x} \quad \text{for } \nu \notin \mathbb{Z}, \tag{2.3}$$

$$W(Y_\nu(x), Y_{-\nu}(x)) = -\frac{2\sin(\nu\pi)}{\pi x} \quad \text{for } \nu \notin \mathbb{Z}, \tag{2.4}$$

$$W(J_\nu(x), Y_{-\nu}(x)) = \frac{2\cos(\nu\pi)}{\pi x} \quad \text{for } \nu + \frac{1}{2} \notin \mathbb{Z}, \tag{2.5}$$

$$W(Y_\nu(x), J_{-\nu}(x)) = -\frac{2\cos(\nu\pi)}{\pi x} \quad \text{for } \nu + \frac{1}{2} \notin \mathbb{Z}. \tag{2.6}$$

The functions in Equations (2.3)–(2.4) are independent provided $\nu \notin \mathbb{Z}$ and the functions in Equations (2.5)– (2.6) are independent provided $\nu + \frac{1}{2} \notin \mathbb{Z}$.

Applying Equation (2.2) in Equations (1.28) and (1.29) gives the two integrals

$$\int \frac{dx}{xJ_\nu^2(x)} = \frac{\pi}{2}\frac{Y_\nu(x)}{J_\nu(x)}, \tag{2.7}$$

$$\int \frac{dx}{xY_\nu^2(x)} = -\frac{\pi}{2}\frac{J_\nu(x)}{Y_\nu(x)}. \tag{2.8}$$

A closely related integral is obtained by noting that from Wronskian relation (2.2) we have

$$\frac{(Y_\nu(x)/J_\nu(x))'}{Y_\nu(x)/J_\nu(x)} = \frac{2}{\pi x J_\nu(x)\,Y_\nu(x)} \tag{2.9}$$

and hence

$$\int \frac{dx}{xJ_\nu(x)\,Y_\nu(x)} = \frac{\pi}{2}\ln\left(\frac{Y_\nu(x)}{J_\nu(x)}\right). \tag{2.10}$$

Equations (2.7)–(2.8) and (2.10) are tabulated in [9]. Equations (2.3)–(2.6) give the additional integrals:

$$\int \frac{dx}{xJ_\nu^2(x)} = -\frac{\pi}{2\sin(\nu\pi)}\frac{J_{-\nu}(x)}{J_\nu(x)} \quad \text{for } \nu \notin \mathbb{Z}, \tag{2.11}$$

$$\int \frac{\mathrm{d}x}{xJ_{-\nu}^2(x)} = \frac{\pi}{2\sin(\nu\pi)} \frac{J_\nu(x)}{J_{-\nu}(x)} \quad \text{for } \nu \notin \mathbb{Z}, \tag{2.12}$$

$$\int \frac{\mathrm{d}x}{xY_\nu^2(x)} = -\frac{\pi}{2\sin(\nu\pi)} \frac{Y_{-\nu}(x)}{Y_\nu(x)} \quad \text{for } \nu \notin \mathbb{Z}, \tag{2.13}$$

$$\int \frac{\mathrm{d}x}{xY_{-\nu}^2(x)} = \frac{\pi}{2\sin(\nu\pi)} \frac{Y_\nu(x)}{Y_{-\nu}(x)} \quad \text{for } \nu \notin \mathbb{Z}, \tag{2.14}$$

$$\int \frac{\mathrm{d}x}{xJ_\nu^2(x)} = \frac{\pi}{2\cos(\nu\pi)} \frac{Y_{-\nu}(x)}{J_\nu(x)} \quad \text{for } \nu + \frac{1}{2} \notin \mathbb{Z}, \tag{2.15}$$

$$\int \frac{\mathrm{d}x}{xY_{-\nu}^2(x)} = -\frac{\pi}{2\cos(\nu\pi)} \frac{J_\nu(x)}{Y_{-\nu}(x)} \quad \text{for } \nu + \frac{1}{2} \notin \mathbb{Z}, \tag{2.16}$$

$$\int \frac{\mathrm{d}x}{xY_\nu^2(x)} = -\frac{\pi}{2\cos(\nu\pi)} \frac{J_{-\nu}(x)}{Y_\nu(x)} \quad \text{for } \nu + \frac{1}{2} \notin \mathbb{Z}, \tag{2.17}$$

$$\int \frac{\mathrm{d}x}{xJ_{-\nu}^2(x)} = \frac{\pi}{2\cos(\nu\pi)} \frac{Y_\nu(x)}{J_{-\nu}(x)} \quad \text{for } \nu + \frac{1}{2} \notin \mathbb{Z}, \tag{2.18}$$

$$\int \frac{\mathrm{d}x}{xJ_\nu(x)J_{-\nu}(x)} = \frac{\pi}{2\sin(\nu\pi)} \ln\left(\frac{J_\nu(x)}{J_{-\nu}(x)}\right) \quad \text{for } \nu \notin \mathbb{Z}, \tag{2.19}$$

$$\int \frac{\mathrm{d}x}{xY_\nu(x)Y_{-\nu}(x)} = \frac{\pi}{2\sin(\nu\pi)} \ln\left(\frac{Y_\nu(x)}{Y_{-\nu}(x)}\right) \quad \text{for } \nu \notin \mathbb{Z}, \tag{2.20}$$

$$\int \frac{\mathrm{d}x}{xJ_\nu(x)Y_{-\nu}(x)} = \frac{\pi}{2\cos(\nu\pi)} \ln\left(\frac{Y_{-\nu}(x)}{J_\nu(x)}\right) \quad \text{for } \nu + \frac{1}{2} \notin \mathbb{Z}, \tag{2.21}$$

$$\int \frac{\mathrm{d}x}{xY_\nu(x)J_{-\nu}(x)} = \frac{\pi}{2\cos(\nu\pi)} \frac{Y_\nu(x)}{J_{-\nu}(x)} \quad \text{for } \nu + \frac{1}{2} \notin \mathbb{Z}. \tag{2.22}$$

Equation (2.19) is tabulated in [9].

For the Bessel equation, Equations (1.22)–(1.23) become

$$y''(x) + \left(\frac{1}{x} + 2\frac{J_\nu'(x)}{J_\nu(x)}\right) y'(x) = 0 \tag{2.23}$$

with a solution

$$y(x) = \frac{Y_\nu(x)}{J_\nu(x)} \tag{2.24}$$

and

$$y''(x) + \left(\frac{1}{x} + 2\frac{Y_\nu'(x)}{Y_\nu(x)}\right) y'(x) = 0 \tag{2.25}$$

with a solution

$$y(x) = \frac{J_\nu(x)}{Y_\nu(x)}. \tag{2.26}$$

Employing Equations (2.23)– (2.24) in Equation (1.2) gives for arbitrary $h(x)$

$$\int xJ_\nu(x)\left(h''(x) + \left(\frac{1}{x} + 2\frac{J_\nu'(x)}{J_\nu(x)}\right) h'(x)\right) Y_\nu(x)\,\mathrm{d}x$$

$$= h'(x) J_\nu(x) Y_\nu(x) - \frac{2h(x)}{\pi} \tag{2.27}$$

and employing Equations (2.25)–(2.26) in Equation (1.2) gives

$$\int xY_\nu(x)\left(h''(x) + \left(\frac{1}{x} + 2\frac{Y_\nu'(x)}{Y_\nu(x)}\right)h'(x)\right)J_\nu(x)\,\mathrm{d}x$$

$$= h'(x)\,xJ_\nu(x)\,Y_\nu(x) + \frac{2h(x)}{\pi}. \tag{2.28}$$

Choosing the fragment

$$h''(x) + \frac{1}{x}h'(x) = 0 \tag{2.29}$$

with solution $y(x) = \ln(x)$ in Equations (2.27)– (2.28) gives the integrals

$$\int J_\nu'(x)\,Y_\nu(x)\,\mathrm{d}x = \frac{J_\nu(x)\,Y_\nu(x)}{2} - \frac{\ln(x)}{\pi}, \tag{2.30}$$

$$\int Y_\nu'(x)\,J_\nu(x)\,\mathrm{d}x = \frac{Y_\nu(x)\,J_\nu(x)}{2} + \frac{\ln(x)}{\pi}. \tag{2.31}$$

Perhaps the most interesting special cases of these integrals are

$$\int J_1(x)\,Y_0(x)\,\mathrm{d}x = -\frac{J_0(x)\,Y_0(x)}{2} + \frac{\ln(x)}{\pi}, \tag{2.32}$$

$$\int Y_1(x)\,J_0(x)\,\mathrm{d}x = -\frac{J_0(x)\,Y_0(x)}{2} - \frac{\ln(x)}{\pi} \tag{2.33}$$

and these results appear to be new. Mathematica is able to evaluate the integrals in Equations (2.32)–(2.33) in terms of the Meijer G function, but is unable to reduce these expressions.

Employing the conventional recurrences in Equation (1.2)

$$J_\nu'(x) = \pm\left(J_{\nu\mp1}(x) - \frac{\nu}{x}J_\nu(x)\right), \tag{2.34}$$

$$Y_\nu'(x) = \pm\left(Y_{\nu\mp1}(x) - \frac{\nu}{x}Y_\nu(x)\right), \tag{2.35}$$

and the fragment

$$h''(x) + \frac{1\mp2\nu}{x}h'(x) = 0 \Rightarrow h(x) = x^{\pm2\nu} \tag{2.36}$$

in Equation (1.2) for $\nu \neq 0$ gives the integrals

$$\int x^{2\nu}J_{\nu-1}(x)\,Y_\nu(x)\,\mathrm{d}x = \frac{x^{2\nu}}{2}\left(J_\nu(x)\,Y_\nu(x) - \frac{1}{\nu\pi}\right), \tag{2.37}$$

$$\int x^{-2\nu}J_{\nu+1}(x)\,Y_\nu(x)\,\mathrm{d}x = -\frac{x^{-2\nu}}{2}\left(J_\nu(x)\,Y_\nu(x) + \frac{1}{\nu\pi}\right), \tag{2.38}$$

$$\int x^{2\nu}J_\nu(x)\,Y_{\nu-1}(x)\,\mathrm{d}x = \frac{x^{2\nu}}{2}\left(J_\nu(x)\,Y_\nu(x) + \frac{1}{\nu\pi}\right), \tag{2.39}$$

$$\int x^{-2\nu} J_\nu(x)\, Y_{\nu+1}(x)\, \mathrm{d}x = -\frac{x^{-2\nu}}{2}\left(J_\nu(x)\, Y_\nu(x) - \frac{1}{\nu\pi}\right). \tag{2.40}$$

Equations (2.32)–(2.33) can be considered to provide integrals (2.37)–(2.40) for the forbidden value $\nu = 0$. Results equivalent to Equations (2.40)–(2.43) can be obtained by a formula given in [9].

The sequence of recurrence relations given in [6] for general cylinder functions $Z_\nu(x)$ can be used to obtain many different integrals. The simplest of these are

$$Z_\nu'(x) + \left(\frac{\nu}{x} - \frac{x}{2(\nu-1)}\right) Z_\nu(x) = \frac{x}{2(\nu-1)} Z_{\nu-2}(x), \tag{2.41}$$

$$Z_\nu'(x) + \left(\frac{x}{2(\nu+1)} - \frac{\nu}{x}\right) Z_\nu(x) = -\frac{x}{2(\nu+1)} Z_{\nu+2}(x). \tag{2.42}$$

Substituting Equation (2.41) into Equation (2.28) with $Z_\nu(x) = J_\nu(x)$ gives

$$\int xJ_\nu(x)\left(h''(x) + \left(\frac{1-2\nu}{x} + \frac{x}{\nu-1}\left(1 + \frac{J_{\nu-2}(x)}{J_\nu(x)}\right)\right) h'(x)\right) Y_\nu(x)\, \mathrm{d}x$$
$$= h'(x)\, xJ_\nu(x)\, Y_\nu(x) - \frac{2h(x)}{\pi}. \tag{2.43}$$

Choosing the fragment

$$h''(x) + \left(\frac{1-2\nu}{x} + \frac{x}{\nu-1}\right) h'(x) = 0 \tag{2.44}$$

gives immediately

$$h'(x) = x^{2\nu-1} \exp\left(-\frac{x^2}{2(\nu-1)}\right) \tag{2.45}$$

and $h(x)$ can be expressed in terms of the incomplete Gamma function with definition [10]

$$\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t}\, \mathrm{d}t \tag{2.46}$$

as

$$h(x) = -2^{\nu-1}(\nu-1)^\nu\, \Gamma\left(\nu, \frac{x^2}{2(\nu-1)}\right). \tag{2.47}$$

These results give the integral

$$\int x^{1+2\nu} \exp\left(-\frac{x^2}{2(\nu-1)}\right) J_{\nu-2}(x)\, Y_\nu(x)\, \mathrm{d}x$$
$$= (\nu-1)\, x^{2\nu} \exp\left(-\frac{x^2}{2(\nu-1)}\right) J_\nu(x)\, Y_\nu(x) + \frac{2^\nu(\nu-1)^{\nu+1}}{\pi}\Gamma\left(\nu, \frac{x^2}{2(\nu-1)}\right). \tag{2.48}$$

A similar integral can be derived using recurrence (2.42), which is

$$\int x^{1-2\nu} \exp\left(\frac{x^2}{2(\nu+1)}\right) J_{\nu+2}(x)\, Y_\nu(x)\, \mathrm{d}x = -(\nu+1)\, x^{-2\nu}$$

$$\times \exp\left(\frac{x^2}{2(\nu+1)}\right) J_\nu(x) Y_\nu(x) + (-1)^{\nu+1} \frac{2^{-\nu}(\nu+1)^{-\nu+1}}{\pi} \Gamma\left(-\nu, -\frac{x^2}{2(\nu+1)}\right). \tag{2.49}$$

The corresponding integrals obtained using Equations (2.26) and (2.28) with $Z_\nu(x) = Y_\nu(x)$ are

$$\int x^{1+2\nu} \exp\left(-\frac{x^2}{2(\nu-1)}\right) J_\nu(x) Y_{\nu-2}(x)\,dx$$
$$= (\nu-1) x^{2\nu} \exp\left(-\frac{x^2}{2(\nu-1)}\right) J_\nu(x) Y_\nu(x) - \frac{2^\nu(\nu-1)^{\nu+1}}{\pi} \Gamma\left(\nu, \frac{x^2}{2(\nu-1)}\right), \tag{2.50}$$

$$\int x^{1-2\nu} \exp\left(\frac{x^2}{2(\nu+1)}\right) J_\nu(x) Y_{\nu+2}(x)\,dx = -(\nu+1) x^{-2\nu}$$
$$\times \exp\left(\frac{x^2}{2(\nu+1)}\right) J_\nu(x) Y_\nu(x) + (-1)^\nu \frac{2^{-\nu}(\nu+1)^{-\nu+1}}{\pi} \Gamma\left(-\nu, -\frac{x^2}{2(\nu+1)}\right). \tag{2.51}$$

3. Associated Legendre functions

The associated Legendre functions obey the baseline differential equation

$$\bar{y}''(x) - \frac{2x}{1-x^2}\bar{y}'(x) + \left(\frac{\nu(\nu+1)}{1-x^2} - \frac{\mu^2}{(1-x^2)^2}\right)\bar{y}(x) = 0 \tag{3.1}$$

with

$$\bar{f}(x) = 1 - x^2, \tag{3.2}$$

and the conventional general solution is

$$\bar{y}(x) = C_1 P_\nu^\mu(x) + C_2 Q_\nu^\mu(x), \tag{3.3}$$

where $P_\nu^\mu(x)$ is the associated Legendre function of the first kind and $Q_\nu^\mu(x)$ is the associated Legendre function of the second kind. Here it is assumed that these functions are those defined with the real axis cut outside the interval $-1 < x < 1$, which is the default in Mathematica [7]. The results presented here apply on the real axis within this interval.

From symmetry the six additional functions $P_\nu^{-\mu}(x)$, $Q_\nu^{-\mu}(x)$, $P_{-\nu-1}^{\mu}(x)$, $Q_{-\nu-1}^{\mu}(x)$ $P_{-\nu-1}^{-\mu}(x)$ and $Q_{-\nu-1}^{-\mu}(x)$ are also solutions of Equation (3.1), but the identity [10]

$$P_{-\nu-1}^{-\mu}(x) = P_\nu^\mu(x) \tag{3.4}$$

eliminates $P_{-\nu-1}^{-\mu}(x)$ as an additional independent solution. The function $Q_{-\nu-1}^{\mu}(x)$ is given in terms of $P_\nu^{-\mu}(x)$ and $Q_\nu^{-\mu}(x)$ as [10]

$$Q_{-\nu-1}^{\mu}(x) = \frac{\sin[(\nu+\mu)\pi]}{\sin[(\nu-\mu)\pi]} Q_\nu^\mu(x) - \frac{\pi\cos(\nu\pi)\cos(\mu\pi)}{\sin[(\nu-\mu)\pi]} P_\nu^\mu(x), \tag{3.5}$$

and on the grounds of space, this function will also not be explicitly considered here. With this restriction, the results presented here largely mirror the corresponding results for Bessel functions.

From Abel's identity, the Wronskian of any two independent solutions is of the form

$$W\left(\bar{y}_1(x), \bar{y}_2(x)\right) = \frac{A}{1-x^2}. \tag{3.6}$$

As $\bar{f}(0) = 1$, the constant A in all of these Wronskians can easily be determined from the values of $P_\nu^\mu(x)$, $Q_\nu^\mu(0)$, $P_\nu'^\mu(x)$ and $Q_\nu'^\mu(0)$, within the interval $(-1 < x < 1)$ enclosed by the singularities of the Wronskian at $x = \pm 1$. These values are given in [10] as

$$P_\nu^\mu(0) = \frac{2^\mu\sqrt{\pi}}{\Gamma\left(\frac{\nu-\mu}{2}+1\right)\Gamma\left(\frac{-\nu-\mu+1}{2}\right)}, \tag{3.7}$$

$$P_\nu'^\mu(0) = \frac{2^{\mu+1}\sin\left(\frac{1}{2}(\nu+\mu)\pi\right)\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\nu-\mu+1}{2}\right)}, \tag{3.8}$$

$$Q_\nu^\mu(0) = -2^{\mu-1}\sqrt{\pi}\sin\left(\frac{1}{2}(\nu+\mu)\pi\right)\frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu}{2}+1\right)}, \tag{3.9}$$

$$Q_\nu'^\mu(0) = 2^\mu\sqrt{\pi}\cos\left(\frac{1}{2}(\nu+\mu)\pi\right)\frac{\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)}, \tag{3.10}$$

and these results give the Wronskian

$$W\left(P_\nu^\mu(x), Q_\nu^\mu(x)\right) = \frac{\Gamma(\nu+\mu+1)}{\left(1-x^2\right)\Gamma(\nu-\mu+1)}. \tag{3.11}$$

Hence, in a manner closely analogous to the Bessel function case, we have

$$\left(\frac{Q_\nu^\mu(x)}{P_\nu^\mu(x)}\right)' = \frac{\Gamma(\nu+\mu+1)}{\left(1-x^2\right)\Gamma(\nu-\mu+1)\left(P_\nu^\mu(x)\right)^2}, \tag{3.12}$$

$$\left(\frac{P_\nu^\mu(x)}{Q_\nu^\mu(x)}\right)' = -\frac{\Gamma(\nu+\mu+1)}{\left(1-x^2\right)\Gamma(\nu-\mu+1)\left(Q_\nu^\mu(x)\right)^2} \tag{3.13}$$

which give the integrals

$$\int\frac{\mathrm{d}x}{\left(1-x^2\right)\left(P_\nu^\mu(x)\right)^2} = \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\frac{Q_\nu^\mu(x)}{P_\nu^\mu(x)}, \tag{3.14}$$

$$\int \frac{\mathrm{d}x}{\left(1-x^2\right)\left(Q_\nu^\mu(x)\right)^2} = -\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\frac{P_\nu^\mu(x)}{Q_\nu^\mu(x)}, \tag{3.15}$$

$$\int \frac{\mathrm{d}x}{\left(1-x^2\right)P_\nu^\mu(x)\,Q_\nu^\mu(x)} = \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\ln\left(\frac{Q_\nu^\mu(x)}{P_\nu^\mu(x)}\right). \tag{3.16}$$

These results appear to be new, and Mathematica cannot obtain them directly. The Wronskians for any other pairs of the independent solutions given above can be obtained by switching the sign of μ as necessary in Equations (3.7)–(3.10), which gives the Wronskians

$$W\left(P_\nu^\mu(x),\ P_\nu^{-\mu}(x)\right) = -\frac{2\sin(\mu\pi)}{\pi\left(1-x^2\right)} \quad \text{for } \mu \notin \mathbb{Z}, \tag{3.17}$$

$$W\left(Q_\nu^\mu(x), Q_\nu^{-\mu}(x)\right) = -\frac{\pi\sin(\mu\pi)}{2\left(1-x^2\right)} \quad \text{for } \mu \notin \mathbb{Z}, \tag{3.18}$$

$$W\left(P_\nu^\mu(x), Q_\nu^{-\mu}(x)\right) = \frac{\cos(\mu\pi)}{1-x^2}\text{for}\mu + \frac{1}{2} \notin \mathbb{Z}, \tag{3.19}$$

$$W\left(Q_\nu^\mu(x), P_\nu^{-\mu}(x)\right) = -\frac{\cos(\mu\pi)}{1-x^2} \quad \text{for } \mu + \frac{1}{2} \notin \mathbb{Z}. \tag{3.20}$$

Equations (3.17)–(3.20) immediately give the integrals

$$\int \frac{\mathrm{d}x}{\left(1-x^2\right)\left(P_\nu^\mu(x)\right)^2} = -\frac{\pi P_\nu^{-\mu}(x)}{2\sin(\mu\pi)\,P_\nu^\mu(x)} \quad \text{for } \mu \notin \mathbb{Z}, \tag{3.21}$$

$$\int \frac{\mathrm{d}x}{\left(1-x^2\right)\left(P_\nu^{-\mu}(x)\right)^2} = \frac{\pi P_\nu^\mu(x)}{2\sin(\mu\pi)\,P_\nu^{-\mu}(x)} \quad \text{for } \mu \notin \mathbb{Z}, \tag{3.22}$$

$$\int \frac{\mathrm{d}x}{\left(1-x^2\right)\left(Q_\nu^\mu(x)\right)^2} = -\frac{2Q_\nu^{-\mu}(x)}{\pi\sin(\mu\pi)\,Q_\nu^\mu(x)} \quad \text{for } \mu \notin \mathbb{Z}, \tag{3.23}$$

$$\int \frac{\mathrm{d}x}{\left(1-x^2\right)\left(Q_\nu^{-\mu}(x)\right)^2} = \frac{2Q_\nu^\mu(x)}{\pi\sin(\mu\pi)\,Q_\nu^{-\mu}(x)} \quad \text{for } \mu \notin \mathbb{Z}, \tag{3.24}$$

$$\int \frac{\mathrm{d}x}{\left(1-x^2\right)\left(P_\nu^\mu(x)\right)^2} = \frac{Q_\nu^{-\mu}(x)}{\cos(\mu\pi)\,P_\nu^\mu(x)} \quad \text{for } \mu + \frac{1}{2} \notin \mathbb{Z}, \tag{3.25}$$

$$\int \frac{\mathrm{d}x}{\left(1-x^2\right)\left(Q_\nu^{-\mu}(x)\right)^2} = -\frac{P_\nu^\mu(x)}{\cos(\mu\pi)\,Q_\nu^{-\mu}(x)} \quad \text{for } \mu + \frac{1}{2} \notin \mathbb{Z}, \tag{3.26}$$

$$\int \frac{\mathrm{d}x}{\left(1-x^2\right)\left(P_\nu^{-\mu}(x)\right)^2} = \frac{Q_\nu^\mu(x)}{\cos(\mu\pi)\,P_\nu^{-\mu}(x)} \quad \text{for } \mu + \frac{1}{2} \notin \mathbb{Z}, \tag{3.27}$$

$$\int \frac{\mathrm{d}x}{\left(1-x^2\right)\left(Q_\nu^\mu(x)\right)^2} = -\frac{P_\nu^{-\mu}(x)}{\cos(\mu\pi)\,Q_\nu^\mu(x)} \quad \text{for } \mu + \frac{1}{2} \notin \mathbb{Z}, \tag{3.28}$$

$$\int \frac{\mathrm{d}x}{\left(1-x^2\right) P_\nu^\mu(x)\, P_\nu^{-\mu}(x)} = \frac{\pi}{2\sin(\mu\pi)} \ln\left(\frac{P_\nu^\mu(x)}{P_\nu^{-\mu}(x)}\right) \quad \text{for } \mu \notin \mathbb{Z}, \tag{3.29}$$

$$\int \frac{\mathrm{d}x}{\left(1-x^2\right) Q_\nu^\mu(x)\, Q_\nu^{-\mu}(x)} = \frac{2}{\pi\sin(\mu\pi)} \ln\left(\frac{Q_\nu^\mu(x)}{Q_\nu^{-\mu}(x)}\right) \quad \text{for } \mu \notin \mathbb{Z}, \tag{3.30}$$

$$\int \frac{\mathrm{d}x}{\left(1-x^2\right) P_\nu^\mu(x)\, Q_\nu^{-\mu}(x)} = \frac{1}{\cos(\mu\pi)} \ln\left(\frac{Q_\nu^{-\mu}(x)}{P_\nu^{\mu}(x)}\right) \quad \text{for } \mu + \frac{1}{2} \notin \mathbb{Z}, \tag{3.31}$$

$$\int \frac{\mathrm{d}x}{\left(1-x^2\right) Q_\nu^\mu(x)\, P_\nu^{-\mu}(x)} = \frac{1}{\cos(\mu\pi)} \ln\left(\frac{Q_\nu^{\mu}(x)}{P_\nu^{-\mu}(x)}\right) \quad \text{for } \mu + \frac{1}{2} \notin \mathbb{Z}. \tag{3.32}$$

For the associated Legendre functions, Equations (1.22)–(1.23) and their principal solutions become

$$y''(x) + \left(2\frac{P_\nu'^\mu(x)}{P_\nu^\mu(x)} - \frac{2x}{1-x^2}\right) y'(x) = 0 \Rightarrow y(x) = \frac{Q_\nu^\mu(x)}{P_\nu^\mu(x)}, \tag{3.33}$$

$$y''(x) + \left(2\frac{Q_\nu'^\mu(x)}{Q_\nu^\mu(x)} - \frac{2x}{1-x^2}\right) y'(x) = 0 \Rightarrow y(x) = \frac{P_\nu^\mu(x)}{Q_\nu^\mu(x)}. \tag{3.34}$$

Applying these equations in Equation (1.2) gives

$$\begin{aligned}\int \left(1-x^2\right)\left(h''(x) + \left(2\frac{P_\nu'^\mu(x)}{P_\nu^\mu(x)} - \frac{2x}{1-x^2}\right) h'(x)\right) P_\nu^\mu(x)\, Q_\nu^\mu(x)\, \mathrm{d}x \\ = \left(1-x^2\right) h'(x)\, P_\nu^\mu(x)\, Q_\nu^\mu(x) - \frac{h(x)\,\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)},\end{aligned} \tag{3.35}$$

$$\begin{aligned}\int \left(1-x^2\right)\left(h''(x) + \left(2\frac{Q_\nu'^\mu(x)}{Q_\nu^\mu(x)} - \frac{2x}{1-x^2}\right) h'(x)\right) P_\nu^\mu(x)\, Q_\nu^\mu(x)\, \mathrm{d}x \\ = \left(1-x^2\right) h'(x)\, P_\nu^\mu(x)\, Q_\nu^\mu(x) + \frac{h(x)\,\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}.\end{aligned} \tag{3.36}$$

Choosing the fragment

$$h''(x) - \frac{2x}{1-x^2} h'(x) = 0 \;\Rightarrow h'(x) = \frac{1}{1-x^2};\; h(x) = \operatorname{arctanh}(x) \tag{3.37}$$

in Equations (3.35)–(3.36) gives the integrals

$$\int P_\nu'^\mu(x)\, Q_\nu^\mu(x)\, \mathrm{d}x = \frac{P_\nu^\mu(x)\, Q_\nu^\mu(x)}{2} - \frac{\Gamma(\nu+\mu+1)\operatorname{arctanh}(x)}{2\Gamma(\nu-\mu+1)}, \tag{3.38}$$

$$\int Q_\nu'^\mu(x)\, P_\nu^\mu(x)\, \mathrm{d}x = \frac{P_\nu^\mu(x)\, Q_\nu^\mu(x)}{2} + \frac{\Gamma(\nu+\mu+1)\operatorname{arctanh}(x)}{2\Gamma(\nu-\mu+1)} \tag{3.39}$$

which are closely analogous to Equations (2.32)–(2.33).

The four conventional recurrence relations for associated Legendre functions are [5,10]

$$R_\nu'^\mu(x) = -\frac{\nu x}{1-x^2} R_\nu^\mu(x) + \frac{\nu+\mu}{1-x^2} R_{\nu-1}^\mu(x), \tag{3.40}$$

$$R'^{\mu}_{\nu}(x) = \frac{(\nu+1)x}{1-x^2} R^{\mu}_{\nu}(x) - \frac{\nu-\mu+1}{1-x^2} R^{\mu}_{\nu+1}(x), \tag{3.41}$$

$$R'^{\mu}_{\nu}(x) = \frac{\mu x}{1-x^2} R^{\mu}_{\nu}(x) + (\nu-\mu+1)(\nu+\mu)\frac{R^{\mu-1}_{\nu}(x)}{\sqrt{1-x^2}}, \tag{3.42}$$

$$R'^{\mu}_{\nu}(x) = -\frac{\mu x}{1-x^2} R^{\mu}_{\nu}(x) - \frac{R^{\mu+1}_{\nu}(x)}{\sqrt{1-x^2}}, \tag{3.43}$$

where here $R^{\mu}_{\nu}(x) = P^{\mu}_{\nu}(x)$ or $R^{\mu}_{\nu}(x) = Q^{\mu}_{\nu}(x)$. Substituting Equation (3.40) into Equation (3.35) for $R^{\mu}_{\nu}(x) = P^{\mu}_{\nu}(x)$ gives the integration formula

$$\int \left(1-x^2\right)\left(h''(x) + \left(\frac{2(\nu+\mu)}{1-x^2}\frac{P^{\mu}_{\nu-1}(x)}{P^{\mu}_{\nu}(x)} - \frac{2(\nu+1)x}{1-x^2}\right)h'(x)\right) P^{\mu}_{\nu}(x)\, Q^{\mu}_{\nu}(x)\,\mathrm{d}x$$
$$= \left(1-x^2\right)h'(x)\, P^{\mu}_{\nu}(x)\, Q^{\mu}_{\nu}(x) - \frac{h(x)\,\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} \tag{3.44}$$

and choosing the fragment

$$h''(x) - \frac{2(\nu+1)x}{1-x^2} h' = 0 \tag{3.45}$$

for which the solution is

$$h'(x) = \left(1-x^2\right)^{-\nu-1}; h(x) = x\,{}_2F_1\left(\frac{1}{2}, \nu+1; \frac{3}{2}; x^2\right) \tag{3.46}$$

gives the integral

$$\int \frac{P^{\mu}_{\nu-1}(x)\, Q^{\mu}_{\nu}(x)}{\left(1-x^2\right)^{\nu+1}}\mathrm{d}x = \frac{P^{\mu}_{\nu}(x)\, Q^{\mu}_{\nu}(x)}{2(\nu+\mu)\left(1-x^2\right)^{\nu}} - \frac{x\,{}_2F_1\left(\frac{1}{2}, \nu+1; \frac{3}{2}; x^2\right)\Gamma(\nu+\mu+1)}{2(\nu+\mu)\,\Gamma(\nu-\mu+1)}. \tag{3.47}$$

Substituting recurrences (3.41)–(3.43) with $R^{\mu}_{\nu}(x) = P^{\mu}_{\nu}(x)$ into Equation (3.35) and choosing $h(x)$ in a similar manner gives the integrals

$$\int \left(1-x^2\right)^{\nu} P^{\mu}_{\nu+1}(x)\, Q^{\mu}_{\nu}(x)\,\mathrm{d}x$$
$$= \frac{x\,{}_2F_1\left(\frac{1}{2}, -\nu; \frac{3}{2}; x^2\right)\Gamma(\nu+\mu+1)}{2(\nu-\mu+1)\,\Gamma(\nu-\mu+1)} - \frac{\left(1-x^2\right)^{\nu+1} P^{\mu}_{\nu}(x)\, Q^{\mu}_{\nu}(x)}{2(\nu-\mu+1)}, \tag{3.48}$$

$$\int \left(1-x^2\right)^{-\mu-\frac{1}{2}} P^{\mu-1}_{\nu}(x)\, Q^{\mu}_{\nu}(x)\,\mathrm{d}x$$
$$= \frac{\left(1-x^2\right)^{-\mu} P^{\mu}_{\nu}(x)\, Q^{\mu}_{\nu}(x)}{2(\nu-\mu+1)(\nu+\mu)} - \frac{x\,{}_2F_1\left(\frac{1}{2}, 1-\mu; \frac{3}{2}; x^2\right)\Gamma(\nu+\mu+1)}{2(\nu-\mu+1)(\nu+\mu)\,\Gamma(\nu-\mu+1)}, \tag{3.49}$$

$$\int \frac{P^{\mu+1}_{\nu}(x)\, Q^{\mu}_{\nu}(x)}{\left(1-x^2\right)^{\mu+\frac{1}{2}}}\mathrm{d}x = \frac{x\,{}_2F_1\left(\frac{1}{2}, \mu+1; \frac{3}{2}; x^2\right)\Gamma(\nu+\mu+1)}{2\Gamma(\nu-\mu+1)} - \frac{P^{\mu}_{\nu}(x)\, Q^{\mu}_{\nu}(x)}{2\left(1-x^2\right)^{\mu}}. \tag{3.50}$$

The corresponding integrals obtained from Equation (3.37) with $R_\nu^\mu(x) = Q_\nu^\mu(x)$ in the four recurrences (3.40)–(3.43) are, respectively,

$$\int \frac{Q_{\nu-1}^\mu(x)\,P_\nu^\mu(x)}{\left(1-x^2\right)^{\nu+1}}\,\mathrm{d}x = \frac{Q_\nu^\mu(x)\,P_\nu^\mu(x)}{2(\nu+\mu)\left(1-x^2\right)^\nu} + \frac{x\,{}_2F_1\left(\frac{1}{2},\nu+1;\frac{3}{2};x^2\right)\Gamma(\nu+\mu+1)}{2(\nu+\mu)\,\Gamma(\nu-\mu+1)}, \tag{3.51}$$

$$\int \left(1-x^2\right)^\nu Q_{\nu+1}^\mu(x)\,P_\nu^\mu(x)\,\mathrm{d}x = -\frac{x\,{}_2F_1\left(\frac{1}{2},-\nu;\frac{3}{2};x^2\right)\Gamma(\nu+\mu+1)}{2(\nu-\mu+1)\,\Gamma(\nu-\mu+1)} - \frac{\left(1-x^2\right)^{\nu+1}Q_\nu^\mu(x)\,P_\nu^\mu(x)}{2(\nu-\mu+1)}, \tag{3.52}$$

$$\int \left(1-x^2\right)^{-\mu-\frac{1}{2}} Q_\nu^{\mu-1}(x)\,P_\nu^\mu(x)\,\mathrm{d}x = \frac{\left(1-x^2\right)^{-\mu}Q_\nu^\mu(x)\,P_\nu^\mu(x)}{2(\nu-\mu+1)(\nu+\mu)} + \frac{x\,{}_2F_1\left(\frac{1}{2},1-\mu;\frac{3}{2};x^2\right)\Gamma(\nu+\mu+1)}{2(\nu-\mu+1)(\nu+\mu)\,\Gamma(\nu-\mu+1)}, \tag{3.53}$$

$$\int \frac{Q_\nu^{\mu+1}(x)\,P_\nu^\mu(x)}{\left(1-x^2\right)^{\mu+\frac{1}{2}}}\,\mathrm{d}x = -\frac{x\,{}_2F_1\left(\frac{1}{2},\mu+1;\frac{3}{2};x^2\right)\Gamma(\nu+\mu+1)}{2\Gamma(\nu-\mu+1)} - \frac{Q_\nu^\mu(x)\,P_\nu^\mu(x)}{2\left(1-x^2\right)^\mu}. \tag{3.54}$$

Equations (3.51)–(3.54) can be obtained from Equations (3.47)–(3.50) by exchanging the two kinds of associated Legendre functions in the formulas and reversing the signs of the terms containing hypergeometric functions. The sign reversals are necessary because these terms are derived from Wronskians, which are antisymmetric on the exchange of the two independent functions.

All of the integrals for associated Legendre functions presented here appear to be new. They are not tabulated in the standard literature [9–11] and Mathematica [7] cannot derive them. Two clearly related results for Airy functions are given in [12].

Disclosure statement

No potential conflict of interest was reported by the author(s).

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