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Short Communication

Optimal nonlinear damping control of second-order systems

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Abstract

Novel nonlinear damping control is proposed for the second-order systems. The proportional output feedback is combined with the damping term which is quadratic to the output derivative and inverse to the set-point distance. The global asymptotic stability, passivity property, and convergence time and accuracy are demonstrated. Also the control saturation case is explicitly analyzed. The suggested nonlinear damping is denoted as optimal since requiring no additional design parameters and ensuring a fast convergence, without transient overshoots for a non-saturated and one transient overshoot for a saturated control configuration.

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1. Introduction

For the second-order systems, it is understood that a linear feedback control [1] inherently poses certain limits in terms of possibility to shape the transient response, exponential convergence of the state trajectories and, as implication, steady-state accuracy of the controlled output of interest. Worth to recall is that the input-output second-order systems encompass a vast number of practical applications. Input voltage to output speed in motors, transfer characteristics of different-type RLC circuits, pressure-flow dynamics in the fluid transport

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systems and, finally, motion dynamics of rigid-body systems, in general sense, can be noted as motivating examples for that.

For linear control systems, an assignment of optimal damping, so as to shape the desired dynamic response, is straightforward through for instance pole placement, cf. e.g. with [1]. Also when allowing for a system damping to be switched once as a function of the system state, an optimal damping ratio for linear second-order systems has been proposed in the past [2]. On the other hand, nonlinear control methodology addressed, since long, the problem of an efficient feedback shaping, while the complexity of associated analysis and control synthesis, availability of the system states, control specification, and type of the system perturbations led to quite different design concepts. Among the well-established are the sliding mode control [3], Lyapunov redesign [4], backstepping [5], and passivity based control [6]. For more details and well-known basics we also refer to seminal literature e.g. [7,8]. Some former examples of the nonlinear feedback stabilization and associated nonlinear damping can be found in e.g. [9,10] to mention few here. A comparative evaluation of different controllers, benchmarked on a most simple second-order plant of double integrator, can also be found in [11].

The need to incorporate nonlinear damping in feedback of the second-order systems, especially for improving the stabilizing and convergence properties, has been (empirically) recognized in already former studies in robotics, thus resulting in e.g. nonlinear proportional-derivative controls [12,13]. While the stability proof has been provided for several ad-hoc nonlinear damping strategies, no optimal convergence and trajectories shaping have been so far elaborated. Here it is also worth to side note that the convergence properties are strongly related to homogeneity of the corresponding dynamics vector-field and, as implication, of the feedback map to be determined, in other words to be assigned. For overview on the use of homogeneity for synthesis in, e.g. sliding mode control, we exemplary refer to e.g. [14]. As another approach, to feedback control problems, it appears that to enter energy into a system, through potential field of the output feedback, is more straightforward than to control its dissipation. The latter should occur in a peculiar way, thus ensuring the desired convergence to an equilibrium. For energy shaping in the feedback regulated Euler–Lagrange systems we refer to e.g. [15,16] and some basic literature [6].

In this paper, we propose a novel nonlinear damping control of the second-order systems in combination with the linear output feedback. Using the fact of conservative energies exchange in an undamped (oscillatory) second-order system, the dissipated energy is shaped in an optimal way with respect to the convergence to zero equilibrium and no transient overshoot independent of the initial state. That way assigned nonlinear damping is quadratic to the output derivative and inverse to the set-point distance, while no free design parameters for the damping term are required. The proposed control is generic and globally asymptotically stable. It also allows for control saturation, that is relevant for applications. The principle analysis of the control behavior, provided below, is focusing on the unperturbed second-order dynamics only. In that was the further aspects of sensitivity and robustness are subject to the future works.

2. Second order system with state-feedback

Throughout the paper we will deal with the feedback controlled second-order systems

$$\dot{x}_1 = x_2, \tag{1}$$

$$\dot{x}_2 = -kx_1 - D, \tag{2}$$

where x_1 and x_2 are the available state variables, $k > 0$ is the proportional feedback gain, and D is the control damping of interest, correspondingly to be shaped. Obviously, the system (1), (2) is a classical double-integrator dynamics, for which a vast number of application examples can be found e.g. in electrical and mechanical systems and combinations of those.

2.1. Optimal linear damping

Using the linear state-feedback damping, the system (1), (2) can be written in a standard form

$$[\dot{x}_1, \dot{x}_2]^T = A \cdot [x_1, x_2]^T = \begin{bmatrix} 0 & 1 \\ -k & -d \end{bmatrix} \cdot [x_1, x_2]^T, \tag{3}$$

where the system matrix A is Hurwitz, for positive damping coefficients $d > 0$, and is already in the controllable canonical form. It is worth recalling that the state-feedback controlled system (3) is equivalent to the proportional derivative (PD) controller for which an appropriate choice of the feedback gains allow for arbitrary shaping the closed-loop response, either in time t - or in Laplace s -domain. Assuming that k is given (by some control specification) and requiring the control response has no transient oscillations or overshoot, meaning the real poles only, one can assign the linear damping term by solving

$$s^2 + ds + k = (s + \lambda)^2 \tag{4}$$

with respect to d . Here the real double-pole at $-\lambda$ determines the optimal linear damping, usually noted as critical damping, since for $d > 2\lambda$ the system behaves as overdamped, while for $d < 2\lambda$ the system becomes transient oscillating. For any non-zero initial conditions $[x_1, x_2]^T(0) \equiv [x_1^0, x_2^0]^T \neq 0$, which can be seen as a set-value control problem, the trajectories are given by

$$[x_1, x_2]^T(t) = \exp(At) [x_1^0, x_2^0]^T. \tag{5}$$

It is obvious that the unperturbed matrix differential equation (3), with two stable real poles, has an exponential convergence property, meaning

$$\|x_1(t), x_2(t)\| \leq \beta \exp(-\gamma t) \tag{6}$$

for some $\beta, \gamma > 0$ constants. From the output control viewpoint that means $x_1 \rightarrow 0$ for $t \rightarrow \infty$.

3. Main results

3.1. Optimal nonlinear damping

The proposed nonlinear damping endows the system (1), (2) to be

$$\dot{x}_1 = x_2, \tag{7}$$

$$\dot{x}_2 = -kx_1 - x_2^2|x_1|^{-1}\text{sign}(x_2). \tag{8}$$

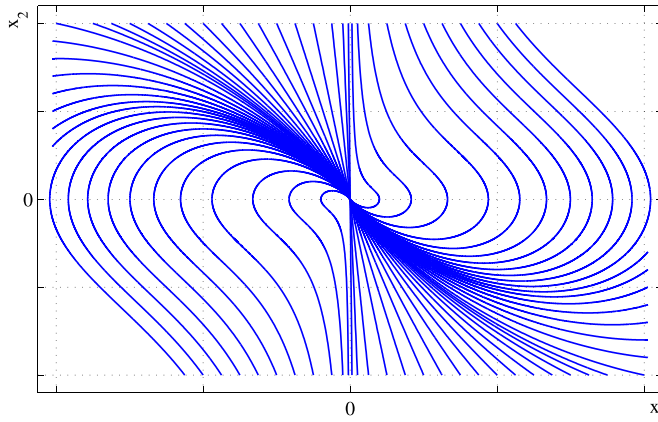


Fig. 1. Phase portrait of the control system (7), (8).

The single control parameter remains the given output feedback gain, while the quadratic damping term yields optimal for all $k > 0$ values. The solution of Eq. (8) is non-singular except in $x_1 = 0$, while the unique equilibrium $(x_1, x_2) = 0$ is globally attractive as will be shown below in Section 3.2. The phase portrait of the system (7), (8) is shown in Fig. 1.

One can recognize that the damping rate, and the required control effort, which is $\sim \dot{x}_2$, notably increases in vicinity of $x_1 = 0$ for $|x_2| \gg 0$. At the same time, the non-singular solution provides the global convergence to origin within the II and IV quadrants without x_1 zero crossing, thus without transient overshoot of the control response. For showing this, consider the region of attraction in vicinity of the origin. For the steady-state, one obtains

$$\mathbf{0} = \begin{bmatrix} 0 & 1 \\ -k & -|x_2||x_1|^{-1} \end{bmatrix} \cdot [x_1, x_2]^T, \tag{9}$$

which results in

$$k|x_1|x_1 = -|x_2|x_2. \tag{10}$$

This can be seen as a trajectories’ attractor in vicinity of zero equilibrium. Rewriting Eq. (10) as

$$kx_1^2 \text{sign}(x_1) = -x_2^2 \text{sign}(x_2) \tag{11}$$

and allowing for the real solution only, results in

$$x_2 + \sqrt{k}x_1 = 0, \tag{12}$$

which is a slope along which the trajectories converge to zero in vicinity of origin, that without crossing the x_2 -axis.

3.2. Global asymptotic stability

Assume the following Lyapunov function candidate

$$V = \frac{1}{2}x_2^2 + k\frac{1}{2}x_1^2, \tag{13}$$

which is positive definite for all $(x_1, x_2) \neq 0$ and also radially unbounded, i.e. $V(x_1, x_2) \rightarrow \infty$ as $\|x_1, x_2\| \rightarrow \infty$. Taking the time derivative and substituting dynamics of the states, i.e. Eqs. (7) and (8), results in

$$\dot{V} = -x_2^3|x_1|^{-1}\text{sign}(x_2) \leq 0, \tag{14}$$

which implies the origin is globally stable. Since the trajectories do not remain on the x_1 -axis when $x_2 = 0$, due to non-zero vector field cf. Eq. (8), and proceed towards origin, cf. Fig. 1, the asymptotic stability of origin can be concluded despite $\dot{V} = 0$ for $x_2 = 0, x_1 \neq 0$. This fact excludes the appearance of invariant sets and ensures $(0; 0)$ is the single asymptotically stable equilibrium.

For addressing the system stability within singularity $x_1 = 0$, we first consider the $x_2 \neq 0$ case and, therefore, potential zero-crossing of the x_1 -axis. It can be shown that when approaching $x_1 = 0$ from the left in the II-nd or from the right in the IV-th quadrant, cf. Fig. 1, the trajectory is always proceeding (asymptotically) to the origin, thus never crossing the x_1 -axis. Equally, when approaching $x_1 = 0$ from the right in the I-st or from the left in the III-rd quadrant, the trajectory is always repulsed away from $x_1 = 0$ and, thus, never crossing the x_1 -axis as well. For a trajectory reaches $(x_1, x_2) = 0$, one can substitute the attractor (12) into (8), correspondingly (14). This results in cancelation of the $|x_1|^{-1}$ term and eliminates a not well defined solution in the origin.

3.3. Closed-loop passivity

For analyzing damping properties of the control system (7), (8) we are to demonstrate the passivity of the closed-loop dynamics

$$\dot{x}_2 + kx_1 = -x_2^2|x_1|^{-1}\text{sign}(x_2). \tag{15}$$

Here the left-hand side can be seen as a conservative (oscillatory) system part, in other words plant, and the right-hand side of Eq. (15) as a stabilizing control input u which provides the closed-loop system with a required damping. Recall that for a system with output y to be passive, the input-output port power should be greater than or equal to the rate of energy stored in the system self, i.e. $uy \geq \dot{V}$. Here the same energy function as the Lyapunov function candidate (13), which is the system’s Hamiltonian, is assumed while x_1 is the controlled system output of interest. The above power inequality (for system passivity) yields

$$-x_2^2 \text{sign}(x_2)\text{sign}(x_1) \geq -x_2^2|x_2||x_1|^{-1}, \tag{16}$$

which results in the following passivity condition

$$\frac{|x_2|}{|x_1|} \geq \text{sign}(x_2) \text{sign}(x_1) \tag{17}$$

for the state-space. Based on that it is evident that the system is always passive in the II and IV quadrants of the phase plane, see Fig. 2. Otherwise, the system becomes transiently non-passive for $x_2 - x_1 < 0$ in the I quadrant and for $x_2 - x_1 > 0$ in the III quadrant (gray-shadowed in Fig. 2). In those non-passive segments, the level of energy stored in the system increases, this way also ensuring the state trajectories always cross x_1 -axis and do not remain at $x_2 = 0$. Following to that, the trajectories always change, upon the velocity zero crossing, to the passive segments of II or IV quadrant, which both act as a control attractor to the globally stable origin.

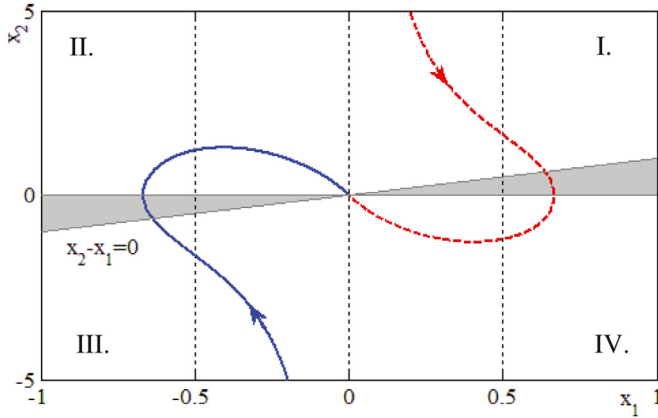


Fig. 2. Regions of system passivity in the phase plane.

3.4. Convergence time

The asymptotic convergence of the state solutions is ensured by $\dot{V} < 0$. On the other hand, in order to ensure a finite-time convergence, one has to show that

$$\dot{V} + \alpha V^{\frac{1}{2}} \leq 0 \tag{18}$$

for some positive time constant $\alpha > 0$. If inequality (18) holds, the finite convergence time t_c is bounded by

$$t_c \leq 2V^{\frac{1}{2}}(0)\alpha^{-1}. \tag{19}$$

Substituting the Lyapunov function candidate (13) and its time derivative Eq. (14) into Eq. (18) results in

$$\frac{x_2^3}{|x_1|} \text{sign}(x_2) \geq \alpha \frac{\sqrt{2}}{2} \sqrt{x_2^2 + kx_1^2}. \tag{20}$$

An explanatory graphical interpretation of inequality (20) is shown in Fig. 3 by two surfaces, one of the energy level and another of its time derivative.

One can recognize that the finite-time convergence can be ensured in vicinity of $x_1 = 0$ and that until certain neighborhood to the origin only (cf. both both red horns above the green cone in Fig. 3). Outside of those regions the inequality (20) becomes violated, cf. Fig. 3, and the control system (7), (8) features the asymptotic convergence. Here it is worth emphasizing that, from the applications' viewpoint, such partial finite-time convergence can be desired and sufficient, since the convergence to absolute zero is inherently restricted by some finite resolution of sensors used in the feedback control.

3.5. Control with saturation

From the applications' viewpoint, where frequently the input limitations have to be taken into account, the control value v with saturation is essential. That means it is to show whether the proposed nonlinear damping control system (7), (8) remains further on performing and,

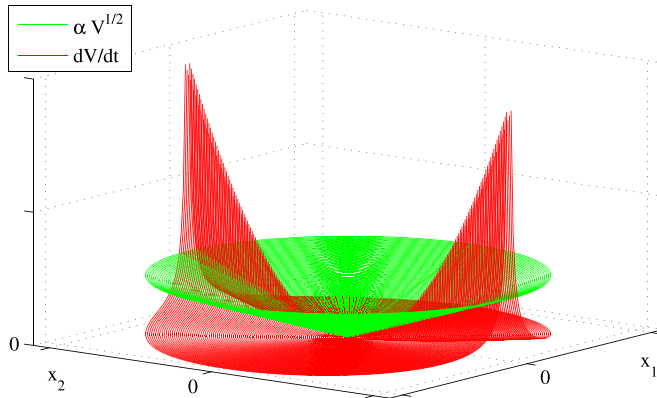


Fig. 3. Surface of time derivative of the Lyapunov function \dot{V} (in red, dark) and square root of the Lyapunov function $\alpha V^{1/2}$ (in green, light). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

above all, globally stable when the overall control input

$$-S \leq v = -kx_1 - x_2^2|x_1|^{-1}\text{sign}(x_2) \leq +S \tag{21}$$

is limited in the amplitude by some positive S . The latter constitutes an inherent control system constraint, correspondingly a given fixed parameter. In the saturated control mode, the system (7), (8) evolves to

$$\dot{x}_1 = x_2, \tag{22}$$

$$\dot{x}_2 = S \text{sign}(v), \tag{23}$$

and it has to be proven whether the control value returns to $|v(t)| < S$ after transients and, therefore, to the nominal dynamic behavior independent of the initial conditions. In this case one one needs to demonstrate that

$$\left| -kx_1 - x_2^2|x_1|^{-1}\text{sign}(x_2) \right| < S \tag{24}$$

can be achieved and will hold for some finite time $t > 0$ for any initial state $[x_1, x_2]^T(0) = [X_1, X_2]^T$. Due to symmetry of the control system and without loss of generality we focus, in the following, on the positive saturation only, while the respective developments for a negative saturation are equivalent when turning the sign and flipping the following inequality. The positive control saturation requires to prove

$$-kx_1 - x_2^2|x_1|^{-1}\text{sign}(x_2) < S, \tag{25}$$

while the saturated control action $\dot{x}_2 = S$ yields an explicit solution of the state trajectories

$$x_2(t) = X_2 + St, \tag{26}$$

$$x_1(t) = X_1 + \frac{1}{2} St^2. \tag{27}$$

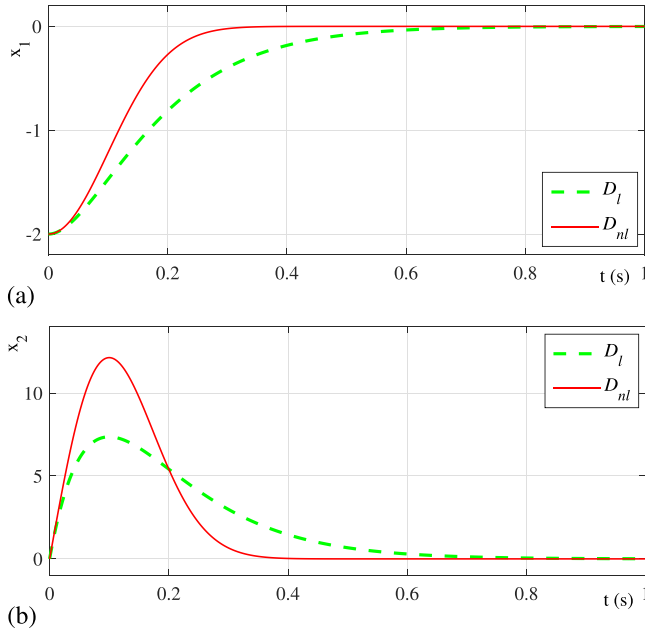


Fig. 4. Convergence of linear and nonlinear damping control.

Substituting Eqs. (26), (27) into Eq. (25) results in

$$-k\left(X_1 + \frac{1}{2}St^2\right) - \frac{(X_2 + St)^2 \operatorname{sign}(X_2 + St)}{\left|X_1 + \frac{1}{2}St^2\right|} < S. \tag{28}$$

While the second left-hand side term of Eq. (28) remains always positive for $t > -X_2S^{-1}$, the brackets of the first term remains also always positive for $t^2 > -2X_1S^{-1}$. This implies that there is a $\tau > 0$ so that the condition (28) holds for all $t > \tau$. This proves the closed-loop control system (22), (23) always returns to a non saturated control mode, i.e. Eqs. (7) and(8), at some $0 < t = \tau < \infty$, and that for all admissible $\{X_1, X_2\}$ initial states and admissible control parameters $S, k > 0$.

4. Comparative numerical study

Two feedback control systems described by Eqs. (1) and (2) are compared: one with the linear damping $D_l = dx_2$ and one with the proposed nonlinear damping $D_{nl} = x_2^2|x_1|^{-1}\operatorname{sign}(x_2)$. The convergence of the state trajectories is comparatively shown in Fig. 4 for the initial values $[x_1^0, x_2^0] = (1, 0)$ and output feedback gain assigned to $k = 100$. The optimal (critical) linear damping factor, cf. (4), is $d = 20$.

As next, the convergence of the controlled output (absolute value) is shown logarithmically in Fig. 5 for both the linear and nonlinear damping. It can be seen that the control with nonlinear damping reaches much faster, in fact quadratically on the logarithmic scale, some low bound of the steady-state accuracy. Different, the control with linear damping converges linearly on the logarithmic scale.

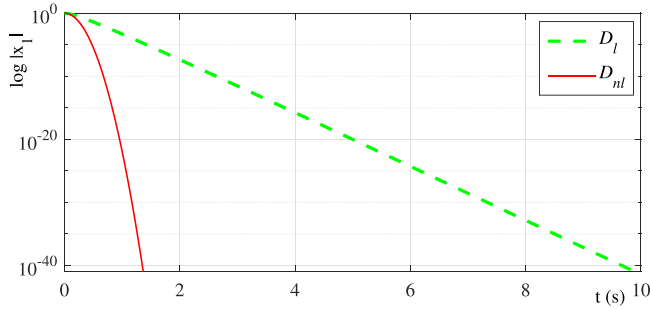


Fig. 5. Logarithmic convergence of the output magnitude value.

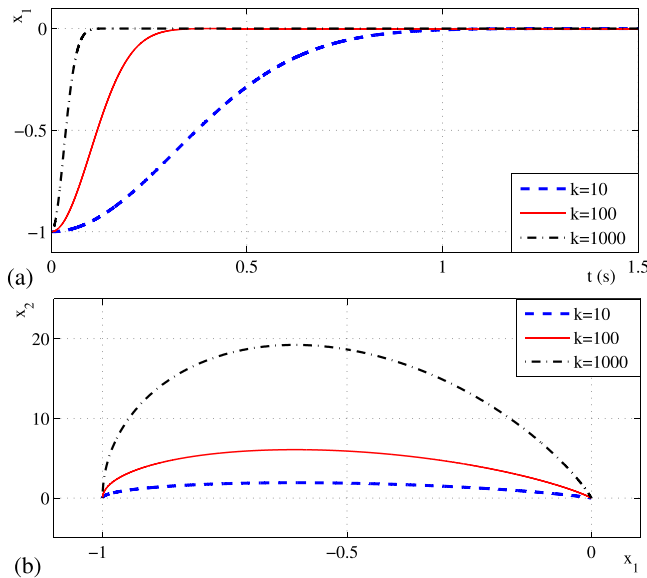


Fig. 6. Output convergence (a) and trajectories phase portrait (b) of the nonlinear damping control with various k parameter.

The output convergence and state trajectories of the nonlinear damping control are shown in Fig. 6 when assuming the varying (by order) values of the feedback gain $k = \{10, 100, 1000\}$. One can recognize a similar (scaled) trajectory shape independent of the control gain value.

Finally, the impact of the control saturation, i.e. of the bounded v -input cf. Section 3.5, is demonstrated for the different feedback gain values $k = \{50, 100, 150, 200\}$ and $S = 25$; for the largest gain $k = 200$ the non-saturated (n.s.) case, i.e. $S = \infty$, is also included for the sake of comparison. The control response and state trajectories are shown in Fig. 7(a) and (b) respectively. The saturation slows down the convergence and leads, in worst case of largest gain, to a single transient overshoot, after which the trajectory converges as expected (cf. with Fig. 1).

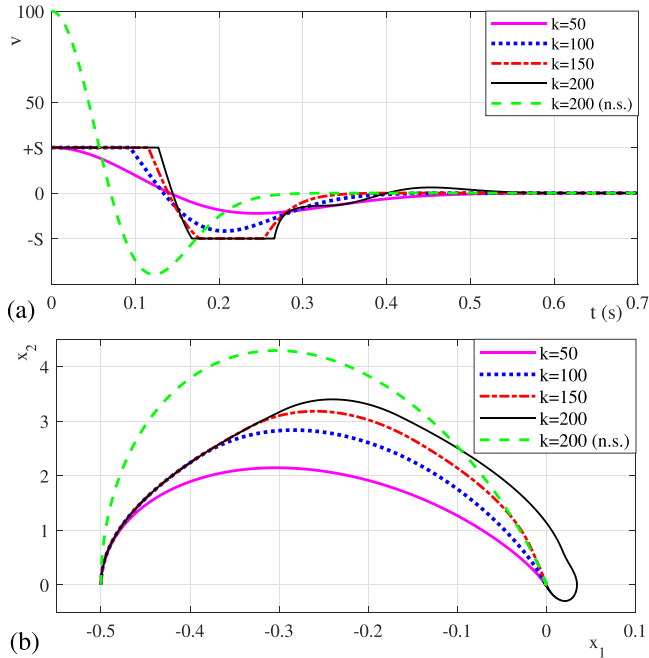


Fig. 7. Control value (a) and trajectories phase portrait (b) for various k -parameter values, with and without control saturation.

5. Conclusions

This paper has proposed a novel nonlinear damping control for the second-order unperturbed systems with output feedback. The control is claimed to be optimal since it does not require any additional parameter and provides a fast (exponentially quadratic) convergence without transient overshoots, when no control constraints. The global asymptotic stability, passivity, and finite-time convergence until certain neighborhood to the stable origin of the state variables have been explored. An enhanced performance has been demonstrated comparing to the linear and optimally (i.e. critically) damped controller. Also the saturated control case, as relevant for applications, was analyzed, regarding convergence, and demonstrated to have no negative impact on the principal control performance. It is believed that the proposed controller may represent an interesting alternative to a conventional proportional derivative (PD) controller.

Declaration of Competing Interest

The author declares that the submitted manuscript does not contain any conflict of interest.

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