Applied Mathematics Letters 105 (2020) 106293

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations

Tongxing Li [a], Yuriy V. Rogovchenko [b,*]

[a] School of Control Science and Engineering, Shandong University, Jinan, Shandong 250061, PR China
[b] Department of Mathematical Sciences, University of Agder, Post Box 422, N-4604 Kristiansand, Norway

ARTICLE INFO

Article history:
Received 23 December 2019
Received in revised form 10 February 2020
Accepted 10 February 2020
Available online 18 February 2020

Keywords:
Asymptotic behavior
Oscillation
Neutral differential equation
Third-order
Comparison

ABSTRACT

By using comparison principles, we analyze the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations. Due to less restrictive assumptions on the coefficients of the equation and on the deviating argument τ, our criteria improve a number of related results reported in the literature.

1. Introduction

Higher-order functional differential equations have numerous applications in engineering and natural sciences, see Hale [1]. For instance, one can describe the behavior of solutions to third-order partial differential equations using information about the asymptotics of solutions to associated delay differential equations; see Agarwal et al. [2] for more details. In this paper, we are concerned with the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations

$$\left(a(t)\left(z''(t)\right)^{\gamma}\right)' + q(t)x^{\gamma}(\tau(t)) = 0, \tag{1}$$

where $t \in \mathbb{I} := [t_0, \infty) \subset (0, \infty)$, γ is a ratio of odd positive integers, $z(t) := x(t) + p_0 x(t - \delta_0)$, $p_0 \geq 0$, $p_0 \neq 1$, and δ_0 are constants, $\delta_0 \geq 0$ (delayed argument) or $\delta_0 \leq 0$ (advanced argument), $a,\ q,\ \tau \in \mathrm{C}(\mathbb{I}, \mathbb{R})$, $a(t) > 0$, $q(t) \geq 0$, $q(t)$ is not identical to zero for large t, and $\lim_{t\to\infty} \tau(t) = \infty$.

Let $t_* := \min\{t_0 - \delta_0, \min_{t\in\mathbb{I}} \tau(t)\}$. By a solution of Eq. (1) we understand a function $x \in \mathrm{C}([t_*, \infty), \mathbb{R})$ such that $a(z'')^{\gamma} \in \mathrm{C}^1(\mathbb{I}, \mathbb{R})$ and x satisfies (1) on $\mathbb{I}$. We consider only solutions of Eq. (1) which satisfy

* Corresponding author.
E-mail addresses: litongx2007@163.com (T. Li), yuriy.rogovchenko@uia.no (Yu.V. Rogovchenko).

https://doi.org/10.1016/j.aml.2020.106293
0893-9659/

$\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq t_0$ and tacitly assume that (1) possesses such solutions. A solution $x(t)$ of (1) is said to be oscillatory if it has arbitrarily large zeros on $[t_x, \infty)$ for some $t_x \in \mathbb{I}$ depending on the solution; otherwise, it is called nonoscillatory.

Usually, a second-order differential equation is called oscillatory if all its solutions oscillate. This is not the case for third-order equations whose solutions often exhibit different asymptotic behavior. Thus a third-order differential equation is called oscillatory if it has at least one oscillatory solution, see Erbe [3] and Hanan [4]. Furthermore, the presence of functional argument in the equation may significantly affect the nature of solutions. For example, a third-order linear differential equation

$$x'''(t) + x(t) = 0$$

has a nonoscillatory solution $x_1(t) = \mathrm{e}^{-t}$ along with the oscillatory solutions $x_2(t) = \mathrm{e}^{t/2}\cos\left(\sqrt{3}t/2\right)$ and $x_3(t) = \mathrm{e}^{t/2}\sin\left(\sqrt{3}t/2\right)$. However, one can completely eliminate all nonoscillatory solutions introducing the delayed argument and considering a third-order linear delay differential equation

$$x'''(t) + x(t-\pi) = 0.$$

By the result due to Ladas et al. [5, Theorem 1], all solutions to the latter equation are oscillatory since the associated characteristic equation $\lambda^3 + \mathrm{e}^{-\pi\lambda} = 0$ has no real roots. We note that such drastic changes in the asymptotic behavior of solutions are not specific for third-order equations and can be observed also for first-order differential equations. Taking into account that under our assumptions differential equation (1) can be both delayed and advanced and that we are concerned in this paper only with the asymptotic behavior of solutions, we tacitly assume that solutions to the equation under study exist and can be continued to infinity.

Numerous researchers analyzed asymptotic behavior of solutions to various classes of functional differential equations; see, for instance, the monographs [6–9], the papers [2,5,10–19], and the references cited therein. Assuming that

$$\int_{t_0}^{\infty} a^{-1/\gamma}(t)\mathrm{d}t = \infty, \tag{2}$$

$$0 \leq p_0 < 1, \tag{3}$$

$$\delta_0 \geq 0, \quad \tau(t) \leq t, \tag{4}$$

and

$$\text{either} \quad a'(t) \geq 0 \quad \text{or} \quad a'(t) \leq 0, \tag{5}$$

Baculíková and Džurina [10], Džurina et al. [14], and Yang and Xu [19] established several sufficient conditions which guarantee that all nonoscillatory solutions to Eq. (1) tend to zero at infinity. Baculíková and Džurina [11] and Li et al. [16,17] studied asymptotics of Eq. (1) for $\gamma = 1$ under conditions (3) and (4). Candan [13] analyzed behavior of solutions to (1) assuming that (3) holds and

$$\tau(t) = t - \tau_0. \tag{6}$$

Finally, Li and Rogovchenko [15] investigated Eq. (1) under conditions (6) and $0 \leq p_0 < \infty$.

The objective of this paper is to analyze the asymptotic nature of solutions to Eq. (1) in the case where condition (2) is satisfied but without assumptions (3)–(6). In the sequel, all functional inequalities are supposed to hold for all t large enough. Without loss of generality, we deal only with eventually positive solutions of (1) since, under our assumption on γ, if $x(t)$ is a solution of Eq. (1), so is $-x(t)$.

2. Auxiliary lemmas

The following lemmas will be used to establish our main results.

Lemma 1. *Assume that condition (2) is satisfied and let $x(t)$ be an eventually positive solution of Eq. (1). Then there exists a sufficiently large $t_1 \geq t_0$ such that, for all $t \geq t_1$, either*

$$z(t) > 0, \quad z'(t) > 0, \quad z''(t) > 0, \quad \left(a(t)\left(z''(t)\right)^{\gamma}\right)' \leq 0, \tag{7}$$

or

$$z(t) > 0, \quad z'(t) < 0, \quad z''(t) > 0, \quad \left(a(t)\left(z''(t)\right)^{\gamma}\right)' \leq 0. \tag{8}$$

Proof. Thanks to condition (2) employed also by Baculíková and Džurina [10, Lemma 1], the proof follows the same lines as in the cited paper since assumptions (3)–(5) are not required here. ■

Lemma 2 (*Győri and Ladas* [8, Lemma 1.5.1])**.** *Let $f,\ g \in \mathrm{C}(\mathbb{I},\mathbb{R})$ and $f(t) = g(t) + pg(t-c)$, $t \geq t_0 + \max\{0, c\}$, where $p \neq 1$ and c are constants. Assume that there exists a constant $l \in \mathbb{R}$ such that $\lim_{t\to\infty} f(t) = l$.*

(S_1) *If $\liminf_{t\to\infty} g(t) = g_* \in \mathbb{R}$, then $g_* = l/(1+p)$.*
(S_2) *If $\limsup_{t\to\infty} g(t) = g^* \in \mathbb{R}$, then $g^* = l/(1+p)$.*

Lemma 3. *Let $x(t)$ be an eventually positive solution of Eq. (1) and assume that $z(t)$ satisfies (8). If*

$$\int_{t_0}^{\infty}\int_{v}^{\infty}\left(\frac{1}{a(u)}\int_{u}^{\infty} q(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u\mathrm{d}v = \infty, \tag{9}$$

then

$$\lim_{t\to\infty} x(t) = 0. \tag{10}$$

Proof. By virtue of inequalities $z(t) > 0$ and $z'(t) < 0$, there exists a constant $z_0 \geq 0$ such that $\lim_{t\to\infty} z(t) = z_0$. We claim that $z_0 = 0$. Otherwise, using Lemma 2, we conclude that $\lim_{t\to\infty} x(t) = z_0/\left(1+p_0\right) > 0$. Then there should exist a $t_2 \geq t_0$ such that, for all $t \geq t_2$,

$$x(\tau(t)) > \frac{z_0}{2(1+p_0)} > 0. \tag{11}$$

It follows from (1) and (11) that

$$\left(a(t)\left(z''(t)\right)^{\gamma}\right)' \leq -\left(\frac{z_0}{2(1+p_0)}\right)^{\gamma} q(t).$$

Integrating this inequality from t to ∞, we conclude that

$$a(t)\left(z''(t)\right)^{\gamma} \geq \left(\frac{z_0}{2(1+p_0)}\right)^{\gamma}\int_t^{\infty} q(s)\mathrm{d}s,$$

which implies that

$$z''(t) \geq \frac{z_0}{2(1+p_0)}\left(\frac{1}{a(t)}\int_t^{\infty} q(s)\mathrm{d}s\right)^{1/\gamma}. \tag{12}$$

Integrating (12) from t to ∞, we have

$$-z'(t) \geq \frac{z_0}{2(1+p_0)}\int_t^{\infty}\left(\frac{1}{a(u)}\int_u^{\infty} q(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u.$$

One more integration from t_2 to ∞ yields

$$z(t_2) \geq \frac{z_0}{2(1+p_0)} \int_{t_2}^{\infty} \int_{v}^{\infty} \left(\frac{1}{a(u)} \int_{u}^{\infty} q(s) \mathrm{d}s \right)^{1/\gamma} \mathrm{d}u \mathrm{d}v,$$

which contradicts condition (9). Therefore, $\lim_{t\to\infty} z(t) = 0$, and the desired property (10) follows now from the inequality $0 < x(t) \leq z(t)$. ■

3. Main results

Theorem 4. *Let conditions (2) and (9) be satisfied and assume that*

$$\delta_0 \geq 0. \tag{13}$$

Suppose that there exists a function $\eta \in \mathrm{C}(\mathbb{I}, \mathbb{R})$ such that $\eta(t) \leq \tau(t)$, $\eta(t) < t$, and $\lim_{t\to\infty} \eta(t) = \infty$. If the first-order delay differential equation

$$y'(t) + \frac{q(t)}{(1+p_0)^{\gamma}} \left(\int_{t_2}^{\eta(t)} \int_{t_1}^{v} a^{-1/\gamma}(s) \mathrm{d}s \mathrm{d}v \right)^{\gamma} y(\eta(t)) = 0 \tag{14}$$

is oscillatory for all large $t_1 \geq t_0$ and for some $t_2 \geq t_1$, then every solution $x(t)$ of Eq. (1) is either oscillatory or satisfies (10).

Proof. Let $x(t)$ be a nonoscillatory solution of (1); assume that it is eventually positive. By Lemma 1, there exists a $t_1 \geq t_0$ such that either (7) or (8) hold for all $t \geq t_1$. For (8), it follows immediately from Lemma 3 that (10) holds and we need to consider the second case. Suppose now that conditions (7) are satisfied. Using the property $(a(t)(z''(t))^{\gamma})' \leq 0$, we conclude that

$$z'(t) = z'(t_1) + \int_{t_1}^{t} \frac{(a(s)(z''(s))^{\gamma})^{1/\gamma}}{a^{1/\gamma}(s)} \mathrm{d}s \geq a^{1/\gamma}(t) z''(t) \int_{t_1}^{t} a^{-1/\gamma}(s) \mathrm{d}s. \tag{15}$$

Integrating (15) from t_2 to t, $t_2 \geq t_1$, we obtain

$$z(t) \geq a^{1/\gamma}(t) z''(t) \int_{t_2}^{t} \int_{t_1}^{v} a^{-1/\gamma}(s) \mathrm{d}s \mathrm{d}v. \tag{16}$$

Since $z'(t) > 0$ and $z''(t) > 0$, there exists a positive constant c_0 (it is also possible that $c_0 = \infty$) such that $\lim_{t\to\infty} z'(t) = c_0 > 0$. Consequently, by Lemma 2, $\lim_{t\to\infty} x'(t) = c_0/(1+p_0) > 0$, and we conclude that

$$x'(t) > 0. \tag{17}$$

Conditions (13) and (17) yield that $z(t) = x(t) + p_0 x(t-\delta_0) \leq (1+p_0)x(t)$, that is,

$$x(t) \geq \frac{1}{1+p_0} z(t). \tag{18}$$

By virtue of the inequalities $\eta(t) \leq \tau(t)$, (17), and (18), we conclude that

$$x(\tau(t)) \geq x(\eta(t)) \geq \frac{1}{1+p_0} z(\eta(t)). \tag{19}$$

Using now (19) in (1), we arrive at

$$\left(a(t) \left(z''(t)\right)^{\gamma}\right)' + \frac{q(t)}{(1+p_0)^{\gamma}} z^{\gamma}(\eta(t)) \leq 0. \tag{20}$$

Combining inequalities (16) and (20), we deduce that

$$\left(a(t)\left(z''(t)\right)^{\gamma}\right)' + \frac{q(t)}{(1+p_0)^{\gamma}}\left(\int_{t_2}^{\eta(t)}\int_{t_1}^{v} a^{-1/\gamma}(s)\mathrm{d}s\mathrm{d}v\right)^{\gamma} a(\eta(t))\left(z''(\eta(t))\right)^{\gamma} \le 0,$$

which means that the function $y(t) := a(t)\left(z''(t)\right)^{\gamma}$ is a positive solution of a delay differential inequality

$$y'(t) + \frac{q(t)}{(1+p_0)^{\gamma}}\left(\int_{t_2}^{\eta(t)}\int_{t_1}^{v} a^{-1/\gamma}(s)\mathrm{d}s\mathrm{d}v\right)^{\gamma} y(\eta(t)) \le 0.$$

An application of the result due to Philos [18, Theorem 1] yields that the associated delay differential equation (14) also has a positive solution, which contradicts the assumption of the theorem. ■

Combining Theorem 4 with [9, Theorem 2.1.1], we derive the following useful result.

Corollary 5. *Let conditions (2), (9), and (13) be satisfied. Assume that there exists a function $\eta \in \mathrm{C}(\mathbb{I},\mathbb{R})$ such that $\eta(t) \le \tau(t)$, $\eta(t) < t$, and $\lim_{t\to\infty}\eta(t) = \infty$. If, for all large $t_1 \ge t_0$ and for some $t_2 \ge t_1$,*

$$\frac{1}{(1+p_0)^{\gamma}}\liminf_{t\to\infty}\int_{\eta(t)}^{t} q(u)\left(\int_{t_2}^{\eta(u)}\int_{t_1}^{v} a^{-1/\gamma}(s)\mathrm{d}s\mathrm{d}v\right)^{\gamma}\mathrm{d}u > \frac{1}{\mathrm{e}}, \tag{21}$$

then conclusion of Theorem 4 remains intact.

Proof. Condition (21) ensures that, by virtue of the result in Ladde et al. [9, Theorem 2.1.1], Eq. (14) is oscillatory. An application of Theorem 4 completes the proof. ■

The next result relates oscillation of (1) in the case when

$$\delta_0 \le 0 \tag{22}$$

to that of an associated first-order delay differential equation.

Theorem 6. *Let conditions (2) and (9) be satisfied and assume that (22) holds. Suppose also that there exists a function $\eta \in \mathrm{C}(\mathbb{I},\mathbb{R})$ such that $\eta(t) \le \tau(t)$, $\eta(t) < t - \delta_0$, and $\lim_{t\to\infty}\eta(t) = \infty$. If the first-order delay differential equation*

$$y'(t) + \frac{q(t)}{(1+p_0)^{\gamma}}\left(\int_{t_2}^{\eta(t)+\delta_0}\int_{t_1}^{v} a^{-1/\gamma}(s)\mathrm{d}s\mathrm{d}v\right)^{\gamma} y(\eta(t)+\delta_0) = 0 \tag{23}$$

is oscillatory for all large $t_1 \ge t_0$ and for some $t_2 \ge t_1$, then conclusion of Theorem 4 remains intact.

Proof. Let $x(t)$ be an eventually positive solution of (1). By Lemma 1, there exists a $t_1 \ge t_0$ such that either conditions (7) or (8) hold for all $t \ge t_1$. If (8) hold, Lemma 3 immediately yields the desired conclusion (10). Next, suppose that (7) are satisfied. As in the proof of Theorem 4, inequalities (16) and (17) hold. It follows now from the definition of $z(t)$ and inequalities (17) and (22) that $z(t) = x(t)+p_0x(t-\delta_0) \le (1+p_0)x(t-\delta_0)$, which implies that

$$x(t) \ge \frac{1}{1+p_0}z(t+\delta_0). \tag{24}$$

Using the assumption $\eta(t) \le \tau(t)$ and inequalities (17) and (24), we have

$$x(\tau(t)) \ge x(\eta(t)) \ge \frac{1}{1+p_0}z(\eta(t)+\delta_0). \tag{25}$$

Substitution of (25) into (1) yields

$$\left(a(t)\left(z''(t)\right)^{\gamma}\right)' + \frac{q(t)}{(1+p_0)^{\gamma}} z^{\gamma}(\eta(t)+\delta_0) \le 0. \tag{26}$$

Combining (16) and (26), we conclude that

$$\left(a(t)\left(z''(t)\right)^{\gamma}\right)' + \frac{q(t)}{(1+p_0)^{\gamma}} \left(\int_{t_2}^{\eta(t)+\delta_0} \int_{t_1}^{v} a^{-1/\gamma}(s)\mathrm{d}s\mathrm{d}v\right)^{\gamma} a(\eta(t)+\delta_0)\left(z''(\eta(t)+\delta_0)\right)^{\gamma} \le 0.$$

This means that the function $y(t) := a(t)\left(z''(t)\right)^{\gamma}$ is a positive solution of a delay differential inequality

$$y'(t) + \frac{q(t)}{(1+p_0)^{\gamma}} \left(\int_{t_2}^{\eta(t)+\delta_0} \int_{t_1}^{v} a^{-1/\gamma}(s)\mathrm{d}s\mathrm{d}v\right)^{\gamma} y(\eta(t)+\delta_0) \le 0.$$

Then, using the result of Philos [18, Theorem 1] once again, we deduce that the associated delay differential equation (23) also has a positive solution, which contradicts the main assumption of the theorem. ■

Combining Theorem 6 with [9, Theorem 2.1.1], we obtain the following result.

Corollary 7. *Let conditions (2), (9), and (22) be satisfied. Assume that there exists a function $\eta \in \mathrm{C}(\mathbb{I},\mathbb{R})$ such that $\eta(t) \le \tau(t)$, $\eta(t) < t - \delta_0$, and $\lim_{t\to\infty}\eta(t) = \infty$. If, for all large $t_1 \ge t_0$ and for some $t_2 \ge t_1$,*

$$\frac{1}{(1+p_0)^{\gamma}} \liminf_{t\to\infty} \int_{\eta(t)+\delta_0}^{t} q(u) \left(\int_{t_2}^{\eta(u)+\delta_0} \int_{t_1}^{v} a^{-1/\gamma}(s)\mathrm{d}s\mathrm{d}v\right)^{\gamma} \mathrm{d}u > \frac{1}{\mathrm{e}}, \tag{27}$$

then conclusion of Theorem 4 remains intact.

Proof. According to [9, Theorem 2.1.1], assumption (27) guarantees that Eq. (23) is oscillatory. A direct application of Theorem 6 yields the desired conclusion. ■

4. Examples and discussion

The following examples illustrate theoretical results presented in the previous section. In both examples, $t \ge 1$ and $p_0 \ne 1$ is a nonnegative real number.

Example 8. Choosing $\eta(t) = t - 1$ in Corollary 5, we conclude that every solution to a third-order neutral differential equation

$$\left(x(t) + p_0 x(t-1)\right)''' + (\mathrm{e}^2 + p_0\mathrm{e}^3)x(t+2) = 0$$

is either oscillatory or satisfies (10). In fact, $x(t) = \mathrm{e}^{-t}$ is an exact solution to this equation satisfying (10).

Example 9. An application of Corollary 7 with $\eta(t) = t$ yields that every solution to a third-order neutral differential equation

$$\left(x(t) + p_0 x(t+1)\right)''' + (\mathrm{e}^t + p_0\mathrm{e}^{t-1})x(2t) = 0$$

is either oscillatory or satisfies (10). As a matter of fact, $x(t) = \mathrm{e}^{-t}$ is an exact solution satisfying (10).

Remark 10. An important feature that distinguishes our results from many related theorems reported in the literature is that we do not impose specific restrictions on the deviating argument τ, that is, τ may be delayed, advanced, or change back and forth from advanced to delayed. On the other hand, we would like to point out that, contrary to Baculíková and Džurina [10,11], Candan [13], Džurina et al. [14], Li and Rogovchenko [15], Li et al. [16,17], and Yang and Xu [19], in our results we do not need restrictive conditions (3)–(6), which is an improvement compared to the results in the cited papers.

Remark 11. Theorems 4 and 6 and Corollaries 5 and 7 ensure that every solution $x(t)$ of Eq. (1) is either oscillatory or tends to zero as $t \to \infty$. Since the sign of the derivative $z'(t)$ changes, it is hard to derive sufficient conditions which ensure that all solutions of Eq. (1) are just oscillatory and do not satisfy (10). Neither is it possible to utilize the method exploited in this paper for proving that all solutions of Eq. (1) only have the property (10). These two interesting problems remain open for now.

CRediT authorship contribution statement

Tongxing Li: Methodology, Investigation, Writing - original draft. **Yuriy V. Rogovchenko:** Conceptualization, Writing - review & editing, Supervision.

Acknowledgments

The authors express their sincere gratitude to two anonymous referees for the careful reading of the original manuscript and useful comments that helped to improve the presentation of the results and accentuate important details. The research of TL was supported by NNSF of P.R. China (Grant No. 61503171), CPSF (Grant No. 2015M582091), and NSF of Shandong Province (Grant No. ZR2016JL021).

References

[1] J.K. Hale, Theory of Functional Differential Equations, Springer-Verlag, New York, 1977.
[2] R.P. Agarwal, M. Bohner, T. Li, C. Zhang, Oscillation of third-order nonlinear delay differential equations, Taiwanese J. Math. 17 (2013) 545–558.
[3] L. Erbe, Existence of oscillatory solutions and asymptotic behavior for a class of third order linear differential equations, Pacific J. Math. 64 (1976) 369–385.
[4] M. Hanan, Oscillation criteria for third-order linear differential equations, Pacific J. Math. 11 (1961) 919–944.
[5] G. Ladas, Y.G. Sficas, I.P. Stavroulakis, Necessary and sufficient conditions for oscillations of higher order delay differential equations, Trans. Amer. Math. Soc. 285 (1984) 81–90.
[6] R.P. Agarwal, L. Berezansky, E. Braverman, A. Domoshnitsky, Nonoscillation Theory of Functional Differential Equations with Applications, Springer, New York, 2012.
[7] R.P. Agarwal, M. Bohner, W.-T. Li, Nonoscillation and Oscillation: Theory for Functional Differential Equations, Marcel Dekker, Inc., New York, 2004.
[8] I. Győri, G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Clarendon Press, Oxford, 1991.
[9] G.S. Ladde, V. Lakshmikantham, B.G. Zhang, Oscillation Theory of Differential Equations with Deviating Arguments, Marcel Dekker, Inc., New York, 1987.
[10] B. Baculíková, J. Džurina, Oscillation of third-order neutral differential equations, Math. Comput. Modelling 52 (2010) 215–226.
[11] B. Baculíková, J. Džurina, Oscillation theorems for third order neutral differential equations, Carpathian J. Math. 28 (2012) 199–206.
[12] B. Baculíková, J. Džurina, Yu.V. Rogovchenko, Oscillation of third order trinomial delay differential equations, Appl. Math. Comput. 218 (2012) 7023–7033.
[13] T. Candan, Oscillation criteria and asymptotic properties of solutions of third-order nonlinear neutral differential equations, Math. Methods Appl. Sci. 38 (2015) 1379–1392.
[14] J. Džurina, E. Thandapani, S. Tamilvanan, Oscillation of solutions to third-order half-linear neutral differential equations, Electron. J. Differential Equations 2012 (2012) 1–9.
[15] T. Li, Yu.V. Rogovchenko, Asymptotic behavior of higher-order quasilinear neutral differential equations, Abstr. Appl. Anal. 2014 (2014) 1–11.
[16] T. Li, E. Thandapani, J.R. Graef, Oscillation of third-order neutral retarded differential equations, Int. J. Pure Appl. Math. 75 (2012) 511–520.
[17] T. Li, C. Zhang, G. Xing, Oscillation of third-order neutral delay differential equations, Abstr. Appl. Anal. 2012 (2012) 1–11.
[18] Ch.G. Philos, On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays, Arch. Math. (Basel) 36 (1981) 168–178.
[19] L. Yang, Z. Xu, Oscillation of certain third-order quasilinear neutral differential equations, Math. Slovaca 64 (2014) 85–100.