3 On Octahedrality and Müntz spaces

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Abstract

We show that every Müntz space can be written as a direct sum of Banach spaces X and Y, where Y is almost isometric to a subspace of c and X is finite dimensional. We apply this to show that no Müntz space is locally octahedral or almost square.

3.1 Introduction

Denote the closed unit ball, the unit sphere, and the dual space of a Banach space X by B_X, S_X, and X^* respectively. Let $\Lambda = (\lambda_i)_{i=0}^{\infty}$, with $\lambda_0 = 0$, be a strictly increasing sequence of non-negative real numbers and let $\Pi(\Lambda) := \text{span}(t^{\lambda_i})_{i=0}^{\infty} \subseteq C[0,1]$, where $C[0,1]$ is the space of real valued continuous functions on $[0,1]$ endowed with the canonical sup-norm $\|\cdot\|_\infty$. We will call $\Lambda = (\lambda_i)_{i=0}^{\infty}$ a *Müntz sequence* and $M(\Lambda) := \overline{\Pi(\Lambda)}$ a *Müntz space* if $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. This terminology is justified by Müntz famous theorem from 1914, which says that $\Pi(\Lambda)$ is dense in $C[0,1]$ *if and only if* $\lambda_0 = 0$ and $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$.

It is known that a Müntz space $M(\Lambda)$ is isomorphic to a subspace of c_0, provided that the Müntz sequence satisfies the gap condition, i.e. $\inf_{k\in\mathbb{N}}(\lambda_{k+1} - \lambda_k) > 0$ ([GL05, Theorem 9.1.6(c)]). In Section 3.2 we show that all Müntz spaces embed isomorphically into c_0. This is done by showing that $M(\Lambda)$ can be written as a direct sum $X \oplus Y$ where Y is almost isometric to a subspace of c and X is finite dimensional.

Definition 3.1.1. Let X be a Banach space. Then X is

(i) *locally octahedral* (LOH) if for every $x \in S_X$ and $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x \pm y\| > 2 - \varepsilon$.

(ii) *octahedral* (OH) if for every $x_1, \dots, x_n \in S_X$ and $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x_i \pm y\| > 2 - \varepsilon$ for all $i \in \{1, \dots, n\}$.

In Section 3.3 we will show that no Müntz space is OH, answering the question posed in [ALMN17] whether Müntz spaces can be OH. A partial negative answer was given in [ALMN17, Remark 2.9] for Müntz spaces with Müntz sequences consisting only of integers, by combining the Clarkson-Erdös-Schwartz Theorem (see [GL05, Theorem 6.2.3]) with a result of Wojtaszczyk (see [Wern00, Theorem 1]).

Definition 3.1.2. Let X be a Banach space. Then X is

(i) *locally almost square* (LASQ) if for every $x \in S_X$ there exists a sequence $(y_n)_{n=1}^{\infty}$ in B_X such that $\|x \pm y_n\| \to 1$ and $\|y_n\| \to 1$.

(ii) *almost square* (ASQ) if for every $x_1, \dots, x_k \in S_X$ there exists a sequence $(y_n)_{n=1}^{\infty}$ in B_X such that $\|y_n\| \to 1$ and $\|x_i \pm y_n\| \to 1$ for every $i \in \{1, \dots, k\}$.

Both ASQ and OH are closely related to the area of diameter two properties, which has received intensive attention in the recent years (see for example [BGLPRZ16] and [HLN18] and the references therein). Trivially ASQ implies LASQ and OH implies LOH.

The area of diameter two properties concerns slices of the unit ball, i.e. subsets of the unit ball of the form

$$S(x^*, \varepsilon) := \{x \in B_X : \ x^*(x) > 1 - \varepsilon\},$$

where $x^* \in S_{X^*}$ and $\varepsilon > 0$. Müntz spaces and their diameter two properties were studied in [ALMN17]. Haller, Langemets, Lima and Nadel [HLLN18] pointed out that the proof of [ALMN17, Theorem 2.5] actually shows that, in any $M(\Lambda)$ we have that for every finite family $(S_i)_{i=1}^{n}$ of slices of $B_{M(\Lambda)}$ and $\varepsilon > 0$, there exist $x_i \in S_i$ and $y \in B_{M(\Lambda)}$, independent of i, such that $x_i \pm y \in S_i$ for every $i \in \{1, \dots, n\}$ and $\|y\| > 1 - \varepsilon$. This property is formally known as *the symmetric strong diameter two property* (SSD2P).

It is known that if a Banach space is ASQ, then it also has the SSD2P. In fact, ASQ is strictly stronger than SSD2P (see [HLLN18, Theorem 2.1d and Example 2.2]). A natural question is therefore if a Müntz space can be ASQ. The results developed in this article will be used to show that this is never the case.

Note that we can exclude the constants and consider the subspace $M_0(\Lambda) := \overline{\text{span}}(t^{\lambda_n})_{n=1}^{\infty}$ of $M(\Lambda)$ and the results of the article still hold true, unless explicitly stated.

We use standard Banach space terminology and notation (e.g. [AK06]), in addition the notation $\|f\|_{[0,a]} := \sup_{x\in[0,a]} |f(x)|$ will be used throughout the paper.

3.2 On embeddings of Müntz spaces

The main results of this article relies on the following results.

Theorem 3.2.1 (Bounded Bernstein's inequality [BE97, Theorem 3.2]). *Assume that $1 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ and $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$, then for every $\varepsilon > 0$ there is a constant c_ε such that*

$$\|p'\|_{[0,1-\varepsilon]} \leq c_\varepsilon \|p\|_{[0,1]},$$

for all $p \in \Pi(\Lambda)$.

Lemma 3.2.2. *Let V be a subspace of $C[0,1]$ such that each $f \in V$ is differentiable. If for every $\varepsilon > 0$ there exists a $K_\varepsilon \in \mathbb{N}$ such that*

$$\|f'\|_{[0,1-\varepsilon]} \leq K_\varepsilon \|f\|_\infty \tag{3.1}$$

for all $f \in V$, then the Banach space $\overline{V}$ embeds almost isometrically into c.

The proof of Lemma 3.2.2 is almost identical to the proof of [Wern00, Theorem 2], however, we do not require V to be closed, but instead require the inequality (3.1).

Proof. Let $\varepsilon > 0$ and choose a sequence $0 = a_0 < a_1 < \cdots < a_i < \cdots < 1$ converging to 1. For each $a_i \in (0,1)$ there exists $K_i > 0$, depending on a_i such that

$$\|f'\|_{[0,a_i]} \leq K_i \|f\|_\infty \text{ for all } f \in V$$

Pick points $0 = s_0 < s_1 < \cdots < s_{n_1} = a_1 < s_{n_1+1} < \cdots < s_{n_2} = a_2 < \cdots$, in such a way that

$$s_{j+1} - s_j \leq \frac{\varepsilon}{K_{i+1}} \text{ for } n_i \leq j < n_{i+1}.$$

Define the operator $J_\varepsilon : \overline{V} \to c$ by $J_\varepsilon(f) = (f(s_n))_n$, thus J_ε is well-defined by continuity of $f \in \overline{V}$. As $||J_\varepsilon f|| = \sup_{n\in\mathbb{N}} |f(s_n)| \leq ||f||_\infty$, for all $f \in \overline{V}$, we have that $\|J_\varepsilon\| \leq 1$. For any $f \in \overline{V}$ let (f_k) be a sequence in V converging uniformly to

f. Let $\delta > 0$ and find $N \in \mathbb{N}$ such that $\|f - f_N\|_\infty < \delta$. Then, for any $s \in [0, 1)$, we have $a_i \le s < a_{i+1}$ for some $i \in \mathbb{N}$. Let $s_m \in [a_i, a_{i+1}]$ be such that $|s - s_m| \le \frac{\varepsilon}{K_{i+1}}$. Then

$$\begin{aligned}
|f(s)| \le |f_N(s)| + \delta &\le |f_N(s) - f_N(s_m)| + |f_N(s_m)| + \delta \\
&\le \sup_{a_i \le t \le a_{i+1}} |f_N'(t)||s - s_m| + \|J_\varepsilon f_N\| + \delta \\
&\le \|f_N\|_\infty K_{i+1} \frac{\varepsilon}{K_{i+1}} + \|J_\varepsilon f_N\| + \delta \\
&\le \|f_N\|_\infty \varepsilon + \|J_\varepsilon f_N\| + \delta \\
&\le (\|f\|_\infty + \delta)\varepsilon + (\|J_\varepsilon f\| + \delta) + \delta
\end{aligned}$$

and therefore

$$(1 - \varepsilon)\|f\|_\infty - \delta(\varepsilon + 2) \le \|J_\varepsilon f\|.$$

Since δ was arbitrary we conclude that

$$(1 - \varepsilon)\|f\|_\infty \le \|J_\varepsilon f\| \le \|f\|_\infty,$$

completing the proof. □

Combining Theorem 3.2.1 and Lemma 3.2.2, we arrive at the following proposition.

Proposition 3.2.3. *Let Λ be a Müntz sequence with $\lambda_1 \ge 1$. Then the associated Müntz space $M(\Lambda)$ is almost isometric to a subset of c. That is, for every $\varepsilon > 0$ there is an operator $J_\varepsilon : M(\Lambda) \to c$ such that*

$$(1 - \varepsilon)\|f\|_{[0,1]} \le \|J_\varepsilon f\| \le \|f\|_{[0,1]}.$$

We will need the following lemma for the coming theorem.

Lemma 3.2.4. *Let $Z = \overline{\text{span}}(z_i)_{i\in\mathbb{N}}$ and let $N \in \mathbb{N}$. If $Y = \overline{\text{span}}(z_i)_{i>N}$ then $Z/Y = \text{span}(\pi(z_i))_{i\le N}$, where $\pi : Z \to Z/Y$ is the quotient map. Consequently Z/Y has finite dimension and $Z = X \oplus Y$ where $X = \text{span}(x_i)_{i\le N}$.*

Remark 3.2.5. For every $N \in \mathbb{N}$ we have that $\overline{\text{span}}(t^{\lambda_i})_{i\ge N}$ is a finite codimensional subspace of $M(\Lambda)$.

By combining Proposition 3.2.3 and Lemma 3.2.4 we obtain the following result.

Theorem 3.2.6. *Every Müntz space $M(\Lambda)$ can be written as $X \oplus Y$ where X is finite dimensional and Y is almost isometric to a subspace of c.*

Corollary 3.2.7. *Every Müntz space $M(\Lambda)$ embeds isomorphically into c_0.*

Remark 3.2.8. From [GL05, Theorem 10.4.4] it is known that no Müntz space of dimension greater than 2 is polyhedral. However, since c_0 is polyhedral ([Klee60, Theorem 4.7]), it follows that any Müntz space can be renormed to be polyhedral.

3.3 On octahedrality and almost squareness of Müntz spaces

The results from Section 3.2 will now be used to derive some results concerning Müntz spaces. $M(\Lambda)^*$ is separable by Corollary 3.2.7, we therefore easily answer the question posed in [ALMN17]. In fact we show more.

Theorem 3.3.1. *No Müntz space $M(\Lambda)$ is LOH.*

Proof. Since $M(\Lambda)^*$ is separable, we can combine [Bour83, Theorem 4.1.3] with [Bour83, Theorem 4.2.13] to see that there exist slices $S(x, \varepsilon)$ of the unit ball of $M(\Lambda)^*$ of arbitrarily small diameter, where x can be taken from $M(\Lambda)$. By [HLP15, Theorem 3.1] this is equivalent to $M(\Lambda)$ failing to be LOH, as claimed. ☐

We finish this article by showing that $M_0(\Lambda)$ fails to be ASQ for any Müntz sequence Λ. Note that $M(\Lambda)$ is trivially not LASQ, just consider the constant function 1. First we show that even more is true for some spaces $M_0(\Lambda)$.

Proposition 3.3.2. *No Müntz space $M_0(\Lambda)$ with $\lambda_1 \geq 1$ is LASQ.*

Proof. Let Λ be a Müntz sequence with $\lambda_1 \geq 1$ and $M_0(\Lambda)$ be the associated Müntz space. Choose some $x \in (0, 1)$. By Theorem 3.2.1 there is a $c \in \mathbb{N}$ such that $\|f'\|_{[0,x]} \leq c$ for all $f \in B_{\Pi(\Lambda)}$. Let $a = \min(\frac{1}{2c}, x)$ and observe that

$$\sup_{f \in B_{\Pi(\Lambda)}} \|f\|_{[0,a]} \leq \frac{1}{2}$$

since

$$|f(t)| = |f(t) - f(0)| \leq \|f'\|_{[0,a]} \cdot |t - 0| \leq c \cdot \frac{1}{2c} = \frac{1}{2}.$$

Recall from [ALV16, Theorem 2.1] that $M_0(\Lambda)$ is LASQ if and only if for every $g \in S_{M(\Lambda)}$ and $\varepsilon > 0$ there exists $h \in S_{M(\Lambda)}$ such that $\|g \pm h\| \leq 1 + \varepsilon$. We claim that no such h exists for $g = t^{\lambda_1}$. Indeed, if $0 < \varepsilon < a^{\lambda_1}/2$ and $h \in S_{\Pi(\Lambda)}$ is such that

$\|t^{\lambda_1} \pm h\| \leq 1+\varepsilon$, then $|h(t)| < 1-\varepsilon$ for $t \geq a$ as $t^{\lambda_1} > 2\varepsilon$ for $t \geq a$. Thus, h must attain its norm on the interval $[0, a]$, contradicting our observation. As $\Pi((\lambda_n)_{n=1}^{\infty})$ is dense in $M_0(\Lambda)$, we conclude that $M_0(\Lambda)$ is not LASQ. □

Proposition 3.3.3. *No Müntz space $M_0(\Lambda)$ is ASQ.*

Proof. Combining Lemma 3.2.4 with Proposition 3.3.2 shows that every Müntz space $M_0(\Lambda)$ has a subspace of finite codimension which is not ASQ. By [Abra15, Theorem 3.6] no Müntz space can be ASQ. □

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