



UNIVERSITETET I AGDER

Using eye-tracking to understand inductive reasoning

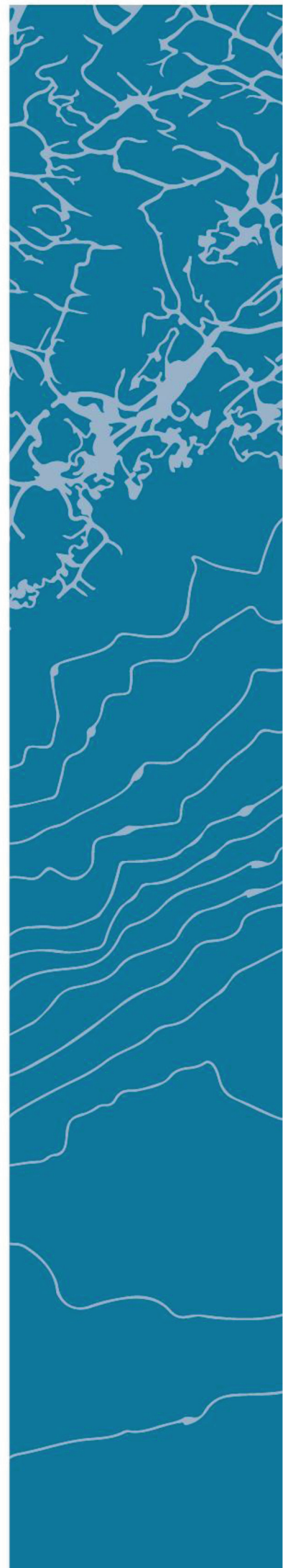
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Preface

Reasoning in general and inductive reasoning, specifically, are part of the “backbone” of mathematics education. Studying inductive reasoning with eye-tracking was a challenge and a chance for me to have insights in the field.

It is an original master thesis, in every component: research topic, question, design, data collection and data analysis and related reflections.

I would like to thank my supervisors prof. Yuriy Rogovchenko and prof. Cengiz Alacaci for suggesting me to use eye-tracking technology and for their support throughout my work and Tue Hvass for his technical advise on iMotions software.

I would also like to thank the Department of Mathematical Sciences for offering me the eye-tracking equipment which was crucial for my study, their premisses to conduct the experiments and book vouchers for the subjects involved in my study.

I would like to thank my parents and my sister for their support given during my master studies in mathematics education at University of Agder and sharing their knowledge in education for the purposes of this study.

Kristiansand, June 2020

Summary

This master thesis explains the work done to answer three research questions which are related to the understanding of inductive reasoning in solving shape pattern tasks.

A short introduction presents the topic of the thesis, the research questions, the challenges faced during the pandemic, the related effects on the quality of the thesis and main discoveries.

An analysis of related literature was done in order to link this research with previous theory in the field of math reasoning, especially inductive reasoning and eye-tracking method to collect and analyse the data. This has served to shape the research design.

The methodology chapter includes the research design, who are the subjects and how they were chosen. It presents the materials and instruments that helped to collect and analyse the data, the method chosen for data collection and the ways these data are analysed. This chapter ends with a brief discussion of the ethical consideration, validity and reliability issues.

This is followed by the chapter of analyses and results which shows detailed analyses of collected data and the results which constitute the main value of this research.

Strengths and limitations are presented in a separate chapter in order to give an idea to the reader of what the strong points that gave value to this study are and what could have gone better. These considerations help future researchers who may want to replicate this research by giving them a lot information about strengths and limitations.

The discussion of the results with support in previous research is done in order to list the arguments that support the results of this research.

The last chapter is of a particular importance as it presents the implication of the results in the field of mathematical education.

The thesis ends with the list of figures and tables as well as references.

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1. Introduction

The mathematical reasoning is crucial for the success of math teaching. The main modes of reasoning are deductive reasoning and inductive reasoning. Inductive reasoning is used mainly as a need to generalize, to have the knowledge for the whole set, by studying, or having evidence only for a part of it.

Teaching inductive reasoning needs theoretical support from different sources. Since it is in the field of the abstract operations of someone's mind, the importance of the behaviour of the person who is in the process of creating inductive reasoning products is often not valued. Therefore there is a prejudice that observing and studying the behaviour of the subject during this process won't bring any benefit to the knowledge we have about inductive reasoning.

This research tries to convince the reader that observing what is visible through eye-tracking technology while solving problems which engage inductive reasoning has some benefits in understanding it deeper.

One of the clear forms of inductive reasoning is seen in shape pattern task. The subject is obligated to use its eyes to get information which is the raw material of its reasoning, but the movements of eyes express the order given by the mind and the study of eye-movement is a mine to have additional knowledge about the reasoning.

It is known that eye movement are driven by the stimulus or from our mind. Discovering the distinction between them in the process of solving pattern problems is not an easy task. Here, the eye-tracking methodology brings many insights for holding wholes (gazing), discerning details, recognizing relationships and perceiving properties as generalities, which are thought as stages of inductive reasoning in solving shape pattern tasks (Rivera, 2015).

The theory of measuring the concept using indicators explained by Alan Bryman in his book "Social Research Methods" (fourth edition), 2012 was useful for this research in deciding the research questions and the research design.

The overall goal

The overall goal of this research is to discover how we can describe the modes of reasoning used by students when they are engaged in solving inductive reasoning tasks, using eye-tracking methodology and to know the characteristics of the skill of generalizing involved in effective inductive reasoning.

After some efforts in clarifying the literature which would serve as a theoretical support and prediction for the time needed for the research some adjustments were necessary. Obviously we studied the topic with eye-tracking technology, however, drawing conclusions of different types would still be difficult. This made it necessary to have concrete research questions which would contribute to the overall goal but not fully, and the research question were defined as follows:

The research questions are:

Research Question 1: What is the ratio of eye-tracking measures of the same type in different areas of interests in inductive reasoning in solving pattern problems?

Research Question 2: How to measure inductive reasoning in solving shape pattern tasks?

Research Question 3: What are the indicators to measure inductive reasoning in solving shape pattern tasks and how do they work?

The answers to the first research question contribute by adding additional knowledge about the mental process of inductive reasoning, for the existence of which we are aware of only by analysing eye-tracking measures.

Measuring the concept is part of the knowledge we have for that concept. The second and third research questions ask for this type of knowledge. The answers are of important value for teaching mathematics related to inductive reasoning.

There are several studies done with eye-tracking that contribute in many aspects of mathematics education. It was difficult to me to find literature closely related to the research questions of this thesis.

It is obvious that the aim of the research is for practical reasons, to use the new knowledge in addressing teaching methodology problems. A chapter in this thesis explains the benefits for a teacher who knows the facts explained here.

This research coincided with the pandemic Covid-19, which caused the lock-down of all educational institutions by the Norwegian government. This stopped the data collection process, reducing the number of subjects being observed with the eye-tracker.

This study was a pilot one in the sense that no other student had experience in research using eye-tracking technology. Despite many challenges encountered, the data collected this way was very helpful to answer the research questions and to have additional hypothesis.

Despite the above challenging situation, I am glad that the data collected was enough for some discoveries and good support for many useful conclusions explained in this thesis. The planned but uncollected data would result in many other conclusions and discoveries.

The research design of this study used unstructured observation as a main method of collecting data. Five subjects, three of them being master students in mathematics education and two of them secondary school students, took part in the study. Each of them dedicated an average of 50 minutes to this study, wearing eye-tracking glasses in order to see 5 pattern tasks displayed on a tablet through them. They interacted with the tablet to write what they wanted to. The eye-tracking glasses were able to record eye movements, and the video of the interaction including audio. The subject were allowed and instructed to think out loud during solving pattern tasks.

The measure of confidence is a concept spontaneously discovered during the data analysis. In the thesis it is shown how to measure it in two ways with many indicators or just with two of them.

The contribution of this research thesis to the knowledge was the finding of the evidence that the time spent for the figure 2 of the pattern and the number of fixations that are revisits for the figure 2, is nearly the sum of time and revisits for the figure 1 and figure 3. This can later be verified or not in additional research with larger set of data. There is evidence which shows that math teachers think that the time spent is directly proportional (in increasing order) with the areas of the figures, therefore the second being greater than the first but less than the third. This research shows that this is not the case.

The research was used to formulate additional hypothesis which need to be verified on further researches.

I hope this thesis will be an additional help for teachers of mathematics not only for the purposes of their work with inductive reasoning but in shaping their need to use eye-tracking methodology in other fields of maths teaching.

2. LITERATURE REVIEW

2. 1 Inductive reasoning

One of the topics in school mathematics is inductive reasoning. In fact, developing general mathematical thinking skills (analyzing, problem solving, generalizing, deductive/inductive reasoning) is a goal of school mathematics, which is necessary for cultivating the mathematical proficiency of students. “Generalizations are the lifeblood of mathematics” (Mason et al., 2010).

2.1.1 Inductive reasoning - the concept, the related skills, and its importance

“Klauer and Phye (1994) define inductive reasoning as the discovery of regularities through the detection of similarities, differences, or a combination of the two with respect to either object attributes or relations between objects.” (Perret, 2015).

Inductive reasoning (IR) can be considered as a form of higher-level cognitive processes. Other cognitive processes are: attention, perception, memory, language, learning. It is obvious that to reach at the level of “higher reasoning” one should go through all the processes. You can’t have good reasoning skills without having had good performance with attention, perception, memory, language, and learning. Inductive reasoning requires moving from the known to the unknown. Because of this the inferences we draw by reasoning inductively may be challenging because “it involves forming and testing hypotheses about rules” (Perret, 2015).

According to Perret (2015), we have inductive reasoning when we:

Attribute a feature we observe in one part to the remaining part, for example, if the first numbers that come from a box are even numbers we predict the following numbers will be even too. This is called feature attribution.

In *analogical reasoning* we “copy” the relation of two concepts as being true for two other concepts. This is an instance of inductive reasoning because there is uncertainty involved.

In *causal reasoning* we try to find the cause-effect relationship between two events. We find the cause in the previous set of events that precede the effect. Perret (2015) claims that we have “probabilistic judgment when we modulate our predictions about future events after taking a sample of similar events into account”.

Inductive reasoning proves to be very important in problem solving because it “lies at the heart of fluid intelligence” (Perret, 2015). It is obvious that creativity, as a high skill which uncovers or invents the unknown often requires using inductive reasoning.

2.1.2 Patterns and inductive reasoning and patterns in the math curriculum

Many inductive reasoning problems are composed of series of shapes, and other visuals or symbols connected with logical and semantic links.

The tasks consists of:

- (a) detecting regularities,
- (b) abstracting relations, and
- (c) deriving general rules.

“It is one of the main levers by which children understand how the world around them works through the abstraction of its underlying rules” (Perret, 2015).

Awareness levels or attentional states applied to pattern generalization are studied by Mason, 2009 et al., Watson, 2009. They list:

- a) holding wholes (gazing),
- b) discerning details (making distinctions),
- c) recognizing relationships (among specific discerned elements) and
- d) perceiving properties (as generalities which may be instantiated in specific situations) and
- e) reasoning on the basis of identified properties. (Rivera, 2015)

Their order is important, i.e you can't recognize relationships before discerning details. The list of attentional states is important for the researcher who aims to explore more on inductive reasoning with eye tracking methodology.

2.1.3 Inductive reasoning and patterns in Norwegian math curriculum

In the 2nd grade, by the age of 7-8, the goal of training for inductive reasoning tasks is: “experiment with counting both forwards and backwards, choosing different starting points and different differences and describing patterns in the counting lines” and “know and describe repeating units in pattern and create own pattern”. That means the young learners are expected to start exploring the patterns, investigating what is the same and what is different between the consecutive terms and find the next term. The last task (find the next term) is an expression of inductive reasoning. In the 8th grade, the goal is to “describe and generalize patterns in own words and algebraically”. The new challenge is to generalize a) in own words and b) algebraically. Describing was as a goal even in grade 2. We can expect to raise the level of difficulty in given patterns in higher grades, where it is not so easy to generalize or to find the general term of the pattern. In the 9th Grade the goal is “to describe, explain and present structures and developments in geometric and numerical patterns” The new challenge is to “explain structures and developments”, “present structures and developments” in “geometric and numerical patterns”. This represents a higher level of difficulty. To summarize, there are curricular expectations in Norwegian math curriculum that involve inductive reasoning, especially in the context of patterns.

2.1.4 Types of patterns

Patterns are regularities that we can perceive . We can perceive patterns with different senses. To discern patterns, we must identify the pattern unit. We need to understand not just the individual elements within this pattern unit, but also how the pattern unit is repeated. (Dreme, 2020)

There are different types of patterns observed in school curriculum.

A visual pattern is a set of ordered pictures or geometric objects with a certain rule. The task usually is to find the rule, the next term or to describe the sequence formed.

A numerical pattern is a set of ordered numbers which follow a certain rule. There is a function between the set of natural numbers in ascending order (which show us the place of a certain term) and its value. Both types of patterns need high skills in observing, interpretation, linking what you see with what you have learned so far, or with previous experience, in predicting, imagination, in calculation, and reasoning.

2.1.5 Research on reasoning with patterns generalization

Radford (2008) believes that the process of pattern generalization involves:

- (1) grasping a commonality,
 - (2) generalizing this commonality to all terms of the sequence, and
 - (3) proving a rule that allows them to directly determine any term of the sequence.
- (Luz Callejo and Zapatera, 2016).

Another way to explain the stages of pattern generalization involves observing:

- a) numerical and spatial structures; “Radford (2014) and Rivera (2010) indicate that coordination between spatial and numerical structures is a cognitive mechanism in pattern generalization” (Luz Callejo and Zapatera, 2016);
 - b) functional relationships
 - c) inverse process needed to identify the position of a figure when it is given the number of elements in it. (Rivera 2010; Warren 2005).
- (Luz Callejo and Zapatera, 2016)

They also propose that the above stages can be used as a taxonomy for evaluating students’ ability in pattern generalization, in which the easiest is to find the next term or to deal with numerical structures, but not spatial ones. More successful are the students who are able to coordinate spatial and numerical structures. The best ones are those who arrive to find the general rules and invert the functional relationship. (Luz Callejo and Zapatera, 2016)

The studies from El Mouhayar and Jurdak 2016; Kuchemann and Hoyles 2001; Rivera and Becker 2008 show the differences in students’ use of reasoning approaches (numerical or figural) is linked to the strategies they use. They think “numerical reasoning requires a surface attention” whereas “figural reasoning is more complex and involves more attention”. (Luz Callejo and Zapatera, 2016)

2.2 Eye tracking

Eye tracking is the technology of tracking a measuring of the eye movements with the help of its hardware and software which enables us to have data sets important to explain sometimes fully and sometimes partially what is happening with the visual attention and other cognitive processes (perception, learning, memory, reasoning).

There are two types of eye trackers:

Screen-based eye trackers (desktop), which allows the recording of eye movements at a distance (nothing to attach to the respondent) and

Glasses (mobile), which can record the eye activity from a close range and it is mounted onto lightweight eyeglass frames.

2.2.1 Key eye-tracking metric

Holmqvist et al. (2011) defined eye-tracking measures as “precisely quantifiable data which can be calculated taking events and/or representations as input” (p. 299, Suvorov, 2013).

They separated the 120 eye-tracking measures that they found into four classes:

- (a) position measures (i.e., measures and properties of gaze at certain spatial locations),
 - (b) movement measures (i.e., measures and properties of eye movements in space),
 - (c) latency measures (i.e., measures of duration between eye-movement events) and
 - (d) numerosity measures (i.e., measures of any countable event involving eye movements)
- (Suvorov, 2013).

The type of eye-tracking measure that a researcher chooses for his/her study depends on several factors depending on their research question and the type of eye-tracking software and hardware. The four common measures used in many eye-tracking studies are listed below:

Saccades - “rapid eye movements used in repositioning the fovea to a new location in the visual environment (Duchowski, 2007, p. 42) They range from 10 milliseconds (ms) to 100 ms.” (Suvorov, 2013)

Fixations - “eye movements that stabilize the retina over a stationary object of interest” (Duchowski, 2007, p. 46) they are not absolutely static, but entail miniscule tremors, drifts, and microsaccades that are believed to occur due to physiological peculiarities of the human visual system (Suvorov, 2013). The mean duration of fixations ranges between 180 ms and 330 ms, depending on the type of visual activity and task (Rayner, 2009)

Smooth pursuits - “are movements of the eyes that follow a moving object, such as a passing car (Holmqvist et al., 2011)” (Suvorov, 2013)

AOI (Area of Interest) are “particular regions in the visual field that are defined by researchers during the analysis of eye-tracking data” (Suvorov, 2013).

According to Suvorov (2013), these four eye-tracking measures are normally reported as evidence of overt visual attention and are used by researchers as “indirect measures of cognitive processes that cannot be directly assessed” (Holmqvist et al., 2011, p. 65), although it is only during fixations that the viewer is believed to actually acquire new information (Rayner, 2009)."

There are two ways in which the eye-tracking data can be visually displayed:

Heat maps which produce an overview of the data by displaying the spatial distribution of eye-tracking measures over participants and time. (Suvorov, 2013)

Scanpath which is a path of the viewer's eye movements in time and space. Scanpaths are dynamic, such as gaze replays or static, such as an image of an eye-movement trajectory over a certain period of time. (Suvorov, 2013).

2.2.2 Eye tracking, visual attention and perception

When reading different research papers, a number of issues to consider emerge:

The study of eye movements as a psychological or physiological phenomenon (Duchowski, 2007). Duchowski's concern is the psychological perspective, which has to do with examining the viewer's attentional behavior, or visual attention (Duchowski, 2007) and not a physiological one, which deals with exploring neural mechanisms of the human visual systems. The main justification for the conduction of eye-tracking studies is the assumption that by tracking the movements of eyes, one can better understand what the viewers draw their visual attention to. (Duchowski, 2007) Visual attention has been studied for over a hundred years. The psychologist William James gave this a good qualitative definition regarding visual attention:

Everyone knows what attention is. It is the taking possession by the mind, in clear and vivid form, of one out of what seem several simultaneously possible objects or trains of thought. Focalization, concentration of consciousness are of its essence. It implies withdrawal from some things in order to deal effectively with others... When the things are apprehended by the senses, the number of them that can be attended to at once is small. (Duchowski, 2013)

When talking about the visual attention, the studies evidence three of its dimensions: “where”, “what” and “how”. The “where” concept (studied by Von Helmholtz’s (1925)) relates to the location where the visual attention is directed to, the “what” of visual attention (James, 1981) concentrates on the actual object of visual inspection; the “how” concept (Gibson, 1941), proposes that attentional behavior leans on the viewer’s reaction and attitude against the object of visual attention. (Suvorov, 2013)

From tracking the eyes we study the visual attention, and from the latter (visual attention) we hope to understand other cognitive skills, which affect and guide the visible behavior.

The study of the relationship between visual attention and eye movements suggests that visual attention is engaged during the foveal direction of gaze, but Posner, Snyder, and Davidson (1980) have found that people are able to pay attention to an object or area of interest while their foveal gaze is directed elsewhere. According to Duchowski, “an eye tracker can only track the overt movements of the eyes, however, it cannot track the covert movement of visual attention”. (Duchowski, 2007, p. 12).

Suvorov (2013) evidences the relationship between eye movements and perception as a crucial point. From its definition “the perception refers to the way sensory information is organized, interpreted, and consciously experienced”. Sensation is considered as a physical process, while perception as a psychological one.

Suvorov (2013) notices that “eye movements recorded with an eye-tracking system contain raw data about the viewer’s interaction with a visual stimulus, but provide no information about visual perception - the meaning concluded by the viewer after receiving the visual stimulus”

Therefore, the research suggests that if we aim to study the visual attention, the perception or other cognitive skills, eye-tracking evidence or data may not be sufficient. We should triangulate the data from tracking eye movements with other types, such as performance data (Bojko, 2006) or self-reported data (suggested by Suvorov, 2013).

2.2.3 Eye movements and cognitive processes

It is true that eye-tracking measures provide us with some information related to the viewer’s cognitive processes. The problem with the use of these measures to study the viewer’s cognitive processes is that they show us only what caught the viewer’s attention in the visual field. These measures don’t give us information about what cognitive process might have influenced a viewing behavior. (Suvorov, 2013)

The concept of cognition associated with the viewing behavior can be quite complicated. To help resolve the complexity, researchers suggest “ supplementing eye-tracking measures with other methods that allow for examining cognitive processes of the viewers occurring during their eye movements” (Duchowski, 2007; Guan, S. Lee, Cuddihy, & Ramey, 2006). (Suvorov, 2013)

Several questions emerge in order to clarify the idea “what exact cognitive processes might have caused a specific viewing behavior” (Suvorov, 2013):

The list of cognitive processes includes: attention, perception, memory, language, learning and higher reasoning, therefore the eye-tracking measures may be linked to each of the cognitive processes: attention (where to see), perception (what is its meaning), memory (recall), language (what is the word/ concept for it), learning and higher reasoning (work out the general term of the series).

2.2.4 Eye Tracking and mathematics

Previous research papers in mathematics education have showed that information about the attentional processes included in solving mathematical tasks can be provided by the eye-tracking technique. (Norqvist etc.)

Despite the fact that video recordings of the tasks solved by students showed the order of the solution steps, eye-tracking gives a more fine-grained access to students' processes involved in solving a task (Schindler, Lilienthal, Chadalavada, and Ögren, 2016).

According to Susac, Bubic, Kaponja, Planinic, and Palmovic, (2014) experts were more focused on directing attention to relevant information (had less fixations) than novices. With increasing task difficulty this difference also increased. This was also confirmed by Lin and Lin (2014) who concluded that the big number of fixations between novices was influenced by the difficulties in relating complex visual stimuli to memorized referents. They also discovered that the time spent on logical structures when reading mathematical proofs was greater by the experts than by the novices, who were mostly focused on surface features. (Inglis & Alcock, 2012)

2.3 Measure of confidence

In their research paper "Measures of the Trait of Confidence" Stankov Et al. (2014) quote: "Shauger and Schohn (1995) defined confidence as a self-perceived sense of competence and/or skill to deal with various situations effectively." Two kinds of assessments of individual differences in confidence are: (1) self-reported and (2) judgments of accuracy after the completion of a task. They capture cognitive aspects, personality, implicitly and motivation.

3. Methods and discussion of methods for data collection and analyses

3.1 Research design

This study is a case study design, because it entails the detailed and intensive analysis of a single case, (Bryman, 2012). It is a single case, because it is concerned with the complexity and particular nature (Stake, 1995) of five subjects, each of them seen as a single case, with no intention to draw conclusions that stand for all. If there are conclusions for the subjects seen as a group, this is done not as part of the study of the group. This is done in order to give examples of how some findings for individuals will work out if the case study would be a group.

The focus of interest in its own right (Bryman, 2012) is the subject's performance in the inductive reasoning tests.

There were five participants (subjects). Three of the subjects were master students in mathematics education. The other two were students of grade 9. The subjects provide a suitable context for the research questions (Bryman, 2012) of this study. They had previous learning and experience with shape patterns tasks, which need inductive reasoning to be solved.

The approach of this study is inductive, because the theory is the outcome of the research.

The research area is inductive reasoning (IR).

Klauer and Phye (1994) defined inductive reasoning as the discovery of regularities through the detection of similarities, differences, or a combination of the two, with respect to either object attributes or relations between objects.

The research questions are:

Research Question 1: What is the ratio of eye-tracking measures of the same type in different areas of interests in inductive reasoning in solving pattern problems?

Research Question 2: How to measure inductive reasoning in solving shape pattern tasks?

Research Question 3: "What are the indicators to measure inductive reasoning in solving shape pattern task and how do they work?"

Research strategy

To answer the first research question, *“What is the ratio of eye-tracking measures of the same type in different areas of interests in inductive reasoning in solving pattern problems”*, quantitative research strategy is used, because the data collected are mainly numbers expressing different eye-tracking measures:

- fixations and gaze points which give information about the position of the eyes;
- heat-maps which show the general distribution of gaze points;
- areas of interest
- time to first fixations;
- time spent;
- fixations sequences;
- revisits;
- average fixation durations.

Linking the numerical data emerging from iMotions software, with the context which was “solving pattern task” gave the opportunity to interpret this quantitative data.

To answer the second question, *“How to measure inductive reasoning in solving shape pattern tasks”*, the qualitative research strategy was used. The data were lines of reasoning of five subjects solving the tasks.

To answer the third question, *“What are the indicators to measure inductive reasoning in solving shape pattern task and how do they work”*, the qualitative research strategy was used. The data were lines of reasoning of five subjects solving the tasks. This data was converted in quantitative data by giving values 0 and 1 to each indicator and then a large set of data became quantitative and analysed as such.

3.2 Participants

This study included two groups of participants. The first group consisted of 3 master students in mathematics education from a university located in Southern Norway (3 girls). The other group was made up of 2 students (2 girls) of grade 9 from an international secondary school located also in Southern Norway. The school offers an international education, the well-known International Baccalaureate (IB).

The background of the students' performance in mathematics was not known before the selection. These students willingly volunteered following the signed consent from their parents to participate in the study. For the purpose of confidentiality, in the discussions and analysis, instead of their real names, the following names are used: Era, Dua, Rita, Lucy, Anne.

How were the master students chosen?

A presentation of the topic of the thesis was done at the university. The participants were master students in didactics of mathematics along with other professors in the field of mathematics education. At the end of the presentation the master students were invited to participate in the study. A letter of informed consent was given to them by the researcher. A period of one week was decided upon. After returning the signed consent, a time was booked for each participant to complete the tests (solve shape pattern tasks). The time was decided by the students themselves depending on the period when they were available.

How were the secondary school students chosen?

Firstly, a letter was sent to the principle of the international school asking for permission to do a presentation of the study in the school with students of grade 10. After receiving the permission, a presentation to the 10th grade students was made and a letter of informed consent was given to them. A period of one week was decided for them to take the decision. Due to the exam period that they were having at the same time, unfortunately none of the grade 10 students accepted to participate. After that, a permission to collaborate with the grade 9 students was requested to the principle. The answer was again positive, so a presentation was done and the students expressed great interest. The signed consents were received and the times were booked for each respondent. In order to conduct the experiment, daylight was needed, secondary school students couldn't come during weekdays, so they were asked to come during the weekend.

All the subjects needed to come to the university to the room where the eye-tracker was.

In order to encourage more subjects to accept the invitation and as a sign of appreciation for participating, bookstore vouchers of value 200 NOK were offered to each participant.

For the purposes of the first research question, only data from the subject Anne was used. For the second and third research questions the data from 5 subjects was used.

Pre-test questionnaire

A pre-test questionnaire was conducted in order to provide background information about the research participants. The collected background information included the participants' name, age, gender, email address and brief math self-efficacy. The questions about the respondent's private information were made only for research purposes, meaning in case something happens to their data or for communication purposes only. The math efficacy questions were made in order to understand if there is a connection between those and the participant's performance in the test.

The data collected are presented in following table:

Age	Gender	Enjoyment working with math problems	General mathematical skills	Solved pattern problems before (math class+ outside school)	Confidence working with pattern problems
23	F	4	4	Yes	3
23	F	4	3	Yes	3
24	F	4	4	Yes	5
15	F	4	4	Yes	3
15	F	5	4	Yes	3

Table 1

3.3 Materials and instruments

3.3.1 Inductive reasoning / visual pattern tasks

Visual patterns tasks provide good examples of inductive reasoning. In this study, five pattern tasks were used, where the pattern figures were taken from the internet and the tasks were created by the researcher.

The reason behind choosing five different shape patterns was to give subjects a reasonable number of chances to measure their inductive reasoning skills.

The common task for all the problems was to find the n^{th} term, and for two of them an additional task was given to find the number of circles in the 5th or 6th term. The first three figures were given in all the tasks.

Task 1: Looking at the three figures below, find the number of grey squares in the n^{th} figure.

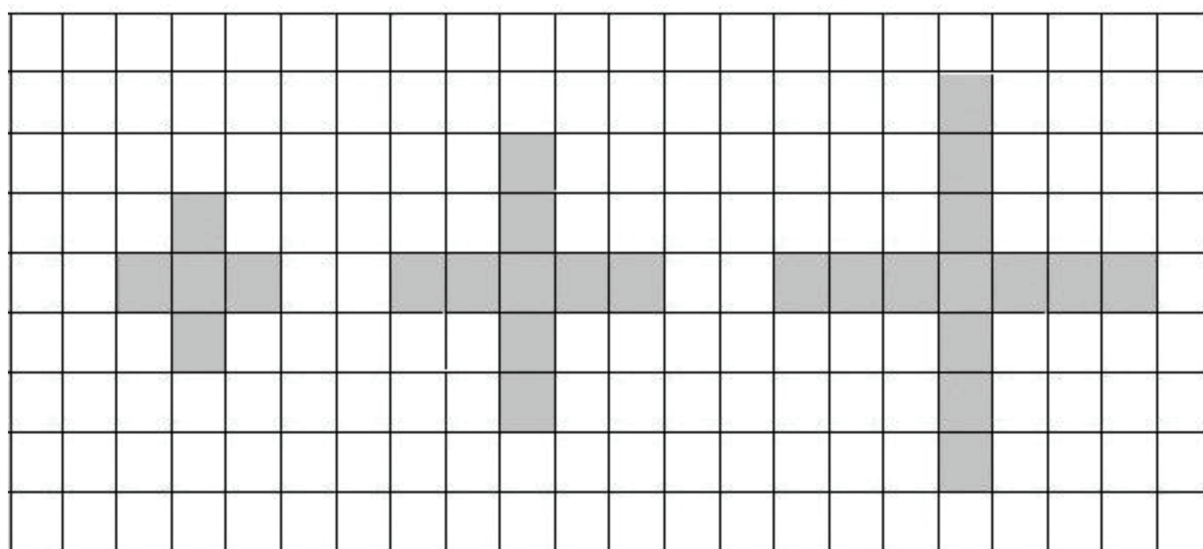


Figure 1

Figure 2

Figure 3

Figure 1

Task 2: Looking at the three figures below, find the number of grey circles in the 6th and nth figure.

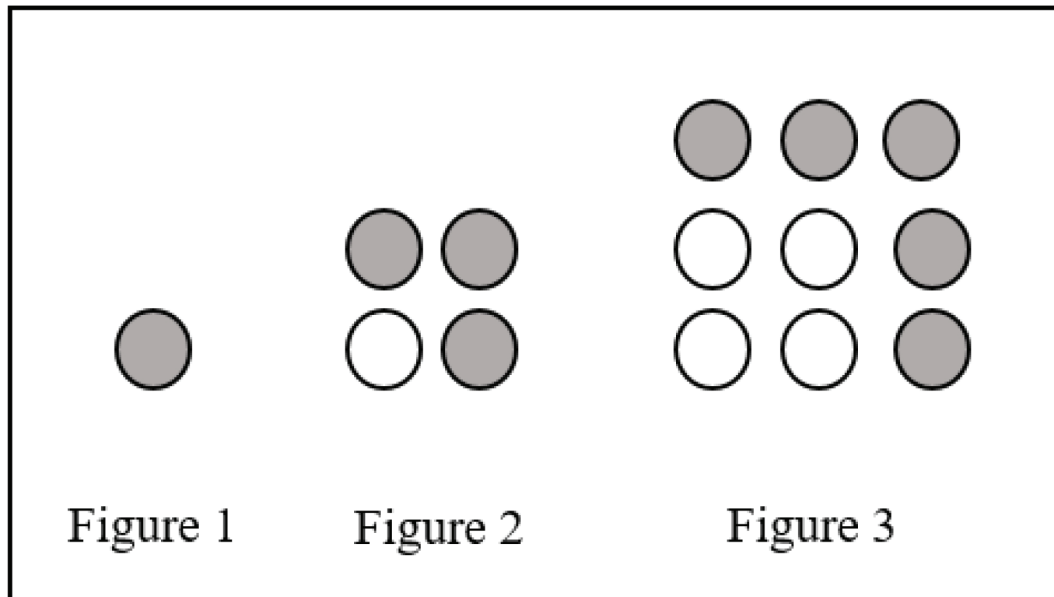


Figure 2

Task 3: The outer borders of the three figures below are made of black matchsticks. Find the number of the black matchsticks in the nth figure.

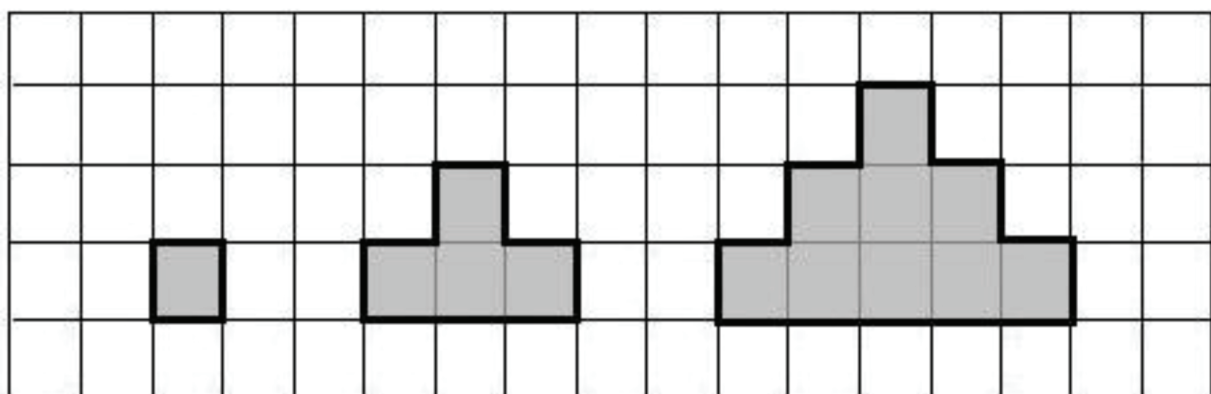


Figure 3

Task 4: Looking at the three figures below, find the number of grey circles in the 5th and nth figure.

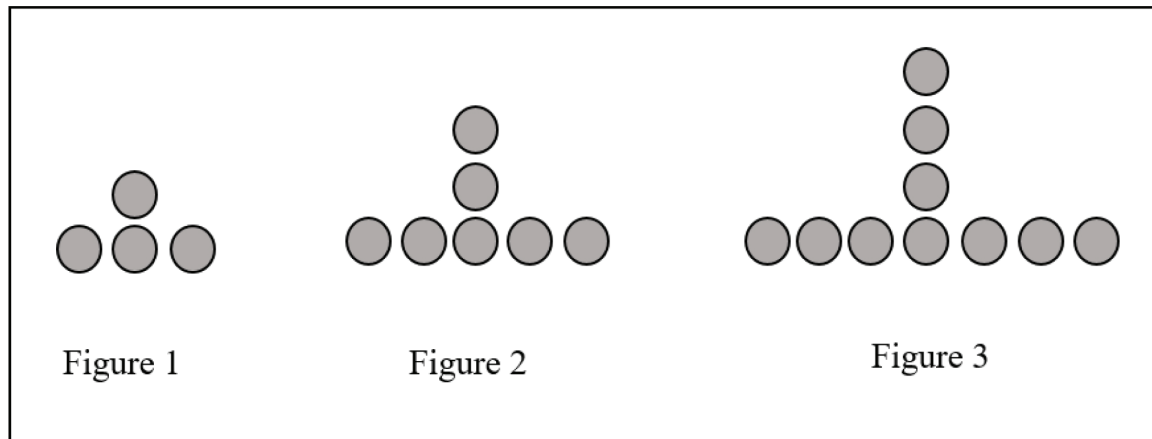


Figure 4

Task 5: Bigger cubes are cut into smaller cubes and the outer faces of some of the smaller cubes are painted dark as shown below. Looking at the three figures, find the number of painted cubes in the nth cube.

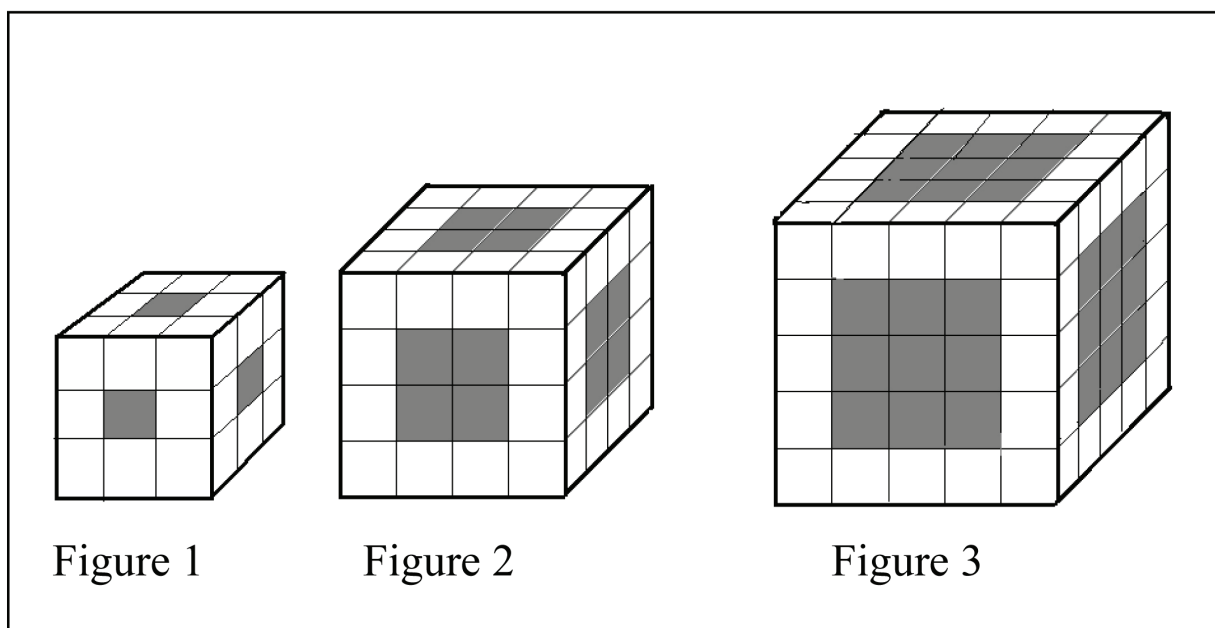


Figure 5

3.3.2 The tablet

The pattern tasks that the students solved were put in a PDF file on the computer by the researcher and imported on the tablet. The “reMarkable” tablet is also called a “the paper tablet”. ReMarkable adds digital forces to notes and writes directly to the page.

Choosing this device was helpful because it lets the participants take notes with the remarkable pen while thinking.

Since the eye-tracking glasses were used, the canvas display simply reflects natural light making it wonderfully gentle on the eyes.

The photo shows the room where the study took place along with the eye-tracking glasses on the left of the tablet.

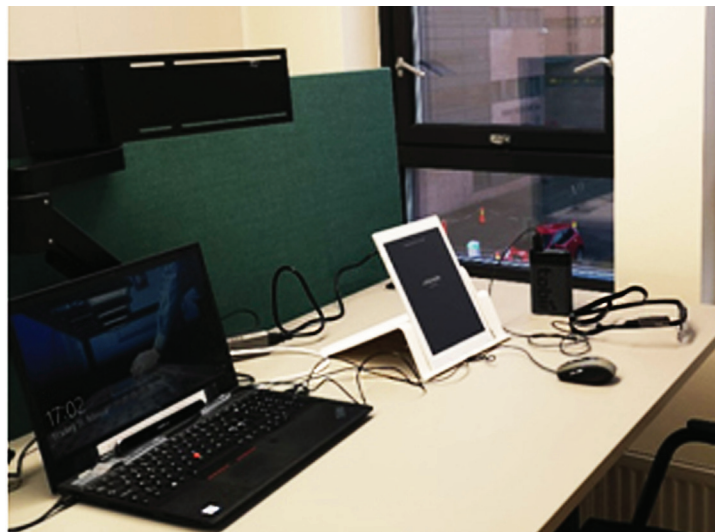


Figure 6

3.3.3 Eye-tracking glasses (Tobii)

In this research, the eye-tracking glasses used were produced by TobiiPro and offered by the university along with the software (iMotions) .

Tobii Pro Glasses gives insights into human behaviour by showing what a person is looking at in real time. The real-time observation of participants’ gaze data provides quick and actionable insights. In order to start collecting eye-tracking data, a calibration card is needed. The subject should focus their eyes in the centre point until the software shows that the calibration process succeeded. After that the researcher presses the icon “start recording” on the software.



Figure 7

3.3.4 iMotions software

The software used in this research to analyse the data is the iMotions software.

iMotions is a software for researches in human behaviour. iMotions provides analysis options for large range of data. Biosensors like eye tracking measure human's eye movement. However, no sensors or stimuli are connected and several software tools are needed to get it all to work.

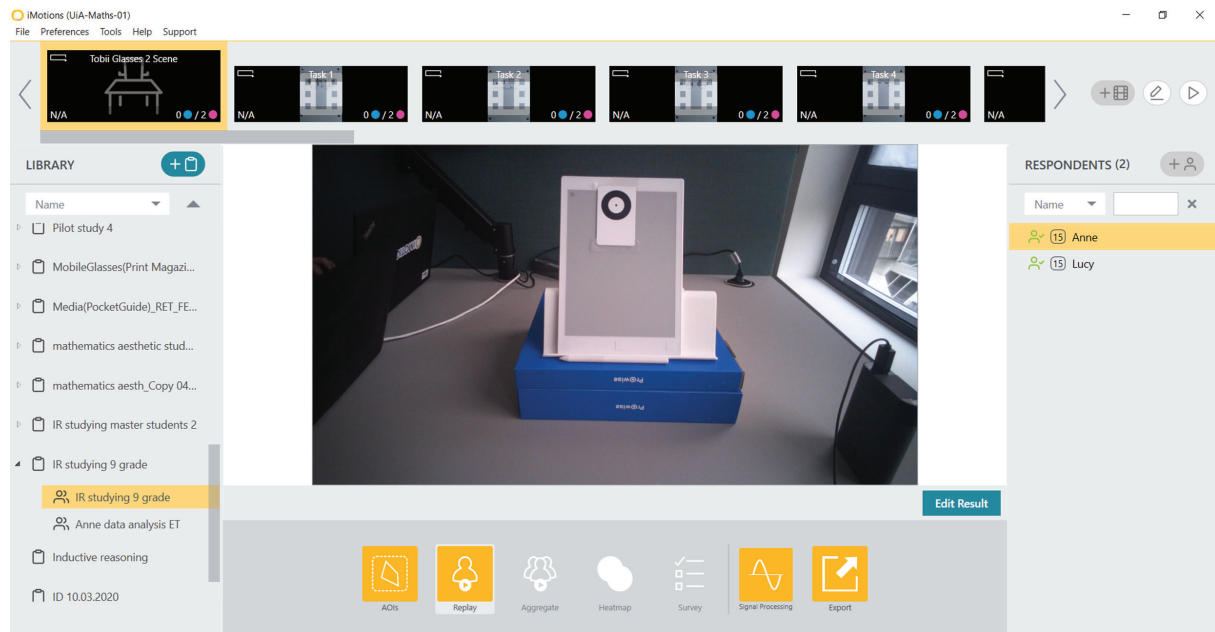


Figure 8

3.3.5 Post test questionnaire

Two months after data collection, a post-test questionnaire was sent by email to the five subjects to complete it. The questionnaire was a Likert Scale questionnaire which consisted of 13 claims to be graded by the subject with 1-5 points, where 1 represented “strongly disagree” and 5 represented “strongly agree”. After discovering 13 indicators, it was interesting to see how the subjects themselves would use these indicators as a self-evaluation tool and to compare these with the results that emerged from data analyses. Another aim was to see the ranking of the indicators by the subjects.

The list of claims given

- I read the task more than once.
- I draw figures related to the task.
- I create a number sequence based on the figures in the pattern.
- I create a list of first values for mapping $n \rightarrow f(n)$.
- I compare the given figures in extension.
- I notice the same feature in different terms
- I look for the growth rule for consecutive terms.
- I find the zeroth term in the numerical sequence.
- I provide the justification of the rule or apply it.
- I conjecture about (guess) the form of the nth term.
- I try different approaches to the solution and assess their usefulness.
- I confirm the formula for the general term.
- I test the formula .

3.4 Methods of data collection

The main method chosen to collect the data was unstructured observation.

The interest of the researcher was to let the subject solve the tasks without any interference. The aim was to record as much details as possible.

An office was offered by the university, where the eye tracking equipment (glasses), the computer with the related software and the tablet was placed, for the purpose of data collection. Only the researcher had access to the room on the days when the data collection process took place.

A time was booked for each subject with a difference of one hour between each other in order to have time for any inconvenience that could have happened. The windows of the office were covered with a translucent bag that prevented the entry or reflection of the natural light (day light) on the tablet. This was important for the process of data collection with the eye-tracking equipment. For each of the subjects the following instructions were given before the data collection process started:

1. Five pattern tasks need to be solved.
2. The time to solve the five tasks is a maximum 35 min.
3. The tasks are going to appear on the tablet one by one.
4. The right button on the tablet needs to be pressed in order to pass on the next task.
5. During the problem solving, you are required to think out loud and write your notes on the blank space of the screen with the tablet's pencil.
6. The eye-tracking glasses should be worn from the beginning of the process until its completion.
7. It is important not to look outside the frame of the glasses.
8. Before starting the process of data collection, the eyes should be focused on the middle black point of the calibration card, until the researcher says "You can continue".
(The calibration card served as a tool to connect the eye-tracking glasses with the iMotion software, where all the data were collected. The subject was requested to look at the card until the software showed that the calibration was successful.)
9. Before starting working on each task concentrate your eyes on the blank space of the screen until the researcher says: "You can continue".
(This was needed to facilitate the process of data analysis later.)
10. Remember that the process can be stopped immediately in case stress is caused by the process and the data gathered will be destroyed at that moment.

After the instructions were given, the subjects were asked to settle on the desk, in front of the tablet. The only person that was present in the room, where the process of data collection took place, was the researcher. The whole process was followed by the researcher on the software. After the subject had finished their work, the import of the related data from the eye-tracking glasses to the software was done. Each of the subjects was offered a snack as a sign of appreciation for participating before leaving the room.

3.5 Methods of data analyses

To answer the first research question, “What is the ratio of eye-tracking measures of the same type in different areas of interests in inductive reasoning in solving pattern problems?” from the iMotions software in spreadsheet, these eye-tracking measures were analysed:

- fixations and gaze points
- heat-maps
- areas of interest
- times to first fixations
- time spent
- fixations sequences
- revisits
- average fixation durations.

The name of the data (for example “gaze points” or “fixations”) shows the nature of data taken. The nature of data along with statistical analyses tools and the type of task (inductive reasoning) gave the possibility to draw conclusions.

In the chapter “Analyses and results”, in the part related to the first research questions, these statistical analyses tools were used: average, arithmetic mean, column chart, maximum, minimum, mode, table of values, pie chart.

To answer the second question, “How to measure inductive reasoning in solving shape pattern tasks”, after the transcription of the video was done, the thoughts or the lines of reasoning of the subjects were classified in groups of statements, claims, reasoning which had something important in common that could be called an indicator for the skill in focus.

A group of claims, or lines of reasoning, contained evidence that the subjects were able to show a partial skill which was useful for the solution of the task. If this skill was verified to be present in many subjects and many tasks, it was given the “status” of the indicator. For example:

Looks for the growth rule for consecutive terms

(Subject: Era) Task 1: “So starting with the first and counting how many, ...five and ... then just adding one more on the outside.” (evidence no. 4) Task 4: “So they are just adding three more outside of each. So it would like increase from 3 here, 3 here.” (evidence no. 60)

(Subject: Rita) Task 3: She writes the equations $F_1 = 4$, $F_2 = F_1 + 6$, $F_3 = F_2 + 6$, $F_4 = F_3 + 6$, while looking at figure 1, figure 2, figure 3 and figure 4 respectively. (evidence no. 59)

(Subject: Dua) Task 3: “So plus 6, plus 6.” (evidence no. 32)

This way, initially there were 24 indicators, but some of them overlapped, which means that some evidence was used twice for two “different” indicators. Some of the “indicators” were removed because there were no partial skills with a direct link to inductive reasoning. For example, “Operates/manipulates well with the formula”. At the end, the list was reduced to 13 indicators.

To answer the third question, “What are the indicators to measure inductive reasoning in solving shape pattern task and how do they work”, the results of the second research question were used, but a process of translating qualitative data into quantitative data took place.

Each indicator was given the value of 0 if this skill had no evidence in the work of the subject and the value 1 was given if the evidence for the indicator existed.

This way a table of 325 values (0 or 1) was produced. This means 5 subjects x 5 tasks x 13 indicators (5 x 5 x 13 = 325).

The arithmetic mean per each task, per each subject had an important value for the analyses and the final interpretation.

Many other operations proved the idea that the measure of indicators worked very well for the interpretation of the work done by the subjects.

The following diagram summarises the method of data analyses.

Figure 9 shows the whole process of analysing the data. The transcription process is related to the 2nd and 3rd research questions, which is based in qualitative data.

The bottom part of the figure relates to quantitative data collected through eye-tracking technology or Likert Scale questionnaire, needed to answer the 1st research question.

3.5.1 THE DIAGRAM OF DATA ANALYSES

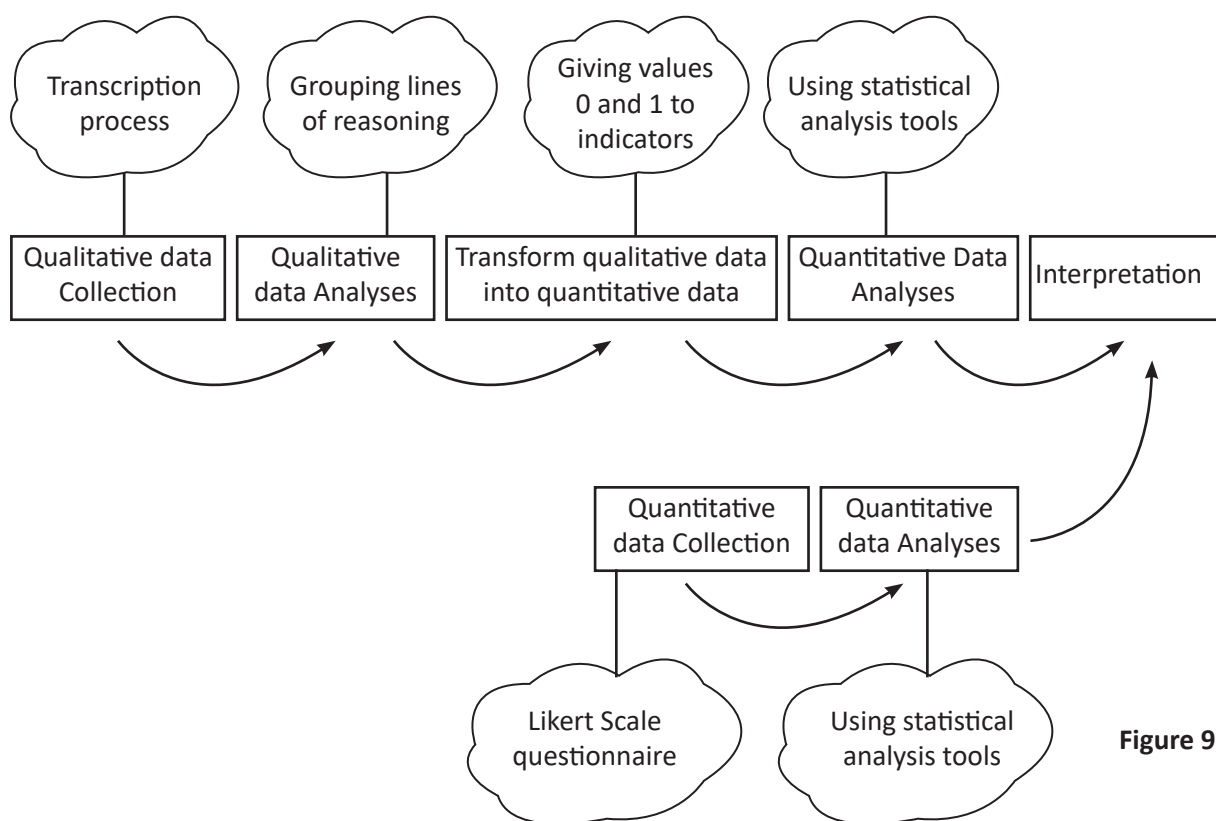


Figure 9

POST TEST QUESTIONNAIRE WITH THE TEACHERS

The “Questionnaire for teachers” (see page 85) aimed to discover if there was a difference between the findings of this research question and what teachers of mathematics thought about the time spent for each figure. Since there wasn’t time to ask a large number of teachers and to ask for permission, a questionnaire was sent to a teacher known by me as a serious person asking to pass the form to other math teachers, but not telling me who they were. The questionnaire had 4 answers in an anonymous way. I do not know who they are. All of them had the same trend of answers. The answers would serve to write considerations about the ecological validity of the research.

From the website of NSD, Data Protection Services, after providing some answers to their questions this text was shown as result: *“You have indicated that neither directly or indirectly identifiable personal data will be registered in the project. If no personal data is to be registered, the project will not be subject to notification, and you will not have to submit a notification form to us.”*

3.6 Ethical consideration, validity and reliability

The project is a study on the master thesis level which has the Norwegian centre for research data (NSD) approval ab initio.

At the start of the research study, the researcher sought collaboration from the principal of the international secondary school in Southern Norway. The principal accepted the request and gave the approval for a meeting with the students. A ten minutes meeting was scheduled with grade 10 students, together with their mathematics teacher, to familiarize with the students, explain the aims of the study and issue them the letter of consent for their parents to sign.

Due to the exam period that they were having at the same time, unfortunately none of the grade 10 students accepted to participate. After that, a permission to collaborate with the grade 9 students was requested to the principle. The answer was positive, so a 10 min presentation was done again, where the students expressed great interest. After one week, five students returned the consent letters signed by their parents where they approved the use of the recording gadgets.

Since all educational institutions in Norway were closed, by the government due to the pandemic Covid-19, two of the students couldn't participate in the study.

The only people that had access to the video and audio recordings were the researcher, the supervisors and the iMotion software specialist. Another name was given to each subject in order to keep their anonymity. The files were named with their "fake" names. In this thesis other names are used, not their real names. The files were encrypted with passwords in the computer provided by the university. To prevent the identities of the subjects, the only thing recorded was their voice and partially the hands, not the faces. No other personal data was required from the subjects, except their e-mail addresses and their phone numbers. The latest were asked only for research purposes and they were deleted from any files and folders.

In the case of the form intended for teachers, no directly identifiable personal data was collected. No background information that may identify individuals was collected. The online form used did not collect personal data.

References for quoted texts, photos, from different studies, books, or websites are provided throughout the report.

The findings of this research study have not been influenced by the researcher's point of view and values.

Reliability

The arguments in favour of the reliability of this study are as follows:

The subjects had previous knowledge and confidence in solving pattern tasks. Three of them were master students in math education. This let the researcher discover the set of indicators that show what should be done if you try to solve such tasks.

In order to get very accurate data from the eye-tracker, a process of fixing the algorithms between the photo of the task and the recorded video from the eye-tracker was done.

Validity

Measurement validity - The eye-tracking measures were taken from iMotions software, which is a reliable software. The indicator's measure has meaningful values of 0 and 1, where 0 is for "non appearing in the subject's work for solving the task" and 1 for "appearing in the subject's work for solving the task". In the data analyses chapter, there are additional explanations about how these measures work, showing the practical value of them.

Internal validity relates mainly to the issue of causality (Bryman, 2012). This research has discovered a causal relationship between two variables: "the way of reasoning" as an independent variable (the cause) and the "number of fixations that are revisits" as a dependent variable.

It is clearly explained in this research: The cause being the fact that figure 2 takes nearly 50% of the total time spent looking at the three figures or 50% of fixations that are revisits, is the way of reasoning, or the type of perceptions the subject has while he/she is holding wholes (gazing), discerning details (making distinctions), recognizing relationships (among specific discerned elements) and perceiving properties (as generalities which may be instantiated in specific situations) and reasoning on the basis of identified properties. (Mason, 2009 et al., Watson, 2009). The type of perception discovered in this research is with pairs of fixations, explained in this thesis in the article "Number of Revisits and Fixations Count".

Linked with the credibility of this research thesis is the chapter of "strength and limitations", which gives details that help the reader have an idea of how credible these findings are.

External validity is concerned with the question of whether the results of a study can be generalized beyond the specific research context (Bryman, 2012). An additional value of this research is that the nature of the findings of this study may inspire other researches. There are some new hypothesis in the research that help other researchers to generalize the findings of this research.

Ecological validity is concerned with the question of whether social scientific findings are applicable to people's everyday, natural social settings (Bryman, 2012). This value is mentioned in the chapter "Implication of the results in mathematics education".

There is enough evidence by the questionnaires that there is difference between the answers of math teachers and this research findings about the question "Which of the figures in the pattern takes more time or more revisits".

The math teachers believe that the amount of time spent is of direct proportion with the figures' areas (which are increasing from 1st to 3rd). This means that they think the second figure takes more time than the first figure, but less than the third, which this thesis finds it is not true, because the 2nd figure takes 50% of time, or the sum of times of 1st and 3rd figure. The scheme of work the math teacher can do with the indicators may result in practical benefits to assess better the students involved in solving pattern tasks.

3.7 Summary

In this section, an account of the student selection, methods used for data collection, brief description of the materials and instruments used, presentation of the research design follows together with the strategy for analysis, ethics, validity and reliability of the study.

The research area is inductive reasoning. The design chosen for this study is a case study design, where each of the subjects is seen as a single case. The focus of this study stands at the performance of each subject in the inductive reasoning test. Since the theory is the outcome of the research and data are collected to build theories, the approach of the study is inductive.

The strategy used in this study is both quantitative and qualitative.

In order to answer the first research question of this study, which is *“What is the ratio of eye-tracking measures of the same type in different areas of interests in inductive reasoning in solving pattern problems?”*, the strategy used was quantitative. The data collected to express different eye-tracking measures were mainly numbers.

In order to answer the second research question of this study, which is *“How to measure inductive reasoning in solving shape pattern tasks?”*, the strategy used was qualitative. The data were lines of reasoning of five subjects solving the tasks.

In order to answer the third research question, which is *“What are the indicators to measure inductive reasoning in solving shape pattern task and how do they work?”*, the strategy used was qualitative. The data once again were the lines of reasoning of five subjects solving the task.

The study included two groups of participants, which consisted of 3 master students in mathematics education and two 9th grade students from an international school. The students willingly volunteered followed by the signed consent from their parents. The “fake” names: Era, Dua, Rita, Lucy, Anne were used in the discussion and analysis for the purpose of confidentiality. In this part, a brief explanation of how the subjects were chosen is provided.

In order to provide some background information of the research’s subjects, a pretest questionnaire was conducted which included four math-efficacy questions.

The materials and instruments used for this study were the inductive reasoning test (visual pattern tasks, the tablet, the eye-tracking glasses and the related software, as well as a post-test questionnaire.

In the study, five pattern tasks were given to the students, where they had to find the n th term. The first three figures were given in all tasks. The five tasks were created as a PDF file by the researcher and imported on the reMarkable tablet. The tablet allowed the subjects to take notes during their work with solving the tasks. The eye-tracking glasses (TobiiPro) were used by each subject, which provided a recording of the subject’s eye movements and voice. The data recorded by the eye-tracking glasses were then analysed with the help of the iMotion software.

After two months of data collection, a Likert scale, post-test questionnaire was given to the subjects to fill. This consisted of 13 claims to be graded by the subject with 1-5 points, where the meaning of 1 was “strongly disagree” and 5 was “strongly agree”.

The method chosen for data collection was unstructured observation. The interest of the study was to let the subject solve the test without any interference from the researcher. The time allocated to solve the tasks was 35 min. The subjects were encouraged to think aloud and write notes on the tablet if

needed. They had to wear and keep the eye-tracking glasses on from the beginning of the study until its completion. It was important for the subject not to look outside of the frame of the glasses and to concentrate their eyes on the tablet. A calibration card was used, where the subject had to concentrate their eyes at the beginning of the process in order to connect the eye-tracking glasses with the related software. The whole process was followed by the researcher on the computer, who had a passive role. The methods chosen for data analysis were several.

To answer the first research question the measures extracted from the iMotion software were analysed. The main measures were: fixations and gaze points • heat-maps • areas of interest • times to first fixations • time spent • fixations sequences • revisits • average fixation durations. To analyse these measures, statistical analyses tools were used.

To answer the second research question, the thoughts or the lines of reasoning of the subjects were classified in groups of statements, claims, reasoning which have something important in common that could be called an indicator for the skill in focus.

To answer the third question, the results of the second research question were used. Here, the process of translating qualitative data into quantitative data took place. Each indicator was given the value of 0 if this skill had no evidence in the work of the subject and value of 1 was given if the evidence for the indicator existed.

A diagram was created by the researcher to better explain the process of how the data was analysed. The chapter ends with a brief discussion about the ethical considerations, the validity and reliability of the study.

The main points mentioned related to ethical considerations were: The study has the approval from the Norwegian Centre for Research Data (NSD); The subjects were given “fake” names in the discussion of the thesis in order to save their anonymity as promised in the consent letter; The files with the data were encrypted with passwords in the computer and the only person who had access to the account on the computer was the researcher; References for quoted texts, photos from different studies, books, or websites are provided throughout the report.

Regarding the validity of the reliability of the study, in order to get very accurate data from the eye-tracker, a process of fixing the algorithms between the photo of task and the recorded video from the eye-tracker was done. Also, the fact that the subject had previous knowledge in solving shape pattern tasks, which helped the researcher discover the set of indicators that give the idea of what should be done if one would try to solve such tasks.

Four types of validity explain why the findings of this research truly represent the phenomenon the researcher claimed to measure. These types of validity were: the measurement validity, the internal validity, the external validity and the ecological one.

4. Analyses and results

The eye-tracker provided two sets of data:

a) eye-tracking measures, such as, number of fixations, patterns, time to first fixations (TTFF), first fixation duration, revisits, areas of interests (AOI), fixations count, heat maps.

b) data taken in the form of video and audio file (the voice of the subjects thinking aloud), video of the movement of the hands writing something on the tablet, and patterns of the eye movements.

The first sets of data served to answer the first research question:

Research Question 1: What is the ratio of eye-tracking measures of the same type in different areas of interests in inductive reasoning in solving pattern problems?

This means finding the ratio of revisits in AOI-1, AOI-2 and AOI-3 or the ratio of time spent in AOI-1, AOI-2 and AOI-3.

When talking about the ratio it is not only about one ratio but a set of ratios, sometimes in the form of a fraction ($1/2$), sometimes in the form of percentages (%), sometimes in a form of pie-chart. This set of ratios gives the teacher the idea of what numbers can be expected when working with shape pattern of this type. In additional researches with more data and more subjects involved the averages may come closer to the truth of the process has but here there are the first steps of the analyses of the ratio eye-tracking measures have.

The second set of data serves as a support for the following research questions:

Research Question 2: How to measure inductive reasoning in solving shape pattern tasks?

Research Question 3: What are the indicators to measure inductive reasoning in solving shape pattern task and how do they work?

The video file provided by the camera of the eye-tracking glasses was played and a transcript for each subject was written. In the transcript, explanations for what the subject was doing while saying something were added.

A table was created for each subject, with 5 columns, to represent the evidence on the transcript:

- a) index number of the evidence
- b) subject is saying ...
- c) subject is doing ...
- d) the researcher's interventions
- e) notes of the researcher

This type of sorting the data accelerated the data analyses because it was easier to concentrate.

After having all the data sorted this way, the analyses began. Firstly, the last column was filled with many notes by the researcher explaining what the subject was doing or saying. These notes helped the researcher better represent and analyse the data.

At this point, it was easier to find those skills that were part of the performance of the overall skill “inductive reasoning in shape pattern tasks” which latter were called indicators.

After the first list of indicators was done for the first subjects, with same list, the transcript of the 2nd subject was analysed to see if these indicators worked as a type of taxonomy. For many of them, it worked well and for some rewording was necessary. Some new indicators were added by the evidence got in the 2nd, 3rd, 4th and 5th subjects.

At the end, after an evaluation of all the drafts for the sets of indicators, a final list of 13 indicators was established.

At this point, the measurement stage started by converting qualitative data into quantitative ones. For each indicator, a value of 0 was given if the subject did not show any evidence and the value 1 was given if the indicator was congruent with the evidence shown by the subject.

A table with 325 values was created, which represented the performance of 5 subjects in 5 tasks by 13 indicators ($5 \times 5 \times 13 = 325$).

At this stage, it was time to analyse quantitative data with statistical tools.

The stage of interpreting the values of the statistical measures started.

4.1 Reading the task

Task 1 had 19 words: Task, 1, looking, at, the, three, figures, below, find, the, number, of, dark, coloured, squares, in, the, nth, figure.

The task is read with 10 fixations (fixations 56-66) in 6000ms (6s), excluding fixation 65 which clearly is not focussed on the words.

The words are read in blocks, and the duration of reading each “group of words” is presented in table “Anne: Reading task 1”. The dimension of the “visual span” is not always clear (which word are read with a fixation), and the table below must be seen as a version.

To find out the “visual span”, we may refer to the average of words $19/10 = 1.9$ words per “visual span”. The average number of characters (without spaces) per fixation is 8.6 (86/10). “The visual span” refers to how many words we can read before and after the currently fixated word. (imotions.com blog).

As it is seen in the table, the group of words, that forms the visual span, that has attracted the attention of the subject for a longer time respectively 1800ms is “in the nth figure”. It is obvious that the “logical stress” of the task is this group of words.

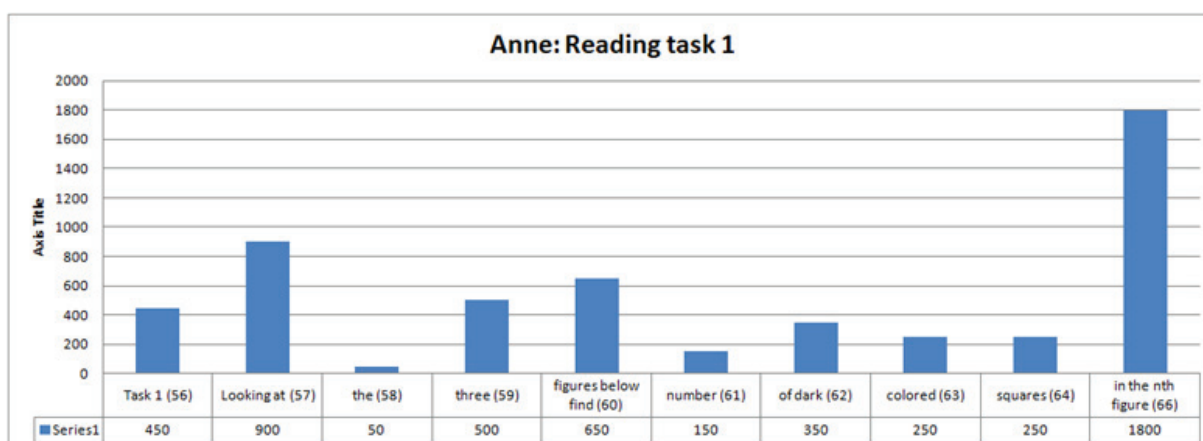


Figure 10

Since the other subjects did not offer a good performance of eye-tracking records, it was not possible to offer comparisons with the others, but the above table is source of following research questions: Do the other subjects provide the same shape of the diagram?

This may show the importance the subjects give to different words, or group of words, and the related links with the solution they give to the task and the success they have.

4.2 Anne: Looking at or analysing the pattern (part 1)

After reading the task (gaze points 56-66), Anne started observing the pattern for 35s750ms with gaze points 67-130, which are 64 gaze points, which have an average of 590ms/gaze points.

At this part of her work, the eye-tracking's gaze points seem not to match the three figures. It seems as a translation. But it allows to understand that Anne is looking at the three figures from left to right and vice versa, which means she is comparing them or having the first idea of term to term rule.

4.3 Anne: Completing the table of values

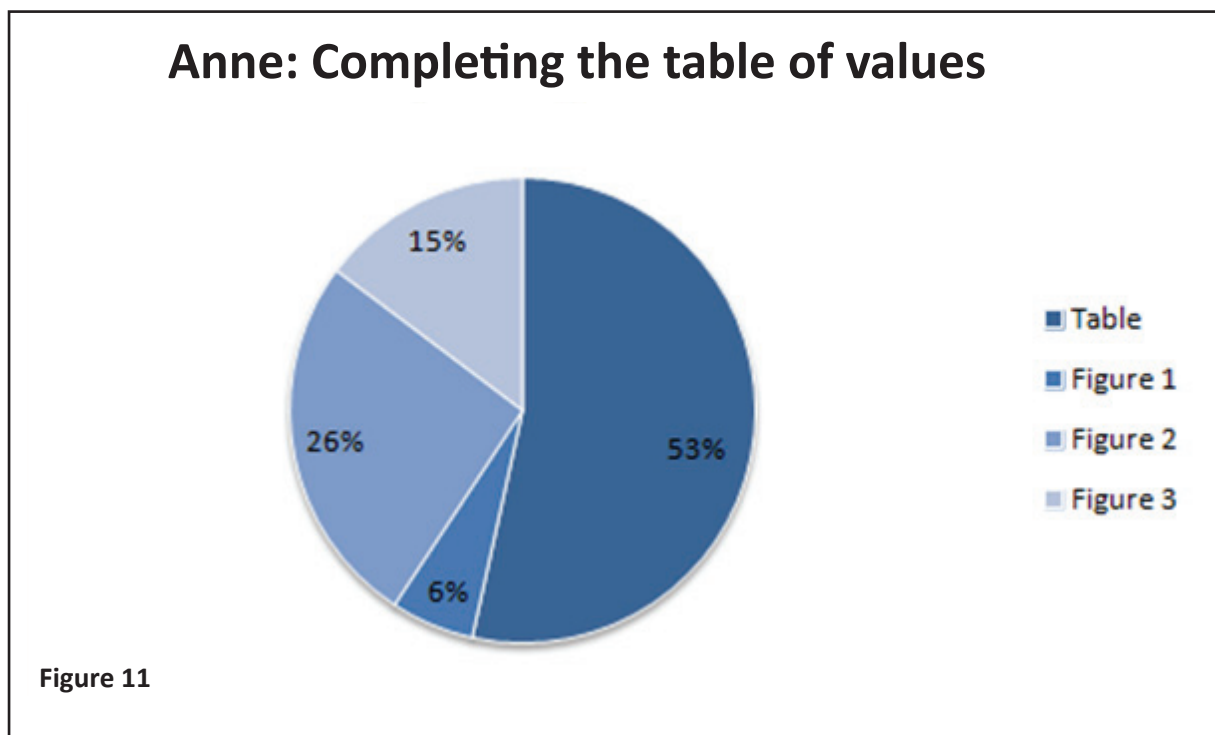
For about 59 s with 77 gaze points, Ann completed this table of values.

The time spent outside the tables or the figures 1, 2, 3 is left out not to be calculated as part of her mental work to complete the following table (see the pictures).

Figure	1	2	3
Dark (squares)	5	9	13

The distribution of time			
Table	Figure 1	Figure 2	Figure 3
27150 ms	2950 ms	13150 ms	7500 ms

Table 2



For Anne, it is observed that the time spent on the second figure of the pattern is higher than the other two. This may be related to the fact that the 2nd figure is the only figure which is part of two pairs of figures (1,2) and (2,3). In each pair the subject tends to find the growth rule between consecutive figures and if the time spent in each pair between the figures is nearly 50% that means that the 2nd figure will have the double. We see that the time spent for the 2nd figure is 26% and for the 1st and 3rd together is 21% (6% + 15%).

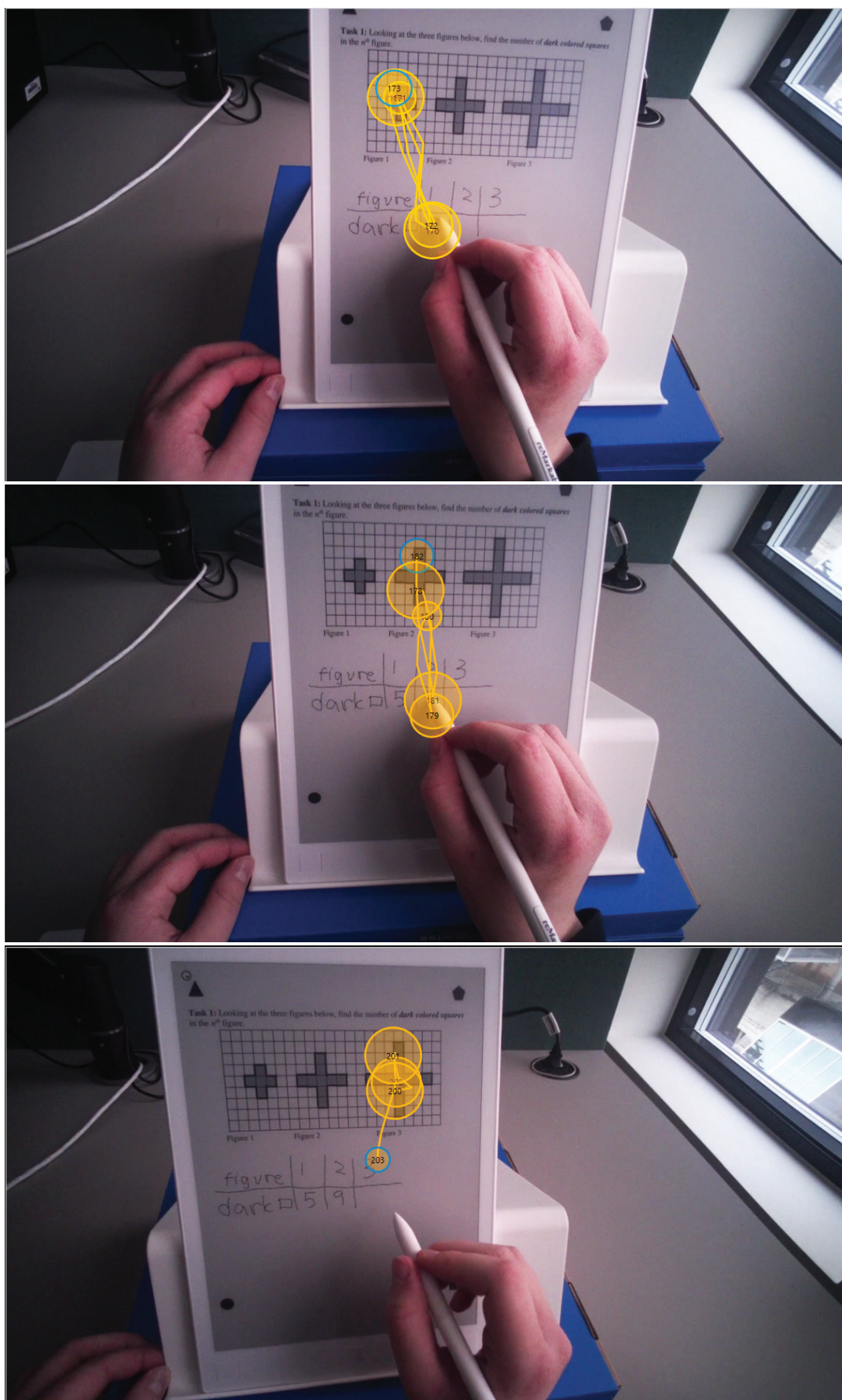


Figure 12

4.4 Analyses of AOI (areas of interest)

Anne, Task 1 - Time spent

The table below shows the time spent on each AOI in percentage. It is noticeable that the time spent in figure 2, is nearly the sum of time spent for two other figures (F_1 and F_3).

With this equation $F_2 = F_1 + F_3$ it is understood the time spent in figure 2 is equal to the sum of the two other figures.

Figure	Time spent-F (ms)
Figure 1	20233
Figure 2	38059
Figure 3	16868

	Time spent-F (%)
Figure 1	7
Figure 2	13
Figure 3	6
Outside figures	74

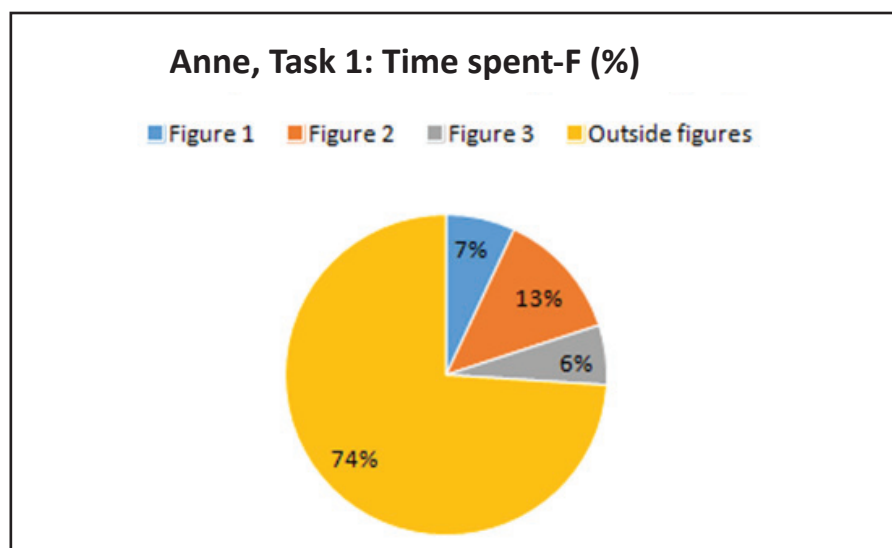


Figure 13

The same equation stands even for the second task (3% for the 1st figure, 12% for the 2nd figure, and 9% for the 3rd figure, $12\% = 3\% + 9\%$). It is nearly the same for the third (3%, 13%, 12%) and fifth task (3%, 6%, 4%), but not for the fourth.

4.5 Number of Revisits and Fixations Count

“Revisit - F” gives information about the number of times the subject has returned looking at a particular AOI after looking outside of that area. On the other hand, “Fixations Count” gives information about the number of fixations on that particular area of interest. The values of these two metrics on each of the decided areas of interest are presented in the table below:

	Revisit-F (Revisits)	Fixations Count
Figure 1	26	30
Figure 2	50	86
Figure 3	24	64

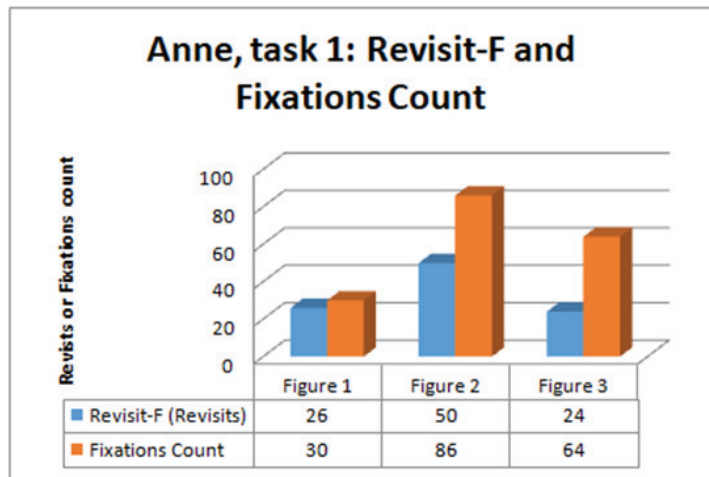


Figure 14

As it is seen from the table above, the same equation, $F_2 = F_1 + F_3$, stands for the number of revisits: $50 = 26 + 24$.

This may be justified by the idea that revisits are part of “pairs of visits” between two consecutive figures, which means that every time the subject tries to make a link (between two consecutive figures), or to understand the relation between two figures, he/she makes two visits in the consecutive figures. This idea is supported by the time spent in each figure, which shows that the time spent in figure 2 is the sum of the two other figures (F_1 and F_3).

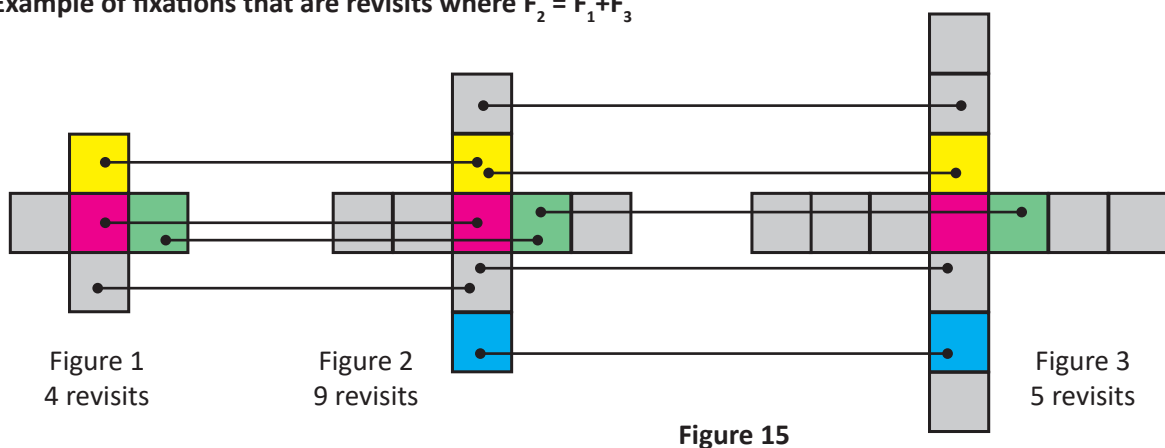
In other words, the time spent in figure 2 and the number of revisits in figure 2 is nearly the double or higher than the average of two other figures (F_1 and F_3).

This may form a hypothesis for additional research to be verified with a large set of data.

The following diagram is created to explain the idea above better:

The subject is supposed to see the same square in two consecutive figures. There is a mapping, which shows that the same number of points (revisits) in two consecutive figures. That is why the second figure has a number of revisits, which is equal to the sum of the number of revisits in two other figures.

Example of fixations that are revisits where $F_2 = F_1 + F_3$



4.6 Fixations that do not count as revisits

If the number of revisits is subtracted from the fixations count, the number of fixations that are not revisits is found. The following values are found:

Figure 1: 4 (30 - 26) Figure 2: 36 (86 - 50) Figure 3: 40 (64-24)

These fixations are eye movements inside the same area of interest (AOI), which may show the amount of work needed to study the features of the same figure. Since the first figure itself doesn't give enough information about the growth rule or the rule of the pattern, it has a small number of fixations that are not revisits. The second and the third figure have much more of such fixations (not revisits), because it is there that the subject understands something about the pattern. It is inside figure 2 that the subject can see the first figure. It is inside figure 3 that the subject sees figure 2 and figure 1. That means that the subject needs more fixations inside the same area of interest to better understand the different parts of the same figures. This results in more fixations that are not revisits.

4.7 The Average Fixation Duration (ms)

From the table of values, it is found that the average fixation duration for the first figure is nearly 1.5 times more than the second figure and the average fixation duration for the second figure is 1.7 times more than that for the third figure.

But this is not observed in other four tasks, where the average fixation duration increases from the first figure to the second and then to the third.

	Average Fixation Duration (ms)
Figure 1	674
Figure 2	443
Figure 3	264

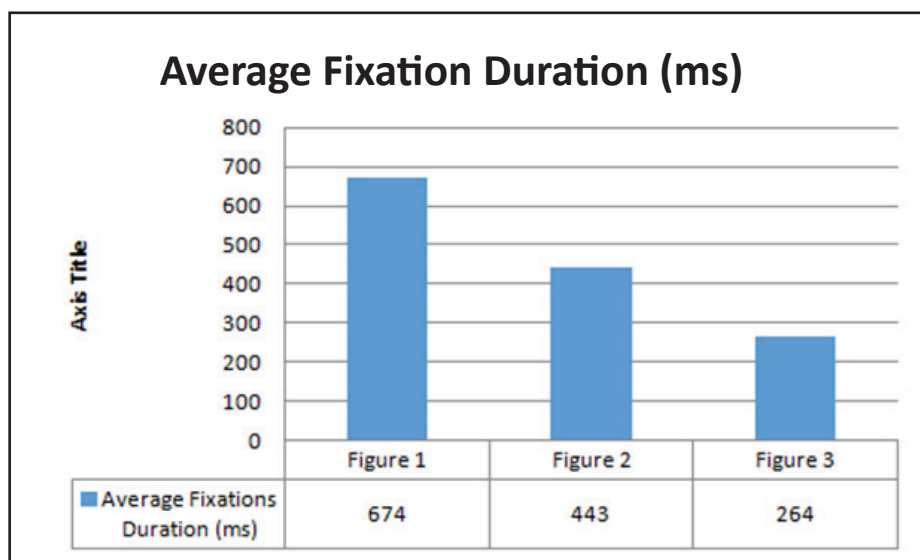


Figure 16

Why these values for the first task?

This may be because of the difficult mental work done in the first figure to understand something without having "help" from a previous figure and it is easier to understand the same in the next figures. That makes the average fixations duration shorter. Another reason may be the fact that with an increasing number of needed fixations, the eyes move quickly between the same figure to understand its inner relations.

4.8 Time To First Fixations (TTFF-ms)

TTFFs give another idea about the subject's work. Nearly 3 seconds after the subject saw the first figure, she fixed her eyes on the second figure. But after a longer period of 19 seconds, she moved her eyes to the third figure. That means that during this time the subject may have dealt with the relationship between the first two figures. It is exactly these two first figures that give the subject the first idea for finding a growth rule between consecutive terms. After having the first idea about the growth rule for the first two figures, there are two ways for continuing the reasoning about the relationship (growth rule) between figure 2 and 3:

- By predicting that the previous rule (from figure 1 to figure 2) will be applied from figure 2 to figure 3 and saying that the third figure must have four more squares and proving that this is true.
- By finding the difference between figure 3 and 2 and comparing it with the first difference between figure 2 and 1 and then drawing the conclusion that it is the same rule.

After that it is time to think about the general rule (the n^{th} term).

The same behaviour is verified with task 4 and 5, where the first fixation for the third figure happens some time after seeing the 1st and 2nd figure. In task 2 and 3, Anne looks at all figures in nearly 2 seconds all the figures, which means that eye movements serve as orientation.

4.9 First Fixation Duration (ms)

There may be several interpretations for these values. One of these may be: The subject uses little time on the first two figures, (about 80 ms on each) and 13 times more (1059 ms) on the third figure. This means that, when seeing the third figure, she has a clear idea on her mind in regard to what is going on. The value 80 ms is close to the value the saccades have (50 ms). This shows that first fixations for the first two figures are needed only to take the coordinates of figure 1 and figure 2. It is known that the first fixations differ within 2.8 seconds. That means that there is not enough time to have a deep thought in 3 seconds. It can be thought that this time is used just for orientation.

The same behaviour, therefore less time for the first fixation duration of the 1st and 2nd figure and more for the third figure is verified on task 2 (220 ms, 520 ms, 860 ms), task 4 (140 ms, 180 ms, 200 ms) and task 5 (900 ms, 540 ms, 1540 ms).

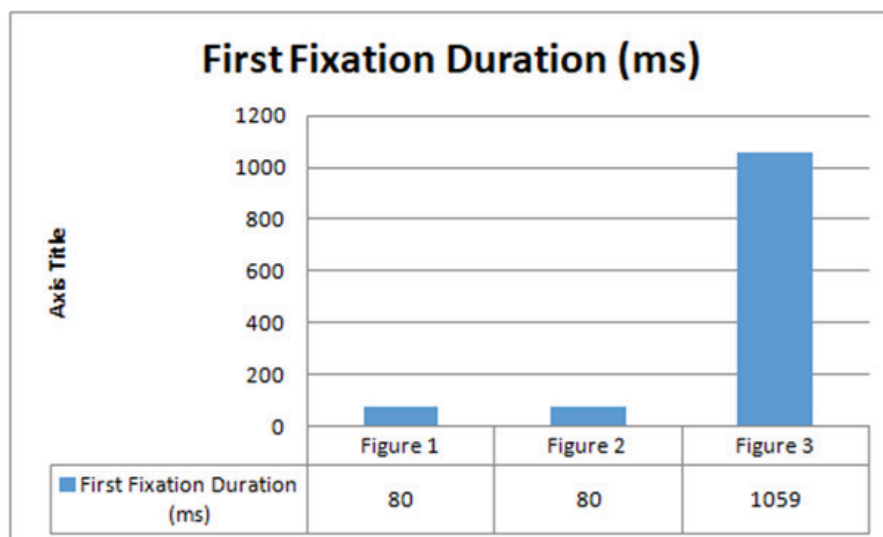


Figure 17

4.10 Heat Maps

Heat maps use the colours red, yellow and green to show the time spent. Red shows where the subjects have spent most of their time, while green shows where they have spent the least of their time.

It is clearly shown with a large and deep red spot that the second figure takes more attention by the subject's eyes and the first figure comes next.

It is also noticed that in each of the three figures, which are the same shape and symmetric, the top arm of the cross takes more attention. This can be explained with the fact that the position of the question is above the figures and that part of the cross is closer to the question. The fact that just one of the arms of the cross takes the attention can be justified by the fact that the figures are symmetric shapes and it is obvious that what is observed in one of the arms, stands for the other arms as well. This means that the information gathered through the sense of sight for one arm of the cross is then processed in their minds.

The green part below the pattern is linked with the notes that the subject has done through the process of solving the task.

The assumption that other subjects' attention will be directed at the top arm/part of the crosses or symmetric shapes when solving these types of tasks, remains unproven.

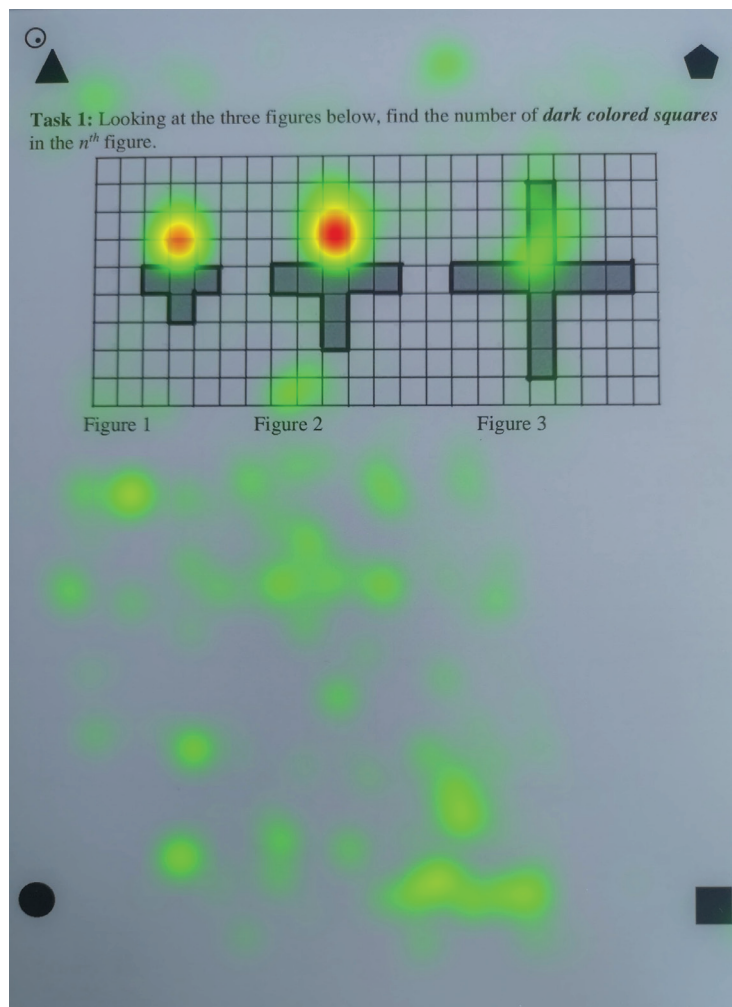


Figure 18

4.11 The concept to be measured: Inductive reasoning in solving shape pattern tasks

While solving 5 tasks with shape pattern, the five subjects gave 25 sets of data including eye-tracking data (video, statistics), written data and audio recording data.

The eye-tracking data were analysed with iMotions software and exported in different types of files: Excel file .xlsx or .mp4 file (video) or .jpeg (heat maps photo).

The data in spreadsheets (excel file) included two types of data:

- a) raw data, also called unfiltered eye-tracking data
- b) fixations data

The videos were used to produce 25 transcripts, (each of the 5 subjects had 5 tasks).

The transcript showed:

- a) where the subjects were looking at, by grouping fixations according to the meaning (seeing figure 1, “counting” the circles, “reading” the task, etc.)
- b) what the subject was saying while seeing something
- c) what the subject was writing
- d) the interventions of the researcher.

A table with 5 columns was made to represent the data above:

- 1st column - the index number of the data
- 2nd column - what the subject says
- 3rd column - what the subject was doing while saying something
- 4th column - the interventions of the researcher
- 5th column - analytic notes of the researcher

4.12 The indicators

After reading all the transcripts, the idea of the indicators for inductive reasoning in solving shape pattern task emerged and a list of indicators was created, as follows:

1. Reads the task more than once.
2. Draws figures related to the task.
3. Creates a number sequence based on the figures in the pattern.
4. Creates a list of the first values for mapping $n \rightarrow f(n)$.
5. Compares the given figures in extension.
6. Notices the same feature in different terms.
7. Looks for the growth rule for consecutive terms.
8. Finds the zeroth term in the numerical sequence.
9. Provides the justification of the rule or applies it.
10. Conjectures about the form of the n th term.
11. Tries different approaches for the solution and assesses their usefulness.
12. Confirms the formula for the general term.
13. Tests the formula.

The above order changes from one subject to another, but the order of many of them are respected by all subjects.

4.13 The measure of indicators and how does it work

For each of the above indicators, in each task, two numerical values were given: 0 (zero) or 1 (one).

The value 0 (zero) was given to the indicator if it was **not** noticed or performed in the work done by the subject. For example, if the subjects did not draw figures related to the task, this indicator was given the value 0 (zero).

The value 1 (one) was given to the indicator if it was noticed or performed in the work done by the subject. For example, if there is evidence that the subject “provides the justification of the rule or applies it”, the value 1 (one) was given to this indicator.

The 6 values scale by using the average (0; 0.2; 0.4; 0.6; 0.8; 1)

An indicator may appear in the work of the subject in a task (value 1) but not in the next one (value 0). The reason behind this may be that the subject does not have that skill that the related indicator presents or does not perform it in that task.

An indicator cannot be measured by giving only the value it has taken in one task. In order to have an accurate measure for an indicator, 5 (five) pattern tasks were given to each subject.

Since the value of an indicator in one task is 1 or 0, in 5 tasks, the average gets the following values: 0; 0.2; 0.4; 0.6; 0.8 and 1.

Since the number of tasks to measure the indicator may vary, it is advisable to find the average of the values 0, 1 and not to find the sum of them. The average remains between 0 and 1, therefore the value of each indicator is better specified.

The number of tasks needed to measure a concept may vary according to the level of difficulty of the concept to be measured, to the nature of the indicator or to the mathematical knowledge or skills the subject has.

This research uses 5 tasks as a reasonable number, in order to not overwhelm the subject.

4.14 The measure of the concept

How multiple-indicators are used to measure “inductive reasoning in pattern tasks”

For each task the presence or not of each indicator is measured by the values 1 or 0. Since the number of indicators is 13, the maximum value of the set of indicators for each task is 13.

This means that the maximum value for the concept in focus of the research “inductive reasoning in pattern task” is 13.

It is better to measure the concept by “the total of 13 values” and not by “the average of 13 values”, because the latest takes the same values (0 to 1) an indicator has, and this is confusing with the measure value one indicator takes.

Another reason is that the sum of values is a better approach to show the number of indicators that are present during the work of the subject solving the task, for example 7 means 7 indicators present during the work.

To measure the concept “inductive reasoning in pattern tasks” better it is important to increase the number of tasks given to the subject. In this research 5 tasks were used.

The measure may be represented by the average of the values (0-13) the tasks show.

4.15 The table of measures for each indicator

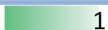

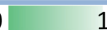












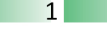


















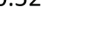









































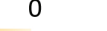





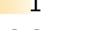





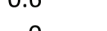























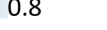











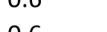



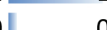

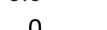





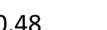











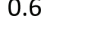

















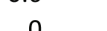













1 Reads the task more than once						
	Task 1	Task 2	Task 3	Task 4	Task 5	Average
Subject 1	 1	 1	 0	 1	 1	 0.8
Subject 2	 0	 1	 1	 0	 1	 0.6
Subject 3	 1	 0	 1	 1	 1	 0.8
Subject 4	 0	 0	 1	 0	 0	 0.2
Subject 5	 0	 0	 1	 0	 0	 0.2
<i>Average</i>	 0.4	 0.4	 0.8	 0.4	 0.6	 0.52
2 Draws figures related to the task						
	Task 1	Task 2	Task 3	Task 4	Task 5	Average
Subject 1	 0	 1	 0	 1	 0	 0.4
Subject 2	 1	 0	 1	 0	 0	 0.4
Subject 3	 0	 0	 0	 0	 1	 0.2
Subject 4	 0	 0	 0	 0	 0	 0
Subject 5	 0	 0	 0	 0	 0	 0
<i>Average</i>	 0.2	 0.2	 0.2	 0.2	 0.2	 0.2
3 Creates a number sequence based on the figures in the pattern						
	Task 1	Task 2	Task 3	Task 4	Task 5	Average
Subject 1	 0	 0	 0	 0	 0	 0
Subject 2	 1	 1	 1	 1	 1	 1
Subject 3	 1	 0	 1	 1	 0	 0.6
Subject 4	 0	 0	 0	 0	 0	 0
Subject 5	 1	 1	 1	 1	 1	 1
<i>Average</i>	 0.6	 0.4	 0.6	 0.6	 0.4	 0.52
4 Creates a list of first values for mapping $n \rightarrow f(n)$						
	Task 1	Task 2	Task 3	Task 4	Task 5	Average
Subject 1	 0	 1	 1	 1	 1	 0.8
Subject 2	 0	 0	 1	 0	 1	 0.4
Subject 3	 0	 1	 1	 1	 0	 0.6
Subject 4	 0	 0	 1	 1	 1	 0.6
Subject 5	 0	 0	 0	 0	 0	 0
<i>Average</i>	 0	 0.4	 0.8	 0.6	 0.6	 0.48
5 Compares the given figures in extension						
	Task 1	Task 2	Task 3	Task 4	Task 5	Average
Subject 1	 1	 0	 1	 1	 0	 0.6
Subject 2	 1	 0	 0	 0	 0	 0.2
Subject 3	 0	 0	 0	 1	 0	 0.2
Subject 4	 1	 1	 1	 0	 0	 0.6
Subject 5	 0	 0	 0	 0	 0	 0
<i>Average</i>	 0.6	 0.2	 0.4	 0.4	 0	 0.32

Table 3 (part I)

Notices the same feature in different terms						
6	Task 1	Task 2	Task 3	Task 4	Task 5	Average
Subject 1	0	0	0	0	0	0
Subject 2	0	0	0	0	0	0
Subject 3	0	0	0	0	1	0.2
Subject 4	1	1	1	0	1	0.8
Subject 5	0	1	1	1	1	0.8
Average	0.2	0.4	0.4	0.2	0.6	0.36

Looks for the growth rule for consecutive terms						
7	Task 1	Task 2	Task 3	Task 4	Task 5	Average
Subject 1	1	0	1	1	1	0.8
Subject 2	1	1	1	1	1	1
Subject 3	1	1	1	1	0	0.8
Subject 4	1	1	1	1	1	1
Subject 5	1	1	1	1	1	1
Average	1	0.8	1	1	0.8	0.92

Finds the zeroth term in the numerical sequence						
8	Task 1	Task 2	Task 3	Task 4	Task 5	Average
Subject 1	1	0	0	0	0	0.2
Subject 2	0	0	0	0	0	0
Subject 3	0	0	0	0	0	0
Subject 4	0	0	1	1	0	0.4
Subject 5	1	1	1	1	1	1
Average	0.4	0.2	0.4	0.4	0.2	0.32

Provides the justification of the rule or applies it						
9	Task 1	Task 2	Task 3	Task 4	Task 5	Average
Subject 1	0	0	1	1	1	0.6
Subject 2	0	1	0	1	1	0.6
Subject 3	1	0	0	0	0	0.2
Subject 4	1	0	1	0	0	0.4
Subject 5	0	0	0	0	0	0
Average	0.4	0.2	0.4	0.4	0.4	0.36

Conjectures about the form of the nth term						
10	Task 1	Task 2	Task 3	Task 4	Task 5	Average
Subject 1	1	1	1	1	1	1
Subject 2	1	1	1	1	1	1
Subject 3	1	1	0	1	1	0.8
Subject 4	1	1	1	1	1	1
Subject 5	1	1	1	1	0	0.8
Average	1	1	0.8	1	0.8	0.92

Table 3 (part II)

Tries different approaches to the solution						
11 and assesses their usefulness	Task 1	Task 2	Task 3	Task 4	Task 5	Average
Subject 1	0	1	0	0	0	0.2
Subject 2	0	1	1	1	0	0.6
Subject 3	1	1	1	1	0	0.8
Subject 4	1	1	0	0	1	0.6
Subject 5	0	0	0	0	1	0.2
Average	0.4	0.8	0.4	0.4	0.4	0.48

Confirms the formula						
12 for the general term	Task 1	Task 2	Task 3	Task 4	Task 5	Average
Subject 1	1	1	1	1	1	1
Subject 2	1	0	0	1	1	0.6
Subject 3	1	0	0	0	1	0.4
Subject 4	1	0	1	1	1	0.8
Subject 5	1	1	1	1	0	0.8
Average	1	0.4	0.6	0.8	0.8	0.72

Tests the formula						
13	Task 1	Task 2	Task 3	Task 4	Task 5	Average
Subject 1	1	0	1	1	0	0.6
Subject 2	0	0	0	0	1	0.2
Subject 3	1	0	0	0	0	0.2
Subject 4	1	0	1	1	1	0.8
Subject 5	0	1	1	1	0	0.6
Average	0.6	0.2	0.6	0.6	0.4	0.48

Table 3 (part III)

4.16 Measuring the indicators and the concept

In the table below, for each subject, there is the total of points for each task (the measure of concept per each task) and the average taken for all the 5 tasks (the average measure of concept using many tasks, five).

The average 6.6 shows the average of points taken from 5 subjects in 5 tasks and in 13 indicators, which means the average of $13 \times 5 \times 5 = 325$ values.

If the average 6.6 (average of the group) is expressed as a percentage, it should be part of the maximum value, 13, for the total of points in 13 indicators, therefore the percentage is 50.769. If rounded, it is 51%.

The same percentage is calculated if we divide **the number of indicators with the value 1** (which in this study is 165) **by 325** (the total), therefore 51% ($165/325 \times 100 = 50.769$).

That means that if the group would be graded from 1 to 10, it would have the grade of 5.1.

The averages for each subjects at the end of the row, show the performance of the subject.

The average for each task at the bottom of each column shows the performance of the group in each task.

TOTAL OF POINTS	Task 1	Task 2	Task 3	Task 4	Task 5	Average
Subject 1	7	6	7	9	6	7
Subject 2	6	6	7	6	8	6.6
Subject 3	8	4	5	7	5	5.8
Subject 4	8	5	10	6	7	7.2
Subject 5	5	7	8	7	5	6.4
Average	6.8	5.6	7.4	7	6.2	6.6

Percentage 50.769

Table 4

The measure of the concept for subject 3 is 5.8, for subject 4, it is 7.2.

The measure of the concept for the whole group is 6.6 and if it is expressed as a percentage, it would be 51% (50.769%) (dividing 6.6 by 13).

4.17 Another measure of the concept, using two indicators

In tables 11 and 12, there is the information for the subjects who tried “different approaches” (table 11) and those who solved the task, in the table 12, “Confirms the formula for the general term”.

If the subject tries different approaches it shows that she is not confident in the way she solved it.

Tries different approaches to the solution 11 and assesses their usefulness						
	Task 1	Task 2	Task 3	Task 4	Task 5	Average
Subject 1	0	1	0	0	0	0.2
Subject 2	0	1	1	1	0	0.6
Subject 3	1	1	1	1	0	0.8
Subject 4	1	1	0	0	1	0.6
Subject 5	0	0	0	0	1	0.2
Average	0.4	0.8	0.4	0.4	0.4	0.48

Confirms the formula 12 for the general term						
	Task 1	Task 2	Task 3	Task 4	Task 5	Average
Subject 1	1	1	1	1	1	1
Subject 2	1	0	0	1	1	0.6
Subject 3	1	0	0	0	1	0.4
Subject 4	1	0	1	1	1	0.8
Subject 5	1	1	1	1	0	0.8
Average	1	0.4	0.6	0.8	0.8	0.72

Table 5

If another table is created “cancelling” the marks in table 12 given for those subjects who solved the task, but with two or more tries, another table of values emerges.

It is the third table. Surprisingly the average of this table is 52%, a value very near to the 51% which was the average of 13 indicators.

This percentage (52%) may be called “the measure of confidence in solving the task”.

Solves the task without many approaches						
	Task 1	Task 2	Task 3	Task 4	Task 5	Average
Subject 1	1	0	1	1	1	0.8
Subject 2	1	0	0	0	1	0.4
Subject 3	0	0	0	0	1	0.2
Subject 4	0	0	1	1	0	0.4
Subject 5	1	1	1	1	0	0.8

Table 6

0.52

In table 12, this measure is 72%. It shows the percentage of subjects who solved the task. The difference with the measure of confidence is +20%.

4.18 HYPOTHESIS

The measure of concept “inductive reasoning in solving shape pattern task” will have nearly the same value if it is calculated as an average of 13 indicators or calculated as the “measure of confidence in solving the task”, with the indicators 11 and 12 (percentage of those subjects who gave the formula but did not try different approaches”.

It is interesting to create a table with both values that are in tables 11 and 12 and calculate their percentages.

The possible pairs are:

- (0, 0) - No different approaches - no solution
- (0, 1) - No different approaches - solution given
- (1, 0) - Presence of different approaches - no solution
- (1, 1) - Presence of different approaches - solution given

Values of tables 11, 12	Task 1	Task 2	Task 3	Task 4	Task 5
Subject 1	(0, 1)	(1, 1)	(0, 1)	(0, 1)	(0, 1)
Subject 2	(0, 1)	(1, 0)	(1, 0)	(1, 1)	(0, 1)
Subject 3	(1, 1)	(1, 0)	(1, 0)	(1, 0)	(0, 1)
Subject 4	(1, 1)	(1, 0)	(0, 1)	(0, 1)	(1, 1)
Subject 5	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(1, 0)

Table 7

Values of the tables 11 and 12

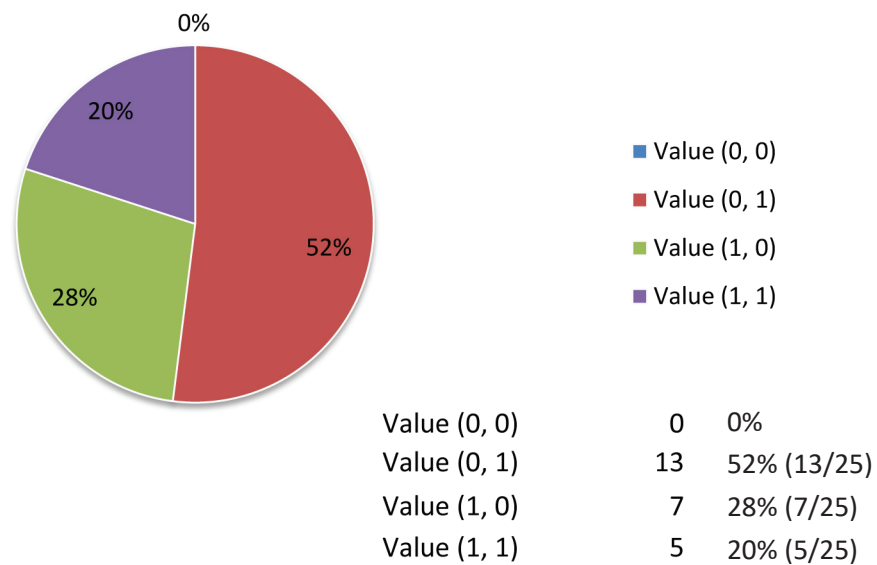


Figure 19

If another cohort of subjects would be tested, the values would be different, but the measures still show important characteristics for them using these two indicators “tries different approaches” and “finds the solution”

4.19 Analysing the task with the lowest measure

From the table of averages, it is understood that task 2 has the lowest average, 5.6.

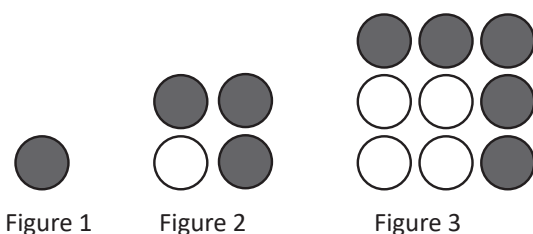
It is interesting to use the measures to analyse it.

For four subjects, task 2 had the lowest average. Only subject 5 had the lowest average in another task.

TOTAL OF POINTS	Task 1	Task 2	Task 3	Task 4	Task 5	Average
Subject 1	7	6	7	9	6	7
Subject 2	6	6	7	6	8	6.6
Subject 3	8	4	5	7	5	5.8
Subject 4	8	5	10	6	7	7.2
Subject 5	5	7	8	7	5	6.4
Average	6.8	5.6	7.4	7	6.2	6.6

Table 9

Task 2: Looking at the three figures below, find the number of **dark colored circles** in the 6th and the nth figure.



The solution is:

“The formula for the nth term is $f(n)=2n-1$ and the sixth figure has $2 \times 6 - 1 = 11$ dark colored circles.”

Figure 20

The table below presents a) (3rd column) the averages of all subjects for “the task 2 in each of 13 indicators, b) (4th column) the averages for all the tasks and all the subjects for each indicator and c) (5th column) the difference between the values of the 3rd and 4th column, which compares the average of the group in one task (task 2) with the average of the same group in all the tasks (5).

From the last column, we notice that +0.32 is the highest difference and it shows that for the indicator number 11, the subjects took more points in Task 2 than the other tasks. This means that they have tried different approaches, which shows that task 2 was the most challenging task.

TASK 2		T2	Ave.	Diff.
10	Conjectures about the form of the nth term.	1	0.92	0.08
7	Looks for the growth rule for consecutive terms.	0.8	0.92	-0.12
11	Tries different approaches to the solution and assesses their usefulness.	0.8	0.48	0.32
1	Reads the task more than once.	0.4	0.52	-0.12
3	Creates a number sequence based on the figures in the pattern.	0.4	0.52	-0.12
4	Creates a list of first values for mapping $n \rightarrow f(n)$.	0.4	0.48	-0.08
6	Notices the same feature in different terms.	0.4	0.36	0.04
12	Confirms the formula for the general term.	0.4	0.72	-0.32
2	Draws figures related to the task.	0.2	0.2	0
5	Compares the given figures in extension.	0.2	0.32	-0.12
8	Finds the zeroth term in the numerical sequence.	0.2	0.32	-0.12
9	Provides the justification of the rule or applies it.	0.2	0.36	-0.16
13	Tests the formula.	0.2	0.48	-0.28

Table 10

4.20 Analysing the indicators

The table below, ranks the indicators by their value, which is the average of 5 tasks for 5 subjects. From this perspective, it is possible to reflect about the benefit each indicator might have.

For the indicators with the maximum value (0.92) or the minimum value (0.2), one can say that these indicators do not serve to differentiate the subjects and for this reason do not measure the concept well. This may be true or false, because other groups should be studied to prove it.

The measure of each indicator		
		Average
7	Looks for the growth rule for consecutive terms	0.92
10	Conjectures about the form of the nth term	0.92
12	Confirms the formula for the general term	0.72
1	Reads the task more than once	0.52
3	Creates a number sequence based on the figures in the pattern	0.52
4	Creates a list of first values for mapping $n \rightarrow f(n)$	0.48
11	Tries different approaches to the solution and assesses their usefulness	0.48
13	Tests the formula	0.48
6	Notices the same feature in different terms	0.36
9	Provides the justification of the rule or applies it	0.36
5	Compares the given figures in extension	0.32
8	Finds the zeroth term in the numerical sequence	0.32
2	Draws figures related to the task	0.2

Table 11

After 2 months of data collection, a Likert Scale questionnaire was given to the same 5 subjects. They were asked if they used the 13 indicators mentioned above. 13 claims were given and a scale of 1-5, where 1 represented “strongly disagree” and 5 represented “strongly agree”. Their responses produced the table below, which ranks the claims by their average score for the whole group. If this study gives 51% as the overall measure of 13 indicators, the subjects in the survey give themselves 82% for the same 13 indicators, which may be seen as a high level of acceptance of the fact that those claims describe very well what they usually do in these types of tasks.

Ranking of the indicators based on survey with subjects		
3	I create a number sequence based on the figures in the pattern.	4.8
1	I read the task more than once.	4.6
9	I provide the justification of the rule or apply it.	4.4
13	I test the formula .	4.4
4	I create a list of first values for mapping $n \rightarrow f(n)$.	4.2
5	I compare the given figures in extension.	4.2
6	I notice the same feature in different terms	4.2
12	I confirm the formula for the general term.	4.2
7	I look for the growth rule for consecutive terms.	4
8	I find the zeroth term in the numerical sequence.	3.8
11	I try different approaches to the solution and assess their usefulness.	3.8
2	I draws figures related to the task.	3.6
10	I conjecture about (guess) the form of the nth term.	3.4

Table 12

4.21 Connection between eye-tracking data and indicators of the concept

To understand the subjects, a lot of information is needed. The information is gathered by hearing to what the subject is saying and recording the data in an audio file, by looking at what the subject is doing (writing, drawing) and recording this data on a video file, by examining the movement of the eyes using the eye-tracker. The importance of each type of data depends on the task or on the meaning that it has.

Using eye-tracking glasses for the research had many benefits because three behaviors were recorded at the same time: the movement of the eyes, what the subject was saying and what the subject was writing on the tablet.

How eye-tracking helped understand the reasoning

- Usually the subject does not say that he/she is reading the task twice, but it can be understood when you see the video that eye-tracker offers. The eye-tracker gives information about the pace of reading and the words they fixate their eyes on.
- Having the confirmation of what is written by the movement of eyes on each figure, it is important to understand the subject's understanding of the pattern.
- Creating the relation $N \rightarrow N$ for the first values is a process of moving eyes quickly to get the information from the figure, therefore the data on the movement of eyes is important.
- It is only from the eye-tracking data that the fact that the second figure takes nearly 50% of the time the three figures of the pattern take, is understood. This is important to analyse how the subject is comparing the figures, and what does one notice during that process.
- The eye movements may create an extra "layer" for the argument and justification of the rule.
- Different approaches to find the solution are sometimes discovered from how the subject moves the eyes according to his/her plan of comparing the figures. This lets you understand his/her approach better.

The following pages 55-78 contain the whole table containing data and analytic notes for one of the subjects (Era)

- index number (1st column)
- what the subject was saying (2nd column)
- what the subject was doing or looking at (3rd column)
- the interventions of the researcher (4th column)
- analytic notes of the researcher (5th column)

4.22 Era’s work

Subject 1 - Era

Imotions file: sc_r001003__s001000 (Time 26min 54sec)

Date/time of the experiment	Date: 06/02/2020 – Time: 12:09		
Subject: Era	Age (24), Gender (female), Education (Master in Didactics of mathematics)		
Materials	Eye tracker & tablet & glasses & digital camera		
Location	Room: J3, 050-051		
Comments for the subject	Positive attitude, willing to collaborate giving detailed descriptions of her thoughts during the work to solve the task.		
	The subject said (<i>Thinking aloud</i>)	The subject actions	The interviewer said/did: Research Comments

Table 13

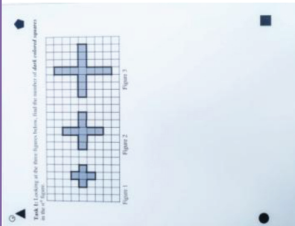
Task 1				
1.		The subject looked first at the blank space of the first page (where the first task is positioned) in order for the researcher to have a full and clear view of the tablet.	Then the interviewer told her to start working. While working the subject was asked to think aloud.	
2.		Era's thinking aloud process of the first task: <i>The respondent is reading the question quietly.</i>	"Just every time think aloud please and try to stay as straight as possible."	
3.	"I'm just reading the task one more time before,...yeah. I'm looking at the first to see how many to compare the three."			She puts the first objectives: "Counting" and "Comparing". Comparing in such tasks is crucial and part of "inductive reasoning". She says "I'm looking" and we may use "eye-tracking device" to track what she is looking at.
4.	"So starting with the first and counting how many, ...five and ... then just adding one more on the outside."	<i>The subject was looking at the second figure.</i>		Observing that one square is added on each arm of the

Table 13

				cross looking figure. It is implicit that "one more" means one more on each arm of the cross.
5.	"And then just adding one more on the next outside"	<i>The subject was looking at the third figure.</i>		She is checking if the rule "one more on the next outside" is true.
6.	"So I think it would be like...if.. and I would do...Should I write like under right under?"		"Yes. Just write everything."	
7.	"Okay. Yeah it's (word) ...hmm, it's like the n, eeee... or no"	<i>The subject erases the letter "n" that wrote on the tablet.</i>		She calls the "n" term as the general term used in such tasks. <i>She changes her mind, because "n" is the index for the general term. She decides for another letter to be used. Because the number of the squares in the n-th term is not the same with n. For example, where $n=3$ is $d_3=13$. We will see she uses n as the index for the d.</i>

Table 13

8.	"Dark colored squares."	The subject gets back to looking at the part of the question where it says dark colored squares.		Repeating "Dark colored squares" focuses her. She observes the "dark colored squares", to be sure what to count or to observe.
9.	"I will call it d, like d ₁ equals 5."	The subject is writing that on the tablet while looking at the first figure		Why did she use "d ₁ , d ₂ , d ₃ " to present the first, second and third figure? May be from the word "difference".
10.	"d ₂ equals 5+4"	The subject is writing that on the tablet while looking at the second figure.		For the moment, she can't see that the first number can be expressed as $5 = 1 + 4$, which has the number 1 (the number for the first term, part of the "domain" 1,2,3,4,...). Therefore she is not used to ask herself to give an expression containing 1 and some operations with that.
11.	"And d ₃ equal 5+4+4"	The subject is writing $d_3 = 5+4+5$ on the tablet while looking at the third figure constantly.		

Table 13

12.	“So then d_n equal $5+...$ ”	The subject is writing that on the tablet while looking at the first three equalities d_1 , d_2 , d_3 and at the same time the second and the third figure.	<p>She uses “n” in d_n. It is the time for the generalization to try finding the general term.</p> <p>Checking? Why? She seems needing help for something? Why did she look only at the 2nd and 3rd figure and not the first? Maybe because only the 2nd and 3rd need the “addend” +4. For her, we have: $5, 5+4, 5+4+4$ and she sees +4 only in the 2nd and 3rd figure. So the change is the next figure, excluding the first. Here “eye-tracking” explains us better her actions and related reasoning.</p>
13.	“No I write 5 there actually, I meant 4.”	The subject laughs and then returns to the third equality d_3 and instead of the second 5, she puts a 4.	
14.	“5, eee..., 4 times...like the number, the n , eee..., $n-1$, like the sort of one formula for it, but it's not the..., you can find the exact one, or you can,	The subject writes the formula $d_n = 5+4*n-1$.	Even though she wrote $5+4*n-1$, I understand she means $5+4(n-1)$

Table 13

	yeah.”			<i>which is the exact expression.</i>
15.	“So n zero would be like just one square. It would be 1.”		<i>The subject writes on the tablet $d_0 = 1$. Then starts looking through the three figures.</i>	She thinks $n_0 = 1$ therefore $n_1 = 5$. Why does she need to recall n_0 ? It is a tool to understand what is the previous term and to see 4 in 5. ($5 = 1 + 4$) She said “n zero” and now she writes d_0 . Does “n zero” mean $n = 0$? If so, she is adding one number (zero) in the set of the “domain”.
16.	“So it would be...Actually, d_1 would be...if I develop it from the one, it would be...”		<i>The subject looks at the first equality that she has written, $d_1 = 5$, and puts an arrow between that and the equality that she is writing at this moment on the right side of the tablet. She continues writing “$d_1 =$”</i>	Now she remembers that d_1 must be expressed as an expression having 1 in it.
17.	“the d_1 would be 1 plus 4 or $1 + \dots n$ times 4. Is that the formula?”		<i>The subject stares for some moment the formula that she has written, $d_1 = 1 + 4 = 1 + n \cdot 4$. Then she erases the n and writes 1 instead.</i>	Now she explains better the situation where 4 is the repeated addend, from one term to the next one to find the general term. At this time she is producing the assumption to be tested, which is the

Table 13

				key part of the inductive reasoning. She finds the formula. The fact that she erases "n" is the substitution process when we try to find the value for $n=1$. She does not need to write in a long way. She uses a short way. Therefore we don't have correcting an error, but producing the value for $n=1$ in the formula.
18.	"Since it's one so that..."		<i>The subject continues writing $d_n = 1+n*4$.</i>	
19.	"Is that correct for everyone or.."			She shows the right knowledge and skills by showing us the right procedure, which means that we should test the formula.
20.	"Okay. And then I can test it for eee...the next numbers so...I would like to (write) for d_4 , that I would know that d_4 would be...by counting it would be 5,6,7,8,9,10,11,12,13..."		<i>The subject is counting the squares in the third figure.</i>	Here the eye-tracking is proving to be a good way to forgive the mistakes (d_4 for d_3)

Table 13

21.	"...and that's (correct) with what I said earlier, no...yes it is"	<i>The subject looks on the formula that she wrote earlier $d_n = 5+4*n-1$.</i>		
22.	"Is 13 in figure 3, so the next would be, in my calculations 17, so d_4 would be $1+4$ times 4, which equals 17, so yeah. I would say that n would be this.	<i>The subject writes the equality $d_4 = 1+4*4 = 17$ and underlines the formula $d_n = 1+n*4$.</i>	"So are you ended with this task?"	She doesn't need to test the formula for $n=4$ (d_4). But she adds another task (next term) which is not required.
23.	"Yeah."		"Okay, can you move to the next one?"	
24.	"Of course. And then stare at the blank space first?"		"Yes."	

Table 13

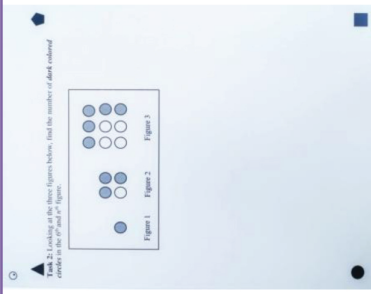
Task 2				
25.		<p>Era's thinking aloud process of the second task:</p> <p><i>The respondents is reading the question quietly.</i></p>	<p>"You can start working."</p>	
26.	<p>"Okay, then I start by looking at figure 1."</p>	<p><i>The subject is looking at the three figures, one after another.</i></p>		<p>Observing How? Analyse it.</p>
27.	<p>"So then I am thinking that, eee..., the white colored circles are the ...all the circles from the last figure...that's my thinking. And then I'll just write what...F₁ equals 1, F₂ equals 3, F₄...oh no sorry, F₃ equals 5, so I would think just by looking at it that the next one, like the fourth would be next odd number so 7. So I'm just gonna try to draw it, like on the side, eee..."</p>	<p><i>The subject draws how the fourth figure would be.</i></p>		<p>She has the idea to think that the previous term converts its colour from "black and white" in only "white" in the next term (figure). After mentioning 1, 3, 5, she finds out that this is the series of the odd numbers and predicts 7 as the 4th term. Later she calls this "thought" as an "assumption". <i>Drawing helps her to</i></p>

Table 13

				solve the task.
28.	“So this would be 3. Oh just, hmmm...can I like make it out or just...is it like a rubber or something? No.”	The subject colors the wrong circle on the fourth figure so she wants to erase it.	“Yeah.”	
29.	“Is in the back?”		“No.”	
30.	“Okay.”		The interviewer shows her how to erase something on the tablet.	
31.	“Thank you. Coloring these.”	After the subject erase the black circle, which she thought it was a wrong decision to color that particular circle, she starts coloring the circles on the first row and on the fourth column of the fourth figure that she drew.		Drawing the 4 th figure.
32.	“And then I see that my assumption was correct, it was 7. So F_4 would be 7. So that I would say that F_n the...like the (discrey) formula or not the actual formula, but like the next one would be the one before, so then it would be like F_{n-1} , eee...noo, stop to think for a bit. That cannot be correct, because that would be ...”	The subject is staring at the fourth figure.		She mention the concept “assumption” as a needed part of her inductive reasoning. She says “it is correct”, therefore “proved” with her sketch.
33.	“...no, it would be correct, like one plus one plus two.”	The subject adds 2 to the equality $F_n = F_{n-1}$, so on the tablet she has $F_n = F_{n-1} + 2$.		It is clear that she thinks of the number of the dark colored circles, where F_n is the number those “dark colored circles” of the n term and F_{n-1} is the number of the previous figure and 2 is the

Table 13

34.	<p>"So the formula could be ...1, n-1 plus 2. No, it would not have been n-1, it would have been just n+2."</p>	<p>The subjects writes on the tablet the formula $F_n = n-1+2$ and then erases the -1. She stares for some time the formulas $F_n = n-1+2$ and $F_n = F_{n-1} + 2$, then she sees the third figure and then the equalities $F_3 = 5$ and $F_4 = 7$.</p>	<p>repeated "addend".</p> <p>From the previous task and this task too, she shows her need for the (n-1) term. That helps her to generalize the relationship between two adjacent figures. She forgets what n represents, because if the formula is $n+2$ for the second figure we will have $2+2 = 4$, which is not true (she said: they are odd numbers).</p> <p><i>It is obvious that without eye-tracking we can't see her movement just from what she writes.</i></p> <p><i>From the two formulas she wrote the first is inaccurate and the second is accurate. The fact that she spends some time looking at the two formulas may show her attempt to understand which one is accurate with the help of information got from the third figure and the formulas she has</i></p>
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Table 13

				<p>written $F_3=5$ and $F_4=7$. If we speculate: she is testing F_3 and F_4 with both formulas: 1st formula $(3-1+2)$ or $(4-1+2)$ and the 2nd formula $(3+2)$ where $3=F_{n-1}$ (2nd figure) or $(5+2)$ where $5=F_{n-1}$ (3d figure).</p> <p>The above prediction of her reasoning is verified with the last actions.</p> <p>Now she is in need for the formula, containing not the previous term but just the variable n.</p> <p>Since she knows that converting the taks in just the series 1, 3, 5, 7, 9, 11, ... the 6th term is right 11.</p> <p>Why does she need to draw other figures? She has no other idea how to solve the general term.</p> <p>Just confirming with sketch.</p>
35.	<p>"But that would not stand, ahahaha, but then it won't be correct, oh no. Emm... this is more correct, like this one, so I will just cross that out."</p>	<p>The subject erases the n on the formula $F_n = n+2$ and then points at the formula $F_n = F_{n-1} + 2$ and says it's correct. Then she puts a line in the middle of the formula $F_n = n+2$.</p>		
36.	<p>"Emm...how can I do this?!"</p>	<p>She stares for some time the fourth figure again.</p>		
37.	<p>"I would assume that F6 would be the six odd number, so it wouldn't be 9, but it would be like 11. So if I just like draw some more..."</p>	<p>The subject draws another row with circles on the bottom of the fourth figure and the fifth column with circles on the right of the fourth figure.</p>		
38.	<p>"This would be the next one, so it would be 9, yeah and this would be like 11."</p>	<p>The subject draws another row with circles on the bottom of the fourth figure with the previous added circles and the sixth column with circles on the right of the fourth figure with the previous added circles.</p>		

Table 13

39.	“That would be correct, so six would be like 11, that would be that one.”	The subject writes on the tablet $F_6 = 11$ and underlines it.	The solution for the first task (the number of dark colored circles in 6 th figure)
40.	“And the n th figure would be the odd number which is in that, () odd number, so the first one is one, the second one is three, the third one is five, so it would like increase, so the difference between them would be, the difference between those two would be two, the difference between those two would be two, the difference between the odd number would be 1, the same one and n one, n two, n three, so that would increase, hmmm...”	The subject looks at the equalities that she has written at the beginning $F_1 = 1$, $F_2 = 3$, $F_3 = 5$ and writes in the middle of these equalities the difference 2. She looks again at those and explains that the difference between the odd numbers is 1. Then she stares at the equalities $F_4 = 7$ and $F_6 = 11$.	She compares the difference 2 between adjacent figures and the difference 1 of the number showing the term. She is trying to solve how to put “n” in the formula when F_n and link the n-“domain” set (1,2,3,4,...) with its F_n -“range” set (1, 3, 5, 7, ...) with its formula $F_n = n \dots$ etc. The pairs (1;1) (2;3) (3;5) (4; 7) ... Being “systematic” and “thorough” in this case would help her to remember to give the right name to the sets of data she has: “domain”, “range”, “pair”, “a formula is an equation” etc. She is confusing herself “the difference between odd numbers” which is not 1 (but 2) with their index number $f_1=1$, $f_2=3$, $f_3=5$.
41.	“The difference between these one	The subject is looking at the index of the $F_6 = 11$	In the pairs (6,11) she

Table 13

	would be 5 and that would be 3."	and explains that the difference between the index and the number is 5 and then looks at the index of the $F_4 = 7$ and explains that the difference between the index and the number is 3.		thinks of 5 as the difference $5 = (11-6)$ and then from the pair (4,7) she makes the difference $3 = (7-4)$. That is why she mentions 5 and 3 while seeing what she wrote $F_4=7$ and $f_6=11$.
42.	"So it would be one less than the actual number, so it would be increased, ..., maybe it's just $n-1$ plus the last number, plus n ."	The subject looks constantly the equality $F_6 = 11$ and then gets back to looking at the formula $F_n = F_{n-1} + 2$ and then writes $n-1+n$. After she looks again the $F_6 = 11$.		The "actual number" for her, it seems to be the index n and the $(n-1)$ is the difference F_n-n or the addend a) if $n=6$ she thinks $(F_6-6)=11-6=5=n-1=6-1$ or b) if $n=4$ she thinks $(F_4-4)=7-4=3=4-1=(n-1)$ From $F_n-n=n-1$ she may find that $F_n=2n-1$. For those who do not remember the general term of series of odd numbers, this is a complicated way to find the general term, but this shows how the mind works in excellent student, who, if forget the past, are able to find quick ways to solve the problem. <i>I am not sure, but I think it is interesting to suppose she is</i>

Table 13

				<p>translating F_{n-1} in $(n-1)$. To remember she thinks $f_6-6 = 11-6=5=6-1=n-1$ that means, for her that $F_n=n-1$, and it is obvious that $F_n = n-1 + n$.</p> <p>It seems there is a chain of reasoning which is hidden. She did something without mentioning it. I am not sure, but I think it is interesting to suppose she is translating F_{n-1} in $(n-1)$. To remember she thinks $f_6-6 = 11-6=5=6-1=n-1$ that means, for her that $F_n=n-1$, and it is obvious that $F_n = n-1 + n$.</p> <p>It seems there is a chain of reasoning which is hidden. She did something without mentioning it.</p>
			<p>While writing this Era is staring at the formula $F_4 = 7$ and then she explains that she considers the n to be the number 4.</p>	
			<p>She says this while staring at the $F_3 = 5$.</p>	
43.	"So $n-1$ is 3 plus n is 4."			
44.	"That's 7, $n-1$ here is 3 minus 1 plus 3, that's correct."			<p>To translate: Maybe, she intends 5 (the outer layer) in the 3rd figure where the circles are $3 \times 3 = 9$ but</p>

Table 13

				the outer layer is (3-1)=2 the upper row (-1) and 3, (the right column) therefore the total of dark coloured circles is 2+3=5. She is saying (n-1) for 2 (3-1) where n=3. We are aware that when we think aloud, we can “cut” or “shorten” part of the real thinking is and stress only what is important for us. Since “thinking aloud” is not like “eye-tracking” which fully captures the movement of eyes, we understand that the subject is telling only a part of her reasoning.
45.	“Yeah, I would say that this is the formula.”	She looks at the $n-1+n$ and writes F_n before it, meaning $F_n = n-1+n$.	“Okay. And for the sixth? Yeah”	She does not simplify it to $2n-1$, which is the standard form of writing the expression, part of the formula for the general formula.
46.	“So this is that one and this is the other one. Okay?”	The subject underlines the formulas $F_n = n-1+n$ and $F_6 = 11$.	“Okay. You can go to the next slide.”	The solution is ok for both tasks.

Table 13

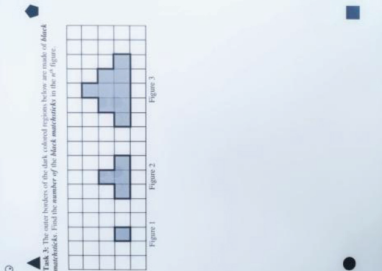
Task 3				
47.		<p>Era's thinking aloud process of the third task:</p> <p><i>The respondents is reading the question quietly.</i></p>	<p>"See the blank space first."</p> <p>"Okay, you can proceed."</p>	
48.	<p>"Okay. So can I just close this?! Yeah. Task3...matchsticks. Okay. So just starting like I did the same the other ones, just like counting how many is in each figure. So F_1 would be 1, 2, 3, 4. F_2 would be 1, 2, 3, 4, 5, 6, 7, 8, 9, 10."</p>	<p><i>Era wrote the equalities $F_1 = 4$ and $F_2 = 10$ underneath the figures after counting the number of matchsticks in each figure.</i></p>		<p><i>After solving two patterns, she is quicker.</i></p> <p>$F_1=4$ and $F_2=10$.</p>
49.	<p>"Am I counting it the right way, like the matchstick is laying like outside of the lines?"</p>		<p>"Yeah."</p>	<p>Asking "Am I counting in the right way" shows an experience in solving such tasks.</p>
50.	<p>"Okay. Hmm... F_3 would be 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16. So..."</p>	<p><i>Era starts looking at the three figures and then the equalities underneath, $F_1 = 4$ and $F_2 = 10$, $F_3 = 16$ and then gets back to figure two and figure three and counts the number of matchsticks</i></p>		<p>$F_3=16$</p> <p><i>Just checking.</i></p>

Table 13

		again quickly.			
51.	<p>"Yeah, just checking if I count well. Yeah, okay. So the next one would probably...because there is just like adding blocks on the outside, so like this would be one more here, one more here, one more here, one more here and one more here. So those matchsticks would be 1, 2, 3...no, just those are the same, hmm..."</p>	<p>She starts coloring the outside squares on the third figure and then start counting the matchsticks of the supposed fourth figure.</p>		<p>She is adding squares to draw the fourth missing figure. That means she is understanding the rule of increasing the figure.</p>	
52.	<p>"...would be like 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22. So F_4 would probably be 22."</p>	<p>She counts the matchsticks again.</p>		<p>$F_4=22$ is ok.</p>	
53.	<p>"Yeah, 22. That's the fourth. So I would like to see the difference, the difference here is 6, 6, 6, so I would think that it would be the sixth that it's like...6 more for each, so like the (discrepancy) formula would be something like the first the figure before like $F_{n-1} + 6$."</p>	<p>After writing $F_4 = 22$, Era starts looking at the equalities $F_1 = 4$ and $F_2 = 10$, $F_3 = 16$ and writes between them the difference, which was 6. Then while looking again at the same equalities, she writes the formula $F_n = F_{n-1} + 6$."</p>			
54.	<p>"Yeah. And the other formula, which you can develop is just....cause this is number 3, how many sixes have we added since 4, we added two sixes, 6+6, so n times, so it's 4 plus n-1 times 6. So you can write it like 4 plus 6 multiplied n-1."</p>	<p>While saying this the respondent was looking first at the formula $F_3 = 16$, then $F_2 = 10$ and then she writes $F_4 = 22$. Then she writes $4 + 6(n-1)$.</p>			

Table 13

55.	"I think, so I can test it."	She writes $F_n = 4 + 6(n-1)$.		
56.	"I would test it in 4 for example. So if F_4 would be $4 + 6(4-1)$, I would say that equals 4 plus 6 times 3, which is 18 plus 4, which is 22. And I would test it for the next one which would be F_5 , ..., we would put in the formula, ..., it would be 4 plus 6 times 4, which would be 24, like 28 and if 22 is the last one, that would be correct. So yeah I would say that the formula for this one is F_n equals 4 plus 6 times $n-1$. Okay?! Do you see it clearly like where it is? ahaha"	The subject writes $F_4 = 4 + 6(4-1)$, $F_4 = 22$ and $F_5 = 4 + 6 * 4$, $F_5 = 28$ on the tablet.	"Yeah."	
57.	"Yeah, I'm a little bit, my brains work, I have to write all over ahahah."		"Okay. Can you move to the next one?"	
58.	"Okay."			

Table 13

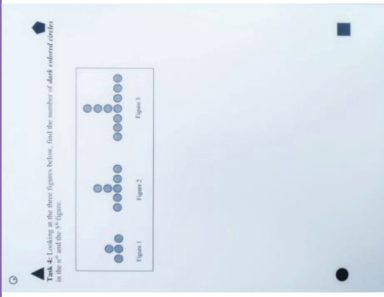
Task 4				
59.		<p>Era's thinking aloud process of the fourth task:</p> <p><i>The respondents is reading the question quietly.</i></p>	<p>"Okay, you can start working."</p>	
60.	<p>"Fifth figure, yeah. I start just writing like I did before, so F_1, \dots the dark colored circles, yeah. F_1 would be 5, no just kidding 4 ahaha, F_2 would be 4 plus 3, so F_2 equals 7, just counting 5, 6, 7, yeah. F_3 would be 7...so it would be 10, 7 plus 3. So they are just adding three more outside of each. So it would like increase from 3 here, 3 here."</p>	<p><i>The respondent starts counting the circles in each figure and writes underneath each of them the equalities $F_1 = 4$ and $F_2 = 7$, $F_3 = 10$. Then she looks at the differences between the equalities and writes 3 in the middle of each.</i></p>		
61.	<p>"I would assume that the next one would just be like this..."</p>	<p><i>By using the third figure, she starts building the fourth by adding 3 circles on each side of the figure. Then she starts writing the corresponding equation F_4.</i></p>		

Table 13

62.	<p>"So I like add three more, so F_4 would be 10 plus 3, which is 13. Yeah, just seeing at the task again to see if I have got it correct"</p>	<p><i>She looks at the question again and then puts a 3, the difference, in the middle of F_3 and F_4.</i></p>		
63.	<p>"So F_5 would be just adding 3, which is 15, no just kidding 16. I have a thing for writing numbers wrong, I say it out loud, I think them and then I just write another one. So yeah it would be 16, it would just be one more, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, yeah. So that would be F_5 would be 16. So I do that first and then I'm going over to the n."</p>	<p><i>After she writes $F_5 = 16$, she uses the fourth figure that she built over the third one, to build the fifth by adding 3 circles on each side of the figure. Then she starts counting the circles of the fifth figure.</i></p>		
64.	<p>"So I would say that F_n would be..., the last figure, so just (write) formula first $F_{n-1} + 3$. And then just the actual formula would be..., let's see if there is some combination to start with 4, so,..."</p>	<p><i>She is looking constantly at the first equation $F_1 = 4$.</i></p>		
65.	<p>"So let's see how many threes do I have to add, so F_{n+1}..."</p>	<p><i>She looks constantly at the first equation $F_1 = 4$</i></p>		

Table 13

		and the first figure.			
66.	"3 plus, no just kidding, let's take that out, it would be,..."		She writes $F_n = 3 +$ and then removes the 3.		
67.	"1 plus as many threes, yeah it would be $1 + 3n$, maybe. I can try that. F_1 would be 1 plus 3, yeah that's right, that's 4. F_2 would be 1 plus 2 times 3, which is 6, is 7. F_3 would be 1 plus 3 times 3, which equals 10. F_4 would be 1 plus 3 times 4, which equals 13. F_5 would be 1 plus 3 times 5, which equals 16. So yeah I would say that's the correct formula I would use. F_n equals $1 + 3n$. And this is the F_5 . Yeah."		The subject writes $F_3 = 1 + 3 \cdot 3 = 10$, $F_4 = 1 + 3 \cdot 4 = 13$ and $F_5 = 4 + 6 \cdot 4$, $F_5 = 28$ on the tablet. Then she writes $F_n = 1 + 3n$ and underlines it. $F_5 = 1 + 3 \cdot 5 = 16$ and then $F_n = 1 + 3n$ and underlines them.	"Okay. You can continue to the next one."	
68.	"Okay."				

Table 13

Task 5				
69.		Era's thinking aloud process of the fifth task:	"Okay, you can start working."	
70.	"It's so funny staring at a space where there is nothing. Big cubes...so...dark painted smaller cubes"	The respondent is reading the question slowly and then she start looking at the three figures.		
71.	"So it's all the sides?"		"Yeah, all the sides."	
72.	"Okay. So you don't have to count those inside?"		"Even the one that are not.."	
73.	"Shown?"		"Every face of the cube"	
74.	"Yeah, so not on the inside? No, just the one that you see colored and then the other ones like behind and under, those count."		"Yeah."	
75.	"So I'll just start like I do with the	The subject writes $F_1 = 6 F_2 = 6 * 4 = 24, F_3 = 6 *$		

Table 13

	other ones, F_1 equals 1, 2, 3, On a cube it is 1, 2, 3, 4, 5, 6 sides, because you use at a game. So 1, 2, 3, 4, 5, 6, so F_1 equals 6. And the bigger cube is F_2 is...I would say like 6 times 4 equals 24. F_3 would be 6 times 9. So it's just square numbers times 6."	9 and $F_4 = 6 * 16$ on the tablet. Then she reads the question again.		
76.	"Yeah, so F_4 would probably be 6 times 16, so then I would say that F_n would just be the square number times 6, so 6 times n in square. Yeah that's my answer."		Era writes the answer $F_n = 6n^2$ and then underlines it.	

Table 13

4.23 Examples of evidence for the indicators

In total, there are 325 cases to check if the subject has evidence of a certain indicator. If there is evidence, the value 1 is given to the related indicator, if not, the value 0 is given to it. Below there are some examples of evidence shown by different subjects.

1. Reads the task more than once

(Subject: Era)

Task 1: "I'm just reading the task one more time before, ...yeah." (no. 3) "dark colored squares" (no. 8)

Task 4: "Yeah, just seeing at the task again to see if I have got it correct." (no. 62)

Task 5: Then she reads the question again. (no. 75)

(Subject: Rita)

Task 3: Rita starts reading the question again, but this time exactly how it is. (no. 36)

(Subject: Dua)

Task 1: Then she reads the question again... (no. 4)

(Subject: Lucy)

Task 3: Then she looks at the question of the task again...(no. 37)

(Subject: Anne)

Task 3: Then she looks at the question again...(no. 21)

2. Draws figures related to the task

(Subject: Era)

Task 2: The subject draws how the fourth figure would look like. (no. 27)

(Subject: Rita)

Task 3: The respondent draws figure 4 on the tablet in the following way: First she draws a square, then adds three squares below that. Then below the second row she draws the third row of squares with the same number of squares as the second row and then adds one more square on each side. Then below the third row, she draws the fourth row of squares with the same number of squares as the third row and then adds one more square on each side. (no. 51)

(Subject: Dua)

Task 5: ...and then to make sure that she hasn't done anything wrong, she draws an open cube and counts the squares ... (no. 58)

3. Creates a number sequence based on the figures in the pattern

(Subject: Rita)

Task 2: The respondent looks at figure 1 first and then reads the part of the question again where it is written “dark coloured circles” and writes 1 above the first figure. Then she looks at the second figure and writes 3 and then she counts the black circles of the third figure and writes 5. (no. 20)

(Subject: Dua)

Task 4: She starts by counting the circles in figure 1 and writes 4 under it and then the circles in figure 2 and writes 7 under it and then the circles in figure 3 and writes 10 under it. (no. 45)

(Subject: Anne)

Task 1:

Figure	1	2	3
Dark squares	5	9	13

(no. 5)

4. Creates a list of first values for mapping $n \rightarrow f(n)$

(Subject: Era)

Task 2: Era wrote the equalities $F_1 = 1$, $F_2 = 3$ and $F_3 = 5$, underneath the figures after counting the number of black circles in each figure. (no. 40)

(Subject: Rita)

Task 3: “She writes the equations $F_1 = 4...$ ” (no. 59)

(Subject: Dua)

Task 3: She writes the equations $F_1 = 4$, $F_2 = 10$, $F_3 = 16$ while looking at the numbers 4, 10, 16, respectively. (no. 33)

(Subject: Lucy)

Task 3: She writes on the tablet the equations $F_1 = 4$, $F_2 = 10$, $F_3 = 16$ and then puts a connection line between them and writes the difference 6. (no. 32)

5. Compares the given figures in extension

(Subject: Era)

Task 4: “So they are just adding three more outside of each.” (no. 60)

(Subject: Rita)

Task 1: “Because you obviously add one each side.” (no. 9)

(Subject: Dua)

Task 4: “okay just by looking at the figures we can see that the number of circles in the figure increases by one in each direction” (no. 48)

6. Notices the same feature in different terms

(Subject: Dua)

Task 5: She looks at the coloured square on the front side of the big cube in figure 2 and writes $4 \times 6 = 24$ under it. Then she looks at the coloured square on the front side of the big cube in figure 3 and writes $9 \times 6 = 54$ under it. (no. 59)

(Subject: Lucy)

Task 1: She first looks at the squares at the top arm of figure 1 and then at the squares at the top arm of figure 2 and at the squares at the top arm of figure 3. After that, she returns to figure 1 and looks at the squares on the right arm and then looks at the squares of the right arm of figure 2 and at the squares on the right arm of figure 3. (no. 9)

(Subject: Anne)

Task 2: She starts looking at the black circles on the first figure, then the second figure, then the third figure. Then she looks at the white circles on figure 3, then figure 1 and then figure 2. Then, her eyes are focused on the top right corner circle of the second figure, then, her eyes focus on the top right corner circle of the third figure. Then she looks again at the dark coloured circles on each figure. (no. 9)

7. Looks for the growth rule for consecutive terms

(Subject: Era)

Task 4: "So it would like increase from 3 here, 3 here." (no 60)

(Subject: Rita)

Task 1: "And then from 5 to 9, you add 4, and from 9 to 13, you add 4. So, you add 4 every time." (no 8)

(Subject: Dua)

Task 3: "So plus 6, plus 6." (no 32)

(Subject: Lucy)

Task 3: "So the common difference is 6 between them..." (no 33)

(Subject: Anne)

Task 1: Then she draws a line between the numbers 5 and 9 and writes 4 under it and a line between the numbers 9 and 13 and writes 4 under it. (no 5)

8. Finds the zeroth term in the numerical sequence

(Subject: Era)

Task 1: "So n zero would be like just one square. It would be 1." (no 15)

(Subject: Lucy)

Task 3: "If I had to follow my previous teacher's, like teachings I would have to find the zero term or figure, which 4 minus 6 is -2." (no 34)

(Subject: Anne)

Task 1: She writes on the tablet "zero term = " and then her eyes go to the table that she draws to the numbers on the first column and then continues writing "zero term = $5 - 4 = 1$ ". (no 6)

9. Provides the justification of the rule or applies it

(Subject: Era)

Task 3: While saying this the respondent was looking first at the formula $F_3 = 16$, then $F_2 = 10$ and then she writes $F_4 = 22$. (no. 54)

(Subject: Rita)

Task 2: "Figure 3...So F_4 would be 5+2, would be 7. And figure 5 would be 9. And figure 6 would be 11. So I figured out the 6th square, whatever you call it." (no. 23)

(Subject: Dua)

Task 1: "Okay so for the nth ... F_1 equals 5, F_2 equals 9, which is the last one plus 4, so this is F_{n-1} plus 2, nei plus 4. And then F_3 is 13, which is the last one plus 4." (no. 5)

(Subject: Lucy)

Task 1: "Oorr... Because the first figure, if I exclude the middle part of the cross from each four ways that the cross goes, it's like in the figure 1 it's one box in figure 2 it's 2 boxes and figure 3 there's three boxes." (no.12) "If I had a way to represent the previous figure and then add 4 to that, but in the general rule maybe that would work." (no. 8)

10. Conjectures about the form of the nth term

(Subject: Era)

Task 2: The subject adds 2 to the equality $F_n = F_{n-1}$, so on the tablet she has $F_n = F_{n-1} + 2$. (no. 33)

(Subject: Rita)

Task 2: "So what I do is, do the previous one, so I do figure $n-1$ and then I add 2. This is for figure n ." (no. 27)

(Subject: Dua)

Task 1: She writes $F_n = F_{n-1} + 4$ on the tablet and then points the fact that she doesn't always know the F_{n-1} , so she puts that on a circle. (no. 9)

(Subject: Lucy)

Task 1: "So if I have y equals...4n plus 1." (no. 13)

(Subject: Anne)

Task 4: "So, 1, the zero term, plus the first common difference, which is 3, times n , which is the figure number." (no. 27)

11. Tries different approaches to the solution and assesses their usefulness

(Subject: Rita)

Task 2: "This is 2 times 2. This is 3 times 3. So you double the figure number cause in figure 2, you can do 2 by 2 minus 1 and you get 3. And on figure 3, you can do ...hmm you can do 3 by 3, which is 9 and minus 1 would give us 8, which is the wrong answer." (no. 24)

(Subject: Dua)

Task 4: "Okay, F_1 equals 12 plus 3, F_2 equals 22 plus 3 and F_3 equals 32 plus 1, F_4 equals 42 minus ... okay this is 4 and this is 7, this is 10 and then 16, this is minus 3...25 minus 9 equals 16. Okay, so here we cannot use anything" (no. 47)

“So if you take...this first one has 4 and then we have 4 plus 3 and then we have 4 plus 6. And this one is 4 plus 9 and then 4 plus...” (no. 49)

“Okay, we can... F_n has to be 4 plus something...but there is no connection here, because in F_2 we have, like n plus 1, which is 3 here and in number 4, nei in number 3 we have + 3 and here we have +5 and F_5 we have +7.” (no. 51)

“So, and we can try, so if we have 4, if we have 5, which we know are 16, we have 4 plus n , which is 5, plus prime number 4, so that’s 8, no this is wrong, because we have plus 12 and here we have 8, we have 13. No, I don’t know.” (no. 56)

(Subject: Lucy)

Task 1: “Emm...it will be n for the figure, and figure plus 4. No, because the n only means that, for example if I have figure 3 than n will be 3, probably.” (no. 7)

She starts by staring at the squares on figure 3 and then figure 2. Then she writes “ $y = nn$ ” on the tablet and then her eyes go to figure 2 and then figure 3. After looking at them, she erases the equation “ $y = nn$ ”. (no. 7)

“Maybe squared, no. If I had 5 squared that would be 25, that does not make sense. And there is 1, 2, 3, 4, 5, 1, 2, 3, 4.” (no. 11)

(Subject: Anne)

Task 5: “I’m trying to see how I can multiply n or the figure number by something to get the number of dark squares. So, 3 times, let’s see, 18 gives us 54. And 2 times 12 gives us 24. 1 times 6. So, so when you multiply the next figure from figure it’s multiplied by the double of what you can multiply the previous figure with. So if it’s 1 times 6, then it’s 2 times 12. No, not the double. Increases by 6 each time, what you have to multiply so 1 times 6, 2 times 12, 3 times 18.” (no. 37)

“To make it easier I will write zero first so I know I have included the zero term plus... I’m seeing how I can use the second common difference to find the numbers, so for figure 1, 12 times 1 is 12, minus 6 and you get the number of matchsticks. For number 2, 12 times 2 is 24 and that’s the number of matchsticks, so minus 0. For 3, 12 times 3 is 36, 36 when you do add 18 in order to get 54. It didn’t give me any conclusion.” (no. 39)

12. Confirms the formula for the general term

(Subject: Era)

Task 1: Then she writes $F_n = 1 + 3n$ and underlines it. (no. 67)

(Subject: Rita)

Task 5: She writes the general formula $F_n = (F_n * F_n) * 6...$ (no. 81)

(Subject: Dua)

Task 5: “Okay, so then we can take in F_1 we have 1 times 6, F_2 we have 4 times 6, F_3 we have 9 times 6, in F_n we have n^2 times 6. Okay.” (no. 61)

(Subject: Lucy)

Task 3: “The general rule is y equals $6n$ minus 2.” (no. 37)

(Subject: Anne)

Task 2: She puts “general rule = $-1 + 2n$ ” on a square and writes “dark circles in the n^{th} figure” under it. (no. 18)

13. Tests the formula

(Subject: Era)

Task 4: "I can try that. F_1 would be 1 plus 3, yeah that's right, that's 4. F_2 would be 1, plus 2 times 3, which is 6, is 7. F_3 would be 1, plus 3 times 3, which equals 10. F_4 would be 1, plus 3 times 4, which equals 13. F_5 would be 1, plus 3 times 5, which equals 16..." (no. 67)

(Subject: Rita)

Task 5: ...and then test it for the third figure by writing $F_3 = (3*3)*6 = 9*6 = 54$. (no. 81)

(Subject: Dua)

Task 1: "Oh...no, okay, so F_1 equals n or 1 times 4 plus 1, F_2 is 2 times 4 plus 1 and F_3 is 3 times 4 plus 1." (no. 11)

(Subject: Lucy)

Task 4: "So if I try on figure 2, y equals 3 times 2, plus 1, y equals 6 plus 1, y equals 7. That's correct. For the first figure, y equals 3 times 1 plus 1, y equals 3 plus 1. That's 4. So that's correct. And then figure 3, 3 times 3, plus 1, that's 9 plus 1, which is 10. I guess that's correct." (no. 43)

(Subject: Anne)

Task 3: "Multiplying 6 by one for the first figure which is 6 and then subtracting two which gives me 4 and said it works for the first figure. For number 2, 2 times 6 is 12, minus 2 gives us 10 which is the amount of matchsticks in number 2. For 3, 3 times 6 is 18, minus 2 gives us 16, which is also the correct number therefore I think this is the answer." (no. 23)

4.24 Questionnaire for the teachers

Title of the questionnaire “Shape pattern and time spent”

The instructions for the questionnaire:

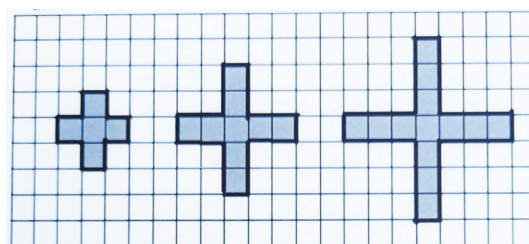
“Imagine the student of your class is solving each of the following shape pattern tasks (find the n -th term, general term) and you are interested on the time spent looking at each figure as a percentage of the total of time spent for the three figures. Please, choose an answer for each figure (the total of three figures should be 100%).

Your answers will be anonymous.

There were 5 questions with the same text and alternatives, but with different figures, which correspond to 5 tasks of the research:

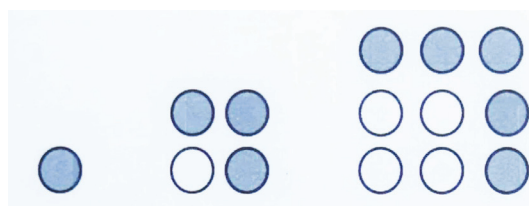
What is the percentage of time spent on each of the figures. (The total should be 100%.)

Figure 1	10%	20%	30%	40%	50%	60%
Figure 2	10%	20%	30%	40%	50%	60%
Figure 3	10%	20%	30%	40%	50%	60%



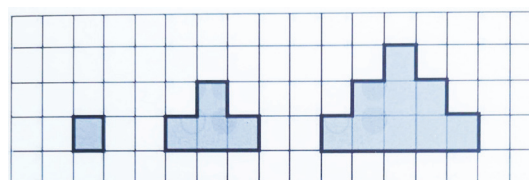
What is the percentage of time spent on each of the figures. (The total should be 100%.)

Figure 1	10%	20%	30%	40%	50%	60%
Figure 2	10%	20%	30%	40%	50%	60%
Figure 3	10%	20%	30%	40%	50%	60%



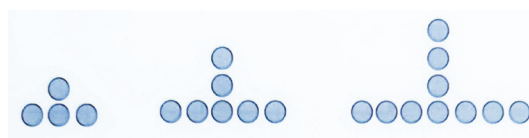
What is the percentage of time spent on each of the figures. (The total should be 100%.)

Figure 1	10%	20%	30%	40%	50%	60%
Figure 2	10%	20%	30%	40%	50%	60%
Figure 3	10%	20%	30%	40%	50%	60%



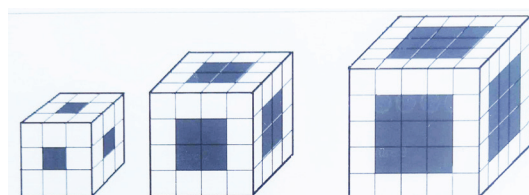
What is the percentage of time spent on each of the figures. (The total should be 100%.)

Figure 1	10%	20%	30%	40%	50%	60%
Figure 2	10%	20%	30%	40%	50%	60%
Figure 3	10%	20%	30%	40%	50%	60%



What is the percentage of time spent on each of the figures. (The total should be 100%.)

Figure 1	10%	20%	30%	40%	50%	60%
Figure 2	10%	20%	30%	40%	50%	60%
Figure 3	10%	20%	30%	40%	50%	60%



This questionnaire was created at the end of the research process, the second half of May 2020. It was the part of research thesis related to ecological validity of the research which is concerned with its applicability to teaching mathematics.

One of the findings of this research was the claim that there is clear evidence that the time spent and the number of fixations that are revisits for the second figure may be equal to the sum of the time spent or the number of revisits for the first and third figures. This is presented with the equation $F_2 = F_1 + F_3$.

It is obvious to ask "What is the importance of this?"

Before answering this question, it must be seen if the teachers know about this equation or, if not, how close they get, by asking those questions that calculate this.

A form was given to four mathematics teachers who completed it in an anonymous way.

All their answers are presented in the table and the related averages for each figure and the average for all teachers are presented in the chart.

	Task 1			Task 2			Task 3			Task 4			Task 5		
	Fig. 1	Fig. 2	Fig. 3	Fig. 1	Fig. 2	Fig. 3	Fig. 1	Fig. 2	Fig. 3	Fig. 1	Fig. 2	Fig. 3	Fig. 1	Fig. 2	Fig. 3
Teacher 1	10%	50%	40%	10%	30%	60%	10%	50%	40%	30%	30%	40%	20%	40%	40%
Teacher 2	20%	30%	50%	10%	30%	60%	20%	30%	50%	20%	30%	50%	40%	30%	30%
Teacher 3	10%	30%	60%	10%	30%	60%	10%	30%	60%	20%	20%	60%	10%	30%	60%
Teacher 4	20%	30%	40%	10%	30%	60%	10%	20%	60%	30%	30%	40%	30%	30%	40%

Table 14

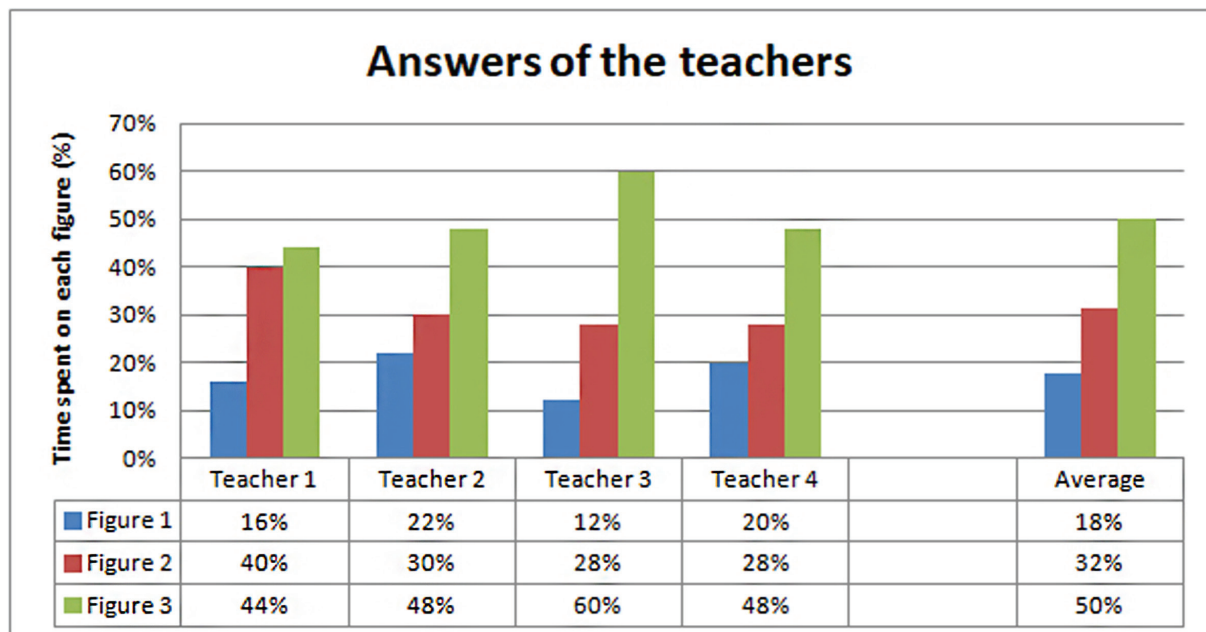


Figure 21

Analyses

The chart clearly shows that all the teachers think that the time spent for each figure is directly proportional to the area of the figures, in increasing order with the 1st figure taking the least time, 2nd figure being in between and the 3rd figure taking the longest time.

The averages of four teachers show that the 1st figure (in 5 tasks) takes 18%, the second, 32% and the third figure takes 50% of the time.

It is clearly the opposite of what this research discovers where the 2nd figure takes more time, more than the 1st figure and the 3rd figure.

Behind the difference between this research and what teachers may think, is the difference in their reasoning of how the students work with shape pattern tasks.

If a teacher mistakenly thinks that the 3rd figure should take more time, he/she may suggest the students to spent more time in figure 3 and not in figure 2, hiding the role of figure 2 in finding the general rule of the pattern.

It was explained before how the subject spends the time, and why figure 2 takes more time, with the following diagram.

Example of fixations that are revisits where $F_2 = F_1 + F_3$

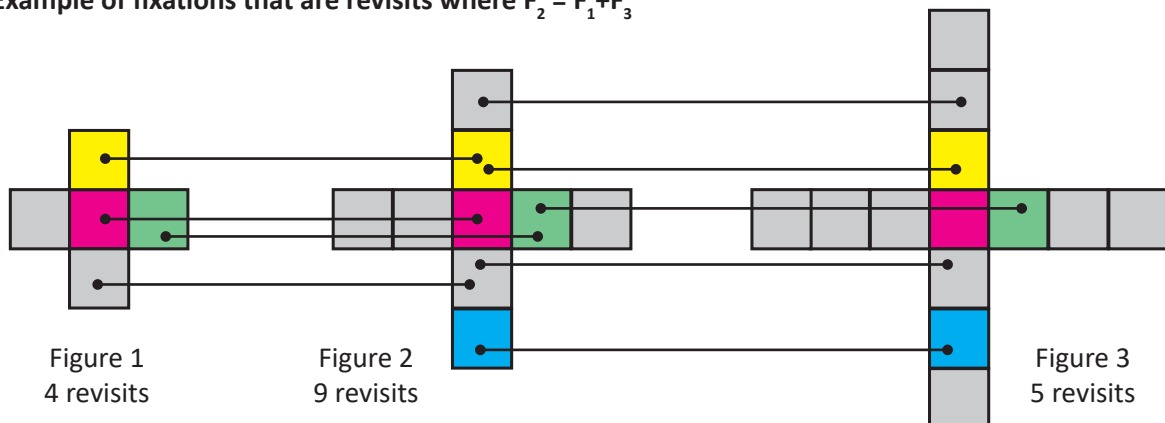


Figure 22

4.25 Conclusions

- The additional insight eye-tracking brought for this study was supporting evidence for the data analysis. Through the eye-movements, it was possible to find out the hidden reasoning of the subject.
- The eye-tracker provided a lot of data which were used to produce 25 transcripts (5 subjects had 5 tasks).
- Using the transcript, there were discovered 13 indicators of performance of inductive reasoning in solving shape patterns.
- To measure the indicators, values 0 and 1 were given.
- To measure the concept “inductive reasoning in solving pattern task”, the scale 0-13 was used. 13 is related to 13 indicators with the value 1.
- To provide a more appropriate measure, the use of the average of many tries (tasks) is advised..
- Two types of measures were discovered. One with multiple indicators (13 indicators) and another one with two indicators. In this research they gave nearly the same values (51% and 52%).
- The measure of the concept with two indicators, also known as “the measure of confidence”, took in consideration only those subjects that found the solution (the general term) but without trying different approaches.
- The indicators function in many ways to describe the level of possession of the inductive reasoning skills. The research gives an example using task 2.
- The research served to produce this hypothesis: “The measure of concept “inductive reasoning in solving shape pattern task” will have nearly the same value if calculated as an average of 13 indicators or calculated as the “measure of confidence in solving the task”, with the indicators 11 and 12 (percentage of those subjects who gave the formula but did not try different approaches).
- The indicators were ranked according to their average value for the whole group of subjects of this research. The ranking lets the reader reflect on the usefulness of each indicator.
- The visual span while reading the task was 8.6 characters.
- Because of the lack of data, it was not possible to prove that all the subjects had the same reading routine. It remains a research task for another study.
- There is evidence that the time spent for figure 2 and the number of revisits is nearly half of time spent for all of the three figures in a pattern, but it must be verified with large sets of data.
- The cause that figure 2 takes nearly 50% of the total time spent looking at the three figures or 50% of fixations that are revisits, is the way of reasoning, or the type of perceptions the subject has while he/she is holding wholes (gazing), discerning details (making distinctions), recognizing relationships (among specific discerned elements) and perceiving properties (as generalities which may be instantiated in specific situations) and reasoning on the basis of identified properties. (Mason, 2009 et al., Watson, 2009). In this research, the type of perception that was discovered involved pairs of fixations, explained in this thesis in the article “Number of Revisits and Fixations Count”.

- The eye-tracker clearly shows the mapping between the figures and different parts of the table.
- The research argued that figure 2 in some tasks has nearly 50% of time and the number of revisits. It was the fact that the time was spent mainly on pairs of figures and not on one figure, therefore there are two pairs of figures (F1, F2) and (F2,F3). Since F2 is in both pairs, it takes more time and revisits (nearly 50% of the total).
- The subject needs more fixations inside the same area of interest to understand better the different parts of the same figures. That is why we have more fixations that are not revisits in figure 2 and figure 3.
- In many tasks the average fixations duration (ms) decreases from figure 1 to 3. This is due to the clarification process. Moving from figure 1 to 2 to 3, the subjects understand more and increase the number of fixations, this decreases the average fixations duration (ms).
- After having an idea about the growth rule for the first two figures, there are two ways for continuing the reasoning about the relationship (growth rule) between figure 2 and 3:
 - a) By predicting that the previous rule (from figure 1 to figure 2) will be applied going from figure 2 to figure 3 by saying that the third figure must have four more squares and proving that this is true.
 - b) By finding the difference between figure 3 and 2 and comparing it with the first difference between figure 2 and 1 and then drawing the conclusion that it is the same rule.
- In many tasks, the first fixation durations for figure 2 and 3 are less than the first fixation duration for the third figure. This can be explained by the need to have some initial orientation orientation with quicker eye movements for the first two figures.
- There is clear evidence from the heat maps that the symmetric shapes are seen and analysed partly just for the fact that they are symmetric. For example, in task 1, the figures are symmetric crosses and the subject analyses more the top part, or one arm out of four of them.

5. Discussion of the results with support to previous research

The results of this research show some findings of the process of inductive reasoning for which Klauer and Phye (1994) give the definition as the discovery of regularities through the detection of similarities, differences, or a combination of the two with respect to either object attributes or relations between objects.

These findings are: The amount of work done for the 2nd figure in respect to the other figures, 1st and 3rd. The finding of the indicators that form a list that can be used as a taxonomy to evaluate students.

Inductive reasoning has a high level of cognitive processes. Other cognitive processes are: attention, perception, memory, language, learning. The results of the research proved some patterns in attention and perceptions through eye-tracking methodology. This brought more insights about the “power” of inductive reasoning, expressed in eye-tracking measures (attention, perception).

Awareness levels or attentional states applied to pattern generalization are studied by Mason, 2009 et al., Watson, 2009. They list: a) holding wholes (gazing), b) discerning details (making distinctions), c) recognizing relationships (among specific discerned elements) and d) perceiving properties (as generalities which may be instantiated in specific situations) and e) reasoning on the basis of identified properties. The research served to bring new discoveries for the first stages of IR holding wholes (gazing) and discerning details (making distinctions) and recognizing relationships. The answers to the first research question, *“What is the ratio of eye-tracking measures of the same type in different areas of interests in inductive reasoning in solving pattern problems”*, make some findings about the “ratio” of the work done in different parts of the pattern that are verified in many tasks.

“Radford (2008) proposes that the process of pattern generalization involves: (1) grasping a commonality, (2) generalizing this commonality to all terms of the sequence, and (3) proving a rule that allows them to directly determine any term of the sequence.” (Luz Callejo and Zapatera 2016.)

These are the key features of inductive reasoning, but this research provides details on what should be observed in the work of a student to evaluate his skills in inductive reasoning. 13 indicators and the concept for their measure and their contribution to the measure of the concept itself (inductive reasoning in pattern problems) are findings which may help in practice.

If the eye-tracker gives insights on the visual attention of a subject. Adding a task to the subjects which employs inductive reasoning observed with eye-tracker gives extra details not just for the visual attention but for the reasoning the subject has.

Suvorov (2013) evidences the relationship between eye movements and perception as a crucial point. From its definition “the perception refers to the way sensory information is organized, interpreted, and consciously experienced”. The study shows that the patterns of the eye-movements while solving a task are of a special “design” because the movement is caused by the order a subject follows. There were examples in the research showing this.

The ideas given in “Measures of the Trait of Confidence” (Stankov Et al., 2014) was of great support for what was discovered in this research about the “measure of confidence” while solving a problem without different approaches.

6. Strengths and limitations

Strengths

There is enough argumentation to support the claim “the concept of inductive reasoning in solving shape pattern tasks can be measured with many indicators with each on one taking the values 0 or 1 and this works very well in assessing the student’s work in many ways”.

The finding of the difference between what this research discovered about the time spent on each figure and what the math teachers predict was a helpful evidence to discover more about the ecological value of this research.

One of the strong points of this research was the fact that all the subjects had a level of 52% in solving the tasks with confidence (without different approaches), and enough experience with such tasks. This gave the research enough data to analyse and to compare the subjects and after that to draw useful conclusions.

The eye-tracking software gave measures which assisted the research questions.

The transcriptions were of high accuracy and very helpful in answering the research questions.

The theory of measuring the concept with the help of indicators was applied very well.

There is a strong link with the theory of social research methods, especially with the topics of research design, research questions, measuring of concepts, transformation of qualitative data into quantitative data.

The amount of qualitative and quantitative data was enough to discover findings of important value and to support the role of indicators in measuring the concept.

The concept “measure of confidence” has an original approach in this research.

A set of Likert scale questionnaires, asked to subjects and teachers, were used successfully to support the conclusions.

Limitations

The pandemic Covid19 prevented the researcher to involve more subjects in the research and to have enough technical support for the eye-tracker issues, since meeting face to face in the faculty wasn’t allowed. After the process of data analyses started, it was understood that when Era, Rita, Dua, Lucy started reading task 1, the eye-tracker showed gazes that did not match with what they were saying. Furthermore, the set of gazes was not in a horizontal line like the text was. There were many differences in the up and down eye movements, which made the researcher understand that the fixations the eye-tracker showed were not a true reflection of what the actual “event” was (reading the task). This reduced the amount of eye-tracking data to be analysed. If the research was not conducted in the circumstances of a lock-down, the study would be repeated with other subjects and asking additional support in order to have accurate calibration.

Some of the data was called unreliable since there were clear signs of a lack of congruence between what the eye-tracker displayed and what was expected to be shown. This was difficult to resolve because of the lock-down.

The fact that eye-tracking technology is a new one makes the literature related to this type of research limited.

7. Implication of the results in mathematics education

The indicators may be used by teacher of mathematics to evaluate in more detail the progression of students given inductive reasoning tasks. The examples given in this research of how to measure the indicators and how to analyse the emerging data is useful in many aspects.

“The measure of confidence” - a concept explained here is a quick way to measure how confident a student or a group of students is with these type of task. The explanations that confidence appears when the solution of the problem is given directly without multiple attempts may be of help.

Measuring the level of a group of students engaged in inductive reasoning with pattern task with 13 indicators may be the same with the value of the measure of confidence which is measured using only two indicators. The difference the teacher may get from measuring with 13 indicators or just with two, in the case of “measure of confidence” may give extra insights for the skills the students show.

Using 13 indicators in different shape pattern tasks and after that analysing the data, gives another approach to evaluate the level of a group or to distinguish the problems that arise.

Using eye-tracking, especially when teaching younger children who have difficulties in learning, may be useful to scan what their behaviour is and what the problems are in reading the task, interpreting the shapes, in organizing the movement of the eyes which show in detail how the students understand certain math concepts.

The time spent and the number of revisits for figure 2 is nearly the sum of time spent and the average for two other figures (1st and 3rd). There is clear evidence, but not enough, that teachers think that the time spent for each figure is directly proportional to the area of the figures, in increasing order with 1st figure being the least, the 2nd figure being between and the 3rd figure taking the longest time. Behind the difference between this research and what teachers may think, is the difference in their reasoning of how the students work with shape pattern tasks. If a teacher mistakenly thinks that the 3rd figure should take more time, he/she may suggest the students to spend more time in figure 3 and not in figure 2, hiding the role of figure 2 in finding the general rule of the pattern. It was explained before how the subject spends the time, and why figure 2 takes more time.

Analysing the ratio between the number of fixations that are revisits and those that are not may help find the differences a students shows if he/she is analysing the relationship between the figures (similarities and differences) or analysing the structure of a figure (different parts of the same figure).

Using heat maps in the case of symmetric shapes is a good way to distinguish if the students are using the fact that the shape is symmetric or not. In the first case, the heat maps show a non-symmetrical area of gaze points, and mostly focused on one side of the figure.

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