# *Bringing Nordic mathematics education into the future*

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# Preface

This volume presents Nordic mathematics education research, which will be presented at the Ninth Nordic Conference on Mathematics Education, NORMA 20, in Oslo, Norway, in June 2021. The theme of NORMA 20 regards what it takes or means to bring Nordic mathematics education into the future, highlighting that mathematics education is continuous and represents stability just as much as change.

NORMA conferences are always organized in collaboration with the Nordic Society for Research in Mathematics Education (NoRME). NoRME is open to membership from national societies for research in mathematics education in the Nordic and Baltic countries.

Inclusive classrooms and "mathematics education for all" have traditionally been at the core of Nordic mathematics education. Currently, the digital development and possibilities for individualized learning activities widen the understanding of adaption in compulsory education. This push and pull between inclusion and adaption bring the possibility of renewing mathematics education, including pre-school and tertiary levels, while still maintaining the principle of student-centred mathematics education. Mathematics education is also changing at the level of teacher education, which is reflected in the conference papers included in this preceeding.

The International Programme Committee (IPC) of NORMA 20 represents all Nordic countries and includes one representative from the Baltic countries, with a mix of junior and senior researchers. The IPC has organized the submission and review process leading to this volume. The members of the IPC were:

- Guri A. Nortvedt University of Oslo (Chair), Norway
- Nils Buchholtz, University of Oslo, Norway, and University of Cologne, Germany
- Janne Fauskanger, University of Stavanger, Norway
- Freyja Hreinsdóttir, University of Iceland, Iceland
- Markus Hähkiöniemi, University of Jyväskylä, Finland
- Britta Eyrich Jessen, University of Copenhagen, Denmark
- Jüri Kurvits, Tallinn University, Estonia
- Yvonne Liljekvist, Karlstad University, Sweden
- Morten Misfeldt, University of Copenhagen, Denmark
- Margrethe Naalsund, NMBU, Norway
- Hans Kristian Nilsen, Universitetet i Agder, Norway
- Guðbjörg Pálsdóttir, University of Iceland, Iceland
- Päivi Portaankorva-Koivisto, Helsinki university, Finland
- Jelena Radišić, University of Oslo, Norway
- Anna Wernberg, Malmö University, Sweden

The first NORMA conference on mathematics education, NORMA 94, was held in Lahti, Finland, in 1994. Four years later, the conference was held in Kristiansand, Norway; since then, it has taken place every third year. After each conference, selected papers are published in a proceeding. Due to the

COVID-19 pandemic, the NORMA 20 conference was postponed until 2021; however, many conference papers were in progress and authors were given the opportunity to continue working on them within the original planned timespan. Traditionally, papers are presented at the conference, allowing the authors to receive feedback that is valuable towards finalizing the paper. Instead, the authors have used two rounds of reviewer feedback to substantially improve their papers. In this process, the NORMA community established in 1995, together with external reviewers who are experts in the different fields studied and presented in the papers, have played an important role in producing the Preceeding.

We believe that the NORMA 20 Preceeding is the first conference preceeding to be published, containing 36 papers from authors representing six countries.

After the conference, a traditional Proceeding will be published, containing papers written by submitting authors who decided to wait until after the conference to finalise their papers, to take advantage of feedback from both conference participants and reviewers when they revise their papers.

The IPC would like to extend our thanks to all authors and reviewers for their efforts towards this volume.

Oslo, January 2021, on behalf of the IPC

Guri A. Nortvedt

# Contents

# A framework for analyzing progress in concept knowledge in mathematics textbooks

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*In this paper we report on a framework for analyzing progress in concept knowledge as represented in textbooks. The framework is based on theories of Vergnaud, combined with a theory of Bruner. Two Swedish textbook series are analyzed with respect to how they introduce mathematical content and how it supports the progress of concept knowledge throughout school years. The analysis is restricted to the conceptual field of quotient constructions. We classified representations in terms of situational, iconic and symbol system representations. Results show that the investigated textbooks provide students with the same limited explanations in iconic representations of quotient constructions from year 3, when it is introduced, to year 9. We argue that our theory for progress in concept knowledge, together with our framework for analyzing progress, can be used by teachers and curriculum developers to organize instruction.*

*Keywords: Mathematics, Textbooks, Progress, Concept knowledge*

## Introduction

Mathematics textbooks play an important role in mathematics teaching and learning around the world. They function as mediators between the intentions of the designers of curriculum policies and the teachers that carry out instruction in the classroom. As a consequence of the great importance of textbooks many and major research studies have been conducted (Fan, Zhu, & Miao, 2013). The investigated areas cover textbook analysis, textbook comparison, the use of textbooks and other areas; for example, teachers' preferences in textbook characteristics. In the area of textbook analysis attention has been given to mathematics content and topics, e.g., fractions (Alajmi, 2012) or proportional reasoning (Ahl, 2016), as well as non topic specific aspects like reasoning and proof (Thompson, Senk, & Johnson, 2012). Also, cognition, pedagogy, methodological matters and conceptualization has gained attention from the research community (Fan et al., 2013).

The presentation of mathematical concepts has been investigated from different viewpoints. For example, textbooks have shown an inadequacy in presenting mathematics content investigated through difficulty levels and non-textual elements, such as pictures and graphs (e.g., Charalambous, Delaney, Hsu, & Mesa, 2010). Another important aspect of textbook design is how the mathematical content supports the progress of concept knowledge throughout school years (Fan et al., 2013). This aspect has however not been systematically addressed in textbook research, since most textbook analysis studies have been conducted at the primary level (Fan et al., 2013). Studies covering progress through both primary and secondary level are rare.

Knowledge of mathematical concepts develops over time. In this paper we will present a framework for identifying progress in concept knowledge. The framework is based on a theory building on Vergnaud (2009), with influences from Bruner (1966) (to be elaborated on in the theory section). We have claimed in our earlier research that a central aspect of progress in concept knowledge is that the meaning of concepts and the relations they entail must initially be presented by situations and iconic

representations, but must at some point be replaced by meaning residing in relations described through mathematical symbol systems (for more information see Ahl & Helenius, in press). The framework is designed to elicit such epistemological shifts.

The framework is applied on two Swedish textbook-series and how they introduce concepts, notations and procedures in compulsory school. Since over 90 % of the Swedish teachers rely on mathematics textbooks for carrying out instruction (Mullis et al., 2012), mathematics textbooks can be used as a proxy to investigate how concepts are presented to students. We will present our framework in detail and the results from the examinations of the two textbook-series, one covering grades 1 to 6 and the other covering grades 7 to 9. The question guiding this study is: *How and when is there a shift in meaning making from the icons and situations, to meaning making by mathematical relations and the symbol systems?*

## Theory

We are interested in how concepts are treated, that is how they are given meaning, how the connection between concepts are described and explained and how the usages of concepts are presented. In a formal mathematical exposition, concepts are uniquely determined by definitions. But as objects of the mind, mathematical concepts are not unique in this sense. And in fact, this holds for formal mathematical expositions too when you compare definitions across different expositions. Let us ponder the sine concept. In one exposition the sine function might be defined with triangles, in another with the unit circle, in a third as a power series and in a fourth as the unique solution to a particular differential equation. As the object of individual or collective minds, most would agree it is the same object that is defined. Thus, a definition in one presentation can be proven as a consequence of the definitions in any of the other presentations. As mental objects mathematical concepts are hence fuzzy and there may be many competing definitions or other characterizations. To account for the fact that even mathematically simple concepts are psychologically complex, Vergnaud (2009) developed the theory of conceptual fields. This theory holds that concepts only exist in networks of other concepts. A goal with the theory of conceptual fields was to analyze the development of students' mathematical competence on a medium- and long-term basis (Vergnaud, 2009). Therefore, it makes sense to use elements from this theory when developing a framework for analyzing progress in concept knowledge and it makes sense to base the analysis on a unit comprising a specific conceptual field.

But what then is progress in concept knowledge? Bruner (1966) characterizes learning, or intellectual growth as he terms it, by six points where the first is that "growth is characterized by increasing independence of response from the immediate nature of the stimulus" (p. 5). There are obvious similarities to Piaget's moves from the operational, over empirical abstraction to reflective abstraction (Beth & Piaget, 1966). But whereas Piaget's notions concern a characterization of the mental abilities, Bruner expresses the growth in requirements on the external stimuli. As we eventually want to operationalize ideas about mental growth in concept knowledge into a framework for analyzing textbooks, it makes sense to use Bruner's characterization as a starting point. Bruner describes three types of external stimuli, that are to be understood as hierarchical levels. *Enactive*, concerning action stimulated by actual objects or recollection of previous such action. *Iconic*, concerning action stimulated by actual or mental images of objects. *Symbolic*, concerning action stimulated by symbols whose meaning must be articulated or defined. A modified version of this hierarchy will form a basis

for our view on progress in concept knowledge. But, before we can do that, we need to invoke some further theory.

Vergnaud (1998) developed a comprehensive theory of representations where his initially Piaget-based work on conceptual fields was complemented with Vygotskian ideas. Concepts, as psychological objects, are in Vergnaud's theorization described as a triplet of sets, $C=C(S,I,R)$, where S is the class of situations where the concept makes sense, I is the set of operational invariants that can be used by an individual to deal with these situations, and R is the set of representations, symbolic, verbal, graphical, gestural etc. that can be used to represent invariants, situations and procedures. In our previous work, we have used this characterization as a stepping stone for concluding that mathematical concepts can be born in three essentially different ways. Invariants can be bootstrapped from classes of situations or from iconic representations or described in terms of relationships in non-iconic symbol systems (for more information see Ahl & Helenius, in press). In Bruner's terminology, situations relate to the enactive, iconic representations to the iconic and (non-iconic) symbol systems to the symbolic. We argue elsewhere that when mathematical concepts grow, they tend to get clustered under more general concepts (Helenius & Ahl, accepted). From this we drew the conclusion that progress in concept knowledge eventually must involve an epistemological shift where the meaning of concepts goes from residing in situations and iconic representations to residing in relationships in symbol systems. This is the foundation for our framework for analyzing progress in concept knowledge.

## Methodological considerations

From the theory described above we singled out two methodological principles for the analytical framework: To use progress through one choice of conceptual fields throughout a book series as our unit of analysis and to classify representations in terms of situational, iconic and symbol system representations. This was operationalized in a qualitative approach as described below.

We first identified instances where the textbook, as an agent for the authors, makes mathematical claims, argues for propositions or gives meaning to concepts related to the chosen conceptual field. For each such instance we evaluated in what form the communication is presented, that is, is it presented as a situation, as an iconic representation or by non-iconic symbols in a symbol system. There are some further intricate details of this classification that we will explain through examples below. We also made a further classification by categorizing what function the claim, argument or description had. After a preliminary empirical examination, we found three types of functions: *concept*, where some concept is given meaning; *procedure*, where some procedure is described; *connection*, where a connection between concepts is explored, explained or described. In summary we hence classify our detected instances in the textbooks in a two-dimensional framework consisting of representation type and function.

In this study we analyze the multiplicative conceptual field, restricted to quotient constructions, which means content related to fractions and division. We further narrow our selection by deselecting instances focused on the positions system (like how to divide by 10 or 100) or additive features (like how to add or subtract fractions). We exemplify our method through four examples.

**39. Division – innehållsdivision**  Film  Lyssna på berättelsen.  A

Charlie lägger
12 bullar i påsar.
Han lägger 4
bullar i varje påse.
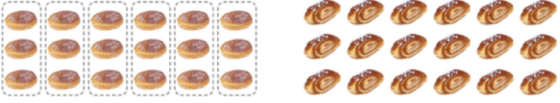
Hur många påsar
behöver Charlie?

Svar: 3 påsar

1. Du lägger 18 bullar i påsar. Hur många påsar behöver du?

a. Det ska vara 3 bullar i varje påse.  b. Det ska vara 6 bullar i varje påse.

Svar: _påsar_  Svar: _____

**40. Att skriva division**  Lyssna på berättelsen.  Film  B

Det finns sex glasskulor.
I varje bägare ska det vara
2 kulor glass. Hur många
bägare behövs det?

Divisionen kan skrivas

$\frac{6}{2} = 3$   $\frac{täljare}{nämnare} = kvot$

**eller**

$6/2 = 3$

täljare/nämnare = kvot

Du säger: Sex delat med två är
lika med tre, eller sex dividerat
med två är lika med tre.

**41. Sambandet mellan division och multiplikation**  Film  Lyssna på berättelsen.  C

Det finns 12 böcker. Hur
många högar får du, om du
lägger 4 böcker i varje hög?

$\frac{12}{4} = 3$

Du kan kontrollera division
med multiplikation.

$3 \cdot 4 = 12$

1. Dividera. Kontrollera division med multiplikation.

a. $\frac{8}{2} = 4$   b. $\frac{6}{3} = $ ___   c. $\frac{12}{6} = $ ___

$4 \cdot 2 = 8$   ___ $\cdot$ ___ $= 6$   ___ $\cdot$ ___ $= $ ___

**Mer om att dividera med bråk**  D

**Inverterat tal**

$\frac{2}{3}$ kallas för det **inverterade talet** till $\frac{3}{2}$.

Om man multiplicerar ett tal med dess inverterade tal
får man alltid produkten 1.

$\frac{2}{3} \cdot \frac{3}{2} = \frac{6}{6} = 1$   $\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba} = 1$

**Division av bråk**

Att utföra divisionen $\frac{1}{2} \Big| \frac{1}{4}$ är detsamma som att
bestämma hur många fjärdedelar det finns i en halv.

$\frac{1}{2} \Big| \frac{1}{4} = \frac{\frac{1}{2}}{\frac{1}{4}} = \frac{\frac{1}{2} \cdot \frac{4}{1}}{\frac{1}{4} \cdot \frac{4}{1}} = \frac{\frac{1}{2} \cdot \frac{4}{1}}{1} = \frac{1}{2} \cdot \frac{4}{1} = \frac{4}{2} = 2$

Förläng med $\frac{4}{1}$ så att nämnaren blir 1.

Att dividera med ett bråk ger samma
resultat som att multiplicera med
bråkets inverterade tal.

$\frac{1}{2} \Big| \frac{3}{4} = \frac{1}{2} \cdot \frac{4}{3} = \frac{4}{6} = \frac{2}{3}$

$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b} \cdot \frac{d}{c}}{\frac{c}{d} \cdot \frac{d}{c}} = \frac{a}{b} \cdot \frac{d}{c}$

$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}$

Figure 1: Examples from the analyzed textbook series

Example A (grade 3): This example introduces the concept division as equal grouping by referring to a situation: *"Charlie places 12 buns in bags. He places 4 buns in each bag. How many bags does Charlie need?"* It is to be understood that division is the mathematical operation that corresponds to this act. The meaning making is hence done by referring to a class of situations. We denote this by the symbol ☺. In example A the situation is also illustrated by an image of 12 buns, grouped into 4-groups in bags. At the bottom of the excerpt the same principle is illustrated by an array model. Both these illustrations are iconic representations, which we will denote by the symbol ♥. The main meaning making argument comes from the situations though, and the pictures of the buns are iconic representations of the situation, rather than arguments in themselves. We will denote this by ☺ ⟶ ♥, the arrow indicating that the representation form that comes after is used as a labelling of reasoning expressed in the representation form that comes before the arrow.

Example B (grade 3): Also here, the main argument for the meaning of the concept of division is given by a situation. *"There are six scoops of ice cream. In each bowl, there should be 2 scoops. How many bowls are needed?"* This is the same type of equal grouping situations as in example A, just in

a different context. Example B also contains symbols from the standard symbol system used to write fractions and division. We denote the existence of non-iconic symbols from mathematical symbol systems with π. No argumentations are carried out through this symbol system though, it is just used to denote the phenomena from the situation. We denote this by ☺ →π.

Example C (grade 3): This example concerns the connection between multiplication and division. It first, exactly like in example B, denote an equal grouping situation with the symbol system. Then it is stated that *"You can check division with multiplication. 3·4=12"* But there is no argument given for this connection between division and multiplication. To find an argument we must instead look at the picture of the 12 books in 3 groups of 4. As we have seen in the other examples, this iconic representation denotes division as equal grouping and also have been denoted by the symbol system expression 12/4. But in previous chapters, multiplication have been described as repeated addition and similar iconic representations has been used to represent (and has been represented by) 3*4=12. Therefore, the argument (albeit very implicit) for the connection between multiplication and division resides in the iconic representation. This is also repeated in task 1 where the 2 by 4 iconic representation represents both a division and a multiplication. We classify this as a concept explanation of type ☺ →π and a connection explanation of type ♥→π.

Example D (grade 8): This excerpt contains both procedural and conceptual information. The procedural information is expressed through the symbol system (π). The conceptual information regarding the inverse is explained through relations expressed in the symbol system: how multiplications of fractions work symbolically and that a fraction with identical numerator and denominator is 1 (π). There is also a description of division that refers back to equal grouping but it is formulated directly in terms of fractions: *"[...] how many fourths there are in a half"*. Though this is expressed verbally, we classify it as expressing relations in the symbol system. The statement is then illustrated with an iconic representation (π→♥). This image is then used to give meaning to the connection between division and fractions (♥→π) but only explains why the answer becomes 2. The rest of the meaning of the calculations and formulas are given purely by using relations in the symbol system (π). A common phenomenon that is not present in any of these examples is when a proposition or procedure is presented in symbolic form, but no reason of given for why the propositions holds or the procedure works. This we denote by →π.

In summary, the analytical framework we propose has the following steps.

1. For the chosen book series, find all instances of the conceptual field you want to analyze.

2. For each instance, classify the function of the reasoning or explanation. In our case, a priori empirical analysis of the material revealed three types, giving meaning to concepts, giving meaning to or illustrating procedures or connections. Other material may contain other functions. The same instance can also have several functions.

3. Classify the representations of each instance. We use three theoretically grounded representations: situations, iconic representations and non-iconic symbol systems. We also analyze if something is explained or given meaning or is just denoted or illustrated.

## Results

Throughout the two book-series, together covering grades 1-9, comprising 15 books and over 3000 pages we found 48 instances from our chosen conceptual field. We present the results in the flowchart in figure 2.

| Grade | Mathematical content | Concept | Procedure | Connection |
|---|---|---|---|---|
| 3 | division, equal sharing | ☺ | | |
| | division, equal grouping | ☺ → ♥ | | |
| | division, equal grouping | ☺ → π | | |
| | multiplication and division | ☺ → π | | ♥ → π |
| | division, equal grouping | ☺ → π | | |
| | proportionality | ☺ → π | | |
| | fractions, geometric part whole | ♥ | | |
| | fractions, geometric part whole | ♥ → π | | |
| | fractions, geometric part whole | ♥ → π | | |
| | fractions, one whole | ♥ → π | | |
| 4 | division, equal grouping and sharing | ☺+♥ → π | | |
| | division, short division | | ♥ → π | |
| | division, short division | | → π | |
| | fractions, equal fractios | ♥ → π | | |
| 5 | fractions, mixed fractions | ♥ → π | → π | |
| | fractions, reducing fractions | ♥ → π | → π | |
| | fractions, reducing to lowest term | | → π | |
| | dividing fraction with whole number | | → π | ☺+♥ → π |
| | multiply fraction with whole number | | → π | ☺+♥ → π |
| | fractions, fraction of numbers | ♥ → π | → π | |
| 6 | fractions, geo. part whole, mixed | ♥ → π | | |
| | fractions, convert to and from mixed | | → π | |
| | fractions, reducing, reducing to lowest term | ♥ → π | → π | |
| | fractions, expanding fractions | ♥ → π | → π | |
| 7 | fractions, geometric part whole | ♥ → π | | |
| | fractions, geometric part hole bigger than one | ♥ → π | | |
| | fractions, size of fractions | ♥ → π | ♥ → π | |
| | fractions, equal fractions | ♥ → π | | |
| | fractions, reduce fractions | ♥ → π | → π | |
| | fractions, expanding fractions | ♥ → π | → π | |
| | fractions, fraction of number | ☺+♥ → π | | |
| | fractions, geometric part whole | ♥ → π | | |
| | fractions, fractions bigger than one | ☺+♥ → π | | |
| | fractions, equal fractions | π , ♥ → π | | |
| 8 | fractions, part whole of numbers | ☺+♥ → π | | |
| | fractions, equal fractions | ♥ → π | → π | |
| | multiply a fraction with a whole number | | π | ☺+♥ → π |
| | multiply fractions | ♥ → π | → π | |
| | multiply fractions | | → π | |
| | fractions | ♥ → π | | |
| | fractions, expand and reduce | ♥ → π | → π | |
| | fractions, reduce fractions with variables* | π | → π | |
| | fractions, division of fractions* | | | ☺+♥ → π |
| | fractions, division of fractions, inverse* | π, π → ♥ | π | π, ♥ → π |
| | fractions, division of fractions with variables* | π | π | |
| 9 | fractions, comparing fractions | ♥ → π | | |
| | fractions, equal fractions, reducing, expanding | ♥ → π | → π | |
| | fractions, multiply fractions | | → π | ♥ → π |

\* From the "red course" with advanced material that not all student will nessecarily encounter.

Figure 2: Results of the study

We will describe patterns in the flowchart relevant for our research question. The flowchart shows that procedures are typically described through symbol systems, but with no explanations given. Often there is a related conceptual content that is explained, but then almost always in terms of iconic representations that in turn are labelled by the symbol system. Early on, meaning making is made through situations. That may be an effect of that division tends to be introduced by situations.

Situations occasionally comes back throughout the grades. There is no clear shift towards letting meaning reside in relations in symbol systems. However, there is one example of symbol system argumentation in grade 7 and four in grade 8, one being example D discussed in the methods section. The four grade 8 examples are from higher level material that not all students typically encounter. This might explain why, in grade 9, we are back to meaning making residing in iconic representations.

## Discussion

The framework we developed concerns progress in concept knowledge as defined by Vergnaud (2009). That is, a growth in competence to operate with concepts in different situations, with different representations by identifying and use the relevant invariants to deal with the situation. In the textbook series we investigated, we found no shifts in meaning making from the icons and situations, to meaning making by mathematical relations expressed in the symbol systems. In fact, only in five instances the textbook designers rely on non-iconic mathematical symbol systems for explaining or giving meaning to concepts, procedures and connections. On a surface level it looks like the textbooks contain quite a lot of symbolic information. By using our framework, we could however show that the symbols are used as labels or representations and that arguments very rarely are carried out by using mathematical information that has already been coded in symbolic form.

Where does the aversion for symbol systems come from? The story may go back to the late 1800 century when the Pestalozzi's concept of "anschauung" spread across Europe. Pestalozzi's method for education of young children stressed that children learn through action and concrete objects (Heafford, 1967/2016). Pestalozzi's principle was that no word should be employed until it was thoroughly understood by concrete observation of a material object or an action as a means of distinguishing one thing from another. We agree with Pestalozzi on that mathematical concepts have to be bootstrapped from situations and iconic representations; but we argue that the "anschauung" concept has been stretched far too long in the textbooks that we investigated. Students in grades 6 to 9 are not young children! By insisting on giving meaning to the complex concept in the quotient conceptual field through iconic representations, we claim that students are not given the opportunity to learn to use the power of reasoning in mathematical symbol systems. Their progress of knowledge of concepts in conceptual fields is restrained by the persistence of explanations in iconic representations.

Our results show that the investigated textbooks provide students with the same limited explanations in iconic representations of quotient constructions from year 3, when it is introduced, to year 9. The conceptual field of quotient constructions are in year 9 expected to be developed to handle a range of different situations, which cannot be explained with iconic representation.

We argue that our theory for progress in concept knowledge together with our framework for analyzing progress can be used by teachers and curriculum developers to plan development; from Pestalozzi's ideas for young children through Bruner's and Vergnaud's ideas of progress in concept knowledge to finally reach the epistemological shift where the meaning of concepts goes from residing in situations and iconic representations to residing in relationships in symbol systems.

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# The nature of mathematical inquiry amongst kindergarten children: more than questioning and verbalisations

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*In this study we problematise the strong associativity of questioning with mathematical inquiry. We argue that in kindergarten we need to adopt a wider analytical focus to capture the complexity of children's mathematical inquiries, i.e. a multimodal approach to mathematical engagement. We examine how processes of mathematical inquiry unfold amongst collaborating children (and adults) engaged in mathematical activities in kindergarten. The study shows that children's exploration of mathematical ideas does not necessarily take place through questioning. Rather, the children burst out with imperatives such as "Look!" or "Wow!", and they initiate mathematical explorations with others through statements about mathematical features they have noticed. Moreover, children's mathematical inquiry is characterised by its multimodal and argumentative nature, as they use artefacts, gestures, and voice to argue their mathematical insights.*

*Keywords: Inquiry, Kindergarten children, Mathematics, Multimodal approach, Questioning*

## Introduction

For years scholars and researchers have argued that mathematical curiosity and exploration are important 'habits of mind' for children to adopt, in order for them to become mathematically skilful and knowledgeable persons (e.g. Clements & Sarama, 2007; Jaworski, 2005). The interest for inquiry-based mathematics education (IBME) has increased over the past two decades. Inquiry, as a normative approach to education has, ever since Dewey (1938), played a significant role in attempts to reform educational practice in school. Inquiry arose as a reaction against the so called 'traditional' teaching approach, where the teacher explains mathematical concepts and procedures and where the students then practice the same procedures by individually working on tasks. A central part of the reaction was that the students were brought to the centre of their own learning process by providing learners' freedom and autonomy to ask their own questions and solve their own problems.

From a dialogic approach, Wells (1999), defines inquiry as "a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them" (p. 121). To adopt an inquiry approach to mathematics thus means that the individual person has to investigate and explore the mathematics in question. Cochran-Smith and Lytle (1999) and Jaworski (2005) extend the construct of inquiry to also comprise a fundamental stance, a way of being and attitude when encountering mathematical problems and phenomena. This view of how to approach and learn mathematics has been adopted in the Norwegian Framework plan (Norwegian Directorate for Education and Training, 2017). Here, mathematics in kindergarten "is about discovering, exploring and creating structures… enable the children to explore and discover mathematics in everyday life… stimulate the children's sense of wonder, curiosity and motivation for problem-solving… It also involves asking questions, reasoning, argumentation and seeking solutions (p. 53).

Artigue and Blomhøj (2013) argue that "[i]nquiry-based pedagogy can be defined loosely as a way of teaching in which students are invited to work in ways similar to how mathematicians and scientists work" (p. 797). They investigate how the concept of IBME resonates with various theoretical frameworks in mathematics education, for example the problem-solving tradition, the modelling perspective, and the dialogical approach to mathematics education. Their conceptualisation of IBME resonates with Cochran-Smith and Lytle (1999) and Jaworski (2005) in advocating "the development of inquiry habits of mind" (Artigue & Blomhøj, 2013, p. 809). Furthermore, Artigue and Blomhøj argue that the nature of the problem is crucial for engaging students in inquiry processes and that mathematical objects (e.g. geometrical objects) themselves may be inquired into.

This study stems from a recent research study which investigated processes of mathematical inquiry in kindergarten (Breive, 2019). Breive found that children rarely asked mathematical questions when they engaged in mathematical learning activities, although they showed eagerness and willingness to solve mathematical problems together with others. Breive further identified that children's mathematical explorations were largely based on other means than verbal language. Rather than exclusively focusing on children's posing of questions and verbal communication, Breive suggests that mathematical inquiry must be conceived in a slightly different manner in kindergarten settings. Although Breive found that children's mathematical inquiry processes must be conceived as multimodal, i.e. based on the use of artefacts, gestures, and voice, including verbal language, it still remains to further scrutinise *how* children actually engage mathematically in a kindergarten setting, *how* they contribute with mathematical ideas and arguments and *how* they communicate mathematics with each other. Thus, we have formulated the following research question for the current study:

> What characterises children's mathematical inquiry when engaging in mathematical activities in a kindergarten setting?

## Questioning and argumentation as part of mathematical inquiry in kindergarten

From various theoretical approaches (constructivist, sociocultural or more pragmatic approaches) questions, especially asked by the students, are emphasised as essential to inquiry (e.g., Alrø & Skovsmose, 2004; Jaworski & Goodchild, 2006; Lindfors, 1999; Roth, 1996; Wells, 1999). Jaworski and Goodchild (2006) argue that "[f]undamentally, inquiry and exploration are about questioning: asking and seeking to answer questions. Together, we ask and seek to answer questions to enable us to know more about mathematics teaching and learning" (p. 353). Others have emphasised the significant role of questions, asked by the teachers, in scaffolding children's learning processes (Carlsen, Erfjord, & Hundeland, 2010; Roth, 1996). For example, Carlsen et al. (2010) showed the significant roles that questions played as a kindergarten teacher (KT) used about 150 questions when teaching a measuring activity. Here questions were used as pedagogical tools to engage children in a mathematical activity and to promote thinking processes and children's learning.

Breive (2019) noticed, in a dataset comprising 37 lessons of duration 40-60 minutes with 5 KTs and their groups of children, that the children rarely asked mathematical questions when they engaged in the mathematical learning activities. Other scholars have noticed that even students rarely ask mathematical questions in teaching-learning situations (Dillon, 1988; Myhill & Dunkin, 2005). Moreover, some argue that the lack of questions asked by the students may stem from the extensive use of questions that teachers ask, and thus that the children get limited space and opportunities to

ask questions (Dillon, 1988). However, Breive (2019) identified that even in the kindergartens, where the KTs gave the children a lot of freedom to act and speak, children seldom asked mathematical questions. Despite the lack of mathematical questions, the children showed eagerness and willingness to solve mathematical problems together with others, i.e. they were 'questioning' the mathematics.

Mathematical argumentation is recognised as another key feature of inquiry-based education (e.g., Alrø & Skovsmose, 2004; Wells, 1999). Researchers have argued that it is important to recognise and facilitate young children's mathematical argumentation to further facilitate their learning (cf. Krummheuer, 2000; Sumpter & Hedefalk, 2015). However, the analytical focus of these studies was mainly the role of children's verbal utterances in their argumentation process. The research is sparse regarding multimodal characteristics of kindergarten children's mathematical argumentation.

## Methodological approach

We acknowledge that multimodality is a fundamental feature of all human cognition, which implies that humans communicate and make sense of the world through various sensuous modes (visual, auditory, kinaesthetic, tactile). Thus, when humans communicate, they (consciously as well as sub-consciously) use various semiotic means, like gestures, body postures, facial expressions, tone of voice as well as spoken language. Each semiotic means brings forth a dimension of meaning (signification), and the complex coordination of all these means (and thus dimensions of meaning) brings forth a (new) complex composite meaning (Roth & Radford, 2011). To identify inquiry segments, we were looking for segments where the children took initiative to mathematical explorations through use of mathematical questions, mathematical claims, and imperatives (e.g. Look! Wow!) mediated multimodally. In our qualitative research study, we draw on the analyses of three cases to exemplify children's mathematical inquiry in kindergarten. These excerpts originated in Breive (2019), where the children participated in playful learning activities through which the children were engaged in problem-solving inquiry processes.

## Example 1. Inquiring into reflection symmetry

This example is taken from a session where the KT introduces the idea of reflection symmetry. At one point in the activity, the children get a drawing of the half of a butterfly, and the KT asks the children to draw the other side of the butterfly. The children were faced with a problem where they were to inquire into the mathematical meaning of reflection symmetry. Elias is very eager and starts to draw at once. He draws the butterfly's feeler a bit bigger than the one on the given picture. Another boy, John, looks at Elias' drawing and interrupts:

| John: | Wow, look at that big eye! |
| --- | --- |
| Jim: | ((Jim is looking at Elias' drawing for a while)). It's not completely equal, (pause) on both sides. [Mumbling]. ((Jim points at the picture while he mumbles)). |
| Elias: | Yes it is! |
| Jim: | Look, it's not equal. |
| Elias: | Yes [Argumentative yes] |
| Jim: | It's not quite equal. ((Jim emphasises "quite" and points at the drawing again)). |
| Elias: | Yes! [Argumentative yes] |

John:          Because, that was a little line. Like that. ((He is pointing at a line on the given picture on the right side. Elias has drawn a bigger line on the left side)).

Elias looks at his drawing for a while, then he looks at John and back at the picture again. Then he wants to draw something on the paper.

John:          You have to make a little, little line. (Pause). And then a little [Mumbling]. ((He emphasises the word 'little' by a substantially pitched voice)).



Figure 1: Jim (left) points at the feeler of the butterfly on the given picture

In the above example, the interaction is not initiated by a question, rather it is initiated by John who burst out, "Wow, look at that big eye!". It seems that John has noticed differences in Elias' drawing compared with the given picture, and he expresses his observation to the others. John's comment initiates the further interaction, where the three boys explore the similarities and differences in the drawing through multimodal argumentation. The argumentation comprises use of verbal language, emphasis on words, intonation, pitch, pointing gestures, etc. For example, when John explains how he thinks Elias must draw the picture by saying "You have to make a little, little line. (Pause). And then a little [Mumbling]", he says "little" in a substantially pitched voice. We argue that the emphasis on "little", made by the pitched voice, mediates that the feeler has to be a lot smaller to match the original picture.

## Example 2. Inquiring into subtleties of geometrical shapes

The KT (called Kari) has had a conversation with the children about various geometrical 2D shapes. Kari is about to finish one part of the activity, and she starts to tide up all the shapes. Eva raises her hand because she has something to say:

Eva:           Kari? ((She raises her hand and tries to get the KT's attention))

Kari:          I think you all have been very clever. Now I would like to hear what you have to say ((she turns to Eva)).

Eva:           Well, isn't it correct that (pause) that one is more acute than that one?

Kari:          Oh, then I have to bring them back again ((she brings the 2D shapes back)).

Eva:           That one is more acute ((points at one of the triangles)).

Kari:          Show me!

Eva:           This one is more acute than that one. Look! ((She puts the two triangles beside each other)).

The KT takes the two shapes, holds them up in front of her and takes a close look at the two triangles. Then she puts them back at the floor again.

Kari:          How can you see that this one is more acute than …. Maybe we should use these ((She chooses to use the other vertex in the triangle with the same angle)).

Eva:          Because that one goes like … it is a little … That one is a little bit more acute and [Mumbles]. ((Eva points at the two different vertices in the two triangles))

Kari:          Yes ((raising intonation)).

Line:          Ah! ((raises her hand quickly)).

Kari:          Line?

Line:          Well…That one looks a little bit more acute!

Kari:          Yes, it does!((still with a raising intonation)).

Line:          It is a little … like sharp in between ((she traces the triangle in the air))

In this example Eva initiates the mathematical interaction by asking "Well, isn't it correct that (pause) that one is more acute than that one?". The question arises from something she has noticed which she wants to express and to get the KT to confirm. Eva has noticed some differences in two of the triangles that they previously worked with. Eva's statement initiates a further exploration of the differences in the two triangles, an eploration characterised by multimodal argumentation for the initial claim. Similar to the first example, the children use various means, like language, gestures and the artefacts themselves, in their argumentation. In this example, gestures play an essential role in the children's argumentation. For example, to understand Eva's argument "Because that one goes like … it is a little … That one is a little bit more acute and [Mumbles]", we have to interpret her gestures. Moreover, to understand Line's argument "It is a little … like sharp in between", we also have to interpret her gestures. The children inquire into the features of the mathematical object at hand.



Figure 2: Contrasting two triangles by pointing at critical features

## Example 3. A lost opportunity for mathematical inquiry

In this example the KT and the children are working on 3D geometrical shapes. Ben, together with the KT, have built two pyramids, one tetrahedron and one squared pyramid. Then Ben bursts out:

Ben:          Look!

KT:          You have two… They were not quite equal, because this one is flat under, and this was… This had four angles, and on this one had only three.

In this example Ben turns to the KT and says "Look!". The KT chooses to use the opportunity to explain the differences between the two shapes. We have chosen to call this example "a lost opportunity for mathematical inquiry", because when Ben turns to the KT with his observation, the KT may have challenged the child to argue for what he had noticed, and thereby mathematically

inquired into the differences and similarities between the two shapes. Instead, the KT chose to explain the differences herself.

## Discussion and concluding remarks

Researchers have argued that to adopt an inquiry approach to mathematics the individual person has to reveal a willingness to wonder and ask questions, to investigate and explore the mathematics through questioning, sharing and explaining ideas (e.g. Wells, 1999; Jaworski & Goodchild, 2006). In this paper we challenge the idea that questioning is exclusively at the core of the interaction identified as mathematical inquiry. Our study shows how mathematical exploration (inquiry) seems to departure from situations where children notice something in their environment (for example, from previous interaction with others). Children turn to others with claims about what they have noticed, and they notify others through use of imperatives like "Wow!" or "Look!", which in turn illustrate children's fascination of what they have encountered and eagerness to reveal their mathematical discoveries. A significant finding is that they seldom turn to others with verbal questions. Moreover, the further interaction is often characterised as argumentation about the claim, argumentation comprising various semiotic means such as artefacts, pointing gestures, word emphasis, intonation, etc., accompanied by verbal use of language. Our study also shows the important role of the KT in being sensitive and responsive to the children. Example 3 illustrates what we call "a lost opportunity for inquiry", because the KT is not responsive and does not use children's inputs as prompts for further inquiry.

Based on these observations we suggest an alternative way of conceiving mathematical inquiry in kindergarten settings, complementing the research literature's emphasis on questioning and verbal contributions. We suggest that the nature of mathematical inquiry in kindergarten is characterised by a multimodal nature in which children contribute with prompts and other ways of contributing with their mathematical insights and notifications, afforded by the KTs' teaching and organisation of the learning environment. Such a stance has analytical as well as pedagogical implications.

Artigue and Blomhøj (2013) suggest various concerns to situate IBME, among them the importance of developing "problem solving abilities and inquiry habits of mind" (p. 809). Another concern is "the 'authenticity' of questions and students' activity in terms of connection with students' real life and link with out-of-school questions and activities" (p. 809). Furthermore, they emphasise the important role of the teacher in scaffolding dialogues and inquiry processes. Although the children in our study do not ask a lot of verbal questions, they seem to be open minded towards their experienced world, which we regard as an important 'inquiry habit of mind'. Moreover, the imperatives children use to notify others, illustrate 'authentic' fascination of the mathematical encounters in their environment. According to Dewey (1938), it is through sensory experiences of the world and from investigations of the environment that children learn. Thus, for education to be meaningful, it should be connected to real-life situations and situations that the students find interesting. In Dewey's view, the indeterminate situation is a pre-condition for inquiry. It is the indeterminate situation that provokes uncertainty and consequently questions to be inquired. It is not the questions itself, but the situation that has this quality. We align with Dewey in that it is through sensory experiences of the world that children encounter mathematical ideas and learn. However, our study indicate that it is not the indeterminate situation that provokes uncertainty and consequently questions to be inquired,

rather *the experience of certainty* (through perceiving some mathematical relations/properties) that children express *fascination* of the situation and eagerness to explain their discoveries.

Although children seldom verbalise questions, this does not mean that they are not 'questioning' (wondering about) the situation. Our study shows how children express fascination of their mathematical encounters. Nevertheless, when they turn to others with for example imperatives like "Look!", it may also be that they request an evaluation from their KT or peers. Thus, we argue that children do 'question' (wonder) what they perceive, but they do not necessarily express this in a well-constructed verbal question. However, we acknowledge the value of verbalising and questioning in mathematics learning and these being important features of mathematical inquiry. Thus, more research is needed to reveal how children may learn to ask verbal questions, and how KTs may learn to use children's inputs as prompts for further inquiry and hence to support mathematics learning. Moreover, more research is needed to understand how mathematical inquiry can be promoted through curriculum and activities in kindergarten. We agree with Chevallard (2015), that we have probably only seen the buds of a paradigm shift, from what he calls the "the paradigm of visiting works" (the 'traditional' teaching approach) to "the paradigm of questioning the world".

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# Undergraduate statistics students' reasoning on simple linear regression

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*Through think-aloud protocols with five undergraduate students, this study investigates what criteria the students emphasize when reasoning on simple linear regression, and how these criteria are linked to their reasoning. Promoting reasoning is important for both formal knowledge and informal meaning-making of statistical data. Five criteria were prominent amongst the students (fitting the overall trend in the data, an equal number of data points over and under the regression line, emphasizing the distance to the points over and under the line, equal sums of deviations over and under the line, and viewing the regression line as some kind of average), and multiple of these criteria were often intertwined in their reasoning. The students built on their statistical foundation when reasoning, shifting their focus between special cases in the data and the aggregate of the data, and some re-evaluated the plausibility of their criteria with a little probing. Nevertheless, our findings also reveal the issue of how oversimplifying formal criteria might challenge students' reasoning.*

*Keywords: Statistical reasoning, simple linear regression, undergraduate students.*

## Introduction

The field of statistics plays an important role in a wide spectrum of disciplines as well as in society in general, as "statistics is a general intellectual method that applies wherever data, variation, and chance appear. It is a fundamental method because data, variation, and chance are omnipresent in modern life" (Moore, 1998, pp. 1254). Technological development has led to statistical procedures and calculations being increasingly transferred to machines, while statistical reasoning and interpretation continue to be human-based (Moore, 1998). This might bring some context to the increasing focus in statistics education on statistical reasoning and thinking, rather than primarily teaching and learning procedural skills (Ben-Zvi & Garfield, 2004).

Being able to properly evaluate data and claims based on data is important for understanding statistical relationships (Ben-Zvi & Garfield, 2004; Garfield, 2002), and reasoning on linear regression models enables students to do so through evaluating signals and noise, association and causation, sample size and representativeness, context, uncertainties in data and claims, as well as consequences when reducing the complexity of a scatter plot to a single, straight line. Such reasoning and evaluation can be based on both formal statistical procedures and informal meaning-making, and in this complex process, one must navigate through great amounts of data to decide what to build one's arguments on. Although there are several studies on students' statistical reasoning, there are, to our knowledge, few empirical studies addressing their emphasized criteria when reasoning on regression modelling. This study will therefore address this, believing that investigating the specific criteria and how they are linked in the reasoning process also will provide deep insight into details of their reasoning. We have conducted in-depth studies of five undergraduate students' reasoning

processes and ask the following question: *What criteria do five undergraduate students emphasize when reasoning on the informal line of best fit, and how are these criteria linked to their reasoning?*

## Students' reasoning and criteria for simple linear regression

In this study, we consider statistical reasoning as the thought processes that occur while working with statistical evidence and ideas to form claims, conclusions and interpretations. This involves the ability to, formally or informally, describe data, explain statistical processes and make interpretations based on different representations, which all together require an understanding of statistical methods and concepts (e.g., Garfield, 2002). We will address three central aspects of statistical reasoning in this study: *statistical foundation, plausibility,* and *an aggregate view.*

The statistical foundation of a reasoning sequence refers to the inherent statistical properties of the elements and concepts involved – formally or informally. This involves describing and relating a problem to statistical ideas and properties in both the process of making, implementing, and evaluating strategies for solving the problem. A statistical foundation requires interpretation and discussion of statistical variables, processes, relations, constraints, and results that are involved, and their adaptations to context and reality (see Garfield, 2002; Garfield & Ben-Zvi, 2008; Gattuso & Ottaviani, 2011). Simple linear regression involves exploring, describing, interpreting, and modelling a possible relationship between two variables, and the statistical foundation is thus important in this reasoning process. The variable data is often illustrated in a scatter plot, which indicates whether there is an overall association between the two variables and the strength of this association. Key to simple linear regression is constructing the line that best fits the variable data as an aggregate, and the most commonly used method for doing so is the *least-squares method*. Thus, the least-squares method is an important part of the statistical foundation in regression modelling. By this method, the line of best fit, the regression line, minimises the sum of the areas of squares with side lengths equal to the vertical residuals between each data point $(x_i, y_i)$ and the line itself. The least-squares method will cause the absolute value of the sum of positive and negative residuals to be equal so that the total sum of the residuals is zero. Also, the statistical distinctiveness when it comes to *uncertainty* should be emphasised: A regression model hardly ever perfectly fits the data, but the process is about adapting the model so that it explains the data as an aggregate in the best way possible.

Statistical plausibility implies that the students include arguments and justifications in their reasoning and that the steps in their reasoning has, for the person doing the reasoning, some logical construction. This can be done both by formally referring to statistical procedures and by informal meaning-making, as long as each reasoning sequence follows the preceding reasoning sequences logically and does not include contradicting arguments or unjustified jumps. Batanero et al. (1994) point out how students, when reasoning about linear regression, use ideas that lead to correct solutions when applied to some problems but are insufficient or unfitting when transferred to more general situations.

Focusing on general patterns in the data, as an aggregate with certain features, is important in exploring and interpreting statistical information (Ben-Zvi & Arcavi, 2001), but learning to look at the data as a whole is a complex process (Ben-Zvi, 2004). Students, across grade levels, often see sample data as a collection of single values or cases (Ben-Zvi, 2004; Ben-Zvi & Arcavi, 2001; Casey, 2014). Moreover, in their study on first year university students, Batanero, Estepa, and Godino (1997) found that the students often used only parts of the data to form their judgment. Avoiding over-

emphasizing single points or cases is important for both the statistical foundation and plausibility of a reasoning sequence, as implementations and interpretations based on single cases can contradict with trends in the aggregate.

In the few studies addressing students' emphasized criteria when reasoning about regression modelling, there are interesting similarities across grade levels and among students in compulsory school (Casey, 2014), pre-service and in-service teachers (Casey & Wasserman, 2015) and students in higher education (Sorto, White, & Lesser, 2011) regarding their conception of, and criteria and methods for, drawing lines of best fit. These studies all used think-aloud protocols followed by interviews where the students were asked to reflect upon their answers. In all the studies, the participants were asked to informally fit a line to a given scatterplot and describe the criteria they used to place the line, and discuss two models consisting of the same scatterplot but with different lines of best fit. The findings from these studies reveal that the participants' criteria and thoughts on the regression line were quite similar: In all three studies, 1) an equal number of data points over and under the regression line was emphasised as a criterion, in addition to 2) viewing the line as some kind of average of the data points. 3) Emphasising the distance from the points to the line was also reported as a focus among the students in the three studies. For the students in higher education, 4) fitting the overall trend in the data was emphasised in addition to the previous criteria (Sorto, White, & Lesser, 2011), and, in the study by Casey and Wasserman (2015) on pre-service and in-service teachers, 5) making the sum of deviations equal for the points on each side of the line was reported as a criteria.

## Method

This study used individual think-aloud protocols followed by an individual short conversation. In addition to provide information about the participant's ability to solve a problem, think-aloud protocols also give access to underlying reasoning processes (Afflerbach & Johnston, 1984; van Someren et al., 1994), and combining think-aloud sessions with conversations (van Somren et al., 1994) enable further exploration of interesting reasoning sequences without interrupting the students' thought processes during the problem solving. Each think-aloud protocol lasted 30 – 45 minutes, including the conversation, and was video recorded.

Five undergraduate students participated in this study. The students had different mathematical background from upper secondary school but had in common that they had just finished their first statistics course at university level. The study was conducted approximately three months after formal instruction on linear regression, and the five students were chosen in collaboration with the lecturer after observing the class. They were all verbal in class, an important criterion for the think-aloud method to function optimally (Afflerbach & Johnston, 1984; van Someren et al., 1994).

Two problems were given to the students. This paper will focus only on the second problem, where the students were given the scatterplot in Figure 1 and asked to draw what they believed would be the line of best fit to the data (by eye) and describe the criteria they used to place the line. By asking the students to draw an informal line of best fit without any technology or calculations to help them, and argue for the placement of their line, our aim was to identify their emphasized criteria with focus on statistical foundation, plausibility and aggregate view. The design of the scatter plot provided room

for reasoning about signals and noise, scattering and outliers, in addition to reasoning on context and interpretations of statistical models.
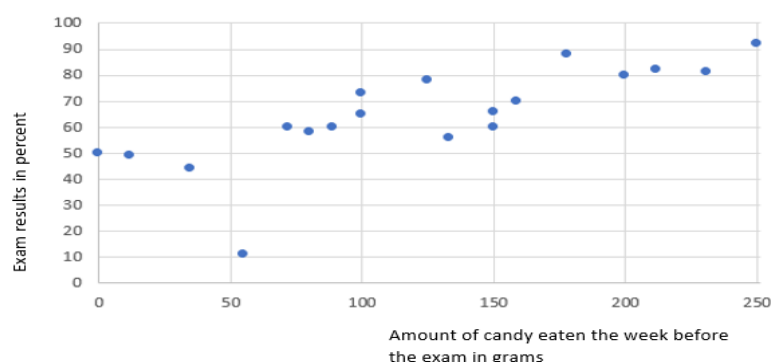


Figure 1: The scatterplot given to the students

The video recordings were transcribed word-by-word before critical events were identified and encoded (Powell, Fransisco and Maher, 2003). A situation was considered a critical event if it included an argument or justification, choice or implementation of strategy, conclusion or interpretation. This wide definition resulted in a high number of critical events. These were further coded according to 1) the three main elements of statistical reasoning and 2) the criteria used for simple linear regression in their reasoning (based on the previous research by Casey (2014), Casey and Wasserman (2015), and Sorto, White, and Lesser (2011): 1) - 5) presented in the theoretical framework. We refer to this numbering in the discussion). Five criteria grew forth from the analysis that were particularly prominent amongst the students, and that further illuminate well the link to their statistical reasoning. These criteria, with associated reasoning processes, are presented and discussed in this paper, and the results include quotes that illustrate the typical reasoning and criteria in a given context. The typicality is explained throughout the following section.

## Results and discussion

In the following extract, Emma had just been given the problem and was sharing her initial thoughts on how to work with it.

Emma:  Oh, well… [reads the problem description out loud] … Ooh, yes, I'm just going to try to… since I'm making a line that I think will fit the best, I want to make sure that there is as many points under and over it.

Solving the given problem by ensuring an equal distribution of points on each side of their informal line of best bit was a common idea among all the participants in this study, and in previous research (Casey, 2014; Casey & Wasserman, 2015; Sorto et al., 2011). The extract also shows how this idea was expressed without much argumentation or justification, which happened in four of the five participants' initially expressed reasoning, despite the problem description specifically asking them to explain and justify their ideas.

The criterion was operationalized by drawing a line, counting data points and optionally redrawing a line if the students were not satisfied with their result. With an odd number of data points (n = 19), the students settled with an uneven, but close, distribution of points on each side. They offered more

insight to their reasoning on this criterion when gently probed. Four of them then related it to making the line fit the data as an aggregate by getting equal sums of distances to the data points on each side of their line. Thus, the students seemed to assume a link between an equal *distribution of points* on each side of the line and equal *sums of deviations* from the line to the points on each side, merging criteria 1) with 4) and 5) (from previous research). 4) and 5) would cause the points to be as close to the line as possible through the concept of the least-squares method. The students seemed to focus on the given data as a whole as they focused on how equal numbers of points over and under the line would make the regression line closest to the aggregate of data points. One might even argue that the focus was too much on the whole, as it seems like the students' primary reasoning did not focus on how the outlier would affect the placement of their line of best fit. As regression modelling emphasizes both the variability in the sample, the effect of any outliers and the general trends in the data as a whole, a flexible shift between giving attention to individual cases and evaluating the overall trends in the data is favourable (Ben-Zvi & Arcavi, 2001), though complex. Formally or informally, the students included and adapted inherent statistical properties in their reasoning, like trying to make their line fit the aggregate of data points, commonly by ensuring equals sums of deviations, pointing towards trying to build a statistical foundation in their reasoning. What seemed to be an obstacle was how their adaptions oversimplified the statistical properties, i.e. from an equal sum of distances to an equal number of points on each side of the line. Batanero et al., (1994) addresses the problem of using statistical ideas that are applicable to some problems but insufficient or unfitting when transferred to more general situations. Though the least-squares method results in an equal *absolute sum of the distance* to the points over and under the regression line, this does not mean that there must be an equal *distribution of points* on each side of the line. Nevertheless, when having nothing else than a scatterplot, a ruler and a pen, this strategy must be viewed as a fair and plausible primary attempt to operationalize the criteria of an equal sum of deviations from the least-squares method. When using (and maybe overly relying on) strategies that enabled them to count data points, the difficulty of the problem decreased, making it more operational.

When being satisfied with their placement of the informal line of best fit, the participants elaborated, either by themselves or after some probing, on whether an equal number of data points on each side of the regression line actually would be a criterion for the best-fitting line. In the following example, Karoline was asked to explain her focus on splitting the data points equally on each side of her line.

> Karoline: Well, why did I want it to be an equal number over and under? So that the line would be, kind of, best fitted for all the observations we have. And you get that by… eh… Oh! It depends, 'cause you kind of have to look at the values so that… this point [the outlier in the scatter plot] will drag it down, so then you have to have some more points over (the line), I believe.

Similar discussions and interpretations were made by two other students in the conversation after they had drawn their own informal line of best fit. Although the students' primary criterion was focusing on splitting the data points over and under their regression line, these three students re-evaluated this when asked to elaborate on their criteria. When trying to combine their emphasized criterion of an equal number of points over and under the line with the criterion of all of the points as close to the line as possible (criteria 1) and 3) from previous research), the three students realised that this could produce a line with an uneven distribution of data points on each side, forcing them to address the

contradictions in their reasoning and re-evaluate their criteria. Not addressing contradictions when insufficient ideas are implemented on new problems can hinder the students' understanding (Garfield & Ben-Zvi, 2008). Using their statistical foundation, they adapted the plausibility of their reasoning through emphasising a regression line closest to the aggregate of points higher than splitting the data points equally. The students all commented both the outlier and how the rather small sample size in their given problem (n = 19) influenced the statistical model and the information they could get from it. Commonly, they stressed that such a small sample size gave them little information about the bigger population, adding representativeness to their statistical foundation. In addition, they argued that an outlier would make a greater impact on this small sample than if they had a scatter plot with more observations. Through their discussion of the sample size, they showed that they were able to point to uncertainties related to the given data and implement this in their discussion of statistical results and representations. Statistics education theories emphasise the importance of model criticism for statistical reasoning (e.g., Garfield & Ben-Zvi, 2008; Gattuso & Ottaviani, 2011), and in this study it contributes to the statistical foundation.

A recurring term in four of the five students' reasoning on the regression line was *average*, similar to the emphasized criterion in the previous research (Casey, 2014; Casey & Wasserman, 2015; Sorto et al., 2011). The four students using this term were asked to elaborate on this view in the conversation, and in the following extract, Susanne elaborated:

Susanne:     Well… If this point is this far from the line [points at one point under her line in the scatterplot], then, another point over the line should be equally far from the line. So that the distances kind of even each other out… like, they become zero. Then the two points kind of become one point that actually lies on the line… in a way.

From the four students' verbalised justifications and arguments, it seemed like they used the term *average* believing that the regression line represents the mean of data points above and below it. In total, three of the students (including Susanne) related their view of the regression line as some kind of average with their previous criterion of an equal distribution of data points over and under the line, making what seems to be a connection between criteria 1) and 2) from previous research. They used arguments related either to making the average distance over and under the line the same or to "mirror" points by having one point over the line and another point under the line with the same distance to the line. The students' argument of making the same average distance over and under the line can be related to a view of the regression line going through imagined midpoints of clusters of data points. The data as a whole is not clearly emphasized in this criterion, as it emphasizes single pairs or clusters of data. Nevertheless, this can be considered as further oversimplifying the common criteria of equal sum of distances on each side of the regression line. Building on their criteria of equal sums of deviations over and under the line and their (not statistically correct) idea that the regression line should split the data points equally (criteria 5) and 1) from previous research), the average distance to the points on each side of the line would be the same. Thus, their reasoning is plausible since it builds logically on their previous reasoning sequences. The second emphasized argument used, mirroring the data points through the line of best fit, suggests a clearly single-case oriented view focusing on pairing data points over and under their line. None of the students who expressed this criterion reflected on how this could (or rather could not) be adapted to making the sum of distances equal. Like the idea of an equal distribution of points on each side of the line and

equal average distances, this criterion points towards the idea of equality from the least-squares method being oversimplified, giving their reasoning some statistical foundation, though not formally correct. A similar relationship between the criteria of an equal distribution of points over and under the line and a view of the line as an average was found by Casey (2014) in grade 8 students in the USA. Our findings indicate that this view also appears amongst older students and that the students were not able to see the conflict between this criterion and drawing the line as close to all points as possible. Not addressing such contradictions can have negative impact on students' understanding of statistical topics (Garfield & Ben-Zvi, 2008). Some of the students reconsidered the use of "average" during the conversation following the think-aloud session, when asked specific questions on what the regression line really was the average of. Hence, the term may also have been used without any reflection on its meaning.

In conclusion, the students actively built on their statistical foundation to reason on the line of best fit through both formal statistical properties and informal meaning-making. All five criteria from previous research (an equal number of data points over and under the regression line, viewing the line as some kind of average, emphasising the distance from the points to the line , fitting the overall trend in the data and making the sum of deviations equal for the points on each side of the line) were prominent, and multiple criteria were used together. As they discussed both their emphasized criteria and how they operationalized them, their focus shifted between the aggregate of the data and the power of single points, helping them develop plausible arguments and in some cases address their own contradictions. Nevertheless, our findings also reveal the issue of how oversimplifying formal criteria might challenge students' reasoning. Several of the criteria above seemed to be somewhat intertwined in the students' arguments for simple linear regression and operationalization of drawing the line of best fit by eye, sometimes without addressing the contradictions this led to in their reasoning. This insight to the students' reasoning, intertwined criteria and the developments and contradictions rising, adds to previous research, and calls for further research on how instruction can be designed to probe the students in their process of statistical reasoning. This can also be considered implications for teaching, highlighting the importance of varied examples and models that facilitates reasoning and evaluation of emphasized criteria. Possible areas of interest are how to link informal meaning-making to formal statistical concepts and how to probe students to critically evaluate criteria and simplifications in their reasoning.

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# Learning modelling with math trails

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*The paper presents a study on the use of math trails in mathematics education. Math trails take students on a guided tour through the city or the schools' surroundings in which they solve different mathematical problems. Ten groups of students from two mathematics classes from Oslo were videotaped on math trails. The study aims to identify students' modelling processes and strategies when working with the problems.*

*Keywords: Math trails, modelling, videography, mathematize, digital tools.*

## Introduction

Mathematics education research shows that students in Norway tend to be relatively unmotivated to learn mathematics, especially in international comparison and when looking at students' transitions from primary to lower secondary level (Kaarstein & Nilsen, 2016). Mathematics is perceived as difficult and very formal, and less cognitively demanding teaching patterns often prevail in Norwegian mathematics education (Alseth et al., 2003; Grønmo & Onstad, 2009). To counteract this, mathematical modelling can provide an application for mathematics, especially when mathematics is used to answer important questions from the environment, culture, or daily life. Mathematical modelling has now become a major part of the new core elements of the 2020 Norwegian mathematics curriculum. However, so far, there are comparatively few comprehensive examples for teachers to follow when integrating mathematical modelling in Norwegian mathematics teaching, and teachers are expected to struggle to incorporate modelling in education (Berget & Bolstad, 2019). Math trails offer an opportunity to learn modelling in extra-curricular learning situations and emphasize playful learning, which is perceived as a useful complement to traditional school teaching patterns. The idea of a math trail is that students work collaboratively on mathematical tasks that are related to objects in the school's or city's surroundings, moving from site to site like in a rally (Shoaf, Pollack, & Schneider, 2004). Tasks are related to different objects and comprise estimating and measuring variables, calculating and comparing sizes, areas, and volumes, and solving problems, thus covering essential elements of mathematizing and mathematical modelling (Buchholtz, 2017). Corresponding research findings on students' modelling processes and strategies when engaging with math trails are scarce. In particular, there has been little research on the specifics of modelling activities on math trails as opposed to modelling activities provided by classical textbook tasks solved in the classroom. However, if modelling activities are to be anchored in the curriculum, it should be clarified what added value the implementation of math trails provides for modelling. The study described here is intended to contribute to filling this research gap.

## Research objectives

In the study "Math & The City" math trails are developed under certain task criteria and subsequently evaluated. The aim of the study is to analyze students' task-specific modelling processes and strategies when working with math trails. This means analyzing their interactions with the objects connected to the different tasks on the trail, as well as their mathematization, interpretation, and validation strategies when estimating and taking measures. Exploratory qualitative research methods

are used to analyze students' modelling processes and thus evaluate the math trails in their function of promoting modelling activities.

The corresponding research questions of the study are:

1. What kinds of modelling processes are enacted by the tasks (e.g., when encountering and interacting with the objects)?
2. What kinds of contextualized mathematization, interpretation, and validation strategies do the students use on the math trails (e.g., measuring, collecting data, use of mathematics)?

## Theoretical framework

The study is situated in the context of mathematical modelling and is theoretically anchored in the notion of mathematizing.

### Mathematizing and the learning of mathematics

The term "mathematizing" originally goes back to Freudenthal (1973) and describes a mathematical structuring in the sense of a transition from the world of life to the world of symbols. In later publications, Freudenthal took up the distinction made by Treffers (1987) between horizontal and vertical mathematization, whereby the former means making a lifeworld problem field accessible for mathematical treatment and continuously schematizing it, and the latter means mathematical processing in the sense of an increasing process of abstraction. In its original meaning, the term refers to basic mathematical operations, such as counting, structuring, and comparing quantities, which are present in almost all mathematical activities (cf. the examples of Freudenthal, 1991, p. 42ff). Both types of mathematizing are important in students' learning. Gravemeijer (1994) explains that within a realistic approach to mathematics teaching, students are presented with contextual mathematical situations that enable them to re-invent mathematics based on their own informal understanding (phenomenological exploration) in order to ultimately gain to a more formal understanding of mathematics.

### Mathematical modelling

In the discussion about mathematical modelling, however, the term mathematizing is nowadays used more specifically in its horizontal meaning for description and translation processes between reality and mathematics (analogous to "interpreting" as a back-translation). Mathematical modelling is, however, not uniquely defined within the mathematics didactics research community and comprises a variety of perspectives and classroom practices (cf. Kaiser & Sriraman, 2006). The transition from reality to mathematics and back again is a central part of modelling. In the literature, modelling is therefore usually described as going through a circular process in which essential activities or sub-competencies play a role—this process consists of simplifying problems, making appropriate assumptions, translating a real problem into mathematics (mathematizing), working mathematically, relating a mathematical result to the real situation, and interpreting and validating solutions (Blomhøj & Jensen, 2003) (cf. Figure 1).
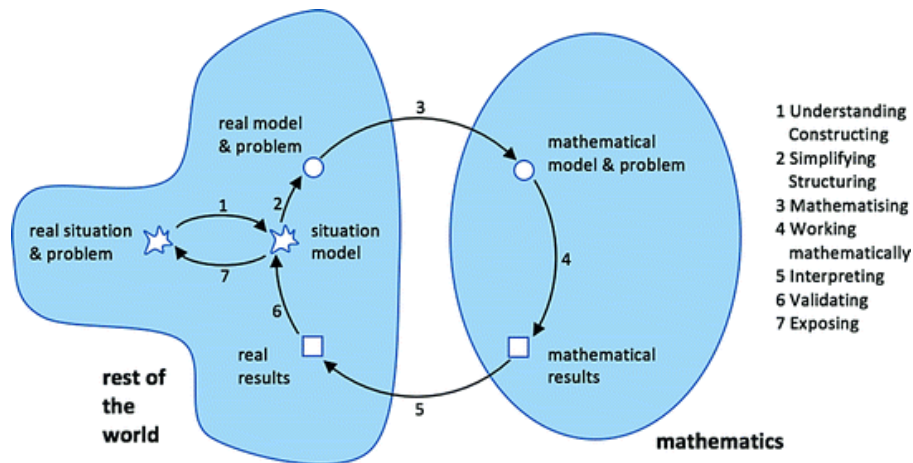
Figure 1: Modelling cycle of Blum and Leiß (2007) as cited in Greefrath & Vorhölter (2016, p. 14)

**Tasks for the math trails**

The process of mathematizing within the tasks takes place in comparatively small steps, and the developed mathematical models are of low complexity. This supports independent mathematizing piece by piece and also generally keeps the processing time of the tasks quite short. Figure 2 shows an example of a task from the math trail.



The well-known Peacock Fountain in Oslo was designed by Carl Nesjar and was inaugurated in 1989.

When the fountain is to be turned on and the pool is filled in the spring, the city council must find out how much water the pool holds.

a)  For this purpose, the perimeter of the pool must be determined.

b)  Next, find the area of the pool.

c)  How much water can the pool hold?

Provide your answer in m$^3$.

Figure 2: Fountain task

The fountain task requires the calculation of the volume of the water in fountain. It is embedded in the context that the fountain must be filled with water every spring, because winters in Oslo have very low temperatures. The city council therefore needs to know the volume of water to be filled. The students must consider how to determine the volume of water in the cylindrical fountain, as the diameter of the basin (7.6 m) is not accessible and therefore cannot be measured. It must therefore be determined by means of an appropriate mathematization using the circumference (24 m) of the fountain. For example, students could take footsteps as a non-standard unit and walk around the fountain, counting the number of steps, and afterwards multiply the length of a footstep accordingly. An additional sub-task in which the necessary steps of mathematization must be arranged to get from the circumference to the surface area serves as a scaffold, providing aid for the necessary step in

solving the task. Care must be taken to measure the stones on the inner edge of the fountain; otherwise, the results will vary considerably. With the corresponding water level of the fountain (0.36 m), the volume of the water (16.5 m$^3$) can then be approximately determined.

## Method

For the evaluation of the math trails, methods of qualitative explorative research will be used. Coming from French sociology, a method from urban city planning developed by Jean-Yves Petiteau in the 1970s can be applied. The method of the narrative walk-in-real-time interview (*Méthode des Itinéraires*) was originally invented to collect and describe the subjective view of pedestrians in order to draw conclusions about urban planning (Petiteau & Pasquier, 2001; Miaux et al., 2010). Central to this method are specific city walks in which the researcher takes a passive role; the researcher is guided by the test person and interviews and audiotapes the test person while a photographer walks behind and takes pictures at each change in direction or emotional change. Different forms of the method exist depending on how determined or open the walk is planned to be and how familiar the test person is with the environment (Evans & Jones, 2011). The method here is adapted for the math trails in order to gather information about the task-specific modelling processes and the strategies employed. The student group describes their experience in "real time," and they are audio- and videotaped while they work by action cameras, which they wear on their body—this makes the data collection minimally invasive. The data of the video material and the post-walk interviews are then analyzed with the help of qualitative content analysis (Mayring, 2014) so that certain aspects of modelling processes can be identified and classified in the data. The qualitative content analysis consists of a deductive and inductive coding process. First, the material is coded according to the theoretical derived categories "understanding the task," "structuring," "mathematizing," "calculating," "interpreting," and "validating," taken from the modelling cycle (Figure. 1). Individual actions or statements are assigned to codes on the basis of a coding guide. In an inductive process, other categories are additionally identified from the material; these contain information about strategies, such as which division of labor the groups chose, how independent their work was, and whether technical aids were used. Figure 3 gives an example of a coded sequence of the fountain task.
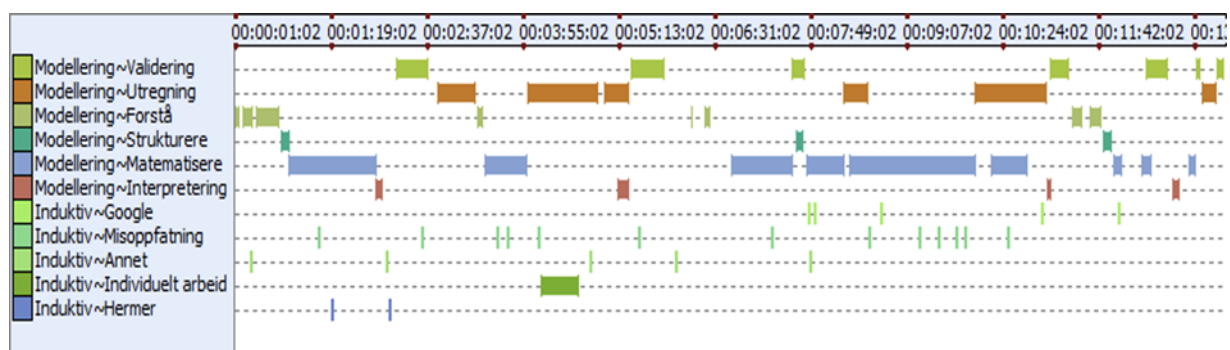


Figure 3: Coded sequence of the fountain task

## Data sources

When conducting the math trails with students, each group of students is equipped with a tape measure and a mobile end device (iPad), which presents the tasks and offers different answer options

according to the task design. The geo-localization of the mobile device leads the students along the route of the math trail. All solutions to the tasks must be submitted in the app. Each group of students is furthermore equipped with an action camera that records the progress of the math trail. The students themselves are responsible for recording the process at the individual stations of the math trail. For the study presented here, two math trails were developed in downtown Oslo. One math trail consists of five tasks on the topic of circle calculation. A second math trail consists of four tasks on the topic of linear functions. In November and December, 2019, the trails were carried out with two school classes (1P) and their mathematics teachers. In each case, five groups of three students each were equipped with cameras and iPads. All the necessary declarations of consent were obtained before the data collection and the study was approved by the Norwegian data protection authority (NSD).

## First results

The recordings show the individual groups of students solving the tasks during the math trail. In contrast to regular modelling tasks in class, the contextualization via the real objects seems to play a special role. For the tasks, students have to measure, scale, count, or estimate quantities and put the determined quantities into a correct mathematical relation or reconstruct or calculate relevant but inaccessible quantities from measured quantities—the actual mathematizing. Conversely, the groups directly checked mathematical results against real objects. Amongst others, we found indications of several direct validation processes (cf. Czocher, 2018), which leads us to assume that students switch back and forth between reality and mathematics more often when modelling in a realistic context than with classic textbook tasks. In Table 1, we present as an example the modelling process of the fountain tasks for the group of students referenced in Figure 3.

| Description of modelling process | Codes and interpretations |
|---|---|
| Read the assignment. Want to find the perimeter of the fountain. | *Understanding the task* |
| First, choose to use the measuring tape, but find that it is difficult to measure from the middle of the fountain due to the water in the pool. Choose to measure around the entire fountain in steps. | *Mathematizing* <br> Use two different approaches to mathematize. <br> Finally, collect data with individual units of measure (steps). |
| The number of steps is the number of meters, so the group answers 37.5 m. Ask a student of another group what they got. The second group got 24 m. | *Validating* <br> Trying to validate by checking what another group got, but do not understand that it is their steps that are too short. |
| Will now find the area. Do not know how to solve the task, but talk about either multiplying or dividing "as far as they know." Guess. Wrong, try again. | *Mathematizing and calculating* <br> This looks to be mainly related to motivation and self-regulation. |
| Loose concentration. Checking social media. | *Other* |
| E1 takes on the task alone and gets the first task done. | *Individual work* |

| | |
|---|---|
| To find the radius, they choose to try and measure with tape measure from the middle of the fountain pool. It doesn't work. | *Mathematizing* <br><br> Try once with the idea they had. |
| Google how to get from perimeter to area. Getting confused about the formula that appeared. Did not include the word "circle" in their Google search and are now searching again to find a circle area formula. | *Google* <br> Searching with the right words isn't easy either, but the group quickly understood what they had done wrong. |
| E2 misinterprets the formula he gets in Google and tells the others that they have to calculate *pi* squared, which he says is 3.14 times 2. Hence, multiply it all by what they think is radius. | *Mathematizing and misconception* <br> E2 may think that Google can help them figure out what to calculate, but interpreting what they get is difficult. |
| E1 looks at the mobile phone and says it is the radius that is squared. | *Interpreting and validating* |
| Correct for this issue, but instead of multiplying *pi* and *r* squared, they now add the two. Wrong answer. | *Calculating and misconception* <br> Does not validate the answer. |
| Will now find the volume. Try to discuss the formula, but do not remember that they can use the area to calculate the volume. Google the fountain to see if it says how much water the fountain pool can hold on the net. | *Google* <br> Does it help to mathematize? <br> Or is it simply cheating? At least the students are using a strategy here. |
| E3 thinks through the numbers they have received as answers to previous calculations. "Which number makes the most sense to be the volume? 46 must be too big, and 6 too small … 46 was the perimeter. Or was that the area?" | *Interpreting* |

Table 1: Modelling activities of Group 2 on the fountain task

Even though misconceptions often arose during the students' processing of the task, different modelling processes and strategies could be identified, such as using steps as a unit to measure distances and lengths (see Figure 4). Subsequent teaching should, in particular, tie in with these uncovered misconceptions (i.e., in the example, strengthen a conceptual understanding of the area and volume formulas of the circle calculation).

## Scientific significance of the study

The study has shown that students perform elementary modelling activities on math trails. If modelling is to be incorporated into the Norwegian curriculum, a practical implication of the study is that math trails will provide alternative teaching practices for integrating modelling into extracurricular teaching. A limitation of the study is that no direct comparison between the solution of tasks on a math trail and in the classroom is possible, so it cannot be said whether this form of modelling is more suitable than traditional teaching with textbook tasks. Future studies on modelling with math trails should therefore take a comparative perspective. From the results of the study, in-depth knowledge of different modelling strategies and mobile learning with math trails can be derived. In the video recordings, the students' approach to the task can be clearly seen in connection with the physical objects; as such, the recordings will lead to deeper insights into situational

modelling strategies. The results of the study can provide the first steps of a theoretical description of situational modelling strategies in extra-curricular learning environments. When fostering modelling competencies in teaching, more than merely the necessary step of mathematizing should come into focus (i.e., the translation between reality and mathematics). Mathematizing is only one sub-competence of mathematical modelling (Blomhøj & Jensen, 2003). There are also other sub-competencies of modelling involved (e.g., simplifying, collecting data, and interpreting solutions and validating) (Kaiser, 2007) that could be further identified in the videos.



Figure 4: Student E1 using steps as measurement unit when mathematising

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# A crowd size estimation task in the context of protests in Chile

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*From a notion of critical statistical literacy, one of the purposes of learning statistics is for citizens to be aware of how statistical information, methods, and arguments shape society. Moreover, they should be able to critique and transform their society through statistics. In this paper, I tell the story of a 12th-grade statistical estimation task, anchored in the context of students' protests in Chile. Students' reflections suggest it is possible to design and engage in a teaching practice that lives up to the ideals of reading and writing the world with statistics. I display the teacher's voice throughout the paper in the form of excerpts from the post-intervention interview. It means to reflect our collaboration in the process of design, implementation and analysis.*

*Keywords: Confidence intervals, critical mathematics education, sample distribution, statistical literacy, statistical estimation.*

## Introduction

One of the shared quests among the Nordic countries, regarding mathematics education, is their push to connect it to democratic life and values. In the pre ICME 10 document, Dahl and Stedøy (2004) tell a shared story that goes from the enlightenment inspiration for equal access to education, to educational objectives involving the development of pupils' critical awareness of the role of mathematics in society. However, as they admit, "it is one thing to have a curriculum; classroom practice might be very different" (Dahl & Stedøy, 2004, p. 8). Concrete links between democratic ideals and actual mathematics educational practices are challenges to be tackled.

A similar drive is taking place in Latin America, since its process of post-dictatorship democratisation in the 1990s. In this context, one possible response arises to the challenge aforementioned, namely the notion of deliberative mathematics education (Valero, 1999). Among other practices, it involves a critical reflection on the shaping of society by mathematical models, by acknowledging how decision-making draws upon mathematical arguments and encouraging students to take part in such type of discussions in the mathematics classroom.

Statistics plays a significant role in how modern societies are read and written. As school mathematics subjects, probability and statistics have grown in relevance and the educational research community agrees on the critical stance all citizens are entitled to engage towards statistics. The role of educational practices in this regard is an ongoing agenda. This paper is embedded in a PhD research project exploring the *problématique* of coherence between the critical justification for the inclusion of statistics in high school, and a teaching practice living up to it.

### The case of estimating crowd sizes

One exemplary case of mathematics in action is the use of estimates of attendees to demonstrations in order to validate – or invalidate – their cause. Organisers and authorities often provide different estimates reported in the media based on non-transparent methods and frame the results according to their respective agendas. This phenomenon provides an opportunity for students to engage in critical reflections about the use of mathematics in society. Recent contingency in Chile provides a rich and

up-to-date Latin American context for the exercise of the democratic right to gather and demonstrate. In late 2019, massive protests have led to starting a new constitutional process in 2020.

The timeline of this study may be relevant to the reader. This study draws the attention to a students' march occurred in April 2018 in Santiago, Chile, as a warning for recently elected government's educational policies. The task was implemented in October 2018, based on that demonstration. One year later, in October 2019, an unprecedented explosion of demonstrations and protests were triggered by high school students. Later, in December 2019, I interviewed the teacher about the task and its potential, considering the new national circumstances. I include excerpts from the post-intervention interview with the teacher about the whole process, positioning the teacher as a 'research subject' as opposed to 'research object' (Skovsmose & Borba, 2004).

The core of the case is a statistical estimation task, based on aerial pictures of a demonstration, implemented and discussed with a high school teacher in Chile. I describe the design process, outcomes of its implementation and its potential to produce critical reflections. The general research question to be addressed is: How and to what extent is it possible to design and implement a teaching practice coherent with critical purposes of high school statistics?

## Theoretical framework: a dual-purpose

Statistics as being relevant to all citizens and justified to be taught for that reason is a common theme in the literature on statistical literacy. Most – if not all – frameworks for statistical literacy include some notion of *critical*, in the form of critical questions as knowledge elements, or a critical stance as a dispositional element (Gal, 2002). Moreover, hierarchical models such as Watson and Callingham's (2003) characterise higher levels involving some critical engagement with context. As a problematisation of the connection between statistical and critical literacies, Weiland (2017) addresses three problems when joining these literacies: the meaning of critical, the distinction between reader and writer of statistical messages, and the role of context. Briefly put, a critically statistically literate citizen can understand how social structures are shaped by statistics in real and eventually divisive contexts and contribute to change them.

Weiland (2017) further proposes a framework for characterising critical statistical literacy in eight aspects, four regarding the ability to read and four to write the world with statistics, respectively. Given the limitations of this paper, I define only four of them, exemplary of reading and writing, which I use later for the analysis. Hereby, an individual is able to read the world through statistics by (R1) "making sense of language and statistical symbols systems and critiquing statistical information and data-based arguments encountered in diverse contexts to gain an awareness of the systemic structures at play in society" (p. 41). He or she can also read the world by (R2) "evaluating the source, collection and reporting of statistical information and how they are influenced by the author's social position and socio-political and historical lens" (p. 41). As for writing the world, critical statistical literacy enables an individual to (W1) "negotiate societal dialectical tensions when formulating statistical questions, data collection and analysis methods, and highlighting such tensions in the results of a statistical investigation" (p. 41). Additionally, one should (W2) "communicate one's social location, subjectivity, and political context to others, and how it shapes one's meaning-making of the world when reporting results of a statistical investigation" (p. 41).

**Inquiry-based task design**

Because of its anchoring in real-life modelling activities, inquiry-based mathematics education is the framework of choice for action. It can be defined roughly as "a way of teaching in which students are invited to work in ways similar to how mathematicians and scientists work" (Artigue & Blomhøj, 2013, p. 797). However, "researching the inner change process that learners' undergo as they develop the 'critical lens' that is part of statistical literacy requires different and less direct approaches" (Petocz, Reid, & Gal, 2018, p. 81), such as open-ended investigations:

> Teacher:     I like structure very much because it allows any person to follow the class. Now, that structure does not have to be so rigid, and one can make changes, because, evidently, excellent ideas will emerge, different from what one had prepared.

Previous studies echo this idea. For example, by asking openly to interpret a chart relating school recess time with race, Brantlinger's (2014) students engage spontaneously in discussions about systemic racism, on the grounds of statistical information. Kuntze, Aizikovitsh-Udi and Clarke (2017) show it is possible to design hybrid tasks that provoke both statistical or critical thinking, employing a "thinking-aloud" task consisting of the evaluation of a claim based on a diagram of births and deaths in Germany since 1945. These studies make use of real data from sensible contexts with problematic representations and a variety of possible interpretations. As a key improvement to the tasks, the authors propose a reflection-oriented framing, i.e. to have the learners ask themselves critical questions after solving the task.

## Methodology: collaborative classroom intervention

As methodological guidance for critical mathematics educational research, Skovsmose and Borba (2004) propose a general model consisting of a triad: *current*, *imagined* and *arranged situations*; (CS, IS and AS, respectively). Critical research possibilities are facilitated by "researching *with,* and not *on* teachers" (p. 220). Therefore, the definition of the CS, IS and AS are done in dialogue and collaboration with the teacher. The CS consists of a 12th-grade class in a school where:

> Teacher:     We work with class participation. However, usually, mathematics is a– among 45 characters you have in the class, only ten really enjoy it. The rest –a bit– suffer it.

The statistics curriculum for that grade includes sampling distributions and confidence intervals and the goal to evaluate information gathered from media outlets critically. Nevertheless, teachers felt that, so far, critical reflections are not possible:

> Teacher:     Because the amount of content we have to cover in all disciplines is excessive. It does not allow for reflection or having like this paused learning, respecting the students, respecting their time.

The IS is that students' own ideas to approach the inquiries lead to statistical notions. As a researcher, I looked for opportunities to allow and provoke such reflections by proposing a set of real-life politically relevant inquiries connected to high school ideas of probability and statistics.

**Production of data: the arranged situation**

> Teacher:     It is useful too because they were fun activities. So you showed us like four activities, and we said: "all right, this is the best".

That chosen inquiry is inspired by the "Counting People" task by Triantafillou, Psycharis, Bakogianni, and Potari (2018). The teacher pointed out that many students had attended the march in April 2018. It also fits the need to cover sampling distributions and confidence intervals.

As for the structure of the sequence, Blomhøj (2016) describes three main stages in inquiry-based teaching: setting the scene, independent work, and joint reflection. Accordingly, the AS starts by students reading two news articles from the same outlet; one where the organisers of the march predicted 60,000 attendees and one after the march including aerial photos, where authorities provide aftermath of 30,000 participants. The general inquiry is given: "How many people attended the students' march?" Students' independent work, guided by a working sheet given by the teacher and researcher, consisted of students making their estimates based on aerial pictures and a map. In Figure 1, students divide a picture in a mesh, take random samples, count and provide the mean to understand how the sampling distribution varies. Later on, they construct a confidence interval for the mean amount of people in a frame. They use Figure 2 to scale up their estimates to the full demonstration. Eventually, they challenge the use of this picture, since many of them attended and knew that the march went beyond what is showed in Figure 2. In the last part of the sequence, representatives of each group of students provide their results. Finally, they are encouraged to share their reflection about the methods used (as suggested by Kuntze et al., 2017), variety of results, the information given in the news articles, and form of work throughout the experience.



Figure 1: Source: Ahora Noticias
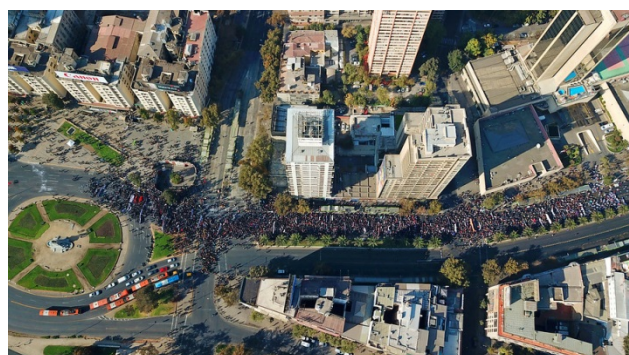
Figure 2: Source: Ahora Noticias

The overall *problématique* this study is embedded in is concerned about possibilities and challenges in actual teaching practice, so the implementation of the task was intended to be done in a real teaching context, as opposed to laboratory conditions. The classroom experience took place in three 12th grade mathematics sessions, during a week where the teacher had planned to teach the statistics part of the programme. In total, 35 12th-grade students (around 17 years old) participated forming six groups, led by both teacher and researcher. Data were collected in the form of audio recordings on each group of students, a video recording as reference for transcription, and scanned written work by the students. Audio transcripts from the final joint reflection are used for the analysis.

**Analysis: reading and writing the world through statistics**

For me to address the research question, aspects of critical statistical literacy must be evoked by the students. By selecting and coding students' interventions into categories by a single researcher, there

is a risk for the method's reliability. To address this issue, in the spirit of the study, I let the voice of the teacher confirm the code indirectly. Her comments during the post-intervention semi-structured interview contribute to a triangulation of the coding process. In some cases, I showed excerpts from the intervention to the teacher to prompt a personal interpretation of the episodes.

Exemplary results for four of the critical statistical literacy aspects are summarised in Table 1. The first column indicates which reading (R1, R2) and writing (W1, W2) abilities are found. The second column shows a brief excerpt of the classroom intervention exemplifying this finding, where R stands for the researcher, T represents the teacher, and GXSY is the Y-th student from group X to intervene in the session. On the third column, there are teacher's comments from the post-intervention interview that provide insight and reliability to the findings.

## Discussion

The task allows a student to make sense of statistical variation in the context of the task (R1). Acknowledging that everyone in the class had different estimates by using the same methods, he refuses to judge the media so lightly and gives them the benefit of the doubt (Table 1, see the first row). The teacher recognises this feature of the task, by explicitly saying that it makes sense to use statistical estimation, compared to the standardised test, they are soon to take finishing high school.

Students also read beyond the data and methods, evaluating the source, collection and its consequences (R2). They make use of their context knowledge (transportation patterns and shadow features) to realise that the pictures are taken too early, and therefore providing an underestimate on any case (Table 1, second row). The teacher was shown a partial transcript of this discussion where one student simply says it was early. Nevertheless, she remembered the observation about the shadows even after a year, making it clear how relevant this argument was.

As for writing the world, students from different groups negotiate dialectical tensions (W1) regarding the scaling of their first estimate to the full demonstration. The key issue, as the teacher also observes (Table 1, third row), is that the march is moving, and the pictures (data) are not. Students tackle this issue by providing results computed under different assumptions of what corresponds to the full picture.

On Table 1, the fourth row, I intended to go deeper and challenge the very importance of crowd sizes when it comes to support or reject a cause, and it is not engaged. This absence shows how a direct question will not necessarily provoke critical utterances if it is not correctly framed within the task. Instead, a student shares a simple reflection on the possibility of being agents and communicating one's social location when reporting their statistical investigation (W2). As can be seen, the teacher was sceptical of her students to achieve this level of reflectivity until I showed this transcript.

| No. | Students' exchange (October 2018) | Teacher's comment (December 2019) |
|---|---|---|
| R1 | G1S1: Beyond casting doubt about the media, and – as it is an own estimation, it cannot be known how many people exactly there are in the march, so I would not stay with that doubt. Since it is an estimate, it can be by itself or not a mistake, so I would not stay with it. | T: I believe everyone is engaged with the activity. I do not know if with learning the confidence interval or with applying it. What happens is that there it (…) made sense using it. Compared to the test, where they are asked for the formula directly. |
| R2 | G2S2: Because at that time, after an hour, they block the transit and no buses pass by. So, in this photo there are buses, it means it was too early.<br><br>G3S1: (…) G2S2 must be right. That about the [photo of the] march being early, because it is like, observing more, yeah, but if it is known that the north is down [in the picture] and the shadows are from there [points left/East], because we are super early, before 12. | T: OK, there are like fun things emerging.<br>R: Yes.<br>T: But there is no mention to that about the shadows.<br>R: No. No, that emerges – the one thing is that here it is said that is early.<br>T: That is early. |
| W1 | G4S2: Yes, that the march was supposed to reach [to Los Héroes], but with the same mass, I mean, "relatively" with the same mass. It is like they were moving all together with the same people.<br><br>G2S1: But we also computed until Torre Entel. More or less we suppose anyway that all this section [is] full of people. So, we computed [as if it were] complete anyway. How much did it give? | T: I mean, I see two things here, like… the thing about them managing to perceive that it is not something constant, that area is not entirely full. Another thing is that they manage to perceive that if it was earlier, there are fewer people, and then more people accumulate. |
| W2 | R: Now that you have these numbers, what – do they produce something in you? Do they change any of your opinions about the march, its causes…?<br><br>G2S2: That oneself can verify whether it is true, and take one's own conclusions (pause) and believe oneself. | T: Now, nowadays, I mean, with all that has happened, I believe that now I could feel that reflection: "yes, it is really important to know a bit of mathematics to be able to be – not to believe so much in what the press tells me". Still, I do not know if that reflection could have been reached. (…) Perhaps I am wrong. |

Table 1: Excerpts from the classroom intervention and post-intervention interview

From the last comment on Table 1, fourth row, the teacher highlights that circumstances have changed, and current social unrest may represent an opportunity to let students reflect on the formatting power of statistics in society. Moreover, group work was innovative for her, but the tides are also changing at a local school level:

Teacher: We just had collaborative work training, and then that disposition [in groups] is quite normal. However, at that time, we had not had such training.

The confluence of evolving circumstances opens new possibilities for transformation. In the language of the methodological approach, a new current situation is reached (CS2). The teacher highlights that Chile's most massive demonstration in history just occurred on October 25th, 2019.

## Conclusion

This paper shows the potential of a statistical estimation task to provoke students' reflections to read and write the world with statistics, addressing the research question of *how* and *to what extent* it is possible to design and implement a teaching practice coherent with a critical statistical literacy.

In principle, this is possible in close collaboration with the teacher throughout the design, implementation and the subsequent reflection process, enabling her to cover mandated mathematical content, namely sampling distributions and confidence intervals. We chose an inquiry-based approach, asking students openly to reflect on their investigation, and a real, contingent and relatable context; a demonstration to which some students, in fact, attended.

The didactical design allowed students to read the world through statistics. They made sense of statistical estimation, and the variation among different social actors' estimates found on the media. Being personally engaged with context enriched the possibilities to evaluate pictures as valid sources of data for a dynamic event such as a march. These tensions were the basis for writing the world with statistics, to the extent of negotiating assumptions to produce different calculations and results. Questioning broader structures – e.g. the use of numbers to stir the adherence to a cause – is a level yet to be achieved and requires more research. Being sensitive to national and local circumstances defines a new concrete situation to investigate, making the experience not only possible to replicate as a new iteration of an ever-transforming process.

Latin America is geographically far from the Nordic countries, but the journey to connect mathematics education and democracy is shared. The critical approach to probability and statistics shaping society is one path that teachers are willing to take:

Teacher: So, with all the revolution currently taking place, (…) we were thinking about which current topics, where there are problems, we could approach from mathematics. So, I do not know, we reached as a conclusion that we could teach about pension fund managers (…) and that is a very mathematical topic (…)

Researcher: And, in particular, related to uncertainty.

Teacher: Yes. And life expectancies!

In Chile, the current shake of the status quo can also respond to the conference theme of projecting the Nordic model into the future.

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# Challenges in enacting classroom dialogue[*]

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*In this article, we report on the challenges of enacting a dialogic approach to teaching mathematics. Using the five principles and three repertoires of dialogic teaching as theoretical framework, we draw on interviews and classroom observations of a secondary-school teacher who aspired to teach mathematics through dialogue. We analysed his accounts and videos of his lessons to identify his strategies for dialogic teaching and challenges in implementing these. We found that the usefulness of strategies for dialogue changed over time and that some challenges were manifestations of tensions between the five principles, and thus intrinsic to dialogic teaching. Specifically, concerns for broad participation needed to be balanced against concerns for mathematical content, while the pursuit of a specific mathematical goal – the purposeful principle – lead to missed opportunities for chaining ideas into coherent lines of thinking and understanding – the cumulative principle.*

*Keywords: Dialogic teaching, cumulative principle, systems of linear equations.*

## Introduction and background

Inclusion of all students is a cornerstone of Norwegian education, encapsulated in *adapted education*, a principle that endured through shifting political discourse (Jenssen & Lillejord, 2009). Interviews with teachers participating in the *Inclusive Mathematics Teaching* project (IMaT) revealed that many considered dialogue as key to both challenging and including every student in quality mathematics. In this study, we aim to understand one teacher's strategies for dialogue and the challenges he met, contributing to research efforts on the implementation of dialogic teaching (e.g. Alexander, Hardman, Hardman, Longmore, & Rajab, 2017; Sedova, Salamounova, & Svaricek, 2014).

## Literature review and theoretical framework

The idea of using language as a tool for learning has given rise to the development of different but interrelated pedagogical approaches involving classroom dialogue, and consequently conceptual confusion over what is meant by 'dialogic teaching' (Kim & Wilkinson, 2019, p.71). We draw here on the conceptualisation of Alexander (e.g. Alexander, 2018) who theorised *dialogic teaching* (DT) from empirical data in a large comparative international study. This emerging pedagogy aims to engage students in genuine dialogue, where participants attempt to take the others' perspective (Alexander, 2018) and it is defined by five principles: *collective* (the classroom is a site of joint learning and enquiry), *reciprocal* (participants listen to each other, share ideas, consider alternative viewpoints), *supportive* (participants do not fear embarrassment over wrong answers, help each other to reach common understanding), *cumulative* (participants build on their own and each other's ideas

and chain them into coherent lines of thinking and understanding) and *purposeful* (classroom talk is structured with specific educational goals in view) (Alexander, 2018, p. 566).

The growing appeal of DT across disciplines creates a need for interrelated research directions, e.g. on operationalisations of the approach, on its impact on student learning, on pedagogical strategies that support or hinder its implementation (the focus of this article). Alexander argues that the realisation of DT depends on exploiting rich *repertoires* flexibly and appropriately. The most important are *organisation* (whole class, groups, one-to-one), *teaching talk* (discussion and dialogue in addition to direct instruction) and *learning talk* (narrate, explain, speculate, imagine, explore, analyse, evaluate, question, justify, discuss, argue) (Alexander, 2018, pp. 567-570). Discussion and dialogue are the most desirable forms of *teaching talk* as they activate a broad range of *learning talk*, while direct instruction only allows students "to tell/narrate and, at a pinch, to explain, but not to speculate, imagine, explore, analyze, argue or ask questions of their own" (Alexander, 2018, p. 569).

A more precise characterisation of these repertoires led to 61 indicators that operationalise DT (Alexander, Hardman, & Hardman, 2017). This extremely detailed decomposition is impractical in classroom observations and more holistic approaches are preferable (Alexander, 2018). Still, existing research (in other fields) found some of the indicators to be more noteworthy. *Higher order teacher feedback* and *open discussions* are considered key to DT, with open discussions identified as the most important variable affecting *student talk with reasoning* (e.g. Sedova, Sedlaeck, & Svaricek, 2016), while *authenticity* (*open-ended questions* aimed at including students' thinking) and *uptake* (building on student responses) correlate most strongly to achievement (Nystrand, Gamoran, Kachur, & Prendergast, 1997). Nonetheless, regarding these indicators in isolation can mislead: Lefstein, Snell and Israeli (2015) found that *open-ended questions* do not guarantee elaborate or thoughtful student responses. As the indicators of the *teaching repertoire* do not necessarily reflect in the *learning repertoire*, we need to understand the factors that affect the impact of pedagogical strategies for DT.

While the idea of DT is seductive, implementation is challenging. We agree with Sedova et al. (2014) in that understanding the challenges of dialogic teaching can inform its implementation making it easier to incorporate in teachers' practice. Some challenges in implementation are evident in Alexander's randomised controlled trial (RCT) in primary schools. The intensive professional development programme transformed both student and teacher talk strikingly, with an improved closed- to open-ended questions rate across all subjects, higher order teacher feedback and a significant increase in discussions and dialogue (Alexander, 2018). In mathematics, student contributions in the intervention group were much more frequent than in the control group, and *the learning repertoire* differed, with students arguing, justifying, imagining, shifting position (Hardman, 2019). Still, the evaluation of this RCT also found that the first three principles of DT were easier to realise, while – even in a carefully designed intervention where the principles were explicit – making teaching *cumulative* and *purposeful* proved harder (Alexander, 2018).

Reviewing research on dialogic teaching – as defined by Alexander – in mathematics education, Bakker, Smit & Wegerif (2015) remark on its surprising scarcity, as well as its promise. Theoretically, quality mathematics teaching is compatible with this pedagogy where the goal is to teach both mathematical concepts and mathematical dialogue (i.e. dialogue is both a tool and an end) (Bakker et al., 2015, p. 1057). Empirical studies support this hypothesis, as investigations of teachers' strategies provide evidence for the impact of dialogue on how students experience mathematics. For instance,

teacher questioning that scaffolds students' engagement in justifying and explaining can give rise to teacher-student dialogue that maintains the cognitive demand of tasks (Estrella, Zakaryan, Olfos, & Espinoza, 2020); responsive teaching strategies, including the provision of prompts that the students could then use in questioning their peers, foster teacher-student and student-student dialogue supporting broad student access to mathematics (Makar, Bakker, & Ben-Zvi, 2015). However, the impact of dialogue is not always positive: an ethnographic study revealed a relationship between the manner in which a mathematics teacher afforded differential access to the whole-class discussion over time and students' agency and identity in mathematics (Solomon & Black, 2008).

Understanding the mechanisms that support or hinder the realisation of DT informs the design of teacher education and professional development. In response to this need, we explore DT in mathematics drawing on interview data and classroom observations from a teacher, Rune, who saw dialogue as essential for quality mathematics teaching. We address the following research questions:

RQ1    What strategies for dialogue did Rune promote and enact? How did they relate to DT?

RQ2    What challenges in realising DT emerged?

## Methodology

The participants were Rune (all names are pseudonyms), a young mathematics teacher with a master's degree in mathematics education and the 24 students (age 15-16) in a mixed ability 10th grade class in a Norwegian urban lower secondary school. Rune was selected because of his aspiration to make dialogue a goal and a tool in mathematics, i.e. an aspiration towards DT (see Bakker et al., 2015). The data consists of two semi-structured interviews and video of three consecutive lessons. The pre-observation interview (I1) addressed Rune's view of quality mathematics teaching, his strategies and his challenges in realising his vision. The lessons (L1, L2, L3) were the formal introduction to systems of linear equations, including eight tasks (2-3 per lesson), six abstract to be solved by substitution and two problem stories. In the post-observation interview (I2), we used stimulated recall to obtain Rune's perspective on the implementation of the strategies, as captured on video.

To answer RQ1 and RQ2, we first identified in the interviews statements on the strategies Rune saw as supporting his goal of dialogue in mathematics and the challenges he reported meeting. We holistically analysed these in terms of their relation to the five principles of DT and the *organizational*, *teaching* and *learning repertoires*. Next, we identified in the video data the enacted strategies and, drawing on the stimulated recall in I2 when possible, we analysed these episodes in terms of the repertoires and principles of DT to identify other challenges. We proceed to the findings before discussing implications for the realisation of dialogic teaching.

## Findings

In this section, we first answer the research questions based on the interviews, where Rune gave his account of the strategies (S1, S2, S3) he used and the challenges (C1, C2) he met in realising a teaching approach centred on dialogue. Then, we turn to video data from the three lessons and to the stimulated recall during I2, and analyse the implementation of these specific strategies during the three lessons, identifying additional challenges (C3, C4) to the realisation of dialogic teaching.

**Strategies and challenges in the interview data**

During interviews, Rune recounted his experience of teaching these students for the past 1.5 years. He described three strategies he used for dialogue in mathematics: (S1) asking open questions, (S2) employing group work, and (S3) using multiple solution tasks. The main types of challenges he reported experiencing over time are (C1) establishing a culture for dialogue in mathematics and (C2) recasting errors as productive. We discuss the findings in terms of the principles of DT.

Rune's goal "to facilitate dialogue among students and between students and me" (I1), has a *collective* quality. For him, "dialogue is central [to quality teaching], how to talk about mathematics":

> [Create] an open forum … What do we think about this [problem]? Everyone can answer in some way, can come with something (…) So maybe, make the questions as open as possible. (I1)

This strategy (S1) had been unsuccessful in the beginning, as students were "very individual [with a] right-answer mentality" (I2) and preferred to work silently on individual tasks (C1):

> They only raised their hand if they had the right answer. If I asked 'What are your thoughts on this problem? About this solution?' it seemed this wasn't mathematics in their eyes. This had to change no matter what since (…) mathematics was for them right answers and 'recipes', algorithms. (I2)

Thus, his *purpose* for dialogue was for students to see mathematics as sense-making rather than right/wrong answers. Even in oral exams he hoped students would tell themselves "I can do it and talk about it, not only set it up and show the answer is right, but also reflect on it, own it." (I2). Initially, asking *open-ended, authentic question* (S1, *teaching repertoire*) had not increased student participation in dialogue and Rune had struggled to engage them in a broader *learning repertoire* in mathematics (C1). To ease their discomfort at speaking in class (C1), Rune varied the *organisational repertoire*, away from individual work and over to pairs ("you force the student to think, it isn't fun to have nothing to say while the other has something", I1) and groups (S2):

> Asking a question to the crowd wouldn't feel as safe [to them]. 'Here I sit alone and think. These are my thoughts. If I come out with them – maybe I misunderstood or I didn't think right.' In pairs, you get either confirmation or refutation, (…) students feel safe. I get much more response (I2).

Thus, Rune reported seeing results in establishing a culture for dialogue (C1): working in pairs allowed students to practice talking about mathematics in low-risk settings, towards a *collective* quality and *reciprocity*, as both partners talked and listened. In addition, he noticed a shift in the content of the exchanges, away from seeing mathematics as just getting the right answers (C2):

> They've gotten better at working [in groups]. They are better at seeing 'OK, that works better here! Your setup works better.' There is more natural equity, because they see what works better. So they take some bits from here, some from there and in the end the group [coalesces]. (I1)

The group's adoption of bits of ideas from everyone speaks to the *collective, reciprocal* and *supportive* interaction. Rune argued that this development was the result of working on multiple solution tasks, as students get a chance to see that all contributions are valued – also those with errors (C2). Looking for (five) solutions, students may speculate, imagine, analyse (*learning repertoire*).

> We [then] look at what the groups did, get [on the board] the different solutions. Nothing is too silly. Mistakes are great because we can use them (…) see why the student thought as he did. (I2)

Rune's account of his strategies and of the challenges encompasses 1.5 years. He reported some progress, but stated that the process was ongoing (I2). We turn to the lessons for the current situation.

**Strategies and challenges in the observed lessons**

The findings from the interview guided the analysis of the videos. We identified episodes representing the two challenges (C1 and C2) and the implementation of three strategies (S1-S3). Analyses in terms of the principles of DT revealed two tensions between conflicting priorities: (C3) broad participation vs. probing student reasoning (C4) building on student contributions vs. avoiding errors.

From the analysis of the strategies S2 (using group work) and S1 (asking open questions) in the lessons, an additional challenge for dialogic teaching emerged: the tension between concerns for broad participation and probing student reasoning (C3). Rune alternated purposefully the *organisational repertoire* between whole-class and groups (S2): pairs discussed specific issues as whole-class discussion paused, while small groups worked on more complex tasks. Twice we observed Rune meet the challenge of establishing a culture for dialogue (C1), and he made sure that students engaged in group-work in a *collective* and *reciprocal* manner, once insisting on four people sitting at a group table so that they see each other (L1) and once reminding a group to interact ("Talk to each other, people!" L1). Otherwise, while the groups (especially the pairs) were working, Rune paced back and forth and mainly waited to be called upon (except for seeking out students for whom he had special concerns). We asked in stimulated recall (I2) if he deliberately stayed away.

> I guess [not engaging] lingers from 8th and 9th grade [when] they turned quiet. And [working in pairs] should be their chat, where they get to use their own words without me disturbing them. (I2)

In other words, his concern was for the *collective* and *reciprocal* exchange between the students, rather than for opportunities to probe student reasoning (C3). When Rune talked to groups, his *feedback* was always positive, but usually not of higher order. He encouraged grit ("You're really on to something there!" L1), praised, confirmed, but he rarely pressed for justifications (C3). Although *supportive*, this practice does not focus on mathematics.

Rune's *teaching repertoire* included some closed questions (driven by a clear *purpose*, e.g. "Can I choose which variable to isolate?" L2), but mostly *open-ended* ones (S2) (e.g. in L3: "What did you talk about?", "Do you have any thoughts about it?", "What do we see?", "What do we feel?", "What do you want to do?"). In these lessons, the *open-ended questions* prompted many students to participate and activated a broad *learning repertoire* (e.g. questioning, imagining, speculating etc.), but seldom *purposeful* in terms of the mathematics and there was limited *uptake*, resulting in many short exchanges lacking in *cumulative* qualities and few opportunities to justify, argue, discuss (C3).

The strategy of using multiple solution tasks (S3) was present in L1, as Rune encouraged multiple informal solutions to the doughnut and soda task (Two doughnuts and three sodas are 85 kroner. Two doughnuts and one soda are 55 kroner. How much for each?) drawn on the board (Figure 1, left). These included guess-and-check and comparing the two situations additively, corresponding to the addition method ("the one to the left is 30 kroner more than the other, so a soda is 15 kroner" L1).
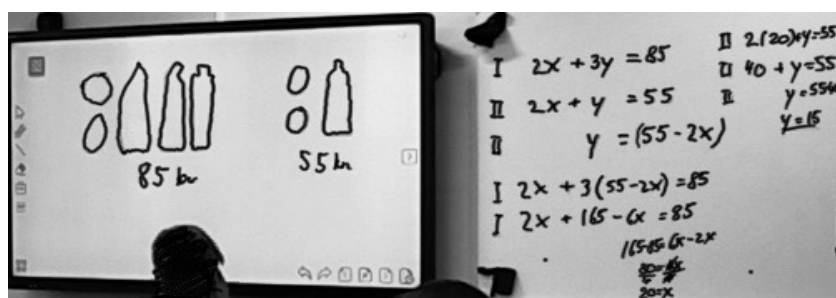
Figure 1: The doughnuts and soda task and its solution by substitution

Once the first equation was set up symbolically (Figure 1, right), and the class was working on setting up the second, Rune ignored twice Nora's suggestion ("$2x + y$ and then minus", L1) that included not only the second equation but also continued with the addition method. He waited until someone suggested $2x + y = 55$, then solved the system by substitution. He then hinted to the addition method ("Do you see any relationship with the drawing? And perhaps how we could reason based only on the two equations? It's a really hard question.", L1). Nora's renewed input ("$2x + 3y$ minus $2x + y$ and then divide by two") was acknowledged ("Hold on to that, Nora") but not built on. A numerical suggestion ("take 85 minus 55") from another student was uses to show arrive at 15 (the value for $y$). Rune then dismissed the addition method as easy "in one particular case" and stated that the substitution method would be used further. During stimulated recall, we asked about the mismatch between Rune's appreciation of multiple solution tasks (S3) and his reluctance to accept here the addition method. He explained choosing the substitution method deliberately, as "it always works" and reduce the risk of weaker students "messing up the signs" (I2). Naturally, Rune's understanding of the addition method shapes his prioritising of the substitution method. Accepting his understanding as a premise, we identify the challenge he experiences between building on relevant student contributions and avoiding methods that can lead to errors (C4). This challenge is an instantiation of a tension between the *cumulative* (connecting the informal solutions and the formal system through Nora's input on the addition method) and *purposeful* (avoiding avenues that can lead to errors).

## Discussion and conclusion

The realisation of DT holds great promise, but is difficult to achieve even for teachers participating in targeted professional development (Alexander et al., 2017; Hardman, 2019; Sedova et al., 2016). In this paper, we explore the pedagogical practices and challenges of a mathematics teacher who, although not aware of the theorisation of dialogic teaching, had aspirations compatible with this approach. Our analysis in terms of the principles of DT (Alexander, 2018) is not an evaluation of this particular teacher, but a means to understanding the challenges in implementing DT. We aim to contribute to the call by Sedova and colleagues to gain such insights with the goal of using them to make the DT more accessible for teachers (Sedova et al., 2014).

We identified four challenges, two from Rune's accounts of working with this class over time, and two from the lessons observed. Rune describes the first – creating a culture for dialogue in mathematics (C1) – as multifaceted: changing established ways of participation (in this case silent, individual work), overcoming the vulnerability of sharing – potentially wrong – ideas with the others and, not least, re-imagining what it means to talk about mathematics. This challenge relates most strongly to the first three principles of dialogic teaching (*collective*, *reciprocal*, *supportive*) but touches on the remaining two (*purposeful* and *cumulative*) as well, as part of the classroom culture is

the meaning attached to talking about mathematics. While primary teachers in Alexander's PD established relatively quickly a practice aligned with the first three principles (Alexander, 2018, p. 581), Rune still had concerns for participation after a year and a half. This challenge was not prominent in the lessons we observed, but his concerns guided his actions, revealing the tension between fostering broad participation versus probing student reasoning (C3). In the lessons, in order to ensure the *collective* and *supportive* quality of the exchanges, Rune asked open (S1) but unfocused questions and avoided higher order feedback. *Open-ended* are key indicators of DT (Alexander, 2018; Nystrand et al., 1997), but our findings add to the evidence that such questions do not necessarily lead to student reasoning (e.g. Lefstein et al., 2015) and bring a new insight about a developing practice; as Rune initially failed to get any response to authentic questions, getting many response signals progress. His concerns for regressing appear to make him wary of systematically pressing for reasoning. This speaks to the contextual factors for the impact of teaching strategies and the need for teachers to continue adjusting them as a class progresses.

Rune's concern with participation at the expense of probing student reasoning (C3) reflect as a tension between the dynamics and the content of dialogue: the three first principles for dialogic teaching have to do with the dynamics of talk, while the last two have to do with its content (Alexander et al., 2017, p. 566). The tension between the dynamics and the content of exchanges mirrors that reported by Sedova and colleagues, who found that insufficient emphasis on rational argumentation was a basic deficit in embryonic forms of dialogic teaching (in Czech language, History or civics) and attributed it in part to the teachers' concern for being *supportive* (Sedova et al., 2014).

The content-related principles (*cumulative* and the *purposeful*) were harder to realise than the others for teachers in Alexander's PD. Alexander attributes the difficulty of the *cumulative* quality to the fact that it depends on the teacher's knowledge of the subject and of the students' pre-existing understanding, as well as on the teacher's interactive skills (Alexander, 2018, p. 566). In our study, we identified a tension between the *purposeful* and the *cumulative*. For Rune, the issue of errors in mathematics resulted in two challenges: his (self-reported) efforts to help students see errors as productive (C2) and his (observed) struggle to choose between building on student contributions or avoiding error-prone methods (C4). Rune prioritised the *purposeful* over the *cumulative*, even though the purpose (consolidating the method he saw as less prone to errors) is in itself in conflict with Rune's goal of projecting errors as productive (C2) and his appreciation of multiple solutions (S3).

Our findings highlight internal conflicts between the five principles of dialogic teaching, conflicts that teachers need to negotiate in the moment. In addition, we identified a shift in the challenges of the classroom (as reported in the past versus as observed) that raises the issue of pedagogical strategies keeping up with this moving target. We will pursue this line of investigation with others IMaT participants invested in dialogic approaches to mathematics to understand how teacher resolve such tensions between the principles of DT. However, we believe that other methodological approaches are needed, including longitudinal studies (to understand potential patterns of action to resolve tensions and the sources informing these) and shifting to a student perspective (to understand the impact). For example, Rune's prioritising of broad participation resulted in some lost opportunities to think deeper about the mathematics, but it may still be beneficial for the class: a quantitative study of dialogic teaching (in languages) showed that 'talkative' classes performed better and there was no connection between utterances with reasoning and class performance (Sedova et al., 2019).

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# An investigation activity as a means of including students in mathematical sensemaking

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*We report on a classroom observation undertaken as part of a larger study of mathematics classroom practice in Norway, focusing on inclusivity. We observed a Ninth grade small-group activity in which the goal was to maximise the volume of a lidless box formed by cutting out the corners from a square piece of paper and folding up the sides. We analysed the teacher-student interactions using Schoenfeld's TRU framework. Although the investigative activity had the potential for cognitive demand and the teacher communicated intentions of exploratory work in general, as this lesson proceeded the potential for cognitive demand seemed to be scaffolded away by the teacher's direction of students' work. Thus, a tension between the teacher's intentions and the actual classroom practice was evident. Our findings suggest potentials for meaningful mathematical engagements.*

*Keywords: Investigative activities, cognitive demand, agency, representation, generalisation.*

## Introduction and literature review

Inclusion of all learners in the form of adapted education is a long-standing aim in Norwegian schooling, building on a legal requirement that every student should receive teaching appropriate to their needs.

This paper is part of the Inclusive Mathematics Teaching (IMaT)[1] project, which aims to investigate how adapted education is perceived and enacted at school and classroom level. We observed a lesson where the students worked on an investigative activity in which a mismatch between the teacher's intentions and actual practice occurred. Such mismatches have also been reported in the research literature (Kleve, 2010; Skott, 2001). In our analysis of the lesson, we consider these tensions as areas with potential for deeper engagement with mathematics.

According to Mason and Davis (2013), exploration needs to be encouraged in order to enable access to conceptual understanding in mathematics and its application in problem solving and reasoning. However, research suggests that changing teachers' practices towards more investigative ways of learning is not straightforward and even when teachers are planning for investigative activities, students' engagement in mathematical sense making is not guaranteed (Kleve, 2010). Thus, investigative activities, as such, are not necessarily an "easy fix" towards meaningful mathematics engagement for all students. According to Henningsen and Stein (1997), high-level cognitive-demand tasks are not only built on students' prior knowledge and appropriate amount of time, they rely on supportive actions by the teacher, such as scaffolding and consistently pressing students to provide meaningful explanations and make meaningful connections. Mason and Davis (2013) highlight the importance of sensitivity to students when responding to students' needs rather than depending on reactions derived from habits. Being present in-the-moment is therefore crucial for the teacher.

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Generalisations and representations, which are widely viewed as central to mathematical problem solving, are emphasised in the renewed Norwegian Curriculum for grades 1-10 as part of the five core elements in mathematics: modelling and applications; reasoning and argumentation; representation and communication; abstraction and generalization; and exploration and problem solving (https://www.udir.no/lk20/mat01-05).

According to Mason, Graham, & Johnston-Wilder (2005), students need experiences of expressing generalities rather than just trying to reproduce techniques. They claim:

> Algebra provides a manipulable symbol system and language for expressing and manipulating that generality. The core pedagogic issue is therefore about enabling learners to employ their natural powers in using algebra to make sense of the world and of other people's use of algebra (p. 2)

In the Norwegian mathematics curriculum, it is recommended that students use various representations when learning algebra. These verbal, numerical, graphical, and algebraic representations have the potential of making the learning of algebra meaningful (Kaput, 2017). In the lesson we studied, the use of a variable became a crucial issue. This was an expression of generalisations from arithmetic and may be considered as a study of functions and relations. A research review of the flexible use of representations shows that students struggle to integrate the various representations and their relations (Nistal, Van Dooren, Clarebout, Elen, & Verschaffel, 2009).

In this paper, we study how the investigative activity developed during the lesson, and the potential for deep mathematical engagement that was provided. More precisely, we focus on how a group of students' engagements with the mathematics in the activity developed during the lesson. Our research questions are: How was the intended investigative activity implemented and enacted? How did students engage in the activity and what role did the teacher's support play?

## Setting the scene: an investigative activity

The aim of the investigative activity analysed here was to maximize the volume of a lidless box formed by cutting out the corners from square piece of paper (24cm×24cm) and folding up the sides. The teacher's presentation of the activity took place with the whole class. As a warm-up, the students were asked to reflect and then suggest how many litres a cardboard box could hold (the teacher had displayed a cardboard box with dimensions 25cm×40cm×40cm). The formula for the volume of a prism was written on the board, and the exact volume of the box was calculated after students had come up with different suggestions for the number of one litre bottles the box could hold.

Students worked in six teacher-assigned heterogeneous groups of three or four. The exact criteria for assigning groups were not clear and some assignments may have been random. They were given a piece of paper and were asked to find how much of the corners they should cut off in order to obtain the maximum volume. They were encouraged to be systematic and strategic, and as they moved into their groups, the teacher said: "What is the variable here?" He also reminded them of an activity they had completed in the previous year where the goal was to maximize a two-dimensional area, and he focused more and more on the variable when he interacted with the different groups.

## Methodology

This research was done as part of a pilot for the IMaT project. We observed three mathematics lessons with one teacher, who had volunteered to participate. The class was a mixed ability grade nine class. The students were usually seated in pairs, but in this lesson, they were placed in groups. Prior to our first classroom observation, we had a semi-structured interview with the teacher. We asked him to describe a typical mathematics lesson, his intentions in teaching mathematics and how he dealt with mixed abilities, adapted education and grouping. We also conducted a post interview with him one month after having observed this lesson.

The lesson started with a 10-minute whole class session (launch), followed by an 80-minute session with group work, and finally with a 20-minute summing-up. The lesson was videotaped, and the teacher wore a microphone. During the group work, the camera followed the teacher as he moved from one group to another. Additionally, the researchers placed an audio recorder with a randomly selected group of 2 boys and 2 girls. The three researchers did not interact with the teacher and students during the lesson and were non-participant observers in the classroom. One managed the video filming, another took general field notes, and the third used the Teaching for Robust Understanding (TRU) framework (Schoenfeld & Floden, 2014) as an observation record.

The TRU framework offers ways to reflect along five dimensions: 1-The richness of the mathematical content, 2-Cognitive demand, 3-Access to mathematics, 4-Agency, Ownership, and Identity and 5-Uses of assessment. We used the scoring rubric for small group work as an observation scheme in the classroom. The focus of the framework is when students are engaged in brainstorming, the role of the teacher is to support students in exploring and justifying. Every 15 minutes, we noted on a line segment (from one to three) a score for each dimension. This gave us the possibility to identify changes in for example cognitive demand. In advance, the researchers had prepared for the application of the framework in order to ensure reliability across the research team in the big research project.

We transcribed the video recording while making notes to obtain an overview of the lesson. Together with the observation schemes this provided us with an overall picture of the development of the teacher's support of groups. We transcribed the audio-recorded data from the group that we focused on in order to analyse how students within the group engaged in the activity. Our aim was to study the teacher's interactions with the group when he "visited" them. Transcripts from these interactions were subject to analysis. Data from pre- and post-interviews with the teacher were used to further illuminate our study and findings.

## Findings

Schoenfeld (2018) emphasises that "The main focus of TRU is not on what the teacher does, but on the opportunities the environment affords students for deep engagement with mathematical content" (p.491). In this section, we will go through the dimensions in the framework in order to identify potential for development. We report only briefly on whole class activities, focusing instead on the teacher's intentions and his interactions with the group's work. Excerpts from the audio recorded data and from the interviews illustrate the analysis.

**Mathematical content: How accurate, coherent, and well justified is the mathematical content?**

In the pre interview, the teacher emphasised that he consciously selected mathematical tasks that were meaningful for the students. Criticizing students' prior schooling he said:

> Many students have negative thoughts about mathematics when starting lower secondary. They have only experienced procedural exercises [ ], it is necessary to unlearn that that is maths. [ ] My teaching reveals a different way of thinking, solving problems in a context. [ ] I don't use a textbook, but I select a specific problem to be solved.

According to what he said here, the task selected for this lesson was well thought through. The task also had a potential for high cognitive demand. His intention was to provide students with opportunities for building a coherent view of mathematics. During the lesson launch, the students were invited to explore. The teacher illustrated the volume by displaying the dimensions of the box, drawing on the board, and calculating the box's volume.

The teacher had planned this lesson with the intention of training mathematical exploration, extending students' algebraic thinking and allowing them to connect several representations. While observing in class, we noted this in the beginning of the lesson with a score between two and three on the TRU observation record. This was confirmed when we watched and analysed the video. However, in the second half of the lesson, the implementation seemed rather procedural, which had been scored close to one on the observation scheme. We explore this further below in the analysis along the dimension of cognitive demand.

**Cognitive demand: To what extent are students supported in grappling with and making sense of mathematical concepts?**

Generally, the teacher initiated the possibilities of conceptual richness as he gave hints and was supportive. He encouraged students to understand the context (close to three on the TRU observation scheme); however, his comments were increasingly characterized by giving the students step-by-step instructions on what should be done, including how to cut out corners from a square piece of paper (close to one on the scheme). Focusing on our selected group, they had talked about cutting 12 cm, but realised that it was not possible (the paper was 24 cm and folding 12 cm would not make a box). They called on the teacher. From the video, we see that the teacher took the paper, told them to listen to him and showed them how it should be folded. He did not comment on why 12 cm was not possible.

| | |
|---|---|
| Teacher: | Ok. Listen to me now. If we cut it off, there are two centimetres here and two centimetres in there. Is it 24, then we must remove …? |
| Students: | Huh? |
| Teacher: | It is 24 cm in width, right? |
| Student 1: | 22, is it? |
| Teacher: | It is 24 cm. Then we cut away two centimetres there and two centimetres here. |
| Students: | 20 |
| Teacher: | 20. Then this [points to one side] will be 20, and this [points to the other side] will be 20. Is it 20 times 20? Will it be right then? |

Other typical teacher input both to this group and the other five groups were: "How is it going?"; "What is the height?"; "Try to be strategic and systematic"; and "What is the variable?" The inputs were directive and discussions with the groups seemed aimed at "answer getting".

Although the given task could be described as a high-level cognitive demand task and had potential for making connections, it seemed not to be implemented as intended. Instead, most of the chances for making connections were transformed into procedural exercises.

By a process of trial and error with numerical values, the groups had no difficulties in figuring out that the maximum volume was 1024 $cm^3$ with corresponding height 4 cm. However, they were overwhelmed by the number of calculations and struggled to get an overview. The teacher told them what numbers to try out and to be strategic and systematic, which led to a decrease in cognitive demand. The teacher introduced the height as a variable, and he focused on finding a correct algebraic expression.

The task offered possibilities of productive engagement or struggle with central mathematical ideas. As observers, we recognized that the teacher's agenda might have been to move between the four different representations (verbal, numerical, graphical and algebraic) in order to work with algebra in a meaningful and coherent way. This was not communicated clearly to the students and seemed not to be obvious. For them, the task was just to find the height providing the maximum volume and folding the paper accordingly.

**Access to mathematical content: To what extent are all students supported in meaningful participation in (group) discussions?**

Analysis of transcribed audio data from the selected group showed that group discussion, for part of the time, was devoted to sharing solution methods or ideas and making sense of them. In the pre-interview, the teacher emphasised the importance for the students to discuss mathematics with each other and then to use mathematical concept correctly: "Students never work alone, always together either in pairs or group of four. They are supposed to discuss and use the mathematical language correctly". However, we did not identify instances where the teacher supported or encouraged students' engagement in student-to-student discussions.

**Agency, authority and identity**: **To what extent did teacher support and/or group dynamics provide access to "voice" for students?**

In this lesson, conversations were teacher-initiated, and students' answers were short, and sometimes seemed to be guesses. According to our field notes and TRU records, some students' chances to explain their thinking were identified. However, the teacher was the driver of the conversations and there were no interactions between students in discussions while the teacher was present.

The teacher provided time for students to develop and express mathematical ideas and to reason during group work. In the pre-interview, he said "I try to give hints so they can think on their own until I come back". This also characterized his interactions, and indicates his intentions were to be supportive. In the post-interview he also emphasised that students should have agency: "In a way it is their agency [] they shall have their own identity and agency and they shall experience an ownership on a meta-level. I want them to define what it implies to have flexible strategies in the subject".

**Use of assessment: To what extent does the teacher monitor and help students refine their thinking within small groups?**

The teacher created a confident atmosphere in the classroom. The students felt free to express their ideas and understandings. However, questions from the teacher were mostly closed and did not seem to expect mathematical justification, and students' algebraic thinking was not challenged. Many teacher-posed questions were answered by the teacher himself. As the teacher did not consider students' reasoning, the task ended up being worked on at a lower level because teacher did not listen actively to students' reasoning and he did not try to assist them based on their pre-existing knowledge. This is exemplified in the episode below. On this fifth (and last) visit, he asked how much they suggested cutting off. They answered "four". The conversation continued:

| | |
|---|---|
| Teacher: | Fantastic! That is right! 1024 cubic centimetres? |
| Student 1: | [proudly] It will be approximately one litre? |
| Teacher: | But can we find another form of expression? Some formula … something for this. Is it possible to do? |
| Student 2: | [giggly] It must be an $x$. |
| Student 3: | [hesitantly] Should we find a formula to... How did we find out? |
| Teacher: | Look here. Now we have found the volume of a box |
| Students: | Length times width, times height. |
| Teacher: | What is the width here or the length? It's going to be... If we call it $x$, then? |
| Student 2: | Won't it be 16 then? |
| Teacher: | Then it gets here 24 minus the length from there... It will be 24 minus … How many $x$'s are we taking away? |
| Student 3: | 2 |
| Teacher: | Minus $2x$... Multiplied by that are … |
| Students: | Mmm $x$... The same |
| Teacher: | The same, correct. Times with that length. This is the length; this is the width |
| Student 3: | Then times with $x$ again. |
| Teacher: | Correct. Then the question is. We were working a bit with algebra last year. Do you remember that? Multiplying parenthesis together and so. So, 24 times 24 it is... 576. 24 times minus $2x$ it is... |

The teacher carried on transforming $(24 - 2x)(24 - 2x)x$ into $576x - 96x^2 + 4x^3$. He thus led the students down a predetermined path and did not solicit or pursue student thinking. In the end, students were left with a task that only required applying a procedure without any idea how to make connections to previous work and underlying mathematical meaning.

## Discussion and concluding remarks

The task was a high cognitive demand task with the potential for meaningful mathematical engagement. However, our findings suggest that the teacher's expressed intentions of doing investigative work were not implemented. Additionally, the students' roles throughout the lesson increasingly became to follow the teacher's instructions instead of investigating and struggling to find mathematical connections. The teacher encouraged students' thinking, but the support he gave became more and more directive. This mismatch between the intentions and implementation may be a potential area for development. This we now discuss illuminated by what the teacher said in the interviews.

In the pre-interview the teacher communicated that he did not consider himself a traditional teacher: "I never teach algorithms. I never do examples on the board followed by students doing same kinds of exercises." He also emphasised that students have to struggle: "I tell students that it takes time doing mathematics. They need to be tired in the same sense as they become tired doing sports. It is through struggling your brain develops." These expressed intentions were underpinned by his selection of a task that had the potential of a high score on all five dimensions of the TRU framework. According to our analysis above, opportunities for building a coherent view of mathematics were provided as students were invited to explore. Additionally, as emphasised by Henningsen and Stein (1997), the students were given enough time to work on the task, mostly in groups, but also when working in pairs. There was a confident atmosphere, and the teacher was available throughout the whole session.

The teacher displayed theoretical knowledge about why students should be supported in group dynamics, and he also expressed importance of students' mathematical agency. However, his interventions constrained students to produce short responses when he interacted with them, or he answered his own questions, not taking what students said into account. The teacher was the driver of the conversations and there were no interactions between students in discussions while the teacher was present.

In the last quoted interaction, in which the teacher suggested an algebraic expression with two parentheses, we see that students' contributions are not explored or built upon.

The students talked about how many litres the box contained, and were thus linking back to the introductory activity, while the teacher focused on algebra, multiplying the factors to get his desired algebraic expression. The conversation between the teacher and the students was a one-way communication. The students were not "on the teacher's track" and for them the teacher's manipulation with symbols did not make sense. The teacher displayed ownership of the mathematics and his goal seemed to be leading the students towards an algebraic representation.

The teacher provided students with possibilities for thinking, but not opportunities for argumentation and reasoning. Since the teacher knew where they should go, he was always ahead of the students, and had difficulties not telling them. The students worked with verbal and numerical representations and did not see the point with algebraic expressions and graphical representations. As Nistal et al. (2009) emphasise, this is a common struggle for students, which the teacher might have noticed. In the whole class summing up, the teacher asked for different representations from the groups and students became engaged in comparing the different representations. He made a table, wrote an

algebraic expression on the blackboard and then using GeoGebra, he drew the graph on the smart board.

Our findings suggest that there was a mismatch between intentions and implementation of the task. We noticed that crucial supportive actions, such as pressing students to provide meaningful explanations or make meaningful connections were not evident. This may be due to the teacher's challenge in having to support all six groups as much as possible, so they were not left unsupported. In the pre-interview we had asked the teacher about what he saw as dilemmas or challenges in teaching mathematics: "I wish I could have smaller classes, however, not according to level, so I could have followed up more". In the post-interview he displayed self-awareness saying: "The problem is that I am so enthusiastic and may tell them too much, so they will not get time to reason, because I think it is so much fun. But I try to give hints so they can reason on their own until I come back". This underpins some conflicting issues mathematics teachers face. Although his intentions may be to give students agency and the possibility to reason on their own, time pressure and stress, and also his expressed enthusiasm in doing mathematics, tempted him to present solutions too quickly and hence he scaffolded the students' struggle away, rather than giving supportive hints as he argued for in the interview. However, his expressed theoretical knowledge and awareness of the mismatch between his intentions and actual practice, we consider a potential for development of students' deep engagement in mathematics so supportive hints will encourage mathematical struggle rather than scaffold students' struggle away.

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# Learning mathematics teaching when rehearsing instruction

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*This study explores rehearsals. Norwegian elementary in-service teachers participated in a practice-based approach to professional development. They collaborated in learning cycles of enactment and investigation, aiming at learning to enact ambitious mathematics teaching practices. The ways in which Teacher Time-Outs (TTOs) in the learning cycles' rehearsals provided the teachers with opportunities to learn multiple practices of ambitious mathematics teaching were explored. The findings revealed that the TTOs provided opportunities for collaborative and simultaneous learning of multiple core ambitious teaching practices, such as whether or not and when to use talk moves to facilitate student talk when eliciting their strategies, parallel to aiming for the goal for the lesson.*

*Keywords: Rehearsing mathematics teaching, learning cycles, professional development*

## Introduction

Based on their review of research on mathematics teachers' professional development, Goldsmith, Doerr and Lewis (2014) conclude that there is a need to explore in-service teachers' (hereinafter teachers) opportunities to learn in and from enactments of teaching. To meet this call, this study addresses teachers' opportunities to learn to enact mathematics teaching practices when rehearsing instruction. The rehearsals were included in *learning cycles of enactment and investigation* (learning cycles, McDonald, Kazemi, & Kavanagh, 2013) in the *Mastering Ambitious Mathematics teaching project* (MAM) research and professional development project.

Ambitious mathematics teaching aims at developing all students' conceptual understanding, procedural knowledge and adaptive reasoning (e.g. Lampert et al., 2013). Among the principles of ambitious teaching are: treating students as sense-makers and engaging deeply with their thinking (Ghousseini, Beasley, & Lord, 2015; Lampert et al., 2013). Examples of core practices of ambitious mathematics teaching are eliciting and responding to students' mathematical reasoning, using representations, aiming towards the goal for a lesson, facilitating student talk and organizing the board (Lampert et al., 2013). Building instruction on students' mathematical ideas (ambitious teaching) is a centrepiece professional development. This is informed by the extensive and growing research base on students' mathematical thinking and development (e.g. Lester, 2007), as well as findings suggesting that gains in student achievement are related to instruction having students' ideas at the core (e.g. Sowder, 2007). Following this, the aim of professional development is to support teachers in learning the demanding endeavour of ambitious teaching practices and rehearsals are found to be contexts for such learning (Kavanagh et al., 2019; Lampert et al., 2013; Wæge & Fauskanger, 2020).

## Previous research on rehearsals

In a rehearsal, a teacher leads an instructional activity with colleagues acting as students. The participants may pause instruction by initiating a *Teacher Time Out* (TTO) during which they discuss how the rehearsing teacher might respond to student contributions and determine the direction of the further instruction (e.g. Gibbons, Kazemi, Hintz, & Hartmann, 2017). Teachers are, thus, provided with opportunities to try out and discuss ambitious teaching practices.

Most research on rehearsals has focused on initial mathematics teacher education in the US context (e.g. Ghousseini et al., 2015; Lampert et al., 2013). From this research we learn that rehearsals are contexts where novices could learn "to do adaptive teaching while developing their knowledge, skill, and identities" (Lampert et al., 2013, p. 238). We also learn that as novice teachers rehearse instruction, they connect their own knowledge and relevant aspects of the context when enacting instruction. There is, however, a need for research exploring if findings from the US context translate into contexts outside the US. More recent research has explored rehearsals in professional development for teachers. Kavanagh et al. (2019) found that by reducing teachers' choices in rehearsals "it was possible to focus more tightly on how best to give full attention to, understand, and respond to student ideas" (p. 11). Based on these findings, Kavanagh et al. (2019) also call for future studies exploring if findings from the US context translate into other contexts.

Building on the findings from Kavanagh et al. (2019), Wæge and Fauskanger (2020) conclude that the rehearsals provided the teachers with opportunities to learn ambitious practices. These researchers point out that their findings indicate that the participants could work simultaneously on multiple practices but that this needs to be explored more thoroughly. Building on this limited research on rehearsals in professional development in the Nordic context, the aim of this study is to answer the following research question: In what ways do TTOs in rehearsals open for teachers' opportunities to learn multiple practices of ambitious mathematics teaching?

Learning is in this study understood as emerging in activities, such as rehearsals. Learning, thinking and knowing are "relations among people engaged in activity in, with, and arising from the socially and culturally structured world" (Lave, 1991, p. 67). From this perspective, teachers' opportunities to learn refer to developing the ability to engage in particular practices. In the learning cycles, the teachers were positioned as responsible contributors. The rehearsals allowed the teachers to pause instruction by initiating a TTO to ask questions, discuss, explain and justify their mathematical and instructional ideas, and thus share in the decision-making (Wæge & Fauskanger, 2020).

## Methodology

In the MAM project, 30 teachers from ten Norwegian elementary schools worked together in repeated learning cycles. Each cycle included preparation, co-planning, rehearsing, co-enacting and making collective analyses of instruction with the aim of learning core practices and principles of ambitious teaching. In order to support the teachers to encourage students to engage in mathematical talk, use of talk moves (Kazemi & Hintz, 2014) was worked on. Two groups of teachers (n=14) participated in the research study reported on here.

A total of 175 TTOs across 18 rehearsals were identified. Building on Lampert et al. (2013), ambitious teaching practices (e.g. elicit and respond, use of representations, aiming towards goals, facilitating student talk) were used as codes for coding the TTOs. In many of the TTOs analysed, the participants worked on two or more ambitious practices in relation to each other. An in-depth analysis of each of the TTOs was conducted in order to pinpoint the ways in which the TTOs in rehearsals provided teachers with opportunities to learn multiple practices simultaneously. This was first done by coding for the a priori codes described above, followed by an analysis of how the participants in parallel engaged in multiple teaching practices. In this paper, one of these TTOs will be used as an illustrative example. This TTO lasted for three 3 minutes and 23 seconds and included a total of 74 utterances.

The entire rehearsal lasted 19 minutes and included seven TTOs. The instructional activity in this rehearsal was quick images (see Figure 1). This activity was chosen as one of the activities in the MAM project because it is designed to allow inquiry into the relationships between practices and principles of ambitious teaching and the mathematical content relevant for the teachers. In addition, given instructional activities are found to support participants in eliciting student thinking and in making judgments on how to respond in principled, instructive ways (Lampert et al., 2013).

## Findings

In the co-planning session prior to the rehearsal, the teachers decided that the learning goal for the lesson should be the distributive property of multiplication and a quick image was selected as instructional activity for this purpose. In the beginning of the rehearsal, the students (the teachers and the teacher educator (TE)) were shown a quick image for a few seconds, and the rehearsing teacher (RT) asked them how many dots they saw and if they could explain how they arrived at their answer. Just prior to this TTO (initiated by the TE), the RT represented two student strategies on the board (Figure 1).
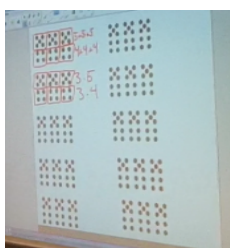


Figure 1: Student strategies represented on the board

| 1 | TE: | I want to take a very little time-out because I see we're getting a little crowded here [on the board]. Do you see that? |
| 2 | RT: | Yes. |
| 3 | TE: | In such a way that it/ |
| 4 | RT: | Should we make them bigger? |
| 5 | TE: | Perhaps, or perhaps we should put them to the side [the second column]? |
| 6 | RT: | These ones [points at the column of quick images on the right side]? |
| 7 | TE: | Yes, these could go, I don't know, maybe. |
| 8 | RT: | So, we could get more room for more along here [points to right side of the first column]? |
| 9 | TE: | Yes [walks towards the Smartboard]. |
| 10 | RT: | Yes, there was more room before. |
| 11 | TE: | Yes, because we started to make this row [column] on the side [presses the board to remove the quick images in the right column]. |
| 12 | RT: | I'll put this down [board pen] here now. |
| 13 | TE: | I think we'll delete these here first [deletes the quick images in the right column], like this. Because then we can write along here, I think so… |
| 14 | RT: | Mm. |
| 15 | TE: | Mm. |

The TE interjected to draw the participants' attention to the board, claiming that they did not have enough space (1). The RT asked if they should make the images larger (4) and the TE suggested that they could remove the quick images in the right column so they could write the whole number sentences (student strategies) next to the quick image (5, 7 and 13). As the RT agreed (8, 10 and 14), the TE deleted the upper two quick images from the right column (11 and 13). As this exchange illustrates, participation in this TTO episode opened for a learning situation for the participants in which they considered how to best organise the board in order to represent the students' strategies in a particular way. The participants were collaboratively making sense of the practices of organising the board and using representations.

The TTO continued as follows:

| 16 | RT: | Yes, is it smart to take even more to begin with, like this? |
| 17 | TE: | Yes. |
| 18 | OT[1]2: | But I thought perhaps that, before they offer their ideas, could it have been an idea to talk together and say something about "what you saw", or is there any point in doing this? |
| 19 | RT: | Turn and talk to begin with? |
| 20 | TE: | Yes, what do you think about that? |
| 21 | RT: | Before we start, yes. |
| 22 | OT2: | Yes. |
| 23 | OT3: | Then we make the process [of eliciting and representing student strategies] longer, I think. |
| 24 | OT2: | Yes, then everyone feels that they have been able to say what they want [their strategy]. |
| 25 | OT3: | Yes, maybe both, but this makes it quite a long sequence for getting to where we want to be. |

In lines 19–22 the participants clarified that OT2's suggestion in line 18 was to use the talk move "turn and talk" before the students presented their strategies. After this clarification, OT3 hesitated by indicating that using this talk move would take (too much) time (23 and 25). OT2 supported her own suggestion in line 16 by highlighting "turn and talk" as providing all students with an opportunity to present their strategies (24). The pros and cons of using the talk move "turn and talk" were part of this TTO discussion. This part of the TTO thus opened up a learning situation for the participants in which they considered if and why talk moves should be used when eliciting students' thinking. All participants were able to engage in the decision-making process, and they were collaboratively making sense of the practices of eliciting student ideas and facilitating student talk. The TTO continued by clarifying how much time they would have for enacting the lesson, and they agreed that they had enough time for "turn-and-talk" discussions (26–34). Due to this agreement, they continued their discussion as follows:

| 35 | RT: | Could they have discussed a little before beginning to tell me [their strategy]? |
| 36 | OT2: | The danger then is that perhaps they just copy what someone else has said. |

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[1] OT is an abbreviation for "Observing Teacher".

| 37 | Several: | Yes. |
|----|----------|------|
| 38 | OT2: | And don't come with their own [strategy], maybe because they see that the other person's [strategy] is quicker and better. |
| 39 | Several: | Yes. |
| 40 | OT2: | And then maybe you don't get more of this "five plus five plus five". |
| 41 | RT: | You might get less variation [in student strategies presented]. |
| 42 | OT2: | Yes. |
| 43 | Several: | Yes. |
| 44 | OT3: | Or maybe not. |
| 45 | OT2: | You could rather… |
| 46 | OT3: | Perhaps rather turn and talk when you have some examples up there and ask 'do you see any connections'? |

When co-planning the lesson, the teachers anticipated that some students would suggest $5 + 5 + 5 + 4 + 4 + 4$ as their strategy. They decided to use this strategy to lead the students towards the goal for the lesson: the distributive property of multiplication (i.e. $3 \times 5 + 3 \times 4 = 3 \times (5 + 4)$). In this part of the TTO, OT2 suggested that some of the anticipated strategies might not be presented if students talk amongst themselves before presenting their own strategy (36, 38 and 40) and that the teachers risked that $5 + 5 + 5$ would not be suggested (40) by the students after talking with their classmates and learning a strategy they might find "quicker and better" (38). Based on their discussion, they agreed that inviting the students to turn and talk might reduce the number of different strategies presented (41–43). Consideration of the  pros and cons of using talk moves was also part of this TTO discussion, but whereas the first part of the discussion (23–34) was about time issues, this part was about the danger of having less variation in the student strategies presented if these were shared by the students when talking with each other. Bearing this in mind, OT3 suggested that they should rather use "turn and talk" "when you have some examples up there and ask 'do you see any connections'?" (line 46). As this exchange illustrates, participation in this TTO discussion opened up a learning situation for the participants in which they considered reasons for whether or not and when to use the talk move "turn and talk" when eliciting student strategies. They also discussed whether the use of turn and talk could support them in reaching the goal or not. The participants were collaboratively making sense of the practices of eliciting student responses, using talk moves and aiming for goals in relation to each other. Their goal for the lesson was implicitly visible in the discussion, but it was also discussed explicitly in the continuation of the TTO:

| 47 | RT: | Because we said we were going to use it [turn and talk] after, or when we begin to get them to focus on our goal [the distributive property of multiplication]? |
|----|-----|------|
| 48 | Several: | Yes. |
| 49 | RT: | But we can agree on that, can't we? |
| 50 | Several: | Yes. |

Some of the TTOs addressed the practice of drawing attention to and leading the students towards the mathematical learning goal for the lesson. In this extract, the RT reminded the participants that they had planned to ask the students to turn and talk when aiming towards the learning goal for the lesson (47). The others remembered what they agreed upon (48 and 50). The analysis indicates that this part of the TTO engaged the participants in discussing the practice of facilitating student talk (i.e. whether

or not and when to use the talk move "turn and talk" when eliciting student strategies) while at the same time aiming for the goal for the lesson. Thus, the TTO opened up a learning situation in which the participants simultaneously worked on these practices. After this exchange, the participants agreed that the rehearsing teacher should ask the students to turn and talk if she felt that some students had "dropped out" (OT4, 52–58). The last part of this TTO proceeded as follows:

| 60 | OT2: | When it comes to writing them [the students' strategies represented in the quick images] like this under each other, perhaps it would be better to write them [the student strategies written as number sentences] along this way? |
| 61 | OT4: | And write it [the number sentence] out. |
| 62 | OT2: | Then we come quicker to the [the distributive property of multiplication]. |
| 63 | RT: | That I write... |
| 64 | OT4: | Three times five plus three times four. |
| 65 | RT: | mm, at once. |
| 66 | OT2: | Yes. |
| 67 | RT: | [Removes $3 \times 4$ written under $3 \times 5$ and writes it next to $3 \times 5$ on the board, see Figure 2] Yeah, because now we have room for it. |
| 68 | Several: | Yes. |
| 69 | TE: | And then I think that it's kind of important for the four plus four plus four that it's written out too. |
| 70 | RT: | [Starts writing]. Should I use parentheses as well? |
| 71 | TE: | I don't think you need to. |
| 72 | RT: | No [Writes $4 + 4 + 4$ after $5 + 5 + 5$ which is already written on the board]. |
| 73 | Several: | mm. |
| 74 | RT: | Yes. |
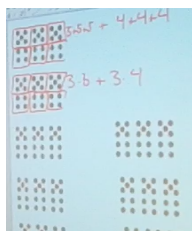


Figure 2: The board work towards the end of this TTO

This part of the TTO started out by focusing on where and how to write student strategies on the board (60–61), seeing this as important when aiming for the goal for the lesson (62). Several teachers (64, 66 and 68) supported the RT (63, 65 and 67) when deciding how and where to represent student strategies on the board. Whether or not to use parentheses when writing student strategies on the board was part of this discussion (71–74). At the same time, suggestions were also made as to which student strategies to focus on when aiming for the goal for the lesson (64 and 69), and the rehearsing teacher wrote the suggested strategies on the board (70). This analysis indicates that the activity enabled the participants to learn where, when and how to write student strategies on the board (organising the board and using representations) in order to reach the goal for the lesson.

This last part of the TTO was followed by the continuation of the rehearsal. The rehearsing teacher asked the other participants (acting as students) if they had other strategies for finding the number of dots in the quick image.

## Concluding discussion

Based on a need to study teachers' opportunities to learn in and from enactments of teaching (Goldsmith et al., 2014), the aim of this study was to explore how TTOs in rehearsals opened for teachers' opportunities to learn multiple practices of ambitious mathematics teaching simultaneously. TTOs in 18 rehearsals were analysed and the findings are presented by using one illustrative example of a TTO. In this TTO, the participants worked on several practices in relation to each other by asking questions, giving each other feedback and offering specific suggestions on what the rehearsing teacher could do in a particular situation, drawing on key principles of ambitious teaching (e.g. Lampert et al., 2013). Participating in TTO discussions when rehearsing instruction opened up learning situations (Lave, 1991) for the participants in which they were discussing and making sense together of multiple practices simultaneously, such as organising the board when representing student strategies so they could make connections between representations (lines 1–15). This supports findings from studies of novice teachers' rehearsals in the US context (e.g. Lampert et al., 2013), which indicates that rehearsals are learning situations for novice as well as experienced teachers. In the TTO, the participants also discussed how decisions concerning one practice could influence another practice. They discussed whether or not and when to use the talk move "turn and talk" to facilitate student talk when eliciting their strategies while simultaneously aiming for the goal of the lesson (lines 16–59). The finding that the participants are provided with opportunities to learn to facilitate student talk from rehearsing instruction supports findings from similar studies in a PD context in the US (Kavanagh et al., 2019). The focus on particular talk moves (Kazemi & Hintz, 2014) in MAM, seems to add to the existing literature on rehearsals as learning situations in PD (e.g. Kavanagh et al., 2019). The participants were provided with opportunities to learn whether or not and when to use talk moves to facilitate student talk. Furthermore, in their argumentation related to one practice, the participants sometimes included considerations about other practices. In lines 60–74, the participants discussed where, when and how to write which student strategies on the board when aiming for the goal of the lesson, while simultaneously considering what student strategies to focus on to reach the learning goal for the lesson. This supports findings from studies of rehearsals in the US context (Kavanagh et al., 2019; Lampert et al., 2013).

Similar to Kavanagh et al. (2019), we found that by reducing the teachers' choices in the rehearsals the TTOs gave the participants opportunities to try out and discuss, and thus learn (Lave, 1991), several teaching practices that are responsive to the students' contributions. In addition, and similar to findings from the US initial teacher education context (Lampert et al., 2013), we found that these practices were worked on simultaneously and in relation to each other. Finally, the findings indicate that learning cycles developed in a US teacher education context (McDonald et al., 2013), and in particular rehearsals, provide teachers in a Norwegian context with opportunities to learn ambitious teaching practices and thus to develop their capacity to enact responsive instruction. These findings have implications for design of professional development as well as for future professional development research in the Nordic countries.

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# Learning professional noticing by co-planning mathematics instruction

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*At the core of ambitious mathematics teaching involves using knowledge of students' mathematical thinking when facilitating and leading mathematical discussions. A teacher's ability to productively use this knowledge depends on his/her noticing expertise. This study explores in-service teachers' opportunities to develop their ability to notice through a practice-based approach to professional development. Fourteen Norwegian elementary-school teachers collaborate with teacher educators in learning cycles of enactment and investigation, where the overarching aim is to learn to enact ambitious mathematics teaching. This study investigates what this innovative approach enables teachers to work on when co-planning to notice. The findings suggest that the co-planning discussions focused on particular students' mathematical thinking (focused noticing) and on both students' mathematical thinking and teacher's pedagogy (extended noticing).*

*Keywords: Professional noticing, co-planning, mathematics teaching, learning cycles of enactment and investigation.*

## Introduction

The aim of this paper is to explore teachers' opportunities to learn professional noticing when collectively planning (co-planning) mathematics instruction. In any profession "we are sensitized to notice certain things" (Mason, 2002, p. xi) and the teaching profession is no exception. This "sensitized noticing" is often referred to as professional noticing, but hereinafter we will use the terms noticing and professional noticing interchangeably.

Informed by the extensive research base on students' mathematical thinking (e.g. Lester, 2007) and reviews of research suggesting that one of the core activities of teaching is "sizing up students' ideas and responding" (Ball, Lubienski, & Mewborn, 2001, p. 453), building instruction on students' mathematical thinking has been endorsed in many reform documents (for the Norwegian context, see Utdanningsdirektoratet, 2019). Students' mathematical thinking (students' thinking) refers to strategies, representations and reasoning students use in an instructional setting. Students' thinking is a coherent and logical approach to reasoning that often differs from the way mathematicians and other adults think (Carpenter, Carpenter, Franke, Levi, & Empson, 2015). Teachers' professional noticing can be defined as an expertise that includes attending to students' thinking, interpreting their understanding and deciding how to respond (Jacobs, Lamb, & Philipp, 2010). For teachers, noticing students' thinking is essential and research has suggested that developing the ability to notice can be learned through scaffolded support and collaboration (e.g. Star, Lynch, & Perova, 2011). The aim of professional development (PD) is therefore to support teachers in learning the demanding endeavour of noticing students' thinking, often referred to as ambitious teaching (Lampert et al., 2013).

Ambitious mathematics teaching aims to develop all students' conceptual understanding, procedural knowledge and adaptive reasoning (e.g. Lampert et al., 2013). Two principles of ambitious teaching are treating students as sense-makers and engaging deeply with their thinking. Examples of core

practices of ambitious mathematics teaching are eliciting, responding to and representing students' thinking and facilitating student talk.

Using student thinking during instruction is valued by the mathematics education community, yet ambitious practices to support such use remain difficult for teachers to enact well, particularly in the moment during whole-class instruction. In the *Mastering Ambitious Mathematics teaching* project (MAM), teachers were invited to collaborate in learning cycles of enactment and investigation (learning cycles) in order to develop their ability to notice students' thinking and to build on students' thinking in their teaching. For the purpose of this paper, the analysis intends to shed light on the ways in which co-planning instruction enabled teachers to collectively learn to notice professionally.

## Professional noticing

The idea of noticing as a discipline sees noticing as a collection of ambitious practices. Each practice is designed to "sensitize oneself so as to notice opportunities in the future in which to act freshly rather than automatically out of habit" (Mason, 2011, p. 35). Noticing builds on the concept of professional vision as a process through which teachers make sense of what occurs during teaching and through which they make plans to respond to students' thinking (Sherin, Russ, & Colestock, 2011). Since ambitious teaching practices are difficult for teachers to enact well, predicting student strategies and discussing "what one wants to notice" (Mason, 2011, p. 48) are important steps in the ability to notice in the moment of whole-class instruction.

Ball (2011, p. xii) sees noticing "as a practice essential to attending to learners, to the domain for which the teacher is responsible, and to connections between the learners and the domain." Noticing is consequential, it is an awareness that enables action (Mason, 2011) and skilled teachers are quicker to identify situations that require intervention (Miller, 2011). Noticing has consequences for what a teacher sees and does not see, and for what a teacher does and does not do. Noticing is thus "a key component of teaching expertise and of mathematics teaching expertise in particular" (Sherin et al., 2011, p. 79) because it can lead to changed practices, where planning for such practices is necessary.

Even though there are various conceptualizations of noticing (Miller, 2011), the two interrelated and cyclical processes of attending to and making sense of particular students' thinking in an instructional setting are often involved. For example, Star et al. (2011) include what a teacher attends to as well as what the teacher decides not to attend to in their conceptualization of noticing. Jacobs, Lamb, Philipp and Schappelle (2011) also include what teachers' plan to respond to in a classroom activity in their understanding of noticing. These researchers thus include the following in their characterization of noticing: how teachers pay attention to a classroom activity, their interpretation of the activity and how they intend to respond. For the purpose of this paper, the term professional noticing is considered to include a) attending to students' thinking when co-planning instruction and b) deciding how to respond based on prediction of students' thinking (Jacobs et al., 2010).

Noticing in teaching is suggested as "special" and "unnatural" (Ball, 2011, p. xxi). When working closely with a group of experienced teachers, Empson and Jacobs (2008) found that the teachers were unprepared to be responsive to students' thinking. Noticing is thus important for professional development (PD). In order to learn to notice students' thinking, an interrelated and situated set of skills for attending to their thinking is required. As these skills are specialized, a significant shift in how teachers conceptualize their role is required (Empson & Jacobs, 2008). Although not usually

developed in teacher education programs (e.g. Ball, 1993), and taking years to learn (e.g. Empson & Levi, 2011), these skills are learnable with sustained PD (e.g. van Es & Sherin, 2008).

In the MAM project, our focus situates teachers in the authentic work of teaching through *learning cycles of enactment and investigation* (learning cycles). Building on the importance of being prepared to notice students' thinking (Mason, 2011), we investigate how co-planning enables teachers to collectively learn to professionally notice their thinking.

## Methodology

The work is informed by social views on teacher learning and a key part of this perspective is to view learning as it emerges in activities. From this perspective, teacher learning includes developing the ability to engage in particular (ambitious teaching) practices (Lave, 1991) in learning cycles in PD. In the MAM research project, fourteen Norwegian elementary-school teachers worked together in two groups in repeated learning cycles with the aim of learning core practices and principles of ambitious teaching. Each group was guided by a teacher educator (supervisor) and the group structure enabled them to work together in co-planning, rehearsing, co-enacting and analyzing instruction. The participants met for nine full learning cycles over the course of two years, resulting in 18 videotaped cycles. In this paper, the analyzed data material is from video recordings of the co-planning sessions where teachers together with their supervisor collectively planned instruction.

A framework developed by van Es (2011) was used to analyze the depth and analytic stance of noticing in teachers' co-planning discussions. This framework includes an identification of "what is noticed and how teachers reason about what they observe", as well as "a trajectory of development in these two dimensions from Baseline to Extended Noticing" (van Es, 2011, p. 138). For the purpose of this paper, the focus is on *what* teachers plan to notice, using the dimensions from van Es's (2011) "*What* Teachers Notice" (Table 1).

Table 1: Framework for learning to notice student thinking (adapted from van Es, 2011, p. 139)

|  | *Level 1 Baseline* | *Level 2 Mixed* | *Level 3 Focused* | *Level 4 Extended* |
|---|---|---|---|---|
| *What* Teachers Notice | Planning to attend to whole class environment, behavior, and learning and to teacher pedagogy | Primarily planning to attend to teacher pedagogy<br><br>Planning to begin to attend to particular students' mathematical thinking and behavior | Planning to attend to particular students' mathematical thinking | Planning to attend to the relationship between particular students' mathematical thinking and between teaching strategies and student mathematical thinking |

We undertook a three-step analysis. First, co-planning sessions were divided into episodes where the change of topic defined a new episode. Second, each episode was divided into sequences which were coded according to the framework of learning to notice (van Es, 2011), including four levels of noticing – baseline, mixed, focused and extended levels. Each level of noticing represents what the teachers in collaboration with the teacher educators plan to notice. Their attention to whole-class

observations or teacher pedagogy represents lower levels of noticing. At higher levels of noticing, the focus is on particular students or connections between teaching and student learning. Descriptive and evaluative comments represent a lower level of noticing, while higher levels of noticing are characterized by a focus on students' thinking. Lastly, a qualitative in-depth analysis of sequences was conducted to identify and explore examples of noticing on different levels. Sequences and not individual utterances were considered as the unit of analysis, and in this in-depth analysis the sequences were explored using van Es's (2011) framework (Table 1). In this study, a representative example from selected sequences in one co-planning session has been chosen to present our findings from the second and third step of the analysis.

## Findings

We have previously identified three visible ambitious teaching practices that were particularly discussed in learning cycles (Fauskanger & Bjuland, 2019). One of these practices – particularly worked on during the co-planning sessions – was to predict student strategies for finding the number of dots in the quick image (Figure 1). The other two practices were to represent student ideas in the quick image and to aim towards a mathematical goal for the lessons. In the present study, we have been particularly concerned with these three practices when digging deeper into the learning cycles using the learning to notice framework (Table 1) as the basis for our analysis, and thereby investigating the teachers' opportunities for learning to notice students' thinking when co-planning instruction.

Throughout the co-planning sessions, there were few sequences where the teachers appeared to be concerned with themselves and their own practices (baseline noticing, Level 1). There were some examples of mixed noticing (Level 2). The major parts of the co-planning sequences focused on particular students' thinking alone or together with teacher's pedagogy (focused noticing and extended noticing, Level 3 and Level 4), highlighting the teachers' opportunities for engaging in these particular noticing practices and thus providing opportunities for learning them (Lave, 1991).

Two related sequences from one co-planning session particularly focusing on predicting student strategies as one ambitious teaching practice will be used to illustrate the focused and extended noticing in the co-planning sessions. The first example is a brief sequence in the co-planning discussion.

### Focused noticing: attending to a student response related to teaching strategy

The participants have been working on predicting student strategies for finding the number of dots in a quick image, as shown in Figure 1. In the continuation of the discussion, one of the teachers, T4, implicitly recapitulates the particular student strategy $3 \times 5$ and $3 \times 4$, paying attention to the relation between this strategy and a prediction of how students will see 15 in the quick image (39):

| 39 | T4: | I think that some [students] would think that they see 15, right? |
| 40 | S: | Yes, that they see the whole [top row]. |
| 41 | T4: | They know, they have played Yatzy and they know that it's 15, it's nothing more than that. |

The supervisor (S) makes a supportive contribution and expresses agreement, seeing the first row (Figure 1) as "the whole".
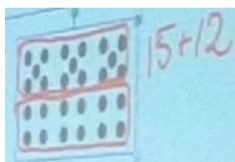
Figure 1: The predicted strategy 15 + 12 represented on the board

We could argue that this brief dialogue on the co-planning discussion illustrates an example of focused noticing (Level 3, Table 1) and planning to attend to particular students' thinking since T4 is paying attention to one predicted student strategy. By predicting that students will see 15 in the quick image as three groups of 5 (3 × 5), T4 also indicates an awareness of the students' familiarity with the game of Yatzy (41). This indicates that T4 might also be knowledgeable about the basis for students' thinking, namely Yatzy.

**Extended noticing: attending to the relation between different student strategies**

In the following, we dig deeper into a longer sequence, illustrating indications of extended noticing (Level 4, Table 1).

| 145 | S: | Yes, but if we look at exactly this picture [points to the quick image where 15 is marked]. If we [discuss] the students who see 15 here. |
| 146 | T2: | But they quickly see 12 on all, yeah. |
| 147 | S: | So you think that [they see] 15 plus 12? |
| 148 | T2: | Yes, they might do that. |
| 149 | S: | Yes [writes +12 on the board next to 15]. |
| 150 | Many: | Yes. |
| 151 | T4: | Then I'm thinking a bit in relation to if you take 15, [you can ask the students] "how do you see 15 here?" |
| 152 | S: | Yes [at the same time frames the three fours]. |
| 153 | T4: | Then this is three times five or five times three, plus and then [you can ask the students] "How do you see 12 here"? |
| 154 | S: | Yes. |

The discussion illustrates how the participants' utterances are related to each other, building on and elaborating on each other's initiatives. Levels of noticing are therefore difficult to code utterance by utterance. However, looking at the whole sequence, we observe how the participants recapitulate predicted student strategies and discuss how these strategies might be represented in the quick image. T4 predicts that some students will see the three fours as 12 (146-148), and this representation is illustrated by the supervisor by circling the three fours as one 12 (152, see Figure 1). The discussion reveals how the participants decide to ask the students how they saw the 15 and the 12 in the quick image. In this sequence of the co-planning discussion, we learn that the teachers and their supervisor attend to the relationship between particular students' mathematical thinking, the relation between predictions of different student strategies and between representing student ideas in the quick image. We observe that the supervisor is the one who points to this relationship between student thinking and the ambitious teaching practice of representing student ideas in the quick image. The crucial role of the supervisor was visible in many co-planning sequences indicating extended noticing.

In the following sequence, the supervisor also challenges the teachers to see the predicted student

strategies in relation to the goal for the lesson by saying:

> S: Yes, fine, but then I think that we have arrived at some possible strategies [points at the predicted student strategies written on the board]. In a way, then the question is: What direction do we want to take [in the instruction]? Because some of the suggested strategies we have might pull us in one direction related to the goal and another [strategy] might pull us in a different direction.

This suggests a focus on particular students' mathematical thinking in relation to the goal for the lesson. It also illustrates the relationship between particular students' mathematical thinking and instruction, and that focused and extended noticing (Level 3 and Level 4) go hand in hand throughout the co-planning sessions.

## Concluding discussion

Noticing is an awareness that enables action (Mason, 2011) and in the co-planning sessions in the learning cycles in the MAM project the predicted student strategies and actions discussed might be such an awareness. These results are interesting and promising. They differ somewhat from studies of teacher noticing in video clubs (e.g. van Es & Sherin, 2008) and in post-lesson discussions in lesson study cycles (e.g. Karlsen & Helgevold, 2019). By using the analytic stance of noticing (van Es, 2011), Karlsen and Helgevold (2019) shed light on teachers' attention to student learning, using notes from classroom observations to identify interactions in post-lesson discussions that can influence teachers' professional noticing. Their findings provide important insights into how such discussions may extend or narrow levels of noticing. In a similar way, our analysis points to the opportunities for learning to notice students' thinking when co-planning instruction. It seems that co-planning in learning cycles invites teachers to learn higher levels of professional noticing. At these higher levels, the participants do not only attend to teacher pedagogy and student behavior, but to particular predicted students' thinking and to teaching strategies building on students' thinking (van Es, 2011). The role of the supervisor seems, however, to be of crucial importance for moving the co-planning discussion to higher levels of noticing.

One element of the MAM project's learning cycles of enactment and investigation, namely co-planning sessions, has been analyzed and we gain insight into how these sessions create learning situations for teachers' collective learning of professionally noticing students' mathematical thinking. In conclusion, while planning to attend to predicted students' strategies, it appears that co-planning sessions in learning cycles are contexts where teachers can learn to size up students' ideas (Ball et al., 2001). When discussing how to respond to students' thinking by representing their ideas in the quick image (Figure 1), the participants also discuss how to base instruction on predicted students' thinking (Carpenter et al., 2015; Empson & Jacobs, 2008) and thus plan to attend to the relationship between particular students' thinking and between teaching strategies and student thinking. Developing the ability to notice and in particular *what* to notice (van Es, 2011) can be learned through scaffolded support and collaboration (e.g. Star et al., 2011). Our analysis indicates that the learning cycles in the MAM project, and in particular the supervisor's active role in these sessions, provide the participants with this type of scaffolded support.

While this study offers some insight into learning professional noticing by co-planning mathematics instruction in the context of PD, more research is needed in this field. Compared to studies of teacher noticing in video clubs (e.g. van Es & Sherin, 2008) and in lesson study cycles (e.g. Karlsen &

Helgevold, 2019), the co-planning in learning cycles seems to invite teachers to learn higher levels of noticing. However, to be able to make clearer conclusions, we need to provide systematic descriptions of each element of the learning cycles and develop understanding of how the different elements enable teachers to collectively learn professional noticing. The role of the supervisor is also yet to be explored. Moreover, studying possible ways in which teachers might learn together how to notice (van Es, 2011) in all elements of learning cycles and how the learning within this project might lead to changes in their classroom practice will also be of importance for future research.

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# Spontaneous mathematical situations with young children

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*Young children are often not considered capable of engaging in mathematical investigations. This is generally connected to the belief that mathematics must be taught, often in relation to what is needed in school. In a collaborative project between researchers and staff members in kindergarten, the features that contribute to spontaneous mathematical situations were explored through a discussion of photos taken by the staff of children at play. Although by their nature, spontaneous mathematical situations cannot be predetermined; staff, preservice teachers, and researchers were able to identify some essential features needed to support children´s mathematical learning through spontaneous situations.*

*Keywords: spontaneous, toddlers, context, mathematics*

## Introduction

In this paper we will explore the features of what we call spontaneous mathematical situations involving toddlers. Spontaneous situations typically occur in everyday life and play situations. They are initiated by the children; hence, the mathematical topics, approaches, examples, or materials cannot be predetermined. Spontaneous situations have many of the hallmarks of free play. Sweden and Norway have a long tradition of focussing on children's initiatives and interests (Ministry of Education and Resesarch, 2017; Skolverket, 2016). In Scandinavia, free play is valued as a learning possibility, while in other countries 'play' is often conceptualised as an activity initiated and controlled by the staff (Bennett, 2005).

However, spontaneous mathematical situations are rarely discussed in the literature. With the pressure for more formal teaching and learning in kindergarten, the importance of these situations may not be recognised before they are lost. Recently, Helenius (2018) discussed how spontaneous situations in Swedish preschools could lead to mathematics learning, which requires decisions about how to act in the moment. Helenius claims that spontaneous situations can be as valuable as planned situations. The Swedish School Inspectorate advocated for more formal teaching in preschools, which illustrates the pressure for more school-like activities in Swedish kindergarten. In the *Norwegian Framework Plan for Kindergartens* (Ministry of Education and Research, 2017), the term 'teaching' is not used. However, this does not mean that there is no tension between preparing children for school by focusing on the mathematics they need to learn and by using children´s play and own interests to support their mathematical curiosity (see Fosse, Lange, Lossius, Meaney, 2018). The *Norwegian Framework Plan for Kindergartens* (Ministry of Education and Research, 2017) emphasises that learning should occur in everyday activities within children's play. The Framework Plan states that 'kindergartens shall highlight relationships and enable the children to explore and discover mathematics *in everyday life* (…)' (2017:34, our emphasis). Furthermore, the Framework Plan (2017, p. 43, our emphasis) identifies the importance of both spontaneous and planned situations:

> The staff shall (…) build on creativity and play and be open to *improvisation* and the children's own contributions, alternate between *spontaneous and planned* activities, support

and enrich the children's initiative (…), support the children's reflections on situations, topics and phenomena and create understanding and meaning together with the children.

Many situations in kindergarten require improvisation, and good improvisation depends on knowledge, preparation, and training (Steinsholt & Sommerro, 2006). Thorsby Jansen (2014) describes a jazz musician who improvises by adapting pre-composed material (Berliner 1994, Jansen 2014). The jazz musician is compared with the pedagogue in kindergarten, who adapts pre-composed material using a repertoire that provides potential alternatives to choose from.

The literature on spontaneous mathematical situation in kindergarten is scarce. Krummheuer (2012) examines unexpected situations that occur in kindergarten and regards improvisation as a condition for early years of mathematical learning for children between the ages of three and ten.

Flottorp (2020) studies how preservice teachers report on engaging with children in spontaneous situations, while reflecting on the situations afterwards. The study reveals that subject knowledge does not automatically leads to ability to follow up the children's ideas. The informants argue that they have limited time to train. In addition, the period with each discipline at campus is short and intense with limited time to reflect and mature the new knowledge. The preservice teachers note the dilemma of being passive or active participants in the children's play. They demonstrate sound pedagogical judgement for the apparently passive stances which often can be labelled as active. For example, giving children time to continue with an activity when it has exceeded the schedule of the day.

Gasteiger (2012) reports from a project in Berlin that includes several kindergartens for a long period. She focuses both on planned and everyday situations. She also highlights the opportunities of free play and everyday situations because such situations often offer ideas and materials to foster mathematical competence.

Because of the limited research on spontaneous mathematical situations in kindergarten, it is difficult to recognize and utilize spontaneous situations to support young children´s engagement with mathematical ideas. It is likely that this will lead to more planned activities and formal mathematical teaching in kindergartens. This seems to be the case in Sweden (Helenius, 2018).

Planned situations with predetermined mathematical topics, goals, examples and materials may leave little space to the toddler's own meaning-making. Such situations might reduce the toddlers' activities to fulfil the goals of the staff. Not all planned situations present these characteristics, but spontaneous situations being initiated and controlled by the children have their interest. Therefore, our research question is:

What do educators notice and recognize as influencing spontaneous mathematical situations with toddlers?

## Perspectives

Some factors are likely to contribute to spontaneous situation becoming mathematical. In order to understand these factors, we draw on the work of Nordin-Hultman (2004, p.92). She analyses how regulations of time and space are restricted in educational environments. Kindergartens have daily schedules and routines. Rooms and premises are organized and intended for specific activities. These regulations decide where children can be at different time of the day, and what kind of material they have access to.

Thus, time and space create restraints on the possibilities the children get to create interesting relationships and find meaningful activities. The regulations may vary from kindergarten to kindergarten, but Nordin-Hultman states that "all educational activities are somehow limited and regulated in time and space" (2004, p. 92, our translation).

Therefore, while considering spontaneous mathematical learning situations, we need to focus on the boundaries affecting the development of the spontaneous activities within time and space. Firstly, Nordin-Hultman (2004) scrutinizes the routines created in kindergarten as a result of time and space regulations, including how these regulations are expressed.

Secondly, she creates a theory of how governance, power, and control work within 'child-centred' businesses. She draws on the work of Bernstein (1983, cited in Nordin- Hultman, 2004) to discuss how the organization of time and space in educational settings are related to power and control. Nordin-Hultman (2004) identifies how time and space in the kindergarten can govern what space they use, which can impact what they can do. She discusses how time boundaries are legitimized in educational environments, even where the children's interest and initiatives are supposed to be highlighted.

Bernstein (1983, cited in Nordin-Hultman, 2004), discusses what it means for the children when different materials are placed in separate rooms. The way of organizing the materials, often determines rules for proper use of the materials. Children may not be allowed to combine different types of material based on their own ideas, and they can become dependent on staff members to gain access to the rooms and equipment for free play.

Nordin-Hultman (2004) highlights that some materials such as water, sand, clay, and utensils seem to provide ideas about experimentation and hypothesis building. From her perspective, dry and moist sand, water tanks, and translucent water hoses all encourage exploration.

## Methodology

The research was undertaken in cooperation with staff members from a kindergarten in the Oslo area, which collaborated with OsloMet in a special program for university teacher education. The focus on spontaneous situations and toddlers arose from initial discussions with the staff members. For this part of the project, our aim is to identify and describe some of the key factors that seem to affect the opportunities for spontaneous mathematical situations to take place.

**Data sources**

The data presented in this paper, is a subset from a larger study investigating mathematics in daily life situations. In this paper we present recordings from a group interview, based on photos taken by the staff members.

The episode occurs outside while children are investigating rowanberries they have picked. It is the staff members and preservice students who chose this situation to the be discussed because it represents a spontaneous situation where toddlers are engaged in mathematical explorations. Three staff members, two preservice teachers and four university teachers participate in a long discussion about the photos. The discussion, lasting for about one hour, are taped, then transcribed.

Our aim is to identify the key factors that the group consider crucial for a spontaneous mathematical situation to take place. We prepared a semi-structured interview guide for the discussions. It is not

followed closely, instead, the participants respond in a variety of ways which advance the conversations into unexpected, but rich areas. In many ways, the discussion results in a *co-constructed* knowledge about the factors that affect the children's possibility to explore the mathematical aspects of rowanberries. *Explorative talk* is a more appropriate term than *interview* to denote the type of discussion that is taking place.

## Transcription and coding

When transcribing the discussions, we indicate the participants by their different roles: pedagogical leader, assistants (one with a diploma-level education), preservice teachers, and university teachers. In the analysis, we refer to these roles only if the utterance is closely linked to a certain role. Otherwise, we describe all the participants as *educators* or *staff members*. The former denotes all the participants in the discussion, including the university teachers. The latter refers to the staff working in the kindergarten: pedagogical leader, assistants, and preservice teachers.

First, we read the transcriptions and made comments on what emerged as interesting. The analysis shows that the staff members repeatedly return to a few themes that we clustered into four categories. 1) Children's interests and the curiosity of the  staff, (2) Material and surroundings, (3) Professional awareness, mathematical competence and verbalisation, (4) Time and number of adults per child.

Our four categories mirror Nordin-Hultmans' theories (2004). *Children's interest and the curiosity of the staff* exemplify how we conceptualize children as individuals who constantly create their own identity. The core of Nordin-Hultmans' work is the *material and environment;* we include *rooms* in this category. Nordin-Hultman work also addresses *professional awareness* explicitly and we included *Mathematical competence and verbalization* which are factors brought up by the staff members. *Time* is an important factor in Nordin-Hultman's work. We interpret the *number of adults* as a part of time since this varies throughout the day in Norwegian kindergartens.

## Description of the episodes with the rowanberries

The starting point is a group of toddlers who are walking in the forest, being amazed by some bright orange rowanberries. They pick some of the berries and take them back to the kindergarten where the berries are placed on a table outside, available for everyone to inspect.

The basis for the discussion is the two photos in Figure 1 and Figure 2, which depict two different situations during the day, involving two different children, both with the rowanberries. Figure 1 depicts an episode in the middle of the day, when a one-year old boy fills a plastic bear mould with berries. Afterward, he recites some number words. When one of the staff members joins him and counts with him, the child quickly loses the interest.

The second situation occurs at the end of the day when another one-year old boy examines the rowanberries in different ways (see Figure 2). A staff member joins him, in the beginning by just observing what he was doing and after a while by repeating the child's actions. The child tastes the berries and discovers that the berries can roll. Then he throws them and finds out that some berries can travel a long distance. Finally, the boy discovers the groove in the bench. He finds out that the small berries can fit in the groove by trial and error. Some berries slip through the groove to the ground below the bench. The situation at the end of the day lasts for a long period of time.

Figure 1. *Rowanberries in a plastic bear mould*



Figure 2. *Rowanberries in a groove on the bench*

## Results and discussion

In the following sections we describe the four categories that emerged as important in the discussion. Each category is linked to discussion extracts and the perspectives of Nordin-Hultman (2004).

### Children's interests and the curiosity of the staff

The educators identify that children's interests need to be a focus in spontaneous situations. One of the educators highlight that it is much easier to achieve good interaction with young children when children´s interests are followed. This applies to all subject areas. Similarly, Gasteiger (2012) considers spontaneous situations to be natural learning situations. From Gasteiger´s perspective, children's interests and meaningful contexts play a crucial role in mathematical learning. In spontaneous situations, learning is to a large extent self-directed and monitored by the interest and motivation of the learner. One staff member said that when children are interested in the situation, they will be excited to keep going. "When the children are very young, you have to follow them where they are, rather than your own interests," the educator says.

To sit down with the children and wonder about what they are focussed on, affects the spontaneous situation and is a core activity, according to the staff members. To accomplish this, the staff members have to be curious about what the children are interested in. "It is so fun, being able to be with the children and forget about everything for a while, just to participate in their play, talk with them, watch them and join in. That is what gives me pleasure about working with children ", one of the staff members says. The staff members mentioned the importance for the adults to be involved in playing and exploring the material with the children. Another theme comes up when the pedagogical leader talks about putting the children's actions into words, to describe what the children were doing.

Thus, for the staff members, one of the key factors is being curious about what the children choose to do. This means, following the initiative of children and watching what they are is trying to find out and accomplish. Just as important, is to restrain from input unless the children want help, which the children often communicate with by body language.

### Materials and surroundings

The child's interest in rowanberries develop because the berries are available to them. One of the preservice teachers points out that the children would not have been drawn to the rowanberries without the availability and access to the materials. The existence of a forest nearby and the willingness of the staff members to allow the children to pick berries and carry them back to the

kindergarten, are other factors that create the context of the spontaneous situation.

Nordin-Hultman (2004) draws attention to the importance of the material being used and how a specific material can capture children's attention, allowing them to work on their differences and relationships. She emphasizes that children should be provided with materials that allow them to conduct experiments and test hypotheses.

The rowanberries provide these opportunities. The boy in Figure 2 uses all his senses when he carries out different experiments to examine the rowanberries. He uses his haptic sense, when he tastes and squeezes the berries, he experiences the meaning of distance when throwing them and the properties of round forms by rolling the berries. Finally, he perceives their size by trial and error, using the groove on the bench to investigate which berries can slip through the groove. These ways of investigating are all *body mind* which is a non-dualistic view of how we experience and learn using both body and mind.

Commercially made mathematical materials are often designed to be used in a specific way. By contrast, natural materials can be used and manipulated in many different ways, none of them predetermined. In the rowanberry situation, the materials are connected to the mathematics. During the discussion, the preservice teachers name important mathematical ideas that the children are engaged in, such as counting, forms, size and distance. In this paper, our focus is on identifying the aspects that influence spontaneous mathematical situations. Thus, we do not elaborate further on the mathematical ideas, mentioned by the preservice teachers.

**Professional awareness, mathematical competence, and verbalisation**

The educators mention verbalization as another key factor in spontaneous mathematical situations. The pedagogical leader emphasizes the importance of describing both what she does, and what the child does:

> When you play with a child, you may say: "Oh, this one rolled" or "now we have tested all those 40, and then they are done" or "that one went far". When you put these actions into words, it becomes a mathematical experience.

The pedagogical leader emphasizes that she recites what she is doing out loud to highlight the important mathematical ideas the child is engaged in. In this way, she also supports assistants and preservice teachers in their professional awareness.

Three of the educators do not have Norwegian as their first language. Due to their differences in upbringings and culture, they have different experiences with verbalization. "In my culture, there is less focus on putting the child's actions into words. (…) You have a strong commitment to language. I see how important this is," one of the educators say.

Additionally, the pedagogical leader stresses that not every situation needs to be described in detail. "You must not talk excessively, with the risk of killing the activity. You must know when to be completely silent," she says. Professional judgement needs to be exercised in order to determine when and how verbalization is desirable.

Part of the professional awareness is how the staff members cooperate. Expressing the same view about the children and having same goal for the work are important factors. The cooperation is expressed by one of the assistants had worked in this specific kindergarten for three months. She summed up her experience, saying: "I have never learned so much in such a short time." The importance of cooperation and having the same goal for the work is evident in situations when one is

engaged in a one- to-one talk with a child. Other staff member must have sensitivity to the situation and take the other children if needed. The same cooperation occurs when an assistant gets get the attention of other staff members by calling out loud: "Now I have a good situation here, and you have to take the other kids". The pedagogical leader explains that after working together for a while, the other staff members recognize a valuable situation, and there is no need to tell them what to do.

The pedagogical leader admits that in spite good cooperation, is not always possible to follow up on in spontaneous situations. They do not always manage to follow up on spontaneous situations. "If you have eight other children to look after and you are alone, it is impossible to focus on only one child," she says. Thus, sufficient staff members are important for following up on spontaneous situations. In order for one staff member to concentrate on one child, someone else must take care of the remaining children – this requires sufficient staff members.

Another factor which influence the possibilities to follow up a spontaneous situation is the time of the day. The staff members say that it is easier during specific parts of the day, for example, at the end of the day which is the case in this episode. At this time, it is often just a couple of children left in the kindergarten, and one single staff member can focus on a single child for a long period.

**Time and number of adults per child**

The number of employees and the way the day is scheduled, are important considerations. The schedules and routines have a strong impact on how the days are organized. How strict the plans are implemented may vary from kindergarten to kindergarten.

It is important that unplanned events can unfold and be given time and space. The ability to provide time for the children's own initiatives when the time frame is tight, is dependent on to the professional competence of the staff members.

Nordin-Hultman (2004) raises the issue of how time and space is organized in educational establishments, especially in kindergarten. She often refers to how the days in Swedish preschools are divided into numerous planned activities and transitions. In heavily regulated Swedish preschools, the children can, to a small extent, decide what to do and influence how things are done. The space for children's free play has also been a matter of concern in Norwegian kindergartens. The pedagogical leader in our study echoed some of the same concerns for the possibility to focus on what the children are interested in.

Therefore, time management and the number of staff members have a significant impact on the development of spontaneous mathematical situations. Although the participants in the discussion recognize the importance of following the children's own interests, there is not always the opportunity to do so, even if the situation appeared to be potentially enriching.

## Conclusions and implications of the study

The study describes some of the key factors that educators consider as influencing the development of spontaneous mathematical situations. Some of them are structural, such as time, space and the number of staff; others are professional characteristics of the staff, such as the ability to cooperate and the curiosity to follow up on the children's own interests.

All this requires respect for the child's abilities to undertake investigations. One of the assistants shares her excitement observing how the children used all their senses and invented different ways to

examine rowanberries. She says: "The children and I are equals, I am not superior to the children regarding the outcome or the answers of the investigations. This makes it fun for the adults."

The study has several implications. Spontaneous situations are based on the children's initiative and can lead to mathematical investigation among very young children, who are already motivated. Such situations occur all day. In contrast, planned situations are often controlled by the staff members, motivated by the goals of the staff, and not initiated by the children's initiative. It is therefore crucial for the staff and the student's teachers to give time and space for spontaneous situations and recognize and respond to them.

The study indicates that the main role of the staff in interaction with very young children might be the *presence* of adults. The aim of the staff is to follow the children with curiosity and verbalisation when appropriate, and interference might not always be the best approach. The role outlined above might appear as a passive one, but it is the opposite: it requires intense presence and close observation.

We need more research to explore the relationship between mathematical knowledge and the educators' ability to follow up toddlers in spontaneous situations. For example, what is the interplay between being present, observing, and the ability to respond to mathematical aspects in spontaneous situations?

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# Milieus of learning in a Norwegian mathematics textbook

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*This paper identifies the types of tasks in a Grade 2 textbook to better understand the potential models children have for their own problem posing, an aspect of mathematical modelling, emphasised in the new Norwegian curriculum. The results show that most of the tasks are set in a pure mathematics or semi-reality context and belong to Skovsmose's tradition of exercises paradigm. Tasks that provide opportunities for children to engage in a landscape of investigation were limited to semi-reality contexts and were much rarer. Suggestions for games and statistical investigations in the textbook provide a real-life context, where children have some autonomy about what they could do. Nevertheless, their presentation in the textbooks suggest that children are unlikely to pose their own questions in these tasks, unless the teacher provides explicit examples of how this could be done.*

*Keywords: Landscape of investigation, exercise paradigm, textbook analysis, problem posing*

## Introduction

In this paper, we identify the types of tasks in a Grade 2 textbook, which could be models for children's own problem posing. Problem posing is an important component of mathematical modelling (English, 2013), emphasised in the new Norwegian curriculum (Kunnskapsdepartementet, 2018). In Norway, the combination of problem posing and problem solving is known as *regnefortelling*, with the nearest English translation being "number story". These stories, written by children, contain mathematical problems and calculations (Botten, 1999; Fosse, 2019; Anderson, 2020) with an expectation that they will draw on children's daily lives. However, previous research on *regnefortelling*, from our wider project (Fosse, 2019; Anderson, 2020), suggests that children in posing their own problems typically set them in semi-reality contexts, which expect one correct answer and need very little investigation to solve. Although problem posing is highlighted as promoting students' mathematical thinking (Christou, Mousoulides, Pittalis, Pitta-Pantazi, & Sriraman, 2005), posing a restricted type of problem may limit these beneficial possibilities.

Therefore, to determine what might influence children to use restricted contexts for their *regnefortelling*, we chose to investigate the models provided in textbooks as many Norwegian mathematics classrooms rely heavily on textbooks. Consequently, our research questions for this paper are: What kinds of tasks are presented in the textbook? Of these, which tasks could potentially support children to connect to their own interests and engage in critical mathematics education issues?

## Theoretical framework

To undertake our textbook analysis, we used Skovsmose's (2001) "milieus of learning". Berisha, Thaçi, Jashari and Klinaku (2013) had previously used milieus of learning in analysing Kosovar textbooks, but we acknowledge that this approach to textbook analysis is uncommon. However, as our focus was on problem contexts, we considered that milieus of learning provided a clear link to critical mathematics education (CME), which is at the heart of our wider project (Lange & Meaney, 2019). The aim of CME is for mathematics to be used to develop critical democratic skills (Skovsmose & Valero, 2002), similar to the emphasis on democratic competence in the new Norwegian mathematics curriculum (Kunnskapsdepartementet, 2018). CME highlights the need to

critique mathematics and how it is used in society (Skovsmose, 2001). *Regnefortelling* provides possibilities for even young children to develop democratic competence using CME themes (Meaney & Lange, 2013). In a small study, Edland (2019) found that Grade 4, Norwegian students could identify many themes related to important decisions, such as the environment, elections, inclusion and exclusion, and social aid. However, the students were unable to connect these themes to mathematics, as mathematics was defined from the perspective of what they had done in school. The mathematics in the themes remained hidden and thus not available as contexts for *regnefortelling*. Edland (2019) suggested that the teacher needed to support students to see the mathematics and the possibility for solving problems connected to these themes.

Milieus of learning were developed by Skovsmose (2001) as part of his philosophy of critical mathematics education. Within milieus of learning, tasks are classified as belonging to one or other paradigms, either sets of exercises, labelled the "tradition of exercises" or the "exercise paradigm", or landscapes of investigation, where learners have the opportunity to pose their own problems around topics. Skovsmose (2001) suggested, "making a critique of mathematics as part of mathematics education is a concern of critical mathematics education. Such concerns seem better taken care of outside the exercise paradigm" (p. 123). In relationship to the two paradigms, traditions of exercises and landscapes of investigation, Skovsmose (2001) identified three types of references: pure mathematics where the tasks were not embedded in a context; semi-reality where the tasks appear related to real-life contexts, but were used only to highlight a particular aspect of mathematics; and real-life contexts where the problems were likely to be similar to ones that students could experience in real life. Critical mathematics education tasks, such as those connected to the themes identified by Edland (2019), would be classified as belonging to the landscape of investigation paradigm with a reference to real life in that children could pose problems, based on their own experiences.

## Methods

To respond to our research questions so that the responses related to our wider project, we analysed the textbook *Multi 2a* (Alseth, Arnås, Kirkegaard & Røsseland, 2011). This textbook series is the most commonly used in Norway (Tokheim, 2015), including at the school where the *regnefortelling* research had been conducted (Fosse, 2019; Anderson, 2020). Grade 2 was chosen because the original collection of *regnefortelling* came from this grade. When the tasks were not self-explanatory in the textbook, we sought further information from the teacher guide. The teacher guide often included suggestions for related tasks, although these were not always closely connected to the textbook tasks.
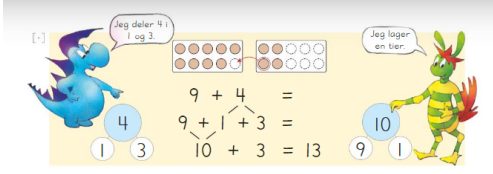
Initially together and then individually, we categorised each of the textbook tasks according to Skovsmose's (2001) milieus of learning. A reliability check by the first author was undertaken, with any differences in classification discussed between the two researchers and a further check of tasks undertaken. To determine trends in the kinds of tasks in the textbook, we defined a task as having a single question or instruction, regardless of the number of examples connected to it. Tasks from the landscape of investigation paradigm usually had a single question or instruction, as the students were expected to determine what they should investigate, whereas tasks belonging to the tradition of exercises paradigm often came with many examples, typically ranging in number from 4 to 15.

In the tradition of exercises paradigm, we identified pure mathematics tasks as those without a context, often only including symbols, such as "8+4=_" or "15=10+__". Tasks with a semi-reality

reference included pictures of potentially real situations, such as of money, which provided a background to the mathematical ideas. The tasks that had real-life references included those where the children had to use their understanding of reality to make sense of the context and respond to the problem (Palm, 2006).

In the landscape of investigation paradigm, children were expected to decide what they should investigate within the boundaries of the task.



Figure 1: A task with both semi-reality and pure mathematics references (Alseth et al., 2011, p. 39)

Although Skovsmose (2001) had used his distinction for tasks used in classrooms, some of which were from textbooks, our analysis suggested that sometimes tasks in *Multi 2A* (Alseth, et al., 2001) belong to more than one type. Figure 1 shows an example which has a semi-reality context, in that it provides drawings of physical things, eggs in egg cartons. However, the exercises required the children to engage with pure mathematics references, with the contexts being of limited importance for completing the task. We, therefore, adapted Skovsmose's (2001) original classification to include a category that had both pure mathematics and semi-reality references (see Table 1). Similarly, there were also some tasks that did not fit easily into Skovsmose's (2001) two paradigms, especially when there was a reference to real life. As a consequence, we added an extra column in between the tradition of exercise and the landscape of investigation paradigms, in Skovsmose's (2001) original table. The adjustments are shown in Table 1 and discussed in the next section.

## Results and discussion

In this section, we present our results from the analysis of the textbook tasks and discuss them in relationship to our second research question, "Which tasks could potentially support children to connect to their own interests and engage in critical mathematics education issues?" Table 1 provides an overview of the number of tasks categorised as belonging to each category, with an example of each type. Given that tasks could involve anything from 1 to 15 examples for the students to complete, these results present a trend in the kinds of tasks presented in the textbook, rather than showing an exact numerical difference between types of tasks.

| | Tradition of exercises | Tasks where students had some autonomy | Landscapes of investigation |
|---|---|---|---|
| References to pure mathematics |  109 |  3 | |
| In between |  13 | | |
| References to semi-reality |  52 |  3 |  10 |
| References to real-life |  3 |  31 | |

Table 1: Milieus of learning in Multi 2A, with examples from Alseth, et al., 2011)

Table 1 shows that the largest number of tasks were in the tradition of exercises paradigm. Of these, most made references to pure mathematics. Similar results have been found in analyses of textbooks in other countries (see for example, Berisha et al., 2013; Zhu & Fan, 2006; Kolovou, van den Heuvel-

Panhuizen, & Bakker, 2011). Only three tasks made references to real-life. These included traditional children's activities, such as dot-to-dot drawings (see Table 1) where the child was expected to draw a line between points that were numbered in a specific order. As these types of activities would be done outside of the school environment, they were considered to belong to the category with real-life references.

However, none of the tasks categorised as belonging to the tradition of exercises paradigm provide models for students to learn how to use mathematics as a democratic citizen (Skovsmose, 2001), rather their focus is on automating students' calculation or counting processes.



Figure 2: Landscape of investigation, semi-reality reference (Alseth et al., 2011, p. 1)

There were very few tasks which were categorised as belonging to the landscapes of investigation paradigm, with most having reference to semi-reality. These types of tasks tended to be the introduction pages to each chapter, which had a picture showing a range of activities, although not every child may have experienced all of them (see Figure 2). It was indicated in the teacher guide that the children should be encouraged to pose their own mathematical problems about what they were seeing in the pictures. It is this expectation of the children posing their own problems, which classifies this task as being an example of a landscapes of investigation.

Although these tasks could connect to children's own interests, none provided opportunities to develop competencies connected to being democratic citizens. Therefore, even if a task did introduce students to a landscape of investigation, it did not necessarily mean they would have opportunities to engage with critical mathematics education.

As noted in the previous section, there were some tasks that did not seem to fit into either of the two paradigms, suggested by Skovsmose (2001) and so were placed in-between (see Table 1). Although these tasks required students to respond in specific ways, children had some autonomy to make decisions about what they could do. These tasks were often: games, where the students were expected to use a spinner or calculate the total from throwing two dice to move around a board; free-choice measurement tasks; or designing and collecting responses to survey questions. For example, in some of the measurement and survey questions in the textbook, the students could make choices about how

many responses to get and how they would collect this information. Sometimes, the textbook provided a choice of possible responses. In the middle column in Table 1, different types of fruit were suggested for inclusion in a bar graph. The task was considered as making reference to semi-reality. More open tasks provided the students with more autonomy. For example, in some measurement tasks, children could decide what to measure and, in some games, they could decide their own strategies. This autonomy suggested that these tasks could not be classified as belonging to the tradition of exercises paradigm. Yet, the tasks did not allow the students to pose their own questions and so they could not be classified as belonging to the landscapes of investigation paradigm.
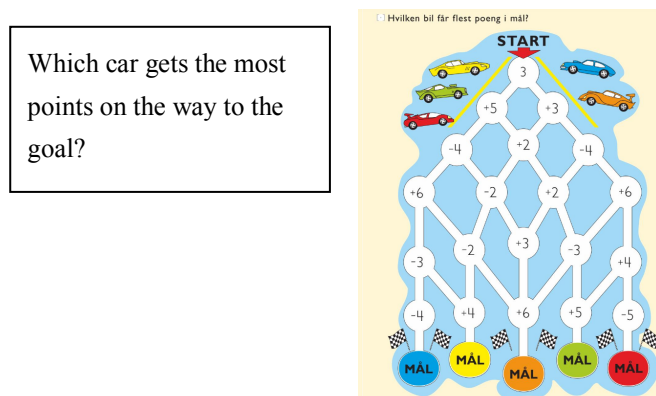


Figure 3: An in-between task with reference to real-life (Alseth et al., 2011, p. 53)

The teacher guide often suggested that the games could be connected to children developing their own *regnefortelling* or to making up their own calculations. One such suggestion occurred in relationship to the car game on page 53 of Alseth et al. (2011) (see Figure 3). It was a task that we classified as real life, because it could be played like a board game, that the children might have played at home. However, it fell in-between the tradition of exercises and the landscape of investigation, because it allowed the children to make some choices about where to move from and to. In the teacher guidebook, it is suggested that the calculations that the children had to do in the board game could be a basis for them writing their own *regnefortelling*, perhaps with the implication that the students could pose their own questions about which path would produce the highest total. Nonetheless, this did not provide explicit links to developing students' democratic citizenship skills.

The results of our analysis suggest that most of the textbook tasks were focused on students' automating their calculation processes. None provided models of problems that could support students to see how mathematics was linked to democratic citizenship. This could explain why in earlier studies in our project (Fosse, 2019; Anderson, 2020), the students' *regnefortelling* referred to mostly semi-reality contexts and were not connected to important themes related to critical mathematics education issues.

Although the tasks linked to the landscapes of investigation paradigm could be related to children's own interests, it was not easy to see how they could be adapted to include critical mathematics education themes. This is because the existing tasks mostly made references to semi-reality, where connections to CME themes were not easily identifiable.

In contrast, the statistical investigations and games seemed to offer possibilities for adapting them to include CME themes. For example, in the car game (Figure 3), the different coloured cars do not have the same possibility for gaining the largest total and so this game opens up opportunities to discuss

issues of fairness that could be connected to democracy. From this, a landscape of investigation task could be developed in which students could pose problems about fairness/unfairness in the game, with the teacher's support, as suggested by Edland (2019). Similarly, games that using a spinner could also be adapted so that students could investigate issue of fairness. These teacher-adapted textbook tasks could provide models for students to produce *regnefortelling* in which they pose and solve questions about the fairness or otherwise of games, using their calculation skills. This would then provide opportunities to discuss how rules in society can be unfair for some groups.

## Conclusions

In this paper, we categorised the tasks in a Grade 2 textbook. Although it may be interesting to analyse other textbooks in the series, the purpose of our study was to provide background to a wider project (Fosse, 2019; Lange & Meaney, 2019) about supporting Grade 2 students to produce *regnefortelling* connected to CME themes. Skovsmose's (2001) distinction between tradition of exercises and landscapes of investigation paradigms provided an appropriate analytical framework for the analysis, although some adjustments needed to be made to reflect the kinds of tasks in the textbook.

The majority of tasks belonged to the tradition of exercise paradigm with reference to pure mathematics. The textbook tasks that seemed to have the most potential for allowing students to connect to their own interests and important CME themes were those classified as making reference to real-life and which gave students some autonomy over what they did. The teacher would still need to support students to use the potential in these textbook models to produce *regnefortelling* connected to CME themes. This is something we will investigate in the next part of our wider research project.

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# Investigating data collection methods for exploring mathematical and relational competencies involved in teaching mathematics

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*This paper examines the methodological issues involved in investigating mathematical and relational competencies relevant to teaching mathematics. In the study, 14 mathematics teachers were asked to reflect on the teaching and learning of mathematics based on a mathematics lesson shown in a video sequence. These reflections were documented in three different ways: some teachers were interviewed individually, some were interviewed in focus groups and some wrote individual reflections on paper. The empirical materials from these three different types of documentation were analysed using the same framework. The results of the study indicate that mathematical and relational competencies came to light mainly in the individual and the focus group interviews. However, this may be due to the analytical framework used, and another framework may be better suited to analysing individual writings.*

*Keywords: interviews, the Knowledge Quartet, mathematical competence, methodology, relational competence*

## Introduction

This paper examines the methodological issues involved in investigating mathematical and relational competencies relevant to teaching mathematics. This methodological interest arises from a proposed study focused on inclusive mathematics education. Of particular concern are the different data collection methods connected to triangulation. The term 'triangulation' often refers to using different data collection methods to capture the core of a study or to validate results (Bryman, 2016). However, triangulation in the sense of using a variety of methods to collect data can also contribute to a wider diversity of findings and help to distinguish the essence of different aspects of results (Skott, Larsen, & Østergaard, 2011).

The rationale for investigating both mathematical and relational competencies for teaching mathematics in a study on inclusive mathematics education is based on previous studies showing that *relational leadership* promotes inclusive mathematics education (Schmidt, 2015). On the basis of a study on inclusive mathematics education, Roos (2019) claims that inclusive mathematics education requires the teacher to possess not only mathematical, didactic and pedagogical skills but also relational competencies in seeing each student as a person and understanding their needs. Thus, besides mathematical and didactic competencies, relational competencies are central to the mathematics teaching profession. Relational competencies have been shown to have a significant impact not only on students' social but also on their content-specific development (Aspelin, 2017; Hamre & Pianta, 2005). Schmidt (2015) describes relational leadership in terms of safeness, whereby students who are in classrooms where teachers practice relational leadership are comfortable in taking risks and in giving unsure answers to questions. In mathematics education, a feeling of safeness is essential because mistakes and errors can be key factors in enhancing learning (Fredriksson et al., 2017). Furthermore, teachers who practice relational leadership have high and positive expectations of the students (Schmidt, 2015).

Even though there are studies showing that mathematical and relational competencies influence inclusive mathematics education (Roos, 2019; Schmidt, 2015) and thus students' content-specific development (Aspelin, 2017; Hamre & Pianta, 2005), I have not found any studies on how mathematics teachers' mathematical and relational competencies for teaching are related. This will be investigated in a study of inclusive mathematics education in primary schools. However, in this paper, the focus will not be on this study as a whole but on *how* to document teachers' mathematical and relational reflections on mathematics teaching within the study. The research question for this methodological paper is: What aspects of teachers' mathematical and relational competencies for teaching mathematics are brought to the surface by the use of different data collection methods?

## Theoretical framework

The framework to be used in the analysis is based on the Knowledge Quartet (KQ) (Rowland, 2013; Rowland, Huckstep, & Thwaites, 2005) to which relational competencies are combined as a network strategy (Bikner-Ahsbahs & Prediger, 2010). The KQ is a conceptual framework of how mathematics subject matter knowledge and pedagogical content knowledge come to action in mathematics teaching. The framework consists of the four categories *foundation*, *transformation*, *connection* and *contingency*. These four categories were derived empirically from observations of student teachers during their school-based teacher training. *Foundation* relates to the teacher's mathematical theoretical background and the mathematical knowledge possessed by the teacher, irrespective of whether it is used in teaching. Foundation underpins the pedagogy used and makes it possible for the teacher to deliberately use mathematical terminology, be aware of the purpose of the lesson, identify errors, etc. Thus, the first category is foundational to the following three, which address the actual mathematics teaching. *Transformation* implies how a teacher's foundational knowledge is transformed into actions when teaching, for example, when demonstrating mathematical content through explanations, chosen examples, instructional materials and mathematical representations. *Connection* implies the connections made by the teacher concerning the coherence of the teaching across shorter or longer timespans, e.g. connections between procedures, concepts and sequenced examples. Connections also include the ability to anticipate complexity. *Contingency* relates to the teacher responding to students' ideas that it would not be possible to plan for in advance, and to deviations from the intended actions in a planned lesson that still make the teaching mathematically meaningful for the students. Each category consists of a number of methodological codes which are to be used to carry out the analysis. The relational competencies that are combined with the KQ are based on Aspelin's (2017) two-dimensional perspective on relational competencies, with one social and one inter-human dimension. The social dimension concerns the teacher's actions on the classroom level, for example, regarding the classroom climate and the relationships between students. The inter-human dimension concerns the teacher's actions on the student level, 'recognising, facing, and responding to the student's situated needs' (Aspelin, 2017, p. 50) as a unique individual. The combination of the two theoretical frameworks (Aspelin, 2017; Rowland, 2013; Rowland, Huckstep, & Thwaites, 2005) implies that the social and the inter-human dimensions in the analysis are combined with each of the four categories from the KQ. This two-dimensional perspective makes it possible to analyse relational competencies regarding teachers' interactions with groups of students as well as with individual students.

## Method

The exploration of different data collection methods presented in this paper is intended as a pilot study to precede the study on inclusive mathematics education mentioned above. The pilot study was conducted to validate the data collection methods to be used in the full-scale study.

### Selection of informants

Purposeful sampling (Bryman, 2016) was used in the pilot study. The sample consisted of 14 mathematics teachers who were studying different mathematics education courses at the second-cycle level. In connection with one of their course seminars, they were asked to participate as informants in the pilot study. The informants had from 5 to 37 years of experience teaching mathematics. Furthermore, they taught mathematics to different grades, from Grade 1 to Grade 12. One of the teachers also had a postgraduate diploma in special education needs in mathematics. Thus, the informants represented a wide range of teaching experiences and were thus expected to serve as a rich dataset. The ethical codex from the Swedish Research Council (2017) was followed.

### Implementation

The informants met once with the researcher (the author of this paper). At the meeting, they were shown a video sequence from a mathematics classroom. The video sequence was taken from the TIMSS video study[1] showing a male mathematics teacher teaching linear equations in a Grade 8 classroom. In the sequence shown to the informants, the students were to collaborate in small groups. On the basis of the task the students worked on, they were to construct a value table out of a given function and mark the coordinates in a coordinate system where a straight line would appear. The chosen sequence was two minutes long and displayed a moment where the teacher stays by a group of four students working jointly on a task. In the sequence, the teacher identifies errors in their answers and starts to interact with the students over the task. The video was chosen because it is from an authentic classroom, rather than having been produced to convey a certain message or instructional method. Furthermore, the selected sequence contains situations where the teacher talks to the students both as a group and as individuals. The informants were shown the video sequence three times. After that, they were asked to answer open-ended questions related to the video sequence they had been shown. The first question was what they had noticed in the video sequence. Next, they were asked to give examples from the video sequence of where they thought the teacher's actions supported or counteracted the students' learning of mathematics. The questions were open-ended to make it possible for the informants to give answers related to both mathematical and relational (or other) competencies. For example, the informants could reflect on how the teacher represented, structured, sequenced or explained the mathematical content in the sequence, as well as on how the teacher adapted his teaching to the group or to the needs of one single student.

### The different data collection methods

Because of the research question of this paper, the informants were divided into three groups (A, B, C) when they were asked to answer the open-ended questions. Different methods of collecting data

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[1]The Third International Mathematics and Science Study (TIMSS) 1999 Video Study has made the videos public and available for education researchers on this website: http://www.timssvideo.com/us1-graphing-linear-equations

were used in the three sub-groups to investigate what aspects of teachers' mathematical and relational competencies for teaching mathematics were brought forth by the use of different data collection methods. The informants in group A were to answer the open-ended questions individually in writing, those in group B were to answer them individually in an interview and those in group C were to participate in a focus group interview. The open-ended questions were the same regardless of the group, but in the interviews, supplementary questions could be asked by the researcher if needed. In addition, in the focus group interviews, the informants could ask each other supplementary questions or elaborate on each other's comments.

**Analysis**

The empirical material from each group (A, B, C) was transcribed and merged as a whole, based on the three data collection methods used. The rationale for this was the focus on each data collection method rather than on each informant as an individual. A two-step qualitative deductive content analysis (Bryman, 2016) based on the previously presented theoretical framework with pre-formulated categories was used to analyse the empirical material. In the first step, the codes from the KQ were used to categorise the data as Foundation, Transformation, Connection or Contingency. In the second step, utterances focusing on relational competencies within these four categories were identified. The criteria for identifying relational competence were utterances containing considerations of the teacher's interactions with individual students (individual level) or groups of students (social level) that were connected to their needs in the current teaching situation. Through the second step, subcategories emerged within the categories based on the KQ. For each of these, the reflections were also categorised based on whether or not they were connected to relational competencies, resulting in two aspects within the informants' reflections.

# Results

In this section, the results of the pilot study are presented. The results are illustrated with excerpts. First, the informants' reflections, categorised based on the four categories of the KQ, are presented. Each category is also divided based on the utterances of the informants, in terms of whether or not it is connected to relational competence or not. Then, the two emerging aspects *the mathematics teaching aspect* and *the relational mathematics teaching aspect* are connected to the three methods of collecting data.

**Informants' reflections related to the Knowledge Quartet and relational competencies**
The two different aspects are presented below as they relate to the categories in the KQ.

*Foundation:* Sections where the informants identify that the teacher in the video possesses mathematical knowledge were categorised as foundation. One example is how the informants write or talk about instances when the teacher in the video sequence identifies errors in a student's solution. One informant describes the teacher in the video sequence as not explaining the misunderstanding, and the utterance below shows the informant identifying the mathematical knowledge possessed by the teacher that came into play.

Written answer:  [The teacher] points out errors without explaining misconceptions. […] When he [the teacher] took the number 0 and put it in [the function]. He [the teacher] gives the answer.

Several answers in this category also emphasise that the teacher takes 'too much space' and that there was a lack of engagement with the students and that their arguments were not asked for or waiting for. The informants comment that the teacher in the video sequence did not investigate situations sufficiently or wait to hear the student's reasoning. For example, one informant talks about how a student had put coordinates wrongly in the diagram and the teacher could have asked for the student's approach to the solution. The utterance below also shows the informant's reflections on the mathematical knowledge possessed by the teacher of linear equations.

> Individual interview:    [The teacher could have asked] if there is something [wrong] with this line. Is it like you expected? No? It would have been a straight line, but now it curves like this. I suspect some points are wrong.

***Transformation:*** Sections where the informants give examples from the video sequence concerning explanations, choices of examples, instructional materials or mathematical representations were categorised as transformation. One informant suggests that the teacher in the video sequence should exemplify how to solve a task, break down the task into smaller pieces or exchange the numbers in the task with similar ones.

> Written answer:    [The teacher could] model approaches to solutions. Give part of a task to the students that they try to solve, for example, another line, or focus the incline or intercept.

Another informant mentions that if the teacher gives the same explanation to different students without adapting it to the students' different needs, this constrains the students' learning in the mathematics lesson.

> Individual interview:  Firstly, I turn to one student. [Shows with her body how the teacher did this.] I do not discuss mathematics, only explain exactly, this is how you are supposed to cope, this is how you do it. Then I turn to the next student and explain to him in the same way.

Thus, the informants' answers diverge in the sense of what the transformation implies; some answers are related to the mathematical content while others are related to adapting teaching to students' different needs.

***Connection***: Answers focused on connections between procedures, concepts or sequenced examples were categorised as connection. For example, one informant describes how the teacher in the video sequence could have compared different procedures to better support students' learning.

> Written answer:    [The teacher can] contrast successful/less-successful ways of solution.

Another informant describes how the teacher could have connected to earlier mathematics lessons as well as to students' solutions.

> Focus group interview: Nevertheless, I think, I suppose they [the students] had been working with the mathematics content before this [lesson], that they had looked more at the graph before they started to draw. One could have done that, [looked] at the ones [graphs] that the students drew as well. What are the graphs saying? How do they slope? And strengthen the mathematical content.

Thus, the informants' answers diverge in terms of what the connections highlighted are related to. Some answers are related to the mathematical content, while others are related to the students' expressed mathematical understanding.

*Contingency:* Answers focused on responding to students' ideas not planned for in advance or deviations from the intended actions that were still mathematically meaningful for the students were categorised as contingency. One informant talks about the teacher's bad manner when finding errors in students' solutions, which according to the informant constrains students' learning.

> Written answer: He [the teacher] is in a hurry and talks a bit patronisingly to the students when they are wrong and do not do what they are supposed to do.

Another informant describes how the teacher could have asked for and encouraged students to give responses that might have provided an understanding of how and why the students did what they did or responded as they did. The utterance below refers to a situation when a student was browsing a book after the teacher had started the lesson by telling the students to put away their books.

> Focus group interview: I perceived him [the student] as trying to find clues in the book. He [the teacher] could have asked him: what do you think? What is it that you do not understand?

The informants' answers in this category diverge in the sense that some answers focus on the teacher's different approaches to students and whether his brusque manner in response to the students' unexpected way of expressing knowledge would make the students withdraw. Other answers focus on how a teacher can invite and encourage students to respond with their questions and reasoning.

To summarise, the results show that the informants' answers diverge into two subcategories within each KQ category. The first subcategory reflects the mathematical content of the lesson and the mathematical competencies of the teacher in the video sequence. These answers can together be categorised as *the mathematics teaching aspect*. The second subcategory reflects the relational competencies of the teacher in the video sequence and how these competencies may – or may not – strengthen the mathematics teaching and thus students' possibilities of learning mathematics. These answers can together be categorised as *the relational mathematics teaching aspect*.

**Different methods of collecting data provide diverse information**
The two subcategories presented above are not equally distributed in the empirical material derived from the three different data collection methods. The *mathematics teaching aspect* emerges mostly in the written answers. In the written documentation, it is described how the actions of the teacher in the video sequence are connected to mathematical content and how instructions or materials can or cannot support students' learning in the mathematics lesson. Several written items of documentation were related to the students' actions as well. The *relational mathematics teaching aspect* emerges in all of the individual and focus group interviews and in some of the written documentation. In these items of documentation, it is described how the teacher in the video sequence needs to take students' perspectives on the mathematical content into consideration.

## Discussion and conclusions
The results show two subcategories regarding how the mathematics teacher in the video sequence acts in a way that supports or counteracts students' learning in the mathematics lesson. Among the

utterances of the informants, two interrelated aspects emerge, both of which are connected to the KQ. The connection involves how mathematical and relational competencies come to action in mathematics teaching. The first aspect is the *mathematics teaching aspect*, where the reflections are grounded in the mathematical content of the lesson. The second aspect is *the relational mathematics teaching aspect*, where the reflections are grounded in both the content of the lesson and in the students' views of and expressed knowledge of mathematics. In this aspect, students' different educational needs are often emphasised. The two aspects are not opposites and indicate that mathematics is foundational for both of them, as it is for the KQ. However, in this study foundation as a possessed knowledge seems to be especially challenging to bring to light, and may require additional interview questions.

The *relational mathematics teaching aspect* is only visible to a limited extent in the written documentation. However, this may be due to the analytical framework used, when another framework may have been better suited to analysing individual writings. Although the utterances of the informants must be seen as examples from a small sample, the results indicate the importance of using different data collection methods to enable different aspects of mathematical and relational competencies for teaching to come to light. If only written documentation had been used, almost no relational competencies would have been made discernible. That could have led to misinterpretations, as becomes clear when the two other types of documentation are available. This is in line with the argument of Skott et al. (2011) for using different methods, as 'they may shed light on decidedly different forms of practice' (Skott et al., 2011, p. 34). Accordingly, different data collection methods may yield diverse types of information. To conclude, an implication of this is that the full-scale study should use individual and focus group interviews as data collection methods to address both mathematical and relational competencies for teaching mathematics.

Several limitations of this pilot study should be taken into consideration in the design of the full-scale study. The different data collection methods can be elaborated to a greater extent to facilitate the uncovering of teachers' mathematical and relational competencies. However, this pilot study indicates the importance of using different data collection methods, as they may capture diverse aspects of mathematical and relational competencies for mathematics teaching. Furthermore, if more than one person had been involved in the process of analysis, inter-rater reliability could have been measured to provide a higher degree of consistency in judgements about categorising data (Bryman, 2016). Lastly, another limitation is the lack of previous research regarding the operationalisation of relational competence specifically connected to mathematics.

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# Realization of the mathematical signifier 25 × 12

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*This paper identifies how a mathematical signifier "25 × 12" is realized through the practice of multiple solution strategies when teaching multiplication. This case study is conducted in a fifth-grade classroom where the teacher employs a context-based explorative teaching approach. Analyses of data from teaching sessions show that the teacher prioritizes effective mathematical communication through representations, gestures and visual mediators. The result shows that the students attained a set of mathematical realizations of the signifier by examining the discursive equivalence established through mathematical communication, gestures, multiple connections and use of visual mediators in the discourse. Possible implications of these findings are discussed.*

*Keywords: Signifiers and realizations, mathematical communication, visual mediators, gestures, multiplication*

## Introduction

For some decades, researchers in mathematics education have worked extensively to develop concepts, frameworks and theories to enhance teaching-learning activities (Adler & Ronda, 2015; Ball, Thames, & Phelps, 2008; Radford, 2003; Sfard, 2007, 2008). The recent years have witnessed that a discourse perspective for making sense of classroom processes has evolved and earned major attention (Adler & Ronda, 2015; Adler & Sfard, 2016; Sfard, 2008). A discourse perspective, related to an interpretive framework for the study of learning is called commognitive, combining the terms cognitive and communicational (Sfard, 2007). In a commognitive framework, Sfard (2007) considers that a mathematical discourse consists of four interrelated discursive features: word use, visual mediators, routines and narratives.

Multiplication is one of the four fundamental operations in arithmetic and mastering it is important to build confidence for higher-level mathematics. For some decades, research has highlighted four models that influence students' understanding of multiplication: equal groups, (rectangular) arrays, rectangular area and multiplicative comparison (for more details, see Greer, 1992). In a Nordic context, a recent Swedish study has investigated two students' multiplicative understanding for multi-digits and decimals (Larsson, Pettersson, & Andrews, 2017). Individual interviews were conducted over five semesters during grades 5-7. The authors found that the two students revealed a robust conceptualization of multiplication as equal groups, but they had difficulties in connecting calculations to models for multiplication.

Developing inaccurate and inefficient counting methods and encountering problems in learning multiplication tables may promote mathematics difficulties (Wong & Evans, 2007). While different models have been adopted in teaching multiplication, these authors suggest that the "use of concrete materials, pictures, diagrams, and discussion increases students' familiarity with the process" (Wong & Evens, 2007, p. 89). This viewpoint agrees with Sfard's commognitive framework which considers visual mediators as "visible objects that are operated upon as a part of the process of communication" (Sfard, 2008, p. 133). Arithmetic deals basically with calculations involving numbers. Multiplication

is one of such calculations that is governed by well-framed procedures. Sfard (2008) considers such calculations "as a discursive sequence built according to well-defined rules that, once uttered or written, counts as a confirmation of the discursive equivalence of two numerical expressions" (p. 52).

In the present study, we consider the two numerical expressions, twenty-five times twelve (or, in written symbolic form, $25 \times 12$) and three hundred (300). These two expressions are discursive equivalents. Discursive equivalence means that we can use these two numerical expressions interchangeably for any communicational purpose. While we try to objectify arithmetical discourse, thereby multiplication, we "need to account for the fact that two different symbolic or verbal strings, count as exchangeable" (Sfard, 2008, p. 52). $25 \times 12$ is a signifier, which is a primary object, used in communication. Signifiers are words or symbols that work as nouns in communication, and a single signifier may have several realizations. Sfard (2008, p. 301) defines a realization as "a procedure that pairs a signifier with another primary object or the product of such procedure". She suggests that the process of achieving the realization in question can be visual (that includes verbal, iconic, concrete and gestural) and vocal.

The major focus of the present study is to explore mathematics communication and paint generic pictures of processes and acts like visual mediators and gestures that arise from mathematical discourses in a classroom. To be precise, we will address the following research question: How is the mathematical signifier $25 \times 12$ realized in a discursive classroom when teaching multiplication in grade 5 in a Norwegian school? To seek an answer to this question, we will analyze two episodes that focus on the use of examples in teaching multiplication to identify students' signs of discursive development. A particular focus will be made on the realizations of the mathematical signifier $25 \times 12$.

## Methodology

This study is a part of a project conducted by the Mathematics Education Research Group (MERG)-2018 at the University of Stavanger. The project was conducted in two fifth-grade classrooms. The mathematics teacher of these classes employed a context-based explorative teaching method. The primary goal of the project was to focus on the mathematical discourse developed in the classroom during the teaching-learning process. The present study focuses on how a signifier $25 \times 12$ was manipulated to achieve its realizations. It has used the qualitative approach with observation, exploration, and interpretation as tools for understanding these discourses. Kieran, Forman, and Sfard (2001, p. 1) see a clear shift from the classical *background-method-sample-finding-discussion* structure to "a distinct and relatively new type of research" in mathematics education which they term as a discursive approach. As a part of this approach and using the commognitive perspective (Sfard 2007, 2008) as our analytical framework, we will focus on the student interactions in small groups, their two-way communication with the teacher and the shift in the discourses as they are challenged with the mathematical tasks.

The empirical material of the MERG project includes observations of 16 teaching sessions captured as video recordings in which all sessions were transcribed. For this study, we have selected one teaching session in which the students applied different solution methods to find the product of two numbers. The task, written on the whiteboard, was to compute $25 \times 12$. The teacher asked her pupils to compute this with different methods. From this teaching session, we have selected two different episodes that showed a clear shift in the discourse (Adler & Ronda, 2015). This shift could be a task

change, a shift in focus from one student group to another one, or a shift in the teacher's presentation. Here, it implies a shift in the multiplication method. These two episodes were then thoroughly analyzed, using the commognitive framework with a focus on visual mediators, revealing the realization of the mathematical signifier embedded in those episodes. As we chose to focus on two episodes of a specific teaching session from 16 such sessions, owing to our methodological choice, we might have missed the opportunity to take into account the mathematical discourses that were developed in other teaching sessions. However, the selected episodes illustrate the specific realizations of the mathematical signifier $25 \times 12$.

## Analyses and results

The first episode accounts for gestures, speech and multiple connections of the realizations, while the second episode focuses on the role of visual mediators in achieving the realizations of the signifier. The analyses will focus on the subjects of the discourse (teacher and/or student/s), but also on the signifier $25 \times 12$, searching for the signs of realizations of the signifier in the tasks and the utterances within or across the episodes. These realizations are achieved through different activities. Both the teacher and the students use symbolic representations across the episodes to perform the mathematical operations. Although the students uniformly use a dot ($\cdot$) to represent multiplication, the teacher often uses a cross ($\times$) while she uses a dot ($\cdot$) sometimes. Despite this ambiguity, no signs of confusion were noticed with the students.

### Episode 1. Gestures, speech and multiple connections

*Sequence 1: The interplay between gestures and speech*

The teacher is here visiting two boys, Tor and Peter who have formed a group. The dialogue shows a verbal conversation between the teacher and Peter since Tor was silent here. The teacher is looking at Peter's notebook, following Peter's strategy of solving the multiplication problem.

| | | |
|---|---|---|
| 069 | Teacher: | 25 times 4 plus 25 times 4 plus 25 times 4 (Moves finger pointing to each term of $25 \cdot 4 + 25 \cdot 4 + 25 \cdot 4$ that the student has written) |
| 070 | Tor: | [Yeah] |
| | Peter: | [Yeah] |
| 071 | Teacher: | Yeah. (3s) Why should (unknown text) times 12 there? (Points to what the student has written from term to term) |
| 072 | Peter: | (Unknown text) (Shows something in the exercise book) |
| 073 | Teacher: | Umm (.) This is not multiplied like this here, 25 times 4 plus 25 times 4 plus 25 times 4. Whether you can think of a different way of writing this? (Moves index finger steadily from left to right and then from right to left pointing $25 \cdot 4 + 25 \cdot 4 + 25 \cdot 4$) |
| 074 | Peter: | 25 times 4 times 3 (Looks at the teacher) |
| | | In the conversation that followed, the teacher asks Peter to write what he said. He writes $25 \cdot 4 \cdot 3$ and the teacher asks if this could be written differently. |
| 080 | Teacher: | Can this be written as a different multiplication than this? Only 2 factors. These here are 2 factors, and these here are 3 factors. (2s). Could it be thought that there is a third way to write the same multiplication? (Peter looks into his book) |

The teacher's utterances (069, 073) are the examples of the use of mathematical objects in the discourse. Although the task is not a descriptive multiplication simplified with the use of the symbols, the correct interpretation of the operation (multiplication here) with the symbols gives the first impression of the use of mathematical objects. As the teacher explains the procedure, she moves her index finger from left to right, pointing each term Peter has written (see Figure 1). Peter looks attentively at what is pointed. The student has written $25 \cdot 4 + 25 \cdot 4 + 25 \cdot 4$.

The teacher (073) utters, *this is not multiplied like this here, 25 times 4 plus 25 times 4 plus 25 times 4* as she moves her index finger steadily from left to right and then from right to left again pointing to the expression $25 \cdot 4 + 25 \cdot 4 + 25 \cdot 4$, indicating that she was talking about the whole expression and not only the specific terms. Peter had written the expression correctly but had made a mistake in his calculation. After the explanation from the teacher, he makes correction in his calculation. The teacher (080) refers to factors of 300, expressing and pointing that *these here are two factors, and these here are three factors*. The pointing gestures the teacher used were coordinated with the speech.
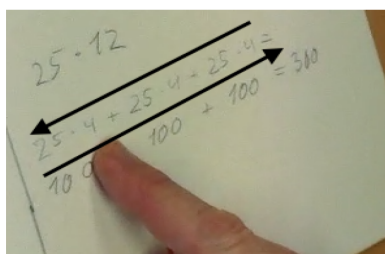


Figure 1: The teacher moves her index finger steadily from left to right and then from right to left again pointing the expression.

*Sequence 2: Establishing discursive equivalence through multiple connections*

The teacher goes around and interacts with her students about their methods. A student (Mia) has performed the multiplication as $25 \cdot 10 + 25 \cdot 2 = 300$. In the communication that followed, the teacher refers to this as Mia's method.

| 140 | Per: | First, it is 10 times 12 (which) is 120. And then it is 10 times 12 again. It is 120. 120 plus 120, it becomes well 240. And then we have what: (.)5 again. Then we have 5 times 12 (.) which becomes well thus 60. 60 plus 240 becomes 300. |
|---|---|---|
| 141 | Teacher: | Was it more difficult than Mia's (method)? |
| 142 | Per: | Can (be) more difficult than Mia's (method). |
| 143 | Teacher: | How is it (the solution) different from Mia's and how is it similar to Mia's? |
| 144 | Per: | We split differently. For example, I split 25 while she split 12. |

In this sequence, the teacher is moving a step ahead. She wants to compare the methods (141) employed by two students: *How is it different from Mia's and how is it similar to Mia's?* (143). Per answers that they split the rectangle differently (144). He wrote 25 (the length of the rectangle) as 10 + 10 + 5 and multiplied each number with 12 (the breadth of the rectangle). He, finally, summed up the products to get 300 (140). Mia, on the other hand, split 12 (the breadth of the rectangle) into 2 numbers as 10 + 2 and multiplied each number with 25 (the length of the rectangle). She then summed up the products to get 300.

The correct symbolic representations for multiplication of the factors are the manipulations of mathematical signifiers, i.e., symbols. The teacher talks about four different symbolic strings: 25×10, 25 × 2, 25 × 12 and 300 and asks if the students consider that the first two together are the same as the third and/or the fourth. Making use of students' solutions, the teacher tries to make multiple connections and account for the fact that 25 × 12 and 300 are discursive equivalents and thus, count as exchangeable.

**Episode 2. Visual mediators**

In episode 2, the teacher and a student (Mads) both use visual mediators to demonstrate the multiplication of 25 × 12.

| 246 | Teacher: | So, you explain how you cut the grid, Mads! |
| 247 | Mads: | How I split? (Low voice) |
| 248 | Teacher: | You did it yesterday |
| 249 | Mads: | Yeah:: I thought 4 grids (Looks at the teacher) |
| 250 | Teacher: | (Unknown text) (Very low voice) |
| 251 | Mads: | Um: instead of 25 times 12, I took 50 times 6 (Draws the figure below the one drawn by the teacher (25 · 12) and plots 50 times 6) |

The dialogue illustrates that the teacher invites Mads to present his solution (246, 248). Mads manipulates the 25 × 12 rectangle by doubling the length and halving the width while keeping the area intact to make a 50 × 6 rectangle (251). Though not included in the dialogue here, Mads goes one step further and makes a 100 × 3 rectangle (refer 248). He explains and performs the manipulation. The iconic visual mediator (drawing of the rectangle here), whose concrete equivalent could be a grid or other 3D objects that could be manipulated to fit into the operation, seems to be an important tool in order for Mads to come up with his solution (see Figure 2).
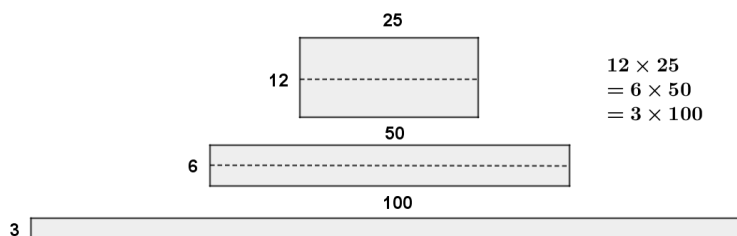


Figure 2: A 25 × 12 rectangle manipulated to 50 × 6 and then to 100 × 3.

Presenting some of the discourses developed in a mathematics teaching session across the two episodes, we have given an insight into a discursive classroom. Figure 3 shows different types of representations that the students used to multiply 25 by 12 (see for example: 069, 080, 251). They have used both symbolic and iconic representations. These different representations are realizations of a single mathematical signifier 25 × 12. Acquiring an abstract mathematical realization of a signifier is not an easy task. But realizing the principal signifier (here 25 × 12) can be made easier by examining the relation of developing an iconic representation using visual mediators which are the grids or the rectangles drawn in students' notebook or on the board in our case here (Heyd-Metzuyanim & Sfard, 2012).
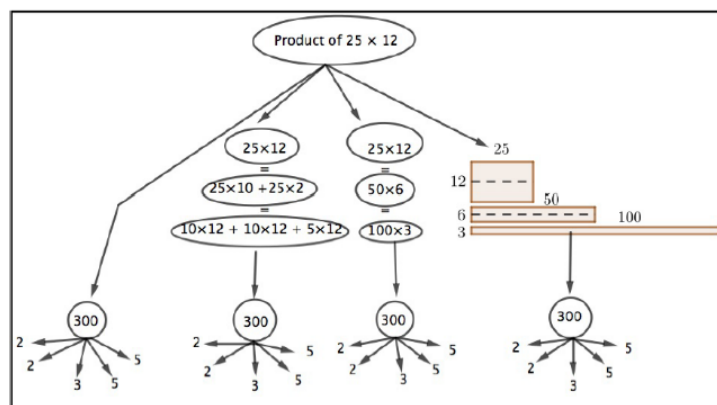
Figure 3: A realization tree of the signifier 25 × 12. The concept of the sketch is derived from Sfard (2008, p. 165)

When we organize these realizations as shown in Figure 3, we tend to construct a tree of realizations. With effective employment of verbal, gestural, iconic and concrete communication, we achieve a series of mathematical realizations. The realization tree that we have achieved by using the solution procedures (each realization) of the students can be extended further. For example, finding the factors of 300 ($= 2 \times 2 \times 3 \times 5 \times 5$) in the third row, we have extended the tree. In other words, "any mathematical realization may be used as a signifier and then realized even further" (Sfard, 2008, p. 165).

## Concluding discussion

The teacher provided her students the opportunity to compute 25 × 12 in different ways in this teaching session. Supplementing the pupils' example models with explanation, she showed how different methods could be used to multiply 25 by 12. Unlike as noted by Zodik and Zaslavsky (2008), the teacher did not use the example models just for the sake of using them. Except for some direct calculations, she adopted explorative teaching and invited her students to the mathematical discourse. Adler and Sfard (2016) argue that only explorative mathematics is a tool for life. It is only in such explorative and interactive classrooms the students can establish generalizations, objectify mathematics and attain mathematical realizations as assumed to have been achieved by the students of this teacher. Using example models and performing the operations by splitting 25 and 12 appropriately, the students got the opportunity to compare the procedures. They also got the opportunity of using procedural knowledge, apply it and make and multiple connections. This formed a means to achieve realizations of the signifier 25 × 12.

By providing some perspectives of a discursive mathematics classroom of a Norwegian primary school, the results from this study contribute to the literature about teaching practices in mathematics (Erath, Prediger, Quasthoff, & Heller, 2018; Shinno, 2018; Williams, 2016). The primary focus of the present study was to analyze how the use of the examples (and example solutions) could be used for acquiring the realizations of a signifier in a discursive mathematics lesson. The results show that the practical use of example solutions is instrumental in developing an understanding of mathematical processes in a general sense. More specifically, it can be pointed out that the students got an opportunity to manipulate the symbols in the operations which acted as a signifier. The students also got the opportunity of using symbolic and iconic visual mediators, thus developing a visual realization. As seen in the discourse, the interplay between gestures and speech (effective communication) played an important role in supporting students' understanding. The vocal mathematical communication signified the importance of mathematical language for formal

mathematical discourse. Summing up, by combining the realizations of the signifier evident from the students' task, a realization tree of signifier could be drawn.

Since, students tend to depend on superficial mathematical reasoning, discursive practice in mathematics teaching is often challenging. Supported by the presented result and the discussion that followed, we consider that the teacher can help the students attain realizations of a mathematical signifier by: 1) converting the abstract mathematical concepts (of multiplication) to concrete numerical strings, 2) using visual mediators like drawings, grids, gestures, and 3) establishing generalization through exemplification and associated explanation. It would be worthwhile to see how the use of colloquial language might enhance the process of attaining realizations of a mathematical signifier. It will be meaningful to consider this question for future research. Not standing totally against the colloquial language, we stand in line with Sfard (2008) and argue that it is important to develop mathematical concepts whenever it is possible as these, which are the part of the formal discourse, signify learning mathematics.

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# Supporting structural development in modelling at first grade

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*This paper reports on a case study following four preservice teachers (PTs) who implemented modelling in their practice teaching at first grade. During the modelling activity the students made, and responded to, different forms of mathematical representations. We have applied a categorical framework to classify the responses into four broad stages of structural development. The aim was to investigate how the PTs supported the students at various stages. One finding was that the PTs had tendency to overlook responses that could be classified into the pre-structural stage. When the PTs themselves offered representations, these were aiming towards a more sophisticated level.*

*Keywords: Modelling, first grade, preservice teachers, representations, structural development.*

## Introduction

Starting from autumn 2020 mathematical modelling will be part of the syllabus at all levels in Norwegian compulsory school. Lesh, Yoon and Zawojewski (2007) stressed modelling as a meaningful activity, because it directs students' attention towards mathematics as conceptualisation, description and explanation. Sevine and Lesh (2018) found that modelling based courses helped preservice teachers to think critically about stereotypical text-book problems and developed their skills towards designing and evaluating realistic problems for mathematics education. Modelling integrates several topic-specific domains, and Mulligan and Mitchelmore (2009) claimed modelling to provide "rich opportunities for students to integrate their mathematical knowledge and use patterns and structure". For primary grade education, teacher educators have little experience with how to advise preservice teachers on developing modelling problems that reflect such aspects of modelling and are feasible to students. Nevertheless, it has been shown that early grade students possess abilities to work with modelling. For example, English (2013) found that school children at fourth grade, in absence of direct instruction, were able to participate in modelling activities involving complex data.

## Literature review and theoretical framework

Several studies problematise teachers' supervision of modelling processes. English and Watters (2005) advised teachers at third grade to make the students familiar with reading tables, collect, analyse and represent data, and work collaboratively before starting a modelling process. Ferri and Blum (2013) investigated how teachers at grade 8–10 intervened, and how they succeeded in realising the balance between students' independence and teachers' guidance in modelling processes. They found that teachers' interventions were mostly intuitive, and not independence-preserving, in the meaning that they were often organisational and did not support or challenge students' own strategies. The interventions were often, unconsciously, related to specific solution strategies or mathematical content. Ferri and Blum (2013) claimed that being supplied by a spectrum of intervention modes, besides reflecting on their thinking styles, teachers would better find appropriate balance between thinking and acting in modelling processes (p.431). Doerr and Ärlebäck (2015) discussed the difference between "telling to initiate", and the action of "eliciting", modelling activities. They defined an "independence-preserving move" to be a teacher move that had the intention to preserve students' independent on-going work, and where this was the consequence of the intervention (p. 857).

It is not obvious how to understand modelling at early grades. Lesh and Fennewald (2013) claimed that especially for young students, interpretations of a modelling task can be situated, piecemeal and non-analytic. They discussed whether interpretations of models that function informally or intuitively, should be referred to as "models". For educational research on models and modelling they presented a "first-iteration definition" of a model (p. 7): *"A model is a system for describing (or explaining, or designing) another system(s) for some clearly specified purpose"*. Thus, the purposeful classification and structuring of different materials or objects originating from real-world contexts, can be interpreted as modelling. Developing visual representations often make systematisation processes become easier. From this we deduce that at early grades, modelling can be understood as work-processes that involve systematisations, and most likely also representations, of a real-world situation. According to the previous literature review there is a need for more research on how teachers can support students' strategies during modelling processes. We therefore rise the following research question: *How can the structural development of early grade students that participate in a modelling activity be classified, and how do PTs support various structural understandings?*

To classify structural development in children Mulligan and Mitchelmore (2009) developed a categorical framework consisting of four broad categories: The pre-structural, emergent, and partial structural, stage, and finally the stage of structural development. These categories were based on interviews with, and representations made by, 103 students aged 5-6, where the representations were developed from a wide range of tasks. At the pre-structural stage, representations lack evidence of numerical or spatial structure, and mostly focus on idiosyncratic features. Then, more formal representations gradually appear. At the emergent stage some relevant elements of the given structure are represented, but their numerical or spatial structure is not represented. Representing at the third stage implies that the most relevant aspects of numerical or spatial structure are shown, but that the representation is incomplete. At the fourth stage, the representation correctly integrates numerical and spatial structural features. Mulligan and Mitchelmore (2009) found that individual students belonged to the same structural stage in the majority of the tasks. Thus, the categorization was regarded as a valid and reliable measure for conceptual understanding of mathematics in grade 1. Mulligan, Hodge, Mitchelmore and English (2013) used the above framework to determine how the structural development in a group of highly able students, at the age of about six, could be classified. Their analysis was based on pictographs made by the students inspired by situations familiar to them. Mulligan et al. (2013) also included a fifth level "advanced", where children could recognize the generality of a structure. By selecting high-performing students they hoped to catch a development that with randomly chosen students could be lengthier (p. 532). One of their findings was the importance of students having ownership to their pictographs during the process.

## Method

To investigate the research question, we followed a random group of four second-year PTs, who through a mandatory assignment were asked to implement modelling in a practicum-class, consisting of 27 first graders aged around six years. The assignment was part of a second, 15 ECTS, mathematics course the PTs were about to perform at the University College. It asked the PTs to direct the students towards systematization and use of representations. To assist the PTs in implementing modelling, the college teachers distributed a popular article suggesting a three-act pedagogical process (Wallace & Jensen, 2017). In the first act, students should be presented to a practical situation in form of a picture,

film or story. From this, they should create a modelling problem and propose hypotheses about its solution. By supervision, the PTs were encouraged to support students in creating their own tables, drawings or informal bar graphs. The PTs cooperated on planning two modelling lessons, in which they decided to facilitate for station work. Later, we followed one PT positioned at a randomly selected station and took recordings and observational notes. We also recorded an open interview with all PTs directly after the lessons. As part of the mandatory assignment, the PTs' wrote a reflective document about the lessons. The content of this document was considered as part of our data.

To answer the first part of the research question, we used the situational data to categorize the students' structural development according to the framework of Mulligan and Mitchelmore (2009). Because the PTs took the leadership for creating most of the representations, we analysed students' utterings to reveal their structural understanding. This approach can have shortcomings, since an utterance that refers to a representation is not as easy to interpret, and not as persistent, as a visual representation. To answer the second part of the research question we applied conversational analysis to the situational data, to investigate how the PTs supported different structural understandings. The literature review, together with transcripts from the interview and reflective document, were used to support this analysis.

## Findings and analysis

When planning the modelling activity, the PTs reported that the goal for the modelling lessons was to make students familiar to sets, magnitudes and counting. They appeared positive to the suggestion of experimenting with the inclusion of structural representations. One of the PTs started the first lesson with a recollection of a fruit salad made in a previous lesson. In line with the three-act modelling, the PTs showed a picture of a fruit salad, and then asked some questions about the previously made salad. It appeared that no specific modelling problem resulted from this activity, but concepts like "more", "less" and some basic calculations were activated. The students were then divided into groups supposed to take turns in station work. One station involved drawing a fruit salad, in the two others the students should pretend to make a fruit salad out of modelling clay and centi-cubes, respectively. Students were to decide for how many persons they would like to draw and "make" the salad. If one interprets the fruit salad as a system and the different representations of the fruit salads as "systems" representing the salad, then, according to the definition in (Lesh & Fennewald, 2013) the station work can be understood as modelling.

The following transcript refers to an episode from the second lesson, which involved a group of five students and one preservice teacher. In cooperation with this PT the students had determined how
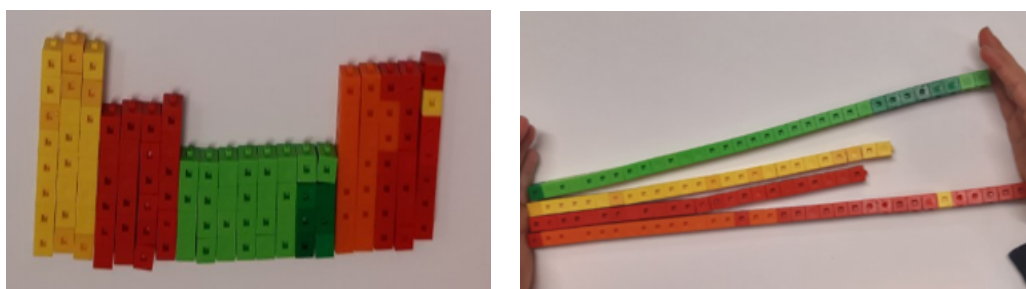


Figure 1 Two different representations made by centi-cubes, left a), right b)

many centi-cubes each type of "fruit" should consist of, and how many "fruits" to be in the "salad". From the PTs' initiative, the group had organized the centi-cubes as depicted to the left in Figure 1. Each column corresponded to one fruit, and the structure of the representation was as follows: Yellow: 3×8, Red: 4×6, Green: 8×4, Orange: 5×7. Yellow columns were supposed to represent grapes, red apples, green kiwi and orange represented oranges. (There were a few shortcomings in the representation, mainly caused by lack of cubes.)

The transcript below is from the phase when the PT and the five students just had finished shaping the representation in Figure 1-a.

| 1 | Student 1: | It looks like a castle it looks like a castle! […] Notice! It looks like a castle! |
| 2 | PT: | What sort of fruit is it the most of, then? […] Kiwi? |
| 3 | Student 1: | It is eight. |
| 4 | PT: | Do you think it is most kiwi? How can we find out what fruit it is the most of? |
| 5 | Student 3: | Counting. |
| 6 | PT: | Do you think we can find out another way? To see how many pieces? Is it an easier way then counting? |
| 7 | Student 1: | Calculating. |

From the first line we notice that Student 1 is occupied with a simulacrum he associates with the representation (a castle). Student 1 is attentive to what the student interprets as an idiosyncratic feature of the representation. According to the framework by Mulligan and Mitchelmore (2009) this interpretation can be categorized as pre-structural. The PT does not respond to Student 1's utterance, instead the PT asks what sort of fruit it is the most of. This question can be interpreted as the PT is thinking about the learning goals from the syllabus concerning familiarity with magnitude. Student 1's answer ("It is eight") refers to a counting strategy. The PTs' later questions "Do you think it is most kiwi", "Do you think we can find out another way" and "Is it an easier way" can be interpreted as attempts to draw students attention away from counting. It was difficult to perceive what fruit it was most of from this representation. Encouraging the students to not think about counting, can be interpreted as the PT wanted to motivate them to replace columns with similar colours on top of each other. We interpret the situation in this way, because later the interview informed that the PT was thinking about a representation similar with Figure 1-b. The PTs' questions can be understood as attempts to draw students' thinking towards the emerging stage of structural development (Mulligan & Mitchelmore, 2009) where one realizes that in addition to counting, visual inspection of the aspect ratios between columns, like the ones in Figure 1-b, is an alternative strategy to determine magnitudes. The PTs' questions can also be interpreted as an attempt to make transition to a new stage in the modelling process, a more systematic representation. The students did not respond to the PTs' attempts but referred to "Counting" (fifth line) and the more advanced concept "Calculating" (seventh line). Our observation showed that later the PT ended up with more direct instruction for the students to follow up on reorganizing the cubes (Ferri & Blum, 2015).

The second line shows that the PT did not pay attention to the pre-structural level of Student 1. Instead the PT attempted to direct the students' attention towards creating the representation in Figure 1-b.

To create a representation like this could have been a problem if the students appeared not to be at a somewhat developed counting level, but at least student 1 had displayed counting abilities (line 3).

Subsequently the students started to count the pieces, and in the below transcript the participants still refer to the representation in Figure 1-a:

| 1 | PT: | But I am thinking of the number of pieces. Where is the largest number of pieces? |
|---|---|---|
| 2 | Student 2: | I think it is this one. |
| 3 | PT: | [Student 2] thinks it is oranges, [Student 1] thinks it is kiwi. Why do you think that [Student 1]? |
| 4 | Student 1: | But the kiwi …*that* is the lowest one [refers to green column]. |
| 5 | PT: | Yes, it is. But is it of importance, then, …the height? |

In the first line, the PT attempts to have the students estimate what fruit it was the most of by studying the heights of the columns in Figure 1-a, instead of counting. This had appeared as difficult from the representation in the figure because of the similarities of the set sizes. The two students had different perceptions about the amount of "fruits". Student 2 had earlier (perhaps because of counting) already suggested oranges (of which there were most pieces). It can be several explanations that Student 1 in the fourth reply believes that kiwi is "the lowest one". Uttering "*that* is the lowest one" stressing the word "that", could mean that the student did not answer the question but pointed to the fact that the green columns were the lowest ones. A second interpretation is that the student had discovered that even if there were most green columns, there were fewer "kiwis" than "oranges". A third interpretation is that Student 1 wondered about kiwi being the fruit most of, since the number of green columns were largest. By asking in the fifth reply if the height was of importance, it seems like the PT could have interpreted Student 1 in this way. This shows the difficulties for PTs in how to support hidden representational interpretations and ways of thinking (Ferri & Blum, 2015).

From the representation in Figure 1-a it was difficult to recognize what colour it was the most of. In addition, counting the cubes were laborious. This situation was occasional, and probably it encouraged to develop the representation in Figure 1-b, which easily could be inspected visually. This situation shows the challenge of preparing for specific scenarios that can arise when using representations freely in modelling. The utterances by Student 2 and 1 (second and fourth line) can be interpreted as they had started to become more attentive to the spatial structure in Figure 1-a. According to the framework by Mulligan and Mitchelmore (2009) the structural development of the students was tending towards the partial structural stage, where most relevant aspects of numerical or spatial structure are shown, but that the representation is incomplete.

Later in the lesson, the PT suggested to organize the centi-cubes on top of each other, as depicted in Figure 1-b. After having illustrated this for one of the four colours, the PT and the students organized the rest colours together (this resulted in some inconsistency compared to Figure 1-a). The structure of the right representation in Figure 1 was as follows: Yellow (grapes): 26, Red (apples): 24, Green (kiwis): 33, Orange (oranges): 35. In relation to Figure 1-b, we transcribed the following dialogue:

| 1 | Student 2: | That was quite high, this was very high! […noise] |
|---|---|---|
| 2 | Student 3: | Measure it! Measure it! […] |
| 3 | PT: | Now you must listen! What kind of fruit do we have the most of now? |

| 4 | Students: | Oranges! |
|---|---|---|
| 5 | PT: | Yes! …and what fruit is it the least of? |
| 6 | Students: | Apples! |
| 7 | PT: | Apples. Did we recognize any difference in magnitude when they [the columns] were like now, and when they were side by side? |
| 8 | Students: | It became higher! |
| 9 | PT: | Yes! But was it easier to consider how many pieces there were? |
| 10 | Student 2: | Yes, then we can count [starts to count the centi-cubes] |

By the first line we notice that the attention of Student 2 is now directed towards the height of the columns. The suggestion by Student 3 to measure the columns (second line), shows that also another strategy (measuring) than counting and visual inspection, was activated. This correlates with the assertion that work with modelling problems shift students' attention towards use of several strategies. In the third line the PT does not follow up Student 3's suggestion but continues the discussion by asking what fruit it is the most of. Students answers "oranges" (fourth line) and "apples" (sixth line) showed that the students leaned on the spatial structure of the representation Figure 1-b to roughly determine the aspect ratio. In line number nine the PT made the students aware also of the numerical structure of the representation ("…was it easier to consider how many pieces there were?") According to the framework by Mulligan and Mitchelmore (2009) the group can be interpreted to reach for the partial structural stage, where one integrates the most relevant aspects of numerical or spatial structure in ones' understanding.

To investigate how the PTs had experienced the tutoring of the modelling activity, we selected transcripts from the interview with all PTs right after the modelling lessons. The first transcript is from the start of this open conversation:

Interviewer: I understood from [practice teacher] that this is how they *usually* work here. So, then […] can I first ask you about […] what do you think is the difference between what you did now, compared to how they *usually* work? And why is this modelling?

PT: It is because we are trying to ask the students' questions and try to let them lead. But, at the same time, it's grade one, so you *can't* do that.

In the first expression the author inadvertently asks two questions: How the previous lesson-activity had diverged from the performance of ordinary lessons, and how the activities could be understood as "modelling". The wording "modelling" was used, because it was the expression used by the PTs when planning the lessons. It is not clear if the PT tried to answer both questions when stating that they were "trying to ask the students questions" and "let them lead". We found that the course-literature emphasized supervision of modelling processes to build upon open questions and minimization of direct instructions. If the PT answered to the question of why this was modelling, it could be that the answer was related to the literature advice. A transcript from the PTs' document tells: *"The teachers had actively planned to interact in students' processes, but not as much as they ended up doing. […] Given this, the teachers become uncertain whether this is actually modelling, considering how actively they had to behave to involve the students."* Here PTs refer to themselves as "the teachers". This content supports the assertion that the PTs seemed to associate modelling with teaching practice connected to learning situations largely managed by students themselves. In the sentence ending with "But at the same time, it's grade one, so you *can't* do that", the PT expressed it

as conflicting to tutor the activity, at the same time leaving a certain amount of independence to students this young. Similar conflicts were reported by Ferri and Blum (2013) for teachers at grade 8–10 and are discussed in (Doerr & Ärlebäck, 2015) for the higher grades.

## Discussion

In attempting to reach for the partial structural stage, the PT typically overlooked utterances that could be categorized into the pre-structural stage. Similar observations were made for all PTs. An aspect here is that utterances at the pre-structural level can have potential for mathematical discussions. In the perspective of mathematics lessons remarks like the one about the castle, can be exploited as starting points for discussions of heights or relative sizes (e.g. of towers). By building on students' own representations it could have been easier for the PT to lead the students towards making a more advanced representation, like the one in Figure 1-b, on their own. Mulligan et al. (2013) considered ownership to a representation as important to develop structural understanding. Support at the pre-structural stage preserve students' independent, on-going work, and can possibly elicit modelling processes, instead of telling how to proceed (Doerr & Ärlebäck, 2015). Our observations showed that the PT ended up with relatively direct instruction for the students to create the representation in Figure 1-b. This is in line with findings in (Ferri & Blum, 2015) which revealed that teachers had tendency to focus on organisation and mathematical content, rather than strategies.

In the interview and document the PTs reported that it was difficult that the students had a fixed interpretation of mathematics as a topic only related to numbers and counting. This had hindered them in supporting the students to focus on the modelling activity and perform transitions to higher stages of structural development. The PTs pointed out that even if the students previously had made the salad, many did not experience ownership to the process of creating a representation of it. A reason for this could have been the way the modelling activity was introduced; as initiation, rather than in the form of eliciting (Doerr & Ärlebäck, 2015). The PTs did not relate these supervision problems to transitions between stages in modelling processes, or phases in the three-act performance. At start of the lesson the practice teacher had informed that this was the way they usually worked. These observations could indicate that the challenges reported by the PTs not necessarily were related to the "modelling" but could also be explained by their inexperience with the students and role as teachers.

## Conclusion

This paper referred to an experiment with preservice teachers acting as tutors for first grade students in a modelling activity directed towards developing structural understanding, which for most children is a lengthy process (Mulligan et al., 2013). The students referred to were chosen randomly and categorized according to their interpretation of informal bar-graphs developed by centi-cubes. They appeared to be at a pre-, and emergent, structural stage of development (Mulligan and Mitchelmore, 2009). The basic conclusions are that, in addition to recognizing students' representation levels, preservice teachers should learn more about how to ask appropriate questions at various structural categorical stages, and at transitions between such stages. Particularly they should learn how to relate to the pre-structural phase, where focus of the child is on idiosyncratic features of the representation.

The interview with the preservice teachers revealed that they had experienced it as difficult to leave autonomy to the students in the supervision of the modelling activity. They did not relate this experience to stages in the students' structural or modelling development. From this, it is reason to

believe that preservice teachers should be relatively familiar with a class before implementing modelling activities. For future research it would be interesting to perform a similar experiment with in-service teachers, to investigate if their teaching experience and knowledge of students have impact on how they act as tutors to modelling activities involving different stages of structural development.

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# Programming in the classroom as a tool for developing critical democratic competence in mathematics

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*In this article, we highlight some critical aspects of students' collaboration, while programming in the mathematics classroom, with the Scratch programming tool. We take a closer look at what students' working together through programming in Scratch can offer mathematics education, in relation to the development of critical democratic competence in mathematics. We study democratic challenges in the learning of mathematics through analysis of pairs of students' dialogue and argumentation as they collaborate on programming in the mathematics classroom. The collaborative process is analyzed in relation to the term empowerment. We indicate from our study that the pair of students are being empowered in the process, but to different degrees. We note that through the empowering process, programming, to a certain degree, can contribute to the student's critical democratic competence.*

*Keywords: Programming, empowerment, critical democratic competence, collaborative learning, critical mathematical reasoning.*

## Introduction

The Norwegian curriculum (Utdanningsdirektoratet, 2020a) general part emphasizes participation, empowerment and democracy as important shared values. How we, through education and training, develop democratic citizens in society is reflected in the general part of the curriculum. In a formation perspective, the school could be seen as a key player in developing democratic thinking and empowered citizens. In the overall part of the curriculum, the assignment in school is formulated as:

The school should be a place where children and young people experience democracy in practice. Students should learn that they are listened to in school life, that they have real influence, and that they can influence what concerns them. They will gain experience with and practice various forms of democratic participation ... (Utdanningsdirektoratet, 2020a, our translation (p. 8).

In a broader context, Børhaug, Christophersen and Aarre (2008, p. 343) argue that in a democratic practice, there should be an opportunity to participate in, and experiencing the influence of democracy, and that a central part of this participation is empowerment. The consequence will then be that empowering will be a goal in itself and that each individual subject carries a responsibility to present the subject's identity in the empowerment. Within mathematics education research, critical democratic competence is described as the ability to criticize, evaluate and analyze applications of mathematics in the society (Blomhøi 1992, Blomhøi 2003, Hansen, 2009 and Hansen, 2010). Hence it is important to construct a lerning setting that promotes democratic competence, leading to critical citizenship. Creation of such setting starts with the premise that people are builders of meaning, understanding, knowledge, cultural expressions and social groups, and that every experience of the people is an opportunity for learning. That include people interaction with themselves or with different mathematical tools such as programming tools.

Programming as a learning activity has been the focus of much attention in recent years, to such a large amount that programming and algorithmic thinking has become an integral part of the Norwegian mathematics curriculum in K-12(13) (Utdanningsdirektoratet, 2020b). In the spring of 2014, a subject review of the mathematics curriculum was conducted. Borge et al. (2014, p. 85) pointed out in their report that digital technology changes the premise of mathematics as a school subject (...) practical programming in a dedicated technology field. It will provide them with a tool that they can use to learn mathematics, and the mathematics subject will become an arena where students apply and further develop their programming skills (Sanne, et al., 2016, p. 61). At the same time, this means that the concept of the algorithmic thinking becomes much more important than before, because we have through the computer been given such a powerful tool to execute the algorithms. Developing new algorithms has therefore become an even more central part of mathematics than before. This is to a small extent reflected in the mathematics taught in school today. (…) The computer is largely used as a tool for doing "classical" mathematics, and the algorithms are hidden behind a button. The central, creative, mathematical activity involved in developing algorithms is largely absent. (Sanne, et al., 2016, p. 89).

The attention to programming stems from "Experimental curriculum in elective programming" (Kunnskapsdepartementet, 2016). The new curriculum states that digital skills in mathematics include being able to use graphs, spreadsheets, CAS, dynamic geometry and *programming* to explore and solve mathematical problems (LK20). The new development in paying attention to programming is in line with Skovsmose's reasoning about the role of technology in further building learning environments for democratic development of students. Skovsmose (2016, p.8) outlines how the rationality of mathematics affects all parts of society and that technology is influenced by this rationality. In this way, it can be argued that knowledge in and about programming will be important for democratic competence.

Critical learning can take place when students are given ownership (Alrø & Skovsmose, 2002, p. 232). If the students accept the offer, the idea is that the intention has changed, and the students will be able to acquire greater ownership in relation to the activity itself. The idea is that ownership can only take place if the activity is not enforced (Alrø and Skovsmose, 2002, p. 232). Participation in a landscape of investigation could then change communication methods and patterns (Alrø and Skovsmose, 2002, p. 51). Skovsmose's idea of a landscape of investigation has one critical requirement for the students work process, namely that the students have to mentally engage in the mathematical project of investigation. At the same time, it is not only the choice to participate in the landscape of investigation that will be decisive for the empowerment of the students. The organization of the activity itself will also play a key role. Depending on the activity, the organization may be more or less influenced by the teacher. Skovsmose and Borba (2004, p. 207) highlight different perspectives on the role technology can have in the classroom. In a democratic perspective, regardless of background, everyone should have access to mathematical ideas through action with mathematics. Furthermore, (p. 207) a critical perspective on communication and access to computers is outlined. In the programming process where two learners are solving problems, that involves dialogue, Alrø and Skovsmose (2004, p. 128) explain that "… dialogue involves making inquiry, running risk, and

maintaining equality" (Alrø & Skovsmose, 2004, p. 128). Computer access is here interpreted as how access to a single computer affects collaboration in a couple.

Empowerment is about power over language and power over skills in the use of mathematics (Ernest, 2002, p. 2). Ernest (2002, p. 1) explains *empowerment* as the process by which individuals or groups establish power. Furthermore, Ernest explains three different but complementary understandings of mathematical empowerment. *Mathematical empowerment* is described as empowerment through actions with mathematics. Mathematical empowerment is about processes where power over language and power over skills in mathematical practices are central (Ernest, 2002, p. 1). In a cognitive perspective, it is about acquiring facts, skills, concepts and structures in particular, and their use in problem solving in general (Ernest, 2002, p. 2). Ernest (2002, p. 2) distinguishes between mathematical empowerment in relation to a cognitive and a semiotic perspective. The cognitive perspective is about the acquisition of facts, skills, concepts and conceptual structures in mathematics and general problem-solving strategies. It also deals with seeking general and specific strategies for planning and approaching mathematical problems. Finally, it is about asking questions and considering suggested solutions. In a semiotic perspective, mathematical empowerment is about knowledge and control over mathematical texts represented by signs, symbols, indices and icons related to context. *Epistemological empowerment* is explained by what Ernest calls personal empowerment which, in addition to personal mathematical empowerment, is an individual growth in which the individual takes control of the development and validation of knowledge. *Social empowerment* through mathematics is about the opportunity to use mathematics to improve life prospects in relation to studies and work. It is also about participating more in society through its critical mathematical participation in society. In this sense, it involves gaining power over a wider social domain, including work, life and social affairs (Ernest, 2001, p. 2).

We offer an imaginary situation to further elaborate the ideas of Skovsmose and Borba (2004), Alrø and Skovsmose (2004), and Ernest (2005), mentioned above. One can imagine that two students are given a collaborative task where they freely can discuss how they will proceed to solve mathematical problems through programming. The students find this to be interesting and initially take "the offer" to take part in the mathematical inquiry process. In the planning phase, both are active, ideas are outlined, and direction is pointed out. Then the programming phase itself is initiated. The programming activity itself is critical in that it presupposes some "rules of the game" that are difficult to deal with. Only one of the students will be able to take control over the keyboard at any time. The ability to take control of the situation can be determined by the degree and type of empowerment which is held by the students. Communication takes place between the two students and one computer. One of the students will have a direct line of communication to the computer. The other student will have only an indirect communication line to the computer. For the collaboration to be fruitful, a great responsibility rests on the leading student´s ability to communicate both ways. The communication pattern will have to relate to some critical rules of play that raise new critical issues. Given that a teacher refrains from allocating time at the keyboard, the situation related to empowerment may be different for the two students. What about the qualities of the communication content? Will the inherent "rules of the game", through collaboration around a computer, lead to discrimination? Can there also be a challenge in that there is no longer a communication between two parties but three, teacher included?

To examine and address these questions, the critical perspective of Skovsmose and Borba (2004, p. 207) is used to inquire what programming in Scratch can offer critical-democratic competence in mathematics education. More specifically we studied students' dialogues and mathematical reasoning in relation to empowerment in collaborative processes. Our research questions are as follows: How students' democratic communication is affected by their participation in interaction of programming with one computer? And if and in what ways their individual empowerment is attained in this interaction? We also had a desire to look more closely at what programming in Scratch could offer in relation to the development of critical democratic competence in mathematics through the critical mathematical reasoning.

## Methodological approach

This article springs from the Learning About Teaching Argumentation for Critical Mathematics Education in Multilingual Classrooms (LATACME) project. The data collection in the LATACME project is complying to the rules of NSD (Norsk Senter for Forskningsdata/Norwegian Centre for Research Data.) In the current article we analyze the students´ dialogues in relation to Ernests description of the concept of empowerment (Ernest, 2001). Two master students collected data (sound recordings and video), where eight students in the seventh grade worked in pairs on problem solving with the use of the programming in Scratch (Lindberg, 2019, Pettersen, 2019). In the NSD-approval of the project, the use of collected data is allowed to be used within the time span of the LATACME-project. The eight students had previously taken part in a Scratch course volunteered to participate in the project. The previous Scratch training, which was recipe-based, was conducted by the Code Club. The Code Club run courses on a voluntary basis. The master project was intended for students to work more exploratory in pairs around a computer. The implementation of the project started with an introductory phase (pilot) in which the students became more familiar with the basic functions and applications of the program. Then the students were given the task of programming geometric shapes. Qualitative data was collected using video and audio recordings in addition to screen recordings. The recordings, together with the observations, were intended to provide both direct and contextual information. The dialogues in the collected videos and recordings were transcribed to text (Lindberg 2019, Pettersen, 2019). The data has been analysed following the Ad-Hoc-method, as described by (Brinkmann & Kvale, 2009).

## Example

In the analysis we have been interested in finding examples of students 'mathematical reasoning, and what the reasoning offered to students' empowerment and further development of their mathematical democratic competence. In the example, the students first draw a pentagon on paper, before they start programming the pentagon in Scratch. The students are in a process where they try to draw the correct size of angles and get a correct number of angles to draw the pentagon.

01    Student 1: Okay, if we delete. [Student 1 pushes Delete]

02    Student 2: and then if we take it right up now, and then to the side. Then we made a house.

03    [Student 1 is pointing on the screen.]

04    Then we can cross the house.

05    [both laugh]

06    Student 1: no… [a little low]

07    [Student 1 scratching her head]

08    Student 1: Okay, wait a minute. If we take the cursor and then "b". [Student 1 pushes "b"]

09    Student 1: aa…? Yes, but it's not a pentagon?

10    Student 2: One, two, three, four, five [pointing at the screen as she counts]

11    Student 1: Then it must be straight up there [pointing at the screen]

12    Student 2: It becomes a hexagon ...

13    Student 1: One, two, three, four ... if we get it up there. How do we do it? [mouse cursor on

14    screen, Student 1 points with the cursor on the screen on both sides of the green line.]

15    Student 2: How did we manage to make it tilt? [Pushes "Delete", moves the hand drawn

16    figure.]

17    Student 1: Diagonally? [soft voice. Student 1 gesticulates the hands to illustrate.]

18    [Student 1 moves and pushes in Scratch.]

19    Student 1: Ehm, … now it has to turn … [gesticulates with the arms] … more than 45 degrees.

20    Student 2: Then it has to be … 90?

21    Student 1: how?

22    [Student 1 pushes "b"-testing the script, and the same thing happens.]

23    Student 1: How does it go up there? (pointing at the screen) It has to turn …(speaking softly)

24    … it is impossible to determine that.

25    [Short pause in silence]

26    If we try one more. Turn … [writes in Scratch block. (Turn # degrees)]

27    Student 1: A little. It is a bit smaller than 90.

28    Student 1: if we … [softly]…

29    Student 2: 85, 80, 85? [Student 1 writes in (Turn 75 degrees)]

30    Student 1: 70 [writes in Scratch block] Now I am only guessing … [Student 1 writes in «Rotate» 75 degrees.]



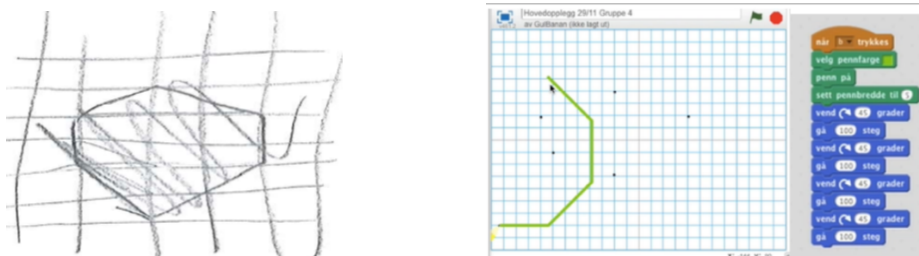Figure 1. The students preparing a model of a pentagon by hand, and trying to realize the model in Scratch

## Analysis and discussion

From the dialogue we can see that the students are well into the process of programming and well into the process of putting the ideas to life. It seems that both students have also addressed the particular challenge they face and are in a phase where they first try to find out what the concept of

"pentagon" consists of. Both students demonstrate through constructive participation in the discussion that they have accepted the "offer" to participate in this inquiry represented by the program development. Through the students' dialogue, it can be argued that the students are mathematically empowered through the participation of the project. Through the project we observe that the students are "gaining power over the language, skills and practices of using and applying mathematics" (Ernest, 2002, p. 1). Both students are mathematically empowered, however to different degrees. Student 1 (as found in lines 11-14 of the dialogue) seems to take the leadership and has a more "correct" mathematical language, and Student 2 has a more informal mathematical language. Through the dialogue, both students contribute to the creation of a pentagon on the screen. Student 1 seems to show more mathematical substance when it comes to creating the pentagon. In line 26-31 we note that both students are moving towards generalization and algorithmic thinking when they try to determine the pattern of how large each angle has to be in a regular pentagon. Both students seek and develop specific strategies to move from planning to solving a problem related to mathematical challenges. In this case, this situation contributes to both the cognitive and semiotic interpretation of mathematical empowerment. By asking questions and considering proposed solutions, both students contribute to solving the isolated problem in this situation. As a group, but also individually, it seems that both validate each other's arguments and establish mathematical knowledge and contribute new arguments. Based on this, we argue that both students have taken ownership, and therefore critical learning has taken place. Although only one student has control over the keyboard, it may indicate that both students have been empowered, both mathematically and epistemologically. When we analyze the example with regards to the different empowerments (mathematical, social, epistemological), we find that the students are empowered in two of the three different empowerments. The social empowerment is about improving life opportunities in relation to study and work in general. This kind of improvement can be achieved through for example scientific and technological skills (Ernest, 2002, p.4) Thus, one could argue that programming can generally offer improvement in professions where these skills are necessary or desirable. In the configuration where we have two students working together on the programming task, it is however difficult to analyze to which degree the student who is not in control of the keyboard is socially empowered in this process. The equality of the mathematical discussion seems to be maintained during the dialogue (Alrø & Skovsmose, 2004). Scratch is a block-based programming environment. Block based programming can be seen as a preliminary stage of text-based programming. Both block-based programming and text-based programming will be a requirement in specific educations and professions. Based on the dialogue in this project, because of the nature of the social empowerment, it is difficult to state whether the students are socially empowered. The dialogues in the project is part of a larger context, and it could be possible that the students are socially empowered through classroom discussions connected to the project. Students development of epistemological empowerment can be linked to participating in Alrø & Skovmose's inquiry (Alrø & Skovsmose, 2002). The fact that students accept the offer to enter an inquiry can be related to epistemological empowerment, in the way that it is a "personal sense of power over the creation and validation of knowledge" (Ernest, 2002, p.2) In the introduction of this text, we referred to Børhaug, Christophersen and Aarre (2008) who state that a central part of the participation and influence of democracy is empowerment of the learners, and that the subject carries a responsibility to present the subjects identity of the empowerment.

## Closing comments

We have studied democratic challenges in the learning of mathematics through analysis of pairs of students' dialogue and argumentation as they collaborate on programming in the mathematics classroom. We have had a desire to shed light as to how the critical aspects of programming could shed light on students' mathematical empowerment in the mathematics classroom. We also had a desire to look more closely at what programming in Scratch could offer in relation to the development of critical democratic competence in mathematics. We have argued that programming (in Scratch) as part of the educational system contributes to the different branches of empowerment of the students. Through the mathematical empowerment the student develops skills and knowledge which makes them able to contribute and critically evaluate mathematical aspects of the society. Through education and professional participation, the student´s epistemological and social empowerment is given a mandate to take part in critical democratic discourse. We have to a certain degree shown, that through programming in the mathematical classroom the subject (mathematics) identity has been cared for. However, we have not found any signs in the data material of social empowerment of the students. This might imply that this aspect of the empowerment can be something to be aware of in the planning of future teaching. We have studied critical aspects of two students collaborating on problem solving on one computer. In spite of inherent difficulties in the collaboration, we argue that to a certain degree, both students are mathematically and epistemologically empowered. Additionally, the student at the keyboard is to a certain degree socially empowered as well.

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# Students' productive struggle when programming in mathematics

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*The paper concerns two seventh grade students who share a computer and program a pentagon. The task proves to be challenging and the students face several kinds of struggles. However, the students are persistent, apply communication qualities, and make continuous refinements in ways that create an interesting interweaving of mathematics and programming.*

*Keywords: Programming, productive struggle, mathematics, middle school, student pairs.*

## Introduction – rationale, positioning, and focus

This paper investigates a pair of seventh-grade students struggling to program a pentagon. Our intention is to contribute to a better understanding of how programming and struggling can facilitate and challenge students' mathematical language and learning. Important inspirations are the motto of a first-grade mathematics class: "If you are not struggling, you are not learning" (Carter 2008, p. 136) and the focus on how students can engage with and express important mathematical ideas through programming in the ScratchMaths project in the UK (Benton, Saunders, Kalas, Hoyles & Noss, 2018).

During the last decade, *programming* is integrated into mathematics (and other subjects) in many countries (Balanskat & Engelhardt, 2015). In the future, Balanskat and Engelhardt argued, students need to not only be consumers but also take part in developing technology. Programming has become an important 21$^{st}$-century skill, and this emphasis is becoming increasingly more evident in national policy documents (Bocconi, Chioccariello, & Earp, 2018). Forsström and Kaufmann (2018) reviewed research on programming in mathematics education and found a limited number of articles showing improved performance and motivation, mainly in geometry. They call for more research on collective learning through programming in mathematics, particularly because programming transforms education – it creates new ways of learning and communicating in the mathematics classroom.

*Productive struggle* can be linked with programming through computational thinking. Computational thinking is an approach that involves systematic steps to solve problems and find solutions, and programming is often required to execute these solutions (Morris, Uppal, & Wells, 2017). Several countries in Europa have integrated computational thinking and programming as a key competence to be acquired in the national curriculums. Two components of computational thinking are debugging (finding and fixing errors) and persevering (keep going), and these are core aspects within productive struggles. According to Hiebert and Grouws (2007), productive struggle concerns students' "effort to make sense of mathematics, to figure something out that is not immediately apparent" and "solving problems that are within reach and grappling with key mathematical ideas that are comprehendible but not yet well formed" (p. 387). It is not about creating needless frustration or give tasks that are beyond reach for the students, rather it is about providing challenges that make sense and facilitates understanding in progress. The choices and understanding during the process of working towards a solution are as important as solving the task. The National Council of Teachers of Mathematics (2014) highlighted that supporting students' productive struggle in learning mathematics is an important teaching practice. Like Mason, Burton, and Stacey (2010) argued: "being stuck is an honourable state and an essential part of improving thinking" (p. viii). Boaler (2016) supported the idea of struggle as

valuable because that is when "the brain sparks and grows" (p. 11). She argued that we do not need students to calculate quickly, we need them to ask good questions, to discuss and reason about complex problems. They should be encouraged to take risks, struggle, fail and feel good about working on hard problems.

Little research is done on the link between programming and struggling. One exception is Kim, Yuan, Vasconcelos, Shin, & Hill (2018) who found that processes of debugging when doing block-based programming promotes the understanding of programming and can facilitate productive struggle. Kapur (e.g. 2014) developed the concept of productive failure based on the value of letting students solve problems before being taught the concepts and procedures, even if it leads to failure. Granberg (2016) built on Kapur's work when she focused on how students' problem solving can generate productive struggles when they use GeoGebra to solve linear function tasks. The following recurring student comment illustrates the significance of students developing their methods and making errors: "We did not actually think until we discovered that we failed" (p. 46). GeoGebra provides instant feedback and visualizations, and this helped the students recall and reconstruct prior knowledge, which was vital for making their struggles productive.

Warshauer (2015) developed a framework for productive struggle in which she identifies four kinds of struggles. The first is to *get started* and concerns confusion about what the task asks for and how to approach it. The second is to *carry out a process* and involves slow progress due to problems with carrying out procedures and making errors. The third is *uncertainty in explaining and sense-making* and concerns challenges in explaining the work and justifying strategies. The fourth is to *become aware of* and *express misconceptions and errors*. Warshauer underlined the importance of offering tasks that challenge students to do mathematics that goes beyond memorization and procedures and how maintaining this level of cognitive demand is dependent on communication qualities. Important examples are teacher responses that provide information and that links students' thinking with prior understanding. According to Warshauer, this involves careful listening, revoicing, and questioning that give direction and demand intellectual work. It is about communicating in ways that make mistakes into sources for learning and sense-making. Alrø and Skovsmose's (2002) concept of *dialogue* can be used to elaborate on these communication qualities. Taking an Alrø and Skovmose perspective, productive struggle is a collaborative activity where processes and answers are not given beforehand. This implies elements of unpredictability and risk and requires an open, equal, inquiring, and learning-oriented dialogue. According to Alrø and Skovsmose (2002), the concept of *intention in learning* plays an important role in a learning-oriented dialogue. Intention means having a purpose and voluntarily be striving for something. Intentions cannot be forced upon someone else, because "intentional orientation must be performed by the person himself or herself" (Skovsmose, 1994, p. 184). Warshauer's framework, together with Alrø and Skovsmoses' concept of dialogue, offer a rationale and an approach for investigating students' collaboration when they strive and struggle.

The perspectives presented above suggest that striving and struggling to solve mathematical problems is necessary for engaging with and expressing mathematical ideas. However, little research exists on students' mathematical struggles in programming settings, and on how such struggles can be productive and promote students' mathematical communication. The research question addressed in this paper is therefore: *what characterizes students' mathematical struggle when programming a*

*pentagon*? The aim is to offer insights into what characterizes students' struggle and identify qualities in what they say and do when they are challenged to program a pentagon.

## Methodology

Eight seventh-grade students divided into four pairs participated in the study. Each pair shared a laptop and did block programming with Scratch. The students had little experience with programming, but Scratch does not require knowledge about programming language/syntax and the interface is transparent and user friendly. The students were challenged to program a quadrilateral, pentagon, a circle, a star, and a house. The data collection was conducted by two master students, and they aimed to facilitate a mathematical focus by making tasks of appropriate complexity so that how to get started and to carry out the process was not immediately apparent for the students.

This paper concentrates on Ida and Knut's effort to program a pentagon. Their work was video recorded perpendicularly from the side, and an external microphone ensured good sound quality. A screen-recording app documented what the students did on the computer, and the recordings were merged into a picture-in-picture film, see Figure 1. This provided multimodal and informative documentation of their collaboration; on how they discuss, use the computer, and react to what happens on the screen. Four excerpts are
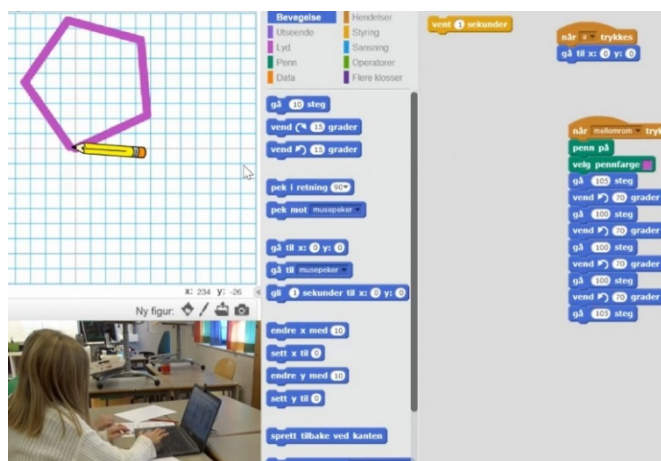


Figure 1: Screen dump from the PIP film

chosen because they are informative and represent well the students' struggles. Warshauer's four kinds of struggles and Alrø and Skovsmose's concept of dialogue are used as analytical lenses to identify key moments and to help understand and describe the students' struggles.

## Analysis: programming a pentagon – managing different struggles

The following excerpts show, chronologically, how Ida and Knut collaborate to program a pentagon. They immediately face challenges with deciding the number of steps for the side lengths and how many degrees to turn for making suitable angles. Prior to this, they had programmed a square.
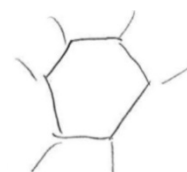


Figure 2: Drawing

The students struggle to get started because they have to discuss what a pentagon looks like. Knut takes a sheet of paper and draws a test shape (Figure 2). He counts the sides, becomes uncertain, and asks Ida if she thinks it is a pentagon. Ida first says yes, then takes a closer look and counts six sides: "That's a hexagon", she says. They are both trying to draw a pentagon by revising Knut's initial drawing, but they get confused: "This looks like a house?" "Then we make a house and a pentagon?" "No, it cannot be correct […] because the angles have to be identical." "Do they?" Their tone of voice is questioning and dwelling, they make multiple drawings, and they question, listen, point at the drafts and invite each other to participate. Ida and Knut show uncertainty; they are struggling to understand – together. The question about identical angles makes them halt. They look at each other, turn and ask one of the master students, but she encourages them to decide the criteria themselves. They carry on, agree on five sides as a key property and revise Knut's drawing to make a sketch similar to a

regular pentagon. They have not yet talked about programming – they have struggled to interpret the task mathematically. They have however made a strategic first step in preparing for solving the task of making a pentagon, and they have started to establish a learning-oriented dialogue. Their struggle now turns into a discussion about the size of the angles, and they start using the computer:

Knut:      How can we do it? Wait, how many degrees are a pentagon?

Ida:       I have no idea. A quadrilateral is 90°. A pentagon, I have no idea. It is at least more than 90°. […] I think we need more than 90°. It has to be more than 90°, kind of, because all of them are obtuse.

Knut:      120?

Ida:       Yes, maybe. I don't know.

Knut:      Let's just try, then.

Figure 3: The first code

Knut shows uncertainty and is inviting when he asks: "How can we do it?" He appears to have the sum of interior angles in mind when he asks how many degrees there are in a pentagon. Ida responds, and she as well expresses uncertainty: "I have no idea." They both dare to take the risk of telling the other that they do not know, and that indicates that they trust each other. Ida follows up and refers to the 90° angles in a quadrilateral and seems to have a regular polygon in mind. Ida goes on to argue that the angles have to be more than 90° by calmly saying "because all of them are obtuse". Knut suggests trying 120°. Ida is not sure, but they decide to start coding, see Figure 3.

The program stops functioning temporarily, and when it starts working again, the students decide to try 160° instead of 120° and get the angle in Figure 4. When the angle is drawn, they say "What?" with a confused tone of voice. They see that 160° is an error and agree that it makes sense to try a smaller angle like 60° and get the angle in Figure 5. The *turn* block does not give them the anticipated interior angle β, rather they get the turning angle α (α and β are supplementary angles), and this adds to their sense-making struggle of finding an appropriate angle. However, their systematic approach to test angles less than and more than 90°, and the way they support each other to address the issue of the number of sides, give them a better understanding of how the program works and make them ready to start extending the code to make a pentagon by inserting more of the *move* and *turn* blocks.
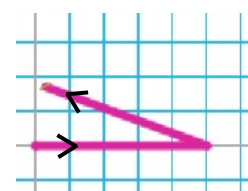
Figure 4: Test angle
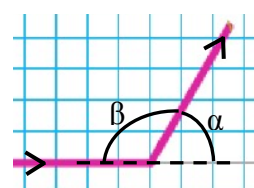
Figure 5: Test angle

Ten minutes earlier, Ida and Knut programmed 90° angles when making a square, and that is probably a reference for them when they test angles larger and smaller than 90°. When challenged to program a pentagon, it becomes evident that they are unsure about what a pentagon looks like. They are told to decide the criteria themselves. Even though they are confused, they make and investigate new drafts, and at the end, they agree to make a pentagon with five sides and the angles should be of equal size. They go on to investigate the programming part, remembering that the square was made by moving a number of steps and turning 90°, but they struggle to become familiar with how the turn block works. Although the Scratch interface is user friendly, it is not straight forward to understand that the turn block draws turning angles and not interior angles. This initial phase has productive

qualities for at least two reasons: the students become aware of what the task asks for and how to approach it, and they apply dialogic collaboration qualities. They are engaged, make drafts, express insecurity, and listen to each other. Their communication shows respect and equality. They run a risk when they show insecurity, make proposals, and investigate a problem to which they do not know the answer. These characteristics, together with their persistence, the willingness to succeed, and how they manage to make a basis for a carry out process, describe their joint learning-oriented intention.

The start is a process of systematic trial and error, demanding and vulnerable, involving Warshauer's (2015) four kind of struggles and in line with what Alrø and Skovsmose (2002) described as dialogue. The dialogical approach continues when the students carry on and revise the code by changing the *turn* blocks to 60°, see Figure 6. They are both focused and lean and point towards the screen. The communication is intense, energetic, they almost speak at the same time, but they manage to reply and continue each other's utterances. They end up with the shape in Figure 7, and the confusion and struggle continue:

| | |
|---|---|
| Knut: | We lack one. Look! |
| Ida: | Yes, I don't quite get it. |
| Knut: | No … |
| | (They click space again, and get the same shape) |
| Ida: | Then we have to take *move 100*. |
| Knut: | No, it makes a hexagon. |
| Ida: | Ok, but then we can't use 60°. Then we must have less. No, more. |
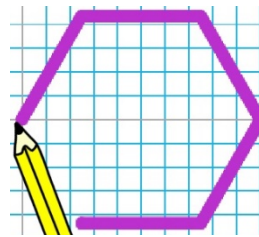


Figure 7: Hexagon?    Figure 6: The code

The shape lacks one side to be closed. They get confused and uncertain, click the spacebar one more time but get the same shape again. They realize that 60° gives a hexagonish shape and try a different angle. Ida suggests that they "must have less" and then corrects it quickly to "No, more". They continue the inquiring process and try 80° and get the shape in Figure 8 where the last side intersects the first. Based on this shape, they quickly decide to try 70°:

| | |
|---|---|
| Knut: | Wait! Isn't that a hexagon? … A pentagon. |
| Ida: | Yes, that is what we want. 70° then. 70 is probably ok since 60 isn't … isn't, let us try 70, he-he. |
| Knut: | Yes, it might be … if it is 70, then I'll be happy! |
| Ida: | Yes, me too … okay. (Clicks the green flag) |
| Ida: | Yees! No, what is that? (Points to the gap) |
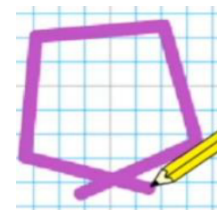


Figure 8: 80° angles

| Knut: | 1, 2, 3, 4, 5 (points to the sides), yes, then we got it! |
| Ida: | But it's not completely … closed … (points to the gap again) |
| Knut: | It is, it is, it is (energetic tone of voice), they can't see that. |
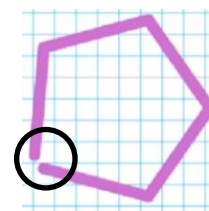


Figure 9: 70° angles

When they change the *turn* block to 70°, they get the shape in Figure 9. It looks like a pentagon, but the last side stops just before it gets back to the starting point, the shape is not closed. Knut wants to accept it, and says, "They can't see that". Ida asks the master students if they can approve it, but the master students challenge them to explain why there is a small hole and give a hint by asking if it could be something with the numbers. Knut acknowledges the room for improvement, and they find yet again energy to revise their code:

| Knut | Can try 102, no 105, on the first and the second. No, on the first and the last. |
| Ida | But maybe it is the degrees? |
| Knut | No, it's not something with the degrees, we have tried that too much. |
| Ida | But maybe some that are longer than others? |

The students are aware of the gap error and adjust the first and the fifth *move* block from 100 steps to 105, click the green flag and get what looks like a closed pentagon (it works because they have pen size 5). The gap is between the first and the fifth side, so it makes a lot of sense to increase the length of these two sides to get a closed polygon. They give a sly smile; point to the screen and comment that the sides are not equal.

## Discussion and concluding comments

Ida and Knut had to inquire what a pentagon looks like because they needed an understanding of its geometrical properties to get started with the programming. This understanding is used as a basis and is further challenged when they face some of the affordances and constraints of Scratch. When they start coding, the code blocks in Scratch facilitate a focus on the length and number of sides and on the size and number of angles. Making continuous adjustments to the angle size in the code involves mathematical understanding about the turning angle and the supplementary interior angle to make sensible adjustments. They struggle, but their working process is supported by how the shapes that represent each code are drawn instantly and provides immediate feedback to the students. In that respect, the computer screen can be regarded as a third participant in the discussion. It is a multimodal context in which mathematical ideas are represented with several different representations: the students' drawing with pencil and paper, the programming part with the codes and the drawings of the shapes, and the student's gestures and oral explanations. Thus, their struggle to program a pentagon can be characterised as a systematic trial and error approach that facilitates and requires an *interweaving of programming and mathematics*.

The struggle to program a pentagon could easily have been a missed opportunity if the teacher or a peer gave Ida and Knut a quick solution for how to program a pentagon: Use a turn block of 72° and a move block of x steps together with a repeat block and voila: a pentagon is drawn. Instead, the students faced all four kinds of Warshauer's struggles. To *get started* they had to discuss the number of sides and if the sides had to be of equal length, and then if the angles had to be equal. Their struggles continued when *carrying out the process* of programming by first understanding how the turn

function worked and thereafter with different angles, narrowing down towards an appropriate angle. On several occasions, the students faced *uncertainty in explaining and sense-making*. They started at 120°, then tried out 160° and got confused. They tried 60° and got a shape that looked like a hexagon but lacked one side. They went on to try 80° and got a shape that they were about to accept as a pentagon, but the last side intersected with the first. Trying 70° gave them a shape that was almost a pentagon, but it had a small gap and was not closed. Knut argued that it was good enough and the task was solved, but when they got the idea of changing some of the side lengths, they got renewed energy. Finally, they managed to make what appeared to be a closed polygon with equal angles but different side lengths. It was not a regular pentagon, but the task never required that, and it was, mathematically speaking, not closed. However, through the process, the students dealt with several mathematical ideas about properties of pentagons: the role of angles, the size of the angles and different types of angles, regularity and connections between types of polygons and the size of angles, and key questions like: can sides intersect and do polygons have to be closed?

Despite their struggles, Ida and Knut kept going and focused on the investigations and procedures they discuss and agree to carry out. They managed to *become aware of and express errors* and evaluate them to make new, adjusted efforts. Their struggling processes can be described as recurring loops: discuss-make new attempts-get negative/positive result-evaluate. The moments when they were ready to test a revised code appear to be critical: They straighten up, look at the screen, hold their breaths, and then click the green flag. When they become aware of an error, they bring forth new energy to start a new loop. Even after the fourth loop, when Knut must acknowledge the tiny gap, they see new possibilities for revising their strategy. Through their revisions, they show a more and more refined understanding of the problem. A key characteristic of their struggle is how it triggers what Alrø and Skovsmose (2002) called *dialogue*: they find strategies, try them out, learn from them to get new ideas for the next attempt, and they express ideas and listen to each other. They develop an open, inquiring, and learning-oriented dialogue and communicate in ways that make risk, uncertainty, and mistakes into sources for learning and sense-making. They strive to get insight into and master the mathematics-and-programming activity, they show *intention in learning*.

The study supports the findings by Granberg (2016) and Kim et al. (2018) that programming can promote productive mathematical struggles. Ida and Knut grapple with the pentagon and programming, both of which are ideas that are comprehendible but not yet well formed for them. It is a process that can make them more open to taking on a more mathematical stringent approach. For instance, they have spent time discussing regularity and are aware of the fact that their polygon is not regular. This can increase their readiness to learn more about regularity and angle sizes. From a teacher perspective, being able to identify and understand what productive struggle looks like can provide insights about what students can be able to achieve. It can however be a fine line between productive and unproductive struggles, so it will require knowledgeable educational choices.

**Acknowledgement**

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# Teachers' views of low performances in mathematics at vocational education

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*This paper reports findings of an ongoing project aiming to deepen the understanding of aspects that affect vocational students' goal achievement in mathematics. The results derive from a study among teachers working with vocational students in risk of not achieving the goals in an upper secondary school in Sweden. The focus was on the teachers' perceptions of difficulties connected to vocational education and the mathematics course (Ma1a). The study analyses the school staff's descriptions based on an inductive approach to identify aspects affecting vocational students' goal achievement. Focus is also on how the student at risk is conceptualised. The results indicate that several of the aspects identified are linked to socioemotional factors and special needs among the students.*

*Keywords: Upper secondary school, special needs, vocational education.*

## Introduction

A final degree from upper secondary school is a key to successful establishment in the labour market and is important to both society and the individual. Without a final degree, young people risk unemployment, financial difficulties, and exclusion, according to both national and international studies (Lamb et al., 2010; Lundahl et al., 2015). It is therefore concerning that approximately 15,000 Swedish students (19.6%) fail to obtain a final degree from upper secondary school each year. For students at vocational education the main obstacle to getting the final degree is the mathematics course (Ma1a) (SNAE, Swedish National Agency of Education, 2017). Ma1a is a mandatory course for all vocational students during their first year of upper secondary school. The course is similar to the mathematics course of lower secondary school, (year 7–9) and does not involve any new areas of mathematical knowledge. In other words, Ma1a is, or at least should be, mostly revision. Without passing Ma1a it is impossible to gain a final degree, despite finishing the three-year long education. The irony of this problem is that even though vocational students have passed mathematics in year 9, a requirement to be admitted, many of them have difficulty in passing a course with almost the same content one year later. What makes mathematics so difficult when they come to upper secondary school?

The study's hypothesis is that the cause of the difficulties should be sought not only in the subject or the didactics but also in other domains of the school. Studies addressing mathematical difficulties in vocational programs are few (Bakker, 2014). Therefore, it is of interest to investigate why a large group of the vocational students have difficulties with Ma1a and what vocational students' goal achievement is in the mathematics course, Ma1a. The overall purpose of the study is to contribute to knowledge and understanding of aspects that affect vocational students' goal achievement in Ma1a.This will answer the following research questions:

– Which aspects with significance for vocational students' goal achievement can be identified in the teacher's descriptions?

– How is the vocational student at risk conceptualised according to the identified aspects?

## Background

The Swedish School Act (SFS 2010: 800) clearly states that students' different needs must be considered; all students should be given support and encouragement to be able to develop as far as possible. The assessment of a student's need for support must be made by teachers or other school staff based on how the student develops in relation to the knowledge goals. In the text below these students, at risk of not achieving the goals in Ma1a, are called students at risk.

Both in practice and research, there are different perspectives how to understand the causes of students' school difficulties. The compensatory perspective and the relational perspective are two clear orientation points in the research field of special needs education (Nilholm, 2007). The compensatory perspective stands for a point of view where students' school difficulties are individualised, and explanations are often based on medical, neurological, and psychological models. It is the individual who needs to be addressed and the student is seen as the carrier of the problems (Nilholm, 2007).

In the relational perspective, the environment is seen as significant for how well the student succeeds. Within the relational perspective, explanations for school difficulties are commonly based on educational, sociological, and philosophical models (Nilholm, 2007). Since the current study concerns low performances in mathematics, theories from the field of special needs education provide tools that afford new insights on the phenomenon. The concepts Subject, Student and Environment constitute the theoretical frame of reference that underlies the analysis in this article: Student relates to the compensatory perspective and Environment connects to the relational perspective. The concept Subject connects in this article to aspects in direct relation to the content and didactics of Ma1a.

**Mathematics, students, and the learning environment at vocational programs**

Studies conducted at vocational programs report difficulties with mathematics as highlighted by both students and teachers (Muhrman, 2016; Möllås, 2009). Studies further show that students avoid seeking help from the teacher when they encounter difficulties as well as teachers having lower expectations and demands on students at a vocational program (Johansson, 2017; Hjelmér & Rosvall, 2017). Students' interest in mathematics, however, seems to vary between different upper secondary programs (Högberg, 2009). Students at care-oriented programs generally make more negative comments about the subject of mathematics (Johansson, 2017). Research shows a relationship between the math grade and the choice of program, and students with poorer mathematics grades seem largely to choose a vocational program (Larson, 2014; SNAE, 2017). This can be compared with the statistics from the national screening (see above) which shows that students in the Health-care program to a large extent finish without a pass grade in Ma1a and a full diploma (SNAE, 2017).

Vocational students identify themselves as "underclass" in comparison to students at study preparation programs and define themselves more as workers than thinkers, which may be compared to the concepts of "book-smart" and "street-smart" in international research (Hatt, 2007). With a greater focus on their vocational orientation, they consider core subjects (Swedish, English and Mathematics) as too theoretical and meaningless with no use in their upcoming profession (Hjelmér & Rosvall, 2017; Johansson, 2017). Research shows that mathematics integrated by workplace

content can have an impact on students' achievement, understanding and motivation (Berger, 2012; Lindberg, 2010).

The empirical results above may, in many parts, be linked to the concept of self-efficacy, which refers to the individual's perceived self-ability to cope with a specific task to achieve a certain goal in their life. According to the theory, different factors affect the individual's perceived self-efficacy, for example, past experiences of failure and success, experiences gained by observing others as well as affirmation or degradation from others. Psychological and emotional factors also affect the self-efficacy belief (Bandura, 1997).

## Method

This study investigates the views of 10 teachers connected to Ma1a in an upper secondary school in Sweden. Teachers' opinions were of interest because they meet the students in relation to the subject, but also in relation to the school as a social context. The selection criteria of the school were as follows: 1) a medium-sized (500-1000 students) upper secondary school; 2) no other upper secondary schools in the municipality; 3) the programs offered at the school would consist of both vocational programs and study preparation programs; 4) the school must offer several different vocational programs. The purpose of the criteria mentioned above contributes to a heterogeneous student composition regarding both socio-economic and ethnic conditions. In Sweden, the principle of free school choice is applied, which means that all students have the right to choose which school they want to attend. Contrary to its intention, the principle of free school choice has led to increased social differentiation in many of the country's upper secondary schools (Fjellman, 2019). This fact makes it difficult to study which aspects affect the student's goal achievement with the aim of reaching a broad general picture of the phenomenon. For example, one might suspect that a study among teachers at a school where most students have a foreign language background would generate aspects that are overshadowed by language difficulties.

The school selected for the current study is located in a sparsely populated municipality in north of Sweden and has about 900 students. It offers 9 different vocational programs of the total 12 vocational programs offered in Swedish upper secondary education. The school also includes study preparation programs and the teachers selected for the study meet students from both program types. Every one of them is a qualified teacher and has experience of working with vocational students from 2 to about 20 years.

The study was conducted in accordance with the guidelines of the Swedish Research Council. The interviews lasted for about one hour on average and took place at the informants' school. They were conducted individually and were semi-structured to avoid limiting or controlling the answers. The aim was to approach the phenomenon being studied from different perspectives to reach as deep and broad an analytical basis as possible. The interview guide consisted of two themes, Challenges and Success factors, and the teachers were invited to talk freely on the themes from their point of view and from the students' point of view. The focus was on the teachers' descriptions of vocational students at risk. The interviews were fully transcribed and coded by the author. All descriptions were sorted into categories on the basis of their similarities by using thematic content analysis (Kvale & Brinkman, 2014). This gave 31 different categories (aspects), which were analysed from a subject didactic and a special educational perspective, using the concepts Subject, Student and Environment.

Categories with direct connection to the subject of mathematics or to subject didactic factors such as the content of the syllabus, mathematics tests, teaching materials and methods were sorted under Subject. Statements about deficiencies, needs and difficulties were sorted under the Student concept. Statements about factors related to time, room, schedule, group size, group composition, as well as teacher's competence and response, were sorted under Environment. Examples of the analysing process are displayed in Table 1.

| Statement | Code | Category | Theoretical concept | Theoretical approach |
|---|---|---|---|---|
| *Students have often had a bad test experience. Depending on the teachers, different degrees of importance are attached to the national tests.* | National test | Test anxiety | Subject | Subject didactic |
| *Those who need the most help find it difficult to accept it.* | Students don't want help | Poor self-confidence | Student | Compensatory perspective |
| *Group size is the biggest concern. Last year, 30 people were in the math group. This year, there are about 25 pupils in the math group.* | Group size | Large classes | Environment | Relational Perspective |

Table 1: Examples of the analysis process

## Results

In the analysis of the teachers' stories, the aspects were identified as expressions (units) describing the student at risk and/or explaining the cause of the low performances in mathematics. The aspects identified are displayed in Table 2, which also shows the scope of units and the distribution across the groups: Aspects related to Student were mentioned to a much greater extent (268 units) than aspects related to Subject (60 units) and Environment (90 units).

| Subject (60 units) | Student (268 units) | Environment (90 units) |
|---|---|---|
| knowledge gaps | disabilities and diagnoses | new unknown contexts |
| poor connection to reality | language deficiencies | several precarious social groups |
| shortcomings in teaching materials | lack of concentration | big classes |
| negative experiences | low interest | mixed classes with different programs |
| test anxiety | negative attitude | no peace and quiet |
| | weak motivation | poor scheduling |
| | poor self-confidence | no access to smaller, quiet rooms |
| | weak ability | time lost due to other school activities |
| | need for more time | prolonged sedentary sitting |
| | weak work performance | poor expectations from teachers |
| | need to be seen | no/little encouragement |
| | need for close individual help | |
| | physical and mental health | |
| | school absence | |
| | teenage identification phase | |

Table 2: All identified aspects in teachers' description of students at risk and the aspects connected to Subject, Student and Environment

**Aspects within the subject perspective**

The aspects within the Subject perspective can be divided into two themes: Content and materials and Negative emotions. The teachers talk about students at risk as having knowledge gaps and negative experiences from lower secondary school, which causes difficulty in achieving their goals. They also talk about the subjects' poor connection to reality as an aspect affecting the goal achievement and the fact that the mathematics books used are abstract and hard to understand for some of the students at

risk. The following excerpt not only expresses the importance of linking the mathematics content to reality, but also hints at the fact that workplace integrated content affects both the effectiveness and the quality of learning: "/…/ if you can somehow have it (mathematics) more connected to their reality then maybe they would understand faster and better" (Teacher 10, i.e.,T10). These results are comparable with previous studies among vocational students (Lindberg, 2010; Muhrman, 2016), which reported increased motivation and understanding when implementing workplace content to the subject of mathematics.

**Aspects within the student perspective**

The aspects within the Student perspective can be divided into four themes: diagnoses and special needs; affective aspects; aspects connected to identification and health aspects and school absence. A relatively small number of the teachers' statements concerned the fourth theme, health aspects and students' absence from school, unlike the other three themes that occupied more than half the total interview time.

The first theme contains aspects related to diagnosis and weak ability, difficulties in language and concentration, as well as students' need for extra time and individual support. Need for extra time is linked to the students' understanding of the mathematics content and to a need for close individual help, shown in this statement: "When you meet such students and sit with them alone /…/ they get another direct feedback…I often hear them say: "Oh, now I understood!" (T1). Having the chance to meet a teacher in addition to the scheduled time in the classroom means more time to process the content, which seems to affect the understanding.

The second theme concerns students' interest and attitude towards the subject of mathematics. The teachers express the students' view of the course as "uninteresting" (T7) and their lack of interest thus: "But why should I do math? I will build houses or I'm going to drive a truck ..." (T10). According to previous studies (e.g., Johansson, 2017) this result is in line with the strong link to benefit and usefulness. Students' lack of interest and motivation is also linked to self-confidence and fear of failure and the teachers' descriptions illustrate that students' perceived ability does not always match their actual ability. Poor self-confidence is also described as a consequence of too little encouragement during compulsory school: "How often has this guy been credited for being good at something?" (T10). The quote comes from a teacher's story about a student who was very talented at various practical parts but who had difficulty in achieving the goals of Ma1a. This result is comparable to previous findings about how vocational students define themselves more as workers than thinkers and also how negative experiences from previous schooling affect self-efficacy (Hjelmér & Rosvall, 2017; Johansson, 2017). Students' attitude, self-image and fear of failure were linked by many of the teachers to identity and social factors which can be recognised to some extent in previous studies (Johansson, 2017; Hjelmér & Rosvall, 2017) but are expanded in this study.

**Aspects within the environment perspective**

The aspects in the perspective Environment can be divided into two themes: aspects related to relationships and aspects related to schedule and organisation. In the first theme, teachers talk about the students meeting with new unknown settings as well as new classmates when coming to the upper secondary level and link this to adolescents' need to find their identity: "/.../at their age it is about identifying themselves. It is that phase of life /…/ so the biggest challenge is about identifying

yourself in the group, in the class, in the new context" (T7). The statement relates to the psychological phase that the teenage period entails. The informants have experienced that this is more evident among vocational students than among students in higher preparatory programs who basically meet the same subjects in the same types of classrooms and arrangements as during previous schooling. Vocational students meet a new context (new subjects, engineering classrooms, working clothes, etc.) with a strong link to future profession. Hatt (2007, p.158) explains this as "the cultural practice we use to give meaning to others and to ourselves". The teachers highlight that this new vocational context contributes to difficulties in the social area, which in turn affects goal achievement; students at risk do not want to "feel stupid in front of their classmates" (T5), which contributes to feelings of "worry and insecurity" (T3), which in turn affects their attitude and motivation. A good personal relationship may counteract the students' feelings of insecurity and create "a safe climate in the classroom" (T3), an aspect with impact on the students' goal achievement identified in the teachers' descriptions. When talking about the fact that the groups and situations are new to the students, several teachers also highlight group size as an aspect affecting students at risk: "not everyone can handle the large group and is able to work in it" (T6). The quote indicates how large classes lead to a poor study environment with the absence of peace and quiet as well as precarious social situations where students do not know each other. Aspects concerning scheduling, such as no access to smaller, quiet rooms, and long, sedentary lessons are also aspects that affect goal achievement, according to the informants' descriptions.

## Discussion

The general purpose of the study is to contribute with knowledge of aspects that affect vocational students' goal achievement in Ma1a. Concerning the first research question, Which aspects with significance for vocational students' goal achievement can be identified in the teachers' descriptions, the result indicates that aspects other than difficulties with the subject's content and didactics may play a major role for students' goal achievement in Ma1a (table 2). At the same time the result draws a picture of the complex relationships between aspects: For example, the teachers' talk about students having knowledge gaps (Subject) link to motivation, interest, self-efficacy and special needs (Student) which demands actions connected to students' future profession (Subject) as well as scheduling for extra time and smaller groups (Environment). Workplace integrated content and aspects connected to the group link to both students' identity work and their attitude, as well as social aspects. Challenges for students at risk connect to both subject aspects and group aspects, and the relation to the teacher as well. The results of this study show that aspects of, or in connection to, an affective and social nature dominate and link to both Subject, Student and Environment. Based on the results, a possible pattern is that socio-emotional aspects connected to group and identity play a large role for the students' situation in upper secondary school. Such aspects may have a greater impact on the vocational students' goal achievement than previously anticipated.

Concerning the second research question, how the vocational student at risk is conceptualised according to the identified aspects, the result indicates that, to a large extent, teachers in this study conceptualise students at risk in line with the compensatory perspective (Nilholm, 2007). Students were described as carriers of disabilities and needs, as well as carriers of knowledge gaps and test anxiety, drawing from both Student aspects as well as Subject aspects (see Table 2). The fact that several teachers linked the phenomenon of not achieving the goals to students' identification phase

also tends to see the students as carriers of the problem, although the identification connected to both classrooms and clothes (Environment) and relationships and social belonging in the group (Environment). The relational perspective among the teachers is shown when they point at the students' poor goal achievement as being due to factors in the learning environment (Nilholm, 2007). Aspects sorted under Environment were fewer and were not mentioned to the same extent as the aspects under Student.

## Conclusions

The result confirms the diversity and complexity of aspects that affect students' goal achievement. Students at risk need to be understood from more perspectives than only the perspective of mathematics content or didactics. Explanations of students' low performances in mathematics must be sought on a broad basis. Solutions to improve students' achievement in Ma1a are likely to connect to both Subject, Student and Environment. Mathematics didactics needs to be combined with a special education approach in both research and practice if we are to succeed in finding ways for more young people to finish upper secondary school without the risk of ending up in exclusion and unemployment. The result also shows that teachers talk about students at risk far more from a compensatory perspective than from a relational perspective, which may also have implications for the support offered to the students. Different perspectives may imply different actions for improvement to the same problem and different actions may have different outcomes. The findings in this study mirror parts of previous research (Johansson, 2017; Muhrman, 2016), but also highlight a need for further research to investigate both social, affective and identity aspects concerning vocational students' goal achievement in mathematics. If support is only provided with links to subject-specific aspects according to students' statements that "math is difficult" (Muhrman, 2016; Möllås, 2009), there is the risk that the support will not be an effective tool to reach a full degree.

## Limitations

The purpose of the study, as well as that of this paper, is to provide a broad picture of the phenomenon. Data from current study did not contain enough ground to give detailed information about the nature of specific aspects and this merits further research to gain a deeper understanding. Both previous research and this study indicate links to social cognitive theory (Bandura, 1997) and further studies among students at risk should shed light on the self-efficacy beliefs among vocational students.

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# Exploring opportunities to learn mathematics in practice-based teacher education: a Norwegian case study

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*This study critically examines the opportunities a group of prospective teachers—participating in learning cycles of enactment and investigation as part of their teacher education—has for developing their mathematical understanding. Using a quick image as an instructional activity, analysis of the prospective teachers' discussion in a co-planning session reveals that the following learning situations are visible: 1) when involved in learning cycles, the prospective teachers got the opportunity to experience how their own mathematical understanding matters for teaching, 2) participation in learning cycles enabled the prospective teachers to apply their own understanding in new contexts and 3) learning cycles provided an opportunity to learn mathematics in a context where the prospective teachers were motivated and convinced about the relevance of the content. Three additional areas for future research are identified and discussed.*

*Keywords: Learning cycles, teacher education, prospective mathematics teachers, planning.*

## Introduction

Whereas teacher education traditionally has focused on *what* prospective teachers should learn in preparation for practice, Ball and Cohen (1999, p. 10) suggested instead to focus on *how* prospective teachers could learn "in and from practice". From their review of literature on practice-based pedagogies in mathematics teacher education, Charalambous and Delaney (2020) note that most of the research has been carried out in a US context, and they call for more studies in other countries. Although results from implementations of practice-based teacher education in the US context are encouraging, Charalambous and Delaney (2020) also report on some potential negative effects. One example is that highlighting teachers and their practices might involve a risk of ignoring students and their learning. We would like to add that highlighting teachers and their practices might also involve the risk of ignoring content. Strong and Baron (2004) also pointed out this risk. In their study of mentoring conversations with beginning teachers, only 2% of the mentors' suggestions were related to content. A lack of focus on content characterized decades of research on teaching and teacher education, until Shulman (1986) identified it as a missing paradigm. When prospective mathematics teachers engage in collective planning, enacting and analysis of teaching as part of their teacher education, there is an inherent risk of focusing more on practical issues than on the mathematics

involved in that teaching. To mitigate this risk, and to call attention to the inherent danger of ignoring content in practice-based pedagogies of teacher education, we critically examine the opportunities a group of prospective teachers have to develop their own mathematical understanding when participating in one particular approach to practice-based teacher education.

## Theoretical background

This study builds on a sociocultural theory of teaching and learning. Within this theory, we consider opportunity to learn in terms of Wells' (1999) spiral of knowing. This model describes four different opportunities for meaning making. The first opportunity lies in the situated *experiences* that individuals get from participation in communities of practice. Second, *information* is considered as the interpretations that people make of their experiences. Wells then describes the third opportunity as *knowledge building*, which entails active involvement in the meaning making process. The fourth and final opportunity is *understanding*, which relates to the relationship between experience and knowledge building. Understanding, then, is the pinnacle of the cycle of knowing (Wells, 1999).

When exploring what learning to teach in and from practice might be, some researchers have proposed to organize teacher education around core or high-leverage teaching practices (e.g., McDonald, Kazemi, & Kavanagh, 2013). There is not yet a common or agreed-upon definition of "core practice", but core practices are often considered to be frequently occurring practices that novice teachers can begin to learn and enact in teacher education (McDonald et al., 2013). Core practices are also research-based, and they have the potential to contribute to student learning. Although only a particular aspect of teaching is singled out in a core practice, core practices still maintain the complexity of teaching (Lampert et al., 2013). One example is the practice of leading group discussions. With the aim of making mathematics teacher education more practice-based, McDonald and colleagues (2013) developed a model where prospective teachers collectively engage in authentic instructional activities in "learning cycles of enactment and investigation". Learning cycle is a framework for learning to enact core practices in teacher education that rests on a sociocultural perspective of learning as collective activity (e.g., Wells, 1999). Under careful supervision of a teacher educator, prospective teachers collaboratively plan, rehearse, enact and analyse a particular instructional activity (McDonald et al., 2013). These instructional activities are pre-defined episodes of teaching that are found to be suitable for learning and enacting core practices.

The instructional activity we focus on in this paper is "quick images". In this activity, students are presented with an image like the one in Figure 1. After having viewed this image for three seconds, the students are asked about the total number of dots. Quick images are designed to help students visualize numbers and form mental representations of a quantity by being invited to explain how they organized and subitized quantities in order to find the total number of dots in the image. Through this activity, where students get the opportunity to learn about the commutative, associative and distributive properties of multiplication (Schumway, 2011), prospective teachers get the opportunity to enact core practices like eliciting student thinking and leading group discussions.
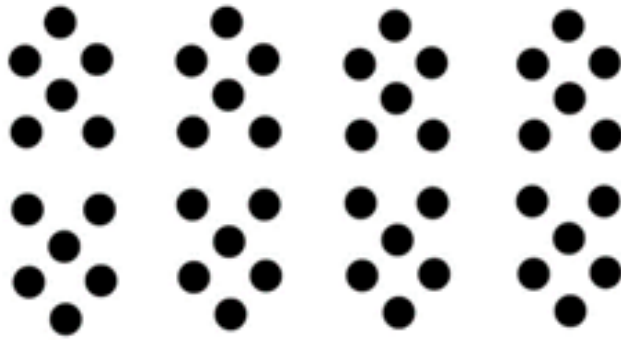
Figure 1: Quick image planned to be used (Matematikksenteret, n.d.)

Recently, the practice-based pedagogy of learning cycles has been implemented in some professional development programs in Norway (Fauskanger & Bjuland, 2019), as well as in some initial teacher education programs (Rø, Valenta, Langfeldt, & Ødegaard, 2019). The present study from the *Practicing dialogue-based mathematics teaching* project (in Norwegian: "Øve på Dialogbasert Undervisning", ØDU) represents another attempt to implement this practice-based pedagogy in a Norwegian teacher education context. Hopefully, such a practice-based approach to mathematics teacher education will better prepare prospective teachers to carry out the complex work of teaching mathematics.

## Method

In the ØDU project, 21 Norwegian prospective primary school teachers (school years 1–7) worked together in three repeated learning cycles (McDonald et al., 2013). The aim was to learn core mathematics teaching practices such as launching mathematical problems, using mathematical representations, aiming towards a mathematical goal, facilitating student talk, and eliciting and responding to students' mathematical ideas (Lampert et al., 2013). The prospective teachers were in the third year of their teacher education program and had elected a 30 ECTS[1] credit specialization course in mathematics[2] (beyond the first 30 ECTS credit course, which is mandatory). They were divided into three groups with seven prospective teachers in each group. A teacher educator guided every group. One of the groups volunteered to participate in the research study, and we videotaped all three cycles in this group. The three cycles were located at two different primary schools. Only the co-planning phase (98 minutes) of one learning cycle is analysed for the purpose of this paper. The following enactment phase was carried out with a group of seventh grade students in their own classroom.

Inspired by Wells (1999), we started the analysis by dividing the co-planning session into episodes according to different thematic foci in the prospective teachers' discussions. An episode represents a part of the session where the prospective teachers have a focused discussion. Another episode begins when there is a clear shift in the focus of the discussion illustrated by an utterance (e.g., a question or a statement). Two of the authors first identified 27 episodes individually before reconciling. There

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[1] European Credit Transfer and Accumulation System (ECTS)

[2] Course literature was based around Kazemi and Hintz (2014)

was total agreement on the episodes. We have selected parts of the prospective teachers' discussions from two episodes for further analysis, aiming to illustrate the prospective teachers' opportunities to learn mathematics in a particular co-planning session. These episodes are analysed in terms of Wells' (1999) model of meaning making.

## Results

The first episode considers the prospective teachers' opportunity to learn in terms of Wells' (1999) spiral of knowing, related to the third opportunity of *knowledge building*. The prospective teachers' uncertainty about the distributive and associative properties of multiplication is visible in the discussion. There is a need for clarification, discussing these properties for their own sake that entails active involvement in their meaning making process. The second episode demonstrates the fourth opportunity, *understanding*, in Wells' model of meaning making. The relationship between the prospective teachers' experiences (participation in the mathematical discussion concerning these properties) and their knowledge building illustrate their understanding of the distributive property. These discussions reveal their opportunity to apply their knowledge about the distributive property of multiplication in new task situations while planning to use in the lesson (see Figure 1).

**Opportunities of knowledge building: having their own understanding of core mathematical concepts challenged**

This dialogue takes place while the prospective teachers are engaged in a discussion of the mathematical content included in the quick image (see Figure 1). In the previous episode, they discussed possible suggestions aiming towards a mathematical goal for the lessons. In the continuation of the co-planning session, the teacher educator says that he is missing something about the distributive property. Astrid[3], one of the prospective teachers, responds that twelve can be written as three times four. The teacher educator challenges her by asking, "What property do you think about now?" Astrid responds, "Isn't this the distributive property [of multiplication]? Or is it maybe the associative [property of multiplication]?" We interpret this response to indicate uncertainty about her own mathematical understanding. The teacher educator follows up with a hesitant, "Ehh…", and we consider this to indicate that he has become aware of Astrid's uncertainty, and that he is considering how to follow up on it. Then, several of the other prospective teachers simultaneously start making suggestions about these properties. The teacher educator decides to intervene by giving an example of the distributive property, pointing at one of the quick images in which they have written $(5 + 1) \times 8$: "Here, I would say that it is appropriate to focus on the distributive property. The distributive property says that (3s) we can multiply (.) distribute this eight on both of these numbers. So, eight times one and eight times five." Astrid points at several copies of the quick image (Figure 1) and says: "This is the same here (2s). Well, (2s) the fours are outside [the parenthesis] and then this is inside." The teacher educator appears to realize that Astrid is mixing up associativity and distributivity, and he points at the quick image, written as $(3 \times 4) \times 4$ while responding: "Yes, I think that's more the associative property. It would be the same [answer] if we calculate that parenthesis first, or if you start with the number outside [the parenthesis]. Because we don't multiply four times four and then four times three. No, this isn't the distributive property."

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[3] All names used in the paper are pseudonyms

The teacher educator continues by giving examples of the associative and distributive properties of multiplication and asks the prospective teachers if they are familiar with these properties or if they would like him to recapitulate the differences between associativity and distributivity. Three prospective teachers, Hedda, Viktor and Tuva, all give brief responses, indicating that there is a need to elaborate on these concepts.

Our analysis illustrates how prospective teachers get an opportunity to develop their understanding of core mathematical concepts after having been challenged by the teacher educator in the co-planning session of the learning cycle. This situation provides an opportunity for their knowledge building, entailing active involvement in their meaning making process (Wells, 1999).

**Opportunities of understanding: applying knowledge about the distributive property of multiplication in new situations**

The second episode takes place at the end of the planning session. The participants have been concerned with the learning goals for the lesson, how to launch the problem, how to elicit and respond to students' thinking, and how they as prospective teachers can lead a targeted discussion regarding the distributive property of multiplication. At this point, the teacher educator's initial plan was to end the planning session. Instead, he asks a final question before the rehearsal, "Would you like to include a summing up session where you illustrate the use of the distributive property (...) for instance if we are to calculate 5 times 16 [$5 \times 16$], then it might be easier to calculate $5 \times (10 + 6)$?" After some discussion among the prospective teachers, Tuva suggests that, "Six times eight is not a difficult task for them [the students]. So why should we do this? But five times sixteen, for instance, is much more difficult to calculate mentally." Tuva's suggestion indicates that she recognizes the power of the distributive property of multiplication in an example that is more complex than the given task in the quick image. Asbjørn follows up on this by suggesting another example that applies the distributive property by introducing twelve times four: "So, if we have given them [the students] a task: twelve times four, do you manage to calculate this? And then it would be easier [for the students] if we write it as $(10 + 2) \times 4$". Asbjørn's suggestion demonstrates knowledge of the distributive property. Implicitly, he also appears to demonstrate knowledge of the associative property since he considers eight times six, which Tuva brings into the discussion, observing that $8 \times 6 = (4 \times 2) \times 6 = 4 \times (2 \times 6) = 4 \times 12$. After this, Astrid draws attention to larger numbers: "It could be completely different numbers as well. If we consider 1072 and show [the students] that they can split it (...) 1072 times five okay, 1000 and 70 and 2 (inaudible)." We observe that Astrid, in the first episode, seemed to be uncertain about the distributive and associative property of multiplication, but here, at the end of the planning session, she is able to apply the distributive property to a self-selected example. From this illustrative example, we notice that at least three of the seven prospective teachers correctly apply the distributive property.

At the end of this episode, the prospective teachers all agree about paying attention to the applications of the distributive property of multiplication when they are engaged with the students in the subsequent enactment of the lesson.

## Discussion

Whereas practice-based teacher education aims at enabling prospective teachers' learning in and from practice (Ball & Cohen, 1999), some teacher educators are concerned that increased emphasis on

practice might lead to a corresponding decrease in the emphasis on content (Strong & Baron, 2004). Our analysis shows that the practice-based approach of learning cycles might provide ample opportunities for prospective teachers to focus on mathematics. The following discussion will highlight three important aspects.

First, when prospective teachers were involved in learning cycles, they got the opportunity to experience how their own mathematical understanding matters for teaching. In a regular mathematics class at the university, they might have been tempted to move on without truly understanding the mathematics. However, in a context where they engage in planning for an instructional activity that they are going to teach themselves, they might experience, like Astrid did, that they *have to* understand the mathematical content. This situation thus provided for opportunities to meaningfully engage with content, and the prospective teachers did not have to question whether or not this knowledge of content was relevant for their own work of teaching.

Second, the context of learning cycles enabled the prospective teachers to apply the mathematical concepts they discussed earlier as part of their collective lesson planning. In the first episode, the prospective teachers found it necessary to revisit the associative and distributive properties of multiplication in order to proceed with their planning. In the second episode, we observe how they took the opportunity to apply these concepts in new and meaningful situations when planning a summing up session. Overall, these two findings could indicate a development, from the first episode to the second, of the prospective teachers' understanding of these mathematical concepts in the context of planning for teaching.

Third, the context of learning cycles provided an opportunity to learn mathematical concepts in a practice-based context. Oftentimes, mathematics teaching is located at the university campus in a situation where prospective teachers sometimes fail to see the connection between the mathematical content they are struggling with and the work they think they are being prepared for (e.g., McDonald et al., 2013). In field placement, the focus is sometimes on other aspects of the work than the mathematical content (for mentoring sessions see e.g., Strong & Baron, 2004), and prospective teachers tend to experience this as a disconnect between these two fields. In the context of learning cycles, the prospective teachers get an opportunity to bridge the gap between theory and practice in a meaningful context.

We are aware that the data material is rather limited, and some critical voices might question the importance of the findings. However, we suggest that the findings reveal some opportunities for prospective teachers' knowledge building and understanding of the distributive and associative properties of multiplication in a co-planning session.

## Conclusion

This study provides existence proof (Schoenfeld, 2007) that prospective teachers might get opportunities to develop their own mathematical understanding in a practice-based context like that of the ØDU project. However, our study does not identify factors that provide these opportunities to learn in the structure of the learning cycles. Below, we point to three plausible factors for further investigation in future research.

The first factor relates to *the enactment phase*. In the ØDU project, the prospective teachers are required to carry out the lesson with actual students after the planning session. In a similar study, Rø

et al. (2019) questioned the importance of the enactment phase and called for more research on this subject. We find it plausible that the close relationship to practice and the enactment with students influence the development of the prospective teachers' mathematical understanding in the context of the ØDU learning cycle. Without enactment, the prospective teachers might miss the opportunities to experience that their own mathematical knowledge matters for teaching. We therefore suggest that practice-based approaches such as the ØDU project should have parts of its instruction in the practice field, but further studies are needed to investigate how the close relationship to practice influences prospective teachers' opportunities to learn.

The second factor concerns *the structure of the planning session*. The prospective teachers are required to go through the mathematical content as there is allocated time for this. The session is divided into topics (with timeframes), and therefore they are able to focus on different parts of the work of teaching in the corresponding part of the session. They are also expected to have worked on the task prior to the session. We think these are important aspects of the structure that influence the prospective teachers' opportunities to learn, and future research should explore how such aspects of the structure might influence the opportunities for prospective teachers' learning.

A third plausible factor is *the teacher educator*. The teacher educators' own mathematical knowledge, their relationship with the prospective teachers, their abilities to orchestrate mathematical discussions, their abilities to seize pedagogical opportunities, and their abilities to assess the prospective teachers' understanding and intervene as they go along, seem important for prospective teachers' learning. Future research should investigate whether and how teacher educators might influence the prospective teachers' opportunities to learn mathematics in the context of practice-based teacher education.

The idea of making teacher education more practice-based is not new (e.g., Ball & Cohen, 1999), and progress has been made to develop and explore practice-based pedagogies in mathematics teacher education (Charalambous & Delaney, 2020). So far, a significant portion of the research on practice-based teacher education has been conducted in a US context, and it is important to study affordances and constraints of various approaches in other contexts. Our study explores how one specific practice-based pedagogy of mathematics teacher education might work in a Norwegian teacher education context. The results from our study indicate that this practice-based pedagogy might provide opportunities for prospective teachers to learn mathematics, but it also points toward a need to further investigate what might be entailed in the work of teaching mathematics in such an approach to teacher education. It could be interesting for future research to investigate if the same (or other) opportunities were present in other co-planning sessions. Another direction for future research could be to investigate how the mathematical issues addressed in the planning session played out in the classroom and in the reflection sessions.

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# Probability explorations via computer simulations in a Norwegian classroom: a discursive approach

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*The commognitive theoretical framework could be called "radical participationism" in the sense that it defines the result of learning as the ability to take part in communicational activities within a given discourse. In this paper, we analyze a teaching situation in a Norwegian classroom in which a computer simulation was used to stimulate whole-class discussions on the concept of probabilities and compound events. We suggest definitions and give examples for three discourse which emerge in the students' speech. We show that these discourses fit the contents of the students' interventions but also reflect an existing tension in the discipline itself.*

*Keywords: Probability, simulation, ICT, discourse, commognition.*

## Introduction and theoretical framework

What we call "chance" in our daily life, as opposed to mathematical probability, is a "fuzzy" notion (Pratt 1997). The development of probability as its own scientific field is fairly recent compared to, for example, arithmetic or geometry. Reasons for such a late emergence of probability as a science are explored by Hacking (2006). In light of these historical perspectives, it should come as no surprise that the transition from informal probability to mathematical probability is also difficult for students. This transition has been studied in the mathematics education literature (Fischbein and Schnarch 1997) or in the psychology literature (Scholz 1991; Hawkins and Kapadia 1984), with a variety of foci such as misconceptions and intuitive reasoning.

In the present study, we give another look at this question: "How do students approach *chance* in the mathematical classroom?". Our point-of-view is that of *discourses*: how do students *communicate* and talk about chance? The context is important: students' previous experiences with the word *chance* in their daily lives intersect with the expectations on language that come in a classroom, and especially in a mathematics classroom.

Our theoretical lens is the framework of *commognition* which was developed by Sfard (2008). It studies *communication* and *discourses*, the main tool being discourse analysis through video analysis. It focuses on the observable aspects of learning and takes the point of view that learning mathematics consists in increasing fluency in a mathematical discourse. It is somehow radical, in the sense that it doesn't see communication as a *manifestation* of some abstract notion (such as understanding or misconceptions). It instead considers our ability to communicate fluently within a certain discourse to be what defines learning, directly. This is both a philosophical and a pragmatic choice: Sfard takes great care to define operationalizable concepts. The difficulty of using *a priori* concepts is that one may need to use some guesswork to properly get access to them empirically. Rather, Sfard defines a system of analysis in which the conceptual components are defined directly from the sets of observable acts of communication to which they refer. Pushing the idea further, this theory claims

that thinking can usefully be regarded as communicating with oneself, hence the term *commognition* was coined as a portmanteau of communication and cognition.

In Sfard's theory, a *discourse* is a set of communicational objects and rules which define the ways of communicating within a group (community of discourse). A community of discourse shares a common language, and *rules of endorsement*: rules which its members follow to accept or reject propositions (*endorsing or rejecting narratives*). Sfard distinguishes *object-level* rules (for example: theorems or propositions) and *meta-rules* which are less prescriptive, often implicit, and concern themselves with how mathematical speech should be constructed to be acceptable ("*patterns in the activity of the discursants trying to produce (…) narratives*", Sfard 2008).

In a classroom, students come equipped with preexisting knowledge and beliefs, which they might use to tackle problems given to them. Our assumption, therefore, is that narratives produced by the students make sense within their worldview, even though experts may label them as misconceptions. To reformulate it in a participationist language: students may try to produce narratives which are endorsable or at least plausible within a discourse they master, even though they are not endorsable within the teacher's discourse.

Our goal is to identify which discourses the students use to approach questions about chance, and contrast them with the *discourse of probability*: the "official classroom discourse" which they are expected to acquire eventually.

## Method

The second author designed a software designed to facilitate probability experimentation and simulations in the classroom. This software (FlexiTree) contains several *microworlds* (see for example Biehler et al. 2013), such as "marble falling in tubes" or "spinners". It allows to simulate events one-by-one, in large numbers, or in "batches" (for example: define the experiment to be "spin a certain spinner five times" then repeat the experiment 100 times). The use of this simulation software and several design choices were documented for example by Iversen and Nilsson (2007).

In 2017, the second author designed a series of lessons in probabilities for a 7th grade class, based on the use of this software. The philosophy behind it was to allow pupils to express their probabilistic intuitions and to test them against trials and simulations. The goal was that, through small-group and classroom discussions, and through experimenting with the software, the pupils could construct many of the central notions and concepts of probability. ICT-based teaching is meant to serve as a facilitator: FlexiTree's microworld plays the role of the "environment" against which students will challenge their beliefs or preconceptions (Pratt, 2000), potentially creating cognitive conflicts.

The lessons were carried out by an experienced 7th grade teacher, to her usual class. These lessons were the first time this academic year that the students worked with probability. The second author of the present paper was present in the classroom with video-, sound- and screen-recording equipment (in order to record the students' work with the software).

The present study investigates in depth the first lesson of this sequence: by design, the notions of "chance", "fifty-fifty" or "probability" were not presented beforehand to the students. We investigate how – especially based on which discourses – the notion of chance emerges in the students' discussions and arguments.

Our study is based on the transcript of the teaching sequence. All utterances were transcribed from the main camera and microphones. Sometimes, significant gestures were noted in the transcript when they were visible and unambiguous. Since not all students were visible on the video, gestures are not used significantly in the study. In the present paper, the transcription is shortened: the omitted parts, marked as *(...)* correspond to teacher interventions when they are limited to encouragements, revoicings or turn-giving.

## Data: a student's discursive explorations

We divide the teaching sequence in two acts: in Act I, the class considers the problem of one marble sliding down a tube system (Figure 1). This first act is marked by three simulations being run along the way. In Act II, the problem is changed to twelve marbles falling in short successions down the same tube system. Act II is concluded by one such simulation being run and discussed.
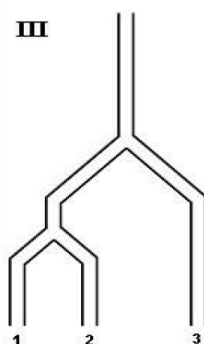


Figure 1: The system presented to the pupils. In the classroom, the outcomes were referred to as A, B, C instead of 1, 2, 3

In presenting our data, we follow one of the pupils, "Charlene", as she expresses her views on the problem – and her views evolve. This choice in the narration of our data should not obscure the deeply social dimension of classroom phenomena: Charlene would probably not have been able to "walk the way" toward constructing a notion of probability without other pupils' contributions.

To begin with, the teacher presents the problem to the class: "If you let go of the marble, then there are three possibilities: A, B and C, as they are called. What do you think are the possibilities… what is the chance that it goes in each?".

As we mentioned, it was a design choice to not even tell that the marble has "equal chance" to fall left or right at each intersection. Instead, the intent was for the pupils to use their intuition of mechanics and the symmetries of the figure to reason. And indeed, several pupils gave reasons grounded in mechanical thinking. When a pupil says that if the marble goes first left, it will go right afterwards, Charlene gives the following argument: the marble may bounce against the wall.

> Charlene: I was thinking… it is bent here and then it goes in a straight wall. And if the marble hits that wall it gets some speed the other way… But… at the same time the marble has speed… also that way… I don't know.

A bit later, a student wonders if it may all depend on how the marble meets the bifurcations: "it depends a bit how you throw the marble…". At this stage, the teacher decides to discourage this direction of inquiry and mentions that the tubes are just as wide as needed, and the marble can't wiggle around: it meets the bifurcations right in the middle (but can't get stuck!).

This clarification leads to Charlene supporting some of the other student's opinion that the outcome is unpredictable, and therefore there is no point discussing it.

> Alice: What we've been discussing all this time now… is that… we can… also it… it is not the marble who decides where it goes. (…) So it can go wherever and we don't know… where it will go…
>
> Charlene: It is very random.
>
> Alice: Yes!

A bit later, Charlene insists on the pointlessness of the discussion.

> Charlene: I think it's random all the same each time, that it's like… that it's no use giving an argument on what will happen. Because… yeah.

However, in the second act, new ideas emerge as the class wonders about what might happen when twelve marbles are allowed to fall in the tubes. A first opinion is "four in each"; some students are concerned, however, whether this opinion is compatible with "fifty-fifty at the first intersection". These concerns eventually crystallize in Charlene's intervention.

> Charlene: That… I think that it's not good to just start with "it's all random because there's nobody who decided where the marble will go". It does fall, it has to go somewhere. But what Alice said is that on top it's fifty-fifty, and… it can make it that most marbles come to C, because it… is… that with A and B they'll split again. If for example half of the mar… marbles get to C and half fall the other way, and they split again at A, B and c… yes, at A and B, then most get to C in the end.

This explanation wins adhesion of the whole class. One of the students who was a vocal supporter of the equiprobability theory ("thirty-three – thirty-three – thirty-three") now changes his opinion to "twenty-five – twenty-five – fifty, fifty in C". Some adjustments are however required for this new narrative to be compatible with a previously accepted narrative: it is *possible* for one basket to get no marble at all, or even for all marbles to go in a single basket. Charlene refines the "25–25–50" theory conditionally as follows:

> Charlene: Yes… that it [can] only get in two or that it only gets in one. But if it gets in all, then (unintelligible) most chance that it is twenty-five – twenty-five – fifty.

Finally, a simulation is run once which leads to the distribution three – one – eight (versus the expected three – three – six). The class expresses surprise.

## Analysis of the data: discourses from which probability is constructed

In the previous section, we unfolded the sequence by following one specific student. We made this choice as Charlene's discursive journey is representative of the different opinions expressed in the whole class. First, she wanted to find reasons why the marble may fall in a given basket. Then, she joined some others in the opinion that the whole exercise was essentially pointless: A, B and C are possible, but no more can be said. Finally, she used a distribution-based reasoning to argue that most marbles might go to C. Her last utterance before the simulation encapsulates much of the tension underlying the notion of chance: she expresses her belief in a rule "25–25–50", while acknowledging the uncertainty of the result (the rule holds "if it [the marbles] gets in all [the baskets]").

In his book *Science et Méthode*, Poincaré (1908) opens a long discussion on the nature of randomness (and its compatibility within a absolutely deterministic view of the physical world) by acknowledging this tension and citing a classic textbook of the time (Bertrand 1889): *"How could one dare speaking of the laws of randomness? Isn't randomness the antithesis of any and all law?"*. In this section, we will analyze classroom discourse through this apparent tension. Student's narratives can be attributed to one of the two discourses which we call "Discourse of determinism and mechanics" and "Discourse of possibilities and uncertainty". The resolution of the tension between these two discourses leads to the "Discourse of probabilities".

### Discourse of determinism and mechanics

Charlene's first intervention above fits in this discourse: it is characterized by several words or expressions such as "speed", "hits the wall". Several students in the class produce narratives belonging to this discourse.

> Cornelius: That… if… it depends a bit how you throw the marble (unintelligible) in a way because if it goes all against this wall here… then you can hit more on that side of the spike and then you go there.

Cornelius' intervention draws upon similar ideas as Charlene's: the path is determined by the shape of the "spike", and the marble future direction is determined by how it meets this spike.

Central to this discourse are the words (speed, etc.), but maybe more importantly the discourse works under the assumption that any effect (such as falling in A, B or C) should follow from a knowable cause (such as the marble being thrown on the right or left side of the tube). This is particularly visible in Charlene and Cornelius' discourses: they look for possible causes to the marble's behavior.

Another aspect of this meta-rule is that, given enough information on the starting conditions, one should be able to determine the outcome for sure. This rule transpires in the following intervention.

> Bernt: There is… There is like fifty-fifty on the first split, but… I don't know why but I believe, eeh… if it goes down to the left here. (…) Yes… it… I don't understand why, but I think it goes in B…

The narrative Bernt considers endorsing is that a turn to the left is followed by a turn to the right ("if left, then right"). It is apparent from the precautions he takes ("I don't understand why") and his hesitations that he is uncertain about this narrative. However, the narrative he considers is deterministic: "if left, then right", and does not have uncertainty built into it (as would for example "if left, then likely right"). Therefore, even though he is uncertain about endorsing this narrative, we view this narrative itself as characteristic of a deterministic discourse.

### Discourse of possibility and uncertainty

This discourse manifests itself in Alice's intervention (punctuated by Charlene's utterance: "It is very random."). The point of trying to predict an outcome is seen as irrelevant: the marble can fall wherever without the class having a say in it ("it is not the marble who decides where it goes").

This discourse focuses on the randomness of the situation: we can't know *for sure* where the marble will fall, and it is therefore meaningless to argue for or against an unknowable outcome. In that way, it reflects very much Bertrand's (feigned) outrage at the very notion of "probability law": we can try

to discover the laws of physics, but the idea of finding laws for random events can appear silly, and it actually led to expressions of frustration from some of the pupils in the class.

This discourse of uncertainty goes together with a discourse of *possibilities*. This is exemplified by the pupils' excited bets just before the second trial was run. In this sequence, pupils didn't speak at their turn, and are therefore not identified. In addition, some simultaneous unrelated utterances were not transcribed. We provide three utterances: the second clearly answered to the first; the third, while it came slightly after, seemed to answer them.

Student 1:    "It will go to A… or B."

Student 2:    "Or C."

Student 3:    *(eruditely)* "It will go to A, B, or C."

Betting on the outcome allows one to guess; however, no outcome can be excluded beyond doubt. The only bet which is guaranteed to succeed (and is seen as "mathematically correct", i.e. which can be endorsed within the usual school mathematics discourse) is the one listing all the possible outcomes: A, B or C.

This discourse has therefore two sides: on the one hand, the description of all possible outcome; on the other hand, the impossibility to prefer one of these over the others – since all are possible.

**Discourse of probability**

The terms "chance" and "fifty-fifty" belong to the probabilistic discourse, and several pupils use them. One pupil, however, proposes a sophisticated (though incorrect) narrative grounded entirely in this discourse.

Alphonse:    *(confidently)* There is more chance that it goes in… in A or B than C. Between A and B, it's random. (…) Since A and b-… If you split up… if you split up the chance of one hundred percent, it makes then sixty-six percent for both A and B – chance that (unintelligible) comes to one of them, and thirty-three… thirty three percent that it comes to C. (…) Because there are two… A and B are two… (…)

Beatrix:    But it is still fifty-fifty on the top one. So it is like no matter if… A and B are further down than the first… in a way…

Teacher:    Are you… does it mean that you agree, then, or disagree?

Beatrix:    Hmmm… no. Because… like A and B… eh… they split further down than… in a way A and B,… and C. Because I think the top (unintelligible) splits also in two and then there are only two of them, not three. (pause) Just that two of them lie there. (…)

Alfonse:    Ok… When I said that I didn't mean… There **is** fifty-fifty chance, and it goes to either right or left. (…) But it's not about whether it goes right or left, what it's about is that there are three alternatives, and two of them are both to the left. So it is more likely that… that it comes in two of three than in one of three.

As we see, Alphonse's discourse uses the vocabulary of probability and percentage points. In this discourse, probability (or "chance") is something which can be "divided up" and added together (A

and B together make 66%). In addition, Alphonse cites explicitly which rules he thinks are relevant when it comes to endorsing his narrative ("what this is about"). The rules to be used, according to him, should not have to do with one bifurcation having two outcomes "right or left", but rather with the three final outcomes A, B and C. Alphonse appears to be applying a counting principle (*à la* Laplace) in a ritual manner: 2 out of 3 is more than 1 out of 3. We hypothesize that he may have some previous familiarity from probability; he is in any case able to identify that the answer to the teacher's original question should be formulated within the probability discourse (he appeared very confident in the first utterances). He also has some sense of the form a correct answer may have.

Interestingly, some of the *routines* which can be practiced when it comes to probabilities (such as dividing them and adding them up) were also applied by Charlene on the marble set ("If for example half of the mar… marbles get to C and half fall the other way…"). This illustrates that a distribution of objects can be a useful metaphor for conceptualizing the probability of events, in a way which makes it more accessible to beginners.

## Conclusion: A successful sequence?

During this computer-aided lesson, students were able to talk about a probability situation using their pre-existing discourses. Through discussions, they could see that their discourses were not satisfactory to make predictions or describe what *might* happen. Finally, using *expected distribution* as a metaphor for chance, they could apply some of the routines of the probabilistic discourse (dividing up marbles as a proxy for doing the same on probabilities). On the face of it, this sequence was successful in getting the students to get a first contact with the probabilistic discourse.

However, the simulation allowed the students to confront immediately their narratives with an outcome. Many students were very surprised – even shocked – to see that the simulated distribution was so different from the one they agreed with. We conjecture that this has to do with the *nature of the narratives* which are produced within the different discourses. Probability produces fundamentally different narratives from the ones usually produced in the rest of mathematics: "We *expect* three–three–six" rather than "It will be three–three–six"; "The outcome is *compatible* with our prediction" rather than "The outcome does / does not match our prediction", etc. This formal difference comes from the necessity for probabilistic narratives to take uncertainty into account. A possible explanation to the students' final surprise is that – while some of their routines changed during the lesson – they still use "deterministic" rules to confront their conjecture with the experimental outcome, and therefore to decide whether the outcome proves or disproves their conjecture. If the simulated outcome was supposed to give the solution it is undeniable that they "didn't get the good answer"! To recognize that probabilistic narratives have both distinct forms and rules of endorsement would require more mastery than they could acquire in one lesson.

While it is too early to make teaching prescriptions based on this work, our results suggest that a reflection on language should fully be a part of lesson planning: for example, asking pupils whether their experiments *strengthen or weaken* their assumptions (as opposed to *confirm or infirm*) could be a way to make them consider the type of narratives of which the discourse of probability is made. Further work needs to be done to document which *metaphors* are used by pupils and teachers as proxies for either objects and rules, investigate their usefulness and their limits.

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# The work of leading mathematical discussions in kindergarten: a Norwegian case study

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*The core practice of leading mathematical discussions has received considerable attention in the school context, but few studies have investigated this practice in a kindergarten context. This study investigates what tasks teachers might be faced with in the work of leading mathematical discussions in a Norwegian kindergarten context. Inductive analyses from a case study of a kindergarten teacher's discussion with a group of four children identify four tasks of teaching in the work of leading mathematical discussions: 1) use of concrete materials to facilitate mathematical play and investigations, 2) use of metaphors to describe mathematical concepts, 3) use of questions to elicit children's mathematical thinking, and 4) use of praise to support children's mathematical self-confidence.*

*Keywords: Mathematical discussions, kindergarten, tasks of teaching.*

## Introduction

Leading discussions is considered a core practice of mathematics teaching and there is a growing body of research on leading mathematical discussions (Jacobs & Spangler, 2017). This research builds on a longstanding interest in communication in mathematics education research. Whereas traditional teaching often follows a pattern of teacher initiation, student response and teacher evaluation (IRE), progressive teaching tends to focus more on discussion (Cazden, 2001). Much has been written about what is involved in teachers' work on structuring and leading mathematical discussions in school (e.g. Kazemi & Hintz, 2014), but much less focus has been placed on what is entailed in carrying out this core practice in the kindergarten context.

The Nordic kindergarten tradition emphasises care and upbringing through free play and everyday activities over school preparation. A few studies have provided insight into the communicative work of teaching mathematics in this context. For instance, Fosse (2016) identified five characteristics of mathematical conversation in a Norwegian kindergarten. Whereas she focused more on conversations among children, Carlsen (2013) targeted the orchestrating work of a Norwegian kindergarten teacher's use of fairy tales. He concluded that concrete materials and questioning were important tools that mediated the discussion. Sæbbe and Mosvold (2016) also found that the use of questions was important in their study of teaching mathematics through play with Lego blocks. They added affirmation as a core aspect in a teaching situation that seemed to be more in line with an IRE pattern than discussion. In a more recent study, Sæbbe and Mosvold (2020) discussed how initiation of mathematical discussions, responding to unexpected questions, dealing with wrong answers, using visual representations and positioning children as valuable contributors were core tasks in the complex work of teaching mathematics in kindergarten. Although communication and discussions were prominent in all of these studies of mathematics teaching in the Norwegian kindergarten context,

none of them had an explicit focus on the practice of leading mathematical discussions. Building on previous research on the core practice of leading mathematical discussions in school, and on previous research on mathematics teaching in the Norwegian kindergarten context, the present study aims to elaborate on the work of leading mathematical discussions in kindergarten. We approach this through the following research question:

> Which tasks of teaching are teachers faced with in the work of leading mathematical discussions in a Norwegian kindergarten context?

We will first provide some theoretical background and define some core terms that have been used in the study before presenting the design and our findings.

## Theoretical background

We define discussion to be a specific kind of communication around a particular content that is distinct from other types of talk. Discussions are oriented towards a particular question or problem that a group investigates to build collective knowledge and understanding by using ideas and input from participants as resources. Whereas the traditional IRE pattern describes exchange between a teacher and individual children, more participants contribute in a discussion by making their ideas public, through active and careful listening or by responding to the contributions of other participants (Jacobs & Spangler, 2017). Mathematical discussions are not only discussions about mathematics, but also support a learning culture where children participate equitably on the joint construction of knowledge.

Focusing on discussion fits well with the Norwegian Framework Plan for Kindergartens, which reflects the social pedagogy tradition and emphasises a child-oriented pedagogy where care, play and development are core foci. The Framework Plan describes promotion of language and communication as a core duty of Norwegian kindergartens, and it specifies that this should be accomplished through dialogue and interaction (Directorate for Education and Training, 2017). Furthermore, the Framework Plan states that kindergartens shall help to create a learning community that values different expressions and opinions. The learning area of "Quantities, spaces and shapes" highlights asking questions, reasoning, argumentation and seeking solutions. It specifies that "staff shall use mathematical terminology thoughtfully and actively" and that "staff shall create opportunities for mathematical experiences by enriching the children's play and day-to-day lives with mathematical ideas and in-depth conversations" (p. 54). The Framework Plan states that kindergarten teachers must work deliberately in monitoring, facilitating and supporting dialogue with and between children. But what does this work of leading mathematical discussion entail? And what do we mean when we refer to it as a "work of teaching"?

Studies of leading mathematical discussions often distinguish between the actions or moves that teachers make when they orchestrate a discussion, and the intentions or goals that point to the intended outcome of the moves (Jacobs & Spangler, 2017). Although such a distinction appears logical, it is often challenging to distinguish the two in practice. For instance, "pressing students for explanations is both a set of moves and a set of goals, each with multiple layers" (p. 779). We use a different approach, where moves and goals are integrated in tasks of teaching (e.g. Ball, 2017; Sæbbe & Mosvold, 2020). An example of a task of teaching is "asking productive mathematical questions", illustrating how moves and goals are integrated in the tasks. Focusing on mathematical tasks of

teaching thus involves an effort to deconstruct the complex work of teaching into problems or challenges that teachers recurrently face in the complex work of teaching (Ball, 2017).

## Methods

To investigate tasks of teaching that might be involved in the work of leading mathematical discussions in the Norwegian kindergarten context, we conducted a case study of one kindergarten teacher. Although a case study is bounded in time and place, it is considered useful for gaining an understanding on a particular issue (Creswell & Poth, 2018) – in our case, what might be involved in the work of leading a discussion on mathematics with young children in a kindergarten context. To recruit participants for the study, we used the network that was available through our work at Norwegian Centre for Mathematics Education, which involves various professional development efforts in kindergartens and schools. In particular, we searched among previous participants in recent professional development projects that focused on mathematical discussions in kindergarten. Two kindergarten teachers accepted the invitation to participate. One of them worked with toddlers, and due to the children's age and language development level we decided against selecting this teacher. Instead, we selected a kindergarten teacher who worked with children who were around four years old as we assumed that these children might be able to participate more fully in a discussion than toddlers. This kindergarten teacher – who had ten years of experience and continuing education in educational-psychological counselling – selected a group of four children to participate in the study. We asked the teacher to carry out an everyday activity or situation in which she felt that mathematical discussion was significant. She thus planned an open, play-based activity that allowed for mathematical investigation without specifying any particular mathematical theme. We received consent from the parents and children and informed all participants that they were free to withdraw from the study at any time.

The first author was responsible for data collection and had the role of a non-participant observer. Since our focus was on identifying tasks of teaching rather than on analysing details of communication, we decided to collect the data through fieldnotes from observations. The fieldnotes were written in three phases: during a conversation with the kindergarten teacher before the discussion, during the mathematical discussion between the kindergarten teacher and the four children gathered around a small table and in a semi-structured interview with the kindergarten teacher after the observation. To support the fieldnotes, we also photographed the room and materials used in the discussion. In the interview, we asked the kindergarten teacher to reflect on some experiences from the discussion. The purpose was to gain insight into reflections, challenges and purpose of choices. The fieldnotes were written in-situ and read through immediately after the observation so that the kindergarten teacher was available to respond to clarifying questions. Some notes were added when necessary.

Where other studies have identified moves and purposes that teachers might use when leading mathematical discussions (Jacobs & Spangler, 2017), our focus was on identifying possible tasks of teaching that might be involved in the work of leading mathematical discussions (e.g. Ball, 2017; Sæbbe & Mosvold, 2020). We thus focused on identifying possible tasks that kindergarten teachers might be challenged with during the work of leading discussions, rather than to describe and evaluate what a particular kindergarten teacher did. The analysis process was open and inductive, where two criteria guided our identification of tasks of teaching. First, in order for something to be identified as

a task of teaching, it had to be *recurrent*. For instance, we noticed several times throughout the activity that the kindergarten teacher used questions. Use of questions therefore seemed to be a recurrent task. Since our analysis was limited to one activity, we sometimes inferred that something was recurrent by observing that children seemed to be familiar with it. Second, tasks of teaching have to be *deliberate*. This was accounted for by a semi-structured stimulated recall interview with the kindergarten teacher directly after the observation. Through this process, four tasks of teaching emerged as prevalent in this activity, and these are presented and discussed below.

## Findings

### Use of concrete materials to facilitate mathematical play and investigations

We have identified using concrete materials to facilitate mathematical play and investigations as a task that might be involved in the work of leading mathematical discussions in kindergarten. Different types of concrete materials are frequently used in kindergarten. A broad distinction can be made between materials that are developed specifically for the learning of mathematics, and materials that were not developed for this specific purpose. In our study, we observed the kindergarten teacher using several types of concrete materials, including Numicon® shapes, pegs in a feely bag and baseboard, counting bears, pattern cards for bears and a balancing scale, and three towers of Lego Duplo® in different colours. Numicon® shapes and counting bears are educational resources developed specifically for learning mathematics, where provided guidelines and web resources elaborate on the intentions behind these materials. Other materials, such as Lego Duplo®, have not been developed for the purpose of learning mathematics, but they can still be used as artefacts to direct children's attention towards mathematics.

Bearing the findings from our analysis in mind, we suggest that there are at least two possible ways in which concrete materials can be used when leading mathematical discussions in kindergarten. First, the concrete materials might serve as a starting point for a mathematical discussion. In our study, we observed how the kindergarten teacher let the children play with concrete materials, and then used their play as a starting point for initiating a discussion by asking them questions related to the materials or what they did with the materials. In this sense, concrete materials might create a common focus. Second, our analysis indicates that the material played a significant role not only for initiating discussions, but also for orchestrating the mathematical discussion. For instance, the kindergarten teacher asked questions about the concrete materials rather than directly about the more abstract mathematical ideas in focus.

There are several reasons why a kindergarten teacher might consider using concrete materials in mathematical discussions. In the interview, the kindergarten teacher in our study explained that the children had been motivated to play with this material in the past. A possible purpose behind the choice of using concrete materials might thus be to increase children's motivation and interest. The kindergarten teacher in this case study decided to make materials with the potential for both play and mathematics learning available to the children with the intention of observing the children's play and interest as a starting point for mathematical discussions. We observe that there is a close connection between the questions of how and why concrete materials might be used here. The task of using concrete materials to initiate mathematical discussions thus involves both moves and purposes (Jacobs & Spangler, 2017).

**Use of metaphors to describe mathematical concepts**

Another task of teaching that we identified as part of the work of leading mathematical discussions in this kindergarten context is the use of metaphors to make mathematical ideas accessible to the children. Whereas the concrete materials discussed above are physical objects that might help direct children's attention towards mathematical ideas, a metaphor is a word or phrase that is used to describe something that is analogous to something else – for instance a mathematical idea.

In our study, we observed that the kindergarten teacher made frequent use of metaphors when leading the mathematical discussion. She used metaphors relating to colour, numbers, shapes, context and more. For instance, when referring to a brown shape, she said: "It looks like chocolate". When referring to a shape with a circle in it, she described it as "an eye". When describing a shape with an odd number of circles, one a protruding extension, she described it as a "diving board". She also extended the use of metaphors to provide a context for the mathematical issue in focus. An example of this was when they were discussing why they should place the counting bears in a particular way in order to continue the pattern. The kindergarten teacher described this to the children as: "the bear is coming to a birthday party".

When it comes to purpose, metaphors can be used to make ideas accessible to the children, but they can also create a low threshold so that more children can participate in the mathematical discussion. In our study, the kindergarten teacher used metaphors with the intention of supporting children's understanding of the ideas they discussed. After the activity, however, the kindergarten teacher reflected on a possible unintended effect of her use of metaphors. She was worried that her use of the birthday party metaphor might have contributed to obscuring the mathematical focus for one of the children. One child selected a bear with a similar colour so that the two bears could "be friends" in the party, rather than selecting a green bear that would continue the intended pattern on the task card. We therefore suggest that metaphors cannot always be used in a straightforward way, but using them constitutes a task of teaching where the kindergarten teacher has to consider and make a decision based on her professional knowledge and understanding of the context, the children and the mathematical content that is in focus in the discussion.

**Use of questions to elicit children's mathematical thinking**

A third task of teaching that we found to be involved in the work of leading mathematical discussions in kindergarten involved using questions to elicit children's mathematical thinking. Questions are common in the more traditional IRE pattern of teaching (Cazden, 2001), but the work of leading mathematical discussions might also involve questions. A discussion is initiated by a question or problem that the group aims at exploring in order to increase their knowledge, but questions might also be used when orchestrating a discussion. Below, we discuss two broad categories of questions that we observed in the orchestration phase and possible purposes or effects of such questions.

The first category of questions might be described as probing questions. These are often "closed", in that they only call for short answers. Examples of such questions that we observed in our study are: "Which piece is this?" and "What size is this piece?" It was interesting to note that the kindergarten teacher in our study was not able to recall examples of such questions in the interview, and we therefore suspect that these questions might have been posed habitually rather than purposefully. However, use of such questions tends to lead to recitation rather than discussion (Cazden, 2001), so

it is important to be aware of the potential effects of such questions. When asked directly about the purpose of such questions, the kindergarten teacher in our study assumed that they helped the children to maintain their attention on the mathematical issue or task in focus.

Another category of questions might be described as eliciting questions. These questions tend to be more "open", and are questions for which the teacher does not know the answer. An example of an eliciting question that we observed in our study was when the teacher asked in the group: "Why did you choose to put that one there?" This is a type of question that invites children to explain their thinking rather than to probe their understanding. When asked why she posed such questions, the teacher admitted that she actually was wondering why the child had done as she did. This indicates that eliciting questions might also be questions where the kindergarten teacher genuinely wonders about the answer herself. Another purpose of asking eliciting questions might be to make children's thinking more publicly available to other participants in the discussion. Examples of such questions that we observed asked in the group are: "How do you know?" and "Can you show me?" The questions support children in explaining and showing their ideas on the table so that the other children can engage in them. Another example of this type of question was when the kindergarten teacher asked one of the children: "Why did you want to have a red one there?" This use of questions is often described as pressing for clarification and reasoning, and researchers have identified specific talk moves that are types of questions that can be used in the work of leading mathematical discussions to elicit children's thinking (e.g. Kazemi & Hintz, 2014). Considering the different purposes of the questions posed, we get a notion of the complexity that is involved in the task of choosing and using different types of questions in mathematical discussions.

**Use of praise to support children's mathematical self-confidence**

A fourth task of teaching that we identified in our analysis relates to the use of praise. Praise is given frequently in the more traditional IRE pattern as feedback or evaluation (e.g. Cazden, 2001), but our analysis shows that it might also be involved in the work of leading discussions. We identified two types of praise in the analysis of our data.

The first type of praise is when the teacher aims at orienting the children towards other children's thinking, and towards understanding that there are many ways of thinking. An example is when the kindergarten teacher in our study said: "Yes, that can be the right solution for you!" The teacher highlights that for the purpose of this move to work as intended, it has to be stated in front of the group, which indicates that the teacher was deliberate about this.

Second, praise might be used in discussions to encourage children or make them feel good about themselves. An example of this is a situation when a child identified a shape with ten holes, and the kindergarten teacher provided the following praise: "You can really count far!" As a result, the children shifted focus and started counting as far and fast as they could. This illustrates how use of praise might have unintended consequences. A kindergarten teacher might consider it important to make sure that children feel smart, and there might be reasons why mathematics is a topic where this type of praise is considered especially important. The kindergarten teacher in our study expressed concern about the tendency of children to be vulnerable to negative feelings about mathematics, and she argued that at this age, building positive attitudes towards mathematics is more important than

doing tasks correctly. It is worth noting, however, that this use of praise is similar to the use of feedback in traditional recitation.

Our analysis indicates that the use of praise can be challenging – particularly in unexpected situations. One example is when a child in our study presented a solution to the pattern task that appeared to be mathematically incorrect. The other children in the group sensed this and made some negative comments. In this situation, the teacher overlooked the comments. In the interview afterwards, the teacher admitted that she had been uncertain about how to respond in this situation, and she decided to deal with it later. Her concern was to avoid making the situation worse, and her plan was to take the necessary actions with the aim of raising the child's status in the group later the same day. This example shows how challenging it can be to respond to children's contributions in the moment. Many discussions start out with children being asked to explain their thinking, and some teachers tend to treat all contributions as equally good. How to follow up on an idea and deciding when and how to use praise to support children's mathematical self-confidence is thus a complex task of teaching that is involved in the work of leading mathematical discussions in kindergarten.

## Concluding discussion

We set out to examine which tasks of teaching are involved in leading mathematical discussions in the Norwegian kindergarten context. Our case study of one kindergarten teacher's practice cannot be generalised to a larger population, but we consider this case study as a proxy for studying what might be involved in the work of leading mathematical discussions in the kindergarten context. Through our analysis, we have identified four tasks of teaching that involve choosing and using appropriate concrete materials, metaphors, questions and praise. We suggest that these tasks of teaching illustrate some of the complexity that is involved in the work of leading mathematical discussions.

Several previous studies have identified questioning as important in the work of teaching mathematics, also in the Norwegian kindergarten context (e.g. Carlsen, 2013; Fosse, 2016; Sæbbe & Mosvold, 2016). In the context of discussion, the use of questions to elicit children's thinking appears particularly important. This corresponds with Fosse's (2016) emphasis on the need for reflection. Our identification of the use of praise appears to correspond with other studies that have pointed out affirmation as a task of teaching in the kindergarten context (e.g. Sæbbe & Mosvold, 2016), but we suggest that considerations of when and how to use praise differ between traditional recitation and discussion. We also consider it important to pay attention to possible negative effects of use of praise. Other studies have identified use of representations or concrete materials as an important aspect of the work of teaching mathematics in kindergarten (e.g. Sæbbe & Mosvold, 2020), but we suggest that there might be something specific about using concrete materials in discussions that needs further investigation. Finally, we suggest that the use of metaphors might be a part of the work of leading mathematical discussions in kindergarten that requires particular attention.

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# "A bit uncomfortable" – preservice primary teachers' focus when planning mathematical modelling activities

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*The purpose of this study is to get insight into the aspects of mathematical modelling (MM) that 16 primary preservice teachers (PTs) focus on when planning teaching for school practicum. The analyzed planning session was part of the PTs mathematics education course in their second year. All the PTs have the concern to choose a topic that interests the students for the modelling activity, and to let students make choices by themselves during modelling. The unpredictability of the MM is experienced as uncomfortable. Three categories are identified in the data: student perspective, teacher perspective and activity perspective. The results are relevant in exploring the gap between the potential that MM holds for the students' learning and the teachers' reluctance to implement it.*

*Keywords: mathematical modelling, preservice teachers, primary school.*

## Introduction and previous research

Judging by the amount of international empirical research, modelling has gained an important place in mathematics education, mostly linked to secondary and tertiary education (Galbraith, 2012; Schukajlow, Kaiser & Stillman, 2018), but English, Arleback and Mousoulides (2016) and Stohlmann and Albarracín (2016) referred to a positive development of MM in the primary school the last years. It is important to work with MM in the classroom at all levels to help students: understand the world, to motivate and help them learn mathematics and develop other mathematical competencies, and have a better image of the mathematics (Blum & Borromeo Ferri, 2009); to understand known mathematics in-depth and to be motivated to learn new mathematics (Zbiek & Conner, 2006). English (2010) showed that MM in primary school helps children to apply mathematics in authentic contexts, develop problem-solving abilities, and "facilitate different trajectories of learning" (p. 295). Given these benefits, and the children's exposure to big amounts of data, modelling in primary school is necessary to develop the students' competence to participate in society (Doerr & English, 2003).

In the new Norwegian mathematics curricula (grades 1-10) to be used from autumn 2020 (Ministry of Education and Research, 2020), modelling and applications are one of the five core elements of school mathematics. The aim is to "gain insight into how mathematics is used to describe daily life, work life and society in general" (p. 2), and it includes mathematizing of a real problem, constructing a mathematical model and evaluating it. Evidencing its position in the curriculum does not mean that MM will automatically gain the same position in the classroom, as was the case with German teachers without a proper mathematics background who were reluctant to implement MM (Borromeo Ferri & Blum, 2013). One challenge for Norwegian teachers is the lack of clarity of the new curriculum on how MM is to be defined and implemented (Berget & Bolstad, 2019).

International research has been conducted about the reasons why MM is not so widespread in classroom practice. Blum and Borromeo Ferri (2009) connected difficulties to implement MM to the cognitive demands of the modelling tasks, to the additional mathematical competencies required, and to the teaching becoming more open and unpredictable. Other difficulties were rewarded to contextual factors such as classroom norms, the teacher collegiate, the school leadership, the national curriculum

and policy, lack of resources (Maass & Engeln, 2018) and lack of time (Biembengut & Vieira, 2013). An important hindering factor is the teachers' lack of experience with MM, in their studies and practice (Biembengut and Hein, 2010). Other researchers emphasized the need for teacher professional development in modelling (Maass & Engeln, 2018), in addition to the opportunities for PTs as future mathematics teachers to deal with MM on a theoretical and a practical level, including experiences with modelling at school (Borromeo Ferri & Blum, 2010; Niss, Blum & Galbraith, 2007).

Given the MM's position in the mathematics curriculum and the PTs' need to prepare for teaching MM in their future, we focus on PTs for grades 1-7 and MM in their studies. The research question posed here is: What aspects of MM do the PTs focus on when planning modelling activities for practicum, and how do they consider those aspects? The aspects are elaborated in the next section.

## Modelling in mathematics education: activities, students and teachers

A modelling process can be illustrated in different ways (e.g. Blomhøy & Jensen, 2003; Blum & Leis, 2007). Common for the different models of the process is the starting point, a real situation to be described with a mathematical model. The process is cyclical and contains several sub-processes such as interpreting, simplifying, mathematizing, evaluating and validating. Under interpreting, simplifying and mathematizing, the students have to choose what to consider for solving the problem.

Students learn by engaging actively in a MM classroom (Maass & Engeln, 2018), and teachers should guide them through the process by giving them freedom and possibilities to choose and adjust their arguments (Siller et al., 2011). In a review of MM studies, Biembengut and Vieira (2013) identified a pattern of teachers' role in the modelling process: starting with a topic of students' interest, asking students to collect data, giving orientations to formulate them using mathematics, and guiding them to solve the task and validate the model. One of the difficulties primary school teachers encounter is the «substantial diversity in thinking» (Stohlmann & Abarracin, 2016, p. 7), requiring them to listen to the students and pose probing questions. In describing quality teaching of MM, Blum and Borromeo Ferri (2009) highlighted four important components: balancing teacher guidance with student independence, mastering different intervention modes, supporting individual modelling routes and multiple solutions, and supporting adequate solving strategies. Regarding ways to achieve the balance between teacher guidance and student independence, they argued for strategic interventions «which give hints to students' on a meta-level» (p. 52). Borromeo Ferri and Blum (2013) used survey categories such as the teachers' role, students' motivation and learning, materials etc., to study barriers and motivations for experienced primary school teachers to implement MM in classroom practice. Barriers were: lack of material, time pressure and difficulties to assess MM.

When analyzing the PTs' planning of MM, we will refer to the three perspectives discussed here: activities, students, and teachers, as well as to different aspects about them as addressed by the PTs.

## Background of the study, participants and data analysis

Our study is part of the research project Learning About Teaching Argumentation and Critical Mathematics Education (LATACME). The project aims to gain insight into what promotes or hinders PTs for grades 1-7 in learning to teach argumentation and critical mathematics education to school students. Modelling is one of the focus areas in the research project, and part of the PTs' curricula.

To find out what aspects of MM the PTs focus on when planning practicum activities, we collected data from 16 PTs taking a mathematics education course in their second year of a 5-year teacher education (grades 1-7). After a theoretical and practical introduction to MM, the PTs in 4 practicum groups had to plan a MM activity for their students, and teacher educators were available to support them. The group discussions of 90 minutes were audio recorded and transcribed, and ethical guidelines were followed. When initially categorizing our data, we started with the Borromeo Ferri and Blum's (2013) categories, such as the teachers' role, students' motivation and learning, materials etc, but were open for new categories which wouldn't fit into the existing ones. In their discussions the PTs were focusing on students' learning and interests, on teacher characteristics and their role on MM, and on characteristics of MM activities and organisatorial aspects. These focuses were then categorized as respectively: the student perspective, the teacher perspective, and the activity perspective. The two authors categorized parts of data individually to then compare them. Most of the discussions about the teacher or activity perspective were centred on the students. We will thus discuss the three categories separately and overlaps between them.

## Results and discussion

Two of the groups (2 and 3) were especially active in their discussions and went through several aspects of MM and its teaching. The extracts chosen here are the richest in terms of describing the categories. Common for all the groups was the idea that students should be engaged by working with a topic that interests them, and should make their own choices.

### Student perspective

All groups had students in the centre when planning as they felt they had to adapt the activities to students' interests. Group 1 discussed several topics that would "bring about engagement" for grade 7 students for the MM activities. Students' active engagement in modelling was also pointed out as important by Maass and Engeln (2018). One PT said, "Fortnite game, that would have been nice, very cool, they would have liked it very much", referring to a popular online video game that the students would apparently like, while group 4 picked "plan a trip to Paris" with a given budget for grade 7. In line with research indicating that the teachers start the MM process with a topic of students' interest (Biembengut & Vieira, 2013), the PTs discussed the potential of several topics to make students interested. They were at the same time having an activity perspective to be presented later.

Another aspect in the student perspective was the students' role during the modelling activities, since in modelling "the goal in itself is that they [the students] have to find out by themselves, that they find themselves things to mathematize, in general" (group 2). According to the PT, students have to be active and "find things to mathematize", as choosing situations they want to mathematize. They have to also "find out by themselves", which can be interpreted as referring to the other parts of the MM process such as interpreting, evaluating etc. By using the word "themselves", the PT pointed to student independence. For group 4, the students "have to choose themselves if they want to spend 2 days or more" in Paris. These aspects were common for all the groups. As research emphasized, students have to work independently with modelling (Blum & Borromeo Ferri, 2009), "direct their own learning" (English 2010, p. 9), and they should be given the freedom and opportunities to make and adjust their arguments in the modelling process (Siller et al., 2011). Group 2 discussed how to involve a water basin near the school for a second grade MM activity, and the kind of support and

guidelines the students needed for working with it. The mathematical topic was measuring, and the students could "choose themselves if they want to use the feet [foot length] or steps [step length] or a rope or…". The PT saw the students' role in the modelling activity as being the ones who decide how to practically measure the basin, and conduct the modelling, by choosing different non-standard units such as the feet, step length or a rope. This group came longer in the planning and accomplished to exemplify how they thought about student independence further in the modelling process.

Group 3, working with grade 3, had another discussion about student independence, "we [the PTs] shall make something overarching, but it is the students who are leading this? The students will lead the way forward?". They said "we", meaning the PTs in the teacher role, taking thus a teacher perspective, which will be discussed next. The PTs were concerned with the idea of who is leading the modelling. They used straightforward statements "but it is the students who are leading this", they did not formulate them as questions, but they added a questioning tone at the end, indicating insecurity. As Blum and Borromeo Ferri (2009) also emphasized, there has to be a balance between teacher guidance and student independence for the modelling to be successful. However, this balance is not as simple to put into practice, as also shown by our examples. Being aware of the need to balance, the PTs questioned how this balance looks like.

**Teacher perspective**

This perspective includes thoughts about the teacher's role and position in the modelling classroom. One aspect had to do with how much should the teacher control and guide the modelling process. Group 3 questioned how much they should do for the students, and consequently, how much responsibility should the students have (student perspective), "…then we should just take care that it shouldn't be so clear guidelines. We shall not direct (lead) it too much, we shall make something overarching". The PT was sharing the idea that teachers have to take care of the modelling by not giving "so clear guidelines", and not directing too much, while group 4 suggested that teachers, "can come up with challenges along the way". Group 3 discussed "planning a birthday party" as an overarching topic for grade 3. Later on, they continued, "then we can guide them toward challenges, ask open questions, but not direct ones". Teacher questions are an important aspect of MM (Stohlmann & Abarracin, 2016). Group 3 exemplified some questions to ask in the "planning a birthday party" activity, e.g. "Won't you have something to drink?". The question was introduced as a "not direct" question, a probing question to only remind students to think about some aspects to consider, without leading too much. It is interesting to notice that PTs were heading towards strategic interventions as in Blum & Borromeo Ferri (2009), by planning to give hints to help the students move forward. They did not want to ask direct questions that could guide students too much so that they miss the independence. This was in line with their discussions about not giving clear guidelines.

Another aspect of the teacher perspective was flexibility, the need for the teachers to adapt teaching to their students. This was especially present in the discussions about the PTs' future students in the practicum period, as they had not met the children yet. Group 2 discussed a specific starting activity:

> Should we take that [a specific activity] as a starting point? Students may be very keen on something else when we go to practicum…once were marbles very popular; it can happen that when we are there [in practicum] we use that…

They emphasized though that if they found out that the students had other interests, they would need to change the topic, in order to engage them. This is a somewhat different focus than the one presented by Stohlmann and Abarracin (2016) where the teacher flexibility was more focused on encountering student diversity in thinking. The PTs referred also to adapting the mathematical level of the tasks to the students' level, such as e.g. when group 3 discussed an activity, "what if the students come up with the craziest numbers, what do we do then?" The "craziest numbers" referred to big numbers, which 3rd grade students could come up with if they were to plan a birthday party by themselves. The PTs were worried about students being able to handle the mathematics they come up with and asked what they, as teachers, could do in case the situations occur. Here the student diversity in thinking (Stohlmann & Abarracin, 2016) was the focus of the PTs' concern.

The teacher position and feelings were also present in the PTs' discussions about the openness of MM, as in Blum and Borromeo Ferri (2009) also, since MM can become unpredictable. "I think it is crazy, a bit uncomfortable, when we don't know the process [that the students will take with the task], we don't see the mathematics in this". It was group 3 who was expressing the difficulties teachers encounter when lacking complete control of the process, not knowing in advance the mathematics students can come up with, or how the process would go about. They referred to this as being "uncomfortable". One factor to explain this could be the PTs' lack of extended previous experience with modelling, as also Biembengut and Hein (2010) and Borromeo Ferri and Blum (2013) suggested.

**Activity perspective**

The PTs focused on the choice of activity for the modelling, its potential and goals, the organisational opportunities and challenges, as well as aspects of the modelling process in the classroom.

One focus in all the groups in this perspective was on establishing some criteria for the topic of the activity and the MM process. As such, the topic had to be interesting for the students to a certain degree. A grade 2 activity involving a children's slide was e.g. discarded by group 2 since the students "will get busy mostly with sliding down", and they would forget the task. Another criterion was for the activity's mathematical requirements to be suitable for the grade level. Such an example was earlier presented in the teacher perspective with the PTs worrying if the grade 3 students came up with the "craziest" big numbers they could not handle, or group 4 that discussed if a certain budget for the trip to Paris was too much for their students in regard to age.

Further criteria were for the activity to be open and to require the students to make choices. In the following extract, when group 2 was discussing a water basin in the school's neighbourhood, one of the PTs described a lesson she witnessed in her previous school practicum. One of her peers asked if the activity she mentioned was a modelling situation, and she answered:

> Yes, that is what the students were told, no exact answer, multiple solutions, positive [the PT's experience] how they found out about it, they use different strategies to find out how much they can afford, they have to prioritize.

The PT was listing what she observed in the situation, and many of the points in her list are characteristics of MM in education. The activity has e.g. to be open and give possibilities to choose, "multiple solutions", the students "have to prioritize" (student perspective), and the possibility to reach the goal in different ways, "different strategies to find out", as well as different answers or "no exact answer". She was "positively" surprised about the students' work with the task. This is in line

with English's (2010) findings according to which MM activities that include such qualities enable students to elicit key ideas and to mathematize in different ways.

PTs' discussions about organisational aspects are also included here. All four groups seemed to agree that MM activities should be done in student groups in combination with whole-class discussions. The students would not necessarily work with only one activity, groups 1 and 2 e.g. proposed to organize several stations with different MM activities to perform. The time needed for the activities was another organisational aspect the PTs were concerned with. Group 3 discussed how to round off an activity since "in theory, a modelling activity can last forever". They did not refer specifically to the modelling cycle, but it can be a possible explanation for this "last forever" given the cyclical nature of the modelling and the fact that a model has to be evaluated and improved continually based on the initial requirements. Time was seen as a hindrance for implementing modelling by experienced teachers in Borromeo Ferri and Blum (2013). In our study, group 3 that came the farthest in their planning, reacted to the tight schedule, but tried to find practical solutions such as «split a session into several days». The discussions in group 3 indicated that PTs were concerned with finding constructive solutions to implement the MM activity within the available time.

## Concluding comments

The PTs' discussions when planning MM activities focused on students, teachers and activity. They touched on several aspects of the MM process such as openness and the balance between student independence and teacher guidance, being at times unsure about to what degree to apply those aspects. The idea of the modelling process going all possible directions, where the teacher has no control, was experienced as uncomfortable by the PTs. The openness and the unpredictability of the MM were emphasized also by Blum and Borromeo Ferri (2009) as a hinder for teachers to implement MM in teaching. This indicates the importance for teacher education to facilitate for PTs to experience modelling situations and handle insecurity in practice. The modelling situations can be provided when PTs plan and implement modelling sessions with students, and when they themselves solve modelling problems, since both kinds of experiences are necessary for PTs (cf. Borromeo Ferri & Blum, 2010).

An important and challenging aspect of MM in the classroom are the teacher's probing questions to help students move forward with modelling while preserving the balance between teacher guidance and student independence. In our study, only two of the groups went as far in their planning as to discuss the activity's mathematical aspects they would have to help the students with, and questions they could ask. Teacher educators should facilitate the discussions so that the PTs spend more time discussing what their chosen tasks require from the students to be solved, how the students can eventually respond to them and how the PTs can formulate probing questions related to the MM tasks.

There is a similarity between the PTs' thoughts about using modelling contexts near to the students' lives and the aim for MM in the Norwegian curriculum to use mathematics to describe daily life. The results from all the groups show that the PTs saw this as connected to having motivated students. Different from previous research, they did not consider the "lack of materials" as a hindrance to implementing MM. The PTs did not stop at the context of the activity; they considered additional factors such as the activity's mathematical potential and pitfalls in the light of the curriculum, the students' abilities and organisational aspects. This shows that the PTs have already an idea of the complexity of the MM and try to consider many of those factors while planning.

Given the importance of MM in teacher education (cf. Maas & Engeln, 2018), our study's relevance consists on identifying areas in which PTs have good ideas to build on, but also areas in which teacher educators can focus on to facilitate the PTs' work with MM, as discussed in the previous paragraphs. Further research could focus on studying ways to facilitate the PTs' work to overcome challenges they experience with MM, such as supporting them in formulating probing questions for students.

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# Teachers' arguments for including programming in mathematics education

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*In recent years, programming has been inserted into mathematics curricula in many countries. This paper reports on interviews with 20 Swedish mathematics teachers who, as early adopters, teach programming within the frames of their ordinary mathematics lessons. Qualitative analyses of data identified four types of arguments for teaching programming in mathematics: to develop computational thinking; to increase engagement; to learn mathematics; or simply because it is a powerful tool. We conclude with some implications of these different arguments.*

*Keywords: Mathematics, computational thinking, programming.*

During the last decade, programming has been given a place in school curricula in many countries. Some countries allocate time for a specific subject (e.g. England with Computing) while others incorporate programming and computational thinking into existing subjects, primarily into mathematics (Mannila et al., 2014). In Sweden, *programming* is to be taught in mathematics and applied in technology, while a more general term in the curriculum is *digital competence* (Swedish National Agency of Education, 2018). The present study is part of a recently started research project regarding the ongoing integration of programming in school mathematics (Bråting et al. 2020). The project as a whole is theoretically embedded in Chevallard's (2006) framework about transposition of knowledge, which describes a praxeology of *what is taught* and *why it is taught* and how this changes as knowledge is transposed to different levels of the educational system. In 2018, when programming was added to the Swedish national mathematics syllabus, all mathematics teachers were obliged to teach programming. Official arguments as to why are scarce, except vaguely that it is to help increase students' digital competence. By exploring how teachers talk about introducing programming, our aim in this paper is to better understand the know-why on the teacher level, and to identify challenges in relation to the integration of programming in school mathematics. We ask the question: What arguments do teachers, who are early adopters of programming in primary and secondary school, give for including programming in mathematics lessons?

## Computational thinking

The term *computational thinking* (CT) was first introduced by Papert (1980) when he developed Logo programming. Although the terms CT and *programming* are used differently in research literature, it is clear that they are closely connected. Moreno-Léon et al. (2019) make a distinction between CT as a cognitive ability and programming as just one way of developing that ability. The Swedish national curriculum states that "pupils should be given opportunities to develop knowledge in using digital tools and programming to explore problems and mathematical concepts, make calculations and to present and interpret data" (Swedish National Agency of Education, 2018, p. 55). The curriculum prescribes the use of programming using non-digital activities and visual as well as text-based environments, but does not mention thinking skills or cognitive abilities. Hence, how teachers

interpret what programming is, and why they should teach it is embedded in the question of transposition of knowledge.

There are many attempts world-vide to conceptualize computational thinking skills and show how digital technologies could enhance cognition. One such framework by Brennan and Resnick (2012) proposes three dimensions of computational thinking: *computational concepts* (sequences, loops, parallelism, events, conditionals, operators, and data), *computational practices* (being incremental and iterative, testing and debugging, reusing and remixing, abstracting and modularizing), and *computational perspectives* (i.e., expressing, connecting, and questioning). We consider this framework useful, as it broadens the perspective on computational thinking in the era of the 21st century. Brennan and Resnick's framework emerged from their studies of young interactive media designers using Scratch, a common environment in educational settings. We will use the term computational thinking (CT) in line with Brennan and Resnick (2012) throughout the paper.

## Research on teacher's views on programming in mathematics

The incorporation of programming in school is a new phenomenon, with roots in early initiatives in the 1980's when Papert (1980) introduced Logo programming as a way to develop mathematical understanding. Little is yet known about what is taught and why, in the revival of programming in schools taking place in the last decade. There are a few studies on teacher's views, mostly based on written surveys, for example by Mannila et al. (2014), who gathered data from 961 teachers in five European countries. The researchers asked explicitly about CT skills and concepts, and concluded that teachers already were involved in activities with potential for introducing aspects of CT. Misfeldt et al. (2019) collected data from 133 Swedish teachers, showing that, although teachers were positive towards working with programming in mathematics, not all could see the relationship between the two, nor the relevance for doing it. From an on-line questionnaire given to Finnish primary school teachers, Pörn et al. (2020) analysed 91 written answers to the questions "What is programming". They found that the teachers primarily emphasized "writing, giving and following instructions", but also mentioned mathematical skills such as logical thinking, problem solving and identifying patterns, as well as using modern technology and preparing their students for future work and studies.

In England, the introduction of computing as a new compulsory school subject in 2014, triggered a large research project about using Scratch in mathematics (Benton et al. 2017). While the English curriculum emphasizes CT skills, the study showed that it is also possible to teach mathematical ideas while teaching CT, highlighting the importance of teachers making explicit links between programming and mathematics.

In a related study to the one presented here, Nouri et al. (2019) conducted interviews with 19 teachers with approximately two years' experience teaching programming in school, although not only in mathematics. They found that the teachers talked about developing skills that corresponded well with Brennan and Resnick's (2012) three dimensions of CT, but identified also four other types of skills that teachers wanted their students to develop; cognitive skills, language skills, creative problem solving skills and collaborative skills. No clear mention is made in the article of developing mathematical understanding through programming.

## Method

Data for this study was collected through semi-structured interviews with 20 teachers, classified as "early adopters" (EA) because they were enthusiastic about the implementation of programming in school and already had experience in teaching it. They were recruited through different teacher networks and identified themselves as early adopters. All the EA's teach mathematics, 15% in grades F-3, 30% in grades 4-6 and 55% in grades 6-9. They have 6–35 years of teaching experience, 75% of them also teach technology and 55% have an extended responsibility to implement digital tools and programming in their schools. Their programming competence is diverse; three have an engineer exam, some have participated in one or two programming courses for teachers, but 40% are self-taught with no programming credentials at all. They work in 14 different municipalities well spread around the country and in schools of various sizes.

The interviews were audio recorded and transcribed verbatim. Two of the authors conducted the interviews, initially doing three together and the rest separately. The interviews took approximately 30 minutes and were structured around eight questions that had been supplied in advance with the intention of capturing different aspects of teachers' talk about programming in mathematics. Following four background questions the interview guide included the following topics: 5) What is the role of programming in mathematics? 6) Where do you find inspiration and ideas? 7) Can you give an example of a good programming activity that you have tried? 8) What programming concepts are important to bring up in mathematics? As data is rich, we will focus in this paper on the explicit connections teachers made between programming and mathematics throughout the interview.

A data-driven analysis of the transcripts was made using NVivo software, identifying and sorting quotes into categories illustrating different types of arguments. Categories were identified through a thematic analysis (Braun & Clarke, 2006) in an inductive bottom-up process. In an iterative process the categories were identified by the first author, then revised several times when discussed, compared and validated within the research team. Once identified, the categories were related to the CT framework of Brennan and Resnick (2012). In each category we tried to capture a unique aspect of what the teachers brought up as reasons to work with, or benefits of, programming. Although distinctly different, the categories are sometimes intertwined. While focus is on the phenomenon, not the individual teacher, each interview could generate several arguments.

## Results

We found four main categories of arguments, with some sub-categories. Each category is described in short and instantiated below in a few typical quotes for each category.

### Category 1. Programming is a potentially powerful tool

Teachers see programming as a pedagogical tool in addition to other tools that are potentially useful in mathematics once you master them sufficiently. They think that teaching programming will change with time, since they and their students initially need to spend time learning how to use the tool.

> You need to think of it as a pedagogical tool, just like some years ago when we were making films. See it as a pedagogical tool, that I could use even if I teach Geography, or History. […] I can use it as a working tool for the children, so that somewhere down the line they need to learn how it works. (EA12)

To use programming as a tool for doing mathematics, like GeoGebra, I think that is great. […] All tools that support learning are good, and I think programming is one of them. (EA18)

Well, on this level you don't really need programming to solve the problems. I think it is more that you see that it is possible to use programming to solve problems […] for the students, maybe paper and pencil is faster than using a computer […] when you get to more complex problems programming could be useful, so then you need to learn it on these easier problems. (EA20)

**Category 2. Programming increases engagement**

Teachers say that students find programming interesting, that they engage better with programming tasks than with traditional mathematics tasks. Students become more engaged because it is interesting and fun (2.1) or connects to reality (2.2). They may not learn mathematics through programming, but implicitly the teachers believe increased engagement will enhance mathematical learning.

*Category 2.1 Programming is interesting and fun*

Some students who become interested in programming, they learn a lot of other things too. […] it definitely increases motivation. […] And to develop that interest they need to learn the mathematics, to be able to move on. (EA01)

Well, content wise, at this level I don't think it contributes much but it does add a new dimension, the students get more inspired, more motivated to do the work. (EA17)

Fun, demanding, challenging and inspiring. (EA19)

*Category 2.2 Programming connects to reality*

That's what I find so fascinating, that it is connected to something they have done before, or seen as useful. Well, maybe not always useful, but they have seen it in reality in some way. (EA13)

The students get inspiration, get really interesting discussion going, for example during my maths lesson, about future occupations. I haven't very often talked about that before […] now they have a feeling that: this is programming, and that's a job I could have. (EA17)

**Category 3. Programming develops computational thinking**

Teachers describe programming as a new way of thinking and working, which is different from traditional mathematics but relevant for mathematics education. More or less explicitly, they refer to the development of computational thinking practices as described by Brennan and Resnick (2012), where being incremental (3.1) is pointed out as a generic practice, not restricted to mathematics.

*Category 3.1 Programming teaches students to break down instructions and problems into small sequential steps.* This category connects to the practice of being incremental.

It is a new way of thinking for the children. They have to get into thinking in small steps and giving instructions. Even if we work a lot with instructions in Swedish language class, writing and giving instructions, here is another connection because they need to be so precise. (EA12)

Well it is a logical thinking process, I think. We work a lot with all these concepts, like loop, algorithms. How to create algorithms, that all goes with programming, but also with Swedish language class, like cutting a story into small parts and then putting them all in the right order.

That's how you get computational thinking. That's how I work with the six steps of computational thinking. (EA15)

To me, an algorithm is like a cooking recipe, it is a step-by-step instruction. That is the thinking behind algorithms, I think, and that is a very mathematical way of thinking. (EA16)

*Category 3.2 Programming encourages testing, debugging and modifying*. Teachers highlight that students learn new practices from programming that are useful when doing mathematics, such as testing, failing, and working iteratively, as well as the importance of being persistent and meticulous.

They think it is easy in programming, looking for mistakes and trying again and again and again, many, many times. I would like them to take that with them when they work with other types of mathematical problems. We talk a lot about that. Because sometimes they have the patience, sometimes they don't. […] It is absolutely a feature of computational thinking, to try and fail, and then try again, make small changes … (EA17)

For once they are in a position where it is acceptable to fail, you need to try, […] In programming some things always go wrong. It is a natural part of the process to "hit the wall", to search for mistakes and that sort of thing. [EA18]

I want my students to understand code, to be able to read code, to modify code. It is the same with GeoGebra, it is all about modifying something. You don't really need to know what happens in the background, but you need to be able to modify it to fit the problem you have right now. That's what you need to know. In Excel too. Here is a context where students really find themselves in a position where it is extremely important to be meticulous, where the tiniest mistake can make it all go wrong. (EA18)

## Category 4. Programming is a way of learning mathematics

Teachers want students to learn mathematics through programming. It can be a tool for learning some specific mathematical content (4.1) or it can be considered as mathematics in itself (4.2).

*Category 4.1 Programming is a tool to learn mathematics*. In this category programming is described as a tool for learning some other mathematical content rather than being fundamentally mathematical in itself. However, the teachers often describe it as an ideal, a difficult goal and not yet reached.

The programming helped [students understand] the concept of variable. (EA11)

In mathematics it is more a tool to reach mathematical understanding. Which makes it more difficult to choose good examples in mathematics. (EA01)

Well, mathematics and programming don't really go together. You can be a good programmer and lousy at maths, or the opposite. But you can use programming as a tool to enforce mathematical concepts, of that I am certain. (EA18)

It [Desmos] is one of the best I think we can find right now for mathematics and programming. Because it focusses on programming to make the mathematics visible. (EA05)

By using Excel, Ipads and GeoGebra, the students could investigate relationships between area and perimeter. (EA03)

*Category 4.2 Programming is a genuine mathematical activity*. One of the teachers talk about programming as a mathematical activity in itself.

> I think teaching has been too instrumental, where you never get to the heart of mathematics, like what is the beauty of mathematics, what it is that is interesting in mathematics. I think it is puzzles and patterns and mysteries. And like how we can figure this out and how this relates to that. And I think programming makes it easier to get to that heart. That's how I think about it. (EA09)

## Discussion

Although we did not explicitly ask the teachers to define programming, a lot of their arguments implied a view quite similar to what was found among Finnish teachers (Pörn et al., 2020), as primarily to learn about *giving and taking instructions expressed in small sequential steps* (see Category 3.1). This is a generic skill, not solely mathematical, although fundamental in mathematics. But should not programming in mathematics teach mathematics? The four types of arguments that emerged from our data highlight an important dichotomy in relation to mathematics.

On the one hand Categories 1 and 2 describe programming as *useful and engaging on a general level*, not necessarily connected to mathematics, in line with the teachers in Nouri et al.'s (2019) study who pointed mainly to student's development of general skills, not to programming as a mathematical activity. Being early adopters, the teachers in this study are positive to programming activities, but like teachers in the study reported by Misfeldt et al. (2019), they do not always see the clear connection to mathematics. In fact, some actually think it is easier to include programming in other subjects, which we see also in category 3.1. This could be a result of the fact that visual environments, such as Scratch, are more closely adapted to storytelling activities and animation than to mathematical activities (see also Bråting et al., 2020). Consequently, promoting visual environments while at the same time placing programming in the subject of mathematics is a challenge. This was also shown by Benton et al. (2017) who emphasized the importance of designing Scratch activities with specific mathematical goals, and the teacher's role in making mathematical connections explicit. Using Scratch is in itself not necessarily a mathematical activity.

On the other hand, arguments in Categories 3 and 4 proclaim that programming could help students learn about mathematical ideas and develop practices that are fruitful for mathematical work. We see some teachers emphasizing the development of computational thinking, actually highlighting that CT skills are valuable for mathematics and may be instrumental in changing students' way of working and thinking in mathematics in a good way. By introducing practices like debugging and modifying, suggested in the Brennan and Resnick (2012) framework of CT, and by encouraging students to see failure as a natural part of a problem solving process, the teachers believe that mathematics learning will benefit from programming activities. It is interesting that only one quote from all the 20 interviews describe programming as a mathematical activity in itself (Category 4.2). We see here a challenge for mathematics education researchers and developers: could we come up with programming activities that are more mathematical in nature?

Considering the arguments in Category 1, we conclude that some teachers trust new technology, embracing arguments about the usefulness of programming from other levels of the educational system, but without actually experiencing its usefulness themselves. They say that at the moment we need to spend time learning the tools, referring to themselves as well as their students. Presently the

whole school system is faced with a big challenge when teachers are asked to teach something they do not yet master. Perhaps really seeing how programming can promote mathematics will only be possible when programming had become a tangible feature of everyday life. Some teachers feel that they need to get to that stage first. However, if the main purpose is to learn to use a new tool, it is questionable if the time spent on programming is wasted from a mathematics education point of view. When computers were introduced as new tools in school, learning to use keyboards, search engines and word processors was not included in the mathematics syllabus.

In accordance with the theory of transposition of knowledge described by Chevallard (2006), we can see that the know-why of programming in mathematics has changes as it moved from the curriculum level to the teacher level. While the curriculum mentions programming to explore problems and mathematical concepts, the early adopters tend to focus more on the tool itself and talk about the benefits of programming in general terms. In schools there are many teachers who know much less about programming than these early adopters and who may well have even more general arguments for, or even against, introducing programming in mathematics. As another part of this research project we are currently collecting data from such teachers as well. Although our early adopters express ideas about programming as a powerful tool for engagement and development of thinking skills, it does not necessarily imply learning in mathematics. To benefit from greater student engagement, teachers need to be able to discern powerful mathematical ideas in programming, as well as computational aspects of mathematics, which will require good skills in both programming and mathematics.

In our analysis we easily found the first two dimensions described in the framework by Brennan and Resnick's (2012); computational concepts and computational practices. Working within the digital era with the 21st century skills (including programming) could initiate a transformation of mathematics education to embrace errors as resources for scrutiny of taken for granted mathematical concepts. Such a transformation would most likely change teachers' epistemological beliefs about mathematics into an experimental and practical curriculum, necessary if computational practices and perspectives are to flourish in school mathematics. But this is demanding, and meanwhile, programming takes time from more traditional mathematics learning. Drawing on the early adopters' arguments, we conclude that programming could, but may not, enhance mathematics, depending on whether or not the teachers look for opportunities where it could, and are open to changes in mathematical thinking practices. This is a great challenge for teachers who are not early adopters.

**Acknowledgment**

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# Facilitating argumentation in primary school

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*In this article, we show some results from an ongoing project in which written dialogues are used as a tool to facilitate exploratory talk and argumentation in mathematics in primary school. Our data material was collected in a 7th grade classroom in Norway. The students were asked to individually write an imaginary dialogue between two fictive students applying ground rules for exploratory talk. Afterwards, the students had a discussion in groups of 3-4 participants about the same problem. Our findings indicate that the use of imaginary dialogues might be a valuable tool that makes students explore the characteristics of exploratory talk and facilitate argumentation in mathematics.*

*Keywords: exploratory talk, argumentation, imaginary dialogues.*

## Introduction

There is an ongoing discussion about the nature of argumentation in mathematics and about how to facilitate argumentation in the mathematics classroom (Reid & Knipping, 2010). The research presented in this article is situated in a Norwegian context, where a new curriculum for primary school is coming up in autumn 2020. This curriculum strengthens key concepts such as reasoning, argumentation, representation, and communication (Ministry of Education and Research, 2019). In this article, we present some results from a project about facilitating argumentation where we look at how to develop exploratory conversations among students in primary school. Following Krummheuer (1995), we consider argumentation primarily as a social phenomenon, in which cooperating individuals try to "adjust their intentions and interpretations by verbally presenting the rationale of their actions" (p. 229). According to Mercer and Sams (2006, p. 510), children "need to be helped to learn how to use language to work effectively together: to jointly enquire, reason, and consider information, to share and negotiate their ideas, and to make joint decisions." Teachers must be aware of the nature of argumentation and how to create an environment in the classroom where everyone feels free to come up with their own ideas even if they are wrong.

Barnes (1976) distinguished between exploratory talk and presentational talk. Exploratory talk is a type of talk that emerges under work in progress when students are working collaboratively in order to develop a solution to a problem. It is a talk in which ideas and partial solutions are presented, discussed, improved or dismissed. These are important actions in mathematical argumentation. In contrast, presentational talk is a type of talk that is often found in school when students are asked to present their results, solutions or projects to the teacher and other students in the class.

In our project, we combined phases of individually developed argumentation with group discussions among students. The individual argumentation was developed by the students in the form of written imaginary dialogues, a sort of mathematical writing that was introduced by Wille (2011, 2017) as a tool to understand students' mathematical thinking. In the current project, we used this method in order to develop students' abilities to engage in exploratory talk which we regard as an ideal type of talk in a collective argumentation. The aim was to encourage the students to use the features of exploratory talk in written imaginary dialogues about a mathematical problem. Afterwards, the students discussed the same problems in groups of 3-4 participants. These discussions were recorded

and analyzed to see to what extent the students got critically, but constructively involved in one another´s ideas. While Askevold and Lekaus (2018) analyzed the type of argumentation in imaginary dialogues that became visible through language and drawings, in this article we focus on how to stimulate argumentation and exploratory talk among students. The research presented in this article was guided by the following research question:

Which role can the writing of individual imaginary dialogues play when facilitating the use of exploratory talk in the mathematics classroom?

## Theoretical framework

In this project, we combined Wille´s (2011) ideas of imaginary dialogues and Mercer´s (1996) theoretical framework about exploratory talk. The aim was to let students explore the features of exploratory talk individually using imaginary dialogues, and then apply these features in group discussions. We used the theory of exploratory talk to analyze both the students' imaginary dialogues and the discussions among peers.

Wille defined imaginary dialogues as "a form of mathematical writing where a *single student* composes a written dialogue between two protagonists who discuss a mathematical task or question» (2017, p. 30). She described imaginary dialogues as a way to activate a person's inner mathematical discourse and make students' mathematical self-communication observable, in which mathematical ideas are developed, reflected upon and revised (Wille, 2011).

While Wille investigated students' individual mathematical thinking, Mercer (1996) elaborated Barnes´s (1976) ideas about classroom talk. Through observational studies of children´s talk in groups, he developed an analytical model with three different ways to categorize talk among peers: disputational, cumulative and exploratory talk.

1. Disputational talk: characterized by disagreement and individualized decision making. Utterances are short and not constructive. If at all, only few explanations and justifications are offered, and few questions are asked to other participants. Participants may make suggestions, comments and also give advice on each other's actions, but the conversational activity is competitive, and participants take turns in their actions.

2. Cumulative talk: speakers build positively but uncritically on each other's utterances. Their talk is characterized by repetitions, confirmations, and elaborations. Participants build knowledge by accumulation, in which others' suggestions are not challenged.

3. Exploratory talk: speakers engage critically, but constructively with each other's suggestions. They share information and explain their thinking, but also challenge others' suggestions in a constructive way, offering justification and alternative hypotheses. The participants' reasoning is visible in the talk. (Mercer,1996, p.369)

In contrast, Barnes defined presentational talk as a talk where "the speaker's attention is primarily focused on adjusting the language, content and manner to the needs of an audience" (2008, p.4). While exploratory talk often happens at the beginning of a discussion, presentational talk tends to happen when the student has come to a solution of a problem.

## Method

Data was collected in a 7th grade multilingual classroom (students aged 12-13 years). After consulting the teacher, we designed tasks related to fractions that required argumentation. Students worked in two phases with each task, first individually and then in groups of 3 – 4 students. The class was divided into two smaller groups of 8 – 10 students for two lessons. In the beginning of the first lesson, both group of students were introduced to some features of exploratory talk. Two researchers dramatized a short conversation between two fictive learners who engaged in a constructive discussion about a fraction problem, with elements of oral conversation, writing and drawing. Some characteristics of the dramatized conversation were discussed, and the researchers asked the students which words, phrases or questions could be appropriate in order to explain one's own thinking or encourage other group members' contributions. The students gave examples such as "I think that", "why do you think so", "can you explain this?", "I don't think this can be true because". These were called "ground rules for exploratory talk" (Mercer et al., 1999). In the individual phase, students were encouraged to write an imaginary dialogue (Wille, 2011), applying these ground rules in order to make both fictive characters active partners in the conversation. We wanted to give all students the opportunity to develop their own ideas, hypotheses and initial explanations and to give them time to practice how to phrase these before they engaged in a group discussion. We hoped that this would help to create a variety of ideas to be built on in the group working phase. The individual phase lasted about 15 minutes, and the collected data material consisted of the students' written imaginary dialogues. In the following group discussions, the students were encouraged to use the previously suggested phrases whenever they explained their thinking or wanted to challenge other speakers' suggestions. These group discussions lasted around 7 ½ minutes. In the beginning, the student groups discussed without interference by the teacher and the researchers. When the teacher or one of the researchers noticed that the groups had stopped their discussion, they asked the students to explain their thinking and occasionally gave a hint that should encourage further investigation, an alternative solution or representation of the problem and sometimes a generalization. The data material collected during this phase consists of sound recordings of group discussions and the researchers' observations.

| Monthly allowance | Find a fraction between the two given ones |
|---|---|
| Per: I have already used half of my monthly allowance this month. | Student1: It is easy to find a fraction between 2/5 and 4/5 because there is 3/5 in between. |
| Mari: I have not used as much yet. I've only used a third. | Student2: Yes, but also ½ is in between the fractions. |
| Per: What have you bought? I went to the cinema. | Student1: Right, there can be several possibilities, but I cannot find a fraction between 1/3 and ¼. |
| Mari: Me, too. But then we have used the same amount of money. How can this be? Because 1/3 is less than ½. | |
| Per: Hmm… | Student2: …………………………. |
| Mari: …………………………. | Continue the dialogue and check if you can find various solutions. |
| Continue the dialogue and help the students to find out about this. | |

Table 1

We looked for tasks that addressed typical misconceptions about fractions and made argumentation necessary. The entire group of students worked with the two tasks "Monthly allowance" and "Find a fraction between the two given ones", shown in Table 1. In the first task, the fictive students refer to different units, while the second task requires skills about equivalent fractions. In the last part of the second lesson, the two groups worked on different tasks. Only the two tasks that the whole class worked with are analyzed in this article.

All group discussions were transcribed. Both researchers controlled the quality of the transcription in order to increase reliability. Afterwards, the written dialogues and the group discussions were coded using the four categories *presentational talk*, *disputational talk*, *cumulative talk*, and *exploratory talk*. Barnes (2008) used the concept presentational talk primarily when students present a solution to the teacher. We also use it in situations when students present their products to their peers. The following characteristics were used in order to identify the different types of talk:

Presentational talk: Students take turns to present their (partial) solutions which they regard as their individual products rather than ideas to be used in a joint discussion. We find expressions like "your turn", "who should start", "I didn't write such a long story". Some students "act" and change their voices like in a dramatization to show which one of the characters is speaking.

Disputational talk: Students are competitive and want the group to use their own ideas, but no reasons are given. Refusal of proposals is not justified either. In the language, this is visible by uses of "no – yes – no – yes" and the lack of words like "because".

Cumulative talk: Students build an argumentation on each other's ideas, but without giving or asking for justification: "ok", "I agree", "I thought so, too", "I did the same".

Exploratory talk: We find a variety of ideas and approaches, various representations of fractions, or of ways to present the solution. Explanations are given and asked for. Both characters in the written dialogues or at least two persons in the oral discussions contribute ideas. Doubts are expressed, for example by "But there is still…". Ideas are refused in a constructive way, offering a justification for the refusal. The students are open for others' ideas, expressed for example by phrases like "good idea". Talk can be exploratory even though the presented ideas are not mathematical.

Usually, we could assign the students' written dialogues as a whole to one of the categories. Some of them were too short and had no clear characteristics of one of the categories such that they could not be placed in any of the categories. These were dialogues consisting of only up to 3-4 sentences that had diverse characteristics concerning the type of talk. Some written dialogues were found to be cumulative except for one single phrase of an exploratory nature (such as "Why do you think so?") taken from the list of phrases written on the white board. These dialogues were placed under cumulative talk, as this was the nature of the major part of these fictive dialogues. The group discussions usually had to be divided into smaller parts since the character of talk changed during the conversation.

## Analysis and discussion

We analyzed 30 written imaginary dialogues about these two tasks. Of these, we classified 12 to be exploratory and 8 to be cumulative, 1 presentational. Another 9 dialogues were short and had no clear characteristics such that they could not be classified. In the next section, we present and analyze two

different examples that are rich in language, use multiple representations, and use conjunctions or open-ended exploratory questions. The examples illustrate dialogues in which both protagonists get involved in argumentation and do not contain examples where one of the fictive students dominates the dialogue. We have looked for examples where the students combine written argumentation with other representations for example symbols and drawings, including those that we think contain a sort of argumentation which is not expressed in words. We have chosen these examples to show the potential of this method, and one has to bear in mind that it was the first time the students tried such activities.

**The written dialogues**

The dialogue shown in Figure 1 results in a complete solution and displays many features of exploratory talk. The fictive students use expressions displaying interest in engaging in a joint investigation like «let us look at it», «we can look at it in another way», «what if» and are trying out various representations of fractions and decimals. They are given equal roles and show a positive attitude towards the other student's proposals, expressed for example by «good idea». They are humble about their own proposals calling them «hypothesis» or saying «I am not sure. Let's calculate". Several times the words «since» and «because» are used in order to offer justification and the fictive student Per displays a critical stance when saying «this was strange since…».
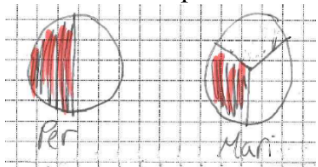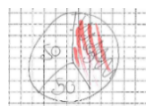
Mari: Let us look at it

Per: If you have used 2/3 of your monthly allowance and I have used ½, we have not used the same amount

Mari: We draw it as slices of pizza



Mari: It is obvious that I have used less money than you

Per: hmm, that is strange since both of us have used the same amount of money

Mari: I have a hypothesis. Because we have both used the same amount of money, but since the diagram/slices of pizza show something else, it might be that we got different amounts of money

Per: We might look at it in a different way. How about trying equivalent fractions and convert to decimals?

Mari: Good idea.
$\frac{1}{2} = 0,5$      $\frac{1}{3} = ^{ca} 0,33$

Mari: I think I got more money than you.

Per: I got 100 kr. How much did you get?

Mari: I am not sure. Let us calculate.
100 – movie (50) = 50 =1/2
Per = 50 "kroner" left

Mari: Now, I understand! Since the movie cost 50, it must be like this:


Per = 100 kr
Mari = 150 kr

Figure 1: Imaginary dialogue between two fictive students

The dialogue shown in Figure 2 has fewer words, but we also classified it as exploratory talk. Phrases such as: «but…, do you have a solution…, maybe…» indicated a critical stance taken by both fictive protagonists. The fictive students find equivalent fractions with the dominator 12, but they notice that there is still no number in between the numerators 3 and 4. They use both fraction and written phrases as a form of representation. Finally, they try a denominator showing more parts and find the solution 7/24. We found the visual representation of interest that accompanies the last part of the dialogue. The drawing of 1/3 and 1/4 does not seem to be random: the rectangular model contains 24 small squares, such that when 1/3 and 1/4 of the rectangle are shaded, this makes 8 and 6 small squares of

the total 24. The drawing thus seems to contain additional justification of why 7/24 is a fraction between the two other fractions, even though the fraction 7/24 is not visible in the drawing. The drawing could be a starting point for group or classroom discussions about how to communicate through a representation. This could make students aware of which information is contained in a representation or only in their own thoughts and how their peers might understand a representation. This could link the focus points argumentation, communication and representation in the Norwegian curriculum.
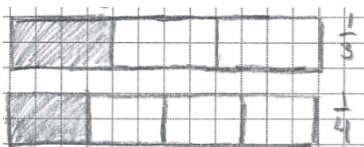
| Student 2: How about finding a common denominator?<br><br>$\dfrac{1}{3}$ and $\dfrac{1}{4}$ $\quad \dfrac{1 \times 4}{3 \times 4} \quad \dfrac{1 \times 3}{4 \times 3}$<br><br>$\qquad \dfrac{4}{12} \qquad \dfrac{3}{12}$<br><br>Student 1: But the difference is only one.<br>Student 2: Do you have a solution? | Student 1: Perhaps a denominator with more parts?<br><br>$\dfrac{4 \times 2}{12 \times 2} \quad \dfrac{3 \times 2}{12 \times 2} \quad \dfrac{8}{24} \quad \dfrac{6}{24}$<br><br>Student 2: Yes, that was smart! 7/24 is in between!<br>Student 1: Or draw it  |

Figure 2: Imaginary dialogue between fictive students

We found other examples of exploratory talk in fictive dialogues, presenting different approaches with various representations or use of percentages and decimals. Not all of these dialogues resulted in a solution of the fraction problem. Few of the examples were "perfect" examples of exploratory talk, but we found potential in many of them such as use of representations, utterances expressing doubts, asking for explanations and introducing justification. The teacher can build on these in either group discussions or in classroom talk. Dialogues and representations can serve as starting point for reflections about when mathematical argumentation is complete and what mathematical argumentation is in contrast to argumentation in general.

**The group discussions**

In the group discussions, we could find all types of talk. The results show however that exploratory talk was more prominent in the imaginary dialogues than in the group discussions among the students. We found some parts with disputational talk, mostly caused by disagreement about details in the formulation of the tasks. The group discussions about "Monthly pocket money" of all but one groups started by the group members taking turns ("I can read mine first", "your turn") and reading aloud their own fictive written dialogue to the other group members. The start of these group conversations thus had the character of presentational talk. Even though these parts of the discussions were not exploratory, we could see that some of the ground rules were applied. All group members were invited to join the conversation ("your turn"), the other group members listened and attempted to understand. This became visible in various ways. For example, we found examples of group members offering mathematical vocabulary ("equivalent fractions"), correcting imprecise use of mathematical language ("double the denominator …" instead of "double the fraction…"), comparing others' solutions to one's own ("that was very different from mine"), or suggesting corrections for a solution ("I think it should be 3 there instead").

Cumulative talk occurred as part of several conversations where the students listened positively but uncritically to each other's solutions. However, a detailed analysis showed that the students were

uncritical because they had come up with the same solution and had found a convincing justification. We saw many valuable features of these conversations as well and some of them started as exploratory talk. For example: "What did you think? – Equivalent fractions. – Yes, I did too.", and the conversation continued with one explaining and the other confirming.

We could see evidence in our data material that different organizational choices affected the qualities of the group conversations. For example, the students had so much time at their disposal during the individual phase of the task "Monthly allowance", that most of the students had arrived at a complete solution when they started the group conversations. Our findings suggest that the students wanted to present their solution to their peers and seemed less willing to engage in others' ideas and explore different ideas. This is in line with Barnes (2008, p.4-5), who found that students were more exploratory when they started to solve a new problem. During the first group conversation, which resulted in mostly presentation of solutions, the students also had access to their written imaginary dialogues. We changed this when students discussed the second task: the individual phase was shortened, and we collected the students' written dialogues before the group discussions, such that the students could not read their ideas aloud, but had to explain them to their peers. Therefore, no presentational talk was found in the group discussions about the second task.

## Conclusion

In the current project, we explore the role of writing individual imaginary dialogues as part of facilitating the use of exploratory talk in the mathematics classroom. We regard exploratory talk as an ideal type of talk in a collective argumentation in mathematics. Mercer (1996) characterized exploratory talk as one in which participants share information, explain their thinking, make their reasoning visible and challenge others' suggestions. We used imaginary dialogues as a method in order to develop students' abilities to engage in exploratory talk. In an imaginary dialogue, a student must take two roles, practicing both reasoning skills, using various representations in their explanations, and getting the opportunity to take a critical stance towards suggestions. In this way, there is a strong connection between reasoning, representation, communication and argumentation in imaginary dialogues. These are elements in the new Norwegian curriculum in mathematics education.

Teachers get access to a written material that can be built on in classroom discussion. The imaginary dialogues can reveal students' ability to use ground rules of exploratory talk. They also show what kind of representations the students use and which explanations they construct around them. Teachers can use these as a starting point to discuss how to communicate through a representation.

The transition between being exploratory in a written conversation and in a conversation with peers is however not automatic. The imaginary dialogues gave all students time to think about one or more solutions, and the data shows that all students entered the group discussions well prepared and were able to contribute. The students also applied many of the ground rules of exploratory talk, including peers in the oral conversations and responding to others' suggestions. But, our data shows that when the students already had a complete solution to the problem after the individual phase, the talk that emerged was mainly presentational, with students taking turns to show their own solutions. Using imaginary dialogues, one must bear in mind the level of difficulty of the tasks and the length of the individual phase. An ideal task should encourage the students to think but not yet come up with a

complete solution as part of the imaginary dialogue. That would support exploratory talk and argumentation in the group discussion.

Our data shows encouraging evidence that the combination of imaginary dialogues followed by exploratory talk in groups might be a tool to make students explore how to use ground rules for exploratory talk, which according to Mercer (1996) take time to establish.

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# Surveying mathematics preservice teachers

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*Preservice teachers begin their mathematics teacher education with a set of understandings about different aspects of mathematics education which will affect their engagement with mathematics teacher education courses. However, very little is known about these understandings. In this paper, the design of a survey to find out about specific aspects, emphasised in the new Norwegian curriculum, is described along with initial results from 96 preservice teachers. The results suggest that using scenarios can provide relevant information about PTs' understanding about different aspects of teaching mathematics. These results may provide teacher educators with potential starting points for planning their own teaching.*

## Introduction

In this paper, we describe the design of a survey and the initial results from its first implementation. The focus of the survey was on preservice teachers' understandings about argumentation, critical mathematics education, ICT and modelling in multilingual mathematics classrooms, which contribute to a 4-year research project, *Learning about teaching argumentation for critical mathematics education in multilingual classrooms* (LATACME). The main aim of LATACME is to document in a systematic way what supports and hinders preservice teachers (PTs), for the first seven years of school, to learn about teaching argumentation for critical mathematics education in multilingual classrooms. By asking the PTs to complete a survey at the beginning of their first compulsory mathematics education course and then again at the end of their second compulsory course, we anticipate being able to document if learning about these aspects of mathematics teaching had occurred. Initial results from the survey also provide teacher educators with input for planning of activities to increase PTs' understanding of these aspects.

LATACME is a response to a perceived need to improve the current teaching in Norwegian schools (Bergem, Kaarstein, & Nilsen, 2016), with a requirement that PTs "have knowledge about and an understanding of multicultural society", including "awareness of cultural differences and being able to use these as a positive resource" (National Council for Teacher Education (NRLU), 2016, p. 9). With a new curriculum coming into operation from August 2020, there is a need for teacher education to provide opportunities for PTs to increase their understanding of aspects of mathematics teaching to do with "reasoning and argumentation" and "modelling and application", two of six "core elements" and "digital skills" as one of five "basic skills" (Utdanningsdirektoratet, 2019).

In the new curriculum, there is also a focus on democratic competencies, which we link to critical mathematics education (CME). In describing CME, Skovsmose (1994) stated that teaching and learning should be oriented towards "the goal of providing students with opportunities to develop their critical competence in the form of qualifications necessary for their participation in further democratisation processes in society" (p. 61). This resonates with the new curriculum, where "critical thinking" is identified as a key value for mathematics and involves "critical evaluation of reasoning

and argument" that "can prepare students to make their own choices and to address important issues in their own lives and in society" (Utdanningsdirektoratet, 2019, p. 1). Although our project focuses on a range of aspects making it complex, teachers are also expected to respond to this range in their own classrooms and so dealing with the complexity is something that we as teacher educators as well as researchers must deal with.

In order to ensure that the survey focuses of the aspects which are the core of our project, in this paper, we describe how we designed the survey and some initial results from its first application. Our aim for doing this is to illustrate some of the complexity in developing a survey that tries to identify what PTs know about a range of aspects to do with mathematics teaching. In another paper at this conference, we provide more extensive details of the results of the first survey.

## Literature review

There is a large amount of research on PTs' mathematical knowledge (Ponte & Chapman, 2008), but less research has been undertaken on PTs'understandings about pedagogical issues connected to mathematics education. This meant that there was limited research, both in Norway and elsewhere, on which to draw in identifying how PTs' understanding of core aspects could be investigated. Due to limited space in this paper, we describe the most relevant of the earlier studies.

In Norway, Thomassen (2016) found that PTs in their fourth year of their teacher education paid attention to and critically reflected on multiculturalism and the education of minority language students, in group discussions. The PTs claimed that they lacked possibilities to focus on this topic in their teacher education, in regard to subject teaching. In Rangnes and Meaney (2021 forthcoming) the PTs noticed how Grade 2 multilingual students, when working with modelling tasks, managed to pose mathematical problems, identify appropriate measurement tools for solving the problems and develop their understandings of different aspects of measurement. In this outdoor modelling activity, the multilingual students were described as skilful when using concrete materials to solve problems, but when the same students worked with textbooks indoors and used concrete materials, they were described as lacking mathematical skills and their home language was not considered a resource.

Stylianides and Stylianides (2009) explored elementary PTs' constructions and evaluation of proofs, which we considered to be related to our focus on mathematical argumentation. Across the semester, the PTs collectively identified criteria for a "good" proof. These criteria included that the proof had to address the question or the posed problem and had to be correct, focused, detailed and precise, as well as clear, convincing and logical. The language, representations, and definitions had to be understood by the people to whom the proof was addressed. Besides, the proof should convince a sceptic and not require the reader to make a leap of faith. In the proof, key points had to be emphasized and pictures or other representations had to be used appropriately.

The situation with PTs and modelling is complex. PTs have been found to lack strong mathematical modelling skills, including how to reflect on and present their results from their models (Sen Zeytun, Çetinkaya, & Erbas, 2017). Yet, Naresh, Poling, and Goodson-Espy (2017) noted that PTs could design modelling tasks based on CME for students from 5 to 14 years old, even though they found this challenging. However, teachers may not be inclined to include mathematical modelling tasks in their teaching as Maaß and Gurlitt (2009) found that teachers preferred tasks that were more likely to provoke one answer and one solution path than tasks that were more open.
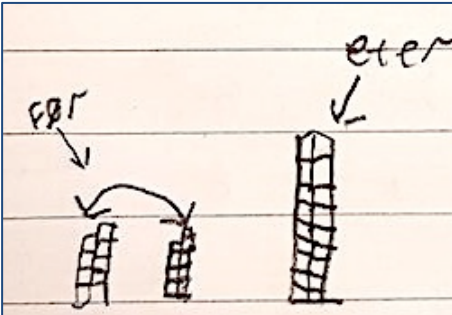
At the same time, concerns have been raised about PTs' understanding about how to use digital technologies in their learning and teaching of mathematics (Starčič, Cotic, Solomonides, & Volk, 2016). After interviewing teacher educators and preservice teachers, Instefjord (2014) suggested that the focus in teacher education should be "towards appropriation of a digital competence that embraces awareness of how technology can be used critically and reflectively in the process of building new knowledge" (p. 328), rather than a focus on the technological aspects of using specific digital tools. There is, therefore, a need to consider how PTs make critical decisions about the integration of digital tools into mathematics education.

Our review of previous research, presented briefly here, suggested that PTs were more likely to show their understandings about the different aspects of mathematics education, if these were presented in scenarios. Therefore in the survey questions, we provided examples of students' work, teacher lessons or policy decisions and then asked the PTs to respond to specific questions about them.

## The survey

The survey was developed across the five stages, identified by Maaß and Gurlitt (2009): determining the rationales for the design of the survey; determining an initial draft set of questions; pilot study of the survey; feedback from an expert group on the questions; and using the survey to determine its reliability.

In the first stage, the rationales for the design needed to be determined. These included focussing on the trends across the cohort, rather than on individual student changes, and so no demographic or personal information was collected. Instead the design focused on determining PTs' understandings about the different LATACME topics. Table 1 provides an overview of the 51 claims connected to the eight themes, from the final version of the survey, with the research which inspired them.

| Theme # | Theme content | | Items |
|---|---|---|---|
| T1 | The understandability, completeness and mathematical correctness of explanations (Stylianides & Stylianides, 2009) from 4 students in Year 4 (9–10 year olds) about why the sum of two odd numbers is an even number, e.g. "I can follow and understand Camilla's explanation". |  **Figure 1 Camilla's explanation (Ure, 2018)** | 1a–1l 12 items |
| T2 | Argumentation tasks in mathematics teaching, e.g. "Students should at least once a week work with tasks that require them to justify a mathematical connection (as for example about odd and even numbers in topic 1)". | | 2a–2d 4 items |
| T3 | Mathematical modelling for students in years 1-7 (Naresh et al., 2017), e.g. "Students in Grades 1-4 are too young for modelling to support them to increase their critical awareness about the use of mathematics in society". | | 3a–3h 8 items |

| Theme # | Theme content | Items |
|---|---|---|
| T4 | Digital tools and mathematics teaching, e.g. "Digital tools improve students' ability to argue in and with mathematics" (Starčič et al., 2016) | 4a–4d<br><br>4 items |
| T5 | Mathematics teaching in multilingual classrooms (Thomassen, 2016), e.g. "Students may use their home language for their learning". | 5a–5h<br><br>8 items |
| T6 | A teaching project for Grade 4 in which students collected and sorted garbage in the beginning and in the end of the year and represented their findings in a diagram (Naresh et al., 2017), e.g. "The next lesson in the class should be about how to make the diagram better by stating the units on the y-axis and separate the bars". | 6a–6c<br><br>3 items |
| T7 | A teaching project for Grade 5 about air pollution in the city (Skovsmose, 1994), e.g. "The project would take too much time from teaching and would not give us the opportunity to get through everything we are supposed to cover in the textbook". | 7a–7i<br><br>9 items |
| T8 | The possible actions PTs would take in an imagined scenario in which the government has decided that students can only speak Norwegian in mathematics classes (Thomassen, 2016), e.g. "you send a written message home to those parents who do not speak Norwegian at home where you describe the change in policy and call upon the parents to speak Norwegian when they help their children with homework". | 8a–8c<br><br>3 items |

Table 1: Themes, examples of claims and the number of items in each topic in the survey

As with all surveys (see Maaß & Gurlitt, 2009), the time needed to complete it had to be limited, while still gaining the most relevant information. We, therefore, chose to ask the PTs to express their degree of agreement with each claim on a Likert-scale comprising the options "disagree completely", "disagree moderately", "neither agree nor disagree", "agree moderately", "agree completely", and "don't know". Christoffersen and Johannessen (2012) stated that Likert-scale questions should have five choices and a "don't know/not applicable" option. This allows the Likert scale to be seen as a "nominal variable with many values" and hence open to a wider range of statistical tools and procedures than for a nominal variable with less-than-5 values (Christoffersen & Johannessen, 2012, p. 135). As it is recommended that sensitive questions should be in the middle (Christoffersen & Johannessen, 2012), we did this both across the survey and within individual questions.

To contribute to the second stage of the survey development (Maaß & Gurlitt, 2009), a small group of the authors of this paper proposed a set of questions. As shown in Table 1, potential questions came from the findings and the identification of relevant scenarios in previous research. For example, to gain insights into PTs' understandings of mathematically correct, complete and understandable argumentation (see T1 in Table **1**), examples of Grade 4 students' argumentation from a Master thesis (Ure, 2018) provided a real scenario from PTs' future work in mathematics classrooms.

The potential set of questions were then taken to a wider group, which included teacher educators who would teach the compulsory mathematics education courses for Grades 1-7 teachers or who had

taught similar courses in the previous teacher education programme. In Maaß and Gurlitt's (2009) process, this is the fourth stage. However, we moved it forward as we wanted to ensure the cohesion of the project by involving as many people as possible in the development of the survey. The wider group was asked to discuss the usefulness of the questions for gaining relevant information from the PTs, if the questions needed to be modified and in what ways. Suggestions from the discussions included the need for an argumentation example written in another language (T1 in Table **1**). It was also suggested that in an example of a mathematics task in which children would consider where they lived in regard to air quality (T7 in Table **1**), that PTs should be asked about ethical aspects involved in CME projects. The wider group' comments contributed to the revisions of the questions.

Step three in Maaß and Gurlitt's (2009) process was a pilot study of the survey. This was undertaken with a group of PTs from an earlier cohort who would not be part of the actual study. The pilot study resulted in an identification of ambiguities in some of the questions. Consequently, these questions were rewritten so that they becamse clearer. It also resulted in the removal of questions that did not seem to give useful information, such as "elevene bør diskutere hva som er en god forklaring i hver undervisningsøkt" (students should discuss what is a good explanation in each teaching session" from T2 in Table 1. The order of the questions was also changed so that there was a reduced chance of PTs being misled to think that a particular response was expected. As well, the number of items were reduced.

The final stage in Maaß and Gurlitt's (2009) process for developing a survey was to use the survey to determine the reliability of the instrument. As our purpose for having PTs complete the surveys were different to those of Maaß and Gurlitt, we instead considered whether the results provided useful information. An initial descriptive analysis of the results is provided in the next section with more detailed results connected to a cluster analysis provided in another paper.

## Data collection

In the teacher education for grades 1-7, PTs have two mandatory mathematics courses of 15 ECTS each. The courses are taught in their 2nd and 3rd semesters and integrate mathematics and mathematics education. The results discussed in this paper come from the survey which was administered at the beginning of the 2nd semester, that is after one semester of teacher education including a practicum period, but before exposure to mathematics teacher education.

The survey was made available electronically and did not provide access to the PTs' IP addresses. The PTs were provided with a link on the Learning Management System in the first week of the semester. Of the approximately 200 PTs in the cohort, 96 chose to complete the survey.

## Results and discussion

In this section, we provide responses to questions that are representative for each of the five foci: argumentation; modelling; digital tools; critical mathematics education; and multilingual classrooms. In the calculations, PTs who chose "Don't know" were not included. To indicate the results for argumentation, we use the example of Camilla's explanation, given in Table **1**, Theme 1.

| Item | Mean | SD |
|---|---|---|
| 1a) I can follow and understand Camilla's explanation | 4.54 | 0.74 |
| 1b) Camilla's explanation is incomplete | 2.85 | 1.17 |
| 1c) Camilla's explanation is mathematically correct | 3.99 | 0.92 |

Table 2: Mean and standard deviation for an argumenation task

The PTs were generally positive about all four of the school students' argumentation examples about odd and even numbers. In Table **2**, they were mostly positive about being able to follow and understand the argumentation, even though Camilla's pictorial argument contained only the words «før» (before) and «et[t]er» (after), as the responses had a very high mean and a small standard deviation. The PTs were neither very positive or negative about the completeness of Camilla's argumentation, with the mean being close to 3 – neither agree nor disagree – and a large standard deviation. These results seem in alignment with the PTs at the beginning of their work on proof in the research by Stylianides and Stylianides (2009).

Theme 7 (see Table **1**) included a lesson plan about a project on air pollution that used a colour-coded map of a city to show air quality in different areas. It also included a question about the use of tolls on roads as a way of reducing the number of cars and improving the air quality. The PTs were asked about different aspects of the project, which covered all five aspects of LATACME.

| Item | Mean | SD |
|---|---|---|
| 7a) This project would be too difficult for students who do not speak good Norwegian because they would not be able to justify their answers. (multilingual classrooms/argumentation) | 3.08 | 0.97 |
| 7b) Students in 5th grade are not able to interpret the map and diagram. (mathematical modelling) | 2.65 | 0.96 |
| 7c) I would not have used this project in my teaching because the students would find it too extensive. (mathematical modelling) | 2.93 | 1.02 |
| 7d) This project will facilitate the students' introduction to basic areas of mathematical modelling. (mathematical modelling) | 3.44 | 0.72 |
| 7e) I would not have used the question of tolls (iii) because it would be too difficult for the students to work out such a plan. (CME) | 3.25 | 1.04 |
| 7f) Students in 5th grade must know or first learn the concepts of average and range in order to understand what the project is about. | 3.43 | 0.93 |
| 7g) It seems strange to let students in 5th grade describe how they would use a PC or tablet in this project (digital tools) | 3.01 | 0.98 |
| 7h) The project would take too much time from teaching and would not give us the opportunity to get through everything we are supposed to in the book. (mathematical modelling) | 2.60 | 1.09 |

| Item | Mean | SD |
|---|---|---|
| 7i) The project is ethically unjustifiable because there may be students in the class living in polluted areas. (CME) | 2.56 | 1.02 |

Table 3: Means and standard deviations for responses to Theme 7

Table 3 shows that the PTs were much more uncertain about how children would respond to this project and its value in regard to different aspects on mathematics teaching that they were to the children's written argumentation. The means are closer to 3 with small standard deviations. In all items, a larger percentage of the respondents replied, "Neither agree nor disagree" or "Don't know". The PTs indicated a slight tendency towards considering the project as being good for supporting students' understanding of modelling as well as needing to understand basic statistical concepts before starting the project. The PTs were only slightly convinced that the project was ethically sound. However, the trends are small. Given that these PTs were at the beginning of their mathematics education courses, their uncertainty about different aspects to do with their teaching is predictable and it will be interesting to see if there are changes at the end of the two compulsory courses. There is some ambiguity in the responses, suggesting that the PTs at this time were conflicted about ensuring that the school students completed work in the textbook and seeing this project as a good introduction to modelling.

Results such as these provide opportunities for developing activities in our teacher education that use these uncertainties to develop rich discussions about how to incorporate the different aspects into mathematics classrooms. The consistencies in the results across the questions related to the same LATACME topic suggests that the questions did provide us with valid responses to determining PTs' understandings about these topics.

## Conclusion

The aim of LATACME is to make changes to our compulsory mathematics education courses for those who want to be teachers of grades 1-7 and so we want to support our PTs to gain the necessary set of understandings about what is required to be a mathematics teacher. In order to know if we have achieved this in our teacher education, we need to find out what changes occur in PTs' understanding about different aspects of mathematics education. Changes in the results from the survey over time provide just one set of data from the range of data that we are collecting during our project. In this paper, we have described the process of developing the survey, following the stages described by Maaß and Gurlitt (2009). We suggest that the transparency in describing each step in the process is important, especially in large projects such as ours where many teacher educators are involved. It is also important for those who may be interested in the results in relationship to their own teacher education courses or who want to adapt it to suit a different set of rationales.

Although our initial results are not startling, they do suggest that using scenarios can provide relevant information about how PTs respond to a range complex issues that confronts them in their process of becoming teachers of mathematics. This provides teacher educators with potential starting points for planning their own teaching, such as discussing how to incorporate mathematical modelling problems so that it covers aspects to do with, for example statistics, that would normally be taught through the textbook.

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# Natural-number bias pattern in answers to different fraction tasks

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*In this paper, we investigate the role of natural-number biases within answers to a computerized fraction test that was conducted by 484 fourth-grade students. A key source of errors or misconceptions in rational-number tasks is the misapplication of natural-number understanding. We hypothesized that students who use this intuitive reasoning from natural numbers have a tendency to do so across different kinds of tasks. We analysed and coded the students' written answers into four different types of natural-number biases and discussed how various types of natural-number biases are related. Our results support the notion that different kinds of natural-number biases can be found within the context of the specific task, rather than there being an overall tendency to create natural-number biases.*

*Keywords: Whole-number bias, Natural-number bias, Fractions, Rational numbers.*

## Introduction

Research has shown that an understanding of fractions in primary school is important for students' overall progression in general mathematics and that proficiency with rational numbers predicts mathematical achievement later in life (Bailey et al., 2012; Siegler et al., 2012). Many students, however, have difficulty developing their understanding of fractions. These difficulties often continue later in school (Lortie-Forgues et al., 2015; Siegler & Pyke, 2013; Tian & Siegler, 2017).

One major difficulty in understanding fractions can be seen as the *natural-number bias*. This bias can be described as a tendency for students to use natural-number reasoning when working within rational numbers; for example, using fractions in an unfitting or incorrect way (Ni & Zhou, 2005; Van Hoof; Vandewalle et al., 2015). This does not mean that the understanding of natural numbers is not essential for the learning of rational numbers; research has shown there is a correlation between competences in natural numbers and fractions (Bailey et al., 2014). However, there seems to be a crucial development in students' number sense when their number knowledge expands to include rational numbers, and this process involves both misconceptions and support from their previous experience with natural numbers. Research has also shown that students often make systematic errors when their problem-solving processes involve reasoning that is not in line with their prior knowledge of natural numbers (Ni & Zhou, 2005; Van Hoof, Verschaffel et al., 2015). Accordingly, the focus of this paper is to investigate how natural-number bias influences the comparison of fractions and the understanding of magnitude.

The primary research question for this paper is: *How are students' different natural-number biases related to each other, and is there a pattern that indicates an overall tendency towards natural-number bias?*

## Theoretical framework

Ni and Zhou (2005) introduced the term *whole-number bias*, and other researchers have since used the term *natural-number bias* with the same meaning (e.g. Van Hoof et al., 2015). Given that our

study is only focusing on natural numbers, we use the term natural-number bias to refer to such misconceptions.

Before students are introduced to rational numbers instruction, they have formed an understanding of numbers as *counting numbers*, and they often have informal experiences with natural numbers (Vamvakoussi & Vosniadou, 2010). Moreover, the development of one's natural-number conceptualization is supported by cultural representations like finger counting (Andres et al., 2008; Carey, 2004). In contrast, there is less support for children to form their own intuition about rational numbers (Greer, 2004; Vamvakoussi & Vosniadou, 2010). Hence, students have developed and constructed an understanding of numbers based on their experience with natural numbers before they are formally introduced to fractions. These understandings influence a student's conceptualization of what a number is, and how it behaves (Gelman, 2000; Vamvakoussi & Vosniadou, 2010). Naturally, when students try to make sense of fractions, it seems they rely on their knowledge of natural numbers, even in contexts where it is not applicable (Ni & Zhou, 2005). In the following subsections, we discuss four aspects[1] of natural-number bias, as described by Van Hoof, Vandewalle, Verschaffel and Van Dooren (2015): representations, size, density and operation.

### Representation

The first aspect, *representation*, can (for example) refer to the fraction notation. Some students might perceive it as two separate integers or two natural numbers under and above a thin line and do not understand it as a rational number (Stafylidou & Vosniadou, 2004). This further leads to students' difficulties in understanding that various symbolic notations can represent the same magnitude, such as $\frac{1}{2} = \frac{2}{4} = 0.5$, while natural numbers have a unique representation of a magnitude. Accordingly, understanding the equivalence of fractions is a new way to interpret numbers (Braithwaite & Siegler, 2016; Kamii & Clark, 1995).

### Size

*Size* is the aspect often described in the context of natural-number bias. It is often connected to the comparison of two fractions' magnitudes, and the bias can be observed in the students' misperception of fractions as "the one with a larger integer" (Van Hoof et al., 2015). For example, when students are asked to determine which fraction is greatest, and the smallest fraction has a larger number in the denominator and the numerator than the fraction it is compared with, such as comparing $\frac{4}{9}$ with $\frac{3}{5}$, research has observed greater errors and longer response times when comparing two fractions (DeWolf & Vosniadou, 2015; Vamvakoussi et al., 2012).

### Density

Research has shown that another aspect of natural-number bias concerns *density*, which means that some students tend to consider fractions, as with natural numbers, as having unique predecessors and successors, for example "$\frac{1}{5}$ comes after $\frac{1}{4}$", or "$\frac{4}{6}$ comes after $\frac{3}{5}$" which, of course, is not the case (Van Hoof et al., 2015). Another way the natural-number bias seems to occur is that some students think that there is a finite number of fractions. As if between two fractions, for example $\frac{1}{5}$ and $\frac{1}{7}$, there is only $\frac{1}{6}$. However, there is an infinite number of fractions between any given different fractions

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[1] Also described as *dimensions* by Obersteiner et. al (2016).

(Vamvakoussi et al., 2011; Vamvakoussi & Vosniadou, 2010). One of the challenges is that the student cannot count which number comes next, which also influences the operation aspect. Thus, the student cannot use the counting strategy when solving addition tasks.

**Operation**

The last aspect, *operation,* is when students use their prior knowledge of natural numbers operations on fraction operations; for example, that multiplication always makes bigger and division always makes smaller, which is not always the case when multiplying or dividing fractions (Stafylidou & Vosniadou, 2004). In addition, students often hold the misconception that when adding two fractions, one adds the two denominators and the two numerators, for example $\frac{1}{2}+\frac{2}{5}=\frac{3}{7}$ (Obersteiner et al., 2016; Siegler et al., 2011). A student's interpretation of denominator and numerator as two separate natural numbers might also be considered a misconception connected to the aspect of *representation*.

We expect that the four different natural-number biases can be closely connected. Our hypothesis is therefore; if the students have a tendency to use their understanding of natural numbers in one context, they are more likely to also use that understanding in others.

In this study, we will investigate if and how the four aspects are related. Finally, we will broaden the discussion concerning the relationships between different natural-number biases, and how this affects the focus of the teaching and the learning of fractions.

## Methods and analysis

The data used in this study consists of answers from 484 fourth-grade students (mean age 10 years and 3 months, std. 0.02) on a computerized fraction test at the beginning of the 2018/2019 school year. In all, 235 girls and 249 boys participated in the study. The test consisted of 36 different items, including problems on part-whole, number line, comparison and fraction addition. The test was time-restricted (10 minutes), meaning that not all students finished the assessment. We therefore chose to include only the 484 students who managed to progress to item 22 on the test (146 students were excluded). The ethnicity of students' participating schools is reported as 92.7% Danish origin, 6.8% non-western immigrants and 0.5% western immigrants. The test was based on the Curriculum Based Measurement framework, which had previously shown adequate psychometric properties (Foegen et al., 2007). From the test, 14 items were selected in which there was an opportunity for the students to compose answers that were influenced by one of the natural-number bias aspects. We renamed the items based on the four aspects: R(representation), S(size), D(density), and O(operations). The students have briefly been introduced to fractions in third grade according to the Danish Mathematic books' trajectories. However, instruction in fraction addition is not introduced before fourth grade. We chose to keep the operation aspect in the analysis, and therefore the students with a good concept of fractions could reason how to add $\frac{1}{5}+\frac{2}{5}$.
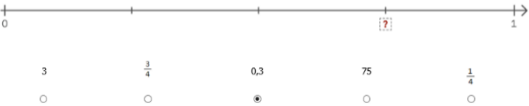
| A. Representation | Translated to: What fraction of the shape is coloured? |
|---|---|
| **Hvor stor en brøkdel er farvet?** $\dfrac{4}{1}$ <br><br> (green pie chart) | **Explained:** <br><br> Counting the green parts and the white part. Looking at the numerator and the denominator as separate integers. |
| B. Size <br><br> **Indsæt det manglende tegn: >, < eller =.** <br><br> Tryk på pilen, og find det rigtige tegn. <br><br> $\frac{1}{2}$ [ < ] $\frac{1}{3}$ | Translated to: Choose the right symbol: >, < or =. <br><br> **Explained:** <br><br> Looking at the denominators as integers and comparing the size of these. |
| C. Density <br><br> **Hvad skal stå i boksen?** <br><br> (number line from 0 to 1 with ? box) <br><br> 3   $\frac{3}{4}$   0,3   75   $\frac{1}{4}$ <br> ○   ○   ◉   ○   ○ | Translated to: What should be in the box? <br><br> **Explained:** <br><br> Counting the lines on the number line. |
| D. Operation <br><br> **Regn opgaverne.** <br><br> $\dfrac{1}{5} + \dfrac{2}{5} = \dfrac{3}{10}$ | Translated to: Calculate. <br><br> **Explained:** <br><br> Adding the denominators. |

Table 1: Example of coded natural-number bias

## Analysis

Data analysis involved three steps: firstly, we coded natural-number biases connected to representation in four items, for instance, a pie chart where four of five pieces were green ($\frac{4}{5}$ coloured; $\frac{1}{5}$ not coloured). If the student answered that $\frac{1}{4}$ or $\frac{4}{1}$ was green, it was coded as a natural-number representation bias: the student counted the four coloured pieces, and the one not-coloured piece, treating the fraction notation as two separate integers. Three items were based on comparing two fractions with each other, for example $\frac{1}{2}$ and $\frac{1}{3}$. If the student chose $\frac{1}{3}$ as the biggest, we coded this as a natural-number bias, based on size. Three items from the test consisted of a multiple-choice number-line task in which the students should identify the right number from five choices. Here we coded a natural-number bias, based on density, if the student chose, for example, 0.3 instead of $\frac{3}{5}$, because they count three lines on the number line and argue that this must lead to 0.3 because it is less than

one. The last aspect included four fraction addition items, which we coded as natural-number bias based on whether operation denominators and numerators were added (for example $\frac{1}{2}+\frac{1}{3}=\frac{2}{5}$) (see Table 1). The second step was to conduct a descriptive statistics analysis of each of the 14 items in order to create an overview and to describe the coding information as simply as possible (e.g. the number of natural-number biases, missing answers) (see Table 2).

In the third step, aggregated variables were obtained by counting the number of natural-number bias errors for each of the four aspects for each student. For example, for the representation aspect, the possible values for the aggregated variable were 0, 1, 2, 3 or 4. The aggregated variables were clearly not normal due to discreteness and also bimodality or skewness. To account for non-normality in the distribution of data, the relations between the aggregated variables for the four natural-number bias aspects were studied using Spearman's rank-order correlation coefficient ($r_s$). This results in six correlation coefficients for each of the six possible pairs of natural-number bias aspects. To adjust for multiple testing, we used a Bonferroni correction to calculate the significance level for testing whether each correlation was zero. Thus, to obtain an overall significance level of 0.05 we used the adjusted level 0.05/6=0.0083 for each individual correlation. The results regarding the correlations are shown in Table 3. The statistical analyses were carried out with Stata (Version IC/15.1).

| Item | type | Correct | | | NB | | | NC | | | DNA | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | N | P | SD | N | P | SD | N | P | SD | N | P | SD |
| R1 | rep | 314 | .65 | .02 | 146 | .30 | .02 | 10 | .02 | .01 | 14 | .03 | .01 |
| R2 | rep | 308 | .63 | .02 | 68 | .14 | .02 | 93 | .19 | .02 | 15 | .03 | .01 |
| R3 | rep | 338 | .70 | .02 | 54 | .11 | .01 | 72 | .15 | .02 | 20 | .04 | .01 |
| R4 | rep | 329 | .67 | .02 | 64 | .13 | .02 | 71 | .15 | .02 | 20 | .04 | .01 |
| S1 | size | 155 | .32 | .02 | 228 | .47 | .02 | 25 | .05 | .01 | 76 | .15 | .02 |
| S2 | size | 56 | .11 | .01 | 261 | .54 | .02 | 79 | .16 | .02 | 88 | .18 | .02 |
| S3 | size | 108 | .22 | .02 | 263 | .54 | .02 | 26 | .05 | .01 | 87 | .18 | .02 |
| D1 | den | 177 | .37 | .02 | 164 | .34 | .02 | 94 | .19 | .02 | 49 | .10 | .01 |
| D2 | den | 180 | .25 | .02 | 180 | .37 | .02 | 132 | .27 | .02 | 49 | .10 | .01 |
| D3 | den | 169 | .35 | .02 | 115 | .24 | .02 | 128 | .26 | .02 | 72 | .15 | .02 |
| O1 | op | 36 | .07 | .01 | 281 | .58 | .02 | 26 | .05 | .01 | 141 | .29 | .02 |
| O2 | op | 0 | .00 | - | 274 | .57 | .02 | 50 | .10 | .01 | 160 | .33 | .02 |
| O3 | op | 4 | .01 | .00 | 287 | .59 | .02 | 34 | .07 | .01 | 159 | .33 | .02 |
| O4 | op | 0 | .00 | - | 284 | .59 | .02 | 40 | .08 | .01 | 160 | .33 | 0.2 |

Table 2 Proportions and Standard Deviations by Problem of Correct Answers. Natural-Number Bias Errors, No Category Errors and Did Not Attempt Errors. *Note.* P = proportion, N = number of answers, NB = natural-number bias errors, NC = no category errors (e.g. students wrote the wrong fractions), DNA = did not attempt, rep = representation, den = density, op = operation

## Results

See Table 2 for a summary of results for the coding of the 14 items. Overall, the majority of the wrong answers can be explained by students having a natural-number bias in their attempt to solve the task.

The highest proportions of correct answers were connected to representation (.60-.70), whereas the lowest proportions of correct answers were found within operations items (.01-.00). The proportion of natural-number bias mistakes connected to representation ranged from .11 to .30 and the proportion of natural-number bias mistakes connected to size ranged from .47 to .54. The proportion of natural-number bias connected to density and operation ranged from .24 to .37 and .57 to .59 each. Overall, when looking at the students' wrong answers (natural-number bias and not coded answers), we see that natural-number bias is the main explanation for the wrong answers on many items. There was no strong correlation found between the four aspects, and none of them were statistically significant (see Table 3).

The highest correlation was found between the two aspects: *representation* and *density* $r_s = .117$ (ns).

| Aspect of natural-number bias | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1. Representation | 1 | | | |
| 2. Size | -.001 (ns) | 1 | | |
| 3. Density | 0.117 (ns) | 0.035 (ns) | 1 | |
| 4. Operation | 0.051 (ns) | 0.050 (ns) | -0.065 (ns) | 1 |

Table 3: Correlation matrix between the four Natural-number aspects. Note: Spearman's rank-order correlation coefficient ($r_s$), Significance levels: $p < 0.05$ (overall) and $p < 0.0083$ (Bonferroni corrected) for individual correlations, (ns): not significant

## Discussion

This statistical investigation of four natural-number bias aspects revealed the following three perspectives:

Firstly, natural-number biases had a greater influence, or explanation degree, on wrong answers given from students who had just started fourth grade. The lowest proportion of answers influenced by natural-number biases (.11 to .30) were found in the four items connected to representation, whereas operations had the highest proportion of answers influenced by natural-number biases (.57 to .59). This can be explained by the fact that the tests were conducted at the beginning of fourth grade when most students had not yet been taught fraction addition. As previously mentioned, fraction addition is often not part of the trajectory, and therefore the students naturally draw more on their knowledge about natural numbers. On the other hand, a conceptual understanding of the fraction magnitude and notation could lead students to solve the addition task; therefore, the concept of addition is well known in fourth grade. The high proportion of incidents of natural-number bias in this group supports the fact that at this stage, students base their solving process on their knowledge of natural numbers (e.g. Ni & Zhou, 2005; Van Hoof et al., 2015).

The three size items also had a high proportion of wrong answers connected to natural-number biases (.47 to .54) and very low proportions of non-coded wrong answers (.05 to .16), meaning wrong answers that cannot be explained by natural-number biases. This is not surprising considering that the students could only choose from three answers (equal, bigger and smaller), and incorrect answers can therefore be easily explained by a natural-number bias; there were not a lot of different ways to give a wrong answer. Secondly, Spearman's correlations (Table 3) indicate that the four aspects of natural-number biases are not related to each other. This, of course, contradicted our hypothesis that

students who have a tendency to apply their knowledge of natural numbers to fractions will repeat this mistake across different tasks. It is therefore important to be able to understand how fractions differ from natural numbers within the context of the task or problem. When we teach fractions, we therefore must consider how students can be given the opportunity to learn how fractions differ from natural numbers within each of the four aspects. Teaching fractions should focus on different aspects of rational numbers and on exploring these aspects with more depth. (Bailey et al., 2014).

The results were not consistent with our hypothesis that the students with a tendency towards natural-number bias in one aspect are influenced by natural-number bias across all four aspects; instead, their misconceptions seem to be connected to the setting defined by the nature of the fraction task.

Thirdly, the non-significant correlations between the four aspects must lead to a further discussion on whether we should look at the four aspects as types instead of aspects of natural-number biases. As a term, aspect indicates a close connection between the four items, which does not seem to be the case. Therefore, it might be better to define them as four different types of natural-number biases. For future research, we will collect qualitative data to study the students' working process in order to overcome the limitations of computer-based testing whose results can only give limited insights into the students' thinking and reasoning and how this is influenced by natural-number bias. In addition, we need to investigate further how students develop their knowledge of fractions over time, and how this is connected to their development within the four natural-number bias types. The test could be further developed to better screen for natural-number biases; in particular, the number-line task could be optimized so the students could write their answers down instead of selecting an answer from a multiple-choice group.

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# Swedish parents' perspectives on home-school communication and year-one pupils learning of mathematics

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*In this paper we explore parents' perspectives on home-school communication with respect to year-one pupils' learning of early numeracy. Constant comparison analyses of semi-structured interviews identified three forms of communication. The first, a weekly information letter, was appreciated and typically functioned as a starter for conversations with children about their learning of early numeracy. The second, the development talk, was appreciated as an indicator of a child's progress, but proved controversial in the presentation of mixed messages to parents and limited with respect to helping parents support future mathematical learning. The third, parent-initiated contact, was discussed in ways that masked parents' reasons for making contact and prevented, therefore, any insights into their contribution to mathematical learning. Some implications are discussed.*

*Keywords: Home-school communication, parent perspectives, numeracy, year-one, Sweden.*

## Introduction

It is widely accepted that a secure understanding of early numeracy (Desoete, Stock, Schepens, Baeyens, & Roeyers, 2009) and appropriate parental involvement influence greatly children's mathematics learning (Skwarchuk, Sowinski, LeFevre, 2014). However, for parents to know how best to support their young children's learning of early numeracy, it is important to understand the forms of communication that exist between schools and parents, particularly in the context of a country like Sweden, where schools are obligated to "work together with and continuously inform parents about the pupil's school situation…" including knowledge acquisition (Skolverket, 2019, p. 14). Based on this, the research reported in this paper focused on addressing the question:

> What are Swedish parents' perspectives on home-school communication and their year-one children's learning of numeracy?

**What is known about home-school communication w.r.t. the learning of early numeracy**

Various scholars have examined parental involvement and found a strong positive correlation between their involvement and their child's achievement, "regardless of the definition of parental involvement" (Wilder, 2014, p. 392). Also, attempts to categorise parental involvement typically conclude that home-school communication is one of the most important forms (Thompson & Mazer, 2012). Home-school communication takes different forms, each offering different perspectives on, for example, the freedom of parents to take initiative and seek resolutions to their concerns. Also, home-school communication can be either synchronous, whereby participants are simultaneously engaged, or asynchronous, whereby participants engage at different times. For example, a parent-teacher conference or a telephone conversation are synchronous, while email or a weekly newsletter are asynchronous. When parents contact schools, they typically prefer asynchronous communications like email when addressing academic matters but synchronous communications like face-to-face

meetings or telephone when addressing matters of social well-being (Thompson & Mazer, 2012). However, research internationally tends to suggest that communication is generally structured by schools in ways designed to "keep parents in their place" (Karlsen Bæck, 2010, p334; Saltmarsh & McPherson, 2019).

A well-known mode of synchronous communication, found in many educational systems of the world, is the parent-teacher conference. Often, as in England (MacLure & Walker, 2000), these conferences are teacher-led, with the passive presence of children arbitrarily permitted. For parents, they "tend to be ritualized occasions", with teachers controlling the conversation in ways that allow them to impart information, smooth over problems and prevent parent input until the meeting's end (Minke & Anderson, 2003, p.50). In the context of Sweden, the aim of the conference is to "inform the parents on the status of their children's schooling… discuss their social and psychological adaptation… (and) areas of improvements which the child needs to work on" (Osman & Månsson, 2015, p. 44). In this respect, the Swedish educational system differs from many others in that the child is present, and, is frequently expected to lead the conversation (Pihlgren, 2013). Research has shown that children typically take between one and two weeks of school time to prepare the presentations they need for the meeting (Pihlgren, 2013). Moreover, to the detriment of other matters, students typically focus on assessment and, depending on individual circumstances, have mixed feelings about the process (Lindh-Munther & Lindh, 2005). All that being said, while positive parental perspectives on home-school communication are related to the provision of numeracy-enriching home environments for kindergarten children (Lin, Litkowski, Schmerold, Elicker, Schmitt & Purpura, 2019), little is known about the same relationship for year-one children.

## Methods

In this paper, by means of data derived from semi-structured interviews, we examine how parents of year-one children construe the relationship between home-school communication with respect to their children's learning of numeracy. The principals of three schools, which we have labelled City, Suburban and Satellite, due to their locations in the centre, suburbs and satellites of the same large city, were contacted about the project. These three schools were approached primarily due to variation in the population demographic profiles. With the principals' permission, parents and carers of year-one children were invited to participate, leading to one, fifteen and nine interviews in Satellite, Suburban and City Schools respectively. With the support of the schools' principals and to facilitate the process, 22 interviews were undertaken in private rooms at children's schools during the same weeks as children's development talks (utvecklingssamtal), while three were held in parent's workplaces. Parents, informed of their rights, gave written consent to participate. Interviews, lasting between 20 and 30 minutes, were recorded for later transcription. The interviews were structured by a series of broad questions focused on how parents construe their roles in relation to their children's learning of mathematics in general and basic numeracy in particular. For this paper, we focus on the parents' response to questions about home-school communication with respect to learning early numeracy, with follow-up questions, when needed, inviting elaboration, particularly with respect to eliciting details about mathematics.

Acknowledging the exploratory nature of the project, data were subjected to the constant comparison analytical process of the grounded theory, whereby a random script was read and utterances related to home-school communication identified and coded. A second script was then read with the aim of

identifying both new codes, in which case the first script was reread, to see if the new codes had been missed on the first reading. The process was repeated until no new codes emerged, after which all codes were placed in the three broad themes presented in the results. In conventional grounded theory research, new data are sought only when needed. Here, we instead were constrained by the availability of parents at children's development talks. Consequently, we had to decide, at the start of the study, how many interviews should be undertaken. Earlier studies with similar idiographic aims have indicated that fifteen interviews should be sufficient to achieve thematic saturation (Robinson, 2014), a total confirmed sufficient for thematic saturation by our own teacher interviews (Sayers, Marschall, Petersson & Andrews, 2019). Thus, acknowledging that with a less homogeneous group than would be the case with teachers, we aimed for a minimum of 20 and achieved 25. Importantly, to minimise the loss of contextual meaning, transcripts were analysed by Swedish members of the project team before quotes were translated into English for inclusion in this paper. With one exception, all informants had academic backgrounds. We conjecture that these well-educated parents, with generally high levels of economic capital, social capital and cultural capital, are more likely to have the confidence to volunteer for interview than less well-educated or minority parents (Vincent, 2017). In the following, we remain mindful of this possibility.

## Results

All parents, to preserve anonymity and make for ease of reporting, were given pseudonyms defined by their child's school and a unique reference number. Thus, the one parent from Satellite School was designated Sat 1. The fifteen parents from Suburban School were designated Sub 2 through Sub 16, while the nine parents from City School were designated Cit 17 through Cit 25.

The analytical process described above yielded three broad forms of home-school communication, succinctly summarised in Sat 1's comment that, "we get information every week, on Friday, with the weekly newsletter. Otherwise, it's the development talks that you learn more from, although you can also come here every day". The first of these, the weekly newsletter, is an asynchronous, school-initiated, teacher-to-many and one-way form of communication with no opportunity for parents to take initiative. The second, the development talk, is a synchronous, school-initiated and two- or three-way communication with opportunities for parents to take the initiative dependent on how individual schools, teachers, parents and children construe the meeting. The third, the parent-initiated conversation, is typically an asynchronous communication with, in principle, wide opportunities for parents to take the initiative. In the following, we present the results for each of these forms of communication in turn before discussing them collectively.

### Communication through weekly letters from school

It was clear from the spread of the informants' responses that all three schools sent out weekly newsletters to parents and that, broadly speaking, parents were satisfied with their content. For some, it acted as an aid to their understanding of what was currently happening in school, particularly when, as mentioned by Sub 10, "if you don't have a talkative child, you don't know which numbers they are learning presently". Moreover, for some, the weekly letter was seen as a means of prompting conversations at home, especially when, as noted by Sub 2, children respond to the question, "what did you in school today?", with "I don't remember". In such circumstances, parents can draw on the content of the newsletter to initiate conversation at home, whether at a general level or a specific

mathematics. For example, with respect to the former, Cit 24 commented that "as one reads the weekly letters one gets information from which one can ask questions of your children about things that one has found out from the weekly letter", while from the latter, Cit 2 spoke of saying to his child that "I read that you worked with this digit in school", thus prompting a quiet child to respond.

It was also clear that, while all three schools set regular reading homework, mathematics homework was a rarity. On those rare occasions, parents spoke of how it was sent directly to them, typically in the newsletter. For example, as Cit 18 recalled, the newsletter can serve multiple mathematics-related purposes, saying that it "encouraged talking about halving… It is their way of putting the information in the weekly newsletter that is special. It may well be a way to connect around the dinner table". Another parent, Sub 13, spoke about an occasion in which he received instructions for a "homework on 10-bonds…we practised the bonds one evening.".

However, a few parents seemed less positive about the content of the newsletter, even though their comments never referred explicitly to mathematics. Typically, they spoke of too much information for them to keep track. For example, Sub 5 spoke of its mainly including "dates and things like that", adding that "you may miss certain things. It's an incredible amount of information", while others, like Sub 13, felt that "it is expected that much information will be shared, which is constantly pumped from school… One needs to keep track of skating that day, and excursion with lunch bag that day, and all such basic information. It takes time to absorb such things".

Overall, though, with respect to mathematics, most parents seemed content with the weekly newsletter, not least because the received view was that schools expected little parental involvement other than to follow the information they send and, as noted by Sub 8, support their child in developing a "positive image of school". In this respect, Sub 13's comment was particularly telling. He said, approvingly, that

> I think that the Swedish school is principally based on everyone having the same opportunity. So, pushing things onto parents and creating expectations that parents should help goes a little against it, because all parents and homes have different conditions. So, I think I'd like to say that the expectation of parents is pretty low.

### Communication in development talks

At both Suburban and City schools, development talks are child-led, while at Satellite school they are teacher-led. Moreover, at City school, these child-led talks are held in parallel, with one teacher circulating between them. However, such distinctions seemed not to influence parents' perspectives on the communication they engendered, which varied considerably within and across schools. Interestingly, and perhaps unsurprisingly, those with most to say were typically those least satisfied and, while all parents were asked to focus on mathematics, most spoke in general terms.

For many parents the development talks were opportunities for them to learn about their children's progress in mathematics. For example, with respect to Suburban school, Sub 14 said that "the children lead (the meeting) themselves … describe what they have done and how they experience what is easy and what is difficult. They are very nervous but very well-prepared". In similar vein, Sub 4 said, of her daughter, that "we got to know what they have been working on and how she wants to develop herself and what she has for goals. And then we got to look in her maths book, because she has been working on numbers". Similar responses emerged from the parents at City school, as with Cit 19,

who said, "from my son's point of view, he has to write what he thinks is good and what he thinks is not, what he likes or dislikes. I think it is quite clear. They don't lie, they just show what they think. I can see exactly what he likes or dislikes at school", while Cit 20 added that, in the particular context of City school "the kids have been preparing. They will paint in different colours how they think they can do this… Then we will set goals together".

However, following their children's presentations, parents' responses varied. On the one hand, Cit 19 spoke positively, saying, "after, I can discuss with the teacher, how to improve or what the problem is, why he doesn't like. Or why he is not good at this and how to improve", while Sub 11 commented that although "the development talks are student-led at this school… if you have something beyond that, you can talk to the teachers themselves about the child's development".  On the other hand, Cit 20 was critical, saying, with respect to goal-setting, that "it should be done together with teachers. But it never happens, because there are so many others in that room who need the teacher. No teacher will ever come to us, so we have to set some goals ourselves that we think seem reasonable".

A second theme, albeit less frequently expressed, concerned an apparent dissonance between different participants' perspectives on a child's development. For example, Cit 22 commented that "we saw the test [during the development talk] and he did correctly on most of it. But it was, kind of, his own impression that he performed poorly". Other parents, such as Cit 18, were more explicit in their uncertainty. She commented that because her daughter "really does not like mathematics, she should choose the colour blue, which means that you are insecure and think it is difficult. But she coloured it green. Then I ask myself; to what extent can a child assess herself? At the age of seven?" Finally, this particular theme was summarised well by Sat 1, who said that

> I'm a little stressed out by the development talks. And it's not just me. There are many, many parents who say the same. Almost every teacher says that he or she (the child in question) … is very capable, and everything is good and so on. But, by the end of the school, by the end of the year, I mean, you know that it is not so good…. So, it would be good if the teacher could say, 100%, how it is with children. If he is good at maths or if he is not good at reading.

**Communication through parent-initiated contact**

Most parents spoke of the means by which they would contact teachers should the need arise. In general, these fell into three forms, catching teachers when leaving or collecting their child, emailing or telephoning. With respect to catching teachers, Cit 24 was among the most assertive, saying that "If I want to talk about something, I just grab a teacher when I am there or book a meeting". Others were more circumspect, as seen in Sub 5's comment that "if I need to talk to the teachers, there is an opportunity for that. I can do it, but I haven't had that need… If I had that need, I would rather send an e-mail… I think that is easier than talking at eight o'clock". However, for many teachers, and for a variety of reasons, such an option does not exist. For example, Cit 20 observed that "when I leave my child in the classroom, there is no chance to talk to the teacher because it is busy and noisy", while at the end of the day, "when I pick him up… there is only the leisure staff". This latter comment reflected those others, as in Sub 9's experience that "they (teachers) are not here, it is only leisure staff" and Cit 18's "when you pick them up in the afternoons, the teachers are not available".

Most of the parents who could not see teachers at the beginning or the ends of the school day, plus many others, spoke of their use of email. For some, it was their preference, as seen in Cit 18's

comment that "I usually mail. It is usually the easiest... I feel e-mails are very good. It's fast and easy. And you can also request your own separate call". In similar vein, reflecting the earlier issues about teacher availability, Sub 8 noted that "if there is something I would like to talk to them about, I think they (the teachers) are accessible and open, but maybe not when they stand in the doorway as you pick up and leave. But I know I could introduce it in an email, which make me feel welcome". Others spoke of their positive experiences of having emailed teachers. In this respect, Sub 5 commented that "I emailed the class manager about something… and the class manager added that my daughter needs to practice ten mates… and I think that's great, because I didn't know that she had a problem with ten-mates". In similar vein, Cit 20, despite her earlier reservations, added that "I emailed once and asked to have a meeting. It was about mathematics. I don't think that was a problem at all. It went great".

Finally, a few mentioned using the telephone. For example, in addition to Cit 18's observation that she can email to request a convenient time to call, Sub 16 mentioned that he "can call or send emails", which he does "three to four times a term", while Sub 9 commented that parents "have access to their (teachers') phone numbers, so we can call them if it is urgent ... Not very often, mostly when something has happened. We call at most once or twice per term approximately".

## Discussion

In this paper, our goal was to explore parents' perspectives on the ways in which home-school communication supports their children's learning of numeracy. Analyses of semi-structured interview data yielded three broad themes similar to those found in the literature. These concern school-initiated newsletters and development talks, and parent-initiated contacts. The weekly newsletter was broadly seen as helpful, particularly when its content informed parents about the mathematics currently being studied and prompted mathematics-related conversations in the home. It was also viewed positively when it included mathematics-related activities parents could do with their children or homework explicitly tied to number-related learning. However, for some parents, although this was never discussed in relation to mathematics, some letters were thought to include too much detail, prompting concerns that parents would 'miss' something important. In sum, this teacher-initiated form of communication was generally appreciated and seen to play an important role in children's learning.

In principle, all parents seemed to value the development talk, seeing it as an essential element of home-school communication. Typically, this was manifested in comments confirming the aims for the talk as summarised in Osman and Månsson (2015), particularly the need to establish learning goals, albeit rarely explicitly tied to mathematics. However, reflecting earlier research (Minke & Anderson, 2003; Osman & Månsson, 2015), some parents were made anxious by teachers portraying falsely positive (or negative) images of children's mathematical competence. With respect to their children's personal presentations, some parents were clearly impressed and felt they had learnt much about their child, while others, particularly where several development talks were held simultaneously, felt abandoned at a time when teacher-input was desired. That said, in contrast with the year-six pupils discussed by Lindh-Munther and Lindh (2005), there was little evidence that parents thought their children found the experience uncomfortable, although some parents questioned their children's ability to evaluate accurately their learning. Overall, the school-initiated development talks were positively viewed in principle but challenged in practice, particularly as it was only in parents' expressed concerns about the process that mathematics and numeracy were discussed.

Finally, while a small number of parents felt able to 'grab' teachers at the beginning or end of the school day, most seemed to accept that teachers were unlikely to be available for individual consultations. In their acceptance, the majority felt they could email teachers and set their own agendas for discussion, confirming earlier research that it is easier to email than to 'grab' a teacher (Thompson & Mazer, 2012). Moreover, parents that had emailed teachers spoke positively of the responses they had received and of teachers' openness. Few chose to use the telephone, although those that did spoke positively of the experience. In sum, with a single exception and despite interviewer prompts, mathematics did not feature in discussions of parent-initiated contact. This may have been a consequence of the ways in which we conducted our interviews, although there was little evidence of any justification for parent-initiated contact.

In closing, this small-scale study has shown the differential impact of three forms of parent-school communication on year-one children's learning of mathematics. The weekly newsletter clearly has a support role, which parents generally value. The development talk, beyond reporting progress, seems to have relatively little impact on parents' ability to support their children's mathematical growth. It may be, we speculate, due to the possibility that children leading the conference discussions of future mathematical learning may be subordinated to the child's perspectives. Parent-initiated contact was discussed in ways that failed to indicate its impact on mathematical growth. Importantly, research internationally tends to suggest that home-school communication is generally structured by schools in ways designed to 'keep parents in their place' (Karlsen Bæck, 2010; Saltmarsh & McPherson, 2019) and this may be true here with respect to the newsletters and development talks. However, this may not by the case with parent-initiated communication, although, since our sample included only well-educated parents, work may need to be done to help all parents ask the right questions when concerns over their children's learning of mathematics arise and they need to contact the teacher.

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# Interactive mathematical maps – a contextualized way of meaningful learning

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*Student teachers are often not able to link mathematical school and university contents. This problem results in the "double discontinuity" that goes back to Klein (1924)[1], which manifests itself in a lack of understanding and significance of university contents on the part of future teachers. Based on modern learning psychology we assume that this issue can be overcome by defragmentation – a thematically meaningful arrangement – of knowledge. For this purpose, we present a "dynamic interactive mathematical map" which should enable "meaningful learning" in the sense of Ausubel (1963)[2]. By giving the learners the possibility to retrace the formation of mathematical subjects in conjunction with the presentation of similar concepts, understanding is deepened and ultimately transfer of learning is facilitated. We also present one of the tentative studies connected with the map.*

*Keywords: Technology, teacher education, maps, joined-up thinking, transfer.*

## Introduction: sham knowledge versus transfer of learning[3]

The numerous advantages of today's almost inexhaustible supply of information are bound up with pedagogical and didactical challenges. Beside some ethical issues Till (2019) also mentions the reduction of "exploration for solutions to hard problems", and the "proliferation of misinformation" (p. 1). However, problem solving is one of the major aspects of science education in many countries (cf. e.g. Saavedra & Opfer, 2012, p. 5). It seems that the flood of information, which is available without greater effort, leads to an uncritical and unreflective attitude. Additionally, student teachers often have problems to "search and study mathematical literature in an autonomous way" (Winsløw & Grønbæk, 2013, p. 15). As a result, often only decontextualized facts presented in university scripts are memorized. Martin Wagenschein (1968) used the term "Entwurzelung" (Uprooting) to describe the consequences of a decontextualized, superficial study of subjects. He argues that the "'stormy' progress of modern natural science [...] can have an uprooting effect where it is a matter of converting

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[1] "The first 'discontinuity' concerns the well-known problems of transition which students face as they enter university. The second 'discontinuity' concerns those […] which return to school as teachers and the (difficult) transfer of *academic knowledge gained at university to relevant knowledge for a teacher*." (Winsløw & Grønbæk, 2013, p. 2)

[2] Ausubel (1963) shaped the term "meaningful learning", which declares that learning is only meaningful (contrasted to rote learning), when connected to past learnings and many different contexts.

[3] We go along with Haskell's understanding of transfer of learning as "our use of past learning when learning something new and the application of that learning to both similar and new situations. Transfer of learning […] is the very foundation of learning, thinking and problem solving." (Haskell, 2001, p. xiii)

knowledge into education" (p. 61; translation by the author). This is called *The Wagenschein Effect* and states that cutting off the historical and practical context creates „anti-educational, unrealistic, uprooted knowledge: sham knowledge" (p. 66; translation by the author).

In contrast stands a dynamic and well-linked system of knowledge. Studies show: learning that promotes *problem solving* and *global competence* is more sustainable and effective than rote learning of isolated facts (cf. e.g. Boix-Mansilla & Jackson, 2011). The educational and brain researcher David A. Sousa (2017) addresses this issue and demands that information has to be organized in a network. This may be a prerequisite, so that knowledge of today enables us to solve the problems of tomorrow. Sousa calls this „transfer from present to future" (p. 165) and thus describes exactly what teacher students need. Winsløw and Grønbæk (2013) show in their study that the second "discontinuity" of Klein (1924) is still a problem for student teachers today. They often fail to transfer their "*academic knowledge gained at university to relevant knowledge for a teacher*" (Winsløw and Grønbæk, 2013, p. 2), especially, when it comes to autonomous work (pp. 7–15). Therefore, the challenge for our teaching is: *How can information be provided in such a way that learners can not only reproduce it but also apply it autonomously for their future needs?* Sousa affirms that the key for successful transfer lies in the defragmentation of knowledge and the meaningful integration of new contents into existing knowledge structures (2017, p. 154). The reason is that the brain is a "pattern seeker, […] wired to use past information and skills to solve new problems" (p. 156). It clusters subjects according to different attributes and tries to solve unknown problems by transferring solution strategies of known similar ones. The design of the interactive mathematical map is intended to facilitate these processes. Consequently, the basis is the encoding of knowledge in the human brain.

## Foundation for learning: neuronal representation of knowledge

Simplified, knowledge is encoded in a highly branched network of neurons that is constantly changing due to the neuroplasticity of the brain. In this way, data are not stored as isolated facts and are not retrieved as such (Hoffmann & Engelkamp, 2017, p. 146). Sousa describes the encoding of new information as follows: "Whenever new learning goes into working memory, long-term memory […] simultaneously searches the long-term storages sites for any past learnings that are similar to, or associated with the new learning" (2017, p. 154). If the brain finds similar contents in the prior knowledge, it comes to "rooting" (Wagenschein, 1968), i.e. an integration into the existing neuron network, which increases meaning and retention (Sousa, 2017, p. 57). This declares why it is important to show students the connections between different topics, contents and contexts. In 2006, Klassen presented a model that takes modern learning psychology into account. In his "story-driven contextual approach", he considers five contexts (affective, theoretical, practical, historical, and social). His model is one foundation for the design of single contents in the interactive mathematical map. Because the design of these contents strongly depends on the target group, the focus in this paper lies on the greater design of the whole map, which includes not exclusively but especially information about the theoretical and historical context.

# Didactical approach: the interactive mathematical map[4]

The interactive three-dimensional mathematical map, which was first presented by Brandl (2009), is based on a constructivist view of learning and intended to "offer the student an optimal solution for establishing successful learning processes" (p. 106). Information should be presented embedded in different contexts so that as many anchor points as possible are found in the existing knowledge network[5]. In the map, nodes in space display mathematical contents (see Figure 1). Edges symbolize historical developments, which emphasize mathematics as an *emerging science*. An advocate of this type of learning, Felix Klein, described it as „intuitive and genetic, i.e., the entire structure is gradually erected on the basis of familiar, concrete things" (1924/2016, p. 9).



Figure 1: Screenshot of the Interactive Mathematical Map with cursor on one node (Edited, Status 28.01.2020)

In this way, university mathematics contents can be understood as a continuation of school mathematics contents and especially the first discontinuity of Klein can be overcome. Therefore, the map uses one dimension and edges to visualize the genetic sequence. To give an additional overview about similarities of subjects, the remaining two dimensions are defined based on thematic relatedness.

## The implementation of the map design

The position of the nodes in space results as follows: The $z$-coordinate is specified by the time of discovery of the mathematical content represented. The $x$- and $y$-coordinates are determined by an algorithm which translates the thematic proximity into a projected spatial proximity. Therefore, we use a force directed method from graph theory. In this context a graph is a set of nodes and edges, which we 'translate' into a physical model. We imagine that nodes are electrically charged and therefore repulse each other quadratic proportional to their distance. Additional there is another force between two nodes illustrated by a spring. Every spring has an ideal spring length according to the thematic proximity of the related nodes. Whether the spring has a repelling or attracting effect on two nodes depends on the ratio of the distance and the ideal spring length. In general, the force affecting the node $v$ is given by

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[4] The interactive mathematical map with all linked functionalities is available under http://math-map.fim.uni-passau.de/. The prototype of the map was created for the subarea Geometry. First steps of development were done by Datzmann and Brandl (2018). Similar maps for Algebra and Analysis will follow.

[5] Since Wood et al. (1976), this is called "scaffolding".

$$F(v) = \sum_{(u,v)\,\in\,V\,\times\,V} k_1 \cdot \log\left(\frac{d(p_u, p_v)}{l_{uv}}\right) \cdot \frac{u - v}{d(p_u, p_v)} + \sum_{(u,v)\,\in\,V\,\times\,V} k_2 \cdot \frac{1}{d(p_u, p_v)^2} \cdot \frac{v - u}{d(p_u, p_v)},$$

where $k_1$ and $k_2$ are constants, $l_{uv}$ is the ideal distance (i.e. the ideal spring length) and $d(p_v, p_u)$ is the Euclidean distance of two nodes $v$ and $u$. After calculating $F(v)$ the node $v$ is shifted a small percentage of the force and the algorithm goes on to the next node. Thus the system will minimize all appearing forces systematically to find an equilibrium, which considers all desired distances.

**Mathematical interdependencies and horizontal cuts**

Information is clustered by our brain according to thematic relationship and similarity (cf. Hoffmann & Engelkamp, 2017, pp. 143–147). As described above the map is also oriented to this property of our knowledge system and uses two dimensions to express the thematic relationship via the Euclidean distance. In order to fasciliate the recognition of similar contents the map offers "horizontal cuts" as a further functionality (Brandl, 2009). After selecting a required time period, all contents discovered in this time period are projected on one level, so that the thematic proximity of individual contents becomes visible. Figure 2 on the left shows the horizontal cut for the time interval from 520 AD to 1593 AD. The ellipse marks contents of the topic Trigonometry, discovered by Arabian mathematicians. Otherwise, little is developed during this period. Looking at the horizontal cut on the right from 1850 BC to 1899 AD similar contents are clustered (highlighted by ellipses manually). On this horizontal cut, the increasing differentiation in mathematics becomes visible. The dashed ellipse marks contents of non-Euclidean geometry, mostly discovered in the 18th and 19th century. The nodes on the left represent rather elementary geometrical contents. They can be accessed again by a simple click.
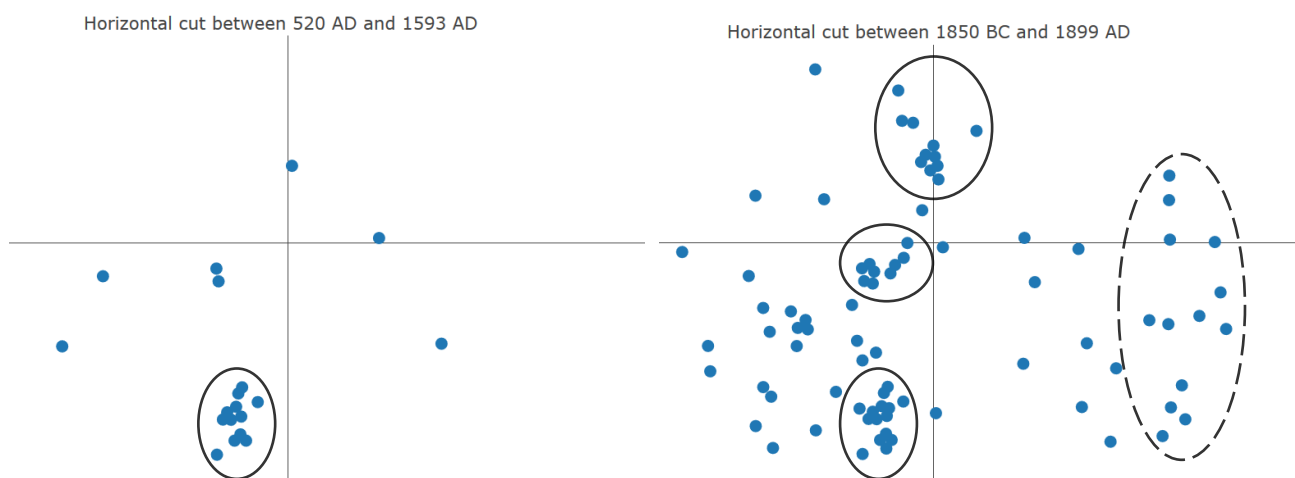


Figure 2: Screenshots of two Horizontal cuts for different time periods (Edited, Status 28.01.20)

**The development of mathematics and vertical cuts**

In order to facilitate understanding, complex contents should be presented as evolving and based on simpler ones. The German mathematician Max Draeger describes the basic idea by saying: „A formal understanding […] is probably possible for the trained mind. A deeper penetration into the subject matter is not, without the knowledge of history" (Draeger, 2012, p. 10; translation by the author). Therefore, the functionality "vertical cut" (Brandl, 2009) is offered. By selecting the node of interest, a new window opens in which this node is coloured and all historically-genetically connected contents including development lines are visible. Thereby, the historical genesis of a content can be retraced.

Figure 3 shows the traditional development of Trigonometry from its beginning when the Egyptians calculate the slope of pyramids. It went on over Thales intercept theorem until the Arabian mathematicians develop the Trigonometry taught in schools today. Eventually with the rise of the analytic geometry, Isaac newton discovered the complex analytic series of trigonometric functions that is taught at universities today. On the left side, important development stages are summarized while on the right side the corresponding vertical cut is depicted.

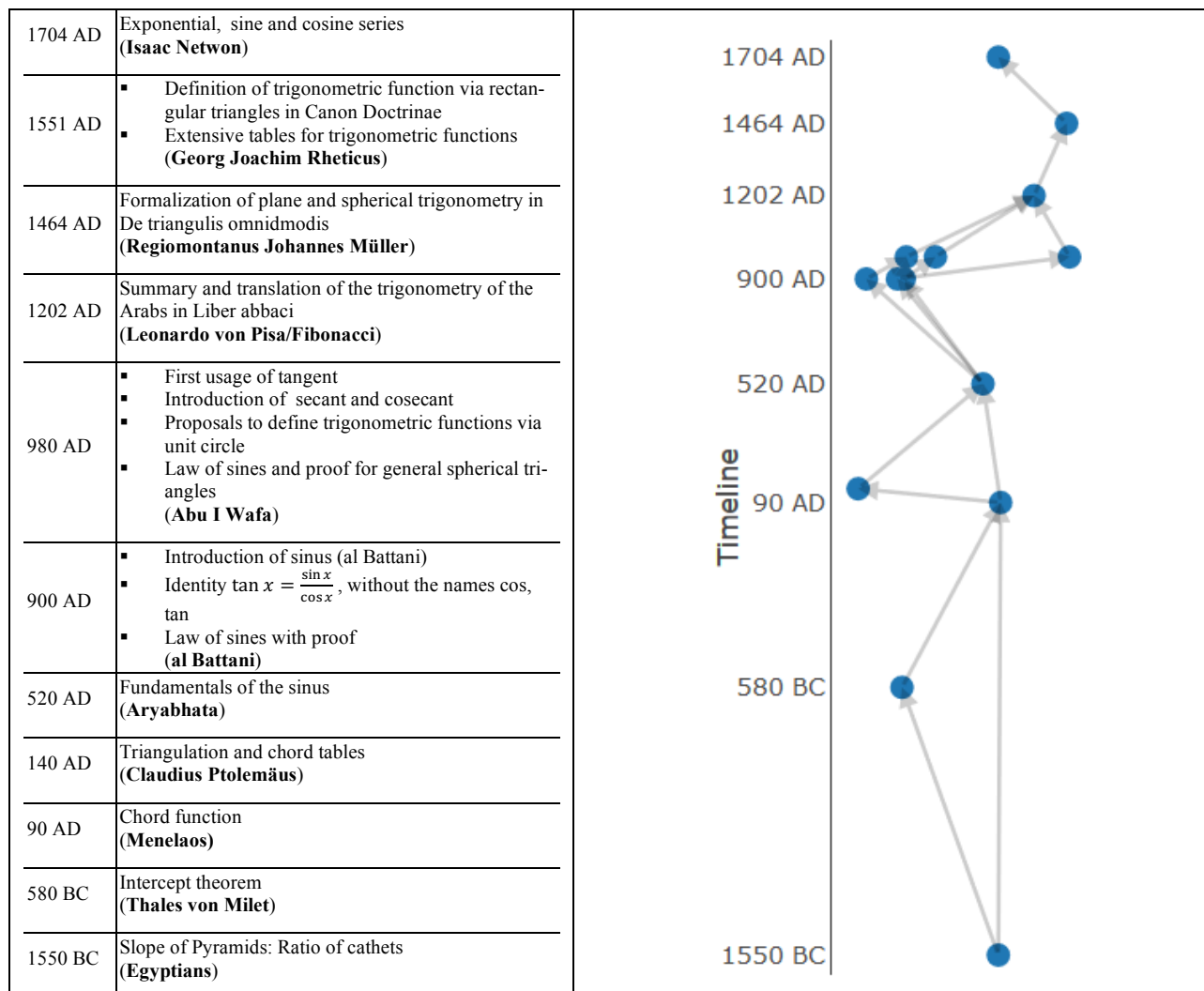| | | |
|---|---|---|
| 1704 AD | Exponential, sine and cosine series <br> (**Isaac Netwon**) | |
| 1551 AD | ▪   Definition of trigonometric function via rectangular triangles in Canon Doctrinae <br> ▪   Extensive tables for trigonometric functions <br> (**Georg Joachim Rheticus**) | |
| 1464 AD | Formalization of plane and spherical trigonometry in De triangulis omnidmodis <br> (**Regiomontanus Johannes Müller**) | |
| 1202 AD | Summary and translation of the trigonometry of the Arabs in Liber abbaci <br> (**Leonardo von Pisa/Fibonacci**) | |
| 980 AD | ▪   First usage of tangent <br> ▪   Introduction of secant and cosecant <br> ▪   Proposals to define trigonometric functions via unit circle <br> ▪   Law of sines and proof for general spherical triangles <br> (**Abu I Wafa**) | |
| 900 AD | ▪   Introduction of sinus (al Battani) <br> ▪   Identity $\tan x = \frac{\sin x}{\cos x}$, without the names cos, tan <br> ▪   Law of sines with proof <br> (**al Battani**) | |
| 520 AD | Fundamentals of the sinus <br> (**Aryabhata**) | |
| 140 AD | Triangulation and chord tables <br> (**Claudius Ptolemäus**) | |
| 90 AD | Chord function <br> (**Menelaos**) | |
| 580 BC | Intercept theorem <br> (**Thales von Milet**) | |
| 1550 BC | Slope of Pyramids: Ratio of cathets <br> (**Egyptians**) | |



Figure 3: Vertical cut: Exponential, sine and cosine series (Screenshot edited, Status 28.01.20)

## Thematic clustering via concept maps

Based on thematic classification of contents (see below), the system offers two-dimensional concept maps. These Concept maps can be created with respect to the underlying geometry and dimensions, the occurring figures, bodies and other thematic characteristics. Nodes are arranged around this according attributes. As an additional tool, this should clarify interdependencies between contents and can increase learning success (cf. e.g. Martinez et al., 2013). Concept maps also increase the effectiveness of teachers (cf. e.g. Beyerbach & Smith, 1990) and have a positive effect on the learning process of students (cf. Nesbit & Adesope, 2006).

**Technical details of the dynamic map, thematic classification and timeline**

In a dynamic map, it should be possible to add, change or delete contents at any time. This is realized via a password-protected interface. All contents need to be classified according to four characteristics:

- Which geometry does the content deal with?
- How many spatial dimensions are covered in the content?
- Which geometric figures/bodies are included?
- How can the type of the content be described?

If necessary, several attributes can be selected for each of these questions, which are differentiated into subcategories for specification. They are used for implementation of all depicted functionalities. If two contents $v$ and $u$ have the same attributes, they are similar in a certain degree. This is measured by a correlation factor $l_{uv}$, which increases for each common attribute and leads to a closer proximity of nodes in space through the algorithm described above. The system automatically updates the location of all nodes when the classification of a content changes. By clicking on a node with the left mouse button, the corresponding content opens on a linked timeline (see Figure 4), where milestones of the considered mathematical area are clearly displayed. Here it is possible to view or download files, links, (embedded) videos, interactive media or other similar items via a further click.
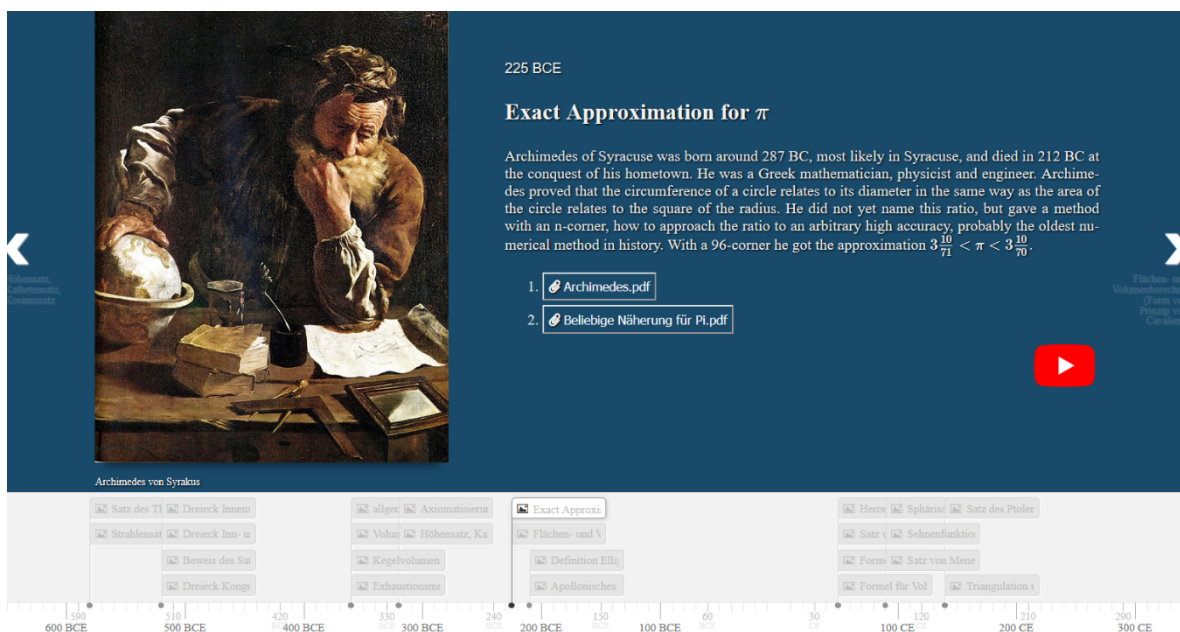


Figure 4: Timeline - Screenshot of Archimedes' Exact Approximation for $\pi$ (Edited, Status 28.01.20)

## Tentative study using interactive mathematical maps

Interactive mathematical maps provide opportunities for studies of learning and teaching on several levels: in-service teachers, teacher students and pupils. Here we outline the first design cycle (cf. Bakker, 2018) for one of the studies planned in this project. It explores the extent to which teacher students can relate university mathematics content to school mathematics content. The aim of the study is to examine whether defragmentation – a thematically meaningful arrangement – of knowledge can build bridges between them and thus overcome the "double discontinuity".

We plan to test the above presented interactive mathematical map constructed for the field of Geometry on teacher students in both University of Passau and Karlstad University. The courses in focus are similar, dealing with Euclidian Geometry. In order to be able to use previous years as a baseline for the study, the course structure and contents remain the same. Only the interactive mathematical map is offered as an additional learning tool. The plan is to collect data in respect to students work with different kinds of exercises and assessments. Examples of such data are, for instance, students' prepared micro-lessons, essays, or examinations. We also plan to collect data describing what content in the maps the student uses. The technical problems are not yet solved, but we tentatively plan for either digital tracking (e.g. mirroring the students' screens) and/or asking the student to keep a log. In order to validate the findings from analysing the collected data, an interview study will be conducted. The study implies informed consent from the participating students following the ethical guidelines in both countries; hence, an ethical approval will be collected from both universities.

In the analysis of the data, we will be able to connect the student use of the map as a tool for learning (e.g., To what extent do the students use the tool, and for what?) to qualities shown in their knowledge (e.g., when transforming the mathematical content into teaching material, etc.). The results from this first design cycle will be used in the development of the interactive map, as well as designing forthcoming studies, in order to develop our expertise on *how information can be provided in such a way that learners can not only reproduce it but also apply it autonomously for their future needs*.

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# Exploring students' metacognition in relation to an integral-area evaluation task

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*Several studies have been conducted to explore students' understanding of the integral-area relationships, however, only a few focused on students' metacognition in relation to this topic. In this study, students' metacognitive knowledge, skills, and experiences in relation to an integral-area evaluation task are explored using semi-structured interviews and think-aloud protocol. The results show that several students developed monitoring strategies in relation to integral-area relationships; however, they do not use these strategies when solving integral-area tasks. The findings suggest that teachers and lecturers could use monitoring strategies more often when solving mathematical questions in class, and encourage students to use these strategies when solving problems.*

*Keywords: Metacognition, metacognitive knowledge, metacognitive experiences, metacognitive skills, integral-area relationships.*

## Introduction

Integral calculus is part of the upper secondary school and undergraduate university mathematics curriculum in many countries as a wide range of real-world problems require an understanding of integral calculus, including a range of contexts in physics and engineering (e.g., Thompson & Silverman, 2008). Many undergraduate and graduate courses in mathematics and the engineering sciences also rely heavily on parts of this topic (e.g., differential equations) (Czocher, Tague, & Baker, 2013). Several studies have been conducted to explore students' mathematical understanding of concepts within integral calculus such as the integral-area relationships (e.g., Jones, 2013; Sealey, 2014; Thomas & Hong, 1996) and the Fundamental Theorem of Calculus (FTC) (e.g., Thompson, 1994; Thompson, & Silverman, 2008). However, only a few studies (e.g., Radmehr & Drake, 2017, 2019, 2020) have explored students' metacognition in relation to integral calculus. Metacognition is "meta-level knowledge and mental action used to steer cognitive processes" (Jacobse & Harskamp, 2012, p. 133). Previous studies (e.g., Radmehr & Drake, 2017, 2019; Dallas, 2014) have highlighted that metacognition "is a driving force in mathematical problem solving" (Czocher, 2018, p. 140); however, many students have not developed well their metacognition in relation to mathematical problem solving (e.g., Radmehr & Drake, 2017, 2019; Jacobse & Harskamp, 2012). In this study, we explore students' metacognition in relation to an evaluation task related to integral-area relationships. We also investigate the differences that might exist between the metacognition of upper secondary and tertiary students in this regard. The research question explored here is: What metacognition do upper secondary and tertiary students reveal in relation to an integral-area evaluation task?

## Facets of metacognition

Three facets of metacognition have been recognized in previous studies (Efklides, 2006, 2008): *metacognitive knowledge*, *metacognitive skills*, and *metacognitive experiences*. In this section, we describe these facets to frame the study. *Metacognitive knowledge* "is declarative knowledge stored

in memory" (Efklides, 2008, p. 278) about factors (i.e., *persons*, *tasks*, *goals*, and *strategies*) that influence cognitive activities (Efklides, 2006, 2008). The *persons* factor relates to how individuals complete or feel about different tasks. For example, you are more confident in finding an enclosed area with respect to the $x$-axis rather than the $y$-axis when both ways can be used for finding the area using integral calculus. The *tasks* factor relates to categories, relationships, features, and how different tasks work. For instance, your knowledge about how a question related to integral-area can be checked. The *goals* factor refers to the goals individuals pursue when engaging in different tasks (e.g., you solve your integral assignment questions to have a better understanding of the topic). Finally, the *strategies* factor consists of strategies that are used for problem-solving, and how, why, and when such strategies should be used (e.g., knowing in the questions related to integral-area, sketching the related graphs of functions will help to determine which method should be used) (Efklides, 2006, 2008).

*Metacognitive skills* are activities that are performed deliberately to help individuals control their cognitive activities (Efklides, 2006, 2008). These activities consist of task orientating, planning, monitoring, regulating, and evaluating (Efklides, 2006, 2008). For example, making a visualization when solving a mathematical problem can help with identifying how the problem can be solved. We need to highlight the difference between metacognitive knowledge and skills. Students might have developed their metacognitive knowledge in different ways; however, they are not using them when engaging in problem solving, indicating lack of their metacognitive skills. For instance, many students know drawing a graph will help in solving mathematical problems, indicating the presence of metacognitive knowledge; however, they might not use this strategy when solving mathematical problems, indicating lack of metacognitive skills.

*Metacognitive experiences* are one's awareness and feelings when engaging in a task and processing its information (Efklides, 2008). Metacognitive experiences include feelings of knowing, familiarity, confidence, satisfaction, and difficulty in relation to different tasks (e.g., I think I can solve this mathematical problem because I saw similar tasks before). It also comprises judgment of learning and the correctness of solutions, and estimating effort and time needed to spend on tasks (e.g., I think I can solve this mathematical problem in 15 minutes or I think I solved this question correctly) (Efklides, 2006, 2008).

## Integral-area relationships and definite integrals

Several studies have focused on students' understanding of integral-area relationships (e.g., Kiat, 2005; Mahir, 2009; Sealey, 2014). These studies have reported that many students have difficulties in finding area using integral, for example, when the function is below the $x$-axis (e.g., Radmehr & Drake, 2019); or the graph of the integrand is not given to students (e.g., Kiat, 2005). For instance, Kiat (2005) reported that 55% of students could not set up the integrals correctly to find a shaded area in a task in which one of the curves was below and one was above the $x$-axis (Kiat, 2005). Furthermore, students' understanding of why integral techniques should be used for finding enclosed area is limited (e.g., Thomas & Hong, 1996). Riemann sums and definite integrals, $\lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x$, entail a number of important concepts (i.e., functions, limits, rate of change, and multiplication) (Sealey, 2014). Previous studies have reported understanding the definite integral as the limit of a sum is a difficult task for many students (e.g., Sealey, 2014). To develop a better understanding how students construct the concept of the Riemann integral, Sealey (2014) has designed a framework to

characterise students' understanding of Riemann sums and the definite integral. Sealey (2014) has identified that understanding the product of $f(x)$ and $\Delta x$ is the most complex part for students.

## Methods

A multiple case study was used to explore students' cognition and metacognition in relation to the integral calculus. Two cases were selected composed of a sample of students that were interviewed in 2014-2015 academic year. Case 1 is one of the top five universities in New Zealand and Case 2 is one of 11 colleges (upper secondary schools) in Wellington city, New Zealand. Nine students from Case 1 and eight students from Case 2 voluntarily participated in the study. All the college students were enrolled in a Year 13 calculus course, and the university students were enrolled in a single variable calculus course (designed for students of the Faculty of Science) when the data collection started. The students of Case 1 were enrolled in a mathematics, statistics, or physics major.

The students participated in a one to one interview with the first author that lasted between 70 to 150 minutes to explore their metacognition and mathematical understanding. Some students completed the interview in one session and some students in two sessions. In the interviews, nine tasks related to integral-area relationships and the FTC were given to students, and they also responded to 14 questions related to their metacognitive knowledge. Students' performance and their metacognitive experiences and skills in relation to seven of these tasks have been reported in Radmehr and Drake (2017, 2019), and their metacognitive knowledge in Radmehr and Drake (2020). Here, we report the results of the following evaluation task that have not described in those studies.

Are these examples solved correctly? Please justify your answer.

**Example 1:** Find if possible, the area between the curve $y = x^2 - 4x$ and the $x$-axis from $x = 0$ to $x = 5$. $\qquad \int_0^5 (x^2 - 4x)dx = [\frac{x^3}{3} - \frac{4x^2}{2}]_{x=0}^{x=5} = [\frac{5^3}{3} - \frac{4(5)^2}{2}] - [\frac{(0)^3}{3} - \frac{4(0)^2}{2}] = \frac{-25}{3}$

**Example 2:** Find if possible, the area enclosed between the curve $y = \frac{1}{x^2}$ and the $x$-axis from $x = -1$ to $x = 1$. $\qquad \int_{-1}^1 \frac{1}{x^2} dx = \int_{-1}^1 x^{-2} dx = [\frac{(x)^{-1}}{(-1)} = \frac{-1}{x}]_{x=-1}^{x=1} = \frac{-1}{1} - \frac{(-1)}{(-1)} = -2.$

Additionally, we have not explored how much of the metacognitive knowledge students possess about a mathematical topic, they have actually used when solving mathematical problems. In other words, this is our first attempt to compare students' metacognitive knowledge with their metacognitive skills and experiences. The first example is designed based on a task from Kiat's study (2005). Students may find an incorrect area for both examples if they do not sketch the graph of the curves. In addition, students who only focus on integration techniques and do not pay enough attention to the integral-area relationship may also make mistakes.

To explore students' metacognition in relation to each facet of metacognition, different approaches have been proposed (Radmehr & Drake, 2018). To explore students' metacognitive knowledge, a set of questions could be asked from students about what they would do when they face specific mathematics-related situations (Radmehr & Drake, 2018). The results in relation to one of these questions are presented in this paper: *How do you check your answers when solving problems involving finding the area enclosed between curves?*

Concerning students' metacognitive skills, students can be asked to verbalize their thought when they are solving mathematical problems using a think-aloud protocol (Radmehr & Drake, 2018; Dallas, 2014; Jacobse & Harskamp, 2012). Then, their verbal thought can be analyzed using the rubrics

designed for analyzing students' metacognitive skills (see Jacobse & Harskamp, 2012; Veenman, Kerseboom, & Imthorn, 2000). For example, Veenman et al. (2000) suggested checking calculation and answer, reflecting on the final answer, and the learning experience as some of the activities that can indicate the presence of metacognitive skills. In this study, from different components of metacognitive skills, students' monitoring strategies and making drawing in relation to the task were analyzed (when students were solving the evaluation task). These two are among the metacognitive activities important for successful mathematical problem solving (Jacobse & Harskamp, 2012).

To explore students' metacognitive experiences, two items from the Visualization and Accuracy instrument (Jacobse & Harskamp, 2012) were used. First, students read the task, and before solving it, were encouraged to answer the following question: *How well do you think you can solve this problem?* They could choose one of three following: *I am sure I will solve this problem*; *I am not sure whether I will solve this problem correctly or incorrectly*; or *I am sure I cannot solve this problem*. After selecting one of the choices, they were asked to provide a reason(s) for their choice. After working on the task, a similar question was asked: *Rate your confidence for having found the correct answer*. Students had similar choices (i.e., *I am sure I solved this question correctly*; *I am not sure whether I solved this question correctly or incorrectly*; *I am sure I solved this question incorrectly*), and they were encouraged to provide a reason(s) for their choice.

## Results

In this section, first, students' monitoring strategies in response to the metacognitive knowledge question are provided. Then, students' pre-judgments, mathematical performances, metacognitive skills, and post-judgments in relation to the evaluation task are described. The university students are indicated by T and the college students with S.

**Students' metacognitive knowledge in relation to integral-area relationships**

Students in Cases 1 and 2 mentioned different strategies for checking their answers in integral-area problems. Three strategies were domain-specific, including approximating the enclosed area using geometric shapes to find out whether the answer makes sense (T: 1, 3, 4, 6, 8, 9; S: 7, 8); checking the area is positive (T: 3, 9; S: 1, 5, 6); and checking the antiderivative by differentiating it (T: 3, 9; S: 1, 5, 6). The other monitoring strategies mentioned by students were general and can be used in different settings. The most common strategy for checking answers was going over calculations (T: 1, 3, 4, 5, 6, 8; S: 2, 4, 5, 7). The other monitoring strategies mentioned by at least two students were: Using Wolfram alpha website to check answers (T: 3, 4, 5, 6, 7); checking answers with classmates (T: 4, 6; S: 6, 7); Using the answers at the end of the textbook (S: 2, 3, 4, 6); Using assignment solutions (T: 2, 3); and using a calculator to check answers/graph of curves (S: 3, 6). These findings show that approximating area using geometric shapes and also using online resources (e.g., Wolfram alpha) were more common within university students for checking answers. In contrast, college students relied more on solutions at the end of their textbook and standard calculators.

**Students' metacognitive experiences (pre-judgments) in relation to the evaluation task**

In relation to the students' pre-judgments, all the students thought they were able to do this task. However, only T5 identified that the first example was solved incorrectly and only two students (T: 5, 9) identified the second had been solved incorrectly. Students made their judgments based on the familiarity of the examples (T: 1, 2, 4; S: 3, 5, 6), saying the integrals are "simple" and

"straightforward to integrate" (T: 2, 8; S: 7), and they knew how to solve these questions (T: 9; S: 2, 4, 8). T5 made that judgment because she thought only one curve was involved in each example: "sure, in both questions, they are only one equation involved so it is not as tricky as the ones with more equations". S1 made that judgment because he thought he did not need to graph the curves: "Also, I do not need to work out the formula from the graph". Two students' responses (T: 6, 7) related to the fact that the task involved evaluating. T7 said, "…easier to find someone wrong than to prove me right", and T6 mentioned, "I think I can find any wrong steps" [in the solutions]. Finally, T3 thought he was able to complete the task as "nothing looks immediately wrong".

### Students' mathematical performance on the evaluation task

As stated earlier, only T5 identified that the first example was solved incorrectly. For the second example, only two students (T: 5, 9) recognized it was solved incorrectly; however, three students (T: 5, 7, 9) showed they had the conceptual knowledge that the integrand should be continuous on the interval being integrated. T5 showed that the area is diverging by using the improper integral. T9 believed that if an integrand is not continuous at a given point, the calculation is not possible: "Why this work, I thought it will be something in the calculation that wouldn't work [sic]". T7 sketched the graph of $y = \frac{1}{x^2}$ and said "we do not have the division by zero problem here", indicating he did not notice that the function is undefined at zero. The remaining students only checked the integration and calculations, indicating procedural knowledge of the integral-area relationship. A lack of numerical proficiency was also observed whilst students were attempting to answer this question. For example, four students from Case 2 (S: 1, 4, 5, 6) used a calculator to check if the bounds were substituted correctly in the integrand, suggesting they were not confident with calculations such as 25×2 and $\frac{5^3}{3}$. Moreover, S1 simplified $\frac{5^3}{3} - 2 \times 5^2$ as $\frac{125}{3} - 20$, then asked to use a calculator to find the end result.

### Students' metacognitive skills in relation to the evaluation task

Regarding making a drawing to solve the task, only two students (T: 5, 7) made a drawing for this task. However, Radmehr and Drake (2019) showed that for more typical integral-area questions (e.g., "Please calculate the area enclosed between the curve enclosed between the curve $x=y^2$ and $y = x - 2$ in two ways. Which way is better to use? Why?" (p. 91)) many students try to draw the enclosed area. T5 made a drawing for both parts of the question and solved the question correctly. T7 made a drawing for the second part, however, could not identify that the function was not defined at zero.

In relation to checking calculations and answers, five students (T: 3; S: 1, 4, 5, 6) undertook some form of check. College students used their calculators to check their calculations, as mentioned above. T3 checked the calculation after he had solved the question, without using a calculator, to explore whether he had substituted the bounds in the antiderivative correctly, because the answers were negative.

### Students' metacognitive experiences (post-judgments) in relation to the evaluation task

In relation to the post-judgments, 15 students were sure they had solved the task correctly; however, only T5 judged the first example correctly, and only two students (T: 5, 9) judged the second example correctly. The other two students (T: 1, 2) were unsure if they had solved the task correctly because they obtained a negative area. In addition, T1 was unsure whether the antiderivative of $\frac{1}{x^2}$ was $\frac{-1}{x}$ or

not, indicating a lack of procedural knowledge for finding antiderivatives. The 15 students were sure for several reasons. 13 students (T: 3, 4, 6, 7, 9; S: 1, 2, 3, 4, 5, 6, 7, 8) made that judgment because they had the same answer (for one or both examples). T8 was "confident" with the way he checked the solutions of the examples. Three students (T: 5; S: 1, 5) were sure because they had found a different answer to that written in the task; however, S1 had found a wrong answer through a wrong calculation and S5 had an error relating to the integral-area relationships.

## Discussion and conclusion

The paper adds to the literature in mathematics education in several ways. First, a literature search revealed that this is one of the first attempts in regards to exploring how much of the metacognitive knowledge students possess about a mathematical topic is actually used by them when solving mathematical problems. Secondly, only a few studies have explored students' metacognitive knowledge, experiences, and skills in relation to upper secondary and tertiary mathematics (e.g., Radmehr & Drake, 2017, 2019), therefore, the findings could help mathematics education researchers to have a better understanding of students' metacognition in relation to learning mathematics at the upper secondary and tertiary levels.

In relation to students' metacognitive knowledge, students highlighted several useful monitoring strategies that can be used for checking answers of certain types of integral-area questions. However, it seems some of the students in the sample did not develop these monitoring strategies. For instance, nine students did not refer to the approximating area using geometric shapes, and thirteen students did not mention differentiating antiderivative as a monitoring strategy in response to the interview question. Secondly, in relation to students' metacognitive skills, when they engaged in the evaluation task, the students did not use these strategies, and therefore, most of them were unable to identify the mistakes in the examples. Only some of them checked their calculations without drawing the graphs. Teaching monitoring strategies could be considered as a key element of teaching mathematics, and ways in which students can check their solutions could be suggested to them (e.g., Goos, Galbraith, & Renshaw, 2002). The active role of the lecturers/teachers in posing questions like "how can we check this answer?", then, discussing and modelling the use of appropriate strategies, more students might learn, and motivate to use monitoring strategies during solving mathematical problems (e.g., Schoenfeld, 1987). In addition, if teachers and lecturers use monitoring strategies more often when solving questions in class, and encourage students to do so as a part of their problem solving, these strategies might be used more often by students. Designing mathematical problem-solving tasks with metacognitive questions at each problem-solving phase (e.g., the post-judgment item), could also support developing students' metacognition (Dallas, 2014). These activities could lead to fewer errors in solving mathematical problems, and students developing greater confidence in their mathematical understanding (Radmehr & Drake, 2019; Dallas, 2014).

In relation to the metacognitive experiences, the fact that the task was an evaluation task did not negatively affect the confidence of students as they were all confident they could solve the question. Students' pre-judgments show that most students based their judgment on familiarity with the integrand and knowing how to find antiderivatives, and not by focusing on the shape of the enclosed area. Most of the students did not make any drawing of the given curves, which resulted in most students being unable to find the mistake in the solution they were evaluating. This suggests that it might be useful for their lecturers and teachers to more strongly highlight the importance of drawing

curves for integral-area problems and help them focus on the enclosed area's shape when trying to solve such questions. This might also help students to not over-simplify a problem. In this study, consistent with previous studies in relation to the integral-area relationships (e.g., Mahir, 2009), students' procedural knowledge was better developed compared to conceptual knowledge. For instance, the fact that the integrand should be continuous was ignored by most students when solving the task. Also, a lack of algebraic manipulation skills and prior knowledge were observed for several students. These findings, also highlighted by previous studies (e.g., Kiat, 2005), suggesting several students would benefit from improving their algebraic manipulation and/or graph sketching prior to starting integral (Kiat, 2005).

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# Surveying preservice teachers' understanding of aspects of mathematics teaching – a cluster analysis approach

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*Preservice teachers begin their teacher education with experiences that affect their possibilities for accessing and integrating new learning into their teaching practices. Yet often mathematics teacher education courses treat preservice teachers as a homogenous group. Responses to an electronic survey from the beginning of two compulsory mathematics education courses showed that preservice teachers could be divided into two clusters. The preservice teachers in each cluster give similar responses to different aspects of mathematics teaching, suggesting they share similar sets of views. These differences should be recognised in their future mathematics teacher education courses.*

*Keywords: Preservice teachers, cluster analysis, argumentation, modelling.*

## Introduction

Teaching is complicated, with teachers having to simultaneously consider a range of aspects. To ease this complexity, teachers often make assumptions about the groups they teach which then affects what they make available and in what ways and this in turn affects the potential learning of the students in their class (e.g. Hansen-Thomas & Cavagnetto, 2010). Similarly, it is important for teacher educators to know what views preservice teachers (PTs) hold on different aspects of mathematics teaching so they can adapt their teacher education appropriately. Traditionally, teacher educators have focused on what PTs should learn about mathematics and mathematics education (Ponte & Chapman, 2008), rather than what they may already know when entering teacher education. In our wider project, *Learning about teaching argumentation for critical mathematics education in multilingual classrooms* (LATACME), the aim is to improve two compulsory mathematics education courses for teachers of grades 1 to 7 in regard to argumentation, critical mathematics education, multilingual classrooms, mathematical modelling and the use of digital tools. The impetus for this comes from the implementation of a new curriculum in August 2020 in which "reasoning and argumentation" and "modelling and application" are two of six core elements and digital skills is one of five "basic skills" (Utdanningsdirektoratet, 2019), and the requirement that PTs have an "awareness of cultural differences and being able to use these as a positive resource" (National Council for Teacher Education, 2016, p. 9). It is important to identify PTs' existing understanding of these different aspects of mathematics education when they begin these compulsory teacher education courses.

This paper is one of two papers presented at this conference (see also Meaney et al., in press), on the results of an electronic survey about the PTs' initial understandings about the specific aspects of mathematics teaching. The survey was completed at the beginning of the two compulsory courses. In this paper, we focus on the questions that allowed us to identify: 1) if there are clusters in the cohort, and 2) if there were clusters, what were the differences between these clusters. The responses to these questions provide background for the planning of our teacher education courses, which will be discussed in a later paper.

## Previous cluster analysis research on preservice teachers

To identify groups requires an investigation into whether there are different sets of understandings about mathematics teaching within a cohort and whether these are held by particular groups of PTs. In mathematics teacher education, cluster analysis has typically been used to identify key aspects of different groups of PTs. For example, using questionnaire data from Finnish elementary PTs' views on mathematics, Hannula, Kaasila, Laine, and Pehkonen (2005) found three main PT profiles. These were characterized by *positive* (43%), *neutral* (36%), or *negative* views (22%) of mathematics. Each group could be split into two sub-groups, based on the amount of encouragement the PTs received from family and on their view of themselves as hard-working. The PTs with a positive view could be split into the two groups *autonomous* (21%) and *encouraged* (22%). The PTs with a neutral view could be split in *pushed* (18%) and *diligent* (18%), and the ones with a negative view could be split in *lazy* (18%) and *hopeless* (4%) – the latter group believing that they could not learn mathematics.

Cluster analysis has also been used to determine whether groups of PTs see language diversity in mathematics classrooms differently (McLeman & Fernandes, 2012; McLeman, Fernandes, & McNulty, 2012). McLeman and Fernandes (2012) investigated the beliefs of 334 PTs from across the USA about the mathematics education of English-as-a-second-language learners (ELs). The PTs completed an online questionnaire in which they ranked 26 items using a five-point Likert scale (strongly disagree to strongly agree). They identified two clusters, which differed according to PTs' beliefs about: parents from some cultures placing a higher value on education than others; limiting mathematics vocabulary to make the content clear to ELs; and creating discussion-rich classrooms as necessary for ELs to learn mathematics. With some variation, the PTs generally expressed deficit views on culture and family and parents' support for their children's education, which seemed to be quite resistant to change by teacher educators and teaching experiences. The PTs did, however, believe mathematics to be an ideal subject to support students who were learning English.

As a follow-up study, McLeman et al. (2012) investigated 292 PTs' conceptions about the mathematics education of ELs, with a similar analysis to that in McLeman and Fernandes (2012). Their results indicated that PTs' exposure to issues to do with ELs affected their conceptions about their mathematics education. They also documented that apart from the item about discussion-rich, mathematics classrooms being beneficial for ELs, the PTs in cluster 1 held views which were not in alignment with research. For instance, the means were low for claims like "Learning English is more important than native language", "Speaking in a language other than English hampers the learning of English", and "Conversational fluency implies capability to learn math like non-English learners".

## Methodology

The PTs in this study have two mandatory mathematics courses of 15 ECTS each. The courses are taught in the second and third semesters and integrate mathematics and mathematics education. The questionnaire was administered at the beginning of the second semester, after one semester of teacher education, including a three week practicum period, but before exposure to mathematics teacher education. Thus, the responses to the questionnaire can be assumed to be related to the PTs' school mathematics experiences and early experiences with teacher education.

A digital Likert-scale survey was designed with the PTs being asked to which degree they agreed or disagreed on a total of 51 claims concerning argumentation, digital tools, mathematical modelling,

multilingual classrooms and critical mathematics education. Of the approximately 200 PTs in the cohort, 96 completed the questionnaire.

On argumentation, the PTs were presented with Grade-4-student responses to the task: "Why is the sum of two odd numbers always an even number?" Ben's answer is shown in Figure 1. The PTs were asked to what extent they agreed to three claims: "I can understand and follow Ben's explanation"; "Ben's explanation is incomplete"; and "Ben's explanation is mathematically correct". The "Don't know" category, together with the "Neither agree nor disagree" one, reduces the extent of random answers and has been shown not to contribute to lower validity (Lozano, García-Cueto, & Muñiz, 2008; O'Muircheartaigh, Krosnick, & Helic, 2000; Wang & Krosnick, 2020).
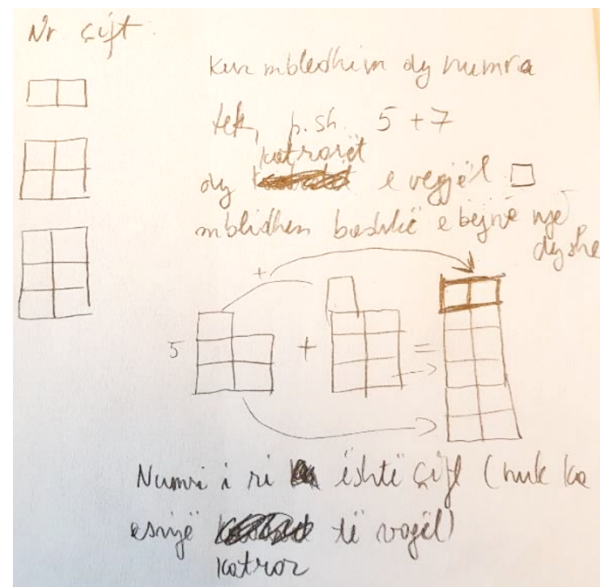


Figure 1: Ben's explanation

In relationship to modelling, the PTs were also asked to rate claims about a project for Grade 5 on air pollution (Figure 2 shows a diagram from the project). Two examples of claims were: "I would not have used this project in my teaching because the students would find it too extensive" and "The project would take too much time from teaching and would not allow us to get through everything we are supposed to do in the book". The PTs were also presented with eight claims about modelling in mathematics teaching. Among these
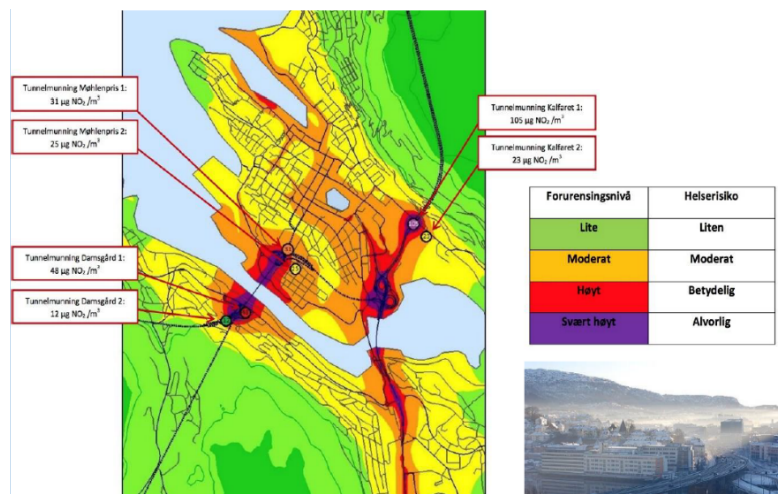


Figure 2: Grade five teaching project on air pollution (Norwegian Public Roads Administration and Bergen Municipality, 2017, p. 15-16)

were "Students in primary school are too young for modelling to contribute to increasing their critical judgment ability about how mathematics is used in society" and "Modelling improves students' attitudes towards mathematics".

A cluster-analysis approach was used to identify connections between responses. This approach makes it possible to identify whether there are groups of PTs, i.e. clusters, that respond in similar ways to sets of questions and whether these clusters differ significantly from one another. To do this, each PT can be considered as a point in a 51-dimensional coordinate system, where each axis corresponds to a claim in the questionnaire. For instance, the first axis corresponds to the claim "I can follow and understand Ben's explanation". If the PT answered, "Completely agree", then "Completely agree" is the value of the coordinate of that point. Thus, 96 PTs yield 96 points (some

perhaps overlapping) in that coordinate system. If some of the PTs commonly give the same answers, the corresponding points would lie close to one another in the coordinate system and yield a cluster.

Kroh (2006) argued that a respondent replying "Don't know" very often has an opinion on the topic, but has either not understood the question or has not taken the time to think about it. Since clustering algorithms will interpret many "Don't know" responses as if the respondents largely agree on a claim, these responses were replaced with random variables where the probability is equally high for each Likert category. This randomising process was repeated 460 times, with the analysis of differences between how the clusters had responded was repeated for each process. Differences between clusters were interpreted as significant only if they were significant for all 460 versions of the data, ensuring that the probability of false positives is less than 0.01. A check of outliers was made, using the Mahalanobis distance (p<0.001). No outliers were found.

To select which clustering algorithm and which number of clusters should be used, we applied the Ratkowsky–Lance criterion for choosing: the clustering method; the similarity measure to use; and the number of clusters to consider (Ratkowsky & Lance, 1978). This criterion has the advantage of also making sense for non-parametric data. Generally, introducing criteria for clustering ensures that a cluster is only split into two smaller clusters if the elements inside the smaller clusters are "sufficiently more" similar to each other compared to the original, larger cluster. However, there must also be a rule for limiting the number of clusters to avoid 96 different clusters with one PT in each.

The similarity measures we tested when applying the Ratkowsky–Lance criterion were the Pearson $\chi^2$ similarity measure and its normalisation, the $\phi^2$ similarity measure. For practical purposes, both methods measure the similarity between claims rather than PTs. Two claims are considered similar if the PTs commonly respond equally to both. However, in the $\chi^2$ measure, the similarity index grows with the number of respondents, so that within a cluster, the size of the cluster would affect $\chi^2$. To fix this, the $\phi^2$ measure is given by dividing $\chi^2$ by the number of respondents (Anderberg, 1973).

The Ratkowsky–Lance criterion defines the "best clustering" as that which maximises a certain index. This index can be described as: For each claim, the average $\phi^2$ similarity between what the PTs have responded and which cluster they belong to, is computed. Then the average for all claims is divided by the square root of the number of clusters. This implies that having too many clusters results in a lower index. Maximising the index yielded that applying the "Within-groups linkage" clustering algorithm together with the $\phi^2$ distance metric and two clusters was, by definition of the criterion, the most ideal approach. In within-groups linkage, the clusters are formed so that $\phi^2$ within each cluster is as small as possible.

When looking for significant differences between the clusters, we applied the Mann–Whitney U test (Mann & Whitney, 1947; Wilcoxon, 1945).

## Findings

Initial findings for the group of PTs as a whole is discussed in the other paper presented at this conference and so the results are only briefly summarised here. The PTs generally chose a positive response to all the claims. They agreed on the need for argumentation in the mathematics classroom, they were largely positive to the idea of modelling, although they placed themselves more in the middle on the claims about the air pollution project. On the other hand, they still showed a tendency towards believing that the project would be good for the students' understanding of modelling.

**Overview of responses and characteristics of the two clusters**

In this section, we present the two clusters and discuss the statistically significant differences between them. By using the "Within-groups linkage" clustering algorithm together with the Ratkowsky–Lance criterion, two clusters were identified. Cluster 1 (C1) consists of 32 PTs, with the remaining 64 PTs forming cluster 2 (C2). Table 1 provides the responses given by the PTs in each cluster on the claims about Ben's argumentation, as well as some claims about modelling and the air pollution project.

| Claims about argumentation (Figure 1) | | Disagree | Neither/nor | Agree |
|---|---|---|---|---|
| 1 a) I can follow and understand Ben's explanation. | C1 | 0.0 | 3.1 | 96.9 |
| | C2 | 25.1 | 15.6 | 56.3 |
| 1 b) Ben's explanation is incomplete. | C1 | 81.3 | 6.3 | 6.3 |
| | C2 | 20.4 | 39.1 | 34.4 |
| 1 c) Ben's explanation is mathematically correct. | C1 | 0.0 | 6.3 | 87.5 |
| | C2 | 4.7 | 42.2 | 42.2 |
| **Claims on modelling in general** | | | | |
| 3 b) Students in primary school are too young for modelling to contribute to increasing their critical judgment ability towards how mathematics is used in society. | C1 | 90.6 | 3.1 | 6.2 |
| | C2 | 45.3 | 39.1 | 6.3 |
| 3 d) Modelling improves students' attitudes towards mathematics. | C1 | 0.0 | 15.6 | 75.0 |
| | C2 | 4.7 | 42.2 | 45.4 |
| **Claims on the air pollution project (Figure 2)** | | | | |
| 7 a) This project would be too difficult for students who do not speak good Norwegian because they would not be able to justify their answers. | C1 | 50.0 | 21.9 | 18.8 |
| | C2 | 14.1 | 34.4 | 43.8 |
| 7 c) I would not have used this project in my teaching because the students would find it too extensive. | C1 | 53.5 | 21.9 | 18.8 |
| | C2 | 17.2 | 31.3 | 34.4 |
| 7 f) Students in 5th grade must know or first learn about the concepts of average and spread measures to understand what the project is about. | C1 | 28.2 | 31.3 | 28.1 |
| | C2 | 6.3 | 20.3 | 56.3 |
| 7 h) The project would take too much time from teaching and would not allow us to get through everything we are supposed to in the book. | C1 | 59.4 | 18.8 | 9.2 |
| | C2 | 34.4 | 28.1 | 20.4 |

Table 1: Overview of how the clusters have responded

In Table 1, "disagree" is the combined number of responses from "completely disagree" and "moderately disagree", and similarly for "agree". "Neither/nor" stands for the response "neither agree nor disagree". The percentages for "don't know" are excluded, for the reasons discussed previously, and, as a result, the percentages do not total to 100.

Table 1 shows that cluster 1 is generally more positive towards Ben's explanation, modelling in general and the air pollution project. Cluster 2's responses are more spread across the different Likert categories and there is a higher percentage of "Neither agree nor disagree". Cluster 1 is more positive towards Ben's explanation even though his explanation is written in a language the PTs are unfamiliar with. Not one PT from cluster 1 disagreed with the claim that "I can follow and understand Ben's explanation". Almost everyone agreed with the claim and among these, 62.5% replied "Completely agree". Only 12.5% of the PTs in cluster 2 chose the same response.

Extreme responses, in which PTs "completely agreed or disagreed" were a particular characteristic of cluster 1. On claim 7 h), nearly a third of the PTs in this cluster "completely disagreed" that the air pollution project in topic 7 would take too much time from teaching. The project provided no information about what the time-frame would be for such a project. Only 3.1% of cluster 2 completely disagreed. For claim 7 c), there are 21.9% of the PTs in cluster 1 who completely disagreed that the air pollution project would be too extensive, while in cluster 2, the percentage is 1.6%.

**Significant differences between the clusters**

We found a significant difference between the clusters on 16 of the 51 claims in the survey. Seven of these were on the claims about student argumentation, two were on modelling in general, and five were on the air pollution project. In addition, one was on argumentation in general and one was on a modelling project concerning composting food waste. Because of limitations on space, we have focused on nine claims where the differences between clusters were the most significant (see Table 1).

As noted in the previous section, the clusters largely differ because the PTs in cluster 1 were more prone to responding on the "completely" end of the scale, while the cluster 2 responses were more evenly spread and with a bigger percentage (compared to cluster 1) who responded, "neither agree nor disagree". This applies in particular for the claims about argumentation. In cluster 1, at least 50% of the PTs answered on the "completely" end of the scale on all except three of the twelve claims. For cluster 2, the same result is only found on two of the claims.

Although the clusters differ, they do largely agree with each other (with only three exceptions among the ones with significant differences), in the sense that if there are more "moderately/completely agree" responses compared to the "moderately/completely disagree" responses in one cluster, then the same is true for the other one. The three claims, where the clusters differ in their responses, are 1 b) on argumentation, and 7 a) and 7 c) on the air pollution project, see Table 1. The difference between the clusters is the largest on 1 b), where 31.3% of cluster 1 completely disagree on the claim that Ben's explanation is incomplete. For cluster 2, the most selected response in 1 b) and 7 c) is "Neither agree nor disagree", and 31% or more selected this for all three claims.

For the remaining 35 claims, the two clusters do not differ significantly. Nevertheless, there is still a trend that indicates that the PTs in cluster 1 were more likely to select the "completely" agree/disagree options, while the PTs in cluster 2 more often selected "Neither agree nor disagree" or "Don't know".

## Discussion and concluding remarks

Previous research on clusters of PTs has focused on the PTs' views on their mathematics skills and beliefs regarding multilingual classrooms (see for example, McLeman & Fernandes, 2012). Given the emphases in the new Norwegian curriculum (Utdanningsdirektoratet, 2019), our survey, instead,

mapped PTs' views on argumentation, modelling, digital tools, critical mathematics education as well as multilingual classrooms. To our knowledge, there has not been research previously on PTs' views across this spectrum of aspects of mathematics education to determine if their responses showed distinct clusters within a cohort. Our results indicate significant differences between the two clusters on argumentation and modelling, but less so in regard to multilingual aspects, critical mathematics education and ICT. Although McLeman et al. (2012) had found differences between clusters to do with multilingual issues, our questions were different and the experiences of the PTs likely to be considerably different to those of PTs in the USA, which could explain why we did not find similar relationships in our data. McLeman et al. identified in their research that demographic factors contributed to the clustering of the PTs, when studying views on English learners in mathematics classrooms. We did not request this data due to privacy issues, as this information could make the PTs' potentially identifiable. Nevertheless, it may be that further research is needed to determine if similar demographic factors could also affect our clustering.

The results are both interesting in themselves and raise issues that require more thought, particularly to do with how to structure our teacher education to reflect the specific needs of the two clusters. It would seem that, on the whole, the PTs in cluster 1 were much more positive about what students could do with non-textbook tasks, but lacked experience in how to utilise the information they do know to make judgments about what to do in their mathematics teaching. Thus, the positivity displayed by cluster 1 might need to be supplemented with critical skills.

On the other hand, the PTs in cluster 2 appear to display some critical skills which can be developed in relationship to engaging in deep-level discussions, surrounding the complexity of bringing together different aspects of mathematics teaching. These skills could be used, for example, to develop the PTs' critical reflection also in the areas that they are not yet showing awareness of, such as what, when and how digital tools should be incorporated into their future mathematics teaching. As well, the PTs in cluster 2 seemed to show that they are willing to be supportive of students' responses and are ambitious for their learning. These are also good skills to have, which perhaps could also be built on in regard to reflecting on the needs of second-language learners in their mathematics classrooms. Further research on the impact of adjustments that are made to our teaching will be published later, as our larger project develops.

### Acknowledgement

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# Using pathologies as starting points for inquiry-based mathematics education: the case of the palindrome

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*Inquiry-based mathematics education (IBME) is an increasingly important ingredient of the mathematics education in the Nordic countries. The central principle of IBME is that the students are to work in ways similar to how professional mathematicians work. In this qualitative case study, we investigate whether mathematical pathologies induce students to work like mathematicians, and thus if pathologies are suitable starting points for IBME. We based our investigations on a little-known pathology: multiplication problems that can be mirrored about the equal sign without altering the answer.*

*Keywords: Inquiry-based mathematics education, mathematical pathologies, palindromes, mirror multiplication.*

The national curricula are essential to the so-called Nordic model of education, and in the new national mathematics curriculum of Norway the Norwegian equivalent of the term *inquiry* is used repeatedly (69 times, to be exact). This is an indication of the rising popularity of IBME: a form of teaching whose guiding principle is that the students are supposed to work in ways similar to how professional mathematicians work (Artigue & Blomhøj, 2013; Council, 2000; Dorier & Maass, 2014). But what kind of tasks can be relied upon to impose questions on the student and thus form suitable starting points for IBME? In this paper, we suggest that one answer to that question lies in the realm of pathological objects.

A mathematical pathology is "an example specifically cooked up to violate certain almost universally valid properties."[1] The history of mathematics abounds with pathological objects, many of which were of crucial importance to the development of one branch or another. Sriraman and Dickman (2017) argue that pathologies can be pathways to divergent thinking and creativity. The chief pedagogical value of pathologies, they claim, lies in "their ability to challenge our perception and intuition, and the resulting advances as we equilibrate" (p. 141).

It seems reasonable to expect that these same question-inducing qualities make pathologies promising starting points for inquiry. In this qualitative case study, we investigate whether mathematical pathologies *are suitable as starting points for inquiry-based mathematics teaching*. To this end, we developed and tried an inquiry-activity around the little-known pathology of multiplication problems like $12 * 63$ that can be reversed ($36 * 21$) without altering the answer. The main purpose of our study is to draw attention to a hitherto perhaps underused resource in mathematics education: the mathematical pathology.

Our main concern shall be whether using pathologies as starting points for inquiry induces students to work in ways similar to professional mathematicians. A source of information on how

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[1] http://mathworld.wolfram.com/Pathological.html

mathematicians work is introspective accounts from professional mathematicians, and we shall largely rely on such in our discussion.

## Methods

The following research question was addressed in this study: *Are mathematical pathologies suitable as starting points for inquiry-based mathematics teaching?* The study followed a qualitative case study design.

### Pathological palindromes

It is a little-known fact that for certain multiplication problems one may mirror the digits about the equal sign without altering the answer. Thus, while it is to be expected that $12 * 63 = 63 * 12$, it is more surprising that we also have $12 * 63 = 36 * 21$. To the best of our knowledge, the only previously published paper on this pathology in English is Manheim (1979), where (amongst other things) the author identifies all such two-digit "palindromes". The most straightforward way of investigating the palindromes is through algebraizing:

$$(10a + b)(10c + d) = (10d + c)(10b + a).$$

This leads to the necessary and sufficient criterion for being a palindrome that $ac = bd$. For more on the palindromes see Manheim (1979) or Roksvold (2018).

### Participants and setting

The participants in this study were convenience sampled and were either teacher students (n=40) or upper secondary school students (n=7). Two of the upper secondary students are referred to by pseudonyms (Alma and Leo) as they were equipped with a GoPro camera and thus followed more closely. Upper secondary and teacher students are different in many respects, but we do not think that these differences are of much consequence to our research question. Accordingly, such differences shall not play a part in the discussion, and we shall occasionally refer to the *students* or the *participants* without further specification. The teacher students all took a course that the first author instructed. The upper secondary students were recruited through their teacher, who is a participant in the SUM-project (Haavold & Blomhøj, in press), in which both authors are involved as researchers. The upper secondary students were in their last year of school, in what would correspond to grade 12 in the K12 system (it is the 13th school year in Norway).[2]

The first two sessions, involving the teacher students, took place at a university in Norway. The third session, involving the upper secondary school students, took place at an upper secondary school in Norway.

### Procedure

All three sessions followed the same basic outline, and each had a duration of 1-2 hours. First, the students were divided into groups of 2-4. Then, the instructor wrote $12 * 63$ in the upper left corner of the blackboard (or whiteboard) and $36 * 21$ in the lower right corner. It was up to the students to take it from there; no further instruction was given, no question was posed, and no direction was suggested.

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[2] The students had R2-mathematics, which is geared towards future studies in the "hard sciences".

We recorded audio from four of the groups of teacher students. In addition, these sessions were observed by the second author, and we collected the teacher students' worksheets. We recorded audio from all three groups of upper secondary students. In addition, their classroom was filmed, and one of the upper secondary students (Leo) had a GoPro camera attached to his head.[3]

## Results

An inductive content analysis (in accordance with Elo & Kyngas, 2008) was performed on the entire corpus of data. The audio and video recordings were transcribed and coded; the codes were then distilled into content-related themes that each relate to the research question. These themes were *engagement and familiarization through examples; collaboration; contingencies and alternative solutions;* and *conjectures and refutations.*

### Engagement and familiarization through examples

The teacher's task in the initiation phase is restricted to somehow displaying $12 * 63$ and $36 * 21$ (preferably with good spacing between them). Some of the students initially seemed unsure about what was probably an unfamiliar situation:

Teacher student 1: What are we supposed to do?

Any hesitation was typically short lived, however, as the two multiplication problems triggered the participants' curiosity:

Teacher student 2: Ok, so these are opposite numbers. I mean … it's the same numbers only in reverse order. We should just calculate both and find the answers.

Seeing that the multiplication problems are mirror images of each other *and* that they have the same answer raised some questions:

Teacher student 3: We should see if this is true for other numbers as well. Two-digit ones. Let's try 28 times 73." And a bit later on: "This is not right; it's not the same.

Teacher student 4: So then we know it's not true in general …

Early on in all three sessions, the students discovered palindromes such as $11 * 33 = 33 * 11$ and $27 * 72 = 27 * 72$ and classified these as special cases fundamentally distinct from the initial example $12 * 63 = 36 * 21$. Having identified these degenerate cases, the students typically continued either by trying to find more non-trivial examples (by chance, even) or by trying to detect some underlying mechanism or pattern in the examples already at hand. Several of the groups focused on factorization and primes, suspecting that a multiplication problem being a palindrome (the students typically did not use this word) had something to do with the factors involved.

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[3] It was our experience that the unobtrusiveness of these cameras made participants less self-conscious about being filmed. Another methodological advantage that the cameras offer is bringing the researcher close to the action, thus perhaps increasing their chances of noticing details. Being attached to the forehead, the camera also conveys the direction of focus of its wearer.

**Collaboration**

The participants of a session collaborated across the groups of 2-4 by sharing new palindromes, approaches or observations with the rest of the "class". New palindromes would help other groups by enabling them to further test their conjectures. The following discussion, between upper secondary students Alma and Leo, illustrates a typical case of the collaboration that took place *within* the groups:

Alma: We need to look for what these [palindromes] have in common.

And, after some trying and failing:

Alma: I think we need to start with one number and then multiply that number with four, for example. If you see what I mean …?

She then explained her idea algebraically by writing $a * 4b = 4a * b$ on the whiteboard and said that they needed to "insert numbers a and b."

Leo: What if we just insert something, like $a$ equals fifteen … well, that becomes fifty-one, so that's wrong …

Alma: But we still need the answers to equal each other after the multiplications are mirrored.

Leo: If it's supposed to be four, then we can't have an odd factor.

Alma: Yes, I agree.

Leo: What if we take twenty-four … well, if we mirror that, it's forty-two, so that's no good …

Alma: But it doesn't have to be four [pointing at the equation $a * 4b = 4a * b$], it can also be two here, see? And twenty-four times two …

**Contingencies and alternative solutions**

The authors, who thought they knew these palindromes, were faced with three surprises. First and foremost, one of the teacher students discovered a new layer of hidden symmetry: In the palindrome $12 * 63 = 36 * 21$, the differences $63 - 12 = 51$ and $36 - 21 = 15$ are mirror images of each other. The students were clearly satisfied with having found something that the teacher (in this case the first author) did not even know was there, and proceeded by inquiring: "Is this always the case?" and, conversely, "Does having this symmetry of differences imply that the multiplication problem is a palindrome?" Testing these newly formed hypotheses on other examples (previously found by the groups' combined efforts), the teacher students soon discovered that the double layer of symmetry unfortunately fails in the case $63 * 24 = 42 * 36$.

Second, a group of teacher students found empirically that the equality $a : d = b : c$ seems to always hold and realized that this should enable them to produce new palindromes. The same students did not proceed to prove the equality algebraically, but it is tempting to suggest that this was a question of having too little time. The relation is equivalent to the product of the "ones" being equal to the product of the "tens".

Third, shortly after the discussion that we reproduced in the previous subsection, it appears that Alma, with the help of Leo and without fully articulating it, discovers a simple and game-like algorithm that was unknown to us: **1.** Start with a number, let's say 41. **2.** Double it: 82. **3.** Then mirror the result: 28. **4.** And finally, divide this last number by two: 14. It turns out; you have just found the palindrome $41 * 28 = 82 * 14$. If the students or the teacher had fully realized this, and had been given more time, it would have been very much in the spirit of IBME to pursue the matter further: "Can you start with any number and end up with a palindrome?" (No.) "Do we get all two-digit palindromes this way?" (Sort of.[4]) And, moreover, "Why does it work?"

**Conjectures and refutations**

Having seemingly forgotten their recent breakthrough, or not recognized it as such, Alma and Leo began forming a series of conjectures based on the growing set of examples that the three groups of upper secondary students had found between them. They tested these conjectures against the examples, and the trials that did not end well led to the formation of a new conjecture through a modification of the old one.

The first in this series of conjectures was that *for palindromes the digit sums on each side of the equal sign must be the same*. This initial conjecture was not long-lived.

Leo:        But … if we are to be completely honest with ourselves … if we just take any product and mirror it, let's say thirty-four times fifty-one and its mirror image fifteen times forty-three, these are not equal to each other, but the digit sums are still the same.

Adapting to this temporary setback and motivated by their own previously found example $12 * 42 = 24 * 21$, they form a new conjecture: *The digit sum of one of the numbers must equal three while the other one's must be divisible by three*. They write this conjecture on the whiteboard and then go about trying to falsify it, by way of counterexamples.

Leo:        But forty-eight times forty-two [an example discovered by one of the other groups] does not have any digit sum equal to three – it has digit sums six and twelve.

Alma:        But those are both divisible by three!

Alma then alters their conjecture by adding the sign "%". The conjecture now reads: *Digit sums must be % 3* - meaning that all digit sums must be divisible by three. Leo then notices that in both $12 * 63 = 36 * 21$ and $42 * 12 = 21 * 24$ one of the numbers has 7 as a factor.

Leo:        And it changes from one number to the other [i.e. the seven is not a factor of sixty-three mirrored, but rather of twelve mirrored]. This I believe is important!

Once again, they alter their conjecture, this time by appending *and there must be a 7 which changes place.* At this point they are interrupted, since one of the other two groups of upper secondary students

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[4] To produce all but three of the 14 non-trivial two-digit palindromes one must sometimes multiply by 3 or 4 in step two of Alma's algorithm (and divide by that same number in step four). Permitting multiplication and division by 3/2 and 4/3 yields the final three as well.

has discovered a promising pattern which the teacher encourages them to share, namely that the product of the ones is equal to the product of the tens. Leo immediately starts calculating $13 * 93$, etc., on the whiteboard, and soon exclaims "Wow! That actually works!" and drops the calculator to the floor.

## Discussion

The first finding was that the pathology seemed to be effective at triggering the students' curiosity and led them to pose both questions and conjectures. Typically, the students' initial investigation was a familiarization through examples. The groups tended to alternate their efforts between looking for new examples and searching for patterns in the examples at hand; this seems to also be the way of professional mathematicians. The late Hungarian mathematician Paul Halmos expressed it like this:

> A good stack of examples, as large as possible, is indispensable for a thorough understanding of any concept, and when I want to learn something new, I make it my first job to build one. (1985, p. 63)

According to Halmos, "the examples should include, whenever possible, the typical ones and the extreme degenerate ones" (1985, p. 62). In the case of two-digit palindromes, the extreme degenerate examples are palindromes such as $11 * 33 = 33 * 11$ and $74 * 47 = 74 * 47$, which participants of all three sessions found early on.

The examples that the groups had at hand at any given time were prodded in the search for different kinds of pattern. Some groups investigated patterns in the prime factorizations, other groups looked at patterns in the digit sums, others again considered patterns of ratio, and so on. The search for and description of patterns is such a significant part of the work of a professional mathematician that mathematics itself is often referred to as the "science of patterns" (see e.g. Devlin, 2000, pp. 7, 72).

Students engaging the material and pursuing their own questions are two of the pillars of IBME (Dorier & Maass, 2014, pp. 301-302). A possible explanation as to *why* the palindromes triggered the participants' investigative spirit is that pathologies in and of themselves pose questions and problems (thus making it less necessary for the teacher to do so); besides, there is something slightly provocative about a mathematical pathology, is it not?

The second finding was that all three sessions were characterized by near constant collaboration, both on a class-level across groups and within each group. This is reminiscent of two levels of collaboration found in professional mathematics.[5] The collaboration that took place on class-level mimics the collaboration of the mathematics community as a whole: When a group discovers and shares a new palindrome that fails to conform to another groups' conjecture, it is similar to what is happening when a counterexample to a mathematician's conjecture is found and published by an unknown colleague on the other side of the world. Likewise, the within-group collaboration mimics mathematicians knowingly collaborating on the proof of a theorem, for example at a congress. Amongst professional mathematicians this type of direct collaboration is the rule rather than the

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[5] For a description of the mathematics community see e.g. Davis, Hersh, and Marchisotto (2012, pp. 9-12). A personal account by a current mathematician is https://cameroncounts.wordpress.com/2009/10/28/collaboration-in-mathematics/.

exception. A vivid illustration of this is the collaboration graph,[6] the study of which has uncovered that in the period 1985-2009 the average number of mathematicians per paper was 1.75 (Brunson et al., 2014). Students collaborating is another pillar of IBME (Dorier & Maass, 2014, pp. 301-302).

The third finding was that several groups formulated conjectures, which they subsequently tested against the examples at hand. Representative of the collaboration that took place within the different groups was the chain of conjectures and refutations that Leo and Alma engaged in. Their dialogue is reminiscent of the fictitious classroom discussion on the Eulerian characteristic of polytopes found in Lakatos's classic *Proofs and Refutations*. Leo and Alma making repeated adjustments to their conjecture to rule out nasty counterexamples is precisely what Lakatos referred to as "monster-barring" – a phenomenon he saw as crucial to the progress and growth of informal mathematics.

On several occasions, students responded with enthusiasm to the ideas and findings of their fellow students – on their own group or otherwise. The learning environment deemed most suitable for IBME is one that values mistakes and contributions (Dorier & Maass, 2014, pp. 310-302). The professional mathematician's appreciation of the necessity of making mistakes is implicit in the words of George Pólya: "Mathematics presented with rigor is a systematic deductive science but mathematics in the making is an experimental inductive science" (1990, p. 117).

The fourth finding was that using pathologies as starting points is likely to increase the number of contingencies, especially so if the pathology is a peripheral one such as the palindromes. The problems most suitable for IBME are open problems with multiple solution strategies: problems that are experienced as real and/or scientifically relevant (Dorier & Maass, 2014, pp. 301-302). Contingencies, although slightly terrifying to the teacher, must be said to add authenticity to the students' mathematical experience. We suggest that students discovering something that the teacher did not know, as was the case on two different occasions with the palindromes, are the ones *most* likely to feel as if they are doing "real" science.

**Conclusions**

Two conclusions can be drawn from this study. The first conclusion is that pathologies can form the basis of teaching that corresponds to the characteristics of IBME as these are described in Dorier and Maass (2014). Most importantly, the pathology induced students to pose and pursue their own questions.

The second conclusion is that pathologies seem to have qualities that help students work in ways similar to those of professional mathematicians. Thus, using pathologies as a starting point is one way to lend an aura of authenticity to the students' endeavor. The students are faced with something that simultaneously awakens their curiosity and causes a cognitive conflict, which they will try to equilibrate from.

The chief practical implication of these conclusions is that teachers doing IBME might have a useful and largely unmined resource in the mathematical pathology. *Useful* because what constitutes a pathology partly depends upon one's mathematical sophistication (see Sriraman & Dickman, 2017, p. 138), which means that mathematical pathologies are plentiful at every level of mathematics.

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[6] See https://oakland.edu/enp/ for information on the The Erdös Number Project and the collaboration graph.

On the theoretical level, this could call for an investigation into the scope of using pathologies in the inquiry-based teaching of specific topics; for example, it does not seem unreasonable to hope that IBME starting from a suitable pathology could play a role in dealing with specific student misconceptions. It is also possible that using pathologies as starting points for IBME can strengthen the students' cognitive flexibility by inducing them to circumscribe and (re)consider the characteristics of the pathology at hand, thereby providing an opportunity to overcome what Haylock (1997) refers to as *content-universe fixation.*

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# How does professional development influence Norwegian teachers' discourses on good mathematics teaching?

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*This study examines how practice-based professional development influences the discourse of a group of Norwegian primary mathematics teachers. Through cycles of enactment and investigation, in-service teachers (ISTs) learn teaching practices that constitute ambitious teaching. The teachers are interviewed before and after professional development and a coding-scheme from the analysis of the pre-interviews is used to analyze the post-interviews to find changes in how teachers conceptualize good mathematics teaching. The findings show that new categories emerge in how the teachers conceptualize good mathematics teaching.*

*Keywords: Professional development, cycle of investigation and enactment, ambitious mathematics teaching, the conceptualization of good mathematics teaching.*

## Introduction

Professional Development (PD) for teachers is an important part of the Norwegian government's desire to improve the quality of education (Kunnskapsdepartementet, 2015). PD can have varying content and can be organized in a number of ways. This study is part of a PD project (the MAM-project) where the main goal has been to support mathematics teachers in the development of their practice to achieve ambitious teaching. Ambitious teaching aims to engage all students in mathematical thinking and develop their conceptual understanding, procedural knowledge and adaptive reasoning through problem-solving (Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2010). Students' ideas are at the core of ambitious teaching, and teachers need to respond to students' in-the-moment thinking respectfully and thoughtfully with the mathematical goal in mind. This study aims to explore how PD can influence the teachers' discourse on good mathematics teaching. This may provide insight into how this kind of PD influences teachers' norms for good mathematics teaching (Krainer, 2005) and will thus contribute to research on PD content and how it should be carried out (Maass, Cobb, Krainer, & Potari, 2019).

There is no common definition of what constitutes good mathematics teaching. There is, however, a growing body of research that examines the features of instruction that supports students' learning (Cai, Kaiser, Perry, & Wong, 2009). Core practices (McDonald, Kazemi, & Kavanagh, 2013), high-leverage practices (Ball & Forzani, 2011) and ambitious teaching practices (Lampert et al., 2013) are terms used to describe research-based practices that give all students opportunities to learn mathematics. Three features are at the core of such practices: shaping a mathematical discourse, developing classroom norms and developing relationships that enable all students to participate and learn mathematics in a supportive environment (Franke, Kazemi, & Battey, 2007). Even though there are some common understandings on features constituting good mathematics teaching, defining what it is seems to depend on the views of teacher educators and teachers. These views might also affect decisions made about instruction (Cai, et al., 2009; Li, 2011).

In recent years there has been a shift from focusing on the development of teachers' knowledge to developing teachers' practice (Zeichner, 2012). PD aims to support teachers in developing ambitious teaching (Lampert et al., 2010; Lampert et al., 2013). The project Learning to Teach In, From and Through Practice (LTP) provided novice teachers with opportunities for learning to enact ambitious teaching through cycles of enactment and investigation (Lampert et al., 2013). Recent research has examined how these cycles can be adapted and implemented with in-service teachers (ISTs) in the Norwegian context (Fauskanger & Bjuland, 2019).

Discourses on good mathematics teaching have been examined in several studies. For example, Hemmi and Ryve (2015) took a discourse-analysis approach to examine how Swedish and Finnish teacher educators conceptualized good mathematics teaching. They used the discourse concept "to refer to ways of interacting with the use of specific words and categories and how these constructions, in turn, produce the social actors of teachers and students within classroom practice" (Hemmi & Ryve, 2015, p. 502). One of the main findings in the study was the difference in how Finnish and Swedish teacher educators explicitly and implicitly position the teacher in the classroom practice. Fauskanger, Mosvold, Valenta and Bjuland (2018) focused on ISTs and how their discourse was constructed. They used Hemmi and Ryve's (2015) definition of discourse, and seven categories emerged in their content analysis of the data. Their findings reveal that Norwegian teachers position a shared responsibility for instruction between teachers and students.

In this paper, I ask the following research question: How does professional development influence teachers' discourses on mathematics teaching? I use directed and conventional content analysis (Hsieh & Shannon, 2005) of the teacher interviews to answer it.

## Methodology

In the MAM project, a model and resources were developed to support the ISTs' professional development. The model builds on learning cycles (Lampert et al., 2013; McDonald et al., 2013) where specific instructional activities are used (Lampert et al., 2010). The structure of the activities offers scaffolding to ISTs for developing ambitious mathematics teaching, for instance, how they can use mathematical representations, teach towards an instructional goal, learn how to present the task, how to organize the blackboard, how to elicit and respond to students' thinking and understanding, and how to help the ISTs to focus on important mathematical ideas (e.g. Lampert et al., 2010). One of the characteristics of the activities is that they are suited for all students. The ISTs learn to teach the activities through learning cycles. Each cycle consists of six stages as described by Fauskanger and Bjuland (2019, pp. 130-131): 1) The ISTs prepare for the cycle by reading articles (e.g. about quick images) and by watching a video showing the enactment of the cycle's activity. Some ISTs try out the activity with their students. 2) One of the supervisors leads a discussion on the literature and the video. 3) The groups of ISTs plan the given activity for the given groups of students, supported by a supervisor. 4) In a rehearsal, one of the ISTs acts as the instructor. The supervisor and the other ISTs act as students. During the rehearsal, all participants can ask for teacher timeouts (TTOs). 5) The same IST enacts the activity with a group of students. All participants can ask for TTOs. 6) The enactment is analyzed by each group of ISTs together with their supervisor. This analysis is followed by a similar analysis with all the participating ISTs, and preparation for the next cycle's activity.

The project runs over four semesters, with three cycles each semester. Thirty teachers from ten schools have been selected by their principals to participate and have been assigned a future role in implementing ambitious teaching practices at their school. Most of the ISTs taught Years 5–7 (11–13 year old students). Their teaching experience varied from one to 30 years. Fourteen of the ISTs were purposefully chosen to be in the two research groups, bearing variation in age, education and teaching practice in mind.

The ISTs were interviewed in these two groups before and after PD (i.e. pre- and post-interviews). By conducting interviews before and after the PD work, it has been possible to look for changes in the ISTs' discourse. Hemmi and Ryve's (2015) definition of discourse, "ways of interacting with specific words and categories", has been used in the analysis. The focus is on how they interact with specific categories. With the analysis of the pre-interviews as the point of departure (Fauskanger et al., 2018), the data material analyzed in this paper comprises the videotaped post-interviews with the two groups of ISTs. After some talk about how they were going to implement ambitious teaching practices at their schools, the teachers discussed the same two questions as used in the pre-interview: 1) How would you characterize a good mathematics lesson? and 2) How would you characterize what for you is a "normal" mathematics lesson?

The videotaped post-interviews were transcribed. The unit of analysis was an utterance, and the coding-scheme used was developed from the analysis of the pre-interviews (Figure 1). This coding-scheme was developed by two researchers who analyzed the pre-interviews, while the other two researchers validated the coding-scheme in the analysis of the pre-interview. (Fauskanger et al., 2018).

| Teacher´s role 1 in mathematics teaching is to | Structure in lessons 2; important to |
|---|---|
| - find a way to present the content in engaging way 1a<br>- find a way to respond to students thinking, build on it towards the learning goal 1b<br>- be a guide, not lecturer 1c<br>- use resources critically 1d | - have clear learning goals – content goals that teacher has (and can reason for), 2a<br>- have content goals that are represented to students so that they know what is to be learned, but without reducing their thinking 2b<br>- have process goals as" discuss patterns" 2c<br>- have variations in lessons 2d<br>- sum up at the end 2e |

Figure 1 Coding scheme, an example of keywords in two of the categories

Using the same coding-scheme as in the pre-interview made it possible to look for changes in how the ISTs interacted with the categories. I started by coding for seven *a priori* categories: 1) teacher's instruction/role, 2) structure in a lesson, 3) differentiation, 4) communication, 5) use of tasks and resources, 6) student engagement and 7) students' learning (Fauskanger et al., 2018). They represent the categories from the study of the pre-interviews (Fauskanger et al., 2018). One utterance could be coded in more than one category. For instance this utterance, "I'm thinking that I almost never, if it is a good task, particularly a problem-solving task (5), feel that I have sufficient time, when we're going to both work on the activity and also put it into words (2)", was coded both in use of tasks – category (5) and in structure of lesson – category (2). The main goal was to look for changes in the use of categories.

After the first coding, not all the utterances fit the categories that were developed in the analysis of the pre-interviews. These utterances were read several times and while reading, emerging ideas were written down. Examples of emerging ideas were the following: teacher knowledge, anticipate student responses and give students the possibility to succeed and develop a classroom discourse where all students can participate. Some of these ideas could fit in the previous categories, for example, the teacher's role and student engagement, but the emphasis was more on teachers' preparation than what takes place in the classroom. For example, the utterance "And of course there is this with the students, that they should experience mastering and all that along the way, and which will create engagement and motivation for the subject" was first coded under student engagement – category (6), but implicitly in this utterance, the teachers take responsibility for giving the students the possibility to succeed. Then this utterance was moved together with the un-coded utterances. This is one example of how teachers talked about their responsibility to enhance their students' learning outcomes, motivate them and give them the possibility to succeed in mathematics. Teachers' planning was a common feature in these utterances and emerged as a new theme. The rest of the utterances that were un-coded were related to how teachers cooperated with parents to obtain a better understanding of how a different way of teaching mathematics would benefit the students.

## Findings

Discourses are referred to as ways of interacting using specific words and categories. The analysis in this study has focused on the specific categories the ISTs use in the post-interviews, revealing that the same categories are present in both the pre- and post-interviews. Not really that surprising perhaps, but two new categories emerged: teachers' planning and cooperation with parents. In the post-interviews, the teachers still talked about their role in the instruction, the structure of the lesson, differentiating to meet students' needs, communication in the mathematics classroom between teacher and students, and critical use of tasks and resources. When they talked about student engagement and learning, they conceptualized this as a result of good mathematics teaching. In the following, I will describe the findings in the post-interviews from each category and give examples of representative utterances from the ISTs.

### Teacher's instruction/role

In the post-interviews, the ISTs point out that they have to listen to the students and appropriately respond to them so that the students orient themselves towards each other and the mathematical content. For example, one of the ISTs says that he/she "thinks a lot about this, how to address these things they are saying. At any rate where I feel that I'm thinking much more about it. How should you respond to things they say, for example? Don't shut them down, or brush aside solutions even if they are incorrect, but rather address them. And a little bit about what to address to move forward." The analysis shows that the ISTs point out that they need to be facilitators and support the students in facing their challenges and talk about how they can use representations when supporting the students and ask questions to help them make progress without leading the way.

### Structure in lessons

The analysis reveals that the ISTs highlight the importance of having a clear goal for the lesson and choosing tasks or activities that support the goal, not the other way around, choosing a task or activity and then deciding the goal for the lesson. This is explained in this utterance: "It doesn't really matter,

in that sense, that this activity is better than that activity, rather it's more about why did you in fact choose that activity?" They also talk about structure in the lessons: inquiry-based activities, discussions and summing up at the end of the lesson. One IST summarizes the structure of the lesson in this way: "A good lesson is well considered then, or you have a goal that you want in there, they can set something, the conclusion, that they find a way to phrase, well, representations of their thoughts."

## Differentiation

From the analysis, it appears that differentiation is easier for the ISTs when they have activities and problem-solving tasks that give different kinds of students the possibility to join in on each their level. As one IST says: "They have something, everybody has something, they can contribute. Everybody can accomplish something". They still think differentiation is difficult because of the diversity of the students, as seen in how this IST thinks: "Yes, my point is, what I think is difficult, that's to adapt the level in a way."

## Communication

The ISTs appreciate when students discuss mathematics with each other. To develop a classroom where students discuss with each other it is necessary to build a positive classroom culture where it is safe for the students to participate. The ISTs point out that talk-moves (Kazemi & Hintz, 2014) have been a useful tool for building a positive classroom culture. They emphasize their role in establishing a community where the students take part in mathematical discussions. As one IST says: "What I am very much focused on is this idea of how to create student activity and conversations between the students and with the students, and the types of questions to be asked, and which I feel are part of the conversation elements."

## Use of tasks and resources

Problem-solving tasks are considered helpful for the ISTs as a way of engaging all the students in mathematical thinking, but they mention the importance of tasks that engage students and that are differentiated. One IST put it this way: "When you have a really good task, right, that engages everybody, on all possible levels, so they can work well with it."

## Student engagement

The analysis indicates that it is important for the ISTs that the students are motivated and interested. The ISTs describe a good lesson as one where there is a lot of student engagement and the students have a feeling of success. The ISTs also talked about the importance of the students being challenged and the way they addressed this point about their students is revealed by this IST, who says: "It's about mastering, and about experiencing being challenged and accepting that, but also having the faith that you can manage it." Student engagement is seen as being a result of the teacher's actions.

## Students' learning

The ISTs point out that the students need to understand basic mathematics, that they learn to learn and can use their knowledge in new situations. This utterance is an example of the idea about the need for students to learn mathematics more deeply. "What I feel is new for me, is this idea about in-depth learning." The ISTs also talk about how they can prepare to enhance their students' learning, as this IST says: "But if you're to be able to guide them on the way, then you, of course, need to have

worked your way through the task first and considered possible strategies. If you just grab a random task lying there and hand it out, then there is no guarantee that you will achieve anything."

In the following, I will give examples of the two categories that were new in the post-interview.

**Cooperation between parents and school**

The analysis reveals that the ISTs experience that parents are critical to the new approach in mathematics teaching or they find it difficult to help their children with their homework. One IST, experienced that parents were critical to her teaching, "Now when we have parent teacher meetings, then I get to hear it, but I have to say that I'm not happy with that there, what was it, one word or another, that she called my teaching, that there is your fantasy teaching, or something like that." Therefore, parents need to be informed about how the ISTs are teaching mathematics.

**ISTs' planning**

When talking about planning, the ISTs talk about their competence and how they can be prepared to help their students achieve the goal for the lesson. One IST expresses it in this way: "We obviously need some knowledge about the mathematics subject, and to be aware of which goals we want the students to reach and which tasks promote these goals." When planning lessons, anticipating students' responses is suggested as an important point, in addition to planning how they will respond so the students can achieve the goal for the lesson that has been highlighted by the ISTs. One IST relates this to his role as a teacher: "If I had not known anything about possible answers for this task, what can be incorrect answers, and how I can move this to the next level, then I would not have been a good teacher". The ISTs point out the importance of being prepared as part of being a good mathematics teacher and implicitly as part of good mathematics teaching. This kind of planning is time-consuming, but the ISTs think it will be more manageable after working in this way over some time as they will have a base of experiences, as one IST puts it: "We need time then, but perhaps most important is we need time at the start before acquiring a base of experiences from things you have tried."

## Discussion and conclusion

The most characteristic finding from studying how ISTs conceptualize good mathematics teaching in the post-interviews is that teacher planning is a part of the ISTs' discourse on good mathematics teaching. This was the most frequent category, and one explanation might be that the ISTs put more emphasis on their planning after they have participated in the MAM project. This indicates that the ISTs put more emphasis on their preparation and put new demands on their knowledge, both in terms of subject matter knowledge and pedagogical content knowledge (Ball & Forzani, 2009). The new demands are expressed in this utterance: "We obviously need some knowledge about the mathematics subject. What we need to keep in mind is which tasks we should present so the students will achieve the goals and be aware of what we want them to accomplish and which tasks will help them do that." This might not be a surprising finding since planning and rehearsal have played an important role in their PD work. Through the co-planning sessions in the PD, they have experienced the complexity of planning a lesson with a specific mathematical goal. This can be related to the learning cycle (Lampert et al., 2013; McDonald et al., 2013) that the PD was built on. In organizing the PD with the learning cycles, co-planning plays an important role, and it might not be surprising that the experiences of co-planning influence how the ISTs conceptualize good mathematics teaching. Through this co-planning

they might develop professional communities where productive discussions on teaching and learning can take place (Gibbons, Kazemi, Hintz, & Hartmann, 2017). The content in the learning cycles may also have influenced why this new category has emerged, referring here both to the articles and instructional activities. Throughout the PD work, the participants have discussed the articles and used them when planning, rehearsing, enacting and analyzing the instructional activities. This close connection between theory and practice emphasizes the importance of judgment and action in the classroom (Ball & Forzani, 2011). To be able to make good judgments and choose the right actions in the classroom, you need to be well prepared.

The new category "teacher's planning" can also tell us something about how teachers position themselves both explicitly and implicitly in the classroom practice. The pre-interviews of these ISTs have indicated that teachers do not emphasize their role in teaching and that there is a shared responsibility between teachers and students for the quality of mathematics teaching (Fauskanger et al., 2018). In the post-interviews, however, the teachers place more emphasis on their role in the classroom and position themselves more centrally in the classroom, giving themselves a more active role. In this way, they are closer to the Finnish teacher educators, where the teacher is described as "a very proactive agent in the classroom" (Hemmi & Ryve, 2015, p. 515). In Fauskanger et al. (2018), the ISTs did not emphasize the teachers' knowledge, preparation and understanding of textbook content. This has changed after the PD, where they now emphasize planning, their knowledge and the critical use of resources. They are more in line with Chinese teachers, who also emphasize preparation, knowledge and understanding of textbook content (Cai et al., 2009; Li, 2011).

The finding of the new categories in the post-interviews indicates that the PD project has influenced how these ISTs conceptualize good mathematics teaching and that their discourse has changed. Bearing this in mind, it might be anticipated that the PD work also influences these ISTs' teaching, where they spend more time planning lessons and thinking through how they best can support their students' learning. Even though there are some changes in how the ISTs conceptualize good mathematics teaching, we cannot say if there is a change and eventually which changes in practice occur. There is reason to believe that the change in how they position themselves will lead to a different dynamic in the classroom (Ball & Forzani, 2011).

This study has only explored the categories teachers interact with in their discourse (Hemmi & Ryve, 2015). In further research, it would be interesting to explore if there is a difference in how they choose their words. Do they use a different language to conceptualize good mathematics teaching? There is also a need to examine how the teachers' discourse is related to their teaching practice when making decisions about the content they want and how to carry out future PD work. This will be important both for teacher educators and policymakers.

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# Structuring activities for discovering mathematical structure: designing a teaching sequence for grade 1

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*This paper reports the results of the initial cycles of a smaller part of an ongoing large-scale design research project, consisting of a five-week teaching sequence for the beginning of grade 1 focussing mathematical patters and structure and additive decomposition of number. The results include both emerging design and emerging insight, and how researcher and teacher knowledge and experiences contributed to those results.*

*Keywords: Design research, Grade 1, pattern and structure, teacher-researcher collaboration*

## Introduction

Early mathematics teaching is seminal for future outcomes in mathematics and education at large (Duncan et al., 2007; Watts et al., 2014). There is extensive research on early mathematics teaching and learning that has supplied profound insights into key areas and aspects of early mathematics learning: early arithmetic and the understanding of number (Clements & Sarama, 2007; Gersten et al., 2005; Jordan et al., 2010), spatial thinking and its links to number sense (van Nes & de Lange, 2007; Moss et al., 2016; Säfström, 2018; Verdine et al., 2014), and pattern and structure (Clements & Sarama, 2007; Mulligan & Mitchelmore; 2013). However, the road from theoretical insights to practical solutions is often long and seldom straightforward (Burkhardt & Schoenfeld, 2003; Gravemeijer et al., 2016). To address this, design research strives to develop both theoretical insights and practical solutions simultaneously (McKenney & Reeves, 2012). As a consequence, it is iterative, going through several cycles of exploration, design, and evaluation over long periods of time (Cobb et al., 2016; McKenney & Reeves, 2012). It is also collaborative, in that it acknowledges the value of teacher and researcher contributions (McKenney & Reeves, 2012), even though descriptions of the collaborative activities are scarce (Pareja Roblin et al. 2014). In this paper, we describe the results of the initial iterations of design of a five-week teaching sequence for grade 1, in terms of both theoretical insights and practical solutions, and how teacher and researcher knowledge and experience contributed to those results.

### The larger design project

The design process described in this paper is part of a larger project aiming to develop, implement and test a teaching material for grades 0–3 (age 6–9), focussing the development of understanding and using numbers. For each grade, the material will cover four modules of five weeks of teaching. This project is primarily a development project in collaboration between the National Centre for Mathematics education and the Swedish Association of Local Authorities and Regions. It does, however, build on several previous research projects in grades 0 and 2 (Sterner, 2015; Sterner et al., 2019; Säfström et al., 2019; Vennberg, 2020).

## The design

The design is guided by four design principles, i.e. recommendations for how a specific class of issues is to be addressed in a range of settings (McKenney & Reeves, 2012). Three of these principles concern the *teaching organisation* and were initially formulated in the grade 0 project (Sterner, 2015). Swedish grade 0 is a unique educational form that lends elements from both preschool and school teaching practices. Hence, these three design principles needed re-specification for the different conditions in grade 2. The fourth design principle concerns the *content* and was initially formulated in the grade 2 project.

### The theoretical basis for the teaching organisation

The teaching organisation in the material is based on a combination of explicit and structured instruction focussing on student-teacher interaction and problem-solving, where teachers' and students' collective reasoning about representations is considered the main vehicle for teaching and learning. One important part of the design is the use of an investigative form of the CRA-sequence (Concrete, Representational, Abstract) to bridge from concrete to abstract mathematics (Witzel et al., 2003). The teaching organisation is based on three fundamental design principles: 1) a systematic and explicit idea of what is to be learned, 2) a systematic and explicit description of activities intended to support learning, and 3) a systematic and explicit instruction in each activity (Sterner et al., 2019). Each week the mathematical activities follow six phases including choruses, pair and individual work, and whole-class discussions about student's own work (Sterner, 2015). The shifts between students' own work and collective reasoning about their work can be linked to Vygotsky's theory (1978) about social interaction between children and adults as the main source for the development of advanced mental functions. Vygotsky (1978) states that all development in the child appears twice: first at a social and then at an individual level. When individual documentation follows whole-class discussions, it is assumed to facilitate students' reflection on the mathematical content they previously reasoned about collectively, but from a different perspective, contributing to the development of thinking. When collective reasoning follows individual and pair work, the teacher's responsibility is to ensure that the students reason about the problem and their solutions, and to challenge the students' thinking by posing questions that will encourage comparison and connection between different ways of reasoning and representing ideas.

### Theoretical basis for the content

We use Mulligan and Mitchelmore's (2013) notions of pattern and structure as an overarching organising idea of mathematical knowledge. A mathematical pattern as "any predictable regularity involving number, space or measure" and structure is "the way in which the various elements [of a pattern] are organized and related" (Mulligan & Mitchelmore, 2013, p 30). The design principle for the content consists of two levels: $4_s$) On the more specific level, the structures focussed in each week's activities are informed by research on young students' development of mathematical knowledge and reasoning, primarily the work of Nunes and Bryant (2015) and Mulligan and Mitchelmore (2013). $4_g$) On a more general level, representations are chosen based on their ability to demonstrate key mathematical structures through visible or audible patterns (Mulligan & Mitchelmore, 2013), what we describe as *their power as models for number*. In this paper, a model for number is a system of representations and rules for constructing and modifying these

representations conserving a subset of number properties. For example, the number line model can be used for constructing a specific number line representation showing 17+6=23. Models include not only objects or drawings but also metaphors expressed in verbal language and gestures (Lakoff & Núñez, 2000). We argue, in line with Lakoff and Núñez (2000), that no model is equal to the idea of number, and therefore that teaching number has to use and explicitly connect different models for students to develop a comprehensive understanding of number.

## Aim and research question

The grade 1 design project aims to extend the range of settings for which the design principles hold— which includes specification and possibly modification—and simultaneously develop a material that realizes these principles, consisting of elaborate lesson plans. In this paper, we focus on the research question: How can these principles be adapted to the specific conditions in grade 1, and realised in a viable teaching sequence?

## Data sources and method

In line with design research methodology (McKenney & Reeves, 2012), different types of knowledge from different contexts were used in the process. Besides the design principles, researchers brought knowledge and experiences from their different backgrounds into the design. Säfström has a background in mathematics and mathematics teacher education, while Sterner has a background in teaching and professional development in mathematics and special education. Both researchers were engaged in the grade 2 project, and Sterner conducted the grade 0 project. In addition, teachers tested the design in their classrooms and provided feedback based on general and specific questions asked by the researchers. The data sources used in the design process include the material and documentation from the grade 0 and 2 projects, relevant research literature, other published designs, and documented researcher-researcher and researcher-teacher communication, including teacher feedback based on classroom experiments. The process can be described as a combination of structured and organic reflection (McKenney & Reeves, 2012), during and after the cycles. Structured reflection should focus on the design challenge and aspects of the integration of the development and research process, while organic reflection requires well-timed breaks with input, unlike-minded sparring partners and engagement in background projects (McKenney & Reeves, 2012). Structured reflection was therefore guided by comparison of the design with both the design principles and the teachers' description of classroom experiments, while organic reflection was facilitated by recurring, spontaneous discussions between the researchers when needed, the different backgrounds of the researchers, and input from the other projects they were engaged in.

## Results from the first iterations

In this section, we will present examples of how different types of knowledge and experience contributed to the process and the results of the initial cycles of the design of the first module for grade 1. In these cycles, three teachers tested and provided feedback. The feedback was consistent across the three teachers, and excerpts from one of the teachers are used to illustrate the content of the feedback below.

**Emerging design**

In the first cycle of design, we focussed on making interpretations of the design principles in relation to our knowledge of the conditions in the early part of grade 1, which differ from both grades 0 and 2. In grade 0, classes are often smaller and taught by several teachers, making it possible to conduct teaching in smaller groups. This is not always the case in grade 1, where one teacher can be responsible for 25–30 students. In grade 2, students are used to the social setting and organisation of school, and able to focus on longer activities, which is not the case in the early part of grade 1. As a consequence, we specified principles 2 and 3 in two ways. First, we divided the lessons into shorter activities and included more physical movement and games than in grade 2 (Chorus, Table 1). Second, we met the challenge of conducting whole-class discussions with up to 30 grade 1 students by limiting the number of issues addressed (Group discussions, Table 1).

| Week | Chorus | Students' own work | Group discussions |
|---|---|---|---|
| 1 | Synchronised movements with units of repeat. | Building, drawing, and symbolizing patterns with a unit of repeat. | Comparing patterns and naming structure verbally and by labelling units of repeat. |
| 2 | Connecting movements to visual patterns made in week 1. | Searching for patterns in the surroundings and depicting them. | Generalizing the notions of pattern, structure, and unit of repeat. |
| 3 | Seeing/hearing and repeating movements, e.g. 5 claps. | Drawing different fives with different patterns. | Seeing and naming structures in fives. |
| 4 | Counting up and down. | Representing the numbers 6–10 in specified patterns. | Five-structure in the numbers 6–10 in different patterns. |
| 5 | The "five plus" game | Numbers as structured building and on the number line. | Five-structure in different models of number. |

Table 1: Main design and key aspects of three types of activities

Regarding the content, we knew that pattern and structure is not always explicitly taught and attended to in regular Swedish classrooms. Therefore, it could not be assumed that neither teachers using the material, nor their students, had a clear conception and awareness of mathematical patterns and structure in advance. In line with principle 1, we, therefore, chose to start the module focussing on the idea of pattern and structure in simple situations (Week 1, Table 2), and follow up with an exploratory activity where the ideas were further challenged and generalised (Week 2, Table 2). Thereafter, patterns and structure were connected to number, emphasising the importance of five-structure and decomposition of number using different representations, applying principles $4_s$ and $4_g$.

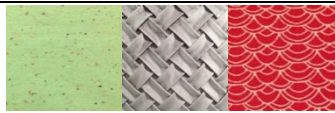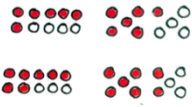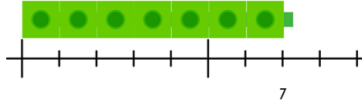| Week 1 |  AABBCAABB( | Repeating patterns: unit of repeat in movements, construction, drawings, and symbols. |
|---|---|---|

| | | |
|---|---|---|
| **Week 2** |  | Pattern hunt: searching for patterns in the surroundings and discussing their structures. |
| **Week 3** |  | Five patterns: different organisations of five objects or elements. Introducing decomposition of five. |
| **Week 4** | $6 = 5+1$   $7 = 5+2$ | Decomposition of five continued. Patterns with fives: seeing fives in the numbers 6–10 in different spatial organisations. |
| **Week 5** |  | Five-structure on the number line and connecting different representations through five-structure. |

Table 2: Main content each week

**Emerging insights**

The classroom experiments conducted by the teachers gave rise to interaction between researchers' theoretical ideas and teachers' practical experience. We found that asking specific questions to the teachers, such as "How did it work to introduce symbols for the patterns? Do we need to do it slower and more carefully?", "Are some activities too fast or too slow?" and "Did something unexpected happen?", facilitated relevant and elaborate feedback, regarding both the teaching organisation and the content. Regarding organisation, teachers verified the challenge of whole-class discussions:

> Teacher:   "It is not possible to let all students present their patterns in the whole-class discussion. My class, which is a well-functioning class, managed 6 presentations."

This feedback resulted in new ideas and designs for whole-class discussions, that were shorter but still involved all students. Three examples were: 1) letting some students present, and then asking follow-up questions to other students, 2) letting all students present a small part of their work, and 3) having an "exhibition" where all students' work is displayed at the same time and everyone searches for similarities and differences and collaboratively group the solutions.

Feedback on organisation also revealed how explicit lesson plans and instructions can be perceived as a restriction or questioning of teachers' practical knowledge and experience, leading to revisions of the application of principles 2 and 3. In this transcript, the teacher refers to a guideline for pair work in the material, stating that it "is not a time for you to explain or demonstrate content, but you can, of course, give students hints or ask questions that help them on with their work."

> Teacher:   "I have a hard time following the idea that you should not teach when the students work in pairs or individually. During the work, I saw that a student, who hadn't understood what patterns are at all, started to build a flag with the cubes. I decided to help him […] it didn't feel ok to let him work alone building pictures and then only later in whole-class discussion discover that what he had done wasn't a pattern. What do you think about that? I feel like it's a big challenge, I understand that it's important to learn from mistakes and that it can be a good basis for discussions, but is it justifiable to let them sit and do such basic errors?"

The teachers' feedback reveals both an agreement with the underlying ideas of the material, and a conflict between the instructions and her practical expertise, which we saw as a call for clarification regarding the guidelines for individual and pair work. The new guideline included both a revised statement: "[It] is not a time for you to explain or demonstrate content, but sometimes you may need to ask a question or explain the task if students do something totally different than what was intended" and concrete examples of students work inspired by the example provided by the teacher.

The feedback on content confirmed the assumption that the ideas of pattern and structure are not familiar to all teachers, but refuted the assumption that the introduction of symbols would be difficult, confirming principle 1 and $4_g$:

Teacher:     "For me, it was new to teach about structure in patterns. … Using letters to describe the structure worked well for the students."

It also provided new insights regarding the conditions necessary for using students' own examples of patterns as a means for discovering and reasoning about mathematical structure, specifying the application of principles 3 and $4_g$:

Teacher:     "Even though I had 21 students building their own patterns, only 2 patterns had the same structure. I think I lost some of the discussion as they did not get the opportunity to see similarities … If I had brought fewer colour on the bricks … or determined a max length on the pattern it might have worked better."

This feedback resulted in additional recommendations in the material, in line with the teacher's suggestions to use fewer colours on the bricks or decide on a maximum length for the unit of repeat.

## Discussion

In this paper, we have reported on how different types of knowledge and experiences from different contexts have contributed to both design and insights in the first iterations of a design process aiming to develop a teaching sequence for the early part of Grade 1. Our results illustrate the complexity of the activities needed in design research and the value and importance of close collaboration between researchers and teachers in the design process. The value of collaboration is often stated when describing design research methodology (e.g. Burkhardt & Schoenfeld, 2003; McKenney & Reeves, 2012), but descriptions of which and how different types of knowledge are used in design is nonetheless scarce in the research literature, which risks hiding research and practice links (Pareja, Roblin et al., 2014).

In addition, our results show that designing teaching focussing on pattern and structure in Grade 1 is challenging in terms of both content and organisation of teaching. Older students have a tendency to imitate given procedures (Sidenvall et al., 2015), which can limit the number of solutions and hamper comparison during whole-class discussions. For younger students, it seems that the opposite can be a problem: too much creativity making it hard to see structure across different patterns.

While beyond the scope of this paper, an inevitable question is whether the material will produce the same positive outcomes as the grade 0 material (Sterner et al., 2019; Vennberg, 2020). Studying this is the key aim of the larger design project.  However, the initial cycles of design and evaluation show that the teaching sequence works in real classrooms, in the sense that they provide opportunities for teachers and students to engage in meaningful activities focussing on  key mathematical ideas.

Without studying these aspects of a design along the way, large-scale implementation risks being futile.

**Acknowledgment**

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# Working with Euclid's geometry in GeoGebra – experiencing embedded discourses

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*This paper presents a finding in an ongoing PhD project revealing that 10-13 year-old students can work with the interplay between original mathematical sources and GeoGebra in ways that support the development of their reasoning competency. The study focuses on how to use the dynamic geometry environments in GeoGebra in combination with original sources with a geometrical content. This finding relates to the fact that Euclidian geometry is embedded in the way GeoGebra is constructed and how this can affect the students' opportunities and challenges when they are engaged in reasoning and proving activities. Two empirical examples involving sixth-grade students working with the interplay between Euclid's proposition 6 and 7, book IV and GeoGebra are analyzed using the concept of reasoning competency and Sfard's theory of discourses.*

*Keywords: Middle school mathematics, geometry, reasoning competency, GeoGebra, original mathematical sources*

## Introduction

Trouche, Drijvers, Guedet and Sacristán (2013) emphasize that the use of digital technology (DT) is increasing in educational contexts, requiring both students and teachers to change their practices and learn how to work in new technological contexts. One kind of change in both students' and teachers' practices could involve focusing more on working with the interplay between original sources (OS) and DT. The literature available in this mathematical educational area underlines that it can be fruitful for students to work with the interplay between the history of mathematics, including original sources (OS), and DT[1]. In this section of the paper, I include previous research within the following areas: 1) the history of mathematics, including original sources; 2) DT; and 3) the interplay between OS and DT.

Working with OS can sharpen and enrich students' mathematical understanding (e.g. Jahnke et al., 2000) and support their development of mathematical competencies (Clark, 2015; Jankvist & Kjeldsen, 2011). DT has a pragmatic value for students when it serves as an effective tool to perform given tasks; and it has an epistemic value when it aids their mathematical understanding (Artigue, 2002). Students are given opportunities to re-experience classical geometrical intuitions in new ways, while they are working with dynamic geometric software (Isoda, 2000). Among other things, this may also give them new opportunities and ways of working with reasoning, axioms and proofs. When the teacher guides students to work systematically to discover and express the connection between Euclidian geometry and the buttons in dynamic geometric software, they can gain insight into justifying and proving—axioms, definitions and theorems—the theory of geometry (Mariotti, 2002). Dragging is a way of getting complex feedback while working with DT (Arzarello, Olivero, Paola & Robutti, 2007), and one "hypothesis to be investigated is that students need to 'internalise' (Vygotsky,

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[1] A list of these articles and studies can be found in: Thomsen, M. (2020). The interactive role of original sources and digital technologies (accepted abstract for the History and Pedagogy of Mathematics conference (HPM), Macao 2021).

1978) the dragging function in order to be able to use it in a productive way" (Arzarello et al., 2002, p. 71). Working with the interplay between OS and DT can induce the hidden math within the program to the students (Olsen & Thomsen, 2017; Thomsen & Olsen, 2019), and enable them to participate in discussions about the issues involved in a reflective way (e.g. Chorlay 2015; Jankvist & Geraniou, 2019; Jankvist, Misfeldt, & Aguilar, 2019; Kidron & Tall, 2015). In the following sections of this paper, the primary focus is on the interplay between OS and GeoGebra (GG).

## Theoretical frameworks

In the context of Danish schools, it is relevant to focus on Niss & Højgaard's (2019) definition of eight mathematical competencies. These eight competencies have a huge influence on the subject's governance documents—curricula—at all levels of the Danish educational system. This paper and the overall PhD project focus in particular on reasoning competency. Sfard states that "Indeed, rules—any kind of rules—would often create mindsets" (Sfard 2000, p. 173). This idea plays an important role in this paper when analyzing and discussing the opportunities and challenges we give students to create their mindsets about reasoning and proving while working with the interplay between OS and GG.

### Reasoning competency – the receptive and constructive facet

Mathematical reasoning competency "deals with a wide spectrum of forms of justification, ranging from reviewing or providing examples (or counter-examples) over heuristics and local deduction to rigorous proof based on logical deduction from certain axioms" (Niss & Højgaard, 2019, p. 16). According to Niss and Højgaard's definition, reasoning competency is one out of eight mathematical competencies. Each competency has both a *receptive* and a *constructive* facet. For instance, the receptive facet of reasoning competency focuses on the individual's ability in terms of "following and assessing an alleged mathematical proof"; while the constructive facet focuses (for example) on the individual's ability with regard to "devising a mathematical proof of a statement" (Niss & Højgaard, 2019, p. 19).

### Discourses, object-level rules and metarules

Sfard (2008) defines thinking as communication and calls this commognition. Communication is perceived as a discourse, and so is mathematics. Commognition involves human beings acting as communication participants in various communities, and it also involves their interpersonal communication. With regard to discourse, Sfard distinguishes between object-level rules and metarules:

> In mathematics, the relevant metarules are those that govern the activity of proving. More generally, *object-level rules are narratives about regularities in the behavior of objects of the discourse, whereas metarules define patterns in the activity of the discursants trying to produce and substantiate object-level narratives.* (Sfard, 2008, p. 201)

Sfard (2000) argues for the importance of working on being more explicit about the metarules that are to be learned even at the most elementary levels in schools. She also emphasizes the importance of introducing students to the metarules of modern mathematics with respect to the students' thinking and understanding of the logic of the subject. The history of mathematics, including OS, can be used to make metarules explicit objects of reflection (Kjeldsen & Blomhøj, 2012).

**One discourse is embedded in another in GG**

OS and GG can be seen as two different discourses (Olsen & Thomsen, 2017; Thomsen & Olsen, 2019). The design of GG is based on Euclidian geometry. Some of the object-level rules are the same within the two discourses, but it could be claimed that the metarules are written explicitly in the OS and embedded and hidden in the construction of GG. The interplay between OS and GG is a teaching resource which can create space to support students when they work with learning, understanding and communication of and about both object-level rules and metarules within different kinds of mathematical propositions. These processes might have different conditions, including different opportunities and challenges, when it comes to supporting students' development of respectively the receptive and the constructive facet of the reasoning competency.

## Data collection and the teaching module

The empirical examples are selected from a pilot study in the author's ongoing PhD project. This pilot study was carried out in sixth grade, involving the participation of 13 students between the ages of 12 and 13 and their teacher. In general, the participants were used to working with GG, having worked with the interplay between GG and one of Euclid's propositions once before[2]. The teaching module alternated between discussions in the classroom and students' work in pairs or groups. Each group had a computer, and there was also a shared computer and projector in the classroom. The entire teaching module consisted of two sessions, one of 2 x 45 minutes and another of 4 x 45 minutes. The module included: 1) Euclid's five postulates related specifically to geometry; 2) Euclid's proposition 6, book IV, *To inscribe a square in a given circle*; 3) the title of Euclid's proposition 7, book IV, *To circumscribe a square about a given circle,* on the basis of which they had to construct their own proofs; and 4) answering questions about how they thought Euclid's proof was structured and why they thought it was a proof. They were given the same questions about their own proofs, and finally they had to explain what the characteristics of a mathematical proof are. Three types of data were collected: 1) film recordings of the classroom dialogues and selected parts of the students' work in groups; 2) sound recordings of selected parts of the work of different groups; and 3) samples of student work—both their handwritten answers and print-outs from GG. Since I only had two cameras and one sound recorder, I chose to record sections of the work done by a variety of groups.

## Empirical examples

The first empirical example concerns students' use of dragging in GG to be convinced of a part of Euclid's proposition 6, book IV. The second example concerns one group's work with the formulation of their own proof based upon the title of Euclid's proposition 7, book IV. These examples are chosen because they present two situations in which the students' learning outcome seemed to be different than expected when planning the activities. This learning outcome seems to be caused by the embedded discourse of Euclidian geometry in GG. The empirical examples are analyzed using the theoretical frameworks presented above. Untapped potentials will also be presented from these perspectives.

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[2] These learning activities are described in Olsen and Thomsen (2017) and Thomsen and Olsen (2019).

**Example 1. Dragging**

This example is linked to the proof part of Euclid's proposition 6, book IV. The proof part of the proposition was divided into smaller pieces. In groups the students had to describe the meaning of Euclid's text and explain why it is correct. One sample of this proposition says: "I say next that it is also right-angled. For, since the straight line *BD* is a diameter of the circle *ABCD*, therefore *BAD* is a semicircle, therefore the angle *BAD* is right"[3]. In this sample, Euclid builds on a proof taken from a previous proposition. So although the students might have found it difficult to understand and explain why this is the case, they could use dragging in GG to visualize that angle BAD is right. While the students were working on this task, the teacher initiated a joint discussion. By questions and hand-drawn illustrations on the blackboard based on the students' responses, the teacher and students demonstrated that points B and D had to be placed on the circle periphery, and that BD also had to be the diameter of the circle. This discussion was led by the teacher. An example of one group's work is shown below. Figure 1 is the students' own explanation of the above written phrase of Euclid's proposition 6, book IV. They might have completed this after the classroom discussion.
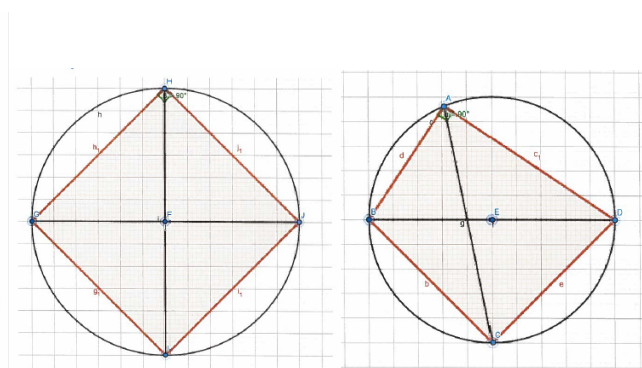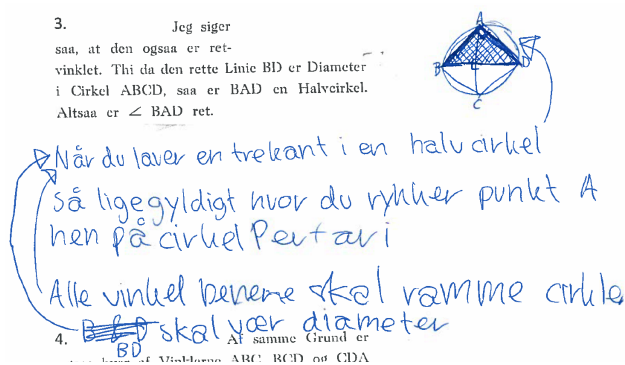


Figure 1: Part of OS - explanations from a group       Figure 2: The group's prints-outs from GG

Figure 1 says (translated from Danish into English): When you create a triangle in one semicircle, then wherever you move point A at the circle periphery. All the sides of the angles have to touch the circle. BD has to be the diameter.

The work sample shows that the students did not mention the right-angle when writing about point A. Based on their drawing at the top of figure 1 and both their constructions in GG (figure 2), the students' work in GG and the class discussion seem to have convinced them of the correctness of this phase of Euclid's proposition. As pointed out above, Euclid could only write like this because he had proved it in an earlier proposition. The visualization of the embedded Euclidian discourse in GG helped the students to gain insight into the object-level rules and touch briefly upon the metarules of that phrase in Euclid's proposition. The task and the class discussion could be characterized as helping the students to continue working on Euclid's proposition and develop the receptive facet of their reasoning competency to some extent. This situation actually had the potential to provide greater support for the students' development of the constructive facet of their reasoning competency by

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3 The students worked with an edition translated into Danish (Eibe, T. (1936). Euklids Elementer. Gyldendals Boghandel – Nordisk Forlag – Copenhagen. This sample is from an English translation taken from this webpage: https://mathcs.clarku.edu/~djoyce/java/elements/bookIV/propIV6.html

working more systematically with the metarules. We could have designed some questions/tasks to help the students gain a deeper understanding in which they could use the constructive facet of their reasoning competency to produce their own explanations of why angle BAD is right when BAD is a semicircle[4]. We could have asked the students questions like: Try to move point A along the periphery a little bit at a time and explain what happens to angle BAD each time—and why does this happen? What happens with the angle BAD if you go very close to point B or D? How do the other angles in the figure move when you drag point A along the circle periphery? Explain your reasoning. Then the teacher and the students could review the work done by the groups on the shared computer screen and compare their reasoning, looking for similarities and differences between them and also between their own explanations and those given in Euclid's earlier propositions. In one way, it could be claimed that working with the interplay between OS and DT might also have the potential to help the students to internalize the dragging functions in ways which would support the development of both the receptive and constructive facet of their reasoning competency. We did not test these kinds of tasks within this teaching experiment. If the students had been given the opportunity to use the constructive facet of their reasoning competency, they might have been able to use this in their work with the task presented in the next empirical example.

### Example 2. Constructing and proving

This example concerns the students' work with the title of proposition 7, book IV, *To circumscribe a square about a given circle.* Here the students were asked to make and present their own proof which they found convincing concerning *circumscribe a square about a given circle*. Firstly, they were asked to describe step by step how they constructed their figure in GG. Then they had to present their own proof for the title of this proposition. Here is an example from one of the groups, based on their first part of their work samples. The group's first work sample reads:

> "Draw a square in 'regular polygon'. Put 2 points and write 4. Create 2 diagonal lines from corner to corner with 'line segment'. Create a circle from the middle of the square and out to the edges, use 'circle from center and point, where the two diameters meet.'" (Translation of the first part of a group's work sample)

On the one hand, you could deduce that the students express themselves in accordance with the Euclidian discourse in describing their work in the discourse of GG. On the other hand, you could say they probably referred to buttons in GG which might be characterized by terms taken from the GG discourse, especially the last statement ("use 'circle from center and point, where the two diameters meet'"). This might be an example where the Euclidian geometry is embedded in the buttons in GG, thereby setting the scene for the way in which the students reason and prove theories while working within the discourse of GG. Here they do not have to be aware and explicitly state the length of the diameters. This may be the reason for their possible confusion about the length of the diameters or the diagonals in the second part (the proof part) of their work sample. This reads:

> "all the angles are right-angles, the diameters have the same length and meet in the middle of the square because we created two right-angled diameters intersecting the square in the middle. The

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[4] Arzarello et al. (2002) show how different kinds of dragging modalities can be used to explore this mathematical assertion.

circle touches the edge of the square precisely because the diagonals have the same length and are right-angles, in other words 90°." (Translation of the second part of the group's work sample)

The students start their argumentation of "all the angles are right-angles" by referring implicitly to their drawing in GG by using "regular polygon" (see the quote above). On the one hand, you could say that they got caught up by the embedded Euclidian discourse in the discourse of GG while they were formulating that part of their proof. In the following part of their proof, on the other hand, you could say that they actually tried to substantiate their argument step by step by trying to transfer some of both the object-level rules and the metarules from their work with the Euclidian way of proving in proposition 6, book IV. You might say that the students focused in this case on the metarules and not so much on the object-level rules. In the classroom they discussed the work of the groups before completing it. They did this by asking selected groups to draw their constructions on the shared screen, presenting their proofs, and discussing with the teacher and their classmates. These discussions showed that some of the groups ran into the same kind of problems with the Euclidian embedded discourse in GG. They did not have time to look at Euclid's proof of proposition 7, book IV, and did not have a class discussion about the differences between proving within the discourse of *Euclid's Elements* and within GG. These examples might show that working with the interplay between Euclid's propositions and GG may make it possible to work systematically to discover and express the connection between Euclidian geometry and the buttons in GG. If the teacher is aware of using these points to show the students that they sometimes—while working with reasoning and proving in GG—get caught within the embedded Euclidian discourse, it may help them to gain insight into the theory of geometry. In such situations, the teacher uses the visible metarules within the discourse of OS and the object-level rules within the two discourses of respectively OS and DT to reflect on and discuss the hidden metarules in GG. This may be one way to help the students to develop the constructive facet of their reasoning competency.

In the first example the main focus was on the receptive facet of the reasoning competency, while the focus was on the constructive facet in example two. Both examples show that it may be possible to support the students' development of their reasoning competency by consciously alternating between these two facets as well as between object-level rules and metarules.

## Conclusion

In this section, and based on the analyses of the two empirical examples presented above, I will attach a few concluding comments to the points emphasized in the previous research presented in the first section of this paper. Working with the interplay between OS and DT seems to have the potential to change teachers' and students' practices and help students to develop both the receptive and the productive facet of their reasoning competency by focusing on both object-level rules and metarules within the mathematics content selected in both discourses. From this perspective, OS can be seen as a possible key to open up *the theory of geometry*, thereby constituting a resource in focusing more on the epistemic value of dragging in GG while working with reasoning and proving. As shown by the first empirical example, it might be possible to use OS to create learning situations in which the students work systematically with dragging and having reflective dialogs about this, while trying to justify a mathematical assertion in a proposition. This might be a way of internalizing the dragging process and using it in a productive way. As shown in example two, working with OS can be seen as a point of departure for managing teaching and learning situations while working with GG, giving

the students the chance to formulate their own proofs and discuss them related to the OS. It might sharpen their reflections on working with both object-level rules and metarules within different discourses. This kind of work in mathematical education allows students and teachers to become aware of the challenges and possibilities that the embedded Euclidian discourse within GG can provide for students engaged in reasoning and proving activities.

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# A novel application of the instrumental approach in research on mathematical tasks

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*In this paper I explore a new approach to analysing tasks in mathematics education. By seeing tasks as an instrument in the activity of learning mathematics, I propose to use the instrumental approach and the notion of instrumental genesis to describe how a student could be able to internalise mathematical knowledge and methods through working with tasks.*

*Keywords: student practices at university level, tasks, competencies, the instrumental approach.*

## Introduction

In the Nordic countries, a focus on the nature of mathematical tasks and their use in education has been a central theme (e.g. Bergqvist, 2007; Lithner, 2017), together with the notion of competence (Haavold, 2011; Lithner, 2017). This has in Norway over the last decades informed both policy-making and the discussion of students' mathematical achievements (Botten-Verboven et al., 2010).

The focus in this paper will be on tasks. I will present a, to my knowledge, new way of applying the instrumental approach (Rabardel, 2000) by describing mathematical tasks as instruments for developing mathematical competence. This is part of a PhD project focusing on first year university calculus and the secondary-tertiary transition. The link between tasks and transition can be seen in, for example (Bergqvist, 2007), where she examines tasks in early calculus courses. Roh and Lee also talk about tasks designed to "bridge a gap between students' intuition and mathematical rigor" (Roh & Lee, 2016, p. 34), pointing towards a connection between how tasks are formulated and presented in upper secondary and in university, and the issue of transition. One of the main research questions in my PhD project, is "How can tasks be used as instruments in developing competences?" I will however, not be able to answer this question fully in this paper. Instead, I will focus on the narrower question "Based on a series of task-based interviews in early university calculus, and using the instrumental approach, what sort of evidence is there for saying that tasks *can* be used as instruments for developing mathematical competence?"

I will divide the argument into two parts. First, I present my theoretical framework. Then I make my case for why tasks can be seen as instruments according to the instrumental approach. The use of the theory will be exemplified through a short case study, selected from one of the interviews.

## Theoretical framework

There have been several ways of describing tasks and describing ways of implementing and working with tasks (Watson & Ohtani, 2015). Tasks have been described as mediating artefacts in teaching and learning mathematics by Clarke, Strømskag, Johnson, Bikner-Ahsbahs and Gardner (2014) and by Johnson, Coles and Clarke (2017). The idea of tasks as artefacts can also be identified in an article by Watson and Mason where they talk about "seeing an exercise as a single mathematical object" (Watson & Mason, 2006, p.91). Tasks and their role in mathematics education is a matter, not only of being the things that one does in class to practice doing mathematics, but they may also act as an

aid in developing deeper mathematical insight. It is not only the content of the task that matters for what is being learned, but also how the task is designed and embedded in the teaching context, and what sort of guidance is given before, during and after solving the task. The last point can be evidenced by the findings of Haavold (2011), where he shows that even high achieving students tend to rely on imitative reasoning rather than creative reasoning, when proper guidance is not given.

In this article I use Activity Theory (AT) (Leont'ev, 1978) to describe the context in which tasks are solved. Through activity, humans try to achieve some *objective*, and this activity is made possible by and mediated through artefacts. Such an activity is said to be *object-oriented*. The artefact plays an important role, as the activity that is conditioned by the artefact could not even be possible without the artefact. In Leont'ev's description, human activity is divided into processes of three different levels, where the activity itself is at the top most level, and is driven by some *motive*, meaning there is a need that must be fulfilled, which *motivates* the activity. The difference between objective and motive is a subtle one. I use the word objective when talking about the concrete end towards which the activity is directed, and motive when talking about the drive towards this objective. Further, each activity consists of *actions*, which have different *goals*. A goal is the concrete end towards which an action is directed. An important distinction between motives and goals is that the subject needs not be conscious about the motive at all times during the activity, whereas the goal is always consciously present during the action. At the bottom level, an individual action is performed through a number of *operations*. The operations are determined by the *conditions*, that is, the material and immaterial resources available to and constraints imposed on the acting subject, both by the environment, but also by the prior knowledge and abilities of the subject itself.

For describing what it means to become competent, one idea that has been guiding me is the competence framework of Niss and Højgaard (2011). In particular, the description of a competency as "a readiness to act" (Niss & Højgaard, 2011, p. 49), stresses that a person is not only *able* to carry out a mathematical procedure, but is also *ready* in the sense of knowing why, when and how the procedure works, as well as having the confidence to be able to perform the procedure.

**Definition of tasks**

I will now use the theoretical approach described above to define what I understand by the word task. First, I describe a task in general, and then I will describe what I define as a formal mathematics task. The description by Watson and Ohtani as the "things to do" (2015, p. 3) highlights the active nature of working with a task. In addition, there should be some sort of obligation connected to a task. At least the person performing the task should have the belief that this is something that he or she should do. And thus, the task should be given by someone, possibly the same person that performs it. Five roles can then be identified in how a person can relate to the task: The roles of designing the task; presenting the task; performing the task; presenting the solution; and evaluating a given solution. In order to distinguish between these roles, I use the words *designer*, *task presenter*, *performer*, *solution presenter* and *evaluator*. I distinguish between these as roles, but it may well be that the same person could play more than one role, or that one role is played by more than one person. The designer could also be the task presenter, and in some cases also the performer, solution presenter and evaluator. I will mainly focus on the three first roles in this text, but for learning, the two last are also important.

Drawing on ideas from AT and the general description of tasks and of competence, I define a *formal task* as a task that fulfils four criteria, to be presented below. A task that fails to fulfil at least one of these criteria will be called an *informal task*.

First, since the motive of the activity is to become more competent, the task should have a *purpose* in achieving this. It is however not necessary that the performer of the task has been informed about this purpose. There might be good reasons for not disclosing the full reasoning behind a particular choice of tasks. For example, if the purpose of a task is to check whether the performer recognises a particular mathematical pattern, informing the performer beforehand about this might void the task of its purpose. The purpose is neither equal to the motive of the activity nor the goal of the action, but is related to answering the question of why this task in particular is chosen. In fact, if the purpose of the task is changed, the task itself changes, as I see the purpose being an integral part of the task itself.

The second criterion is that the purpose should be known to the designer of the task securing that the designer can state the reasons for, and therefore also argue for why someone should solve such a task. A task fulfilling this criterion is called *formulated*. This criterion is similar to the first one, but by securing that the designer knows the purpose of the task, it is possible for someone else to find this purpose without having to solve the task themselves. Thus, a way of selecting tasks is possible informed only by the objective of the activity and the intended purpose of the tasks.

The third criterion is connected to whether the task has an endpoint attainable within a predictable time limit. Such a task is called solvable. The stricter case, where the task has a clear and single answer, as for instance calculating a sum, will be called answerable. The criterion of solvability will exclude many tasks that are in some sense open ended. For example, one can imagine the task of finding an *exhaustive* answer to a *why* question, where one can continually probe deeper into the explanations, without being able to know whether a solution exists. Nevertheless, such a task can still often be divided into solvable subtasks. Still, many tasks considered open can possess the criterion of being solvable, as long as there is a possibility for the performer to be satisfied that the task has reached a solution. If for example the task is *not* to continually probe deeper into a why question, but rather to arrive at an explanation, based on some finite number of assumptions and preconditions, it can be possible to find a solution to the task. It is not, however, necessary for the task itself to provide a systematic way of validating the solution. The solution can still be invalid, but it must be clearly distinguishable as a plausible solution. A trivial example might be the solving of a simple equation, where the answer is a number. Any number could be *plausible*, depending on the knowledge of the performer, but something which is not a number, will not be a solution.

The fourth criterion is inspired by Niss and Højgaard's statement that an answer "must be produced by calculations, that is by a mathematical procedure and not by measurement" (2011, p. 94). A procedure is a way of arriving at a solution through logical inference, possibly as simple as counting on fingers for adding numbers. A task where there exists a way of arriving at the solution in this way, will be called proceduriseable. This does not mean, however, that guessing or recollecting has no place, but there should at least be some way of "sifting" the solutions by means of inference.

In addition to these criteria, tasks can also be composed. A highly composed task will have many subtasks that are more or less necessary for the whole task to be solved.

**The instrumental approach**

I will here present a short description of the instrumental approach.

The instrumental approach (Rabardel, 2000) has, from the introduction to mathematics education, been connected to digital tools (Trouche, 2004). A tool is seen as "something which is available for sustaining human activity" (Trouche, 2004, p. 282). The main idea of the theory is that an instrument is an object consisting of the tool, together with usage patterns and mental schemes connected to the tool. Without the usage patterns and mental schemes, the tool will not yet be useful to the tool user, and will need to go through a process, called an instrumental genesis to become an instrument (Rabardel, 2000). This process consists of two parts: *instrumentalisation* and *instrumentation.*

Through the process of instrumentalisation, the artefact becomes an instrument. That is, the subject personalises the artefact, and creates an instrument by appropriating it into the subject's activity. This might happen through three phases: discovery and selection of relevant functions of the artefact; personalisation, where the user finds the preferred way to apply the functions; and transformation of the artefact, where modifications are made to the artefact to fit the user's usage patterns (Trouche, 2004). Through instrumentation, the subject also changes, to become a tool user. The usage patterns are internalised and the activity of using the tool is conditioned by the artefact. This instrumental genesis is dependent upon the properties of the artefact, or its constraints and potentialities, and it is also dependent upon the subject, its activity, prior knowledge and working methods (Trouche, 2004).

The instrumental approach is also applicable to tools other than digital. A non-physical tool can become an instrument when the tool changes from being a mere artefact into something that in the mind of the tool-user has a purpose and can help in achieving some goal or objective. Indeed, according to Lagrange et al. (2001, p. 6), "While the artefact refers to the objective tool, the instrument refers to a mental construction of the tool by the user". Moreover, a tool "can be material or cultural" (Trouche, 2004, p. 282). Tools need not be understood as physical entities, but can also be abstract, such as formulas, algorithms, and as I will argue, tasks.

## Tasks as instruments

For many students, the most immediate goal while working on a task is to get the task done. This, however might not be the most effective way of achieving competence in mathematics. In my view, using Leont'ev's three levels of activity, it makes more sense to see the task itself as a *tool* in the activity, with the objective of becoming competent in mathematics. Solving tasks are then actions in this activity, and the different operations done to solve the task corresponds to usage patterns.

Since I cannot provide an exhaustive account of the different ways a task can be used as an instrument within the confines of this paper, I will in the rest of the text argue that the idea of tasks as instruments is viable and observable. The core of the argument will be the short case study, but a note about the process of instrumental genesis is needed to argue observability.

In order for a task to become an instrument, it needs to go through an instrumental genesis. As the performer solves a task, I assume that the task "acts upon" the person solving it, when learning happens. The subject is changed by the artefact, through instrumentation. Whether this has taken place can be seen for instance through tests or exams, as the observation that a student becomes more secure and their success rate increases in solving a particular type of task is a sign of this. But in

addition, an instrumentalisation process is also necessary. The task needs to be appropriated by the task performer in order for it to become an instrument. A key to the observation of this is in the three different stages of instrumentalisation, found in (Trouche, 2004). In order to demonstrate how one could observe these different stages, I present an example from an interview. This is selected from a series of video recorded interviews conducted over the course of one semester, in their first university level calculus course. The purpose was to track the development of how six different students would reason during solution of tasks and how they might use tasks for learning.

**Example tasks**

In the first interviews, the students were given tasks on integration, and then told to freely describe their reasoning while solving the tasks. They were themselves responsible for stating when they considered the task solved, and the interviewer would only intervene when the students were silent, by asking them to continue talking. Two of the tasks given are shown in Figures 1 and 2. They will be analysed below using the four criteria of a formal task, and then a case study of one interview where these tasks were used, will be presented, together with observations from the other interviews.

Let $G(x)$ be defined as

$$G(x) = \int_{x-1}^{x+1} f(t)dt$$

What can you say about $G'(x)$?

Figure 1: Task 1

Let $G(x)$ be defined as

$$G(x) = \int_{x-1}^{x+1} f(t)dt$$

where $f(t)$ is a periodic function with period 2, so $f(t) = f(t+2)$ for all $t \in \mathbb{R}$
What can you say about $G'(x)$?

Figure 2: Task 2

The criteria for formal tasks can be applied to these two tasks individually. The tasks have a potential to demonstrate properties of the definite integral, and in the second task, also of periodic functions. This leads me to say they are both purpose oriented, as the purpose of giving such tasks to a student could be to highlight these properties. In addition, taken together as two parts of a composed task, they could serve as a way of highlighting for the student some possible false perceptions about the fundamental theorem of calculus (FTC).

They are both solvable. In Task 1, a possible solution could be to find the algebraic expression $G'(x) = f(x+1) - f(x-1)$, and be satisfied with this as a solution, but because of the vagueness of the question, it is not strictly answerable. Another plausible solution could be to state that $G'(x)$ represents a change in area. In contrast, these solutions would probably not be considered a satisfactory solution to task 2, since it does not take the periodicity of the function into account. Here the intended conclusion would be the observation that $G'(x) = 0$. They are both proceduriseable as well, since applying the FTC could be one such procedure.

For the students, these tasks were difficult, and not all managed to give answers that they themselves were satisfied with. It is worth noticing that although the question of saying something about $G'(x)$ was identical in both of the tasks, the answers the students gave to each of the two tasks varied considerably between the tasks, which correspond well with the pre-analysis of the tasks.

**Case study**

One student had a particularly interesting approach to solving in particular Task 2, using what he himself calls "an extreme example". Moreover, he was the only one who solved Task 2 correctly, although not entirely rigorously. This student had already followed an algebra course at another university while he was still in secondary school, after having finished the highest level of secondary school mathematics a year early. Task 1, he solved relatively quickly, stating that $G'(x)$ relates to the area of the graph between $x - 1$ and $x + 1$. On Task 2 he spent more time. The excerpt below shows his description of his own thinking at the moment when he arrived at his conclusion:

Student: I want to say that $G'(x)$ is zero … but that is for the most part a gut feeling … [partially inaudible] … Okay. Then I'll go for an extreme example. That usually works. [draws a graph]. 2 in the centre … that area [points to the graph] is equal to that area. Yeah, I just want to … I'm going to think about this when I get home …

Interviewer: Just say what you are thinking.

Student: I think $G'(x)$ is … zero … that there is no change in area. Since the period is 2, that is it repeats itself, so we know at least that $f(t)$ on both ends are of equal height. That is given. And then we know also that it will be relatively symmetric. We can always do a translation... or a reflection. For example, I could take that part, move it over there [draws an arrow from the centre of the graph over to the right]. If we had moved that one a little bit to the left, then we would have gotten more of that one and less of that one [pointing at the two maxima on the graph], then we could move that one over there. Yes! $G'(x) = 0$. That is my final answer [drops the pen].
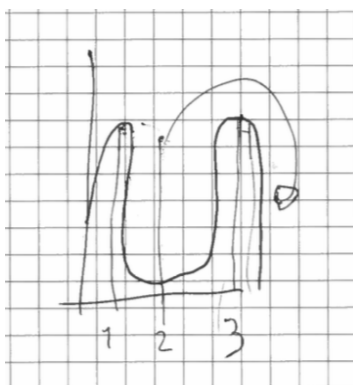


Figure 3: Graph of the "extreme example" drawn by the student.

Several parts of the instrumental genesis process can be observed here. From the triumphant drop of the pen, and declaration that he had found the answer, it seems reasonable to suggest that he has discovered something new by solving the task, and that he has learned something. It is worth noticing that the solution he arrived at is not a rigorous description of the general principle, but it is strong enough to convince him of the truth of his conclusion, and thus induce learning. Steps towards an

internalisation of the usage pattern are taken, as the student might have discovered some relation between a periodic function, and the change of area under its graph.

Different stages of the instrumentalisation can also be observed. The discovery phase can be seen, as he tries different operations on the task. In this excerpt, he tests two operations. He constructs an "extreme example", as he refers to it, in the form of the graph of a function, and he operates on this graph by performing an imagined moving of one part of the graph to another part. Personalisation can also be seen in the statement that an extreme example "usually works". This shows a personal preference to certain operations, and the fact that the example made it possible to convince himself of the solution likely strengthens this preference. Transformation can also be seen in two ways. First, as the two tasks are similar, this constitutes an example of such a change, from the most general case to the periodic, albeit not performed by the student. But the student makes a similar change by further limiting the scope of the task, from finding the solution of the periodic case, into solving the extreme example, and thereby using the now changed task as a tool for exploring the more general case.

Other students also tried similar strategies. Especially the selection of an example function, and then exploring the implications that the question would have to this case. The difference would be that the other students would rather choose a *typical* periodic function, like a trigonometric function, and often explicitly defined, rather than an unspecified periodic function that lies at a perceived extreme. They also in many cases did not make the connection between $G'(x)$ and the change of area. Nevertheless, attempts at exploring and changing of the scope of the task can still be observed also in these cases.

## Final remarks

Some consequences for how we use task seems to emerge from this. Since task are central to learning mathematic, and since tasks can be seen as instruments, as I have argued, this suggests that we need to take the different phases of the instrumental genesis into account when designing and using tasks. One such way may be to give students the opportunity to explore and discover the different ways a task can be used, and what sort of changes can be done to the task. This might be done simply by providing the time and opportunity for such discovery, but also by providing good examples that show how tasks can be used for different purposes, and how to change tasks in order to achieve this.

There are however still more questions to be answered. First, in this paper I have only demonstrated that tasks *can* be seen as instruments, but there is the need to examine *different ways* tasks can be used as instruments. The question of transition seems also likely to be connected to this point, as diversification of the way tasks are used would constitute one way in which the complexity increases. Finally, these observations might also have consequences on the design of tasks, and possibly give rise to some design principles. Different types of tasks might very well lend themselves to use as instruments in different way, possibly also dependent upon and conditioned by the sort of guidance provided by the presenter of the task.

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# Students' perceptions and challenges regarding mathematics teaching cycle in practices of historical and philosophical aspects of mathematics course

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*This paper explores master students' perceptions and challenges when designing a mathematics lesson in the Historical and Philosophical Aspects of Mathematics course following the perspective of the mathematics teaching cycle. The sample consists of thirteen students who are enrolled in this course that is taught within a mathematics education master's program at a large Norwegian university. The data come from students' productions related to their practice assignment, post-classroom reflections, and group and individual interviews. The data are triangulated and analysed by the lens of the components of the mathematics teaching cycle. According to the findings, master students' major perception regarding the mathematics teaching cycle is that the hypothetical learning trajectory is a prediction of student thinking. Their main challenge is the lack of information regarding student pre-knowledge to express hypotheses and making the assessment.*

*Keywords: Mathematics teaching cycle, Hypothetical learning trajectory, Lesson plans to incorporate the history of mathematics, Master students.*

## Introduction

The professional development of preservice mathematics teachers (PMTs) has received particular attention from mathematics educators (Rowland, Huckstep, & Thwaites, 2005; Wilson, Mojica, & Confrey, 2013). The core content in this direction can be described with developing subject matter knowledge and pedagogical content knowledge (Ball, Thames & Phelps, 2008). Regarding these contents, some of the recent research in Scandinavia has focused on PMTs' pedagogical content knowledge occurring in teaching in practice schools. Munthe, Bjuland and Heldevold (2016) studied how to shift the focus of Norwegian preservice teachers when conducting field practice from surviving in the classroom to observing pupils' learning and making their learning visible. Hemmi and Ryve (2015) explored effective mathematics teaching as constructed in Finnish and Swedish teacher educators' discourses with PMTs. In this paper, we particularly focused on professional development by looking at the link between a master course and its teaching practice with a perspective of Mathematics Teaching Cycle (MTC) (Simon, 1995). Primarily, we trained 13 PMTs in a Historical and Philosophical Aspects of Mathematics (HPAM) course about MTC and its specific component, the notion of Hypothetical Learning Trajectory (HLT). Students referred to these notions to create their lesson content and to implement their lesson plans. We report here students' perceptions and challenges regarding MTC (as well as HLT) after they carried out their teaching in practice schools.

## Theoretical framework and research question

The notion of MTC is elaborated by Simon (1995) in order to address the core role of a lesson and task *design* in mathematics teaching. Emphasis is given to HLTs in MTC, which mainly refers to

anticipated student learning steps under the teacher's formulation of an explicit learning goal and her careful plan for teaching activities. Here the mathematical knowledge for teaching (Ball, Thames & Phelps, 2008), both subject matter knowledge and pedagogical content knowledge, is of crucial importance. The teacher blends her subject matter knowledge with students' pre-knowledge and creates a plan for activities through hypothesized students' thinking. Therefore, HLT is here referring to the teacher's prediction regarding students' learning and thinking, such as which steps they would follow, which misconceptions could occur, and then which strategies could be followed. Simon (1995) puts forward HLTs and assessment of students' knowledge as the core parts of the mathematics teaching process. MTC and associated components are rendered in Figure 1.
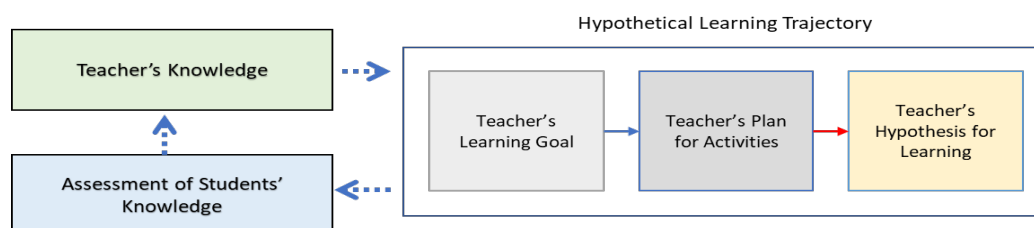


Figure 1. Mathematics Teaching Cycle (adopted from Simon, 1995, p. 136)

In the related literature, the notion of HLT has been receiving particular attention from several researchers. On the one hand, it has been considered to be a design heuristic in designing teaching-learning environments (Huang, Zhang, Chang, & Kimmins, 2019); on the other hand, it has been exploited as an educational tool for teachers' professional development on the development of knowledge of students' thinking (Wilson et al., 2013). For instance, Wilson et al. (2013) observed that the (hypothetical) learning trajectory perspective contributes to teachers' model-building skills regarding students' possible learning paths and it also contributes to teachers' understandings of mathematics itself. Similarly, in the present paper, we refer to the notion of MTC, particularly HLT, as a professional development tool to improve master students' task design skills and focus on the following research question: *What are the perceptions and challenges of master students in mathematics education regarding mathematics teaching cycle in the Historical and philosophical aspects of mathematics course*?

## Research context and methods

This research is part of the pilot study entitled "Understanding student teachers' learning and development in the Master course of HPAM". The HPAM course is designed as a master-level course for PMTs under a two-fold aim. On the one hand, the aim is to provide information about historical, ontological and epistemological foundations for basic mathematical concepts and algorithms with a focus on algebra and geometry, as well as knowledge on what mathematics is, how its nature and methods developed, and what it constitutes today. On the other hand, it is aimed to develop PMTs' skills in transforming knowledge of the history of mathematics into didactical and pedagogical designs for teaching mathematics from 5th to 10th grade. Along the latter direction, the students (in groups) prepare pedagogical designs for field practice, they implement their designs and write formal practice reports, which is one of the compulsory tasks to pass the course.

Along with the pilot study, we observed that the students had issues in designing lesson plans and incorporating the history context meaningfully. To prepare the students better, two lecturers of the course (i.e., the authors of this paper) organized in autumn 2019 a workshop (lasting approximately

4 hours) two weeks before their school practice in two interrelated directions: the first is introducing students to the notions of MTC and HLT, and the second is discussing, contributing and exchanging ideas about students' initial designs for practice. In this way, we aimed to improve the PMTs' professional development in designing a lesson that incorporates the history of mathematics. After we introduced the notions of MTC and HLT with examples, the students worked in four groups for a while and focused on lesson (and practice) plans. They got feedback from the lecturers and then they finalized their plans and presented them. Thereafter, the students got final feedback from the lecturers and their peers and they continued to work on lesson plans. While updating their documents, they communicated with the lecturers a few times. Afterwards, they carried out the teaching and wrote reports about it as an academic text and uploaded them to the course's online portal. The timeline of the process is rendered in Figure 2.
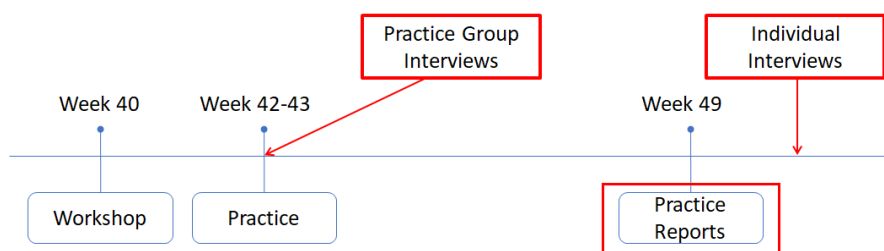


Figure 2. Timeline of research

The participants of the course were in their first year of a two years master program for grades 5-10 at the department of teacher education. The group consisted of 13 (5 females and 8 males) PMTs who all volunteered to be part of the research. Eleven of them started the master program straight after their bachelor studies; two (named D and L; both with ten-year teaching experience) were working as mathematics teachers at public schools and were back at the university to complete a master's degree. Students studied mathematics didactics four out of six semesters during their bachelor years and mathematics comprised 50% of their workload during these semesters. Those mathematics didactics courses are more focused on didactics and less on advancing the students' subject matter knowledge. However, they did not know the notions of MTC and HLT before the lecturers introduced them, as they were trained in their earlier study years to use the didactic relationship model (Bjørndal & Lieberg, 1978) and it's simpler form, known as "hva-hvordan-hvorfor" (what-how-why) scheme, for their lesson plan designs in previous practices.

The data come from practice groups interviews, students' practice reports, and individual interviews, whose time location is shown in Figure 2 (through red rectangles). Students arranged four practice groups (we call them G1 - G4) themselves and they implemented their teaching in these groups in four different practice schools. For the sake of anonymity, we coded students' names as follows: Group 1: A, B, C; Group 2: D, E, F; Group 3: G, H, I, J and Group 4: K, L, M. We conducted group interviews with two randomly selected practice groups (G1 and G3) and three individual interviews, where individuals are coming from G1, G2 and G3. All the interviews were audio-recorded and transcribed, and they lasted about 25 minutes. Practice group G1 interviewees were B and C, interviewees from G3 were only G, H and I, as not all the students were available. Because of their good communication skills, we invited students B and G for individual interviews, and we have included (due to the amount of teaching experience) student D as well. Practice groups interview question was: to what extent was the session about designing a mathematical lesson useful for you?

What did you learn and how did you use it in your teaching in the practice school? And individual interview questions were: (1) What do you think, is there any difference between the notions of HLT and MTC and the way you have been taught/requested to prepare a lesson by the practice office/teachers? (2) In your practice description, did you refer to HLT? Why? Explain… (3) Which part(s) was (were) hard for you to write the HLT? (4) Did your lesson plan work? What happened?

The practice report task included many points: (*a*) explain why your group has planned/chosen this teaching activity and your main aims for using this teaching activity in the classroom, (*b*) clearly explain your learning goal, classroom activities, hypotheses for learning - HLT, planned assessment of your practice, as well as how selected historical context should support students' learning and why? (*c*) give a short description of the class or the group of students, (*d*) report your findings, (*e*) give a reflection and evaluation of the teaching activity and the result of the work in the classroom. We analysed the data by following the steps of Braun and Clarke (2006); familiarisation of data, generation of initial codes, search for themes, review of themes, defining and naming themes and producing the report.

## Findings

This section is divided into four subsections according to three different (data) sources and later we provide an overview regarding perceptions and challenges of MTC and (particularly) HLT. Because of page constraints, we briefly present dialogues.

### Analysis of practice group interviews

The practice group interviews were conducted in practice schools and interviews started after the interviewer asked the question about lesson design. The master students, G, H and I (belonging to G3) immediately expressed their views regarding MTC and HLT that were already elaborated in the workshop. The following excerpts are drawn from their discussion:

I: … I didn't really understand what was the meaning of the planning, cause we know … we have learned how to plan a lesson from before, so I didn't understand what was the difference in this way of planning...

G: … that was just more of a bit deeper into some things what we already knew ... maybe ... and so it wasn't really a lot of new information, I would say ...

It is obvious from the lines above, that I and G think that the notion of MTC is not completely different from their pre-knowledge on (and way of) planning of a mathematics lesson; it is planning but with not fully new information. At the same time, H pointed out that there were too many slides in the workshop, it could be a word document to work on since they are so experienced in preparing plans, it could be rather briefly summarised by giving just instructions through a list of steps. Thereafter, I and G also confirmed this notion. Later, H and I expressed that the notion of MTC is more or less the same with their way of planning (didactic relationship model), just the format is different. Regarding the other practice group (G1) interview, we observe that B (like C) encounters the notions of MTC and HLT for the first time and she thinks that the notions are useful '*because it makes me think more about what I want pupils to learn. What I expect them to learn*'. Further, B felt that the description of MTC during the workshop was a little bit '*jumpy*' and '*not like a specific list of things*', even though she found the PowerPoint file adequate.

**Analysis of student practice reports**

We analysed the participants' practice reports with a four-dimensional framework as ingredients of MTC. Table 1 presents our coding (i.e., precise, not precise, detailed and/or not clear) with our comments associated with each column.

| Group No | Learning Goal | Plan of Activities | Hypotheses for Student Learning | Plan of Assessment |
|---|---|---|---|---|
| **G1** | not precise | detailed | -list of learning steps<br><br>-some of the possible difficulties and how to solve them | detailed with several different techniques |
| **G2** | precise and thought through | detailed and ambitious | not formulated explicitly, but reference to it in the final reflection | not clear |
| **G3** | confusing, not precise | detailed | partial, not completed | not clear |
| **G4** | precise | detailed | vague | short without details |

Table 1: An overview of students' practice reports regarding MTC

Table 1 shows that although the plan of activities appeared in each group report as adequate, the description of learning goals and formulating hypotheses for student learning seemed to be problematic. There was also a lack of a plan for assessment and its techniques in the students' practice reports.

**Analysis of individual interviews**

The first (individual) interview was conducted with D, who belongs to G2. Regarding the notion of HLT, it was new for him and in a situation when he was going to teach a topic which was also new for him, it was hard to know what to expect from the students, what difficulties they might have, what they will learn. As he says '... *I always think about the learning goals, how are they going to get to those learning goals …*'. Further, regarding hypothetical learning, he pointed out that he never wrote something similar before. Following this, he underlined that he could not imagine what would happen in the classroom in this practice assignment. He means that it was very hard to formulate students' anticipated learning steps because they for the first time considered such a task in the classroom: '... *But the hard thing was to discuss when we discussed actually what are the students going to learn. Because I haven't used any task like this in class. So I didn't know what to expect from the students to understand ...*'. Regarding assessment, he underlines the time issue to spare some time at the end of the lesson: '*I'm bad at evaluating what the students have learned. I think one of the reasons is the time or I don't put enough time at the end of the lesson to evaluate*'.

The second interview was with G, who belongs to G3. He perceived the concept of HLT as something very theoretical, but with a practical application: '... *it's pretty much just saying what do you think will happen ... So /laughing/ that's ... hurdle ... it's just kind of a fancy way to say it*'. He understands it as potential paths of students' learning (i.e., thinking): '... *If they manage this then we can go further*

*and if not then ... kind of like a programming language /laughing/ if-then-else …*'. He emphasized the importance of classroom discussions, but on the other hand, he questioned the engagement of students. With better knowledge of students it would be easier to know what to expect from them: '... *is just about the relation, the teacher-student relationship, that builds up over time ... better you know the students, the easier it will be and more accurate it will be* (*HLT*)…'.

The third interview was with B, who belongs to G1. According to her, her (existing) lesson planning schema is parallel to the learning goal and the teacher's plan for activities of HLT. She finds that the whole HLT is an extended form of her schema because HLT has a (third) part where '*we have to project what the pupils might misunderstand or don't understand*'. Next, she clearly expresses the relation between her pre-knowledge about the what-how-why approach and learning goal and planning activities: '... *I have to do hva* (*what*)*, hvordan* (*how*)*, hvorfor* (*why*) *to make the learning goal and plan for the activities. I have to keep that in mind when I'm doing those two…*'. She considers the prediction of students' difficulties – a hypothesis for students' learning, as a very useful extension of her scheme '*this one ... takes it a little bit further ... I think I can just merge them and expand it ... it's kind of like an additional thing that I find very useful …*'. It was new for her, '... *it was a little bit different, we haven't done anything like this before …*', and a little bit difficult too, because her group did not know what students knew. In sum, she addresses that, in their group work, they mostly spoke about hypotheses for students' learning. She always plans how to assess students' learning: '... *it always has been part of the summarization*', but in this practice assignment they didn't have time for it, '... *when we came to class we didn't have as much time ... so it was just like a short summary ...*'.

**Overview of perceptions and challenges**

In order to articulate our findings, we overview major perceptions and major challenges (with participants' codes) observed in three previous sections through Table 2.

| Perceptions | Challenges |
|---|---|
| – The notions of MTC and HLT are different ways of designing a lesson (G, H, I) | – Time issue for assessing in the classroom (B, D) |
| – HLT is just formulating of learning goals and hypothesis for student learning (D) | – Lack of a word document as a list of necessary steps to transform MTC and HLT into practice (B, G, H, I) |
| – HLT is a kind of fancy way of approaching teaching (G) | – lack of information regarding students' pre-knowledge to express hypotheses step-by-step (D) and making an assessment (B) |
| – HLT is composing plans like programming: if the students manage to do or they cannot (G) | |
| – HLT is having backup plans at stake (B) | |
| – HLT is predicting what students will find difficult (B) | – Considering a completely new task in the classroom (D, G) |
| – HLT is useful to project student view (and like to-do list) and thinking about their studentship (B) | – Focusing only on the HLT part, forgetting about assessment (B, D) |

Table 2: An overview of perceptions and (interrelated) challenges regarding MTC and HLT

As seen from Table 2, MTC is not at the fore while perceptions for HLT appear more frequent. Also, we see that the participants' common perception of HLT is thinking and predicting students' thinking rather than considering all three components of HLT. Regarding challenges, PMTs addressed interrelated major points: time issue for assessing in the classroom, not having information about students, for example, what they know and what they don't know to formulate concrete hypothetical learning steps and a lack of a list of necessary steps for MTC. The latter seems to indicate that PMTs are looking for a recipe for how to plan a lesson rather than a heuristic (the MTC) that must be fleshed out by them for each topic and class.

## Discussion

In this paper, we focused on students' perceptions and challenges regarding MTC proposed by Simon (1995). Our findings revealed that some of the participants think about MTC as something in a different form but with the same aim of planning a lesson. This conclusion could be due to students' experience coming from the "what-how-why approach" of the didactic relationship model (Bjørndal & Lieberg, 1978) that they referred to many times to prepare lesson plans with justifying each step. However, some of the master students perceive MTC and HLT notions as new, however, they addressed a particular focus on (only) writing hypotheses for students' learning not together with learning goals and/or plan for activities. The main resource for this phenomenon could be due to students' tendency to consider curriculum competencies as learning goals. Even though our research was short for a professional development program and with a small sample of PMTs, as similarly reported in Wilson et al. (2013), it could be said that inclusion of MTC in the HPAM course contributed to students' awareness regarding hypotheses for students' learning. Concerning this, however, participants reflected on our training with presentation tools pointing out that a description of MTC and HLT through PowerPoint, showing them examples and classroom work on a lesson plan with their discussion, is not enough for making sense of the core ideas. This is in line with Simon (1995) and Wilson et al. (2013) which address that the continual change regarding MTC takes more time than expected. Participants also address that the lack of students' pre-knowledge is a barrier for them to formulate hypotheses for students' learning, considering new task situations and making a good assessment. All conclusions together imply that PMTs lacked a well-developed subject matter knowledge and pedagogical content knowledge needed for a meaningful HLT for students whose prior knowledge is unknown to them. Bearing in mind these conclusions, we plan to follow MTC as a heuristic tool in the forthcoming semesters and focus on the development of PMTs teaching designs step by step. Regarding assessment, it is not at the fore, to participants, due to time issues. We speculate that formative assessment techniques could be key and heuristic tools as integration of MTC and HLT notions.

### Acknowledgement

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# Whiteboards as a problem-solving tool

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*Previous research shows there is more discussion, participation and persistence when pupils work on vertical whiteboards. In this study, we investigate how neighbouring whiteboards support two pupils solving problems in mathematics. We analyse classroom observations using characteristics of "thinking classrooms" to explore the pupils' work on whiteboards, examining what they notice, what they use from other pupils' whiteboards and how they process this in their work. The findings indicate that whiteboards give pupils opportunities to a) discuss different elements of the task, b) support each other during the problem-solving process and c) work independently to find knowledge in interaction with classmates and classmates' work.*

*Keywords: Problem solving, vertical whiteboards, communication, thinking classroom*

## Introduction

Problem solving has long been considered an important element in both teaching and learning of mathematics. This has had an impact on mathematics curricula around the world where the aim is that pupils learn to solve problems and learn through solving problems (Liljedahl, Santos-Trigo, Malaspina, & Bruder, 2016). Problem solving is becoming a key part of teaching in Norwegian schools as it is one of the core elements in the new mathematics curriculum (Kunnskapsdepartementet, 2018).

The new mathematics curriculum will place demands on developing a classroom culture where communication will have an important focus. Pupils will interact verbally, and think creatively and productively together (Littleton & Mercer, 2013). Important elements of mathematical conversations include explaining, arguing and defending mathematical ideas (Walshaw & Anthony, 2008). These components may be closely linked to new core elements in mathematics, such as communication, reasoning, argumentation, abstraction and generalisation.

In the 1990s Wells started to use whiteboards in physics classes with the aim of strengthening and developing discussion among the pupils (Wells, Hestenes, & Swackhamer, 1995). Whiteboards, which had already been used in some mathematics courses at universities in Australia in the mid-1970s (Forrester, Sandison, & Denny, 2017), can support the development of understanding concepts, the use of multi-representations and the development of academic discussions in the classroom (Wenning, 2005). Reports on the use of whiteboards have so far referred to using them lying on the pupils' desks.

It appears that whiteboards had their renaissance in 2016. Liljedahl (2016) used vertical whiteboards to change a classroom culture from what he calls a non-thinking classroom into a thinking classroom. He finds that pupils get down to work quicker, have better persistence, are more eager and collaborate better when working with vertical whiteboards. Megowan-Romanowicz (2016) shows that

whiteboards make it possible to change the focus from finding the correct answer to creating meaning. Our study refers to pupils using vertical whiteboards.

Forrester et al. (2017) explore what pupils think about working on whiteboards and present research on how whiteboards are used. The pupils report that they have more motivation and a higher activity level in the subject, even if a few of them state that they are uncomfortable when their work is visible to all the others. When Liljedahl (2016) uses the vertical whiteboards to change a classroom into a thinking classroom, he rearranges the room. The pupils` working space shifts from sitting at their desks to standing and working on vertical whiteboards. By making this change, Liljedahl discovers that students work more perseveringly, they start working more quickly, they participate more actively in mathematical activities and there is greater mobility of knowledge in the classroom. The focus of this study is on how the pupils use the information from other whiteboards. We will look into what kind of information they use and how they use this information in their work. This leads to our research question: *How can (neighbouring) whiteboards contribute in pupils` mathematical problem-solving processes?*

This study is part of a larger research project where the aim has been to explore how pupils and teachers benefit from working on vertical whiteboards. The focus of this study is to examine how two pupils, in collaboration, use input from other pupils' whiteboards in their work. To answer our research question, we will analyse and discuss episodes from pupils' work on whiteboards, using characteristics of thinking classrooms. Pupil engagement is decisive in a thinking classroom, where proxies for engagement include: Time before pupils get started on their tasks, and time before the initial mathematical notation is made, pupil eagerness to get started, discussion, participation, persistence, non-linearity of pupil work and knowledge mobility between the groups (Liljedahl, 2016).

We have chosen to use four of the proxies in our analysis to help us understand the work process of pupils as a whole. The proxies we have excluded are related to minor parts of the work process. We use knowledge mobility, discussion, participation and persistence. Knowledge mobility means that pupils construe knowledge together, and this knowledge is dispersed amongst them in the classroom. It also refers to the interactions of pupils across groups. Participation and discussion refer to the degree to which the group members take part while working on the tasks and in the discussions, and how they discuss the task in their group and between groups. Persistence refers to how long pupils work with a focus on the task, how they work with challenges without giving up and how they try out new approaches if they get stuck (Liljedahl, 2016). Pupils show resilience and rarely quit out of boredom and frustration (Liljedahl, 2018). We use descriptions of the proxies for engagement to describe the students' problem-solving process.

## Methodology

The term whiteboard is used about different types of non-permanent writing surfaces in the research literature. In our study, whiteboards are large (approximately A2 size) static electric surfaces hanging on the wall with similar functions as other whiteboards. It is easy to write, erase and write over again. Due to their static-electricity property, they can be attached to walls and windows, and they can easily be moved.

**Data collection**

The data material for this study comprises video and audio recordings of one teaching session, lasting around 90 minutes. Pupils in Year 7 (12-13 years old), worked with the task *Talltårn* (Number towers) on whiteboards. We were responsible for the data collection. As we both are mathematics teachers, Kristin, the teacher, wanted us to participate in the teaching while collecting the data. We have in part been participants-as-observers in work with the task (Gold, 1958). The pupils were informed that we might ask them questions and have brief conversations with them. The video material was transcribed and together with the audio material recorded into a chart with columns for the time, what was happening, what was said and what was written.

The given task was selected in collaboration with the mathematics teacher, Kristin (all names are fictitious), and us. Kristin planned and carried out the teaching with input from us. The teacher and her class have thus been selected through purposeful sampling (Creswell & Poth, 2018). We wanted to examine pupils who were accustomed to cooperating on mathematical problem solving, and to collaborate with a teacher who wanted to develop her teaching practice. We also wanted to focus on pupils who were able to cooperate and verbalise how they were thinking, and who would not be too shy in front of the camera. As Kristin knew the pupils best, she chose two *focus pupils*, Mari and Johan, before the teaching session. According to Kristin, Mari and Johan had varying mathematical competence. Kristin picked randomly collaboration partners for all the pupils. Mari and Hanne formed one *focus pair* and Johan and Colin another.

Kristin presented the task orally, formed random collaboration pairs and the pupils started working with the task. After a while, Kristin stopped and asked them to go to their neighbouring group. The pupils were to look at how this group had worked, comparing solution strategies and solutions with their own work. They wrote similarities on a green slip and differences on a pink slip. After some minutes at the neighbouring whiteboard, the pupils returned to their whiteboard and continued with the task. This step in the work process differs from Liljedahl`s (2016) work. We included this step to encourage the pupils to use the neighbouring whiteboards. We monitored each our focus pair with a video camera and audio recorder throughout the lesson.

This paper explores one teaching session, the second in a series of five. The pupils have thus worked on whiteboards earlier, are familiar with the tool and know that while working they should compare what they are doing with what their neighbours are doing. The pupils are working with *Number towers*, a task with relatively many solutions and two main strategies.

**The task**

The task *Number towers* has been derived from mattelist.no (Norwegian Centre for Mathematics Education – Matematikksenteret, 2019). The task is introduced by starting with four whole positive numbers on the ground floor (zero cannot be used). Two and two neighbouring numbers are added, and the sum is placed immediately above the two numbers on the next floor. The same procedure is repeated until there is one number at the top (see Picture 1).



Picture 1: Number tower example
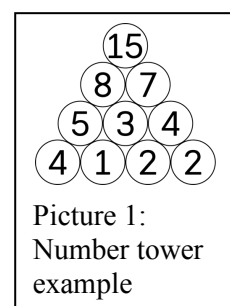
The task is to choose four start-numbers which will give the number 15 at the top. The questions the pupils must answer along the way are: "How many solutions can you find?" and "Can you find

systems that can help you in your work?". There are two main strategies for solving the task. The pupils may either choose four start-numbers to arrive at 15 or start with 15 at the top and work down. If pupils start with 15 at the top, they may make discoveries that can help them find additional solutions. They may find, for example, that they cannot have all possible subdivisions of 15 on the third floor, and that 1 can only be placed on the ground floor. Another discovery is that the number in the middle of the first floor is used twice to form the numbers on the second floor. For that reason, the highest number must be placed on the outermost side of the floor. Pupils who start from the top may also find that four start-numbers in different orders will yield different *number towers*.

Pupils starting on the ground floor will be more dependent on a trial and error strategy and must adjust their start numbers when they find numbers that do not yield 15 on top. They may make the same discoveries as those who start at the top, but these findings are not as obvious. Different premises for the task lead to several different solutions, such as that 4-2-1-2 and 2-2-1-4 may be considered two different sets of start numbers, or they may be viewed as the same four start-numbers and give one solution.

In this paper, we have chosen to study how pupils work with number towers because the task can be solved using two main strategies, the task has several solutions and the pupils have the opportunity to make different discoveries while working.

**The analysis**

To identify episodes in the data material where the pupils used their neighbouring whiteboards as tools we used open coding with the constant comparative analysis method (Postholm, 2019). In this section of the analysis we found two main categories relating to pupils` use of neighbouring whiteboards, either encouraged by the teacher or based on their own need. After completing this step of the analysis, we decided to focus on Colin and Johan's work on the task. The main reason for this choice was that the boys tried both strategies, one in the initial phase of their work, and the other after having studied their neighbours' whiteboard. The episodes during which the boys used the neighbouring whiteboard in some way, were analysed in more detail to interpret and describe how whiteboards supported the pupils' work on solving mathematical problems. We studied what the pupils talked about when looking at other whiteboards, what they saw and how they applied this in their subsequent work.

To increase the credibility of the analysis, we have watched the video material, transcribed, read, noted and commented on the material separately. Where we have disagreed, we have discussed together to arrive at a shared understanding of what takes place in the different episodes. We have examined the data material over and over, reflected and remained sceptical to our first impressions during the analysis process (Stake, 1995).

## Findings and discussion

We will now present excerpts from the dialogue between Colin and Johan, where we find the characteristics of thinking classrooms. We present our findings according to the three chosen proxies for engagement: participation and discussion, persistence and knowledge mobility (Liljedahl, 2016).

**Participation and discussion**

In the following sequence, the pupils study the whiteboard of the neighbouring group, and they are to note down what is similar between their work and the work of the neighbouring group on a green slip, and differences are to be written on a pink slip.

At the neighbouring whiteboard, the following takes place:

The boys write "They work from the top" on the pink slip and "they have found the same solutions as we have" on the green slip. Here is the subsequent conversation between the pupils and Ingunn, the teacher/researcher:

1 Ingunn: What do you think about what this group has done?

2 Johan: It looks right.

3 Colin: It's a bit special that they have started from the top.

4 Ingunn: Do you think that you could have used this strategy?

5 Johan: Yes, because then they'll certainly find what could become 15, what can become eight and what can become 7. It's pretty smart.

6 Colin: They take, like, they have like 15, which is the answer. They don't make any mistakes.

7 Ingunn: Could you look at the first and second floors, whether there is something they have found out?

8 Johan: They have used 7 and 8 quite a lot.

9 Colin: They have used 10, 0 was not allowed.

10 Johan: But that was on the ground floor. We have also used 10.

11 Ingunn: Do you think that they have found all the solutions with 7 and 8?

12 Johan: No. We have just as many correct ones as they have.

According to Liljedahl (2016), this excerpt from the work session shows that both boys take an equal part in the discussion and that they find support in the problem-solving process by looking at the whiteboard of their neighbouring group. Ingunn supports them in this process of investigating the neighbouring whiteboard (1), (4), (7), (11), inviting the boys to consider the work of the neighbouring group. The boys assess the strategy of starting from the top, and it appears that they consider this strategy with curiosity and scepticism. They also see that the strategy has a strong side to it; they cannot find any number tower which will not result in 15 on the top floor. Using this strategy, they will end up with four start numbers that will guarantee 15 at the top (5).

**Knowledge mobility**

We identify two different types of knowledge mobility in Colin and Johan's work (Liljedahl, 2016). One type comes about when the pupils are instructed by the teacher to study the whiteboard of their neighbouring group and then attempt to test the strategy this group has used:

1 Johan: Should we start with this, then (pointing to the pink slip)? Then we have to write 15. Should we try something we don't have (on the second floor), 3 and 12? No, 3 doesn't

work because 1 can't be there (points to the first floor). What is max on the second floor is 4 and 11. 1 can only be at the bottom, because 0 isn't allowed, and then you can't divide 1 (if it is on the first floor).

Johan replaces 3 and 12 with 4 and 11 on the second floor. On the floor below he enters 2-2-9.

2 Colin:   But we already have this (points to a solution on the whiteboard).

In this excerpt, Colin and Johan test the strategy of starting at the top of the number tower. They make some discoveries. They see that 3 and 12 cannot be on the second floor because that gives 1 on the first floor. Here it appears that the pupils are using information from the neighbouring whiteboard that they did not comment on while studying the neighbouring group's work. The neighbouring whiteboard said: "Cannot have 1 on the first floor".

The pupils use what they have seen but not discussed. Johan states (1) that 1 cannot be on the first floor. The boys do not dwell on this at all, and it may thus appear that they have noticed the information given on the neighbouring whiteboard, using it now when they see that it provides meaning. This information is important if one starts from the top of the number tower and works downward, but since the boys always have started from the bottom, they have had no need for this information. They have never had 1 on the first floor.

We see another form of knowledge mobility further into the work session. The pupils are struggling to make progress in their work. On their own initiative, they turn away from their whiteboard and look at other whiteboards in the classroom:

1 Johan:   I'm not sure whether there are more (solutions).

Colin turns to a whiteboard on the other side of the classroom.

2 Colin:   OMG, they have a lot of solutions, really, take a look. They have a lot!

3 Johan:   But the blue ones are wrong!

4 Colin:   Is it allowed just to switch the placement of the bottom numbers (addressing Ingunn)?

5 Ingunn:  Look how the number towers will look like then. Try it!

Colin tries an example.

In the excerpt above the boys are looking for confirmation that they have found all the solutions. Turning around, they look at the whiteboards of classmates and find that one of the groups has found many more solutions than they have. Discovering this, they start comparing the different solutions, seeing that they are not working according to the same premises as the other group. The other group has interpreted 4-2-1-2 and 2-2-1-4 as two different sets of start numbers, while Colin and Johan have deemed them to be the same start numbers (4).

We see that knowledge mobility helps the pupils to be more independent in the problem-solving process as they find knowledge in interaction with classmates and in the work of their classmates. If the boys had been working in their workbook, they might have asked the teacher whether they had finished instead of finding out for themselves. By studying, interpreting and discussing the work of other pupils, they acquire the opportunity to construe knowledge together (Liljedahl, 2016). They

look for tips, ideas, strategies and procedures that may be effective, and which may contribute to their problem-solving process.

**Persistence**

Throughout their work, Colin and Johan are on the brink of giving up or running out of ideas several times, but they keep finding new numbers they can try. When one of them states that he is giving up, the other one offers new ideas:

1 Colin:     Nothing, absolutely nothing. It doesn't work so well when we've come this far.

2 Johan:     What if we try 3 and 6 there now, (pointing to the ground floor of a number tower on the whiteboard). I'm not saying that it'll work. 3+1 is 4, 2-7, 6-9, 15. Like that!

Later in the session:

3 Johan:     I don't know. I'm really out of ideas.

4 Colin:     What if we start at the top, and then do it with seven?

5 Johan:     Let's try it. This is the easiest one.

Towards the end of the work session:
The pupils carry on working for some time using their initial strategy. After around ten minutes' work they are becoming uncertain as to whether there are more solutions. Colin turns towards a whiteboard on the opposite side of the classroom.

6 Colin:     OMG, they have a lot of solutions, really, take a look. They have a lot!

According to Liljedahl (2018), Colin and Johan do not quit even though they are frustrated. They help each other to remain in the problem-solving process by offering several challenges (2), (4), and they use their neighbouring whiteboards as inspiration and guides in terms of whether they can claim to have finished the work (6). These findings correspond to Liljedahl's (2016) description of persistence. The pupils work on the task for 45 minutes.

## Concluding remarks

The analysis of the work performed by Colin and Johan indicates that whiteboards contribute to their problem-solving process in different ways. They use the neighbouring whiteboards as inspiration; they find ideas and compare the number of solutions with their findings. In line with Liljedahl (2016), we find that whiteboards support the problem-solving process by opening for knowledge mobility.

The boys are highly active and participate in their problem-solving process. Ingunn supports them in using the neighbouring whiteboard by asking questions to help them specify what they are looking for. The boys discuss the strengths and weaknesses of various strategies and premises for the task, show persistence throughout the work session, do not give up and try again if they get stuck. According to Liljedahl (2016) and Megowan-Romanowicz (2016), Colin and Johan used their classmates' work as a guide for their problem-solving process. These findings indicate that whiteboards may support a teacher's work with problem solving and communication in mathematics classrooms as also found by Wells et al. (1995).

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