Adaptive Tracking Control of Nonlinear Time-Varying Systems with Unknown Control Coefficients and Unknown Time-Varying Parameters

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Abstract— **This paper investigates the tracking control of a class of strict-feedback uncertain nonlinear systems in the presence of unknown signs of control coefficients and unknown time-varying parameters as well as unknown disturbances. A robust adaptive controller and a new decoupled backstepping approach to stability analysis are developed by constructing a new compensation scheme. By introducing a Nussbaum function and a new type of hyperbolic tangent function, the effects of unknown time-varying parameters and unknown control coefficients are effectively compensated. By using the decoupled backstepping technique, it is proved that under the proposed control, all closed-loop states are uniform ultimate bounded. A numerical example is presented to demonstrate the effectiveness of the proposed control scheme.**

I. INTRODUCTION

Adaptive control of strict-feedback nonlinear systems has received a lot of attention since the appearance of recursive backstepping design in [1] and a great deal of work has been done for this class of systems in the past decades, see for examples, [2], [3], [4], [5], [6], [7], [8] and many reference therein.

Time-variations in dynamical systems occur in many physical systems. The development of adaptive control schemes for uncertain time-varying nonlinear systems has been a task of major practical interest as well as theoretical significance. Several results are available for nonlinear systems with time-varying parameters and/or without the knowledge on the sign of the term multiplying the control [4], [5] and the high frequency gain in the case of linear systems [9].

When the signs of virtual control coefficients or high-frequency gain are unknown, the adaptive control problems are quite involved and Nussbaum-type functions are normally adopted. In [3], the problem of adaptive control with unknown sign of high-frequency gain for linear time invariant systems was studied. In [4], Nussbaum gain incorporating with the backstepping technique was used to design adaptive output stabilizer for high order uncertain time invariant nonlinear systems with unknown sign of high-frequency gain in the absence of external disturbances, where the nonlinearities considered should satisfy sector conditions. In [10], disturbance decoupling was addressed for nonlinear time invariant systems with known sign of the high frequency gain. In [11], a flat zone was used to handle the problem of nonlinear time invariant systems with unknown sign of high frequency gain in the presence of disturbances. In [12], an adaptive output-feedback controller for uncertain linear systems without knowledge of the plant high-frequency-gain sign was proposed. Output feedback control was studied for time-varying systems with or without the knowledge of the sign of high-frequency gain, for examples, [13],[14] and [15]. In [5], the adaptive control was considered for uncertain time-varying nonlinear systems with time-varying control coefficients by using the Nussbaum function, where the time-varying control coefficients were assumed to take value in a bounded interval. When this condition is not satisfied, for example, the control coefficients are time-varying functions of the states with unknown signs, the problem was solved recently by proposing a novel Nussbaum function in [8]. However in[8], the unknown parameters considered are constants and only stabilization is achieved.

This paper investigates adaptive control for a class of uncertain nonlinear systems in the presence of the unknown time-varying control coefficients and unknown time-varying parameters, as well as unknown disturbance. The control coefficients are time-varying functions of the states with unknown signs and unknown parameters are time-varying. By introducing a new hyperbolic tangent function and incorporating a Nussbaum function, the effects of unknown control coefficients, unknown time-varying parameters and disturbance are effectively compensated. By proposing a decoupled backstepping approach to stability analysis, it is shown that the proposed controller can guarantee the whole system uniformly stable. A numerical example is presented to illustrate the effectiveness of the proposed control scheme. The main contributions of this paper can be summarized as follows.

1) A decoupled backstepping approach to stability analysis is developed which avoids to considering of the summation of multiple Nussbaum-type functions in the Lyapunov stability analysis.
2) A Nussbaum function is proposed to handle the unknown signs of high frequency gains.
3) A novel hyperbolic tangent function is proposed and used in the control strategy, which gives great convenience to the stability analysis.
4) Estimation of bounds of time-varying parameters and disturbances are developed by incorporating with the hyperbolic tangent function.
5) Asymptotic tracking control is achieved in the presence of unknown time-varying control coefficients,

unknown time-varying parameters, and external disturbance.

This paper is organized as follows. In Section II, the problem formulation and the preliminary result are given. In Section III, the illustration of the design of an adaptive backstepping control scheme is presented and the proof of the stability of the closed-loop system is shown in Section IV. An illustrate numerical example is shown in Section V. Finally, we draw the conclusions in Section VI.

II. PROBLEM FORMULATION AND PRELIMINARY RESULTS

A. Problem Formulation

Consider the following class of single-input-single-output (SISO) nonlinear time-varying systems in the feedback form

$$
\begin{aligned}
\dot{x}_i &= b_i(t)\beta_i(\bar{x}_i,t)x_{i+1} + \theta_i(t)^T\psi_i(\bar{x}_i) + \phi_i(\bar{x}_i) + d_i(t) \\
\dot{x}_n &= b_n(t)\beta_n(x,t)u(t) + \theta_n^T(t)\psi_n(x) + \phi_n(x) + d_n(t) \\
y &= x_1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1)
\end{aligned}
$$

where $x = [x_1, \cdots, x_n]^T \in R^n$, $u \in R$ and y are system states, input and output respectively, $\bar{x}_i = [x_1, \cdots, x_i]^T \in R^i$, $\beta_i(\bar{x}_i,t) \neq 0$ and $\phi_i(\bar{x}_i)$ are known smooth functions, $\psi_i(\bar{x}_i) = [\psi_i^1(\bar{x}_i), ..., \psi_i^{p_i}(\bar{x}_i)]^T \in R^{p_i}$ is known smooth function vectors, $d_i(t)$ denotes unknown time-varying bounded disturbances, $\theta_i(t) = [\theta_i^1(t), ..., \theta_i^{p_i}(t)]^T \in R^{p_i}$ are vectors of uncertain time-varying parameters belonging to known compact sets Ω_{θ_i} with unknown bounds, $b_i(t) \neq 0$ are uncertain time-varying parameters belonging to known compact sets Ω_{b_i} with unknown bounds and they are referred to as virtual control coefficients. In particular, $b_n(t)$ is referred to as the high-frequency gain.

For the considered system in (1), the following assumptions are imposed.

Assumption 1. The reference signal y_r and its $(n-1)$th order derivatives are assumed to be known and bounded.

Assumption 2. The uncertain time-varying parameters $b_i(t) \neq 0$ and $\theta_i(t) \in R^{p_i}$ are inside the compact sets Ω_{b_i} and Ω_{θ_i} with unknown bounds.

The control objective is to design an adaptive controller for system (1) satisfying Assumptions 1-2 such that the closed-loop system is stable and the system output can asymptotically track a given reference signal $y_r(t)$.

Remark 1: Similar class of strict-feedback nonlinear systems to (1) was considered in [8]. However in [8], only stabilization is achieved and the unknowns b_i and θ_i are constants, which makes the control design more simple because the derivatives of the unknown time-varying functions are needed to be considered. In this paper, the control coefficients and the unknown parameters considered are time-varying and the asymptotic tracking is achieved. As far as we know, the asymptotic tracking control of time-varying nonlinear systems remains unsolved when the control coefficients are time-varying functions of the states with time-varying parameters and unknown signs.

B. Preliminary Results

In order to cope with the unknown control coefficients, the Nussbaum-type function is exploited in this paper, which has the following properties

$$
\lim_{s\to\infty} \sup \frac{1}{s} \int_0^s N(\chi)d\chi = \infty \qquad (2)
$$

$$
\lim_{s\to\infty} \inf \frac{1}{s} \int_0^s N(\chi)d\chi = -\infty. \qquad (3)
$$

Commonly used Nussbaum-type functions include $e^{\chi^2}\cos(\frac{\pi}{2}\chi)$, $\chi^2\sin(\frac{\pi}{2}\chi)$, $\cosh(\chi)\cos(\frac{\pi}{2}\chi)$, etc.
The following Lemma will be employed in later analysis and is presented here.

Lemma 1: Let $V(t)$ and $\chi(t)$ be a smooth function defined on $[0, t_f)$ with $V(t) \geq 0$, $\forall t \in [0, t_f)$, and $N(\chi)$ be an even smooth Nussbaum-type function. If the following inequality holds:

$$
V(t) \leq f_0 + e^{-f_1 t}\int_0^t \left(g(\tau)N(\chi)+1\right)\dot{\chi}e^{f_1\tau}d\tau \quad (4)
$$

where constant $f_0 > 0$ and $f_1 > 0$, $g(\tau)$ is a time-varying parameter which takes values in the unknown closed intervals $I_1 = [g^-, g^+]$ with $0 \notin I_1$, and f_0 represents a suitable constant, then $V(t), \chi(t)$ and $\int_0^t g(\tau)N(\chi)\dot{\chi}d\tau$ must be bounded on $[0, t_f)$.
Proof: See [6], [14].

Remark 2: In the existing papers [5], [6], [14] which use Nussbaum-type functions to deal with unknown signs of control coefficients, the proof of stability relies on a single Nussbaum-type function or multiple Nussbaum-type functions with the same control signs. In order to compensate for the effects from multiple Nussbaum-type functions with different unknown control signs, a novel Nussbaum-type function was proposed in [8]. In this paper, we will propose a new approach to stability analysis that the common Nussbaum-type functions satisfying properties (3)-(4) are able to deal with this problem.

Remark 3: If a reference signal does not satisfy Assumption 1, for example that a ramp function used in start up of industrial systems, we can smooth the reference signal at first and then apply the proposed backstepping control scheme.

III. ADAPTIVE CONTROL DESIGN

A. Design Procedure

In this section, we present the adaptive control design using the backstepping technique in n steps. We take the change of coordinates as follows,

$$
\begin{aligned}
z_1 &= y - y_r \qquad\qquad\qquad\qquad\qquad (5) \\
z_i &= x_i - \alpha_{i-1}, \quad i = 2, 3, \ldots, \rho, \qquad (6)
\end{aligned}
$$

where α_{i-1} is the virtual control at each step and will be determined in later discussions.
We now illustrate the backstepping design procedures incorporating with Nussbaum function to avoid using the control coefficients, with details given for the first step.

Step. 1

It follows from (1) and (5) that

$$
\begin{aligned}
\dot{z}_1 &= b_1(t)\beta_1(x_1,t)(z_2+\alpha_1)+\theta_1(t)^T\psi_1(x_1)+\phi_1(x_1) \\
&\quad +d_1(t)-\dot{y}_r \tag{7}
\end{aligned}
$$

Without using the sign of $b_1(t)$, the following virtual control law α_1 is designed

$$
\begin{aligned}
\alpha_1 &= N_1(\chi_1)\bar{\alpha}_1 \tag{8} \\
N_1(\chi_1) &= exp(\chi_1^2)\cos(\frac{\pi}{2}\chi_1) \tag{9}
\end{aligned}
$$

where χ_1 is generated by

$$
\dot{\chi}_1 = \beta_1(x_1,t)\bar{\alpha}_1 z_1 \tag{10}
$$

and $\bar{\alpha}_1$ is designed as

$$
\begin{aligned}
\bar{\alpha}_1 &= \frac{1}{\beta_1(x_1,t)}\Big[c_1 z_1+\frac{1}{4}\beta_1^2(x_1,t)z_1-\dot{y}_r+\phi_1(x_1) \\
&\quad + \hat{D}_1\tanh\left(\frac{z_1}{\varrho}\right)+\hat{\Theta}_1^T\tanh(\frac{z_1\psi_1(x_1)}{\varrho})\psi_1(x_1)\Big]
\end{aligned}
\tag{11}
$$

and the parameter estimators are designed as

$$
\begin{aligned}
\dot{\hat{\Theta}}_1 &= \Gamma_1\text{Tanh}\left(\frac{z_1\psi_1(x_1)}{\varrho}\right)\psi_1(x_1)z_1-\Gamma_1 k_{\theta_1}\hat{\Theta}_1 \tag{12} \\
\dot{\hat{D}}_1 &= l_1 z_1\tanh(z_1/\varrho)-l_1 k_{d_1}\hat{D} \tag{13}
\end{aligned}
$$

where $\text{Tanh}(\cdot)$ is defined as a p_1-by-p_1 diagonal matrix a $\text{Tanh}\left(\frac{z_1\psi_1}{\varrho}\right)=\text{diag}\left[\tanh\left(\frac{z_1\psi_1^1}{\varrho}\right),...,\tanh\left(\frac{z_1\psi_1^{p_1}}{\varrho}\right)\right]$, $\tanh(\cdot)$ is a hyperbolic tangent function c_1, l_1, k_{θ_1},and k_{d_1} are positive constants, Γ_1 is a positive definite matrix. $\hat{D}_1$ is the estimate of D_1 and D_1 is the bound of disturbance $d_1(t)$, such as $|d_1(t)| \le D_1$. $\hat{\Theta}_1$ denotes the estimate of $\Theta_1=[\Theta_1^1,...,\Theta_1^{p_1}]^T$, where $\Theta_1^j, j=1,...,p_1$ is the bound of each component of the time-varying parameter vector $\theta_1(t)$, ϱ is a positive constant. The hyperbolic tangent function $\tanh(\cdot)$ has the following property.

$$
0 \le X - X\tanh(X/\varrho) \le 0.2785\varrho, \quad \forall\ X \in \Re. \tag{14}
$$

Remark 4: A hyperbolic tangent function is introduced in the virtual control (11) and parameter updating laws (12) and (13). Note that a positive constant ϱ is introduced in the hyperbolic tangent function,

Then from (7) and (11) we have

$$
\begin{aligned}
&\dot{z}_1 - \beta_1(x_1,t)\bar{\alpha}_1 \\
&= b_1(t)\beta_1(x_1,t)z_2+(b_1(t)N_1(\chi_1)+1)\beta_1(x_1,t)\bar{\alpha}_1 \\
&\quad + \theta(t)^T\psi_1(x_1)-\hat{\Theta}_1^T\text{Tanh}\left(\frac{z_1\psi_1(x_1)}{\varrho}\right)\psi_1(x_1) \\
&\quad - c_1 z_1 - \frac{1}{4}\beta_1^2(x_1,t)z_1+d_1(t)-\hat{D}_1\text{Tanh}\left(\frac{z_1}{\varrho}\right) \tag{15}
\end{aligned}
$$

where $\tilde{\Theta}_1=\Theta_1-\hat{\Theta}_1$, $\tilde{D}_1=D_1-\hat{D}_1$. To proceed, we define the Lyapunov function

$$
V_1 = \frac{1}{2}z_1^2+\tilde{\Theta}_1^T\Gamma_1^{-1}\tilde{\Theta}_1+\frac{1}{2l_1}\tilde{D}_1^2 \tag{16}
$$

Then the derivative of V_1 along with (8) and (15) is given by

$$
\begin{aligned}
\dot{V}_1 &\le -c_1 z_1^2 - \frac{1}{4}\beta_1^2(x_1,t)z_1^2 \\
&\quad +b_1(t)\beta_1(x_1,t)z_1 z_2+(b_1(t)N_1(\chi_1)+1)\dot{\chi}_1 \\
&\quad +|z_1|D_1-\hat{D}_1 z_1\text{Tanh}\left(\frac{z_1}{\varrho}\right)-\frac{1}{l_1}\tilde{D}_1\dot{\hat{D}}_1 \\
&\quad -\tilde{\Theta}^T\Gamma_1^{-1}\dot{\hat{\Theta}}+|z_1\theta_1(t)^T\psi_1(x_1)| \\
&\quad -\hat{\Theta}_1^T\tanh\left(\frac{z_1\psi_1(x_1)}{\varrho}\right)\psi_1(x_1)z_1 \tag{17}
\end{aligned}
$$

Using Young's inequality, the following properties are obtained and used later.

$$
\begin{aligned}
&|z_1|D_1-\hat{D}_1 z_1\tanh\left(\frac{z_1}{\varrho}\right) \\
&\le D_1 z_1\tanh\left(\frac{z_1}{\varrho}\right)+0.2875\varrho D_1-\hat{D}_1 z_1\tanh\left(\frac{z_1}{\varrho}\right) \\
&\le \tilde{D}_1 z_1\tanh\left(\frac{z_1}{\varrho}\right)+0.2785\varrho D_1 \tag{18}
\end{aligned}
$$

and

$$
\begin{aligned}
|z_1\theta_1(t)^T\psi_1(x_1)| &\le |z_1\Theta_1^T\psi_1(x_1)| \\
&\le \Theta_1^T\text{Tanh}\left(\frac{z_1\psi_1(x_1)}{\varrho}\right)z_1\psi_1(x_1) \\
&\quad +0.2785\varrho\ \|\ \Theta_1\ \| \tag{19}
\end{aligned}
$$

and

$$
b_1(t)\beta_1(x_1,t)z_1 z_2 \le \frac{1}{4}\beta_1^2(x_1,t)z_1^2+\bar{b}_1^2 z_2^2 \tag{20}
$$

where $\bar{b}_1$ is a positive constant, such as $|b_1(t)| \le \bar{b}_1$. Then the derivative of V_1 is derived as

$$
\begin{aligned}
\dot{V}_1 &\le (b_1(t)N_1(\chi_1)+1)\dot{\chi}_1-c_1 z_1^2+\bar{b}_1^2 z_2^2 \\
&\quad + M_1\varrho+\frac{1}{l_1}\tilde{D}_1\left(l_1 z_1\text{Tanh}\left(\frac{z_1}{\varrho}\right)-\dot{\hat{D}}_1\right) \\
&\quad + \tilde{\Theta}_1^T\Gamma_1^{-1}\left(\text{Tanh}\left(\Gamma_1\frac{z_1\psi_1(x_1)}{\varrho}\right)z_1\psi_1(x_1)-\dot{\hat{\Theta}}_1\right) \\
&\le -c_1 z_1^2+(b_1(t)N_1(\chi_1)+1)\dot{\chi}_1+\bar{b}_1^2 z_2^2(t)+M_1 \\
&\quad -\frac{k_\theta}{2}\|\ \tilde{\Theta}_1\ \|^2-\frac{k_d}{2}\tilde{D}_1^2 \\
&\le -\sigma_1 V_1+(b_1(t)N_1(\chi_1)+1)\dot{\chi}_1+\bar{b}_1^2 z_2^2+M_1 \tag{21}
\end{aligned}
$$

where

$$
M_1 = 0.2785\varrho(D_1+\|\ \Theta_1\ \|)+\frac{k_{\theta_1}}{2}\|\ \Theta_1\ \|^2+\frac{k_{d_1}}{2}D_1^2 \tag{22}
$$

$$
\sigma_1 = \min\{2c_1, k_{d_1}k_1, k_{\theta_1}\lambda_{min}(\Gamma_1)\} \tag{23}
$$

Remark 5: In the cancellation based backstepping design, the coupling term $b_1(t)\beta_1(x_1,t)z_1 z_2$ in (17) will be compensated for in the next step by augmenting the Lyapunov candidate then. In decoupled backstepping design, we use the Young's inequality to transform this coupling term to two terms as in (20). Thus there is only the decoupled term $\bar{b}_1^2 z_2^2$ left in (21), where the boundedness of z_2 will be proved in the next step. According to Lemma 1, if we

could prove that z_2 is bounded, then the stability of z_1 is apparent and easy. It is this fundamental change that makes control system design for this problem solvable.

Multiplying of (21) by e^{ft} and integrating both sides over the interval $[0,t]$ gives

$$
\begin{aligned}
\int_0^t \dot{V}_1 e^{f\tau} d\tau &\leq \int_0^t (b_1(t)N(\chi_1) + 1)\dot{\chi}_1 e^{f\tau} d\tau \\
&\quad - \int_0^t \sigma_1 V_1(\tau)^2 e^{f\tau} d\tau + \int_0^t M_1 e^{f\tau} d\tau \\
&\quad + \int_0^t \bar{b}_1^2 z_2(\tau)^2 e^{f\tau} d\tau
\end{aligned}
\tag{24}
$$

This yields

$$
\begin{aligned}
V_1(t) &\leq V_1(0) + e^{-ft}\int_0^t (b_1(t)N(\chi_1)+1)\dot{\chi}_1 e^{f\tau} d\tau \\
&\quad + M_1 \int_0^t e^{-f(t-\tau)} d\tau + \int_0^t \bar{b}_1 z_2^2 e^{-f(t-\tau)} d\tau
\end{aligned}
\tag{25}
$$

Since $\int_0^t \varrho(\tau)d\tau$ is bounded and therefore $\bar{M}_1 = V_1(0) + M_1 \int_0^t \varrho e^{-f(t-\tau)} d\tau$ is bounded. If there is no extra term $\int_0^t \bar{b}_1^2 z_2(\tau)^2 e^{-f(t-\tau)} d\tau$ in the inequality (25), together with Lemma 1, we can conclude that $V_1(t)$ and $\chi_1(t)$, hence z_1, $\hat{\Theta}_1$, and $\hat{D}_1$ are bounded.

Step.i $(i = 2, \ldots, n)$
The virtual control α_i is designed as

$$
\alpha_i = N_i(\chi_i)\bar{\alpha}_i \tag{26}
$$

$$
N_i(\chi_i) = exp(\chi_i^2)\cos\left(\frac{\pi}{2}\chi_i\right) \tag{27}
$$

$$
\dot{\chi}_i = \beta_i(\bar{x}_i,t)\bar{\alpha}_i z_i \tag{28}
$$

$$
\begin{aligned}
\bar{\alpha}_i = \frac{1}{\beta_i(\bar{x}_i,t)}&\left(c_i z_i + \frac{1}{4}\beta_i^2(\bar{x}_i,t)z_i + \bar{\phi}_i \right. \\
&\left. + \eta_i + \hat{D}_i^T \text{Tanh}\left(\frac{z_i h_i}{\varrho}\right)h_i \right. \\
&\left. + \hat{\Theta}_i^T \text{Tanh}\left(\frac{z_i \bar{\psi}_i}{\varrho}\right)\bar{\psi}_i\right)
\end{aligned}
\tag{29}
$$

and the parameter estimation laws are designed as

$$
\dot{\hat{\Theta}}_i = \Gamma_i \text{Tanh}\left(\frac{z_i \bar{\psi}_i}{\varrho}\right) z_i \bar{\psi}_i - \Gamma_i k_{\theta_i}\hat{\theta}_i \tag{30}
$$

$$
\dot{\hat{D}}_i = l_i \text{Tanh}\left(\frac{z_i h_i}{\varrho}\right) z_i h_i - k_{d_i}\hat{D}_i \tag{31}
$$

where c_i, l_i, k_{d_i} and k_{θ_i} are positive definite constants, Γ_i is a diagnal positive matrix, $\hat{\Theta}_i$ is the estimate of $\Theta_i = [\Theta_i^1, ..., \Theta_i^{\bar{p}_i}]^T$, Θ_i^j which is the bound of $\bar{\theta}_i^j$, $j = 1, ..., \bar{p}_i$, $\hat{D}_i$ is the estimate of $D_i = [D_i^1, ..., D_i^i]^T$, D_i^j which is the bound of $\bar{d}_i^j$, $j = 1, ..., i$, $\text{Tanh}\left(\frac{z_i\psi_i}{\varrho}\right)$ and $\text{Tanh}\left(\frac{z_i\beta_i}{\varrho}\right)$ are $\bar{p}_i$-by-$\bar{p}_i$ and i-by-i diagonal matrices, respectively, $\bar{p}_i =$

$(\sum_{j=1}^i p_j) + i - 1$, and

$$
\bar{\theta}_i = [\theta_i, \theta_{i-1}, ..., \theta_1, b_{i-1}, ..., b_1]^T \in R^{\bar{p}_i} \tag{32}
$$

$$
\begin{aligned}
\bar{\psi}_i = \Big[\psi_i, &\frac{\partial \alpha_{i-1}}{\partial x_{i-1}}\psi_{i-1}, ..., \frac{\partial \alpha_{i-1}}{\partial x_1}\psi_1, \\
&-\frac{\partial \alpha_{i-1}}{\partial x_{i-1}}x_i, ..., -\frac{\partial \alpha_{i-1}}{\partial x_1}x_2\Big]^T \in R^{\bar{p}_i}
\end{aligned}
\tag{33}
$$

$$
\bar{d}_i = [d_i, d_{i-1}, ..., d_1]^T \in R^i \tag{34}
$$

$$
h_i = [1, -\frac{\partial \alpha_{i-1}}{\partial x_{i-1}} ..., -\frac{\partial \alpha_{i-1}}{\partial x_1}]^T \in R^i \tag{35}
$$

$$
\bar{\phi}_i = \phi_i - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_j}\phi_j \tag{36}
$$

$$
\begin{aligned}
\eta_i = &-\sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{\Theta}_i}\dot{\hat{\Theta}}_i - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{D}_i}\dot{\hat{D}}_i \\
&-\sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial y_r^{(j-1)}}y_r^{(j)}
\end{aligned}
\tag{37}
$$

The final adaptive controller $u(t)$ is given by

$$
u(t) = N_n(\chi_n)\bar{\alpha}_n \tag{38}
$$

The control Lyapunov function is chosen as

$$
V_i = \left(\frac{1}{2}z_i^2 + \frac{1}{2}\tilde{\Theta}_i^T\Gamma_i^{-1}\tilde{\Theta}_i + \frac{1}{2l_i}\tilde{D}_i^T\tilde{D}_i\right) \tag{39}
$$

Following the similar procedure in step 1, the derivative of Lyapunov function V_i satisfies

$$
\begin{aligned}
\dot{V}_i \leq &-\sigma_i V_i^2 + (b_i(t)N_i(\chi_i) + 1)\dot{\chi}_i + \bar{b}_i^2 z_{i+1}^2 + M_i \\
&i = 1, ..., n-1
\end{aligned}
\tag{40}
$$

where $\bar{b}_i$ is the unknown bound of $b_i(t)$ and

$$
M_1 = 0.2785\varrho(D_i + \| \Theta_i \|) + \frac{k_{\theta_i}}{2}\| \Theta_i \|^2 + \frac{k_{d_i}}{2}D_i^2 \tag{41}
$$

$$
\sigma_i = \min\{2c_i, k_{d_i}k_i, k_{\theta_i}\lambda_{min}(\Gamma_i)\} \tag{42}
$$

Remark 6: Unlike the normal stability analysis for backstepping control design, the decoupled backstepping technique will do the backward stability analysis for the single Lyapunov function V_i at each step. It is the fundamental change to solve the multiple Nussbaum-type functions with different signs of control coefficients.

IV. STABILITY ANALYSIS

Theorem 1: Consider the time-varying nonlinear system (1) satisfying Assumptions 1-2, with the application of the controller (38), virtual control laws (8)-(11) and (26)-(29), the parameter updating laws (12), (13), (30), (31). All signals contained in the closed-loop systems are uniformly bounded.

Proof: In the decoupled backstepping stability analysis, we will do the backward analysis. From the last step n, the derivative of Lyapunov function V_n satisfies

$$
\dot{V}_n \leq -\sigma_n V_n^2 + (b_n(t)N_n(\chi_n) + 1)\dot{\chi}_n + M_n \tag{43}
$$

Multiplying of (43) by e^{ft} and taking the integration on both sided gives

$$
\begin{aligned}
V_n(t) &\leq e^{-ft}\int_0^t (b_n(t)N_n(\chi_n)+1)\dot\chi_n e^{f\tau}d\tau \\
&\quad +V_n(0)+M_n\int_0^t \varrho e^{-f(t-\tau)}d\tau \\
&\quad -\int_0^t \sigma_n V_n e^{-f(t-\tau)}d\tau \\
&\leq e^{-ft}\int_0^t (b_n(t)N_n(\chi_n)+1)\dot\chi_n e^{f\tau}d\tau + \bar M_n
\end{aligned}
\tag{44}
$$

where

$$
\bar M_n = V_n(0)+M_n\int_0^t e^{-f(t-\tau)}d\tau \tag{45}
$$

Since $\bar M_n$ is bounded. Together with Lemma 1, we can conclude that $V_n(t)$ and $\chi_n(t)$, hence z_n, $\hat\Theta_n$, and $\hat D_n$ are bounded.

Applying Lemma 1 for $(n-1)$ times backward, it can be seen from the above mentioned design procedures that $V_i(t)$, $z_i(t)$, and hence $\chi_i(t)$ are bounded. Thus the solution of the closed-loop is bounded. The results established is concluded in Theorem 1. ∎

Remark 7: The difficulty to achieve the control objective is to handle the effects of unknown time-varying parameters, unknown signs of control coefficients and unknown disturbances. By applying the decoupled backstepping technique and introducing a hyperbolic tangent function, a Nussbaum-type function, and a new estimation of parameter bound, this new control strategy achieves the goals of stabilization and asymptotic tracking for the uncertain time-varying nonlinear systems (1).

Remark 8: To compensate for the effects of unknown time-varying parameters and disturbance, new estimation methods are developed to estimate the bounds of time-varying parameters in (12) and (30) and the bounds of disturbance in (13) and (31).

Remark 9: A Nussbaum-type function is used in virtual control laws α_i (8) and final control law u (38) to deal with unknown time-varying parameter with unknown signs of control coefficients.

V. AN ILLUSTRATIVE EXAMPLE

For illustration of the proposed scheme, an example is considered. The results of simulation will verify that our adaptive controller makes the system stable. We consider the following second-order system

$$
\begin{aligned}
\dot x_1 &= b_1(t)\beta_1(x_1)x_2 + \theta_1(t)x_1^2 \\
\dot x_2 &= b_2(t)\beta_2(x_1,x_2)u + \theta_2(t)(x_2+x_1)+d(t) \\
y &= x_1
\end{aligned}
\tag{46}
$$

where $\theta_1(t)=0.5+\cos(t); \theta_2(t)=1+\cos(t)$, $b_1=b_2=1.5+0.2\cos(t)$, $d(t)=0.1\sin(2\pi t)$, $\beta_1(x_1)=1, \beta_2=10.2x_1^2+x_2^2$ actually these parameters and disturbances are

not needed to be known in controller design. The objective is to control the system output $y(t)$ to follow a desired trajectory $y_r = 0.2 - 0.2\cos(3\pi t)$.

In the simulation, the design parameters were set as $c_1 = c_2 = 2$, $k_1 = k_2 = 1$, $l_1 = l_2 = 1.5$. The simulation results are shown in Figures 1-3. Figure 1 shows the system states x_1 and x_2. Figure 2 shows the tracking error $y(t)-y_r(t)$ converges to 0. Figure 3 shows the control input $u(t)$. Clearly, simulation results verify the effectiveness of proposed scheme.
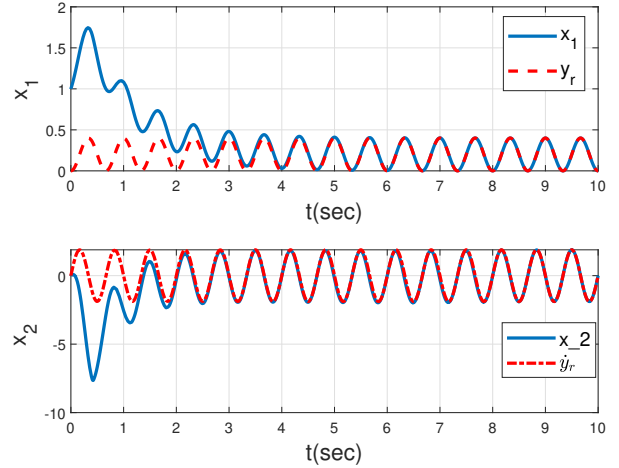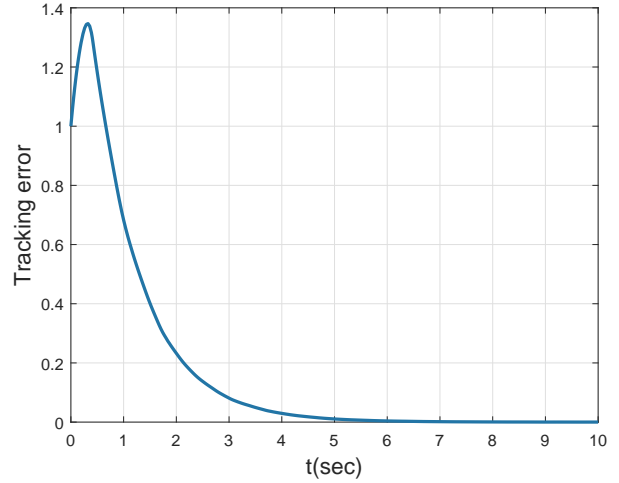

Fig. 1. System states x_1 and x_2.

Fig. 2. Tracking error $y - y_r$

VI. CONCLUSION

In this paper, an adaptive backstepping control scheme is proposed for uncertain time-varying nonlinear systems in presence of unknown control coefficients which are functions of states and unknown time-varying parameters with unknown signs and functions, unknown time-varying parameters as well as unknown bounded disturbances. The proposed
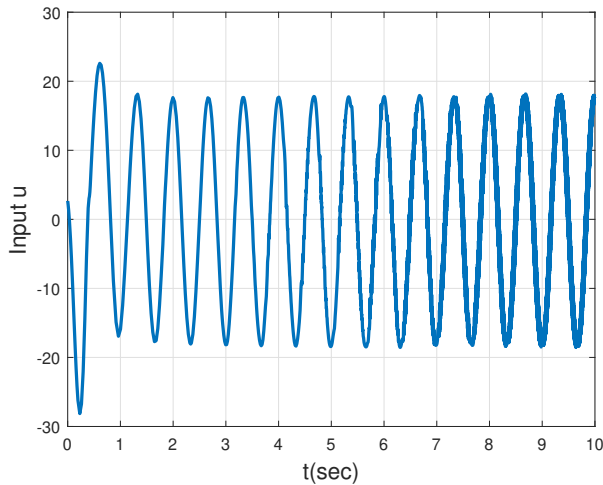
Fig. 3.   System input u.

robust adaptive controller is designed by incorporating new hyperbolic tangent functions, Nussbaum-type functions, and new estimations of parameter bounds. Two adaptation laws are developed for estimation of bounds of unknown time-varying parameters and unknown disturbances. A decoupled backstepping approach to stability analysis is proposed. By using the decoupled backstepping technique, it is proved that under the proposed control, all closed-loop states are uniform ultimate bounded.Simulation results illustrate the effectiveness of the proposed adaptive control scheme. The future work may be the output feedback control of nonlinear systems with unknown control coefficients.

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