DELTA- AND DAUGAVET-POINTS IN BANACH SPACES

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Abstract. A Δ-point x of a Banach space is a norm one element that is arbitrarily close to convex combinations of elements in the unit ball that are almost at distance 2 from x. If, in addition, every point in the unit ball is arbitrarily close to such convex combinations, x is a Daugavet-point. A Banach space X has the Daugavet property if and only if every norm one element is a Daugavet-point.

We show that Δ- and Daugavet-points are the same in L_1-spaces, L_1-preduals, as well as in a big class of Müntz spaces. We also provide an example of a Banach space where all points on the unit sphere are Δ-points, but where none of them are Daugavet-points.

We also study the property that the unit ball is the closed convex hull of its Δ-points. This gives rise to a new diameter two property that we call the convex diametral diameter two property. We show that all $C(K)$ spaces, K infinite compact Hausdorff, as well as all Müntz spaces have this property. Moreover, we show that this property is stable under absolute sums.

1. Introduction

Let X be a real Banach space with unit ball B_X, unit sphere S_X, and dual X^*. Recall that X has the *local diameter two property (LD2P)* if every slice of B_X has diameter two. Recall that a *slice* of B_X is a subset of the form

$$S(x^*, \varepsilon) = \{x \in B_X : x^*(x) > 1 - \varepsilon\},$$

where $x^* \in S_{X^*}$ and $\varepsilon > 0$.

For $x \in S_X$ and $\varepsilon > 0$ denote

$$\Delta_\varepsilon(x) = \{y \in B_X \colon \|x - y\| \geq 2 - \varepsilon\}.$$

We say that $x \in S_X$ is a *Δ-point* if we have $x \in \overline{\operatorname{conv}}\, \Delta_\varepsilon(x)$, the norm closed convex hull of $\Delta_\varepsilon(x)$, for all $\varepsilon > 0$. The set of all Δ-points in S_X is denoted by Δ, i.e.

$$\Delta = \{x \in S_X \colon x \in \overline{\operatorname{conv}}\, \Delta_\varepsilon(x) \text{ for all } \varepsilon > 0\}.$$

2010 *Mathematics Subject Classification.* Primary 46B20, 46B22, 46B04.

Key words and phrases. Diametral diameter two property, Daugavet property, L_1-space, L_1-predual space, Müntz space.

R. Haller and K. Pirk were partially supported by institutional research funding IUT20-57 of the Estonian Ministry of Education and Research.

We will sometimes need to clarify which Banach space we are working with and write $\Delta_\varepsilon^X(x)$ and Δ_X instead of $\Delta_\varepsilon(x)$ and Δ respectively.

The starting point of this research was the discovery that if a Banach space X satisfies $B_X = \overline{\operatorname{conv}}\,\Delta$, then X has the LD2P.

We study spaces that satisfy the property $B_X = \overline{\operatorname{conv}}\,\Delta$ in Section 5. The case $S_X = \Delta$, i.e. $x \in \overline{\operatorname{conv}}\,\Delta_\varepsilon(x)$ for all $x \in S_X$ and $\varepsilon > 0$, has already appeared in the literature, but under different names: The diametral local diameter two property (DLD2P) ([5]), the LD2P+ ([1] and [2]), and space with bad projections ([13]). We will use the term DLD2P in this paper. From [18, Corollary 2.3 and (7) p. 95] and [13, Theorem 1.4] the following characterization is known.

Proposition 1.1. *Let X be a Banach space. The following assertions are equivalent:*

(1) X has the DLD2P;
(2) for all $x \in S_X$ we have $x \in \overline{\operatorname{conv}}\,\Delta_\varepsilon(x)$ for all $\varepsilon > 0$;
(3) for all projections $P : X \to X$ of rank-1 we have $\|Id - P\| \geq 2$.

Related to the DLD2P is the Daugavet property. We have (cf. [18, Corollary 2.3]):

Proposition 1.2. *Let X be a Banach space. The following assertions are equivalent:*

(1) X has the Daugavet property, i.e. for all bounded linear rank-1 operators $T : X \to X$ we have $\|Id - T\| = 1 + \|T\|$;
(2) for all $x \in S_X$ we have $B_X = \overline{\operatorname{conv}}\,\Delta_\varepsilon(x)$ for all $\varepsilon > 0$.

Clearly the Daugavet property implies the DLD2P, but the converse is not true [13, Corollary 3.3].

We will say that $x \in S_X$ is a *Daugavet-point* if we have $B_X = \overline{\operatorname{conv}}\,\Delta_\varepsilon(x)$ for all $\varepsilon > 0$. Every Daugavet-point is a Δ-point, but the converse might fail (see Example 4.7 for an extreme example of this).

In our language, [18, (7) p. 95] states that a Banach space X the DLD2P is equivalent to:

($\mathfrak{D}$) For all projections $P : X \to X$ of rank-1 and *norm-1* we have $\|Id - P\| = 2$.

This statement is repeated in [2, Theorem 3.2] and used in the argument of [2, Theorem 3.5 (i) $\Leftrightarrow$ (iii)]. In the case of the Daugavet property, it *is enough* to consider only norm 1 operators T. This follows by scaling (see the argument below Definition 2.1 in [18]). However, a scaled projection is not a projection. Upon request, neither the authors of [2] nor [18] have been able to give a correct proof that ($\mathfrak{D}$) is equivalent to the DLD2P. Thus the validity of this equivalence is still an open question.

Through an investigation of Δ- and Daugavet-points in concrete spaces, we have been able to show that for $L_1(\mu)$-spaces, where μ is a σ-finite measure on an infinite set, and for $L_1(\mu)$-predual spaces, the property in ($\mathfrak{D}$) *is* equivalent to the DLD2P (and even to the Daugavet property) (see Theorems 3.3 and 3.8 below).

In connection with the open problem just mentioned, it is worth noting that for $X = \ell_1$ a pointwise version of the property ($\mathfrak{D}$) holds for some $x \in S_X$ even though S_X has no Δ-points (see Proposition 2.3 and Theorem 3.1).

In the following we will bring in our main results. In Section 3 we look at the Δ- and Daugavet-points in $L_1(\mu)$ spaces when μ is a σ-finite measure, preduals of $L_1(\mu)$ spaces for such measure μ, and a big class of Müntz spaces. We prove that Δ- and Daugavet-points are the same in all these cases (see Theorems 3.1, 3.7, and 3.13).

In Section 4 we show that there are absolute normalized norms N, different from the ℓ_1- and ℓ_∞-norms, for which $X \oplus_N Y$ has Daugavet-points, and also such N for which $X \oplus_N Y$ fails to have Daugavet-points.

In Section 5 we introduce the convex diametral diameter two property (convex DLD2P) defined naturally using Δ-points. We show that this property lies strictly between the properties DLD2P and LD2P (see Corollary 5.6). We give examples of classes of spaces with the convex DLD2P, more precisely we show that all $C(K)$ spaces, K infinite compact Hausdorff, as well as all Müntz spaces, have this property (see Proposition 5.3 and Theorem 5.7). We also prove that if X and Y have the convex DLD2P, then the sum $X \oplus_N Y$ has this property whenever N is an absolute normalized norm (see Theorem 5.8).

2. Preliminaries

We start this section collecting some characterizations of Δ- and Daugavet-points from the literature.

Lemma 2.1. *Let X be a Banach space and $x \in S_X$. The following assertions are equivalent:*

(1) x is a Δ-point, that is $x \in \overline{\operatorname{conv}}\, \Delta_\varepsilon(x)$ for every $\varepsilon > 0$;
(2) for every slice S of B_X with $x \in S$ and for every $\varepsilon > 0$ there exists $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$;
(3) for every $x^* \in X^*$ with $x^*(x) = 1$ the projection $P = x^* \otimes x$ satisfies $\|Id - P\| \geq 2$.

Proof. The equivalence of (1) $\Leftrightarrow$ (2) is proved using Hahn-Banach separation.

The equivalence (2) $\Leftrightarrow$ (3) is a pointwise version of [13, Theorem 1.4] and the same proof works. □

Lemma 2.2. *Let X be a Banach space and $x \in S_X$. The following assertions are equivalent:*

(1) x is a Daugavet point, that is $B_X = \overline{\operatorname{conv}}\, \Delta_\varepsilon(x)$ for every $\varepsilon > 0$;
(2) for every slice S of B_X and for every $\varepsilon > 0$ there exists $y \in S$ such that $\|x - y\| \geq 2 - \varepsilon$;

(3) *for every nonzero* $x^* \in X^*$*, the rank-1 operator* $T = x^* \otimes x$ *satisfies* $\|Id - T\| = 1 + \|T\|$*;*
(4) *for every* $x^* \in S_{X^*}$ *the rank-1 norm-1 operator* $T = x^* \otimes x$ *satisfies* $\|Id - T\| = 2$.

Proof. The equivalence (2) $\Leftrightarrow$ (3) is a pointwise version of Lemma 2.2 in [14]. The equivalence (1) $\Leftrightarrow$ (2) follows by Hahn-Banach separation as observed by [18, Corollary 2.3].

While (3) $\Rightarrow$ (4) is trivial the implication (4) $\Rightarrow$ (3) follows by scaling as explained in the paragraph following Definition 2.1 in [18]. $\square$

The next proposition shows that we cannot add a version of Lemma 2.2 (4) to Lemma 2.1. In fact, we will see in Theorem 3.1 that no point on the sphere in ℓ_1 is a Δ-point.

Proposition 2.3. *Let* $X = \ell_1$ *and* $x = (x_i)_{i=1}^\infty \in S_X$ *a smooth point with* $|x_1| > 1/3$*. Then:*

(1) *for* $x^* \in S_{X^*}$ *with* $x^*(x) = 1$*, the projection* $P = x^* \otimes x$ *satisfies* $\|Id - P\| = 2$*;*
(2) *the projection* $P = x_1^{-1} e_1^* \otimes x$ *satisfies* $\|Id - P\| < 2$.

Proof. Write $x = (x_i)_{i=1}^\infty$. Let $x^* := (\operatorname{sign} x_i)_{i=1}^\infty \in S_{X^*}$ and $P := x^* \otimes x$. Observe that $x^*(x) = 1$. If e_n is the n'th standard basis vector in X, then

$$\begin{aligned}\|(Id - P)(e_n)\| &= \|e_n - \operatorname{sign} x_n x\| = |1 - (\operatorname{sign} x_n)x_n| + \sum_{i \neq n} |x_i| \\ &= 1 - |x_n| + \|x\| - |x_n| = 2 - 2|x_n|,\end{aligned}$$

and since this holds for all n we get $\|Id - P\| = 2$.

Let $P := x_1^{-1} e_1^* \otimes x$, where e_i^* is the i'th coordinate vector in $X^* = \ell_\infty$. Observe that $x_1^{-1} e_1^*(x) = 1$, so that P is a projection. If $y \in S_X$ we get

$$\begin{aligned}\|(Id - P)y\| = \|y - x_1^{-1} y_1 x\| &= \sum_{i>1} |y_i - x_1^{-1} y_1 x_i| \\ &\leq \sum_{i>1} |y_i| + |x_1|^{-1} |y_1| \sum_{i>1} |x_i| \\ &= 1 - 2|y_1| + |x_1|^{-1}|y_1| \leq 1 + \left|2 - |x_1|^{-1}\right| < 2,\end{aligned}$$

so $\|Id - P\| < 2$, and we are done. $\square$

Let us note that both the DLD2P and property $(\mathfrak{D})$ pass from the dual to the space.

Proposition 2.4. *Let* X *be a Banach space. Then:*

(1) *if* X^* *has the DLD2P, then* X *has the DLD2P;*
(2) *if* $\|Id_{X^*} - P\| = 2$ *for all norm-1 rank-1 projections* P *on* X^**, then* $\|Id_X - Q\| = 2$ *for all norm-1 rank-1 projections* Q *on* X.

Proof. The second statement is trivial, while the first one only requires a bit of rewriting: If Q is a rank-1 projection on X, then $Q = x^* \otimes x$ with $x^* \in X^*$, $x \in S_X$, and $x^*(x) = 1$. Then

$$P = Q^* = x \otimes x^* = (\|x^*\|x) \otimes \frac{x^*}{\|x^*\|}$$

is a rank-1 projection on X^* and by assumption $\|Id_{X^*} - P\| = \|Id_X - Q\| \geq 2$. $\square$

As we noted in the Introduction, we do not know if the property in ($\mathfrak{D}$) is equivalent to the DLD2P. We end this section by observing that, just like the DLD2P, property ($\mathfrak{D}$) implies that all slices of the unit ball of both the space and its dual have diameter two. (See [13, Theorem 1.4] and [2, Theorem 3.5] for the corresponding DLD2P result.) The following result also shows that despite of Proposition 2.3, ℓ_1 is not a candidate for separating property ($\mathfrak{D}$) and the DLD2P since ℓ_1 does not have the LD2P.

Proposition 2.5. *Let X be a Banach space. If $\|Id - P\| = 2$ for all norm-1 rank-1 projections P on X, then X has the LD2P and X^* has the w^*-LD2P.*

Proof. Let $x^* \in S_{X^*}$ and $\varepsilon > 0$ define a slice $S(x^*, \varepsilon)$. Let $\delta > 0$ such that $\delta < \frac{\varepsilon}{2}$. Find $y^* \in S_{X^*}$ such that y^* attains its norm on B_X and $\|x^* - y^*\| < \frac{\varepsilon}{2}$. Let $y \in B_X$ be such that $y^*(y) = 1$ and define $P = y^* \otimes y$. Then $\|Id - P\| = 2$ by assumption and we can find $z \in S_X$ such that

$$\|z - P(z)\| = \|z - y^*(z)y\| > 2 - \delta.$$

We may assume that $y^*(z) > 0$. We have

$$y^*(z) = |y^*(z)| = \|P(z)\| \geq \|P(z) - z\| - \|z\| > 2 - \delta - 1 > 1 - \frac{\varepsilon}{2}.$$

Hence

$$x^*(z) = y^*(z) - (y^* - x^*)(z) > 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = 1 - \varepsilon,$$

i.e. $z \in S(x^*, \varepsilon)$, and

$$\|z - y\| \geq \|z - y^*(z)y\| - \|y^*(z)y - y\| > 2 - \delta - |y^*(z) - 1| > 2 - 2\delta.$$

This proves that X has the LD2P.

To show that X^* has the w^*-LD2P we start with a w^*-slice $S(x, \varepsilon)$, where $x \in S_X$ and $\varepsilon > 0$. Then we find a $y^* \in S_{X^*}$ where $\|Id^* - P^*\|$ almost attains its norm. The proof is similar to the LD2P case. $\square$

3. Δ- and Daugavet-points for different classes of spaces

In the first two parts of this section we study Δ- and Daugavet-points in Banach spaces X of the type $L_1(\mu)$, $C(K)$, and $L_1(\mu)$-preduals. Crucial in our study is the discovery that a Δ-point $f \in S_X$ can be characterized in terms of properties of the support of f (see Theorems 3.1 and 3.4). These characterizations of being a Δ-point are easy to check, and we use them to prove that Δ- and Daugavet-points are in fact the same in all such spaces X. For example, if $X = C([0, \omega]) = c$ then the Daugavet-points are exactly the sequences with limits ± 1.

In the last part of the section we study Δ- and Daugavet-points in Müntz spaces X of the type $M_0(\Lambda) \subset M(\Lambda) \subset C[0,1]$ (see Subsection 3.3 for a definition of a Müntz space). Our initial motivation for doing this, was the known fact that such spaces X are isomorphic, even almost isometrically isomorphic in the case $X = M_0(\Lambda)$, to subspaces of c (see [17] and [16]). Based on this, the results from [3], and other results from [16] one could expect similar results for Müntz spaces as for c. And, indeed, this is the case, at least for $X = M_0(\Lambda)$ (see Theorem 3.13). In this class of Müntz spaces the Δ- and Daugavet-points are the same and the Daugavet-points are exactly the functions $f \in S_X$ for which $f(1) = \pm 1$.

3.1. **$L_1(\mu)$ spaces.** Let μ be a (countably additive, non-negative) measure on some σ-algebra Σ on a set Ω. We will assume that μ is σ-finite even though it is not strictly necessary in all the results. As usual an *atom for μ* is a set $A \in \Sigma$ such that $0 < \mu(A) < \infty$, and if $B \in \Sigma$ with $B \subsetneq A$ satisfies $\mu(B) < \mu(A)$, then $\mu(B) = 0$.

In this section we consider the space $L_1(\mu) = L_1(\Omega, \Sigma, \mu)$.

Theorem 3.1. *The following assertions for $f \in S_{L_1(\mu)}$ are equivalent:*

(1) f is a Daugavet point;

(2) f is a Δ-point;

(3) $\operatorname{supp}(f)$ does not contain an atom for μ.

Proof. (1) ⇒ (2) is trivial.

(2) ⇒ (3). Fix $f \in S_{L_1(\mu)}$. Let A be an atom in $\operatorname{supp}(f)$. Note that a measurable function is a.e. constant on an atom. We may assume that $f|_A = c$ a.e. for some positive constant c. Fix $0 < \varepsilon < 2c\mu(A)$.

Let $g \in B_{L_1(\mu)}$ be such that $\|f - g\| \geq 2 - \varepsilon$. We have $g|_A = d$ for some constant d. Note that

$$\begin{aligned}
2 - \varepsilon &\leq \int_\Omega |f - g| d\mu = \int_{\Omega \setminus A} |f - g| d\mu + \int_A |f - g| d\mu \\
&\leq \int_{\Omega \setminus A} |f| d\mu + \int_{\Omega \setminus A} |g| d\mu + \int_A |f - g| d\mu \\
&\leq 1 - \int_A |f| d\mu + 1 - \int_A |g| d\mu + \int_A |f - g| d\mu \\
&= 1 - c\mu(A) + 1 - |d|\mu(A) + |c - d|\mu(A).
\end{aligned}$$

Therefore

$$c\mu(A) + d\mu(A) \le |c-d|\mu(A) + \varepsilon.$$

If $c \le d$, then $|c-d| = d-c$ and we get $c \le \frac{\varepsilon}{2\mu(A)}$, and this contradicts our choice of ε. Thus we have $c \ge d$, and hence $|c-d| = c-d$ and $d \le \frac{\varepsilon}{2\mu(A)} < c$.

If $g_1, \dots, g_m \in \Delta_\varepsilon(f)$, then

$$\|f - \sum_{i=1}^m \frac{1}{m} g_i\| \ge \int_A |f - \sum_{i=1}^m \frac{1}{m} g_i|\, d\mu \ge (c - \frac{\varepsilon}{2\mu(A)})\mu(A) > 0.$$

This shows that $f \notin \overline{\text{conv}}\, \Delta_\varepsilon(f)$ for this choice of ε.

(3) $\Rightarrow$ (1). Let $f \in S_{L_1(\mu)}$ such that $\operatorname{supp}(f)$ does not contain atoms. Let $\varepsilon > 0$, $\delta > 0$, and $x_0^* \in S_{L_1(\mu)^*}$. By Lemma 2.2 we need to find $g \in S_{L_1(\mu)}$ with $\|f-g\| \ge 2-\varepsilon$ such that $g \in S(x_0^*, \delta)$.

Since μ is σ-finite (so that $L_1(\mu)^* = L_\infty(\mu)$) we can find a step-function $x^* = \sum_{i=1}^n a_i \chi_{E_i} \in S_{L_1(\mu)^*}$ such that $\|x^* - x_0^*\| < \delta$ (and the $E_i \cap E_j = \emptyset$ for $i \ne j$).

We may assume that $|a_1| = 1$. Find subset a A of E_1 such that $\int_A |f| d\mu < \varepsilon/2$. Define

$$g := \frac{\operatorname{sign}(a_1)}{\mu(A)} \chi_A \in S_{L_1(\mu)}.$$

Then

$$x^*(g) = \sum_{i=1}^n \int_{E_i} a_i g d\mu = \frac{1}{\mu(A)} \int_A a_1 \operatorname{sign}(a_1) d\mu = 1,$$

$$\|f-g\| = \int_{A^c} |f| d\mu + \int_A |f-g| d\mu \ge |f| + |g| - 2\int_A |f| d\mu \ge 2 - \varepsilon,$$

and finally

$$x_0^*(g) = x^*(g) - (x^* - x_0^*)(g) > 1 - \delta$$

as desired. □

Lemma 3.2. *If μ is a measure with an atom, then $L_1(\mu)$ does not have the LD2P.*

Proof. Assume that A is an atom and consider $\chi_A \in L_1(\mu)^*$. We have $\|\chi_A\| = 1$. If $f \in S(B_{L_1(\mu)}, \chi_A, \varepsilon)$, then

$$f(t) > \frac{1-\varepsilon}{\mu(A)} \qquad \text{for a.e. } t \in A,$$

and

$$f(t) \le \frac{1}{\mu(A)} \qquad \text{for a.e. } t \in A.$$

Hence $\|f|_A\| > 1 - \varepsilon$ and $\|f|_{A^C}\| < \varepsilon$.

Thus for $f_1, f_2 \in S(B_{L_1(\mu)}, \chi_A, \varepsilon)$ we have

$$\begin{aligned}\|f_1 - f_2\| &\leq \int_{A^c} |f_1 - f_2| \, d\mu + \int_A |f_1 - f_2| \, d\mu \\ &\leq \|f_1|_{A^c}\| + \|f_2|_{A^c}\| + \int_A \frac{\varepsilon}{\mu(A)} \, d\mu \leq 3\varepsilon,\end{aligned}$$

so this slice does not have diameter 2. □

Theorem 3.3. *Consider $X = L_1(\mu)$. The following assertions are equivalent:*

(1) $\|Id - P\| = 2$ for all norm-1 rank-1 projections on X;
(2) X has the Daugavet property.

Proof. If (1) holds, then X has the LD2P by Proposition 2.5. From Lemma 3.2 we see that X does not have atoms. By [6] (see also [7] for the explicit statement for $L_1(\mu)$ spaces) X has the Daugavet property.

The other direction is trivial. □

3.2. **$C(K)$ and $L_1(\mu)$-predual spaces.** In the following we explore the Δ- and Daugavet-points in the class of $L_1(\mu)$-predual spaces and $C(K)$ spaces. We start with a characterization of both Daugavet and Δ-points in $C(K)$ spaces.

Theorem 3.4. *Let K be an infinite compact Hausdorff space. The following assertions for $f \in S_{C(K)}$ are equivalent:*

(1) f is a Daugavet point;
(2) f is a Δ-point;
(3) $\|f\| = |f(x_0)|$ for a limit point x_0 of K.

Proof. (1) $\Rightarrow$ (2) is trivial.

(3) $\Rightarrow$ (1). Let $f \in S_{C(K)}$ and assume that there is a limit point x_0 of K such that $|f(x_0)| = 1$. We will show that f is a Daugavet-point. Fix $g \in B_X$, $\varepsilon > 0$, and $m \in \mathbb{N}$. Consider a neighbourhood U of x_0 such that $|f(x_0) - f(x)| < \varepsilon$ for every $x \in U$. Since x_0 is a limit point, we can find m different points $x_1, \ldots, x_m \in U$ and corresponding pairwise disjoint neighbourhoods $U_1, \ldots, U_m \subset U$. For every $1 \leq i \leq m$ use Urysohn's lemma to find a continuous function $\eta_i \colon K \to [0,1]$ with $\eta_i(x_i) = 1$ and $\eta_i = 0$ on $K \setminus U_i$. Define $g_i \in B_{C(K)}$ by

$$g_i(x) = \big(1 - \eta_i(x)\big) g(x) - \eta_i(x) f(x_0).$$

From $g_i(x_i) = -f(x_0)$ it follows that

$$\|f - g_i\| \geq |f(x_i) - g(x_i)| = |f(x_i) + f(x_0)| > 2 - \varepsilon.$$

Hence $g_i \in \Delta_\varepsilon(f)$. Note that $g - g_i = 0$ on $K \setminus U_i$, and consequently

$$\|g - \frac{1}{m} \sum_{i=1}^m g_i\| \leq \frac{1}{m} \max_{1 \leq i \leq m} \|g - g_i\| \leq \frac{2}{m}.$$

We thus get $g \in \overline{\operatorname{conv}}\, \Delta_\varepsilon(f)$, and so f is a Daugavet-point.

(2) $\Rightarrow$ (3). We assume that there is no limit point x of K such that $|f(x)| = 1$ and show that f is not a Δ-point. Define

$$H := \{x \in K : |f(x)| = 1\}.$$

Then H is a set of isolated points. By compactness, H is finite since otherwise it would contain a limit point. Note that H is (cl)open hence $\delta = 1 - \max_{x \in K \setminus H} |f(x)| > 0$. Let $\varepsilon_h := \operatorname{sign} f(h)$ for all $h \in H$. Since $H \neq \emptyset$ we can define

$$\mu = \frac{1}{|H|} \sum_{h \in H} \varepsilon_h \delta_h,$$

where $\delta_h \in S_{C(K)^*}$ is the point evaluation map at h. We have $\|\mu\| = 1$ and $\langle \mu, f \rangle = 1$, hence $P = \mu \otimes f$ is a norm 1 projection.

Let $g \in B_{C(K)}$ and consider $\|(Id - P)g\| = \|g - Pg\| = \|g - \langle \mu, g \rangle f\|$. For $x \notin H$ we have

$$|g(x) - \langle \mu, g \rangle f(x)| \leq 1 + 1 - \delta = 2 - \delta.$$

While for $x \in H$ we use that

$$\langle \mu, g \rangle = \frac{1}{|H|} \sum_{h \in H} \varepsilon_h g(h)$$

and $\varepsilon_h f(h) = |f(h)| = 1$, so that

$$\begin{aligned} |g(x) - \langle \mu, g \rangle f(x)| &= |g(x) - \frac{1}{|H|} \sum_{h \in H} \varepsilon_h g(h) f(x)| \\ &= \left| \left(1 - \frac{1}{|H|}\right) g(x) - \frac{1}{|H|} \sum_{h \in H \setminus \{x\}} \varepsilon_h g(h) f(x) \right| \\ &\leq \left(1 - \frac{1}{|H|}\right) + \frac{|H| - 1}{|H|} = 2 - \frac{2}{|H|}. \end{aligned}$$

With $\varepsilon = \min\{\delta, 2/|H|\}$ we have $\|(Id - P)g\| \leq 2 - \varepsilon < 2$ for all $g \in B_{C(K)}$ hence $\|Id - P\| < 2$. $\square$

Let X be a Banach space such that X^* is isometric to an $L_1(\mu)$-space, that is, X is a Lindenstrauss space. For such spaces we have X^{**} is isometric to the space $C(K)$ for some (extremally disconnected) compact Hausdorff space K (see [15, Theorem 6.1]). Our next goal is to show that for such spaces Δ- and Daugavet-points are the same. We first need a lemma.

Lemma 3.5. *Let X be a Banach space and let $x, y \in S_X$. The following assertions are equivalent:*

(1) $y \in \overline{\operatorname{conv}}\, \Delta_\varepsilon^X(x)$ for all $\varepsilon > 0$;
(2) $y \in \overline{\operatorname{conv}}\, \Delta_\varepsilon^{X^{**}}(x)$ for all $\varepsilon > 0$.

Proof. (1) $\Rightarrow$ (2) is trivial as $\Delta_\varepsilon^X(x) \subset \Delta_\varepsilon^{X^{**}}(x)$.

(2) $\Rightarrow$ (1). Let $\varepsilon > 0$ and $\delta > 0$. Find $y_n^{**} \in B_{X^{**}}$ such that $\|x - y_n^{**}\| \geq 2 - \varepsilon$ and $\|y - \sum_{n=1}^m \lambda_n y_n^{**}\| < \delta$.

Define $E := \operatorname{span}\{x, y, y_n^{**}\}$. Let $\eta > 0$ and use the principle of local reflexivity to find $T : E \to X$ such that

(i) $T(e) = e$ for all $e \in E \cap X$.
(ii) $(1-\eta)\|e\| \leq \|Te\| \leq (1+\eta)\|e\|$.

Then $\|x - Ty_n^{**}\| = \|T(x - y_n^{**})\| \geq (1-\eta)\|x - y_n^{**}\| > 2 - \varepsilon$ if η is small enough. Also, if η is small enough,

$$\|y - \sum_{n=1}^{m} \lambda_n T y_n^{**}\| \leq (1+\eta)\|y - \sum_{n=1}^{m} \lambda_n y_n^{**}\| < \delta.$$

□

Remark 3.6. The argument shows that the conclusion in Lemma 3.5 also holds in the more general setting of X being an almost isometric ideal (see [4] for a definition) in Z, replacing X^{**} with Z.

Theorem 3.7. *Let X be an (infinite dimensional) $L_1(\mu)$-predual and $x \in S_X$. The following assertions are equivalent:*

(1) x is a Δ-point;
(2) x is a Daugavet point.

Proof. (1) $\Rightarrow$ (2). By Lemma 3.5 we get $x \in \overline{\operatorname{conv}}\, \Delta_\varepsilon^{X^{**}}(x)$ for all $\varepsilon > 0$. Since X^{**} is isometric to a $C(K)$-space, we get from Theorem 3.4 that x is a Daugavet-point in X^{**}, that is, $B_{X^{**}} = \overline{\operatorname{conv}}\, \Delta_\varepsilon^{X^{**}}(x)$ for all $\varepsilon > 0$. Using Lemma 3.5 again we get the desired conclusion.

(2) $\Rightarrow$ (1) is trivial. □

Theorem 3.8. *Let X be an $L_1(\mu)$-predual. The following assertions are equivalent:*

(1) $\|Id - P\| = 2$ for all norm-1 rank-1 projections P on X;
(2) X has the Daugavet property.

Proof. (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2). If $\|Id - P\| = 2$ for all norm-1 rank-1 projections, then X^* has the w^*-LD2P by Proposition 2.5 which is equivalent to X having extremely rough norm. By [7, Theorem 2.4] this implies the Daugavet property for $L_1(\mu)$-predual spaces. □

3.3. **Müntz space.** Now we explore Δ- and Daugavet-points in the setting of Müntz spaces. Let us first clarify what we mean by such spaces.

Definition 3.9. Let $\Lambda = (\lambda_n)_{n=0}^{\infty}$ be an increasing sequence of non-negative real numbers

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$$

such that $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty$. Then $M(\Lambda) := \overline{\operatorname{span}}\{t^{\lambda_n}\}_{n=0}^{\infty} \subset C[0,1]$ is called the Müntz space associated with Λ.

We will sometimes need to exclude the constants and consider the subspace $M_0(\Lambda) := \overline{\operatorname{span}}\{t^{\lambda_n}\}_{n=1}^{\infty}$ of $M(\Lambda)$.

In order to prove a result about the Daugavet points in Müntz spaces, we need the following result.

Lemma 3.10. *For all $\varepsilon > 0$ and $\delta > 0$, there exist $k, l \in \mathbb{N}$ with $k < l$ such that for $f = (t^{\lambda_k} - t^{\lambda_l})/\|t^{\lambda_k} - t^{\lambda_l}\|$ one has $f \geq 0$ and $f|_{[0,1-\varepsilon]} < \delta$.*

Proof. Fix positive numbers ε and δ. Let k be such that

$$t^{\lambda_k}|_{[0,1-\varepsilon]} < \frac{\delta}{2}.$$

Choose $l > k$ such that $\|t^{\lambda_k} - t^{\lambda_l}\| > 1/2$. Then

$$\frac{t^{\lambda_k} - t^{\lambda_l}}{\|t^{\lambda_k} - t^{\lambda_l}\|} < \frac{\delta/2}{1/2} = \delta$$

for any $t \in [0, 1-\varepsilon]$. □

Proposition 3.11. *Let $X = M(\Lambda)$ or $X = M_0(\Lambda)$. If $f \in S_X$ satisfies $f(1) = \pm 1$, then f is a Daugavet point.*

Proof. Fix $f \in S_X$ with $f(1) = \pm 1$ and $\varepsilon > 0$. We show that any $g \in S_X$ can be approximated by the elements of $\operatorname{conv} \Delta_\varepsilon(f)$. For this purpose, fix $g \in S_X$, $\delta > 0$ and choose $m \in \mathbb{N}$ with $m \geq 2/\delta$.

Let $t_1 \in (0,1)$ be such that $|f(1) - f(t)| < \delta$ and $|g(1) - g(t)| < \delta$ for all $t \in [t_1, 1]$. We use Lemma 3.10 to obtain f_1 such that $f_1|_{[0,t_1]} < \delta/2$.

Let $t_2 \in (0,1)$ be such that $f_1|_{[t_2,1]} < \delta/2$. We use Lemma 3.10 again to obtain f_2 such that $f_2|_{[0,t_2]} < \delta/2$.

We continue finding $t_0 < t_1 < \cdots < t_m < t_{m+1} =: 1$ and $f_1, \ldots, f_m$. Define $g_i := g - [g(1)+1]f_i$ for $i = 1, \ldots, m$. Then $\|g_i\| \leq 1+\delta$. Indeed, for $t \in [0,1] \setminus [t_i, t_{i+1}]$ we have that $f_i(t) < \delta/2$ and therefore

$$|g_i(t)| \leq |g(t)| + (1 + g(1))f_i(t) < 1 + 2\frac{\delta}{2} = 1 + \delta,$$

while for $t \in [t_i, t_{i+1}]$ we have

$$\begin{aligned}
|g_i(t)| &\leq |g(1) - [g(1) + 1]f_i(t)| + |g(t) - g(1)| \\
&\leq |g(1)|(1 - f_i(t)) + f_i(t) + \delta \\
&\leq 1 - f_i(t) + f_i(t) + \delta = 1 + \delta.
\end{aligned}$$

Denote by s_i the unique point in (t_i, t_{i+1}) where $f_i(s_i) = 1$. We have

$$\begin{aligned}
\|g_i - f\| &\geq |g_i(s_i) - f(s_i)| \\
&= |(g(s_i) - (g(1) + 1)) - f(s_i)| \\
&\geq |1 + f(s_i)| - |g(1) - g(s_i)| \\
&\geq 2 - \delta - \delta = 2 - 2\delta.
\end{aligned}$$

Hence

$$\|(1+\delta)^{-1}g_i - f\| \geq \|g_i - f\| - \|(1+\delta)^{-1}g_i - g_i\| \geq 2 - 3\delta$$

since

$$\|(1+\delta)^{-1}g_i - g_i\| = |(1+\delta)^{-1} - 1|\|g_i\| \leq |(1+\delta)^{-1} - 1|(1+\delta) \leq \delta.$$

We get that $(1+\delta)^{-1}g_i \in \Delta_\varepsilon(f)$ whenever $3\delta < \varepsilon$. Finally

$$\begin{aligned}\|g - \sum_{i=1}^{m} \frac{1}{m}(1+\delta)^{-1}g_i\| &= \|(1-(1+\delta)^{-1})g + (1+\delta)^{-1}[g(1)+1]\sum_{i=1}^{m}\frac{1}{m}f_i\| \\ &\leq \frac{\delta}{1+\delta}\|g\| + \frac{(g(1)+1)}{m(1+\delta)}\|\sum_{i=1}^{m} f_i\| \\ &\leq \frac{\delta}{1+\delta} + \frac{2}{m}(1+(m-1)\frac{\delta}{2}) \\ &\leq \delta + \delta + \delta \leq 3\delta.\end{aligned}$$

Hence $g \in \overline{\operatorname{conv}}\,\Delta_\varepsilon(f)$. $\square$

Proposition 3.12. *Let X be a Müntz space $M_0(\Lambda)$ with $\lambda_1 \geq 1$. If $f \in S_X$ with $|f(1)| < 1$, then $f \notin \Delta$.*

Proof. First note that from the full Clarkson-Erdös-Schwartz theorem (see [10]), f is the restriction to $(0,1)$ of an analytic function on $\Omega = \{x \in \mathbb{C} \setminus (-\infty, 0] : |z| < 1\}$. Let I be the set of points in $[0,1]$ where f attains its norm, and put $I^{\pm} = \{x \in I : f(x) = \pm 1\}$. From the assumptions we have $I \subset (0,1)$ since every $g \in M_0(\Lambda)$ satisfies $g(0) = 0$.

Suppose I is infinite. Then either I^+ or I^- is infinite. Suppose without loss of generality that I^+ is. Then I^+ must have an accumulation point a in $[0,1]$. By the continuity of f we must have $f(a) = 1$, so $0 < a < 1$. Since f is analytic on Ω and I^+, I^+ has an accumulation point in $(0,1) \subset \Omega$, we must have $1 - f = 0$ everywhere, which is a contradiction.

Suppose I is finite and that f attains its norm on $(y_k)_{k=1}^m \subset (0,1)$ with $0 < y_1 < y_2 < \cdots < y_m < 1$, i.e. $1 = \|f\| = |f(y_k)|$ for every $k = 1, \ldots, m$. By density it suffices to show that there is $\varepsilon > 0$ such that $f \notin \overline{\operatorname{conv}}(\Delta_\varepsilon(f) \cap P)$ where $P = \operatorname{span}(t^{\lambda_n})_{n=1}^\infty \subset X$. To this end, let s be a point satisfying $(1+y_m)/2 < s < 1$. By the Bernstein inequality [9, Theorem 3.2], there exists a constant $c = c(\Lambda, s)$ such that for any $p \in P$

$$\|p'\|_{[0,s]} \leq c\|p\|_{[0,1]}.$$

Since $f \in C[0,1]$ there exists $\delta > 0$ such that for all $x, y \in [0,1]$

$$|x - y| < \delta \implies |f(x) - f(y)| < 1.$$

By choosing δ smaller if necessary we may assume that $c\delta < 1/2$ and that $y_m + \delta/2 < s$. Let $I_{k,\delta} := (y_k - \delta/2, y_k + \delta/2)$. Note that f does not change sign on any $I_{k,\delta}$.

Put $I_\delta := \cup_{k=1}^m I_{k,\delta}$, and $M := \sup\{|f(y)| : y \in [0,1] \setminus I_\delta\}$. Since $[0,1] \setminus I_\delta$ is compact and since f is continuous, the value M is attained and thus $M < 1$. Let $0 < \varepsilon < \min\{1/(2m), 1-M, 1/4\}$. Then

$$|f(x)| \geq 1-\varepsilon \implies x \in I_\delta.$$

Assume that $p \in \Delta_\varepsilon(f) \cap P$. Since $\|f-p\| \geq 2-\varepsilon$ the norm is attained on I_δ. Therefore there exist k and $x \in I_{k,\delta}$ such that

$$|f(x) - p(x)| \geq 2 - \varepsilon.$$

Since $|f(x)| \geq 1-\varepsilon$ and f does not change sign on $I_{k,\delta}$ we must have $|f(x) - f(y_k)| \leq \varepsilon$ hence

$$\begin{aligned} |f(y_k) - p(y_k)| &\geq |f(x) - p(x)| - |f(y_k) - f(x)| - |p(x) - p(y_k)| \\ &\geq 2 - 2\varepsilon - \|p_i'\|_{[0,s]}|x - y_k| > 3/2 - c\delta > 1. \end{aligned}$$

Now, let $n \in \mathbb{N}$ and $p_1, \ldots, p_n \in \Delta_\varepsilon(f) \cap P$. Find $r \in \mathbb{N}$ such that $(r-1)m < n \leq rm$. By the pigeonhole principle, there is an interval $I_{j,\delta}$ where at least r of the polynomials $(p_i)_{i=1}^n$ satisfy $|f(y_j) - p_i(y_j)| > 1$. Put

$$L := \{i \in \{1, \ldots, n\} : |f(y_j) - p_i(x)| > 2 - 2\varepsilon, x \in I_{j,\delta}\}.$$

We get that

$$\begin{aligned} |f(y_j) - \frac{1}{n}\sum_{i=1}^n p_i(y_j)| &\geq |f(y_j) - \frac{1}{n}\sum_{i\in L} p_i(y_j)| - \frac{1}{n}\sum_{i\notin L} |p_i(y_j)| \\ &> 1 - \frac{1}{n}\sum_{i\notin L} 1 \geq \frac{r}{n} \geq \frac{1}{m} > \varepsilon. \end{aligned}$$

Hence $f \notin \overline{\mathrm{conv}}(\Delta_\varepsilon(f) \cap P)$. $\square$

Theorem 3.13. *Let X be a Müntz space $M_0(\Lambda)$ with $\lambda_1 \geq 1$. The following assertions for $f \in S_X$ are equivalent:*

(1) f is a Daugavet point;
(2) f is a Δ-point;
(3) $\|f\| = |f(1)|$.

Proof. (1) $\Rightarrow$ (2) is trivial, (2) $\Rightarrow$ (3) follows from Proposition 3.12, and (3) $\Rightarrow$ (1) is Proposition 3.11. $\square$

4. Stability results

Let us recall that a norm N on $\mathbb{R}^2$ is *absolute* if

$$N(a,b) = N(|a|,|b|) \quad \text{for all } (a,b) \in \mathbb{R}^2$$

and *normalized* if

$$N(1,0) = N(0,1) = 1.$$

If X and Y are Banach spaces and N is an absolute normalized norm on $\mathbb{R}^2$, then we denote by $X \oplus_N Y$ the product space $X \times Y$ with the norm $\|(x,y)\|_N = N(\|x\|, \|y\|)$.

In this section we analyze how Δ- and Daugavet-points behave while taking direct sums with absolute normalized norm N. First note a useful result that simplifies the proofs.

Lemma 4.1. *Let $m \in \mathbb{N}$. Then for all $\varepsilon > 0$, all $\lambda_i > 0$ with $\sum_{i=1}^m \lambda_i = 1$, there exists $n \in \mathbb{N}$, $k_1, \dots, k_m \in \mathbb{N}$ such that*

$$\sum_{i=1}^m \left|\lambda_i - \frac{k_i}{n}\right| < \varepsilon \qquad \text{and} \qquad \sum_{i=1}^m k_i = n.$$

In particular, every convex combination of elements in a normed vector space can be approximated arbitrarily well with an average of the same elements (each repeated k_i times). Furthermore, given two such convex combinations, we can express them both as an average of the same number of elements.

Proof. By Dirichlet's approximation theorem given $N \in \mathbb{N}$ there exist integers $k_1, \dots, k_m$ and $1 \le n \le N$ such that

$$\left|\lambda_i - \frac{k_i}{n}\right| \le \frac{1}{nN^{1/m}}.$$

Then

$$\left|n - \sum_{i=1}^m k_i\right| = n\left|\sum_{i=1}^m \lambda_i - \sum_{i=1}^m \frac{k_i}{n}\right| \le n\sum_{i=1}^m \frac{1}{nN^{1/m}} = \frac{m}{N^{1/m}}.$$

By just choosing N so large that $N^{-1/m} < \varepsilon$ and $mN^{-1/m} < 1$ we get the desired conclusion. By choosing $\varepsilon > 0$ smaller if necessary we can make sure that $k_i \ge 0$ for $i = 1, \dots, m$. $\square$

It is not hard to see that if a Banach space X has a Δ-point, then $X \oplus_N Y$ has a Δ-point too for any Banach space Y. Moreover, if $x \in \Delta_X$ and $y \in \Delta_Y$, then for all $a, b \ge 0$ with $N(a,b) = 1$ we have $(ax, by) \in \Delta_Z$ (see the proof of Theorem 5.8). This implies that if X and Y both have the DLD2P then $X \oplus_N Y$ has the DLD2P for any absolute normalized norm N on $\mathbb{R}^2$ (this was shown in [13] using slices). In contrast, there are absolute normalized norms N for which the space $X \oplus_N Y$ has no Daugavet-points. Therefore there even exists a space where every unit sphere point is a Δ-point, but none of them are Daugavet-points. However, the matter of the existence of Daugavet-points in direct sums is more complex as can be seen from the following propositions.

Definition 4.2. An absolute normalized norm N on $\mathbb{R}^2$ is *positively octahedral* [12] if there exist $a, b \ge 0$ such that $N(a,b) = 1$, and

$$N\left((0,1) + (a,b)\right) = 2 \quad \text{and} \quad N\left((1,0) + (a,b)\right) = 2.$$

Proposition 4.3. *Let N be a positively octahedral norm on $\mathbb{R}^2$. If X and Y are two Banach spaces that both have Daugavet-points, then $X \oplus_N Y$ also has a Daugavet-point.*

Proof. Let X and Y be Banach spaces and N a positively octahedral absolute normalized norm. Denote $Z = X \oplus_N Y$. Let $x \in S_X$ and $y \in S_Y$ be Daugavet points. Since N is positively octahedral, there exist $a, b \geq 0$ such that $N(a, b) = 1$ and $N((a, b) + (c, d)) = 2$ for every $c, d \geq 0$ with $N(c, d) = 1$. We will show that (ax, by) is a Daugavet point.

Let $\nu := N(1, 1)$. Fix $\varepsilon > 0$, $(u, v) \in S_Z$, and $\delta > 0$. First consider the case $u \neq 0$ and $v \neq 0$. Since $u/\|u\| \in \overline{\operatorname{conv}}\,\Delta^X_{\varepsilon/\nu}(x)$ and $v/\|v\| \in \overline{\operatorname{conv}}\,\Delta^Y_{\varepsilon/\nu}(y)$, we have $x_1, \dots, x_m \in \Delta^X_{\varepsilon/\nu}(x)$ and $y_1, \dots, y_m \in \Delta^Y_{\varepsilon/\nu}(y)$ such that (here we use Lemma 4.1 to get the same number of vectors in X and Y)

$$\left\|\frac{u}{\|u\|} - \frac{1}{m}\sum_{i=1}^m x_i\right\| < \delta \quad \text{and} \quad \left\|\frac{v}{\|v\|} - \frac{1}{m}\sum_{i=1}^m y_i\right\| < \delta.$$

Therefore

$$\begin{aligned}
&\left\|(u, v) - \frac{1}{m}\sum_{i=1}^m \Big(\|u\|x_i, \|v\|y_i\Big)\right\|_N \\
&= N\Big(\|u\|\left\|\frac{u}{\|u\|} - \frac{1}{m}\sum_{i=1}^m x_i\right\|, \|v\|\left\|\frac{v}{\|v\|} - \frac{1}{m}\sum_{i=1}^m y_i\right\|\Big) \\
&\leq \delta N\Big(\|u\|, \|v\|\Big) = \delta.
\end{aligned}$$

Note that

$$\|ax - \|u\|x_i\| \geq a + \|u\| - \varepsilon/\nu$$

and

$$\|by - \|v\|y_i\| \geq b + \|v\| - \varepsilon/\nu$$

by the reverse triangle inequality. This implies that $\Big(\|u\|x_i, \|v\|y_i\Big) \in \Delta^Z_\varepsilon(ax, by)$ since

$$\begin{aligned}
&N\Big(\,\|ax - \|u\|x_i\|\,, \|by - \|v\|y_i\|\,\Big) \\
&\geq N\Big(a + \|u\| - \varepsilon/\nu, b + \|v\| - \varepsilon/\nu\Big) \\
&\geq N\Big(a + \|u\|, b + \|v\|\Big) - N\Big(\varepsilon/\nu, \varepsilon/\nu\Big) = 2 - \varepsilon.
\end{aligned}$$

If $u = 0$ or $v = 0$, the proof is simpler. □

Definition 4.4. We will say that an absolute normalized norm N on $\mathbb{R}^2$ has property (α) if for every $c, d \geq 0$ with $N(c, d) = 1$, there exist $\varepsilon > 0$ and neighbourhood W of (c, d) in $\mathbb{R}^2$ such that:

- if $a, b \geq 0$ satisfies $N(a, b) = 1$ and
$$N((a, b) + (c, d)) \geq 2 - \varepsilon,$$
then $(a, b) \in W$;
- either $\sup_{(a,b)\in W} a < 1$ or $\sup_{(a,b)\in W} b < 1$.

Remark 4.5. The ℓ_p-norm, $1 < p < \infty$, on $\mathbb{R}^2$ has property (α).

Given $c, d \geq 0$ with $\|(c,d)\|_p = 1$ for all $\delta > 0$ there exists $\varepsilon > 0$ such that for all (a,b) with $\|(a,b)\|_p \leq 1$ and $\|(a,b) + (c,d)\|_p \geq 2 - \varepsilon$ we have $(a,b) \in B((c,d),\delta) =: W$. Choosing δ small enough we have either $\sup_{(a,b)\in W} a < 1$ or $\sup_{(a,b)\in W} b < 1$.

Similarly, any strictly convex absolute normalized norm N on $\mathbb{R}^2$ has property (α).

Proposition 4.6. *Let X and Y be Banach spaces and N an absolute normalized norm on $\mathbb{R}^2$ with property (α). Then $X \oplus_N Y$ has no Daugavet points.*

Proof. Let X and Y be Banach spaces and N an absolute normalized norm on $\mathbb{R}^2$ with property (α). Denote $Z = X \oplus_N Y$ and let $z = (x,y) \in S_Z$.

Let $(c,d) = (\|x\|, \|y\|)$. From the definition of property (α) there exists $\varepsilon > 0$ and a neighbourhood W of (c,d). Without loss of generality we may assume that $\sup_{(a,b)\in W} a < 1$ since the case $\sup_{(a,b)\in W} b < 1$ is similar. Choose $\delta > 0$ such that $\sup_{(a,b)\in W} \leq 1 - \delta$.

Assume that $(u,v) \in \Delta_\varepsilon(z)$. Then

$$2 - \varepsilon \leq N(\|u - x\|, \|v - y\|) \leq N(\|u\| + \|x\|, \|v\| + \|y\|),$$

hence $(\|u\|, \|v\|) \in W$ from property (α). In particular, $\|u\| \leq 1 - \delta$.

Let $w \in S_X$ and consider $(w,0) \in S_Z$. Given $(x_1,y_1), \dots, (x_n,y_n) \in \Delta_\varepsilon(z)$ we have $\|x_i\| \leq 1 - \delta$ for each $i = 1, \dots, n$ and

$$\begin{aligned} \Big\|(w,0) - \frac{1}{n}\sum_{i=1}^n (x_i,y_i)\Big\|_N &\geq \|w - \frac{1}{n}\sum_{i=1}^n x_i\| \geq \|w\| - \frac{1}{n}\sum_{i=1}^n \|x_i\| \\ &\geq 1 - \frac{1}{n}\sum_{i=1}^n (1-\delta) = \delta. \end{aligned}$$

Using Lemma 4.1 we see that this means that $(w,0) \notin \overline{\operatorname{conv}}\, \Delta_\varepsilon(z)$, and we conclude that z is not a Daugavet-point. $\square$

Example 4.7. *Consider the space $X = C[0,1] \oplus_2 C[0,1]$.*

$C[0,1]$ has the Daugavet property and in particular the DLD2P, hence X has the DLD2P [13, Theorem 3.2]. *But, by Proposition 4.6, X has no Daugavet-points even though every $x \in S_X$ is a Δ-point.*

5. The convex DLD2P

In this last section we consider Banach spaces X with the property that $B_X = \overline{\operatorname{conv}}(\Delta)$. We show that this property is a diameter two property that differs from the already known diameter two properties. We also give examples of spaces with this new property.

Definition 5.1. Let X be a Banach space. If $B_X = \overline{\operatorname{conv}}(\Delta)$, then we say that X has the *convex diametral local diameter two property (convex DLD2P).*

Proposition 5.2. *Let X be a Banach space. If X has the convex DLD2P, then X has the LD2P.*

Proof. Let $x^* \in S_{X^*}$, $\varepsilon > 0$, and consider the slice

$$S(x^*, \varepsilon) = \{x \in B_X : x^*(x) > 1 - \varepsilon\}.$$

Pick some $\hat{x} \in S(x^*, \varepsilon/4)$. Choose $(x_i)_{i=1}^n \subset \Delta$ and a convex combination $x := \sum_{i=1}^n \lambda_i x_i$ with $\|x - \hat{x}\| < \varepsilon/4$. Now at least one of the x_i's must be in $S(x^*, \varepsilon/2)$ otherwise

$$x^*(x) = \sum_{i=1}^n \lambda_i x^*(x_i) < \sum_{i=1}^n \lambda_i (1 - \varepsilon/2) < 1 - \varepsilon/2$$

which contradicts the fact that $\hat{x} \in S(x^*, \varepsilon/4)$ and $\|\hat{x} - x\| < \varepsilon/4$. Now let x_k be one of the x_i's which are in $S(x^*, \varepsilon/2)$ and use the same idea as above to produce some $y \in \Delta_\varepsilon(x_k)$ such that $y \in S(x^*, \varepsilon)$. Since $x_k \in S(x^*, \varepsilon/2) \subset S(x^*, \varepsilon)$ and $\|x_k - y\| > 2 - \varepsilon$ we are done. $\square$

Proposition 5.3. *If K is an infinite compact Hausdorff space, then $C(K)$ has the convex DLD2P.*

Proof. We only need to show that $S_{C(K)} \subset \overline{\operatorname{conv}}\,\Delta$. Let $f \in C(K)$ with $\|f\| = 1$. If $|f(x)| = 1$ for some limit point of K, then $f \in \Delta$ by Theorem 3.4. Assume that $|f(x)| < 1$ for every limit point of K and let x_0 be a limit point of K.

Let $\varepsilon > 0$ and choose a neighbourhood U of x_0 such that $|f(x) - f(x_0)| < \varepsilon$ for every $x \in U$. We use Urysohn's lemma to find a function $\eta : K \to [0, 1]$ such that $\eta(x_0) = 1$ and $\eta = 0$ on $K \setminus U$. Define

$$f^+(x) := (1 - \eta(x))f(x) + \eta(x)(1),$$
$$f^-(x) := (1 - \eta(x))f(x) + \eta(x)(-1).$$

Then $f^\pm \in B_{C(K)}$ and both are in Δ by Theorem 3.4. Let $\lambda := \frac{1+f(x_0)}{2}$ and consider

$$g(x) := \lambda f^+(x) + (1 - \lambda)f^-(x).$$

Then

$$g(x) = \begin{cases} f(x) & x \in K \setminus U, \\ (1 - \eta(x))f(x) + \eta(x)f(x_0) & x \in U. \end{cases}$$

We get

$$\|g - f\| \leq \max_{x \in U} |\eta(x)(f(x) - f(x_0))| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we get that $f \in \overline{\operatorname{conv}}\,\Delta$. $\square$

Corollary 5.4. *Both $c = C([0, \omega])$ and $\ell_\infty = C(\beta\mathbb{N})$ have the convex DLD2P.*

Remark 5.5. In c the points in Δ are exactly the sequences with limit 1 or -1. For ℓ_∞ we have that Δ consists of all sequences $(x_n) \in \ell_\infty$ such that $|\lim_{\mathcal{U}} x_n| = 1$, where $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$. In particular, none of these spaces have the DLD2P.

For c_0 we have $\Delta = \emptyset$ since Δ-points in c_0 have to be Δ-points in ℓ_∞ by Lemma 3.5. Hence the convex DLD2P is not inherited from the bidual unlike the LD2P. The convex DLD2P is also not inherited by subspaces of codimension 1, since c_0 is of codimension 1 in c.

Considering the facts that ℓ_∞ does not have the DLD2P and c_0 has the LD2P, Remark 5.5, and Corollary 5.4, we can conclude that the convex DLD2P is a new diameter-2 property, different from the ones observed so far.

Corollary 5.6. *Let X be a Banach space. Then*

$$\textit{DLD2P} \implies \textit{convex DLD2P} \implies \textit{LD2P},$$

where the implications cannot be reversed.

Our next aim is to show that Müntz spaces also have the convex DLD2P.

Theorem 5.7. *Let $X = M(\Lambda)$ or $X = M_0(\Lambda)$ be a Müntz space. Then X has the convex DLD2P.*

Proof. It is enough to show that $S_X \subset \overline{\operatorname{conv}}\,\Delta$. Since $P := \operatorname{span}\{t^{\lambda_n}\}$ is dense in X, it is enough to show that if $f \in B_P$ with $\|f\| = 1 - s$ for some $0 < s < 1$, then $f \in \operatorname{conv}\Delta$. To this end, given $n \in \mathbb{N}$ we define

$$f_n^+(x) = f(x) + (1 - f(1))x^{\lambda_n}$$

and

$$f_n^-(x) = f(x) - (1 + f(1))x^{\lambda_n}.$$

From Proposition 3.11 we see that $f_n^\pm$ are candidates for being Δ-points since

$$f_n^\pm(1) = f(1) \pm (1 \mp f(1)) = \pm 1.$$

If we define $\mu = \frac{f(1)+1}{2}$, that is, $2\mu - 1 = f(1)$, we have a convex combination

$$\mu f_n^+(x) + (1-\mu)f_n^-(x) = f(x) + \big(2\mu - 1 - f(1)\big)x^{\lambda_n} = f(x).$$

We need to show that when n is large enough we have $f_n^\pm \in S_P$.

Since $f \in P$ we can write

$$f(x) = \sum_{k=0}^{m} a_k x^{\lambda_k}.$$

Now, f, f', and f'' are all generalized polynomials so by Descartes' rule of signs, see e.g. [11, Theorem 3.1], they only have a finite number of zeros on $(0,1]$. Hence there exists $t_0 \in (0,1)$ such that neither f' nor f'' changes sign on $(t_0, 1)$. Without loss of generality we may assume that $f' < 0$ on $(t_0, 1)$. (If $f' > 0$ on $(t_0, 1)$ we consider $-f$.)

There exists N such that

$$t_0^{\lambda_n} < s/2 \quad \text{for } n > N. \tag{5.1}$$

For $n > N$ we get

$$|f_n^-(x)| \leq 1 - s + (1 + f(1))s/2 \leq 1$$

on $[0, t_0]$ and on $[t_0, 1]$ we have

$$\frac{d}{dx}(f_n^-(x)) = f'(x) - \lambda_n(1 + f(1))x^{\lambda_n - 1} < 0.$$

We have $|f_n^-(x)| \leq 1$ in both endpoints of $[t_0, 1]$. Hence $\|f_n^-\| \leq 1$.

It remains to find $n > N$ such that also $f_n^+ \in S_P$. We consider two cases.

Case I: Assume there exists $0 < t_0 < 1$ such that $f' < 0$ and $f'' > 0$ on $(t_0, 1)$. For $n > N$ we have $d^2/dx^2(f_n^+) > 0$ on $(t_0, 1)$, hence f_n^+ is convex on $[t_0, 1]$ and (by using (5.1))

$$\|f_n^+\| \leq \max(f_n^+(t_0), f_n^+(1)) \leq \max(1 - s + (1 - f(1))t_n^{\lambda_n}, 1) \leq 1$$

since also $f_n^+(x) > f(x) \geq -1$ for all $x \in [0, 1]$.

Case II: Assume there exists $0 < t_0 < 1$ such that $f' < 0$ and $f'' < 0$ on $[t_0, 1]$.

Let $\delta := f(t_0) - f(1) > 0$. Define

$$t_n := \sqrt[\lambda_n]{1 - \frac{\delta}{1 - f(1)}},$$

that is

$$t_n^{\lambda_n} = \frac{1 - f(1) - \delta}{1 - f(1)}$$

Note that $t_n \to 1$.

Write $g_n(x) = (1 - f(1))x^{\lambda_n}$. Then $g_n'(x) = (1 - f(1))\lambda_n x^{\lambda_n - 1}$ and

$$\begin{aligned} g_n'(t_n) &= (1 - f(1))\lambda_n \frac{1 - f(1) - \delta}{1 - f(1)} \left(\frac{1 - f(1) - \delta}{1 - f(1)}\right)^{-1/\lambda_n} \\ &= \lambda_n(1 - f(1) - \delta) \left(\frac{1 - f(1) - \delta}{1 - f(1)}\right)^{-1/\lambda_n}. \end{aligned}$$

Note that $g_n'(t_n) \to \infty$ (since we assume that $\sum_{n=1}^{\infty} \lambda_n^{-1} < \infty$). Let $M := \max_{x \in [t_0, 1]} |f'(x)|$. Choose $n > N$ such that $t_0 < t_n < 1$ and

$$g_n'(t_n) > M.$$

Then for $x \in [t_n, 1]$ we have

$$\frac{d}{dx}(f_n^+(x)) = f'(x) + \lambda_n(1 - f(1))x^{\lambda_n - 1} > -M + g_n'(t_n) > 0$$

hence $f_n^+(x) \leq f_n^+(1)$ on $[t_n, 1]$.

For $x \in [t_0, t_n]$ we get

$$\begin{aligned} f_n^+(x) &= f(x) + g_n(x) \leq f(1) + \delta + (1 - f(1))t_n^{\lambda_n} \\ &= f(1) + \delta + (1 - f(1) - \delta) \leq 1. \end{aligned}$$

While on $[0, t_0]$ we have, by using (5.1),

$$|f_n^+(x)| \leq \|f\| + 2 \cdot s/2 \leq 1.$$

Hence $\|f_n^+\| \leq 1$. ☐

It is known that given Banach spaces X and Y, they have the Daugavet property if and only if $X \oplus_1 Y$ or $X \oplus_\infty Y$ has Daugavet property (see [14, Lemma 2.15] and [8, Corollary 5.4]). For the DLD2P we have that for any absolute normalized norm on $\mathbb{R}^2$, both X and Y have the DLD2P if and only if $X \oplus_N Y$ has the DLD2P [13, Theorem 3.2]. The following theorem shows that the convex DLD2P also behaves well under direct sums.

Theorem 5.8. *Let N be an absolute normalized norm on $\mathbb{R}^2$. If X and Y have the convex DLD2P, then $X \oplus_N Y$ has the convex DLD2P.*

Proof. Assume that X and Y are Banach spaces with the convex DLD2P. Denote $Z = X \oplus_N Y$.

Claim: If $a, b \geq 0$ with $N(a, b) = 1$, $x \in \Delta_X$, and $y \in \Delta_Y$, then $(ax, by) \in \Delta_Z$.

Proof of claim. Let $\varepsilon > 0$ and $0 < \gamma < \varepsilon$. Since $x \in \Delta_X$ and $y \in \Delta_Y$, we have $x_1, \ldots, x_m \in \Delta_\varepsilon^X(x)$ and $y_1, \ldots, y_m \in \Delta_\varepsilon^Y(y)$ such that (using Lemma 4.1)

$$\Big\|x - \frac{1}{m}\sum_{i=1}^m x_i\Big\| < \gamma \quad \text{and} \quad \Big\|y - \frac{1}{m}\sum_{i=1}^m y_i\Big\| < \gamma.$$

Note that

$$\begin{aligned}\Big\|(ax, by) - \frac{1}{m}\sum_{i=1}^m \big(ax_i, by_i\big)\Big\|_N &= N\Big(a\Big\|x - \frac{1}{m}\sum_{i=1}^m x_i\Big\|, b\Big\|y - \frac{1}{m}\sum_{i=1}^m y_i\Big\|\Big) \\ &\leq N(\gamma a, \gamma b) = \gamma N(a, b) = \gamma,\end{aligned}$$

and

$$\begin{aligned}\Big\|(ax, by) - (ax_i, by_i)\Big\|_N &= N(a\|x - x_i\|, b\|y - y_i\|) \\ &\geq N(a(2-\varepsilon), b(2-\varepsilon)) \\ &= (2-\varepsilon)N(a, b) = 2 - \varepsilon.\end{aligned}$$

This concludes the proof of the claim.

Now let $(x, y) \in S_Z$. We will show that $(x, y) \in \overline{\mathrm{conv}}\, \Delta_Z$.

Let $\delta > 0$. First consider the case $x \neq 0$ and $y \neq 0$. Then $\frac{x}{\|x\|} \in \overline{\mathrm{conv}}\, \Delta_X$ and $\frac{y}{\|y\|} \in \overline{\mathrm{conv}}\, \Delta_Y$ by the assumption; hence there are $x_1, \ldots, x_n \in \Delta_X$ and $y_1, \ldots, y_n \in \Delta_Y$ such that (here we use Lemma 4.1 again)

$$\Big\|\frac{x}{\|x\|} - \frac{1}{n}\sum_{i=1}^n x_i\Big\| < \delta \quad \text{and} \quad \Big\|\frac{y}{\|y\|} - \frac{1}{n}\sum_{i=1}^n y_i\Big\| < \delta.$$

By the claim above we have $(\|x\|x_i, \|y\|y_i) \in \Delta_Z$. All that remains is to note that

$$\begin{aligned}
&\Big\|(x,y) - \frac{1}{n}\sum_{i=1}^{n}\Big(\|x\|x_i, \|y\|y_i\Big)\Big\|_N \\
&= N\Big(\|x\|\Big\|\frac{x}{\|x\|} - \frac{1}{n}\sum_{i=1}^{n} x_i\Big\|, \|y\|\Big\|\frac{y}{\|y\|} - \frac{1}{n}\sum_{i=1}^{n} y_i\Big\|\Big) \\
&\le N\Big(\delta\|x\|, \delta\|y\|\Big) = \delta N\Big(\|x\|, \|y\|\Big) = \delta.
\end{aligned}$$

Now consider the case where $y = 0$ (a similar argument holds for the case $x = 0$). We have

$$\|(x,0)\|_N = N(\|x\|, 0) = \|x\|,$$

so that $(x,0) \in \overline{\operatorname{conv}}\,\Delta_Z$ follows from $x \in \overline{\operatorname{conv}}\,\Delta_X$ since the claim above shows that $(x_i, 0) \in \Delta_Z$ when $x_i \in \Delta_X$. □

Remark 5.9. Let X and Y be Banach spaces. If X has the convex DLD2P and N is the ℓ_∞-norm, then $X \oplus_N Y$ has the convex DLD2P.

Although we have mostly settled the results about the question whether the direct sum with absolute normalized norm has a Δ-point/a Daugavet-point/the convex DLD2P (there are some norms left to look at in the Daugavet-point case), the results about the components of a direct sum with a given property having the same property, are all still unknown.

Problem 1. *Given $X \oplus_N Y$ with a Δ-point/a Daugavet point/the convex DLD2P, does X have a Δ-point/a Daugavet point/the convex DLD2P?*

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