# LEARNING TO TEACH A MATHEMATICAL EMOTIONAL ORIENTATION

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I do not believe that anyone is a born teacher or a born mathematician. This means that I face the problem of explaining how one *becomes* a teacher or a mathematician. The change of being/doing/knowing that is involved in becoming a teacher or a mathematician can be called *learning*, and I am interested in this process. Here I will recount some stories of learning and use them to explore my current thinking about becoming a mathematics teacher.

I begin by exploring the claim that *knowing is being is doing* and how this claim applies to being a teacher. I then explore the process of becoming a mathematician, and the nature of mathematics. I claim that mathematics is defined by a peculiar criterion for the explanations that are considered acceptable, that I refer to as a 'mathematical emotional orientation'. I close with some speculations on how one might influence others to share the mathematical emotional orientation, thus becoming mathematicians, and hence knowing mathematics.

## Becoming a teacher <level 1 heading>

Laurinda Brown has a story of her first experience in initial teacher education, after years of working with inservice teachers. She began by using the same prompts she had learned in those years, and her students responded to her prompts with stories about all sorts of things she did not expect. Afterwards she commented to her colleague John Hayter that she now knew that the course did something to create teachers by the end of it, but she did not know what.

This story interests me because it describes what happens in teacher education as a change in being, not as the acquisition of knowledge. And I suspect many teacher educators could tell similar stories. We observe novice teachers making decisions that seem odd to us, and describe the event as the novice not yet thinking like a teacher, as opposed to not yet having knowledge a teacher has. Teacher educators describe learning teaching as *becoming* a teacher.

How do we recognise that someone has become a teacher? In Laurinda's story, she had developed prompts that reliably provoked the responses she expected from teachers, but not from her students.

> If someone claims to know algebra, that is, to be an algebraist, we demand of him or her to perform in the domain of what we consider algebra to be, and if according to us she or he performs adequately in that domain, we accept the claim. (Maturana, 1987, p. 325)

If someone claims to know teaching, that is to be a teacher, we provide a prompt to action in the domain of teaching.

# Becoming a mathematician [1] <level 1 heading>

In recent years the main focus of my teaching has been teaching mathematics to future primary school teachers in Germany. My focus is on thinking mathematically, a focus Laurinda observed in Alf Coles' teaching (Brown & Coles 2008) and which Alf associates with 'becoming a mathematician' (Coles 2013). While Alf and I associate different doings with being a mathematician, we both see teaching mathematics as changing our students' beings, not as imparting knowledge.

For Alf, doings that are associated with being a mathematician are "asking questions, spotting patterns, making conjectures or predictions [and] giving reasons or justifications" (p. 6). I agree that mathematicians do all these things. But if I observe a person doing these things, do I observe a mathematician? Historians ask questions, see patterns, make conjectures and give reasons. These doings seem to me to be common to any science (broadly meant, like *Wissenschaft*, to include all systematic inquiry).

Some might say that what makes the mathematician different from the historian is not what s/he does, but what s/he does it to. This is typical of dictionary definitions that say mathematics is the science of number and space. I find this unsatisfactory. For one thing, this list of objects of mathematicians' doings is clearly incomplete. Mathematicians explore many other objects. In fact, *anything* can be mathematised, and so become an object of mathematics, from juggling to lumber milling. For me trying to identify the nature of the objects of mathematics is the wrong approach. Instead, I prefer to look more carefully at the kind of science mathematics is.

Maturana (1987) claims "the intention of doing art is to generate an aesthetic experience, and the intention of doing technology is to produce, the intention of doing science is to explain" (pp. 326–327). Hence, he refers to the sciences as 'explanatory domains'. He outlines four 'operational conditions' for the validation of scientific explanations:

> 1. The specification of the phenomenon to be explained, by specifying what an observer must do to observe the phenomenon.

> 2. A generative mechanism or explanatory hypothesis that gives rise to the phenomenon, but which operates at a meta-level to it.
>
> 3. The deduction or prediction of other phenomena from the generative mechanism that an observer should be able to observe, and stipulation of what an observer must do to observe these phenomena.
>
> 4. The actual witnessing of the predicted phenomena. (paraphrased from p. 327)

These four conditions correspond well to Alf's "spotting patterns" (1), "making conjectures or predictions" (3) and "giving reasons or justifications" (2). What is interesting is that (4) is not a doing associated with being a mathematician, nor is stipulation of what an observer must do to observe predicted phenomena (from 3). A mathematician's deductions are not predictions to be tested; they are new phenomena. Mathematics is the science that does not test its predictions [2].

I find it unsatisfying to describe mathematics as a science lacking in a stage of validation other sciences have. I would prefer a more positive description. The key to such a description is the observation that without an empirical way to validate generative mechanisms, mathematicians instead seek to deduce them from other, somehow more fundamental, generative mechanisms. We call this proving theorems, and it is, I feel, what make mathematics unique.

# Learning about proof <level 1 heading>

As a master's student at Concordia University, some decades ago, I shared a room with the back issues of FLM. I spent a lot of time reading them. One thing I read in that time was Efraim Fischbein's 1982 article *Intuition and proof,* which appeared in FLM **3**(2). I was a different person then, obviously, and so when I read the article then it was a different article than when I read it now. I recall then being annoyed that some trivial numerical datum had altered slightly from its presentation in an earlier PME paper. I am not sure the heart of the article (as I read it now), *intuition and proof*, touched me at all.

A bit later I latched onto a phrase that occurs just after the key comments on *intuition and proof*, "to believe (fully, sympathetically, intuitively) in the *a priori* universality of the theorem guaranteed by the respective proof" (p. 17). Having recently become Lakatosian and sceptical about "a priori universality" I quoted this phrase as an example of a way of looking at proof to avoid. What I missed is the key point that learning about proof is not just about knowing, it is about being:

> In order to really understand what a mathematical proof means the learner's

> mind must undergo a fundamental modification. Of course he can learn proofs and he can learn the general notion of a proof. But our research has shown that this is not enough. A profound modification is required. A new completely non-natural "basis of belief" is necessary, which is different from the way in which an empirical "basis of belief" is formed. (p. 17)

I claim that mathematics is defined by Fischbein's "basis of belief".

I connect Fischbein's 'basis of belief' with what Maturana calls an 'emotion of acceptance' or an 'emotional orientation'.

> What distinguishes an observer in daily life from an observer as a scientist is the scientist's emotional orientation to explaining his or her consistency in using only the criterion of validation of scientific explanations for the system of explanations that he or she generates in his or her particular domain of explanatory concerns. (Maturana, 1988, p. 36)
>
> Whether an observer operates in one domain of explanations or in another depends on his or her preference (emotion of acceptance) for the basic premises that constitute the domain in which he or she operates. (1988, p. 33)

To operate in the mathematical domain of explanations means that one has accepted the basic premises that constitute the domain, that one has a mathematical emotional orientation, that one believes in proofs.

This is the message in Fischbein's article on *Intuition and proof* that it has taken me a long time to understand. Not that proof gives us access to absolute truth, but rather that proof gives us a feeling of certainty that is peculiar to mathematics, and that having that feeling is part of understanding proof, and indeed, understanding mathematics. This peculiar feeling is special to mathematics and makes mathematics special.

## How does one come to believe in proofs? <level 1 heading>

There are actually a few different questions here. First, there is a way of reasoning, often called 'deductive', that is the only way of reasoning used in a finished proof. Finding a proof, of course, involves many different kinds of reasoning [3]. But believing in proofs requires, at least, being able to reason deductively. So the first question is how does one come to reason deductively. Second, proofs are usually presented and interpreted through language, and peculiar forms of language are often used. So a second question is how one learns to interpret these peculiar forms of language. If both deductive reasoning and the ability to interpret the form of a proof are present, there remains the issue of whether one accepts proofs as secure evidence or convincing explanation; whether one has a mathematical emotional orientation or basis of belief, an acceptance of the basic premises that constitute the domain.

My answer to the first question is that children (at least those tested by psychologists) are capable of deductive reasoning, and I believe there are reasons why the human species as a whole should have learned to reason in this way a long time ago. I have discussed elsewhere [4] my reasons for believing this, and for now I wish to take it for granted. As for learning the peculiar forms of language employed in proofs, this is undoubtedly an issue, but people have demonstrated considerable capacity to learn other peculiar forms of language, from everyday speech to musical notation, and I do not see the language of proofs as an insurmountable obstacle to learning to prove. It is a third question that I find most interesting, the question of how one learns an emotional orientation.

I have found this question interesting from the beginning of my teaching career. When I was studying to become a mathematics teacher, I met a woman who was studying to be a music teacher. She was convinced teaching music would be simple. She would play a piece by Beethoven, for example, and the students would be captured by its intrinsic beauty. She could then engage them in further reflections on the piece. “You,” I thought, “are going to be sorely disappointed.” I do not know if she was, but I certainly was when I went out to a school and tried the mathematical equivalent. I was asked to take over part of another teacher’s Grade 7 class for one day, as he had to do some special activity with the rest of the class. So I had a small group and freedom to choose what I taught. I chose to expose them to the intrinsic beauty of the classic proof of the irrationality of the square root of two.

This proof is included in every collection of beautiful proofs. But my students did not see its beauty. As a new teacher this surprised me. I now know, having researched this myself since, that the reasoning involved in a proof by contradiction was not the issue. Much younger children can handle that. And the algebra involved should have been understandable to them. In fact, other researchers have had similar difficulties with this proof and university students, for whom the language is more familiar. But I suspect a mathematical emotional orientation is needed to find this proof explanatory, convincing, and perhaps even beautiful.

So, how does one *learn* an emotional orientation? I do not know, but I have some ideas.

## Teaching through proving <level 1 heading>

I strongly suspect that approaches to teaching mathematics prevalent in the schools I am familiar with are not helping. To paraphrase Maturana:

> What distinguishes an observer in daily life from an observer as a mathematician is the mathematician's consistency in using only mathematical proofs as explanations in mathematics.

However, most students experience a dozen years of learning mathematics in which the mathematics they learn is explained in other ways, by reference to authority, by pattern spotting, or by simple repetition.

These ways of explaining are not wrong. There are domains in which they are the appropriate ways to explain things. But they are not the appropriate ways of explaining in mathematics. That students do not learn that explanations in mathematics are of a certain kind is not surprising, because they are almost never offered such explanations.

Dropping in the occasional proof does not really help. In other domains it is also sometimes possible to use deductive reasoning to explain, but only in special circumstances. If students experience mathematics as a domain in which occasionally proofs are used to explain, we should not be surprised that they would see nothing very different about mathematics in comparison to other sciences.

I have come to believe that students could learn the mathematical emotional orientation by experiencing mathematics as a domain in which only mathematical proofs are offered and accepted as explanations.

Inspired by the work of Gila Hanna, Magdalene Lampert, and Howard Fawcett, in 2011 I coined the name 'proof-based teaching' to describe a possible future approach to teaching mathematics in which students learn mathematics by proving [5]. A few years later I had the good fortune to meet Estela Vallejo-Vargas, who had been teaching divisibility to third graders in a way that seemed to capture what I meant by 'proof-based', and she was already researching her practice. Since then the two of us have been exploring further the nature and challenges of proof-based teaching.

## Learning to teach through proving <level 1 heading>

One challenge we have been addressing brings me back to the beginning. For a teacher to change her teaching approach, her doing, requires a change of being. Estela can teach in a proof-based way as she is a teacher and a mathematician. But most primary school teachers, and even secondary school teachers in many places, do not think of themselves as mathematicians, and rightly so if they do not have a mathematical emotional orientation. The first step, in their learning to teach in a proof-based way, is learning the mathematical emotional orientation. So I am back to the question, how does one *learn* an emotional orientation?

Maturana makes a suggestion:

> The children do not learn mathematics in school; they learn how to live together with a mathematics teacher. Perhaps they will one day carry on this enjoyable and exciting kind of being together independently—and become mathematics teachers or mathematicians themselves. Teachers do not simply transmit some content; they acquaint their pupils with a way of living. In the process, the rules of arithmetic, the laws of physics, or the grammar of a language will be acquired. My claim is: *Pupils learn teachers.* (Maturana, in Gumbrecht, Maturana & Poerksen, 2006, p. 26)

Estela has been working with inservice primary school teachers in Peru. She teaches them explanations for principles of divisibility they already 'know' but have never explained mathematically, in a proof-based way. That is to say, she shares with them her way of living with divisibility and her way of being a teacher. This includes deriving three key notions from everyday experiences of fair sharing: that all the shares are equal, that nothing is broken, and that as many as possible are shared out. From these three key notions properties such as the remainder being smaller then the divisor are deduced, using a mixture of concrete models and verbal arguments (see Vallejo-Vargas & Ordoñez-Montañez, 2015).

While they are learning about divisibility the inservice teachers reflect on what they are learning and the way they are learning with reference to the learning of children they will soon be attempting to influence. Estela then follows them to their classrooms, observes their teaching, and reflects on it with them. There are limits to the extent of the change of being/doing/knowing that Estela can observe in the short time she works with the teachers, but so far the results have been encouraging, with some of the teachers beginning to consider how they might approach other areas of their teaching differently.

I also work with primary school teachers, but at the very beginning of their university teacher education programme. This means that, unlike in Estela's case, they are not yet teachers. They have (mostly) just finished school, and my initial focus is on teaching them algebra in a proof-based way, so that they can explain mathematically the rules and techniques they 'know' from school. Through this experience I hope they also begin to expect mathematical explanations in other areas of mathematics (and we later prove things in geometry, combinatorics, *etc*.) The context in which I do this is not ideal as there are a lot of students and I see them for only a couple of hours a week.

One thing I do not like about this context is that I have limited insight into what my students are learning. One day, however, I did at least get some access into the learning of a teaching assistant who works with me, who attended the same schools as my students. In Germany students learn in school two formulae for solving quadratic equations, the '*pq* formula' and the '*abc* formula' [6] (which, when I went to school in the US, was called the 'quadratic formula'). In one lecture I use geometric materials to physically 'complete the square' for several specific quadratic equations, and then I use one of these as a generic example to derive the *pq* formula.

After that lecture 'Mike', my teaching assistant, walked with me back to my office, and told me a bit of his school history. He remembered learning completing the square as an algebraic procedure in school, one which he understood and accepted. His teacher then presented the class with the *pq* formula, without any explanation as far as Mike could recall. Mike refused to use the *pq* formula, solving any quadratic equations he encountered by completing the square. He could remember the formula, but without an explanation, a generative mechanism, a proof, he was not willing to use it. In my lecture he had seen the connection between the formula and completing the square for the first time, and now was willing to use the formula, having seen it deduced from a procedure he accepted.

I find this interesting because to me it shows that Mike had a mathematical emotional orientation all along. He was never offered a mathematical explanation of why the *pq* formula works and so he rejected it. Seeing the connection between the *pq* formula and completing the square explained it, and changed his emotion towards it. The formula went from something for which he had negative feelings, a way of solving quadratic equations he knew of, but avoided, to one he now recognised as a different representation of a familiar procedure. That change of feeling, rather than any change in the commodity 'knowledge', is a change in his being.

Maturana notes that when an utterance "is accepted and becomes an explanation, the emotion or mood of the observer shifts from doubt to contentment, and he or she stops asking over and over again the same question" (1988, p. 28). This seems to capture what happened to Mike with regard to the *pq* formula. Something similar happened with Estela's teachers when they observed their students engaged in the mathematical activity that Estela had described, but which the teacher had never before seen in their classrooms. Here there is also a shift from doubt to contentment.

## Why should one learn the mathematical emotional orientation?
<level 1 heading>

Once upon a time mathematics was suggested as an explanatory domain that all others should imitate. This idea runs through writing from Descartes' 1637 *Méthode* to Fawcett's 1938 *The Nature of Proof.* We now live in an age that is sceptical of such grand, universal narratives, and so it might be asked why I am interested in exposing all children to the mathematical emotional orientation. I do not claim that mathematics is a better explanatory domain than others. I do not wish (as Descartes did) to apply deductive reasoning to all aspects of human activity. I merely claim that mathematics is a *unique* explanatory domain, and hence affords learners some unique opportunities. I wish to insist that explanations in mathematics be restricted to deductive reasoning, because it is the only domain in which deductive reasoning is the only appropriate way of explaining. And this way of explaining, as Fischbein notes, can give rise to a feeling of certainty unlike the feelings of certainty that arise in other domains. Of course it is not really certain, but nonetheless I believe it is an important feeling to experience, if only to cast other feelings of certainty in a different light. I believe it is important to recognise that there are different domains of explanation, with different feelings of certainty. Mathematics is special, and one way in which it is special is that its criteria for explanations are so well defined that they can be turned on themselves. In mathematics, one can reason about reasoning; one can prove what can be proven. And its way of explaining can be empowering.

Estela told me a story that illustrates this empowerment. She was observing in the classroom of one of the teachers she works with. The students had worked in small groups on an activity and then discussed it as a whole class, and out of that discussion the following property and its justification was written down on a big sheet of paper that was placed on one of the classroom's walls:

> **Property:** "In a division the remainder must be smaller than the number of people. The maximum remainder would be equal to the number previous to the number of people"
>
> **Justification:** "Because if the remainder is bigger than the maximum remainder then it could still be distributed and it would not be a remainder"

To Estela something seemed not quite right here. She asked, "What happens when the remainder is equal to-" but was interrupted by students saying (almost shouting) "It can still be distributed!". The teacher asked Estela where the error was, perhaps interpreting Estela's question as a polite way of indicating to the teacher that an error has been made. This sometimes happens when an expert, brought into the school to advise the teachers, wants to point out an error in a way that allows the teacher to correct it without losing her own authority in the classroom. Following Estela's instructions, the teacher 'corrected' the justification:

> **Justification:** "Because if the remainder is bigger than, or equal to, the maximum remainder then it could still be distributed and it would not be a remainder"

The students objected, pointing out that only if the remainder is equal to the number of people does a problem occur, but "if it would be equal to the maximum remainder, there would not be any problems". The students, Estela and the teacher argued for some time (27 turns in the transcript) before Estela and the teacher proposed a concrete example, through which they finally saw what the students had been pointing out all along. In a community that shares a mathematical emotional orientation, that agrees on the criterion of validation of mathematical explanations, eight year olds can argue successfully with an expert. I think that is an important experience for them to have, and mathematics is one of the few domains where this is genuinely possible. That is why it is important to learn, and teach, the mathematical emotional orientation.

## Notes

[1] I feel a tension as I write this, between my thoughts and the language I am expressing them in. I wish to write about verbs: being, doing, knowing, teaching, … . That ellipsis marks where I was going to continue the list with the verb for what mathematicians do. ‘Teaching’ is a verb, and so above it was fairly straightforward to link being a teacher with teaching. Interestingly, ‘knowing teaching’ does not come so easily. With mathematics I have a different problem. ‘Mathematics’ is a noun, a ‘thing’ one knows, so being a mathematician links easily to knowing mathematics, but what is the verb, akin to ‘teaching’ for ‘doing mathematics’?

Words have been coined, for example ‘mathing’, but they do not enjoy the currency of ‘teaching’. Better known words like ‘mathematising’ have evolved to refer to something a little different. This limitation of my language limits my thinking, probably in some ways I am not aware of. The equating of being, doing, knowing is helpful in making me aware of some ways of thinking my language makes difficult.

[2] Some readers will be objecting, “but Lakatos shows that mathematics is a quasi-empirical science of proof and refutations!” Not quite. Lakatos offers convincing (to me) narratives that suggest that *some* mathematical discovery occurs through cycles of proof and refutations, but that does not mean all mathematical discovery occurs in the same way. And note that the counter-examples in those narratives were not observed because a specific prediction had been made that was then tested using those examples. The examples were stumbled upon, for the most part. Even when mathematics is quasi-empirical, it does not make predictions in order to test them.

[3] At least for humans, and I would question if a proof found by a computer counts before it is read by a human whose reasoning and feelings are modified by it.

[4] Reid, D. (2013) The biological basis for deductive reasoning. In Ubuz, B., Haser, Ç, Mariotti, M.A.(Eds.) *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education,* pp. 206–215. Ankara, Turkey: ERME.

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[6] They are:

<insert Reid Equation 1 here> and <insert Reid Equation 2 here>

The *pq* formula is a rearrangement of the quadratic formula when $a = 1$. For some reason $b$ then becomes $p$ and $c$ becomes $q$.

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