UNIVERSITETET I AGDER

# How upper secondary students solve algebraic word problems in the area of mathematical modelling

A case study of one group of Norwegian students

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# Preface

This thesis marks the end of the two years master's degree programme in Mathematics at the University of Agder (UiA) in Kristiansand, within the specialization Mathematics Didactics. The thesis was performed throughout my third and fourth semesters of the master, autumn 2019, at the Department of Mathematical Sciences under the Faculty of Engineering and Science.

There is this well-known quote in Norway that, "there is no bad weather but bad clothing". This saying inspired me in adjusting to all situations and preparing well for all the challenges that came my way. A lot of work and time have been invested in the process of coming up with this thesis. Without the support of some people, it would have been much more difficult to overcome the difficulties and stress in this process. I am privileged to have met many excellent persons who in diverse ways held my hand in my quest for knowledge. My primary dept of gratitude goes to God as my source of strength and guide. I much appreciate the study opportunity and the experiences I had in Norway.

I further wish to express my deepest and sincere gratitude to my adviser Professor John David Monaghan, whose expertise, understanding, generous guidance and support made it possible for me to work in this research area. I would like to express my sincere appreciation to the mathematics teacher (Trine Einerkjær Harbo) and the students who participated in this study. I am also grateful to my colleague, Kristoffer Heggelund Knutsen, for going through my codes during the inter-coder reliability process. I thank the many excellent professors and students at UiA whom I have learned much from during my time in Kristiansand.

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# Abstract

In order to meet new challenges in school, work, and life, students may have to adapt and extend whatever mathematics they know. This will require students' ability, to some extent, to relate school mathematics to real-life context and vice versa. Mathematical modelling is one of the appropriate media that support the link between school mathematics and the real-life. Research has shown that there is a strong bias in the curriculum against mathematical modelling and that high-level mathematics (theorems, proofs, formulas, and among others) is rather considered important, mostly at the upper secondary level. Again, research has shown that students find difficulties in abstract (factual and hypothetical) algebraic word problems than concrete (factual and hypothetical) algebraic word problems. This study aims to investigate how upper secondary students solve algebraic word problems in the area of mathematical modelling, where we examine how upper secondary students justify their strategies in solving algebraic word problems and how these students interpret their findings after solving an algebraic word problem in a form of mathematical modelling. Twenty-three first year students (14-16 years old) from an upper secondary school located in the Southern Norway participated in this study. The main source of empirical data was interviews (one student representing his/her group) and students' worksheet (students working in groups to solve four algebraic word problem tasks and one modelling tasks).

The findings of the study suggest that some students have difficulties in comprehending abstract algebraic word problems although most of the students regard the very first task they solve as the most difficult task. The results indicate that students resort to an arithmetic method if they cannot solve the equation developed from the word problem. The findings also show that most of the students formulated algebraic equations in solving the modelling task but had at least one incorrect equation. However, these students compared their modelling computations with reality, making their mathematical solution unique. The results indicate that most students want the inclusion of more modelling activities in school giving varying reasons. As a didactical implication, the inclusion of a variety of abstract algebraic word problems in student's mathematics experience may be worthwhile, and teachers should encourage the use of an algebraic and graphic way of solving word problems. Also, the inclusion of modelling activities in the curriculum might help students to view all the mathematics subject as important since they can relate each subject to the real-world through the teacher's guidance. Further research on the use of technological tools for modelling realistic problems is recommended.

*Keywords*: algebraic word problem; mathematical modelling; upper secondary students

## Sammendrag

For å imøtekomme nye utfordringer på skolen, i jobb og i livet, kan elever ha behov for å tilpasse og utvide den matematikken de kan. Dette krever at elever evner, i noen grad, å sette skolematematikken i forbindelse med hverdagslige kontekster og motsatt. Matematisk modellering er en av de hensiktsmessige mediene for å støtte sammenhengen mellom skolematematikken og virkeligheten. Forskning har vist at det er en sterk tilbøyelighet i pensumet mot matematisk modellering og at i stedet høynivå matematikk er ansett som viktig, mest på videregående nivå. Igjen, forskning viser at elever har større utfordringer med abstrakte (fakta og hypotetiske) algebraiske tekstoppgaver enn konkrete (fakta og hypotetiske) algebraiske tekstoppgaver. Denne studien har som mål å undersøke hvordan elever i videregående løser algebraiske tekstoppgaver innen matematisk modellering, hvor vi vil undersøke hvordan elever på videregående forklarer strategiene de bruker for å løse algebraiske tekstoppgaver, og hvordan elevene tolker funnene etter å ha løst en algebraisk tekstoppgave ved bruk av matematisk modellering. Tjuetre elever fra første året (14-16 år) på en videregående skole i Sør-Norge deltok i denne studien. Hovedkilden til empiriske data var intervjuer (en elev representerte hans/hennes gruppe) og elevenes oppgaveark (elevene arbeidet i grupper for å løse fire algebraiske tekstoppgaver og en modelleringsoppgave).

Funnene i denne studien viser at noen elever har utfordringer med å forstå abstrakte algebraiske tekstoppgaver, selv om de fleste elevene omtalte den første oppgaven de løste som den vanskeligste oppgaven. Resultatene indikerer at elevene tyr til en aritmetisk metode hvis de ikke klarer å løse likningen utviklet fra tekstoppgaven. Funnene viser også at de fleste elevene formulerte algebraiske likninger da de løste den modelleringsoppgaven, men de hadde minst en likning som var feil. Samtidig sammenliknet elevene sine modelleringsutregninger med virkeligheten, som gjorde deres matematiske løsninger unike. Resultatet indikerer at de fleste elevene ønsker mer modellerings aktiviteter på skolen med ulike begrunnelser. Som en didaktisk implikasjon, kan inkluderingen av varierte abstrakte algebraiske tekstoppgaver i elevers matematiske erfaringer ha betydning. Lærere bør oppmuntre til bruk av algebraiske og grafiske måter å løse tekstoppgaver. Samtidig å inkludere modelleringsaktiviteter i pensumet, kan bidra til at elever ser på matematikk som et viktig fag, ettersom elevene kan relatere et hvert fag til virkeligheten gjennom lærerens veiledning. Videre forskning om bruk av teknologiske verktøy for modellering av realistiske problemer anbefales.

Nøkkelord: algebraiske tekstoppgaver; matematisk modellering; elever i videregående

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# 1 INTRODUCTION

In order to meet new challenges in school, work, and life, students may have to adapt and extend whatever mathematics they know. This will require students' ability, to some extent, to relate school mathematics to a real-life context and vice versa. Erling et al. (2016) argues that students and even some adults often find mathematics very difficult for the reason that they are not able to relate mathematical facts to a real-life context and vice versa. Relating mathematical facts to a real-life context and the other way round requires realistic considerations and in particular, Verschaffel et al. (2000, 2009, 2010) argue that application word problems (that is word problems in the form of mathematical modelling) requires more of realistic considerations. In considering the mathematics curriculum at the upper secondary level, Stillman (2007) argues that there is a strong bias against mathematical modelling and that high level mathematics (theorems, proofs, formulas and among others) is rather considered important. With this background, the current study aims to investigate how upper secondary students' solve algebraic word problem in the area of mathematical modelling.

This chapter starts with the statement of the problem that drove the study. The next section gives an account of the research question after which a discussion of the significance of the study follows. Afterwards, the motivation of the researcher for conducting the research is presented. The chapter then ends with the outline or structure of the thesis.

## 1.1 Statement of the Problem

In the last decades, there have been a lot of educational research literature in the area of algebra and word problems unlike other areas like trigonometry which is sparse. Algebraic word problems runs through all levels of mathematics curriculum since it forms an integral part of mathematics learning. Verschaffel et al. (2000) defines word problems as mathematical exercises that present significant background information on the problem as text, rather than in the form of mathematical notation. Erling et al. (2016) argues that word problems are often seen as a way of bridging the divide between real life and the mathematics classroom.

Morales et al. (1985) argues that one of the most problematic areas of the mathematics curriculum involves the solution of word problems. They go on to claim that even though students have mastered the technical competencies of doing the mathematical operations involved in the word problems, they experience considerable difficulty with simple word problems that require application of those techniques. On the other hand Lewis and Mayer (1987) also argues that effectively solving a mathematical word problem does not depend only on the students' ability to perform the required mathematical operations but the extent to which students' are able to accurately understand the text of the word problem is relevant. Boonen et al. (2016a) also add that effectively solving a mathematical word problem and the understanding of the text are related in such a way that developing a deeper understanding of the text of the word problem serves as a crucial step before the correct mathematical computations can be performed.

Word problems are usually an example of mathematical modelling[1]. Erling et al. (2016) argues that it is relevant for students to be aware that word problems are a case of modelling mathematical ideas. Word problems in the form of mathematical modelling support the inquiry-based[2] activities in the classroom environment. Erling et al. (2016) further points out that it is important to remind the students that through mathematical modelling is how a lot of mathematics is used in careers beyond school, to model what happens (or may happen) in the world so that complex situations can be manipulated more simply and solutions to problems found. Hernández et al. (2017) on the other hand argues that students who have engaged in the modelling process appreciate the opportunity to use their own ideas in creating a mathematical solution to a real-world problem and have experiences that help them regardless of what college or career path they follow.

In solving algebraic word problems in the area of mathematical modelling, 'understanding' is very relevant in the whole process of finding a suitable solution. The National Council of Teachers of Mathematics (2000, p. 20) asserts that students must learn mathematics with understanding and actively build new knowledge from experience and prior knowledge. Hence, based on this assumption this current study represents an attempt to investigate the underlying understanding of students as they solve algebraic word problems in the area of mathematical modelling. In particular, the study looks at: how students explain their working processes; the construction of a mathematical model; their computational activities; and how they interpret their findings.

[1]Mathematical modelling is defined and explained in Section 3.3.2.
[2]Inquiry-based learning is defined and explained in Section 3.1.1.

## 1.2 Research Question

The purpose of this research study is to address these questions:

1. How do upper secondary students' justify their strategies for solving algebraic word problems?

2. How do these students interpret their findings after solving an algebraic word problem in a form of mathematical modelling?

The first question seeks how the students justify the known techniques they use in their working processes and also to find the specific errors made by the students and the conceptions they have in making such errors. These errors are generally seen in the process of comprehending word problems, translating word problems into equations, and the transfer[3] of solving word problems. The two main questions one could ask from the first research question is:

- What is a strategy?

- How do students justify their strategies?

Strategy in the first research question is explained as a plan of action designed by the students to achieve the desired solution of an algebraic word problem. In this regard, the students particularly transform the algebraic word problem into a linear equation or a simultaneous set of two linear equations and then use addition/elimination or substitution method to solve it. The justification of students' strategies is about how they mathematize the algebraic word problems. For a student to justify his/her strategy for solving an algebraic word problem, the student go through the process of mathematizing the word problem as they use a known technique to solve the equation derived from the word problem. Mathematizing an algebraic word problem usually involves: Understanding the word problem; Devising a plan (that is translate the problem using variables and setting up an algebraic equation); Carrying out the plan (that is solving the equation using a known technique); Looking back; and Presenting the final answer.

The second question on the other hand, seeks how the students interpret their findings. An interpretation of their findings may depend on: the initial understanding of the problem; the construction of a mathematical model; the actual computational activities; and the evaluation of the outcome of these computations. The question also seeks how the students

[3]Transfer in solving algebraic word problems is defined and explained in Section 3.3.1

use the various mathematical representations[4] in constructing a mathematical model and also interpreting their findings. On the whole, these research questions tries to find out whether the students have conceptual[5] understanding (based on reasoning, interpretation and also using different representations for clarity) or procedural understanding (usually following set of rules without knowing why).

## 1.3 Significance of the Study

This study is significant for the reason that it has the potential to contribute to the literature on the issues that characterizes students' solutions of algebraic word problems in the area of mathematical modelling. I also hope that the outcome of this study could help in suggesting some ways teachers could adapt in helping students develop conceptual understanding in their problem solving processes.

Research on algebraic word problems in the area of mathematical modelling, particularly in the Norwegian context, is sparse and quiet limited. Stillman (2007) argues that in some European countries, there is a strong bias against mathematical modelling in the upper secondary level. Artaud (2007) also argues that if mathematical modelling is added to the ordinary didactical system, then the teaching process must be accorded extra time. He further points out that it will be difficult to obtain this (the inclusion of mathematical modelling in the ordinary didactical system) in the general teaching system and also argues that based on its limitations, it is usually provided for students who are supposed to need it, for instance engineering students and among others. Nevertheless, Erling et al. (2016) argues that mathematical modelling is the means by which much of mathematics is used in careers beyond school. Hence the idea of mathematical modelling should not be limited to some specific students. The pressure of time and the many demands in the daily work of teachers usually does not help with the introduction of mathematical modelling and also analyzing students specific errors during the process. Espeland (2017) points out that quiet often as a teacher she experienced her students making errors, but to find the basis for the problems students revealed was difficult under pressure of time and the many demands in the daily work.

The present study tries to address such issues by investigating how upper secondary students' solve algebraic word problems in the area of mathematical modelling based on their

[4]Mathematical representations is defined and explained in Section 3.3.3

[5]Conceptual and procedural understanding is defined and explained in Section 3.2

performance in two different tasks (algebraic word problems and a modelling task). The students are first given some algebraic word problems (matching four problem categories described by Caldwell and Goldin (1979, 1987)) to solve and an analysis of their working processes is made based on how they justify their strategies used in solving the problems. These problems are not very different from what they usually encounter at school. Secondly, the student solve a well designed mathematical modelling task that offers a low floor allowing even low performing student to engage with minimal prerequisite knowledge and skills, and a high ceiling providing opportunities to explore more complex concepts. I hope the outcome of the performances of students' in both tasks will help teachers either to include more of the modelling task in class activities despite the pressure of time and the many demands in the daily work or stick to the usual algebraic word problems. Garfunkel and Montgomery (2016) argues that small modelling activities can be used to reinforce new concepts and to illustrate their application whilst more extended modelling activities help students pull together ideas from different parts of a course and from different courses. Olsen (2006) argues that the Nordic countries have similar achievement profiles in mathematics at the level of lower secondary school and, as such, the results of this study may be relevant for the other Nordic countries as well.

## 1.4 Motivation for the Study

My personal motivation for the study mainly stems from my past experiences. A course (MA-424) I took in my first year master program at the University of Agder sparked my interest in the field of mathematical modelling. In reflecting on a modelling project "Cell Phone Revolution" I did with a colleague during the first year, it was interesting how we were able to pull together ideas from the previous mathematics courses and other different courses we engaged in. As an individual who believes learning is social (that is, it usually occurs when there is an interaction between students in a particular field of study), I have the opinion that inquiry-based activities supports learning in the social setting. And that mathematical modelling activities at one point help students to engage with each other as they pull together ideas from the previous mathematics courses and other different courses in finding a suitable model.

While I was deciding what to study for my master thesis, it was not very surprising that I chose the field of mathematical modelling. Most people with less mathematics background I usually encounter complain about the difficulty of the mathematics subject and its unusefulness in their daily activities. In my opinion, if such people have engaged in mathematical modelling at one point in time at school, maybe they wouldn't see it as uninteresting and also see the importance of it in the real world. The idea of studying algebraic word problem in the area of mathematical modelling was a suggestion by my supervisor upon realizing my interest in the field of mathematical modelling. My interest in mathematical understanding was another motivating factor, that is, I wanted to do something about students' understanding as they justify their strategies used in their solution processes.

A thorough background knowledge was acquired in the process of finding and studying articles. I was, to some extent, motivated as a results of the literature I read. The reading of this literature helped me attain a good understanding of the scope of my topic. Some of the literature I read was not directly of use in the study, however this literature, to some extent, helped me in developing my research design.

## 1.5 Structure of the Thesis

This thesis consists of seven chapters. The second chapter following this introduction chapter presents the education and research setting of the study. This includes a brief description of the Norwegian education system, mathematics education in Norway and word problem in the Norwegian mathematics curriculum, as well as the school where the research took place and the research participants.

The third chapter presents the theoretical underpinnings of the study. The chapter contains a review of literature and related topics that provide a basis for the whole research study. This includes a brief description of socio-cultural theory, and continues with mathematical understanding. A presentation on problem solving which entails algebraic word problem, mathematical modelling and mathematical representations then follows.

The fourth chapter presents the methodology used in this study. This includes a presentation of the case study strategy, the research design which entails the research method, a description of the research participants, the specific methods used in the data collection, the management of the data and the strategy used for the analysis. This chapter also provides details on the ethical considerations as well as the validity and trustworthiness of the study.

The fifth chapter includes the data analysis and the main findings. This entails the analysis of the students' worksheet together with the interview transcripts. The sixth chapter presents the discussion of the research findings. The discussion involves the link between the researcher's interpretations and the literature review. The research questions are being addressed in the discussion chapter. The discussion of this study entails students' prior knowledge, types of word problems, the mathematization of word problems, students' modelling activities, and students' conceptions on both modelling in school activities and group work.

The seventh and final chapter presents the summary of results and discussion, followed by the limitations of the study, an implication of the study for teaching and also suggestions for further research.

# 2 EDUCATION AND RESEARCH SETTING

This chapter presents the education and research setting of the study. The study was carried through with upper secondary school students in Kristiansand, Norway. Hence, there is a need to look briefly at the Norwegian educational system in the first section. A presentation of the Norwegian mathematics education and the algebraic word problem in the Norwegian mathematics curriculum follows afterwards since it relates the scope of the research reported in this thesis. The chapter then ends with a description of the school where the research took place together with the cooperating teacher as well as a description of the research cohort. All information about the Norwegian education system is strictly taken from the Norwegian Ministry of Education & Research (2007), Nuffic (2015), and Onstad and Kaarstein (2015).

## 2.1 The Norwegian Education System

Norway is a unitary constitutional monarchy with a parliamentary system of government. The country is divided into 19 provinces, with several hundred municipalities. The Norwegian education system[6] is governed by national legislation. The Ministry of Education and Research (*Kunnskapsdepartementet*) is responsible for carrying out national educational policy in all levels of education, including pre-school (for children up to age five). The Norwegian Parliament (the Storting) and the government is responsible for formulating education policy, and determines the broad contours of the educational frameworks. Municipal and local authorities ensure that the guidelines and outcomes established by the government are realized. The responsibility for setting up educational programs within higher education is largely delegated to the higher education institutions themselves, and responsibility for quality assurance in higher education lies with the Norwegian Agency for Quality Assurance in Education (NOKUT).

The Norwegian educational system is organized under the levels: primary, lower secondary, upper secondary and tertiary education (see Figure 1, a general structure of the Norwegian educational system). Education is compulsory for children aged 6 to 16 and also free. Compulsory education is divided into two main stages: Primary school (grades 1–7) and lower secondary school (grades 8–10). The subjects of primary and lower secondary education includes: Norwegian; Mathematics; Social Science; Christianity, Religion and Ethics Edu-

[6] More information can be found at Onstad and Kaarstein (2015); Norwegian Ministry of Education & Research (2007); Nuffic (2015); Feagles and Dickey (1994)

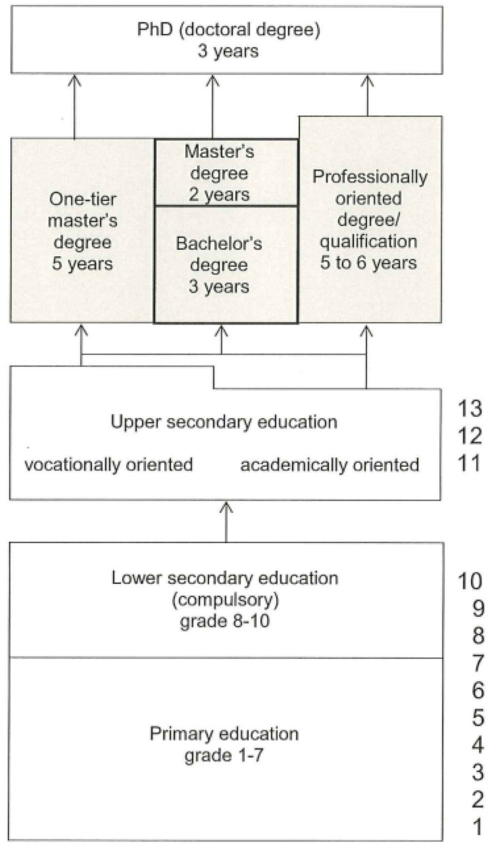



**Figure 1:** A general structure of the Norwegian educational system (adapted from Norwegian Ministry of Education & Research 2007, p. 25).

cation (CREE); Arts and Crafts; Natural Science; English; Foreign Languages/Language In-depth Studies; Food and Health; Music; Physical Education; Student Council Work; and Optional Program Subject. The upper secondary education is divided into two categories: vocationally and academically oriented.

### 2.1.1 Overview of the Norwegian upper secondary school system

Upper secondary education and training comprises all courses leading to qualifications above the lower secondary level and below the level of higher education. Young people who have completed primary and lower secondary education, or the equivalent, have a right to three years' upper secondary education and training leading either for admission to higher education, for vocational qualifications or for basic skills. Pupils under the vocational education and training can achieve the qualifications necessary for admission to universities and university colleges (university admissions certification) by taking a supplementary programme for general university admissions certification.

The education and training normally takes three years, which is divided into three levels: Vg1[7], Vg2 and Vg3 (in a few cases four years with a Vg4). Vocational education and training which usually takes place after two years in school and one year in-service training in an enterprise leads to a craft or journeyman's certificate. In-service training at a training establishment is usually combined with one year's productive work, so that the apprenticeship takes two years in all, however if it is not possible to provide enough training places, and county authorities are obliged to offer Vg3 in school, in which case there is no productive work. General studies on the other hand takes three years which leads to general university admissions certification. The upper secondary education and training is organized in twelve different education programs, that are

Programs for General Studies:

- Programme for specialization in general studies
- Programme for sports and physical education
- Programme for music, dance and drama

Vocational Education Programs:

- Programme for building and construction
- Programme for design, arts and crafts
- Programme for electricity and electronics
- Programme for health and social care
- Programme for media and communication
- Programme for agriculture, fishing and forestry
- Programme for restaurant and food processing
- Programme for service and transport
- Programme for technical and industrial production

The pupils are tested on their knowledge throughout the year, and final exams are taken at the end of each year, however at the end of the third and final year, the pupils take national examinations in addition to their final school exams.

[7]Vg1: first year, Vg2: second year, Vg3: third year

## 2.2 Mathematics Education in Norway

Mathematics is one of the prominent subject in the Norwegian school curriculum. It is also one of the core subjects covered on national examinations in the $10^{th}$ grade. The mathematics curriculum is organized under subject domains: the domains for Grades 1 to 4 are Numbers, Geometry, Measuring, and Statistics; the domains for Grades 5 to 7 are Numbers and Algebra, Geometry, Measuring, and Statistics and Probability; the domains for Grades 8 to 10 are Numbers and Algebra; Geometry; Measuring; Statistics, Probability, and Combinatorics; and Functions (Onstad and Kaarstein, 2015; Norwegian Ministry of Education & Research, 2013).

Longitudinal studies such as TIMSS[8] and PISA show a setback in mathematical performance among students in both lower and upper secondary school, and particularly in the topics of number and algebra (Lie et al., 1997; Grønmo et al., 2004; Kjærnsli et al., 2004, 2007; Grønmo and Onstad, 2009; Grønmo et al., 2010, 2012). Espeland (2017) argues that the Norwegian results from the various comparative studies are generally considered unsatisfactory and have stimulated much debate in Norway.

### 2.2.1 Mathematics education at the upper secondary school

Mathematics is a common core subject for all the education programs in upper secondary education, however it is not compulsory after the first year in upper secondary school. In the first year (Vg1), there are two subject cirrula namely 1T, which is more theoretical, and 1P, which is more practical. Both variants qualify candidates for higher education together with either the common core programme subject Mathematics at level Vg2 (2P) or the programme subject Mathematics (R1/S1). The pupils taking the vocational subjects have their Vg1 curriculum in 1P or 1T, which is 1P-Y or 1T-Y (see Table 1 for an overview of the main subject areas at the Vg1 level).

### 2.2.2 IT Mathematics

Theoretical mathematics (1T) is taken by students under the vocational education programme and the education programme for general studies. According to the Norwegian

[8]TIMSS-Trends in International Mathematics and Science Study and PISA-Programme for International Students Assessment, are internationally standardized assessment which allows participating nations to compare students' educational achievement across borders.

| Year level | Main subject areas | | | | |
|---|---|---|---|---|---|
| 1T | Numbers and algebra in practice | Geometry | Probability | Functions | |
| 1P | Numbers and algebra in practice | Geometry | Probability | Functions | Economics |
| 1T-Y | Numbers and algebra in practice | Geometry | | Functions | |
| 1P-Y | Numbers and algebra in practice | Geometry | | | Economics |

**Table 1:** Overview of the main subject areas (Norwegian Ministry of Education & Research, 2013, p. 3).

Ministry of Education & Research (2013), a solid competence in mathematics involves using problem-solving techniques and modelling to analyze and transform a problem into mathematics form and then solve the problem and evaluate the validity of the solution. Some competence aims after both 1T-Vg1 and 1T-Y-Vg1 programs are listed below (a detailed information can be found at the Norwegian Ministry of Education & Research (2013, pp. 10-13)).

- Numbers and algebra in practice: interpret, process and assess the mathematical content in various texts;convert a practical problem into an equation, an inequality or an equation system, solve it with and without using digital tools, present and provide rationale for the chosen solution and assess the validity of the solution.

- Geometry: elaborate on the definitions of sine, cosine and tangent and use trigonometry to calculate length, angles and area of triangles; use plane geometry to analyse and solve composite theoretical and practical problems connected to lengths, angles and areas.

- Functions: explain the concept of functions and be able to convert between different representations of functions; make, interpret and explain functions that describe practical questions, analyse empirical functions and find expressions for an approximate linear function, with and without using digital tools.

- Probability: make examples and simulations of random events and explain the concept of probability; calculate probability by counting all favourable and all possible results based on tables and by systematizing counts using cross tables, Venn diagrams and the addition rule and the multiplication principle in practical contexts (only in 1T-Vg1).

Appendix C, on page 154, gives a summary of the course content used by the upper secondary school in which this study was conducted.

## 2.3 Word Problem in the Norwegian Mathematics Curriculum

In the Norwegian curriculum for the common core subject of mathematics, reading is among the basic skills. Reading as a basic skill is defined in the curriculum as:

> Being able to read in Mathematics involves understanding and using symbolic language and forms of expression to create meaning from texts in day-to-day life, working life and from mathematics texts. The subject matter of Mathematics is characterised by complex texts that may include mathematical expressions, graphs, tables, symbols, formulas and logical reasoning. Reading in Mathematics involves sorting through information, analysing and evaluating form and content, and summarising information from different elements in the texts. The development of reading in Mathematics begins with finding and using information in the texts by means of simple symbolic language and moves toward finding meaning and reflecting on complex professional and technical literature with advanced symbolic language and concepts (Norwegian Ministry of Education & Research, 2013, p. 5).

Word problems runs through the curriculum from grade 1 to the upper secondary level and, of course, being able to read in mathematics is important in dealing with word problems. Opsal and Tonheim (2018) argues by citing Nortvedt[9] that, there is a strong positive correlation between numeracy and reading comprehension. Word problems usually falls under the topic number and algebra, and the TIMSS report shows a setback in mathematical performance among Norwegian students, mostly in the topics of number and algebra. Morales et al. (1985) argues that one of the most problematic areas of the mathematics curriculum involves the solution of word problems.

Pedersen (2015) argues that analyses have revealed that the Norwegian curriculum for

[9]She studied on how 8$^{\text{th}}$ grade students in Norway responded on a multistep arithmetic word problem in the national test in numeracy and compared the results with students respond in the national test in reading comprehension (Nortvedt, 2010).

upper secondary school mathematics places a greater emphasis on applying procedures and methods. And that, to a far lesser extent, the curricular objectives describe activities such as analyzing, investigating, assessing, discussing, proving, modeling, and generalizing. This does not give students much opportunity to develop their own methods in problem solving as they use their prior knowledge to investigate some problems. Pedersen (2015) again argues that Norwegian upper secondary school students' tend to perform weakly on items that place high demands on symbol manipulation. The Norwegian Ministry of Education and Research has currently decided on the core elements of each subject (see the link[10]), where the core elements of mathematics are: exploration and problem solving; modelling and applications; reasoning and argumentation; representation and communication; abstraction and generalization; Mathematical knowledge areas. The change in the curricular permits students to work more with methods and ways of thinking so that they gain a greater understanding of the mathematics subject. Also, numbers and numerical understanding are the foundation in what students will master during elementary school. Further, personal finance, measurement and statistics are important areas where figures are used in realistic contexts, whilst programming and algorithmic thinking will be part of the mathematics subject.

In this present study, students are given an open algebraic word problem task (modelling task), which has two sets of linear equations where one of the equations has a third variable, that demands the manipulation in finding the break-even point (the point at which the two linear equations are equal) of the two equations. The students performance in this task is compared to their performance in the algebraic word problem tasks they solve involving the use of known techniques such as additions/elimination or substitution method.

### 2.3.1 Mathematical modelling at the upper secondary school

According to Antonius (2004) "the Norwegian curriculum does not use the term modelling explicitly but it says that students should work with problems in a realistic context". He further points out that modelling in the Swedish and the Norwegian curriculum for upper secondary is explicitly connected to information technology, that is the appropriate use of graphic calculators and computers by students in the modelling process.

Stillman (2007) argues that at the upper secondary level, particularly in some European countries, there is a strong bias against mathematical modelling and greater attention is

[10] https://www.regjeringen.no/no/aktuelt/fornyer-innholdet-i-skolen/id2606028/?expand=factbox2606064

given to high level mathematics (theorems, proofs, formulas, and among others). Artaud (2007) also points out that the teaching process must be accorded extra time if mathematical modelling is added to the ordinary didactical system. Antonius (2004), on the other hand, argues that it is still hard to find time for modelling activities which are time consuming to a very high degree in the Nordic context. He further points out that one major challenge of mathematical modelling in this level is the unfamiliarity of students' and teachers' new roles, where students have to make their own choices and argue for those choices whilst the teachers role is to be a guide and not the person with the correct answer. The Norwegian Ministry of Education and Research has currently decided on new core elements for mathematics, which give room for modelling and applications. However, the final part of the new curriculum is yet to be formally approved (will take into effect in the autumn 2020, see the link[11]).

## 2.4 School and Cooperating Teacher

This study was conducted at Kristiansand Katedralskole Gimle[12] (KKG), which is an upper secondary school located in Kristiansand. The school has over 1,400 students divided into education programs for study specialization, sports subjects, service; transport; health; and youth education, IB Diploma programs, work life training, everyday life training and courses for adults. KKG is a modern school with contemporary classrooms and laboratories, and also the teachers are academically updated and participate in collaboration with the University and other upper secondary schools both in Norway and abroad.

The school was chosen by the researcher for two reasons. First, the researcher has little knowledge in Norwegian and KKG is a public school that have some of it courses taught in English (especially the 1T mathematics). A public school will give a good representation of the general mathematics setting of Norway. Secondly, the researcher already had contacts with the mathematics teacher of the 1T mathematics class through the Mathematical Sciences Department of the University of Agder. Further, the teacher cooperated with the researcher due to the teacher's interest in educational research and was also interested in this study.

[11] https://www.regjeringen.no/no/aktuelt/fornyer-innholdet-i-skolen/id2606028/?expand=factbox2606064

[12] Southern Norway's oldest and largest high school. Detailed information can be found in the school's website http://kkg.vgs.no/

### 2.4.1 Research cohort

The students were in their first year and first semester at the upper secondary school. The number of students was 23, including 17 females and 6 males aged between 14-16. The students are categorized in three different levels based at their performance at the lower secondary level: high-performing students; medium-performing students; and low-performing students. Based on this categories the teacher formed 7 groups with an average of 3-4 students per each group and that every group have a mixture of the three categories. During every mathematics class the students work together in this groups created by the teacher. The students had their 1T mathematics lessons in English as a results of their interest in taking the IB Diploma program the following year. For this reason some of the classroom activities are in relation to the settings in the IB Diploma program. The cohort was chosen because of the acquaintance of the researcher with their mathematics teacher as mentioned in the previous section.

#### 2.4.1.1 IB Diploma program .

According to the International Baccalaureate Organization (2014), the IB Diploma Programme is a rigorous, academically challenging and balanced programme of education designed to prepare students aged 16 to 19 for success at university and life beyond. The IB Diploma Programme higher level mathematics course focuses on developing important mathematical concepts in a comprehensible, coherent and rigorous way, achieved by a carefully balanced approach. Students are expected to reason or give a justification and proof of results in the development of each topic.

Students who follow the mathematics higher level course of the IB programme are expected to demonstrate the following; knowledge and understanding, problem-solving, communication and interpretation, technology, reasoning and inquiry approaches. The assessment model described by the International Baccalaureate Organization (2014), to some extent, helps students to develop a good understanding of the subject if the model is effectively put into practice. Saxton and Hill (2014) argues that IB students usually engage in critical thinking, seeking out primary sources and continually questioning and challenging. They further point out that students under the program perform at a high level at the university and also go off into the world with skills and knowledge.

The next chapter presents the theoretical framework for this study.

# 3 THEORETICAL FRAMEWORK

This chapter provides a theoretical base for the present study, that is, a theoretical background is presented from the most general to the most specific issues. First, the socio-cultural theory which constitutes a general perspective is discussed. Then, in view of assessing students' understanding in relation to the justification of their solution strategies, the meaning of (mathematical) understanding is presented. A presentation of relevant literature on problem solving within the study is presented at the end. This section also entails a discussion on algebraic word problems, mathematical modelling and mathematical representations.

## 3.1 Social Perspective

This study is underpinned by a perspective based on the socio-cultural theory. There are many things that individuals can do on their own initiative or alone, however we can not specifically point out that learning is a lonely act of an individual, even when it is undertaken alone. Scott and Palincsar (2013) argues that

> The work of socio-cultural theory is to explain how individual mental functioning is related to cultural, institutional, and historical context; hence, the focus of socio-cultural perspective is on the roles that participation in social interactions and culturally organized activities play in influencing psychological development. (p. 1)

Learning occurs when there is an interaction between students in a particular field of study, that is there is an exchange of ideas and the individual in this setting tends to make the practices and ideas of others their own. Wells (1999) argues that the process of appropriation (of making practices and ideas our own) does not involve transfer from outside, but the gradual construction on the part of the learner of actions equivalent to those manifested in the verbal and other behavior of others and an increasing ability to carry them out independently. Säljö (2000) on the other hand also argues that, appropriation is the knowledge from our fellow human beings in situations of interaction.

Cooperative learning[13] is an essential asset in inquiry learning. Gillies (2016) describes cooperative learning as a pedagogical practice that encourage socialization and learning among students from pre-school through to tertiary level and also across different subject domains. Working in groups not only provides students the opportunity to share

[13] An educational approach which aims to organize classroom activities into academic and social learning experiences.

with one another their ideas and opinions but also helps in individual cognitive development. Students feel more comfortable in a cooperative learning environment and by this motivation they turn to ask questions and also express their ideas. Gillies (2016) again points out that, in cooperative learning students work together to achieve common goals or complete group tasks, that's goals and tasks that they would be unable to complete by themselves. Nuangchalerm and Thammasena (2009) on the other hand argues that, the inquiry-based learning[14] activities promotes cognitive and analytical thinking developments and also learning satisfaction of the participants in their research responded at high level. They also suggested that teachers should play varied roles in supporting students' development of inquiry skills. Despite the importance of cooperative learning, it also has some limitations that could cause the process to be more complicated than first perceived. Sharan (2010) describes the constant evolution of cooperative learning as a threat and also shed light on some of the challenges that are often encountered in the process. Sharan (2010) also argues that, teachers implementing cooperative learning may also be challenged with resistance and hostility from students who believe that they are being held back by their slower teammates or by students who are less confident and feel that they are being ignored or demeaned by their team.

### 3.1.1 Inquiry-based learning

Saragih and Napitupulu (2015) argues that, the lack of teachers' attention on the independence of students' thinking builds student's logical thinking in a mode of remembering mathematical concepts and procedures in order to answer questions. In this approach students do not usually have other alternate thinking in responding to questions and when difficulties emerge, the students do not have another grip as guidance. They further emphasize that most teachers teach mathematics by explaining the concepts and mathematical operation, give examples to answer questions, a little bit debriefing, and then the session continues with students being asked to answer similar questions with what the teacher has explained. The learning of mathematics in such a condition is highly procedural and not well adapted to using and applying mathematics in science and engineering and the wider world. In my experience (from the lower grades to the initial stages of my tertiary education in Ghana), the mathematics classroom environment is usually ordered and orderly to the extent that one can observe a quiet and calm mathematics classroom with students sitting in rows or small groups, usually watching the chalkboard or working through ex-

[14] Inquiry-based learning is defined and explained in section 3.1.1.

ercises. This kind of traditional setting in the mathematics classroom environment does not usually help in bridging the divide between real life and the mathematics classroom. The inquiry-based learning activities has the capabilities of bridging this divide although it may have some challenges.

Inquiry-based learning (IBL) is a student centered pedagogy (learners being active). Bruner (1961) defines it as a pedagogical method developed (during the discovery learning movement of the 1960's) as a counter response to traditional forms of instruction, where people were required to memorize information from instructional materials. Linn et al. (2004, p. 4) defines inquiry as "the intentional process of diagnosing problems, critiquing experiments, distinguishing alternatives, planning investigations, researching conjectures, searching for information, constructing models, debating with peers, and forming coherent arguments". An aspect of their definition which is of interest is 'debating peers and forming coherent arguments'. This is observed in the current study where students work on specific tasks in groups and also justify their strategies at the end. Artigue and Blomhøj (2013, p. 797) on the other hand defines inquiry-based pedagogy loosely "as a way of teaching in which students are invited to work in ways similar to how mathematicians and scientists work".

Nardi and Steward (2003) in their engagement with some students at the secondary level found out that, several of the students under the study have the desire or wish to enjoy mathematics but instead see it as boring due to the use of route learning (rule-and-cue following) teaching method. Boaler (1998) on the other hand argues that, students who learned mathematics in an open, project-based environment (inquiry-based learning) developed a conceptual understanding that provided them with advantages in a range of assessments and situations whereas students who followed a traditional approach developed a procedural knowledge which is of limited use to them in unfamiliar situations. She also emphasize that students under the project-based environment are being apprenticed into a system of thinking and using mathematics that will help them in both school and non-school settings. Inquiry-based learning have some challenges despite it's importance of bridging the divide between real life and the mathematics classroom. Bell et al. (2010) argues that inquiry-based learning takes a lot of planning before its implementation and also providing learners with exactly the support they need (that's the balance between open-ended exploration and the guidance for individual learners) is sometimes very challenging.

## 3.2 Understanding

The National Council of Teachers of Mathematics (2000, p. 20) asserted that students must learn mathematics with understanding and actively build new knowledge from experience and previous knowledge. They again argue that, the vision of school mathematics is based on students' learning mathematics with understanding but unfortunately, learning mathematics without understanding has long been a common outcome of school mathematics instruction. In this present study of how upper secondary school students solve algebraic word problems in the area of mathematical modelling, the understanding of students' is measured based on their justification of strategies used in the problem solving, the interpretation of their findings, and the use of different representations in explaining their findings.

Skemp (1976) compared between two forms of understanding in the context of instrumental understanding and relational understanding. He argues that the instrumental understanding consists of the learning of an increasing number of fixed plans, by which students can find their way from particular starting points to required finishing points. Thus following set of rules without actual meaning or learning through rote memorization and drill. On the other hand, he argues that the relational understanding consists of building up a conceptual structure from which its possessor can produce an unlimited number of plans for getting from any starting point within his/her schema to any finishing point. That is the relational understanding requires conceptual connections and explaining why the rules work. Skemp also talked about some advantages of these two kinds of understanding. Pointing out that, within a limited context the instrumental understanding can be beneficial for a short-term learning, whereas in a broader context the relational understanding is beneficial for long-term learning. In effect, Skemp placed relational and instrumental understanding as two extremes separate from each other.

Hiebert and Carpenter (1992) on the hand presented a different perspective from that of Skemp. Hiebert and Carpenter (1992) describes understanding based on mental connections;

> A mathematical idea or procedure or fact is understood if it is part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections. (p. 67)

In their view, understanding applies to both procedure and relational aspects. That is, from Skemp's point of view, instrumental understanding entails procedures without connections however, in Hiebert and Carpenter's view the understanding of procedures also requires connections in the internal network as well. Van de Walle et al. (2007) also defines mathematical understanding based on connections. They define understanding as "a measure of the quality and quantity of connections that an idea has with existing ideas" (p. 23). The terms *quality* and *quantity* here are used in a similar way to the terms *strength* and *number* in Hiebert and Carpenter's explanation. Nevertheless, this views points out that understanding is not of two extreme kinds of either relational or instrumental but a combined issue. Van de Walle et al. (2007) further describes understanding as a model that "exists along a continuum from a relational understanding—knowing what to do and why—to an instrumental understanding—doing without understanding" (p. 23). They also placed understanding of mathematics into the categories of conceptual and procedural understanding.

Conceptual understanding consists of relationships constructed internally and connected to already existing ideas. In this category students are able to find or identify connections among mathematical concepts, that is procedures, semantic features, vocabulary, ideas, among others are all connected. In contrast, procedural understanding is task-oriented knowledge which may or may not be connected to conceptual understanding. This category too often devolves into rote memorization, rules without understanding, and often leads to frustration when not connected with concepts. Van de Walle et al. again points out that, a struggling learner might identify symbols and might manipulate algorithms (procedural knowledge) without a deep enough understanding of how ideas (conceptual knowledge) are connected to what the child already knows.

National Research Council (2001), presents a comprehensive work combining different aspects related to understanding. They define five components or strands of mathematical proficiency:

1. *conceptual understanding*—comprehension of mathematical concepts, operations, and relations
2. *procedural fluency*—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
3. *strategic competence*—ability to formulate, represent, and solve mathematical problems

4. *adaptive reasoning*—capacity for logical thought, reflection, explanation, and justification

5. *productive disposition*—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (p. 116).

They also argue that these five strands are interwoven and interdependent in the development of proficiency in mathematics. (Skemp, 1976) argues that learning mathematics with understanding is much relevant and that it makes it easier for students to remember things more easily. Van de Walle et al. (2007) on the other hand argues that there is less to remember if someone understands a mathematical idea. Hiebert and Wearne (2003) also argues that understanding can provide students with the idea that mathematics is useful.

Understanding in this study will be based on the connections among mathematical ideas, concepts and procedures. In particular, understanding based on connections during the process of problem solving; the initial understanding of the problem, the construction of a mathematical model, the actual computational activities, and the interpretation and evaluation of the outcome of these computations.

## 3.3 Problem Solving

Mathematics education aims to equip children or students to solve problems (problems in relation to school, work and life). In order to meet new challenges in work, school, and life, students will have to adapt and extend whatever mathematics they know, and doing so effectively lies at the heart of problem solving.

The National Council of Teachers of Mathematics (2000) explains that problem solving is the process of engaging in a task for which the solution method is not known in advance and in order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understanding. They further claim that without the ability to solve problems, the usefulness and power of mathematical ideas, knowledge, and skills are severely limited. For example students who can efficiently and accurately solve a simple linear equation but who cannot identify situations that call for the use of linear equation are not well prepared. On the other hand, they argue that students who can both develop and carry out a plan to solve a mathematical problem are exhibiting knowledge that is much deeper and more useful than simply carrying out a computation. That is the facts, concepts, and procedures students know are of little use unless they can solve problems by applying what they know. The National Council of Teachers of Mathematics again argues that since problem solving is an integral part of all mathematics learning, it should not be an isolated part of the mathematics program. However, the instructional programs in all the levels of mathematics education should enable all students to:

- build new mathematical knowledge through problem solving;
- solve problems that arise in mathematics and in other contexts;
- apply and adapt a variety of appropriate strategies to solve problems;
- monitor and reflect on the process of mathematical problem solving (pp. 52-55).

Schroeder and Lester (1989) studied approaches to teaching problem solving found in school mathematics curricula since the early 1980s. They argue that problem solving serves as a way for learning new mathematical ideas and skills and the most important role for problem solving is to develop students' understanding of mathematics. They identify three approaches to problem solving instruction: teaching *about* problem solving, teaching *for* problem solving, and teaching *via* problem solving.

- Teaching *about* problem solving—In this approach, the component parts of the process of problem solving can be taught and learned separately (that is, students are taught a number of strategies from which they can choose or which they should use in devising and carrying out their problem-solving plans). However, the component parts can be combined for students to solve real problems after they learn the parts. Some strategies that are typically taught includes looking for patterns, solving a simpler problem, and working backward. Teaching about problem solving also involves a great deal of explicit discussion of, and teaching about, how problems are solved. Hembree and Marsh (1993) also found that problem solving practice without direct instruction on strategies did not produce improvement.

- Teaching *for* problem solving—In this approach, the teacher concentrates on ways in which the mathematics being taught can be applied in the solution of both routine and non-routine problems (that is, teachers must be very concerned about students' ability to transfer what they have learned from one problem context to others). Students in this context are given real-world problems that can be solved using a newly developed skill. A typical challenge in this process is about how students are able to translate the real-world situation into a representation, for example, an equation that can be solved to answer the question in the problem.

- Teaching *via* problem solving—In this approach, the problems are valued not only as a purpose of learning mathematics but also as a primary means of doing so. Grouws and Cebulla (2000) found that students are able to learn new skills and concepts while they are solving challenging problems, and it is not necessary for teachers to focus first on skill development and then move on to problem solving (both can be done together). Schroeder and Lester (1989) claims that the goal of learning mathematics at this point is to transform certain non-routine problems into routine ones, that is, the learning of mathematics can be viewed as a movement from the concrete[15] to the abstract[16] (pp. 32-33).

Schroeder and Lester (1989) again propose two models of the process of solving mathematics problems, that is, models suitable for routine and non-routine problems respectively.

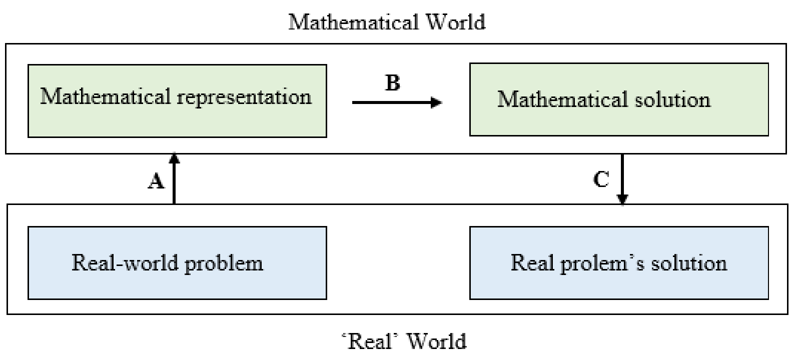



**Figure 2:** A simplistic model of the process of solving mathematics problems (Schroeder and Lester, 1989, p. 35).

They argue that the process of the model in Figure 2 begins with a problem posed in terms of the everyday physical reality, the problem solver first translates (arrow A) the problem into abstract mathematical terms, then operates (arrow B) on the mathematical representation to come to a mathematical solution of the problem, which is then translated back (arrow C) into the terms of the original problem. They point out that by the model in Figure 2, mathematics can be, and often is, learned separately from its applications. However, the difficulty with this proposed model is that it applies to routine problems better than to non-routine ones.

They again argue that for more challenging problems (process problems), the problem solver has no single already learned mathematical operation that will solve the problem. However, a non-routine problems demand usually more complex processes, such as planing,

[15]A real-world problem that serves as an instance of the mathematical concept or technique
[16]A symbolic representation of a class of problems and techniques for operating with these symbols

selecting a strategy, identifying sub-goals, conjecturing, and verifying that a solution has been found. In Figure 3, the Y arrows that points upwards indicates that the problem solver is learning to make abstract written records of the actions that are understood in a concrete setting, whereas the arrows pointing downward might also suggest that a problem solver who had forgotten the details of a mathematical procedure would be able to reconstruct that process by imaging the corresponding concrete steps in the world in which the problem was posed. They also point out that the collection of Y arrows in Figure 3 illustrate the correspondence between the process of solving the problem in concrete terms (labeled X) and the parallel, abstract mathematical process (labeled X'). In a nut shell, a problem solver can typically move back and forth between the two worlds—the real and the mathematical—as the need arises (pp. 35-36).

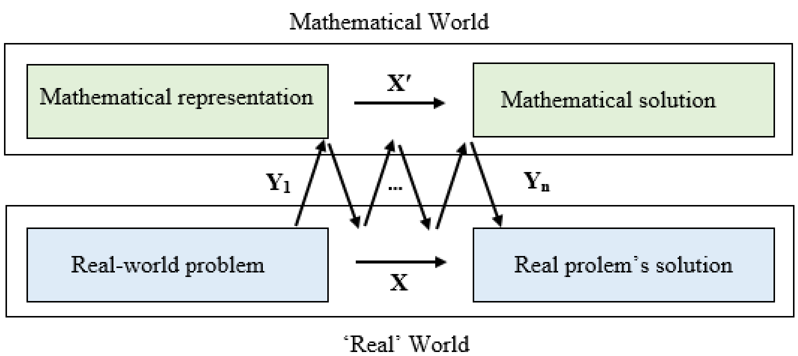



**Figure 3:** A model of the process of solving process problems (Schroeder and Lester, 1989, p. 36).

Schroeder and Lester (1989) asserted that the primary reason for school mathematics instruction is to help students understand mathematical concepts, processes, and techniques. They also believe that understanding should be the focus and goal of mathematics instruction, teachers, textbooks authors, curriculum developers, and evaluators instead of making problem solving the focus. On the other hand Hiebert et al. (1997) also believe that, if we want students to understand mathematics, it is more helpful to think of understanding as something that results from solving problems, rather than something we can teach directly.

Polya (2004) describes mathematical problem solving as finding a way around a difficulty or obstacle, and finding a solution to a problem that is unknown. He studied more on problem solving skills/strategies and articulated a problem solving process as one involving this series of phases which is interdependent: Understanding the problem, Devising a plan, Carrying out the plan, and Looking back.

- Understanding the problem—The student should understand the problem and also desire its solution. The problem however should be well chosen, that is, not too difficult and not too easy but natural and interesting. The verbal statement of the problem must be understood and in this context the student should be able to point out the principal parts of the problem, the unknown, the data and the conditions available.

- Devising a plan—One has a plan when he/she knows, or know at least in outline, which calculations, computations, or constructions he/she has to perform in order to obtain the unknown. Devising a plan is the process in which one finds the connection between the data and the unknown. In conceiving the idea of a plan, the idea may emerge gradually or after apparently unsuccessful trials and a period of hesitation, it may occur suddenly, in a flash, as a 'bright idea'. It is hard to have a good idea if a person has little knowledge of the subject, and impossible to have it if one has no knowledge. However, good ideas are based on past experience and formerly acquired knowledge.

- Carrying out the plan—Carrying out the plan is much easier than devising a plan and conceiving the idea of the solution which usually involves formerly acquired knowledge, good mental habits and concentration upon the purpose. The main point here is that the student should be honestly convinced of the correctness of each step.

- Looking back—Most students miss an important and instructive phase of their work due to the fact that, when they obtained the solution of the problem and written down the argument neatly, they intend shut their books and look for something else. Students could consolidate their knowledge and develop their ability to solve problems by looking back at the completed solution, by reconsidering and reexamining the results and the path that led to it. A student should however have good reasons to believe that his/her solution is correct. Nevertheless, errors are always possible, hence verification's are desirable (pp. 6-19).

Polya also argue and illustrate some set of questions he believed are important and that teachers need to ask students as they go through the four phases of problem solving. For example: What is the unknown? What is the data? What is the condition? Have you seen it before? Do you know a related problem? Can you see clearly that the step is correct? Can you prove that it is correct? Can you check the results? Can you derive the result differently? Can you use the result, or the method, for some other problem?

Mason et al. (2011) also did comprehensive work about the process of tackling a mathematical question. They came up with three phases that an individual goes through when tackling a mathematical question: the Entry phase, the Attack phase, and the Review phase. They explained that work in the Entry phase prepares the ground for an effective attack and it is therefore relevant that adequate time is devoted to it. The Entry phase is usually about formulating the question precisely and deciding exactly what the individual wants to do. On the other hand, one needs to handle the question in two ways: by absorbing the information given and by finding out what the question is really asking. Another activity which often takes place during the Entry phase is to make some technical preparations for the main attack, such as deciding on a notation or a means of recording the results of specializing[17]. Mason et al. claims that it is helpful to structure work in the Entry phase by responding to these three questions: What do I KNOW[18]? What do I WANT[19]? and What can I INTRODUCE[20]?

They further explain that in the Attack phase several different approaches may be taken and several plans may be formulated and tried out. The Attack phase involves conjecturing[21] and justifying your conjecture. Justifying ones conjecture also involves two different activities: seeking why—involves getting a sense of some underlying reason for the truth of your conjecture—and explaining why—involves convincing yourself and, more importantly, convincing others that you can justify your arguments. One major thing that usually happens during the Attack phase is STUCK. Everyone gets stuck and this cannot be avoided, however it is an honourable and positive state, from which much can be learned. Lastly, the Review phase is the time for looking back at what has happened in order to improve and extend ones thinking skills, and for trying to set ones resolution in a more general context. This phase also involves both looking back, to CHECK[22] what you have done and to REFLECT[23] on key events, and looking forward to EXTEND[24] the processes and the

[17]Trying some specific cases. By doing examples an individual makes the question meaningful to himself/herself and may also begin to see an underlying pattern in all the special cases which will be the clue to resolving the question completely.

[18]read the question carefully; specialize to discover what is involved; what ideas/skills/facts seem relevant?; do I know any similar or analogous questions?

[19]classify and sort information; be alert to ambiguities; specialize to discover what the real question is

[20]images, diagrams, symbols; representation, notation, organization.

[21]A conjecture is a statement which appears reasonable, but whose truth has not been established, that is, it has not been convincingly justified and yet it is not known to be contradicted by any examples nor is it known to have any consequences which are false.

[22]calculations; arguments to ensure that computations are appropriate; consequences of conclusions to see if they are reasonable; that the resolution fits the question.

[23]on key ideas and moments; on implications of conjectures and arguments; on your resolution: can it be made clearer?

[24]the results to a wider context by generalizing; by seeking a new path to the resolution; by altering

results to a wider context.

### 3.3.1 Algebraic word problem

Algebraic word problems runs through all levels of mathematics curriculum since it forms an integral part of mathematics learning. Verschaffel et al. (1994) argues that the importance of introducing word problems in schools is to train students in applying the formal mathematical knowledge and skills learned at school, in real-world situations. Verschaffel et al. (2000) define word problems as mathematical exercises that present significant background information on the problem as text, rather than in the form of mathematical notation. According to Kieran (2007), word problems requires practice in translating verbal language into algebraic language (mathematical notation). Although the introduction of word problems can help to bridge the divide between real life and the mathematics classroom, students' still encounter some challenges when solving them. Morales et al. (1985) argues that one of the most problematic areas of the mathematics curriculum involves the solution of word problems. They went on to claim that even though students have mastered the technical competencies of doing the mathematical operations involved in the word problems, they experience considerable difficulty with simple word problems that require application of these techniques.

Newman (1983) analyze errors found when students solve word problems and came up with these six potential areas of difficulty:

1. Difficulty in reading the text.
2. Difficulty in comprehending the text.
3. Lack of suitable strategies to handle the problem.
4. One may not be able to transform the information in the text into mathematical forms.
5. Lack of computational skills.
6. One may not be able to use computation results to solve the problems.

These errors can be put into three different context; comprehending word problems, translating word problems into equations, and the transfer of solving word problems.

some of the constraints.

| Problem | Algebraic solution | Arithmetic solution |
|---|---|---|
| To rent a car from Tiger costs \$100 per day and 20 cents per km. How far can I drive, if the most I can afford to pay is \$240? | Let $x$ be the number of kilometres that can be driven. $0.20x + 100 = 240$<br>$0.20x = 140$<br>$x = 140 \div 0.20$<br>$= 700$<br>I can drive up to 700km | \$240-\$100=\$140 (money to spend on kilometre charge). Cost per km is \$0.20. Number of kilometres that can be driven = money available ÷ cost per km = \$140 ÷ \$0.20 = 700. I can drive up to 700km |

**Table 2:** A Problem with the Mathematical Structure of a Linear Equation (Stacey and MacGregor, 1999, p. 27).

Comprehending word problems in written form, an individual needs to be able to read and understand the text that describes the task (Kyttälä and Björn, 2014). Lewis and Mayer (1987) argues that effectively solving a mathematical word problem does not depend only on the students' ability to perform the required mathematical operations but the extent to which students' are able to accurately understand the text of the word problem is relevant. Boonen et al. (2016a) also added that effectively solving a mathematical word problem and the understanding of the text are related in such a way that developing a deeper understanding of the word problem serves as a crucial step before the correct mathematical computations can be performed.

Roth (1996) in his investigation about the problematic nature of context when students were tasked to solve contextual word problems found that, students' find it quite complex when they move from a text (word problem) to an inscription, such as a data table, graph, comparison of means, or equation. He then argue that, the required transformation (eg. from text to equation) can be made when one sees the various inscriptions as mathematical objects in themselves that can be transformed into one another. Stacey and MacGregor (1999) on the other hand, argues that learning algebra requires making a sometimes difficult transition from the way of solving problems in arithmetic to a conceptually new algebraic way. They further point out that although translating word problems into an equation can be challenging, but the use of algebraic equations to solve problems about real situations is important. That is, problems relating to equations with the unknown on only one side (see Table 2, an example of a problem with both an algebraic and arithmetic solutions)—are easy to solve without algebra, but others—those relating to equations with the unknown on both sides—require hard thinking if algebra is not used (p. 28).

The transfer[25] of solving word problem is one of the challenges that students face when solving problems. Hung (2013, p. 27) defines learning transfer "as applying previously learned knowledge with various degrees of adaptation or modification of that knowledge in completing a task or solving problems". Cree and Macaulay (2000, pp. 1-2) on the other hand defines transfer of learning, in a broad way, "as prior learning affecting new learning or performance". Fuchs et al. (2004) argues that mathematical problem solving is a transfer challenge requiring children to develop schemes for recognizing novel problems as belonging to familiar problem types for which they know solutions. Reed et al. (1985) also reports that most students are not able to apply the problem solving skills they learned recently to solve similar algebraic word problems.

Despite the usefulness of transfer of learning, Cree and Macaulay (2000) argues that research to date has not empirically proved or disproved that transfer of learning exists. Lave and Wenger (1991) on the other hand also provides a critique of ideas of learning transfer that suggest that the mathematics learned in school is then applied to new situations in a different context. Monaghan (2014) argues that school mathematics in many instances is unfitting to out-of-school practices and that in some cases the problems that students encounter in out-of-school mathematics are only apparently similar to school mathematics problems and however in reality there is a range of explicit restrictions which makes school methods unsuitable and thus other methods are used. Lecoutre et al. (2004) also argues that students have difficulty in transferring knowledge and that transfer occurs only for the fraction of students who performed correctly on the training problems of the learning phase. The traditional teaching practice can also be a contributing factor in the failure of transfer. Hung (2013) argues that the issue of teaching and learning knowledge in abstract[26] forms somehow leads to the cause of students' failure to apply and transfer knowledge. The application of knowledge, however, requires more than just acquisition and comprehension of the knowledge.

Cree and Macaulay (2000) puts transfer into two categories, positive and negative. They point out that positive transfer occurs if what is learned in one context enhances learning in a different setting whilst negative transfer occurs if what is learned in one context hinders or delays learning in a different setting. Schunk (2012) on the other hand lists some types

[25] Using previously acquired knowledge and problem solving skills or strategies to solve familiar problems whose context are different from those problems solved before.

[26] In the classrooms or training settings, the focus of teaching is on the explanations of the concepts, principles, or theories, followed by demonstrating a few examples of applying the concept or principle to solve textbook types of problems and also the focus of learning is to memorize the definitions of the concepts and theories, as well as comprehend them.

of transfer (for example, near, far, and among others). He argues that in near transfer there is much overlap between situations (that is, the original and transfer contexts are highly similar) whilst in far transfer there is little overlap between situations (that is, the original and transfer contexts are dissimilar). Hung (2013) argues that far transfer presents many more challenges for students and also "far transfer requires a higher degree of modification of the original knowledge than near transfer to adapt to the requirements or constraints of the target learning transfer condition" (p. 29)

Although transfer of learning is a problematic construct, nevertheless this study focuses on transfer between solved algebraic word problems and new algebraic word problems with similar/different story context and similar/different equations. Reed (1998) gives four main categories of transfer which constitute the transfer of what an individual has learned about solving a word problem to other related word problems. In each category, one can observe a possible relation between an example problem during school practice and a test problem, and the idea underneath is whether the example problem and the test problem share a common story context and whether they share a common solution procedure (a common equation). The categories are:

- Equivalent problems—common story context and common solution procedure (similar context and similar equation).
- Similar problems—common story but different solutions (similar context and different equation).
- Isomorphic problems—different story context but a common solution (different context and similar equation).
- Unrelated problems—which share neither story context nor a common solution (different context and different equation).

The following algebraic word problems illustrates the relation between an algebraic word problem and its corresponding equivalent, similar, isomorphic, and unrelated problem. Algebraic word problem:

> The Kristiansand zoo and amusement park sells two kinds of tickets. Tickets for children cost 15kr. Adult tickets cost 40kr. On a certain day, 278 people entered the park. On that same day the admission fees collected totaled 7,920kr. How many children were admitted on that day? (adapted from Kushman et al., 2014, p. 271).

Equations:

$$\begin{aligned} x + y &= 278 \\ 15x + 40y &= 7920 \end{aligned}$$

Equivalent problem:

The restaurant near the Kristiansand zoo and amusement park have two different prices for their services. A meal for children cost 150kr and that of adults cost 400kr. On a certain day, 278 people entered the restaurant. On that same day the money received at the restaurant totaled 79,200kr. How many children were in the restaurant on that day?

Equations:

$$\begin{aligned} x + y &= 278 \\ 150x + 400y &= 79200 \end{aligned}$$

Similar problem:

The restaurant near the Kristiansand zoo and amusement park have two different prices for their services. A meal for children cost 150kr and that of adults cost 400kr. On a certain day, 560 people entered the restaurant but 282 of them went out because there was no table for them. On that same day the money received at the restaurant totaled 79,200kr. How many children were in the restaurant on that day?

Equations:

$$\begin{aligned} x + y &= 560 - 282 \\ 150x + 400y &= 79200 \end{aligned}$$

Isomorphic problem:

Three rulers and two pens cost 375kr. One ruler and one pen cost 143kr. Find the cost of one ruler?

Equations:

$$\begin{aligned} x + y &= 143 \\ 3x + 2y &= 375 \end{aligned}$$

Unrelated problem:

Currently the subscription to a gym for a single member is 1000kr annually while family membership is 1500kr. The gym is considering raising all membership fees by same amount. If this is done then the single membership will cost $\frac{5}{7}$ of the family membership. Determine the proposed increase?

Equation:

$$1000 + x = \frac{5}{7}(1500 + x)$$

Perkins et al. (1992) argues that near transfer is a transfer between very similar contexts (for instance when students in the process of taking an exam face a mix of problems of the same kinds that they have practiced separately in their homework) whilst far transfer refers to transfer between contexts that, on appearance, seem remote and alien to one another. However, taking into account context only regardless of the equation involve, then the equivalent and similar problems can be categorized as near transfer and also the isomorphic and unrelated problem can be categorized as far transfer. The difference or change of equation does not affect the classification of near (analogous) or far transfer (Catrambone and Holyoak, 1989; Reed et al., 1985; Gick and Holyoak, 1983, 1987). Reed (1998) argues that the transfer to isomorphic problems is a complex and difficult process and that it takes place under certain experimental circumstances and also after students have been involved in well designed forms of training. He also points out that in similar problems, students sometimes find it difficult to adapt solutions and often rely too much on the example solution failing to make the necessary adjustments.

The algebraic word problem tasks given to the students in this study are isomorphic to the word problems they usually solve at school. It has a different story context but a common solution (similar equation). The students have some specific techniques they use in solving algebraic word problems that involves simultaneous sets of two linear equations or linear systems with two variables, and that, the choice of the tasks used in the study will reveal the kind of technique(s) the students revert to if their usual technique fails. The transfer of problem solving between the word problems the student solve at school and the tasks they solve in this study will also reveal the specific errors made by the students, although the students have learned some specific technique(s) of doing the mathematical operations involved in the word problem.

#### 3.3.1.1 Categories of word problems .

Word problems can be put into two main categories; routine and non-routine problems (eg. see Table 3, an example of routine and non-routine problems). Routine problems usually involves finding a solution to word problems through a straightforward translation of the problem text into a mathematical model without the need for developing a conceptual understanding of the word problem context. It also involves the use of at least one of the four arithmetic operations and/or ratio when solving the problem. On the other hand, non-routine problem solution usually requires students to develop an adequate understanding of the situation described in the word problem text before deriving a mathematical model. This to some extent makes the solution process more complex and however, it appears to be more difficult than routine word problems (Elia et al., 2009; Boonen et al., 2016b).

Berry et al. (1999, p. 105) on the other hand views routine problems as problems "for which students may be expected to execute a rehearsed procedure consisting of a limited number of steps". They also argue that routineness is located in a question and by their analysis, students score substantially more marks on what they designate as routine parts of the questions given to students. Boaler (1997) also divides problems into the categories procedural and conceptual, which Berry et al. (1999) views as a form of the routine/non-routine division. Boaler (1997, p. 77) defines procedural problem as "those questions that could be answered by a simplistic rehearsal of a rule, method or formula". That is, these problems do not require a great deal of thought if the correct rule/method had already been learned. An example of procedural problems would be: "Calculate the mean of a set of numbers". If students had learned the method used in calculating the mean, then they do not have to decide upon a method to use. They again view conceptual problems, in contraposition, as problems for which "the use of some thought and rules or methods committed to memory in lessons would not be of great help in this type of question". An example of conceptual problems would be: "A shape is made up of four rectangles, it has an area of 220cm$^2$. Write, in terms of $x$, the area of one of the rectangles (a diagram was given)". In a nutshell, if a problem could be answered from memory alone,

| Type | Word Problem |
|---|---|
| Routine | Andreas has 8 pairs of socks and Kristoffer has 7 pairs of socks more. How many pairs of socks does Kristoffer have? (correct answer: 15). |
| Non-routine | Two pens and three rulers cost 139kr. One ruler and one pen cost 54kr. Find the cost of one ruler and one pen. (correct answer: One ruler cost 31kr and one pen cost 23kr). |

**Table 3:** An example of routine and non-routine problems.

it is procedural; if is also, or instead, requires thought then it's conceptual. Boaler (1997) argues that conceptual problems are more difficult (for students) than procedural problems, and descriptive statistics support this view.

Riley et al. (1983) in their work on the 'development of children's problem-solving ability in arithmetic', put word problems into four categories. These categories are *change*, *combine*, *compare* and *equalizing* (see Table 4 for an example of change, combine, compare and equalizing word problems). The equalizing category described by Riley et al. results from the work of Carpenter and Moser (1982). These four problem types are routine word problems since they usually involve the use of at least one of the four arithmetic operations. Riley et al. (1983) in their explanation noted that the change and equalizing problem categories describe addition and subtraction as actions that cause increases and decreases in some quantity. That is, from Table 4 under the change category, the initial quantity of Joe's three marbles is increased by the action of Tom giving Joe five more marbles. They also point out that the combine and compare problem categories involves a static relation between quantities. That is, from Table 4 under the combine category, there are two distinct quantities that do not change (these are Joe's three marbles and Tom's five marbles) and the problem solver is asked to consider them in combination. The difficulty level of the problems varies among the four problem categories and also the level of the children, for example, according to Riley et al. (1983), some compare problems are usually more difficult than some change problems for first-graders whilst some combine problems are in general more difficult than some change problems for kindergartners and also first-graders as well. Riley et al. (1983) argues that successful problem-solving performance by children depends on the conceptual knowledge or understanding for problem representation leading to the appropriate selection of a specific technique for solution. On the other hand,

| Action | Static |
|---|---|
| CHANGE:<br>Joe had 3 marbles. Then Tom gave him 5 more marbles. How many marbles does Joe have now? | COMBINE:<br>Joe has 3 marbles. Tom has 5 marbles. How many marbles do they have altogether? |
| EQUALIZING:<br>Joe has 3 marbles. Tom has 8 marbles. What could Joe do to have as many marbles as Tom? (How many marbles does Joe need to have as many as Tom?) | COMPARE:<br>Joe has 8 marbles. Tom has 5 marbles. How many marbles does Joe have more than Tom? |

**Table 4:** Types of word problems (Riley et al., 1983, p. 160).

Cummins et al. (1988) uses the same word problems described by Riley et al. and suggests that much of the difficulty children experience with word problems can be linked to the difficulty in comprehending abstract or ambiguous language. Cummins et al. also found solution errors by the children to be correct solutions to miscomprehended problems, that is, word problems that combined abstract or ambiguous language tends to be miscomprehended more often than ones with simpler language, whilst correct solutions in their work was associated with accurate recall of the problem structure and with appropriate question generation.

Although the work of Riley et al. (1983) is mainly on arithmetic word problems, they also argue that their principles and findings are also relevant for word problems in algebra, and that students uses little conceptual knowledge, focusing instead primarily on syntactic information to translate the English problem statement directly into a corresponding set of equations, when solving algebraic word problems.

Caldwell and Goldin (1979, 1987) also put word problems into four categories as they investigate the variables affecting word problem difficulty in both elementary and secondary school mathematics. These categories are *abstract factual* (AF), *abstract hypothetical* (AH), *concrete factual* (CF), and *concrete hypothetical* (CH) (see Table 5, an example of AF, AH, CF and CH word problems).

Caldwell and Goldin (1979) define and explain the four problem categories as follows;

> An abstract word problem is defined as a word problem involving a situation that describes only abstract or symbolic objects, whereas a concrete word problem is defined as one describing a real situation dealing with real objects. For example, a problem about digits in a number is abstract, whereas a problem about baseballs is concrete. A factual problem is defined to be one that merely describes a situation. A hypothet-

| Type | Word Problem | Comments |
|---|---|---|
| Abstract Factual | There is a certain given number. Three more than twice this given number is equal to 15. What is the value of the given number? (correct answer: 6) | No change is described. |
| Abstract Hypothetical | There is a certain number. If this number were 4 more than twice as large, it would be equal to 18. What is the number? (correct answer: 7) | The number is not really 4 more than twice as large. |
| Concrete Factual | Susan has some dolls. Jane has 5 more than twice as many, so she has 17 dolls. How many dolls does Susan have? (correct answer: 6) | No change is described. |
| Concrete Hypothetical | Susan has some dolls. If she had 4 more than twice as many , she would have 14 dolls. How many does Susan really have? (correct answer: 5) | Susan does not really have 4 more than twice as many dolls. |

**Table 5:** Types of word problems (Caldwell and Goldin, 1979, p. 325).

> ical problem is one that not only describes a situation but also describes a possible change in the situation. This change does not really occur within the context of the problem. In solving the hypothetical problem, the problem solver must consider not only the situation that occurs within the context of the problem but also the described alteration that does not occur. (p. 325)

Caldwell and Goldin (1979) in their research context define abstract, concrete, factual and hypothetical, however based on these definitions we can also define or explain AF, AH, CF and CH. That is: AF are word problems involving situations that describe only abstract or symbolic objects and again has the property that it also merely describes a situation; AH are word problems involving situations that describe only abstract or symbolic objects and also has the property that it does not only describes a situation but describes a possible change in the situation as well; CF are word problems that merely describe real situations dealing with real objects; and CH are word problems that not only describe situations but also describes possible changes in a real situation dealing with real objects.

Caldwell and Goldin (1979) argues that concrete problems (factual and hypothetical) are the least difficult which is followed by abstract hypothetical and abstract factual problems at the elementary school level. Caldwell and Goldin (1987) on the other hand again argues that at all levels, concrete hypothetical problems are more difficult compared to concrete factual problems in a large effect, however at the elementary school level and to a lesser extent at the junior high school level, abstract hypothetical problems are actually less difficult than abstract factual problems. They further point out that the greater difficulty of hypothetical problems occurs almost exclusively in concrete contexts with the reason that the students' internal representations are such that the distinction between the situation described and the hypothetical change in the situation is more difficult to maintain.

The AF, AH, CF and CH types of word problems are used in this current study to examine the students, looking for the underlying conceptual understanding or knowledge as the students justify their strategies when solving these types of word problems.

#### 3.3.1.2 Setting up equations .

In Newman's (1983) error analysis, the two common errors among others made by students when solving word problems are: the lack of computational skills; and the inability to transform the information in the text into mathematical forms. Kieran (2007) on the other hand, in general, points out that there are two phases involved in the solving of word problems, that is the setting up of an equation to represent the relationships inherent in the word problem and the actual solving of the equation. Setting up an equation from word problems is much relevant since one can have an accurate computation if the right equation is setup.

Polya (2004) in his book 'How to solve it' gives a brief account on setting up equations from word problems. He argue that "to set up equations means to express in mathematical symbols a condition that is stated in words; it is translation from ordinary language into the language of mathematical formulas" (p. 174). Setting up equations comes along with some difficulties which is usually the difficulties of translation. To overcome such difficulties Polya (2004) argues that one must thoroughly understand the condition and also be familiar with the forms of mathematical expression. He further emphasized that in easy cases the verbal statement splits almost automatically into successive parts (in which each part can be immediately written down in mathematical symbols) whilst in a more difficult cases, the condition in the problem has parts which cannot be immediately translated into mathematical symbols. If the later is the case then one must pay less attention to the verbal statement and concentrate more upon the meaning. Nevertheless in both easy or difficult cases, one must firstly understand the condition.

Polya (2004) also illustrates some examples of setting up an equation for both the easy and difficult cases. These examples are presented below;

*Example 1: Find two quantities whose sum is 78 and whose product is 1296* (p. 175).

**Stating the problem**

| in English | in algebraic language |
|---|---|
| Find two quantities | $x, \quad y$ |
| whose sum is 78 and | $x + y = 78$ |
| whose product is 1296 | $xy = 1296.$ |

**Figure 4:** Setting up an equation for an easy case (Polya, 2004, p. 175).

One can observe from Figure 4 (an example of setting up an equation of an easy case, where the verbal statement is split into appropriate parts in one side with the corresponding algebraic signs on the opposite side) that the verbal statement splits almost automatically into successive parts for which each of the parts can be immediately written down in mathematical symbols.

*Example 2: Being given the equation of a straight line and the coordinates of a point, find the point which is symmetrical to the given point with respect to the given straight line* (pp. 176-177).

The word problem example given by Polya is a plane analytic geometry problem. Polya outlines three questions that must be considered before tackling the problem: *What is the unknown?* That is, a point with coordinates $p$ and $q$; *What is given?* That is, the equation of a straight line, $y = mx + n$, and a point with coordinates $a$ and $b$; and *What is the condition?* That is, the points $(a, b)$ and $(p, q)$ are symmetrical to each other with respect to the line $y = mx + n$. Polya further argues that after going through the three outlined questions, one reaches the essential difficulty which is to divide the condition into parts, however one must ensure that these decomposition into parts of the condition must be fit for analytic expression. The definition of symmetry must be understood in order to find such a decomposition but also one must keep an eye on the resources of analytic geometry. That is, in this situation: What is meant by symmetry with respect to a straight line?; and What geometric relations can we express simply in analytic geometry? He also points out that one must concentrate upon the first question but however should not forget the second. Figure 5, gives a detailed outline of the decomposition of the condition, where we have the verbal statement on one side and the corresponding algebraic signs on the opposite side. According to Riley et al. (1983) must students uses little conceptual knowledge , focusing instead primarily on how to translate the English problem statement directly into a corresponding set of equations, when solving algebraic word problems. It is

| | |
|---|---|
| The given point | $(a, b)$ |
| and the point required | $(p, q)$ |
| are so related that | |
| first, the line joining them is perpendicular to the given line, and | $\frac{q-b}{p-a} = -\frac{1}{m}$ |
| second, the midpoint of the line joining them lies on the given line. | $\frac{b+q}{2} = m\frac{a+p}{2} + n.$ |

**Figure 5:** Setting up an equation for a difficult case (Polya, 2004, p. 177).

with this very approach that students usually encounter difficulties when dealing with this difficult cases described by Polya.

### 3.3.2 Mathematical modelling

Application word problems are another kind of non-routine word problems that require realistic considerations (Verschaffel et al., 2000, 2009, 2010). Verschaffel et al. (2000) argue that the solution to application word problems requires students to develop a proper situation model and not just directly translating the text into mathematical symbols. This type of word problem is what Polya (2004) classifies as a difficult case in his presentation about setting up equations from word problems. The open nature of the application word problems require students to develop a model for the situation and the process of developing such a model is termed mathematical modelling. Mathematical modelling is the modelling of mathematical ideas and its inclusion in the school curricular helps to bridge the gap between real life and the mathematics classroom. According to Erling et al. (2016), it is important to remind students that it is through mathematical modelling that a lot of mathematics is used in careers beyond school, to model what happens (or may happen) in the world so that complex situations can be manipulated more simply and solutions to problems found.

One of the pioneers in the field of applications and modelling in mathematics education, Pollak (2007), describes mathematical modelling as a process of formulating a problem from outside of mathematics, understanding the problem, visualizing, and solving it. Garfunkel and Montgomery (2016, p. 8) on the other hand defines mathematical modelling as "a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena". Confrey and Maloney (2007) also proposed a more comprehensive description of mathematical modelling, stating that;

> Mathematical modelling is the process of encountering an indeterminate situation, problematizing it, and bringing inquiry, reasoning, and mathematical structures to bear to transform the situation. The modelling produces an outcome - a model - which is a description or a representation of the situation, drawn from the mathematical disciplines, in relation to the person's experience, which itself has changed through the modelling process. (p. 60)

The mathematical experiences of students are improved through the modelling processes since through these processes that they practically applied the mathematics they've studied. Hernández et al. (2017) argues that

> students who have engaged in the modelling process appreciate the opportunity to use their own ideas in creating a mathematical solution to a real-world problem and have experiences that help them regardless of what college or career path they follow. (p. 342)

Teachers are also to help their students by transforming more of the mathematics problems to modelling problem, as it helps in bridging the divide between real life and the mathematics classroom. Garfunkel and Montgomery (2016) argues that, when labels are added to mathematics problem they become a word problem and when meaning is added to a word problem they then become an application problem and lastly when interpretation is added to an application problem, we end up having a modelling problem (see Figure 6, an illustration of a way of transforming a mathematics problem into a modelling problem). Erling et al. (2016) also argues that one of the difficult ideas encountered in word problems is about translating the words into algebra and algebra into words. They further point out that students should be encourage to make their own word problems from some algebraic equations, that is by modifying some expressions or equations (mathematics problem) to make them relevant. Kajamies et al. (2010) on the other hand conducts an intervention study aim at developing the mathematical word problem solving of low performing students. They carefully design word problems (application problems) combine with intensive, systematic, and explicit teacher scaffolding, which help in the improvement of students' word problem solving performance. The shift from word problems to application problem can contribute, to some extent, in the problem solving performance of students. Verschaffel and De Corte (1997) also use application word problems that help in promoting students' realistic mathematical modelling in an intervention study. They suggest that it is possible to improve realistic modelling and reasoning skills when we include more application problems and less word problem (word problems usually offered in traditional mathematics classroom) into mathematics lessons.

Mathematical modelling tasks or problems should be interesting and meaningful that majority of the students will want to participate or solve them. Educators and teachers must also ensure that the modelling task given to students offers a low floor allowing even a low performing student to engage with minimal prerequisite knowledge and skills, and a high ceiling providing opportunities to explore more complex concepts. In this regard, the task does not limit high performing students to fully explore their potentials and also not very difficult for low performing students that they can not comprehend.

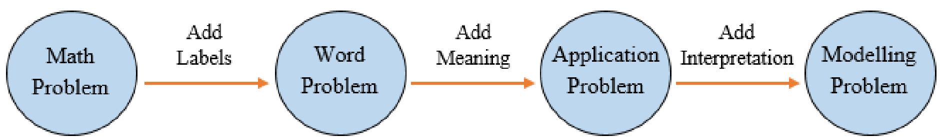



**Figure 6:** One way of transforming a mathematics problem into a modelling problem (Garfunkel and Montgomery, 2016, p. 12).

#### 3.3.2.1 The mathematical modelling process .

In mathematical modelling, students usually go through some processes in deriving a suitable model, they at first try to understand the situation and then find some mathematical representation which they solve afterwards and lastly followed by verifying whether that is the suitable solution or not. Garfunkel and Montgomery (2016) describes some components of the modelling processes (see Figure 7, an illustration of the components forming the modelling process). They also argue that the modelling process is often pictured as a cycle, since one frequently needs to come back to the beginning and make new assumptions in order to get closer to a usable result. Giordano et al. (2013) on the other hand comprehensively presents a procedure that is helpful in constructing models. The various steps (components) in the procedure is as follows;

- Identify the problem: What is it you would like to do or find out?
- Make assumptions: Capture the important factors influencing the problem that has been identified. One should assume relatively simple relationships.
  - classify the variables: What things influence the behavior you identified in the first step (dependent and independent variables).

- determine interrelationships among the variables selected for study: Additional simplifications (sub-models) before one can hypothesize a relationship between the variables.

- Solve or interpret the model: Put together all the sub-models to see what the model is telling you.
- Verify the model: Before you use the model, you must text it out.
  - Does it address the problem?
  - Does it make common sense?
  - Test it with real-world data?
- Implement the model: Unless the model is placed in a 'user-friendly' mode, it will quickly fall into disuse. Explain the model for users to understand.
- Maintain the model: One needs to know if the original problem has changed in any way or have some previously neglected factors become important?

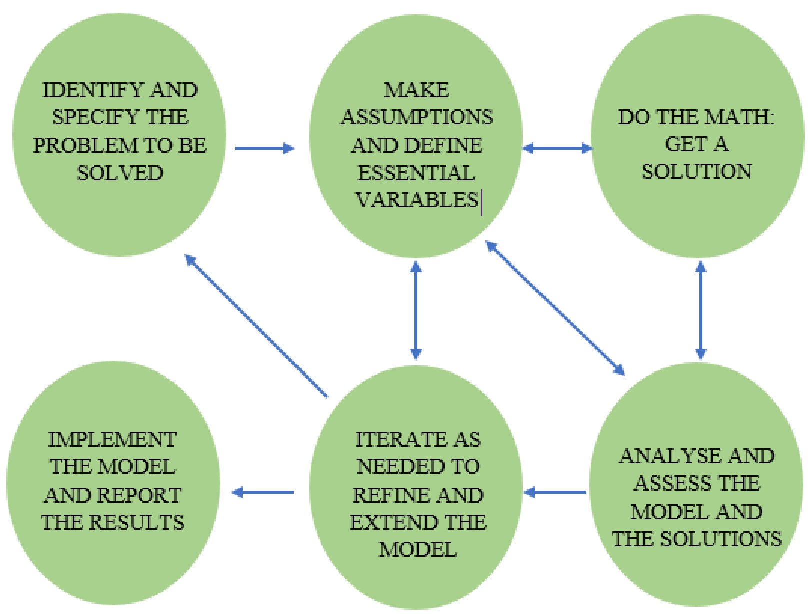



**Figure 7:** A mathematical modelling process (Garfunkel and Montgomery, 2016, p. 13).

Giordano et al. (2013) also argues that one should consider the feasibility of his/her model by considering these properties;

- Fidelity: The preciseness of a model's representation of reality.

- Cost: The total cost of the modelling process.
- Flexibility: The ability to change and control conditions affecting the model as required data is gathered.

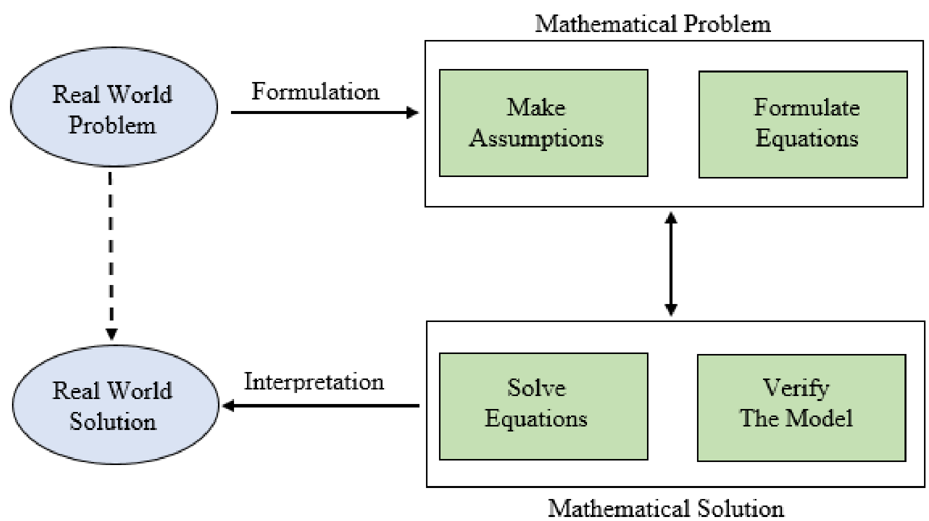



**Figure 8:** A simple view of the mathematical modelling process (adapted from Ang, 2001, p. 64).

A simpler version of the mathematical modelling process adapted from Ang (2001) is shown in Figure 8. In this process, there is a conversion of the real world problem into a mathematical problem through the establishment of some assumptions (important factors influencing the problem) and the formulation of equations. The mathematical problem can then be solved using whatever known techniques (depending on one's mathematical skills) to obtain a mathematical solution, that is, solving the equations and verifying one's model. This solution is then interpreted and translated into real terms (that is a real world solution), however it is also important to employ the use of the various mathematical representations[27] (tables, graphs, equations and among others) to explain the model (solution) for the users to have a better understanding of the model.

#### 3.3.2.2 Modelling at high school and college levels .

Mathematical modelling can be exercised at any school level, yet majority of the research concentrates on high school, college and undergraduate levels. Niss et al. (2007, p. 5) argues that in one category mathematical modelling at these higher levels of education, the learners "focuses on learning mathematics so as to develop competency in applying

[27]Mathematical representation is defined and explained in section 3.3.3

mathematics and building mathematical models for areas and purposes that are basically extra-mathematical". In another category, they argue that applications and modelling may also be a vehicle that facilitates and supports the students' learning of mathematics as a subject. The development of such competencies in both categories requires the inclusion of mathematical modelling activities explicitly in the school curriculum, for the purpose of teaching and learning of mathematics. According to Artaud (2007), modelling is an excellent method in making obvious the mathematics that is implicit in the real world and that it is very important for all students. Garfunkel and Montgomery (2016) also argues that models in the high school setting

> can be used as motivation for learning new techniques and new content; small modelling activities can be used to reinforce new concepts and to illustrate their applications; more extended modelling activities help students pull together ideas from different parts of a course and from different courses. (p. 45)

The mathematics tasks at school are usually close-ended task that requires a specific technique. Such tasks are sometimes uninteresting, I remember asking myself the question "when am I ever going to use this in the real world" as I continuously engaged in close-ended tasks that requires the use of a specific technique at the high school level. Kolis (2011) argues that the study of mathematics is not about memorizing math facts, theorems and solving proofs, and that when mathematics is taught with the focus only on the isolated facts, it lacks context and any connection to students' lives outside of school. When the mathematics taught at school has some connection with the students' lives, they might come to realize the importance of mathematics to their daily lives. A study by Matthews (2018), shows that mathematics teachers infrequently connect their instruction to the real world. They further suggests that teacher's messages about the real world relevance of mathematics matters in shaping how students value mathematics.

Despite the importance of modelling in the school curriculum, Stillman (2007, p. 464) argues by citing Artaud that, "at the upper secondary level, especially in some European countries, there is a strong bias against mathematical modelling as the prevalent attitude is that high level mathematics (theorems, proofs, formulas, and among others) is what is important". Artaud (2007) on the other hand argues that, the teaching process must be accorded extra time if mathematical modelling is added to the ordinary didactical system. However, this would be difficult to obtain in the general teaching system, nevertheless modelling could perhaps be provided for students who are supposed to need it, for instance, engineering students and among others.

| Representations | Advantages | Disadvantages |
|---|---|---|
| Verbal | • Usually used in posing a problem and is also needed in the final interpretation of the results obtained in the solution process.<br>• Creates a natural environment for understanding its context and for communicating its solution. | • The use of verbal language can also be ambiguous and elicit irrelevant or misleading associations. |
| Numerical | • Familiar to students at the beginning algebra stage.<br>• Numerical approaches offer a convenient and effective bridge to algebra and frequently precede any other representation.<br>• The use of numbers is important in acquiring a first understanding of a problem and in investigating particular cases. | • Its lack of generality can be a disadvantage.<br>• A numerical approach may not be very effective in providing a general picture. |
| Graphical | • Effective in providing a clear picture of a real valued function of a real variable.<br>• They are intuitive and particularly appealing to students who like a visual approach. | • Lack the required accuracy.<br>• Frequently presents only a section of the problem's domain or range. |
| Algebraic | • It is concise, general, and effective in the presentation of patterns and mathematical models.<br>• The manipulation of algebraic object is sometimes the only method of justifying or proving general statements. | • An exclusive use of algebraic symbols may blur or obstruct the mathematical meaning or nature of the represented objects and may cause difficulties in some students interpretation of their results. |

**Table 6:** Advantages and disadvantages of the various mathematical representations (Friedlander and Tabach, 2001, p. 173–174).

### 3.3.3 Mathematical representation

The two phases involved in the solving of algebraic word problems described by Kieran (2007), is the setting-up of an equation to represent the relationships inherent in the word problem and the actual solving of the equation. Most students encounter a lot of challenges to arrive at a desired answer or solution as they go through these two phases. Kieran (1992) argues that the presentation of algebra (as the study of expressions and equations) by a vast number of teachers and researches can pose serious obstacles in the process of effective and meaningful learning. As a pressing concern, mathematics educators, National Council of Teachers of Mathematics (2000), recommends that from the initial stages of learning algebra, students must be encourage to use the various (mathematical) representations.

Friedlander and Tabach (2001, p. 173) argues that, "the use of verbal, numerical, graphical, and algebraic representations has the potential of making the process of learning algebra meaningful and effective". They also point out that each representation has it own advantages and disadvantages, however their combined use can cancel out the disadvantages and prove to be an effective tool (see Table 6 for an example of the advantages and disadvantages of the various mathematical representations).

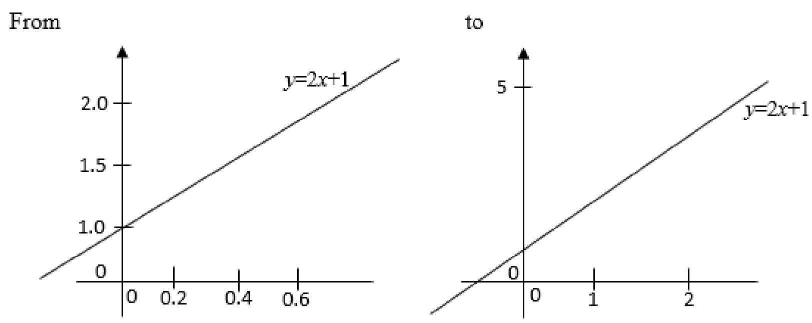
National Council of Teachers of Mathematics (2000, p. 281) on the other hand argues that, "students will be better able to solve a range of algebra problems if they can move easily from one type of representation to another". Since the use of representations improves the understanding of students, it is important for teachers and researches to design a task that will encourage the simultaneous use of several representations. For example designing an algebraic word problem in the area of mathematical modelling (open-problems) where students will have the opportunity to use which ever representation they want and also moving from one type of representation to another. Friedlander and Tabach (2001, p. 184) again in their analysis point out that, "the choice of a representation can be the result of the task's nature, personal preference, the problem solver's thinking style, or attempts to overcome difficulties encountered during the use of another representation".

According to Duval (1999, 2006), the transformation between different representations can be divided into two categories;

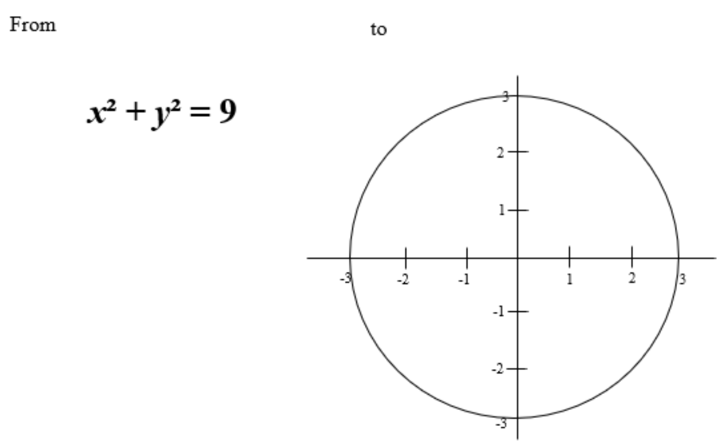
- Transformation within the same register[28] (Treatments)
  - Example 1:

$$\text{From} \quad y = 2x + 5 \quad \text{to} \quad y - 2x - 5 = 0$$

  - Example 2:




- Transformations between different registers (conversions)
  - Example:




Duval (1999) argues that, being able to shift between representations is important in learning, so that students avoid confusing the mathematical object with it's representation. Duval (2006, p. 112) also argues that "conversion is a representation transformation, which is

[28]A certain type or category of semiotic representations, Example: graphs, tables (numerical), verbal description, algebraic expressions (formulae)

more complex than treatment because any change of register first requires recognition of the same represented object between two representations whose contents have very often nothing in common". Duval (1999, p. 10) puts conversion into two categories, congruent or non-congruent (that is any conversion can be congruent or non-congruent). "When a conversion is congruent the representation of the starting register is transparent to the representation of the target register" and non-congruent is the opposite. One can view congruent conversion as an easy translation unit to unit and non-congruent conversion could also be very challenging. He further argues that the congruence or the non-congruence of any conversion depends on its direction and that a conversion can be congruent in one direction and non-congruent in the opposite direction. An example is moving from fractions to decimals and from decimals back to fractions: $\frac{1}{6} \rightarrow 0.1666666666666666 \approx 0.167$ and $0.167 \rightarrow \frac{167}{1000} \neq \frac{1}{6}$.

Janvier (1987) on the other hand came out with a table (see Figure 9, a table with the transformation from one representation to the other with the required translation skills) with different representations of functions. The table pinpoints the different translation skills (translation processes[29]) required in order to be able to move from one representation to another. Janvier (1987) also argues about the direct and indirect translation from one

| From \ To | Situations, Verbal Description | Tables | Graphs | Formulae |
|---|---|---|---|---|
| Situations, Verbal Description | | Measuring | Sketching | Modelling |
| Tables | Reading | | Plotting | Fitting |
| Graphs | Interpretation | Reading off | | Curve fitting |
| Formulae | Parameter Recognition | Computing | Sketching | |

**Figure 9:** Janvier Table: Translation Processes (Janvier, 1987, p. 28).

representation to the other. The direct translation is what we actually see in Figure 9, whilst the indirect translation adds a few arrows to account for alternative ways to achieve translations. For example, the translation 'table → formula' is often carried out as 'table

[29] The psychological processes involved in going from one mode of representation to another, for example, from equation to a graph

$\rightarrow$ graph $\rightarrow$ formula' and 'formula $\rightarrow$ graph' as 'formula $\rightarrow$ table $\rightarrow$ graph'.

The mathematical modelling task used in this study gives the students the opportunity to use the various mathematical representations. In particular, the task requires the translation from a verbal description or situation to an algebraic expression (formulae) by the process of modelling. It also requires the use of verbal description in the interpretation of the final results obtained in the solution process, and some students may further want to make their solution more clearer by the use of graphical representation. This will require the translation from an algebraic expression (formula) to the graphical representation by computing some real numbers they insert in the formulae and plotting this numbers (numerical/table) to obtain the desired graph. In a nutshell, the task offers students' the opportunity to move easily from one type of representation to another.

The methodology of this research study is discussed in the next chapter.

# 4 METHODOLOGY

This chapter presents the methodology of the study. Firstly, the case study strategy is discussed. A presentation of the research design which entails the research method, participant selection, data collection, data management and the strategy for analysis then follows. Finally the ethical considerations, validity and trustworthiness of the study is presented.

## 4.1 The Case Study Strategy

Bogden and Biklen (1982, p. 58) defines a case study as "a detailed examination of one setting, or one single subject, or one single depository of documents, or one particular event". Bryman (2016) on the other hand argues that the basic case study entails the detailed and intensive analysis of a single case. These case include research on: a single community; a single school; a single family; a single organization; a person; a single event. Case studies may be single or multiple. The unit in this study is a one group of first year upper secondary school students, in the Southern part of Norway, taking the 1T mathematics course. The selection of these group of 1T students is a case study on the issues on modelling activities in both the Nordic and European context. In this case study, there is an investigation of both students' conception and performance in two different tasks (algebraic word problems and a mathematical modelling task).

Paré and Elam (1997) argues that case study research strategy makes the capture and understanding of context possible. Zainal (2007) on the other hand argues that, detailed qualitative accounts often produced in case studies help to explain the complexities of real life situations which may not be captured through experimental or survey research. Despite the advantages, case study research has been subject to criticism on the grounds of non-representativeness and lack of statistical generalisability. Yin et al. (1984) points out that, case studies are often tagged as difficult to conduct and also produces a massive amount of documentation (in particular, case studies of an ethnographic or longitudinal nature). Zainal (2007) also argues that case studies provide very little basis for scientific generalization since they use a small number of subjects.

As the research questions requires students justification and interpretations, it is believed that a case study approach is the appropriate research strategy for this study. The research questions could be answered using a survey designed for the purpose of generalization. In this regard, one will consider a good number of upper secondary school students in different schools and find out statistically their performance in both the algebraic word problem and mathematical modelling task. However, this might not reveal in detail the unique experiences of individuals and the kind of conceptions they have.

## 4.2 Research Design

The research design gives the framework which is created to find answers to the research questions. The design of this study includes the research method, description of the participants, the data collection and methods, data management and strategy for the analysis.

The researcher takes the position of interpretive research paradigm in this study. Walsham (2006, p. 320) defines that "interpretive methods of research start from the position that our knowledge of reality, including the domain of human action, is a social construction by human actors". Thanh and Thanh (2015, p. 24) argues that "interpretive paradigm allows researchers to view the world through the perceptions and experiences of the participants". Bryman (2016), on the other hand, argues that interpretivism is characterized by explanations and interpretations through appropriate frameworks. Bryman also notes that there is double interpretation, that is the researcher providing an interpretation of others' interpretations and also a third level of interpretation whereby the researcher's interpretations have to be further interpreted in terms of the concepts, theories, and literature of a discipline. The interpretive paradigm again recognizes that the researcher has an impact on the findings.

### 4.2.1 Research method

A qualitative research method was used in this study. Bryman (2016) defines qualitative research method as a research method that usually spotlight words rather than quantification in the collection and analysis of data. Qualitative research has characteristics that are appropriate for small samples but it's outcomes are not quantifiable. Miles et al. (1994) list some features of qualitative research:

- It offers a rich description and analysis of a research subject.
- The researcher may only know roughly in advance what he/she is looking for.

- The researcher is the data gathering instrument (The researcher becomes an instrument through the relationships he/she builds with research participants).
- The data is more 'rich', time consuming, and less able to be generalized.

Bryman (2016) also outlines the main steps involve in qualitative research:

1. General research question(s)
2. Selection of relevant site(s) and subjects.
3. Collection of relevant data.
4. Interpretation of data.
5. Conceptual and theoretical work.

Bryman (2016) provides some critique of qualitative research. That is, qualitative research is: subjective; difficult to replicate; has problems of generalization; and lacks transparency.

The first step in qualitative research described by Bryman is general research question(s). In this study the following research questions are addressed:

1. How do upper secondary students' justify their strategies for solving algebraic word problems.
2. How do these students interpret their findings after solving an algebraic word problem in a form of mathematical modelling.

These research questions influenced the selection of the site and subjects, and also the collection of data.

### 4.2.2 The participants

Twenty three out of 26 students participated in this study. These students were first year students aged between 14-16 in an upper secondary school located in the Southern Norway. The students were put into seven different groups of about 3-4 persons in a group by their teacher. These groupings was as a results of their performance from the lower secondary level. Each group consist of the three categories: high-performing students; medium performing students; and low-performing students. The students follow an inquiry-based model (during class activities) since fifty percent of them are preparing to take the International Baccalaureate programme the following year whilst the others follow the normal Norwegian programme.

Students have already studied geometry and trigonometry, algebra and some sections of function analysis before the data collection of this study (that is from 1.1 - 3.6, see Appendix C on page 154 for the summary of the 1T course content). In a normal class section, the teacher gives the students some introductory tasks (see Appendix F on page 181, for an example of introductory task on simultaneous sets of two linear equations). The students work in their respective groups to complete the introductory task without the teacher's help. At the end of this process, the teacher summarizes all their findings on the board for the purpose of generating a general rule for solving the tasks. The teacher again gives the students some set of questions to solve after the introductory questions, and at this stage they work individually whilst still seated in their respective groups. The teacher walks around the class whilst the students solve these new questions and also give some hints to individuals within the group that calls for help or get stuck.

The students willingly volunteered to participate in this study. These group of 1T students were chosen because of the acquaintance of the researcher with their mathematics teacher. Seven participants from the various group willingly volunteered to participate in the interviews. Pseudonyms are provided for each of the seven participants, particularly 5-letter names were given to these students (Bjørg, Julie, Hilde, Eirik, Tonje, Helge and Arvid).

### 4.2.3 The data collection

In answering the research questions:

- How do upper secondary students' justify their strategies for solving algebraic word problems?
- How do these students interpret their findings after solving an algebraic word problem in a form of mathematical modelling?

both interviews and students' worksheets were the main source of empirical data. Two different tasks were given to the students who were instructed to solve the tasks in as much detail as possible so that the researcher can understand how they understood the question, why they think their solution is correct and how they interpret their final results. The students worked in groups to solve the various task (7 groups of about 3-4 persons per group). The interview method was also used to find out students' conception about both task. One person from each group was interviewed. Table 7 presents the time-line of the research process.

| Time frame | Procedure | Students involve |
|---|---|---|
| Last week of September 2018 | Students were informed about the research study. | 26 students |
| NSD approval of data collection (first week of November) | | |
| Last week of November 2018 | The participants signed the consent forms. | 23 students |
| First week of December 2018 | The students solve both the algebraic word problems (40min) and the mathematical modelling task (45min). | 23 students |
| Second week of December 2018 | Individual interviews | 7 students |
| In-Depth analysis of interviews and students' worksheets. | | |

**Table 7:** Time-line of the research process ($*$ NSD = The Norwegian Centre for Research Data).

#### 4.2.3.1 The algebraic word problems and the modelling task .

The algebraic word problem tasks which were given to the students were adapted from Caldwell and Goldin (1987) (see Figure 10, the algebraic word problem task used in the study). The four categories of word problem (Abstract factual, Abstract hypothetical, Concrete factual and Concrete hypothetical) were considered and for that reason four word problems in theses categories were given to these students. Task 1, 2, 3, and 4 are concrete factual, concrete hypothetical, abstract factual and abstract hypothetical respectively. These questions are no different from the algebraic word problems that the students solve at school. The algebraic word problem tasks used in the study are similar and also isomorphic (as described by Reed (1998)) to the problems that are usually solved by the students at school.

To answer the first research question, we first answer the questions: What is a strategy?; and How do students justify their strategies? Strategy in this study is explained as a plan of action designed by the students to achieve the desired solution of an algebraic word problem. In this regards, the students particularly transform the algebraic word problem into a linear equation or simultaneous set of two linear equations and then use addition/elimination or substitution method to solve it. The justification of students' strategies in this study is about how they mathematize the algebraic word problems. For a student to justify his/her strategy for solving an algebraic word problem, the student goes through the process of mathematizing the word problem as they use known techniques to solve the equation(s) derived from the word problem. In this study Polya's (2004) four

**Algebraic Word Problem**

*Instructions: Solve the following word problems in as much detail as possible so that I can understand how you understood the question, why you think your solution is correct, and how you interpret your final results.*

1. Marius, a young farmer, has 13 more hens than goats. Since hens have two legs each, but goats have four legs each, all together the animals have 146 legs. How many animals in all does Marius have?

2. There are seven more girls in a Mathematics class than boys. If there were thirteen times as many girls and twice as many boys, there would be 346 pupils. How many pupils are there in the Mathematics class?

3. The value of a given number is fifteen more than the value of a second number. The sum of two times the first number and four times the second number is 162. What is the sum of the two numbers?

4. A given number is six more than a second number. If the first number were four times as large and the second two times as large, their sum would be 126. What is the first number?

**Figure 10:** Algebraic word problem task (adapted from Caldwell and Goldin 1987, p. 189).

interdependent phases of problem solving is considered as a measure of mathematizing the algebraic word problem. That is, mathematizing the algebraic word problem involves: Understanding the word problem; Devising a plan (that is, translate the problem using variables and setting up an algebraic equation); Carrying out the plan (that is, solving the equation using a known technique); Looking back (verifying the answer and also presenting the final answer).

A Polyaian way of mathematizing Task 1 in Figure 10 is illustrated below:

Linear equation:

*Step 1*: Summarize the information in a table. That is, translate the problem using variables.

| | legs | number of animals | Total number of legs (146) |
|---|---|---|---|
| hens | 2 | 13 + x | $2(13+x)$ |
| goats | 4 | x | $4x$ |

*Step 2*: Set up an equation

Total number of legs $= 2(13 + x) + 4x = 146$

Total number of animals $= (13 + x) + x = 13 + 2x$

*Step 3*: Solve the equation

$$\begin{aligned} 2(13 + x) + 4x &= 146 \\ 26 + 2x + 4x &= 146 \\ 6x &= 120 \\ x &= 20 \end{aligned}$$

$13 + 2x = 13 + 2(20) = 53$

*Step 4*: Present the final answer

Marius have 20 goats and 33 hens. Altogether Marius have 53 animals.

Simultaneous set of two linear equations:

*Step 1*: Translate the problem using variables

Let the number of goats be $x$ and the number of hens be $y$.

*Step 2*: Rewrite the information in terms of the variables.

Number of hens, $y = 13 + x$ ............ (1)

Total number of legs of the animals, $2y + 4x = 146$ ............ (2)

*Step 3a*: Solve the equation simultaneously using the substitution method

Substitute equation (1) into equation (2)

$$\begin{aligned} 2(13 + x) + 4x &= 146 \\ 26 + 2x + 4x &= 146 \\ 6x &= 120 \\ x &= 20 \end{aligned}$$

Therefore, $y = 13 + 20 = 33$

*Step 3b*: Solve the equation simultaneously using the addition/elimination method

Multiply equation (1) by -2 and rearrange

$-2y + 2x = -26$ ............ (3)

Add equation (3) and (2)

$$6x = 120$$
$$x = 20$$

Therefore, $y = 13 + 20 = 33$

*Step 4*: Present the final answer

Marius have 20 goats and 33 hens. Altogether Marius have 53 animals.

The Polyaian way of mathematizing Task 2, 3 and 4 in Figure 10 are presented in Appendix E on page 178.

The modelling task used in this study was chosen to help answer the second research question. The mathematical modelling task which was given to the students was adapted from Garfunkel and Montgomery (2016) (see Figure 11, the mathematical modelling task used in the study). The modelling task used in the study is an open-ended and meaningful

**Mathematical Modelling (Algebraic Word Problem)**

*Instructions: Solve the modelling task below in as much detail as possible so that I can understand how you understood the question, why you think your mathematical model is correct, and how you interpret your final results.*

The holidays are approaching and your best friend Kristin would like to make some money to purchase gifts. She found one job that will pay 20kr/hr above the minimum wage. Another job offers to pay half the minimum wage plus commission in the amount of 20kr per item she sells. Which job is better? To help Kristin understand your analysis, include a useful representation (e.g. tables, graphs, equations, etc.) to help her make the decision.

**Figure 11:** Mathematical modelling task (adapted from Garfunkel and Montgomery 2016, p. 9).

problem that offers a low floor allowing low-performing students to engage with minimal prerequisite knowledge and skills, and high-performing students to explore more complex concepts. The equations generated from the modelling task is no different from the linear equations the student have being working with at school. However, the modelling task does not require a specific technique or strategy for solving it, unlike the use of linear equations or simultaneous set of two linear equations to solve algebraic word problems by the students.

Looking at the task in Figure 11, students may find it difficult to answer the question. They will have to compare two equations and find the 'break-even' point, that is the number of items their friend will have to sell every hour in order to earn minimum wage. They will also have to think about whether it is likely their friend would sell that many items, which probably depends on the items and her personality. According to Garfunkel and Montgomery (2016), the research into a context and assumptions about the context are both components of mathematical modelling. Finding the break-even point is just one aspect of the problem that students will have to think about making decisions in the face of uncertainty. Garfunkel and Montgomery (2016) points out that a risk-averse[30] student might advise the friend to take the first job because the pay is decent and guaranteed whilst a risk-seeking student might advise the friend to take the second job for the possibility of making much more money. In this sense, the opinions of the students matter and may influence their answer to the question. In summary, the student would have to do the same mathematics to answer the question but they are forced to reconcile their answer with reality, which makes the mathematics more relevant. According to Garfunkel and Montgomery (2016), making judgments about what matters and also assessing the quality of the solution are components of mathematical modelling.

To answer the second research question, we look at how the students interpret their findings after solving an algebraic word problem in a form of mathematical modelling. An interpretation of their findings may depend on: the initial understanding of the problem; the construction of a mathematical model; the actual computational activities; and the evaluation of the outcome of these computations. In their modelling process, the use of the various mathematical representations for clarity and better interpretation of the findings is also considered in this study. The modelling process in Figure 8 is used as a measurement for the students' solutions in this study. The researcher also adds some dimensions to the modelling process based on the outcome of the students solutions.

The solution of the modelling task in Figure 11 used as a measure of the students' solutions is illustrated below:

Real world problem → Mathematizing

[30] Kahneman and Tversky (2013) argues that risk-aversion is a preference for a certain outcome over a gamble with higher or equal expected value whilst risk-seeking is the rejection of a certain thing in favor of a gamble of lower or equal expected value.

*Make assumptions*

Let the minimum wage be $a$ or Let the minimum wage be 100, 150, ...

Let the working hours/day be $x$ or Let the working hours/day be 7.5, 8, ...

Let the number of items sold per hour be $z$.

*Arithmetical calculation*

$y_1 = (100 + 20) \times 7.5 = 900.$

$y_2 = (50 + 20 \times z) \times 7.5 = 375 + 150z.$

For $y_2$ to be the same as $y_1$, then $z$ must be 3.5 items in every hour.

*Formulate equations*

$y_1 = (a + 20)x$ or $y_1 = (100 + 20)x = 120x.$

$y_2 = (\frac{1}{2}a + 20z)x$ or $y_1 = (50 + 20z)x = 50x + 20zx.$

Mathematizing → Mathematical solution

*Solve the equation*

At the break even point

$$\begin{aligned}(a + 20)x &= \left(\frac{1}{2}a + 20z\right)x \\ a + 20 &= \frac{1}{2}a + 20z \\ a + 20 - \frac{1}{2}a &= 20z \\ \frac{1}{40}a + 1 &= z\end{aligned}$$

If the minimum wage is 100kr then in job 2; Kristin has to sell an average of $z = \frac{1}{40} \times 100 + 1 = 3.5$ items to receive same salary as job 1. Another student can find out the pattern if the minimum wage is below 100 or above it.

*Verify the model*

If $z = 3.5$, minimum wage $a = 100$ and working hours per day $x = 7.5$, then

Job 1: $120 \times 7.5 = 900$

Job 2: $(50 \times 7.5) + (20 \times 3.5 \times 7.5) = 900$

Mathematical solution → Real world solution

*Interpretation of model*

Looking at the equations for Job 1 and 2, if a person sells 3.5 items then Job 2 will be same as Job 2. However, if the person sells less than 3.5 or more than 3.5 items per hour then Job 2 will either have less or more salary than Job 1 respectively (see both Table 8 and Figure 12, the numerical and graphical interpretation of the model).

| **Time/Hour** $x$ | **Job 1** $y = 120x$ | **Job 2** $y = 50x + 20zx$, $z = 2$ | **Job 2** $y = 50x + 20zx$, $z = 4$ |
|---|---|---|---|
| 1 | 120 | 110 | 130 |
| 2 | 240 | 220 | 260 |
| 3 | 360 | 330 | 390 |
| 4 | 480 | 440 | 520 |
| 5 | 600 | 550 | 650 |
| 6 | 720 | 660 | 780 |
| 7 | 840 | 770 | 910 |
| ….. | …… | ….. | ….. |

**Table 8:** Numerical (table) representation of the mathematical modelling task.

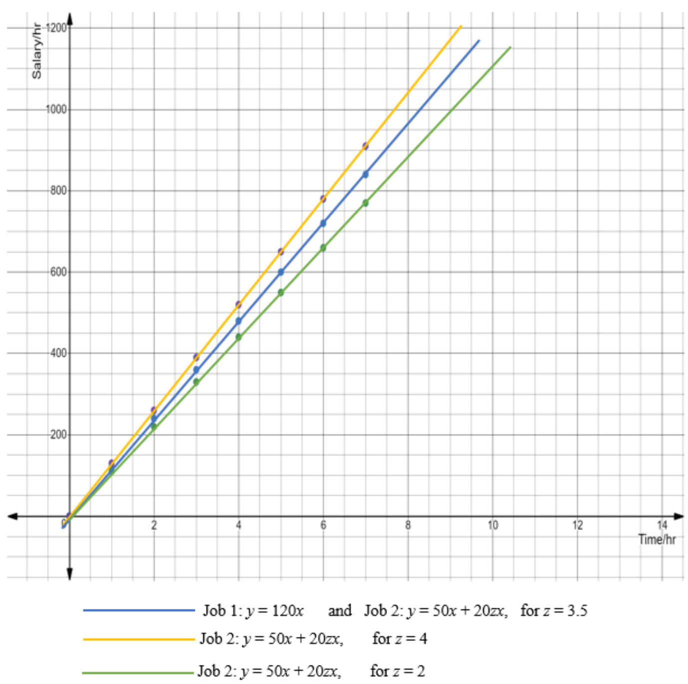



**Figure 12:** Graphical representation of the mathematical modelling task.

*Compare solution*

> The possibility for Kristin to sell that many items may probably depend on the kind of item and her personality. Kristin can take the first job because the pay is decent and guaranteed. however, Kristin can also take the risk for the possibility of making much more money.

The students writing their solution on a sheet of paper will not be enough to answer both research questions. Some group may choose to only write the mathematical aspect of the solution and leave the interpretation and/or the presentation of the final answer. Hence, an interview will be necessary to seek more information about the students' conceptions on both tasks used in the research study.

#### 4.2.3.2 Semi-structured interview .

For the purpose of this study, interviews was used as a method of data collection. It is through the interviews that the students gave account of how they justified their strategies and also how they interpret their findings after the modelling task.

A semi-structured interview was used in this study. Ayres (2008) defines semi-structured interview as a qualitative data collection strategy for which the researcher ask informants a series of predetermined but open-ended questions. She further points out that, in semi-structured interview the researcher has more control over the topics of the interview than in unstructured interviews, but in contrast to structured interviews (usually used in quantitative research) that use closed questions, there is no fixed range of responses to each question. Denscombe (2014) also asserts that the answers to semi-structured interviews are open-ended and that there is more emphasis on the interviewee elaborating points of interest. Bryman (2016) notes that in semi-structured interviewing: there is much greater interest in the interviewee's point of view; the researcher wants rich, detailed answers; the interviewers can depart significantly from any schedule or guide that is being used (that is, new questions that follow up interviewee's replies can be asked). Despite the importance, Denscombe (2014) again argues that analysis of data collected with semi-structured interviews can be difficult and time-consuming. He also points out that the impact of the interviewer and of the context means that objectivity and consistency maybe hard to achieve, and also the data from interviews are based on what people say rather than what they do.

Ayres (2008) points out that an interview guide is prepared in advance by the researcher who uses semi-structured interviewing. Bryman (2016) asserts that the formulation of questions for an interview guide involves: General research area; Specific research questions;

Interview topics; Formulate interview questions; Review/revise interview questions; Pilot guide; Identify novel issues; Revise interview questions; Finalize guide. In the formulation of the questions for the interview guide used in this study, the general topic, the research questions, and the task given to the students were considered. The interview guide was not piloted and revised in this study. The interview guide was prepared in a way which gives the researcher all the necessary information that the students worksheet could not provide (see Appendix A on page 149, for the interview guide used in this study) . The form of semi-structured interview used in this study was one-to-one, involving the researcher and an informant representing his/her group. Group interviews could be useful in this study since in one-to-one interview, there is a limit of the number of views and opinions available to the researcher. However, according to Denscombe (2014), one-to-one interviews are relatively easy to arrange, is relatively easy to control, and it is far easier to transcribe a recorded interview when the talk involves just one interviewee. These reasons and also the time frame of the master thesis influence the choice of one-to-one semi-structured interviewing in this study.

The author/researcher took the role of interviewer for the students interviews and asked some scripted questions. Follow-up questions were also asked when needed as the researcher seeks to gain better insight into students' thinking. Hints and clarifications were given to students where necessary. The students responded to the questions both verbally and sometimes also use paper-and-pencil during the interviews. A small and quiet study room within the upper-secondary school was booked and used for the interviews. The students were cooperative and not nervous during the interviews since the researcher had met them on several occasions before the interviews and also anonymity was guaranteed. The students who took part in the interviews willingly volunteered to represent their respective groups. All the interviews were audio recorded.

The kind of questions used in the study and the justification of using such questions are presented below:

**1.** *Have you solved or encountered similar problems like these algebraic word problems at class before? Were there any difference? Can you tell me what the difference was?*

> The rationale behind the use of this question is to find out how the previous experiences of the students influence their working processes. The question also seeks to address the transfer of problem solving ability of the students. The word problem task given to the student was different in context but similar in terms of equations (that is, the translated equations from the word problem) when

compared to the usual task solved by the students at school. According to Fuchs et al. (2004), mathematical problem solving is a transfer challenge requiring children to develop schemes for recognizing novel problems as belonging to familiar problem types for which they know solutions. Reed et al. (1985) on the other hand also argue that most students are not able to apply problem solving skills they learned recently to solve similar algebraic word problems.

**2.** *Which of the questions among the algebraic word problem task, was the most difficult? Can you tell me why? How about the easiest? Why?*

In this question type, the difficulty level is measured within the four categories; Abstract Factual, Abstract Hypothetical, Concrete Factual and Concrete Hypothetical described by Caldwell and Goldin (1979, 1987). Caldwell and Goldin (1987) argue that at all levels (both elementary and secondary school), concrete hypothetical problems are more difficult compared to concrete factual problems in a large effect, however at the elementary school level and to a lesser extent at the junior high school level, abstract hypothetical problems are actually less difficult than abstract factual problems.

**3.** *Can you tell me the processes you went through in solving these algebraic word problems? (**a**) How did you know your answer was right? (**b**). Do you know of any other method used in solving these algebraic word problems? Can you tell me? (**c**). Was it helpful when you solved these questions in a group? How? How about solving them individually?*

**4.** *Questions about specific errors found in their solutions to the algebraic word problem tasks.*

These questions mainly help to answer the first research question "How do upper secondary students justify their strategies for solving algebraic word problems?" The standard process that will be used as a measurement of students' justification is Polya's (2004) description of the four stages one goes through when solving a problem. That is Understanding the problem, Devising a plan, Carrying out the plan and Looking back. The third sub question under Question 3, seeks to find out how working in a group influence their working processes. Question 4 however seeks the explanation and conceptions of students about specific errors found in their solution to the algebraic word problem tasks.

**5.** *Can you see any connection between the mathematics you learn at school and the outside world? Why?*

**6.** *How often do you solve mathematical modelling task at school? Any reason?*

The questions seek to find out if the students are able to relate or see the connection between the mathematics they learned at school and the real world. According to the National Council of Teachers of Mathematics (2000), in order to meet new challenges in school, work and life, students will have to adapt and extend whatever mathematics they know. The questions also seek to find out the students' previous experiences in the solving of mathematical modelling task. The experiences of the students can be related to their current performance in the modelling task.

**7.** *What is your opinion about the modelling task you solved recently? Does this modelling task has any connection between the mathematics at school and the outside world? Why?*

The question seeks to find out the opinion of the students about the modelling task and if there is any connection between the mathematics and the real world. They can also talk about the difficulty level of the task, which gives an idea if the task offers low floor allowing even below-average performing students to engage with minimal prerequisite knowledge and skills and high ceiling providing opportunities to explore more complex concepts and representations.

**8.** *Can you tell me the processes you went through in solving this modelling task? How did you know your answer was right?*

**9.** *Please, interpret your modelling results to me?*

**10.** *Questions about specific errors found in their solution to the modelling task.*

The questions seek to find out the modelling processes of the students. According to Garfunkel and Montgomery (2016) and Giordano et al. (2013) the processes; Identify the problem, Make assumptions, Solve or interpret the model, Verify the model and among others are helpful steps or stages when one is solving a mathematical modelling task. So, to what extend did the students use this processes, directly or indirectly? At another point, these questions help to answer the second research question "How do upper secondary students interpret their findings when solving an algebraic word problem in a form of mathematical modelling?" That is, their use of the different representations to explain or justify their solution or model. Question 10 however seeks the explanation and conceptions of the students about specific errors found in their solution to the modelling task.

**11.** *Will you want more of the modelling task at school? Why?*

The rationale behind this question is to find out whether the student found the modelling task interesting and are likely to engage themselves with more of the modelling task. According to Stillman (2007), at the upper secondary level (especially in some European countries) there is a strong bias against mathematical modelling as the prevalent attitude is that high level mathematics is what is important. Garfunkel and Montgomery (2016) on the other hand argue that modelling in the high school setting can be used as motivation for learning new techniques and new content; small modelling activities can be used to reinforce new concepts and to illustrate their applications; more extended modelling activities help students pull together ideas from different parts of a course and from different courses.

### 4.2.4 Data management

Bryman (2016) argues that in qualitative research, the interview is usually audio-recorded and transcribed whenever possible on several occasions. The interviews in this study were audio-recorded and the interview recordings of each participant took approximately an average of 14 minutes. Bryman (2016) points out that the problem concerning transcribing interviews is that, it is very time-consuming and that it is best to allow around five-six hours for transcription for every hour of speech. Listening to the audio-recordings after the interviews was the first step and then a thorough transcription of an average of one hour thirty minutes were made.

Van den Eynden et al. (2011) argues that data storage strategy is important for the fact that digital storage media are inherently unreliable. In order to prevent unauthorized persons from accessing the raw data, the interview recordings were stored on the University of Agder server (for which the researcher was the only person that had access). Pseudonyms were used during the transcriptions for the purpose of anonymity. In the final process, the audio-recordings will be completely deleted within a period of six months after the research is done.

### 4.2.5 Strategy for analysis

Denscombe (2014) defines qualitative data analysis as the process of bringing order, structure, and interpretation to the mass of collected data. He further points out that the analysis of qualitative data can take a number of forms that reflects the particular kind of data being used and the particular purposes for which they are being studied. In this sense, there is no single approach to the analysis of qualitative data that covers all situations. In this study a qualitative data analysis approach was used. Denscombe (2014) lists some advantages of qualitative analysis: there is a richness and detail to the data; there is tolerance of ambiguity and contradictions; and there is the prospect of alternative explanations. In contrast to the advantages he argues that: the data might be less representative; there is a possibility of decontextualizing the meaning; and there is also the danger of oversimplifying the explanation.

If the process by which people went about analyzing their data or the kind of assumptions that informed their analysis is not known, then according to Braun and Clarke (2006, p. 7) "it is difficult to evaluate their research, and to compare and/or synthesise it with other studies on that topic, and it can impede other researchers carrying out related projects in the future". The analysis of this study was driven by the research questions and that thematic analysis method was the main tool used in analyzing the data. Braun and Clarke (2006) defines thematic analysis as a method for identifying, analysing, and also reporting patterns (themes) within data. They also point out that a theme (as in the definition) captures something which is important about the data in relation to the research question, and also represents some level of patterned response or meaning within the data set.

In this study, the six-phases of analysis by Braun and Clarke (2006) was used as a framework for conducting the thematic analysis. This phases are: familiarizing yourself with your data; generating initial codes; searching for themes; reviewing themes; defining and naming themes; and producing the report.

Reading and re-reading the transcripts and the students' worksheets was the first step taken in this study. Notes were made and early impressions were jotted down during the readings. In the second stage, the data was organized in a meaningful and systematic way. Each segment of data that was relevant to or captured something interesting about the research question was coded. The transcripts and worksheets were coded manually by writing notes on the texts, using highlighters and coloured pens in indicating potential patterns. Interesting aspects in the data items that formed the basis of repeated patterns (themes) across the data set were identified during the third stage (see Table 9, a table with the preliminary themes). In this third stage, the codes had been organized into themes that seemed to say something about the research questions. In the fourth stage, the preliminary themes were reviewed, modified and developed. That is, main themes, sub-themes and sub-subthemes were developed from the preliminary themes (see Table 10, a table with the main themes and sub-themes).

<table>
<tr>
<td>Theme: Prior knowledge/transfer of problem solving.<br>Codes:<br>equivalent problem<br>similar problem<br>isomorphic problem<br>unrelated problem<br>student sees similarities<br>student sees differences</td>
<td>Theme: Type of word problem and difficulty level of word problem.<br>Codes:<br>abstract factual<br>abstract hypothetical<br>concrete factual<br>concrete hypothetical<br>difficult<br>not difficult</td>
<td rowspan="3">Theme: Mathematizing the word problem.<br>Codes:<br>understanding the problem<br>misinterpretation of task<br>drawing a picture<br>solve the problem arithmetically<br>translate problem into variables<br>setup an algebraic equation<br>simultaneous set of two linear equation<br>linear equation/substitution before expression<br>solve the equation<br>substitution method<br>addition/elimination method<br>looking back/verify answer<br>comparing answers<br>presenting the final answer<br>computational error<br>expression/equation incorrect</td>
</tr>
<tr>
<td>Theme: Mathematical representations.<br>Codes:<br>graphical representation<br>numerical representation<br>algebraic representation<br>verbal representation</td>
<td>Theme: Real world problem----- mathematizing.<br>Code:<br>make assumptions<br>arithmetic calculations<br>formulate an equation<br>incorrect expression/equation<br>computational error</td>
</tr>
<tr>
<td>Theme: Mathematizing ------ mathematical solution.<br>Codes:<br>solve the equation<br>verify the model</td>
<td>Theme: Mathematical solution ----- real world solution.<br>Codes:<br>interpretation</td>
</tr>
<tr>
<td>Theme: Real word problem ---- real world solution.<br>Codes:<br>compare solution</td>
<td>Theme: Importance of group work.<br>Codes:<br>Group works helpful<br>Group works not helpful</td>
<td>Theme: Modelling in school activities<br>Codes:<br>difficult task<br>task not difficult<br>realistic task<br>task unrealistic<br>more modelling task<br>less modelling task<br>no modelling task<br>helps learning process<br>fun to solve</td>
</tr>
</table>

**Table 9:** Preliminary themes.

In the fifth stage, the essence of what each theme is about was identified (that's the final refinement of the themes). During the process of the fifth stage, the following questions questions were considered: What is the theme saying?; How do the subthemes interact

| | | |
|---|---|---|
| **Theme: Algebraic word problem**<br><br>**Subtheme: Prior knowledge/transfer of problem solving**.<br><br>equivalent problem = **EP**<br>similar problem = **SP**<br>isomorphic problem = **IP**<br>unrelated problem = **UP**<br>student sees similarities = **SSS**<br>student sees differences = **SSD**<br><br>**Subtheme: Type of word problem and difficulty level of word problem.**<br><br>abstract factual = **AF**<br>abstract hypothetical = **AH**<br>concrete factual = **CF**<br>concrete hypothetical = **CH**<br>difficult = **D**<br>not difficult = **ND**<br><br>**Subtheme: Mathematizing the word problem.**<br><br>understanding the problem = **UTP**<br>misinterpretation of task = **MT**<br>drawing a picture = **DP**<br>solve the problem arithmetically = **SPA**<br>translate problem into variables = **TPV**<br>setup an algebraic equation = **SAE**<br>simultaneous set of two linear equation = **SLE**<br>linear equation/substitution before expression = **LE**<br>solve the equation = **SE**<br>substitution method = **SM**<br>addition/elimination method = **AEM**<br>looking back/verify answer = **LB**<br>comparing answers = **CA**<br>presenting the final answer = **PFS**<br>computational error = **CE**<br>expression/equation incorrect = **EI** | **Theme: Mathematical modelling**<br><br>**Subtheme: Mathematical representations.**<br><br>graphical representation = **GR**<br>numerical representation = **NR**<br>algebraic representation = **AR**<br>verbal representation = **VR**<br><br>**Subtheme: Modelling in school activities**<br><br>difficult task = **DT**<br>task not difficult = **NT**<br>realistic task = **RT**<br>task unrealistic = **UT**<br>more modelling task = **MMT**<br>less modelling task = **LMT**<br>no modelling task = **NMT**<br>helps learning process = **HLP**<br>fun to solve = **FS**<br>Student sees connection = **SCO**<br>Connection depends on subject = **CDS**<br>No connection = **NCO**<br>Student sees imaginary connection = **SIC**<br>Other subject important = **OSI**<br>Don't know = **DK**<br><br>**Subtheme: Modelling process/cycle**<br><br>**Sub subtheme: Real world problem----- mathematizing.**<br><br>make assumptions = **MA**<br>arithmetic calculations = **AC**<br>formulate an equation = **FE**<br>incorrect expression/equation = **IE**<br>computational error = **MCE** | **Sub subtheme: Mathematizing ------ mathematical solution.**<br><br>solve the equation = **STE**<br>verify the model = **VM**<br><br>**Sub subtheme: Mathematical solution ----- real world solution.**<br><br>Interpretation = **IP**<br><br>**Sub subtheme: Real word problem ---- real world solution.**<br><br>compare solution = **CS**<br><br>**Theme: Importance of group work.**<br><br>Group work helpful = **GWH**<br>Group work not helpful = **GNH**<br>Individual work helpful = **IWH**<br>Verify answer with others = **VAO** |

**Table 10:** Themes at the end of the fourth phase.

and relate to the main theme?; and How do the themes relate to each other? Under each individual theme a detailed analysis was conducted. In the final stage, the final analysis and write-up is presented. A detailed account of the final analysis is given in the next chapter (See Tables 12, 13 and 14 for the summary of results with codes created for both the algebraic word problem tasks, mathematical modelling task and the interview transcripts).

## 4.3 Ethical Considerations

This study was subject to certain ethical issues. In the start of the research, permission to have contact with the students was sought from the head of the natural science department at the upper secondary school located at the Southern Norway. The mathematics teacher of these students also granted approval for the research activities after several communications. The students voluntarily opted to participate in the research study without any influence from their teacher, after explaining to them what the study was about. The head of department, teacher and students were informed about the approval of the data collection by the Norwegian Centre for Research Data-NSD (see Appendix B on page 150, for all the information given to the students prior to the research activities).

All participants reported their acceptance regarding their participation in the research, through a signed consent and information letter. The aim of the letter was to reassure participants that their participation in the research is voluntary and that they were free to withdraw from it at any point and for any reason. The superiors of participants under age 16 also signed the consent and information letter. The participants had the chance of asking any question regarding the research activities and that they were reassured that their answers were treated as confidential and used only for academic purposes and only for the purposes of this particular research. For the purpose of anonymity, pseudonyms (Bjørg, Julie, Hilde, Eirik, Tonje, Helge and Arvid) were used when writing the research report. Participants were not harmed or abused, neither physically nor psychologically, during the conduction of the research. In contrast, the researcher attempted to create and maintain a climate of comfort. The voice recordings were deleted five months after the end of the research study. References from all the literature used in this study are provided throughout the text, to avoid any form of plagiarism.

## 4.4 Validity and Trustworthiness

The findings of this research study have not been solely influenced by the researcher's point of view and values. The interpretations of the analysis are emphatically not the ultimate and that others could interpret the empirical data differently to some extent but not completely. In view of this the researcher used the inter-coder reliability[31] to find out the extent where an independent person agrees on the coding used in this study with an application of the same coding scheme. The approach used in this study can be applied

[31] Inter-coder reliability is defined and explained in the next section

to some areas of mathematics other than algebra. The findings could be more interesting if there were observations of both the teacher and students during mathematics lessons and also video recordings during the research activities. This could give more information about the students' discussions and details of their working process. Since there was no opportunity for lessons observation, any discussion on the mathematics lessons at school was based on the information given by the teacher. In view of this, we can not strongly argue about the techniques the students used in this study in relation to what they normally do at class. We can not also generalized the findings or results since only seven students were interviewed and a total of 23 students participated in this study. In conclusion, the results of this study is only meant to be suggestive and any conclusions drawn are tentative.

### 4.4.1 Inter-coder reliability

Inter-coder reliability refers to the extent to which two or more independent coders agree on the coding of a data of interest as they apply the same coding scheme (Lavrakas, 2008; Lombard et al., 2002). Kristoffer[32] and I had a meeting that lasted approximately three hours, for the reason of checking the reliability of the coding schema that I had developed and applied during the analysis of the data. During the meeting, I first explained to Kristoffer what the research study was about and the kind of data I collected. I also explained to him the codes I developed from the students' worksheet and the interviews (see Table 9 for the preliminary codes) during the early stages of the analysis. Kristoffer studied the definitions and the table and when he confirmed that he had understood the codes, he was given the students' worksheet and the interview transcripts (that presented the coding I had already made) to code. Kristoffer agreed with the codes developed from the students' worksheet because they were standard codes following the four phases of problem solving by Polya (2004) and the modelling cycle by Garfunkel and Montgomery (2016).

Kristoffer did not agreed to some of the codes developed from the first five interview transcripts because he saw them as contradictory. Table 11 below shows a comparative table of my coding and Kristoffer's coding of the first five interview transcripts. From Table 11, Kristoffer introduce three new codes (Shown in a red colour: DI[33], VAO[34] and WNB[35]). At the end of the process I used 44 codes in my coding of the first five interview

[32] A master student at the Department of Mathematical Sciences, University of Agder
[33] Different interpretation
[34] Students verified answer with other group members
[35] group work helpful but it might not be the best

| | Codes | Researcher | Kristoffer |
|---|---|---|---|
| Transfer of word Problem | SSS | 4 | 4 |
| | SSD | 1 | 1 |
| Difficulty level of word problem | D | 3 | 3 |
| | ND | 5 | 4 |
| | DI | 1 | 1 |
| Verification of answer | LB | 4 | 4 |
| | CB | 1 | 1 |
| Importance of group work | GWH | 3 | 1 |
| | GNH | 2 | 1 |
| | IWH | 3 | 3 |
| | VAO | 2 | 2 |
| | WNB | 1 | 1 |
| Link between school mathematics and the outside world | SCO | 4 | 4 |
| | CDS | 0 | 0 |
| | NCO | 1 | 1 |
| | SIC | 2 | 2 |
| Modelling in school activities | MMT | 4 | 4 |
| | LMT | 0 | 0 |
| | NMT | 0 | 0 |
| | HLP | 1 | 1 |
| | FS | 0 | 0 |
| | OSI | 1 | 1 |
| | DK | 1 | 1 |
| **Total** | | **44** | **40** |

**Table 11:** A comparative table of the researcher's and Kristoffer's coding on the first five interview transcript.

transcripts whereas Kristoffer used 40. In the end, the percentage of our coding agreement for the first five interview transcripts was 90.91% ($\approx \frac{40}{44} \times 100\%$).

It can be seen in Table 11 that the only difference in our coding was the four codes that Kristoffer missed: 1 ND, 2 GWH and 1 GNH codes. I had a discussion with Kristoffer afterwards, where I explained why I coded those four codes at those moments but he agreed to some point and suggested that I should use the codes I developed together with what he found for clarity. In the end I only included the code VAO and left DI and WNB out. The reason for leaving these codes out was because the DI is embedded in both D[36] and

[36] word problem difficult

ND[37] whereas WNB is embedded in both GWH[38] and IWH[39].

The data analysis and findings of this study is presented in the next chapter.

[37]word problem not difficult
[38]Group work helpful
[39]Individual work helpful

# 5 DATA ANALYSIS AND FINDINGS

In this chapter, the analysis of data regarding the two research questions is presented. The purpose of the study was to investigate, how upper secondary students' justify their strategies when solving algebraic word problems and also how these students' interpret their findings after solving a mathematical modelling task. The analysis involves students' responses to both the worksheets and the interviews for the purpose of answering the research questions.

This chapter consists of three sections where the first section presents the summary of results from the students' worksheets and the interviews in a table form. The next section presents the analysis on algebraic word problem, including the students' prior knowledge, the type of word problem and how the students mathematize the various algebraic word problem tasks. Finally, the last section presents the analysis of mathematical modelling which also entails the modelling process/cycle of the students, the various mathematical representations used, students' interpretations of their model, students' conceptions of modelling in school activities and the usefulness of group work.

Pseudonyms are given to the students representing each respective group (Bjørg, Julie, Hilde, Eirik, Tonje, Helge and Arvid). I repeat that, the results from this study can not be generalized but are only meant to be suggestive and any conclusions drawn are tentative.

## 5.1 Summary of Results

The summary of results from both the students' worksheets and the interviews are all presented in a table form. The codes used in the various summary of results tables are defined in Table 10 (see page 71).

Table 12 below presents the summary of results from the students' algebraic word problem worksheets. Recall that the students worked together in groups (7 groups) to complete the algebraic word problem tasks. In each algebraic word problem task, a summary of results of how the students mathematize the word problem task is presented. From Table 12, each task has it own results for the 7 groups. The empty spaces (or dash) in the table represents no response. I will further explain this table in Section 5.2.3 with some excerpts from the students' worksheets and interview transcripts as evidence.

| Task | Codes | Group 1 | Group 2 | Group 3 | Group 4 | Group 5 | Group 6 | Group 7 | Total |
|---|---|---|---|---|---|---|---|---|---|
| 1 | UTP | UTP | UTP | UTP | UTP | UTP | UTP | UTP | 7 |
| | MT | - | - | - | - | - | - | - | 0 |
| | DP | DP | - | - | - | DP | - | - | 2 |
| | TPV | TPV | - | TPV | TPV | - | TPV | TPV | 5 |
| | SPA | - | SPA | - | - | SPA | - | - | 2 |
| | SAE | SAE | - | SAE | SAE | - | SAE | SAE | 5 |
| | EI | - | - | - | - | - | - | - | 0 |
| | SLE | - | - | - | - | - | - | SLE | 1 |
| | LE | LE | - | LE | LE | - | LE | - | 4 |
| | SE | SE | - | SE | SE | - | SE | SE | 5 |
| | SM | SM | - | SM | SM | - | SM | - | 4 |
| | AEM | - | - | - | - | - | - | AEM | 1 |
| | CE | CE | - | - | - | - | - | - | 1 |
| | PFS | PFS | PFS | PFS | - | - | - | - | 3 |
| 2 | UTP | UTP | UTP | UTP | UTP | UTP | UTP | UTP | 7 |
| | MT | - | - | - | - | - | - | - | 0 |
| | DP | DP | - | - | - | - | - | - | 1 |
| | TPV | TPV | TPV | TPV | TPV | TPV | TPV | TPV | 7 |
| | SPA | - | - | - | - | - | - | - | 0 |
| | SAE | SAE | SAE | SAE | SAE | SAE | SAE | SAE | 7 |
| | EI | - | - | - | - | - | - | - | 0 |
| | SLE | - | - | - | - | - | - | SLE | 1 |
| | LE | LE | LE | LE | LE | LE | LE | - | 6 |
| | SE | SE | SE | SE | SE | SE | SE | SE | 7 |
| | SM | SM | SM | SM | SM | SM | SM | - | 6 |
| | AEM | - | - | - | - | - | - | AEM | 1 |
| | CE | - | - | - | - | - | - | - | 0 |
| | PFS | PFS | PFS | PFS | - | - | PFS | - | 4 |
| 3 | UTP | UTP | UTP | UTP | UTP | UTP | UTP | - | 6 |
| | MT | - | - | - | - | - | - | MT | 1 |
| | DP | - | - | - | - | - | - | - | 0 |
| | TPV | TPV | TPV | TPV | TPV | TPV | TPV | TPV | 7 |
| | SPA | - | - | - | - | - | - | - | 0 |
| | SAE | SAE | SAE | SAE | SAE | SAE | SAE | SAE | 7 |
| | EI | - | - | - | - | - | - | EI | 1 |
| | SLE | - | SLE | - | - | - | - | SLE | 2 |
| | LE | LE | - | LE | LE | LE | LE | - | 5 |
| | SE | SE | SE | SE | SE | SE | SE | SE | 7 |
| | SM | SM | SM | SM | SM | SM | SM | - | 6 |
| | AEM | - | - | - | - | - | - | AEM | 1 |
| | CE | - | - | - | - | - | - | - | 0 |
| | PFS | PFS | PFS | PFS | PFS | - | PFS | PFS | 6 |
| 4 | UTP | UTP | UTP | UTP | UTP | - | UTP | - | 5 |
| | MT | - | - | - | - | MT | - | MT | 2 |
| | DP | - | - | - | - | - | - | - | 0 |
| | TPV | TPV | TPV | TPV | TPV | TPV | TPV | TPV | 7 |
| | SPA | - | - | - | - | - | - | - | 0 |
| | SAE | SAE | SAE | SAE | SAE | SAE | SAE | SAE | 7 |
| | EI | - | - | - | - | EI | - | EI | 2 |
| | SLE | - | SLE | - | - | - | - | SLE | 2 |
| | LE | LE | - | LE | LE | - | LE | - | 4 |
| | SE | SE | SE | SE | SE | SE | SE | SE | 7 |
| | SM | SM | SM | SM | SM | SM | SM | - | 6 |
| | AEM | - | - | - | - | - | - | AEM | 1 |
| | CE | - | - | - | - | - | - | - | 0 |
| | PFS | PFS | PFS | PFS | PFS | PFS | PFS | PFS | 7 |

**Table 12:** Summary of results from the students' algebraic word problem worksheets (Codes defined in Table 10 on page 71).

| GROUP | Codes | MA | AC | FE | IE | STE | MCE | VM | IP | CS | GR | NR | AR | VR |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | MA | | FE | IE | | | | IP | CS | GR | | AR | VR |
| 2 | | MA | | FE | IE | STE | | | IP | CS | GR | | AR | VR |
| 3 | | MA | | FE | IE | STE | | VM | IP | | | | AR | VR |
| 4 | | MA | AC | | | | | VM | IP | CS | | NR | | VR |
| 5 | | | | FE | IE | | | | | | GR | | AR | |
| 6 | | MA | | FE | IE | | | | IP | CS | GR | NR | AR | VR |
| 7 | | MA | | | | | | | IP | CS | GR | | | VR |
| **Total** (out of 7) | | 6 | 1 | 5 | 5 | 2 | 0 | 2 | 6 | 5 | 5 | 2 | 5 | 6 |

**Table 13:** Summary of results from the students' modelling worksheets (Codes defined in Table 10 on page 71).

Table 13 also presents the summary of results from the students' modelling worksheets. We again recall that the students worked together in groups (7 groups) to complete the mathematical modelling task. The empty spaces in the table represents no response. If you look at Table 13, Group 7 did not formulate an equation (FE) however their equation was seen through their graphical representation (GR). All the groups that formulated an equation for the second job described in the modelling task (on page 60) had an incorrect equation (IE). The equation for the second job by Group 5, 6 and 7 were similar whereas that of Group 1 and 3 were also similar. Only Group 2 had both equations to be incorrect. Group 5 did not clearly state any assumption (MA) but they indirectly considered some assumptions that was only seen in the equation they used. I will further explain this table in Sections 5.3.1 and 5.3.2 with some excerpts from the students' worksheets and interview transcripts as evidence.

| Interview Question | Codes | Task | Group 1 | Group 2 | Group 3 | Group 4 | Group 5 | Group 6 | Group 7 | Total |
|---|---|---|---|---|---|---|---|---|---|---|
| Q1 Have you solved similar algebraic word problems at school before? | SSS | 1 | SSS | SSS | SSS | SSS | - | SSS | SSS | 6 |
| | SSD | | - | - | - | - | SSD | - | - | 1 |
| | SSS | 2 | SSS | SSS | SSS | SSS | SSS | SSS | SSS | 7 |
| | SSD | | - | - | - | - | - | - | - | 0 |
| | SSS | 3 | SSS | SSS | SSS | SSS | SSS | - | SSS | 6 |
| | SSD | | - | - | - | - | - | SSD | - | 1 |
| | SSS | 4 | SSS | SSS | SSS | SSS | SSS | - | SSS | 6 |
| | SSD | | - | - | - | - | - | SSD | - | 1 |
| Q2 Which of the algbraic word problem task was difficult? | ND | 1 | ND | - | - | ND | - | ND | - | 3 |
| | D | | - | D | D | - | D | - | D | 4 |
| | ND | 2 | ND | ND | ND | ND | ND | ND | ND | 7 |
| | D | | - | - | - | - | - | - | - | 0 |
| | ND | 3 | ND | ND | ND | ND | ND | - | ND | 6 |
| | D | | - | - | - | - | - | D | - | 1 |
| | ND | 4 | ND | ND | ND | ND | ND | - | ND | 6 |
| | D | | - | - | - | - | - | D | - | 1 |
| Q3a How did you know your answer was right? | LB | | - | LB | LB | LB | LB | LB | LB | 6 |
| | CA | | CA | - | - | - | - | - | - | 1 |
| Q3c Was it helpful when you solved these questions in a group? | GWH | | - | GWH | GWH | - | GWH | GWH | GWH | 5 |
| | GNH | | GNH | - | - | GNH | - | - | - | 2 |
| | IWH | | IWH | - | IWH | IWH | - | - | IWH | 4 |
| | VAO | | VAO | - | VAO | - | - | - | - | 2 |
| Q5 Can you see any connection between the mathematics at school and the outside world? | SCO | | SCO | SCO | SCO | - | SCO | - | - | 4 |
| | CDS | | - | - | - | - | - | - | CDS | 1 |
| | NCO | | - | - | - | NCO | - | NCO | - | 2 |
| | SIC | | SIC | - | - | - | SIC | - | - | 2 |
| Q6 How often do you solve modelling task at school? | MMT | | - | - | - | - | - | - | - | 0 |
| | LMT | | LMT | LMT | LMT | - | LMT | LMT | LMT | 6 |
| | NMT | | - | - | - | NMT | - | - | - | 1 |
| Q7 What is your opinion about the modelling task you solved? | DT | | DT | DT | DT | DT | DT | - | DT | 6 |
| | NT | | - | - | - | - | - | NT | - | 1 |
| | RT | | RT | RT | RT | RT | RT | RT | RT | 7 |
| | UT | | - | - | - | - | - | - | - | 0 |
| Q11 Will you want more of the modelling task at school? | MMT | | MMT | MMT | - | MMT | MMT | MMT | MMT | 6 |
| | LMT | | - | - | - | - | - | - | - | 0 |
| | NMT | | - | - | - | - | - | - | - | 0 |
| | HLP | | HLP | - | - | - | - | - | - | 1 |
| | FS | | - | - | - | - | - | FS | - | 1 |
| | OSI | | - | - | OSI | - | - | - | - | 1 |
| | DK | | - | - | DK | - | - | - | - | 1 |

**Table 14:** Summary of results for the interviews (Codes defined in Table 10 on page 71).

Table 14 presents the summary of results for the student interviews. Results of each interview question (see page 65 for the interview questions) are presented in the table. Questions 3, 4, 8, 9 and 10 are not included in the table. These questions where specifically asked for clarity on what the students presented in their worksheets. Questions 4 and 10 respectively asked about the specific errors the students made in both solutions to the algebraic word problem and the mathematical modelling task. Questions 3 and 8 respectively asked about the processes the students went through in solving both the algebraic word problem and the modelling task whereas Question 9 asked about the students' interpretations of their modelling results. Recall that one student from each group was interviewed. The title 'Group' used in Table 14 represents the individual students representing their respective group. The results from the interviews are the conception of an individual representing his/her group. The empty spaces (or dash) in the table represents no response. I will further explain this table in Sections 5.2.1, 5.2.2, 5.3.4 and 5.3.5 with some excerpts from the interview transcripts as evidence.

The next two sections present a detailed account of the results on both the algebraic word problem task and the mathematical modelling task. Nevertheless, evidence (that is, 'excerpts' from the transcriptions and students' worksheets) is provided in the form of an analysis or interpretation from the responses of the participants.

## 5.2 Algebraic Word Problem

This section first starts with an analysis of the prior knowledge or the transfer of problem solving of the students. Next to this, is an analysis on the type of word problem and finally an analysis of how the students mathematize the algebraic word problem is presented.

### 5.2.1 Prior knowledge/Transfer of problem solving

The algebraic word problem tasks given to the students were isomorphic[40] to the word problems they usually solve at school, that is the word problem is different in terms of the story context but similar in terms of the equations involve (see Figure 10 on page 58 and Appendix F on page 181, for the comparison between the word problems). If you look at Figure 10 on page 58, the four word problem tasks given to the students have the same equations but are different to each other in terms of the story context. For instance: Task 1 is isomorphic to Task 2, 3 and 4; Task 2 is isomorphic to Task 1, 3 and 4; and Task 3 is equivalent to Task 4.

The students interviewed gave their thoughts about the similarities or differences of the tasks they solved, compared to what they normally solve at school. Results of the first interview question (Q1) in Table 14, which asked students whether they have solved or encountered similar algebraic word problems before, generally revealed that the students have some prior knowledge about the tasks they solved. This conclusion is derived from the fact that most of the students see similarities (SSS) but not differences (SSD) in all the tasks. Arvid from Group 7 did not only answer whether he sees similarities or differences but also provided the reason why he sees similarities and not differences. Arvid offered the following response:

> 661. Teacher: Have you solved or encountered similar problems like these [Points at the algebraic word problem task] algebraic word problems at class before?
>
> 663. Arvid: Yes, we have looked at similar task.

[40] The different types of problem solving transfers are defined and explained in Section 3.3.1 (see page 33).

664. Teacher: Was it very similar?
665. Arvid: Umm, quiet but it wasn't like in text form, it was like only umm.. equation.

**Excerpt 5.2.1.0**

In Arvid's response, he was able to identify that the word problems they solve at school were similar in terms of the equations involve but different in terms of the story context, compared to the task they solved. One student found differences in the first task whilst another student also found some differences in Task 3 and 4. Tonje and Helge from Group 5 and 6 respectively offered the following response:

452. Teacher: Have you solved or encountered similar problems like these [Points at the algebraic word problem task] algebraic word problems at class before?
454. Tonje: Yes, the three last ones, I would say but not the first one. But they are all the same though, like kind of.. we have been through this in pre-school or high school [Laughs]. No, middle school.
568. Helge: Umm.. Yes, in algebra. So, we have encounter several of these.. umm, number one um number two, I was not as familiar with the third one um and I was not as familiar with the fourth one, but the first and second one...

**Excerpt 5.2.1.1**

Tonje specifically indicated that she was not familiar with the first task but rather the three others. She also told me of her experiences in solving such task at the middle school level. Helge on the other hand was not familiar with the third and fourth tasks. On the whole, this results shows that all the students have some prior knowledge about the tasks they solved. It is not surprising that the students performed well, although few of the students were unfamiliar with some of the algebraic word problem tasks given.

In the next section, the analysis of the type of word problems and students' conceptions about the difficulty of the various tasks is presented.

### 5.2.2 Types of word problems

The algebraic word problem tasks in Figure 10 on page 58 given to the students were concrete factual[41], concrete hypothetical, abstract factual and abstract hypothetical respectively. In the first and third tasks no change is described (all the information given in these questions are factual). In the second task, the girls are not really 13 times as many and the boys are also not really twice as many (all the information given in the question are hypothetical), the same applies to the forth task.

[41] The types of word problems are defined and explained in Section 3.3.1.1 (see page 36).

The students interviewed gave their thoughts about the difficulty of the tasked they solved. Results of the second interview question (Q2) in Table 14, which asked students to comment on the most difficult task among the algebraic word problem tasks (see page 66 for the second interview question), revealed that most students (4 out of 7) see the first task as a difficult (D) task. On the other hand, only two students from Group 1 and 4 among the others saw all the tasks as not difficult (ND). Among these two students Bjørg from Group 1 provided the reason she failed at the first try although she sees all the tasks as not difficult. Bjørg offered the following response:

12. Teacher: Ohk!.. So, when you went through the questions, which of the questions among the algebraic word problem task was the most difficult?
14. Bjørg: Umm.. None of them actually. It was like same form.. It was kind of same questions. It had similar, I don't know, equations, but um I kind of forgot to read through whole text, like whole problems, so I forgot to sum up in the end, so I got failed at the first time, but when I went through it like, I read the problem, then I got the right answer as the others in my group.

**Excerpt 5.2.2.0**

In Bjørg's response, the task given were of the same form, that is the task had similar equations. However, she forgot to read through the whole text and as a results she failed in her first attempt. And then, when she reread the problem fully, she had the same answer as the others in her group. In her account, the tasks itself is not difficult but the problem was about not reading the whole text before solving it.

From Table 14, four of the students (from Group 2, 3, 5 and 7) interviewed pointed out that the first task was the most difficult among the other tasks. Tonje from Group 5 provided the reason why she sees the first task as difficult and the impact the first task had on the other tasks. Tonje offered the following response:

464. Teacher: Ohk! So, which of the questions was the most difficult?
465. Tonje: The first one, because we had to like get into the way of calculating the other task as well. So, when we first solve the first one, we were kind of in the game, so we knew how to solve the others [Laughs]. The first one is like a very difficult warm up whilst the three last ones were kind of like, ohk we've done this, it's ohk now.

**Excerpt 5.2.2.1**

In Tonje's response, the first task was the most difficult since they have to get used to the way of solving it, and the solution of the others followed easily after solving the first task. One student, on the other hand, pointed out that the third and the fourth task were the most difficult tasks. Helge from Group 6 offered the following response:

571. Teacher: So, when you went through the questions, which of the questions among the algebraic word problem task was the most difficult?

573. Helge: Umm, definitely these two [Points at Task 3 and 4], number three and four, because I didn't really know how to solve them as um, I had solved some tasks that were kind of some link, but I didn't really solve the exact same ones, umm and so those were the most difficult ones, definitely.

**Excerpt 5.2.2.2**

Helge sees the third and fourth tasks as the most difficult, since he had not really solved tasks that are similar to these tasks. On the whole, most of the students view the concrete factual word problem task (Task 1) as the most difficult. And their reason was that, since it was the first task they have to get used to the way of solving it (For example, see Excerpt **5.2.2.1**) . By assumption, if the tasks were rearranged in terms of numbering, the students may then have chosen the first task as the most difficult task per the reason they gave earlier.

The next section presents the analysis of how the students mathematize the algebraic word problem tasks. Excerpts from the students' worksheets will reveal how they justify their strategies in solving the algebraic word problem tasks.

### 5.2.3 Mathematizing the word problem

If you look at Table 12, one can observe that under each task most of the students justify their strategies for solving an algebraic word problem by mainly using known technique, the substitution method (SM), to solve the algebraic equation(s) derived from the word problem. The process of mathematizing begins with understanding the word problem, translating the word problem using variables, setting up an algebraic equation, solving the equation using a known technique, verifying and presenting the final answer. The students usually use the substitution or additions/elimination techniques to solve linear equations or simultaneous sets of two linear equations derived from algebraic word problem tasks at school. The analysis presented in this section is based on the group work and not individual work of the students.

The solution to the first task on students' worksheet in Table 12, revealed that 5 groups used the usual known technique (SM and AEM) to mathematized the first task successfully. However, there was a computational error in the solution of one of these 5 groups' worksheet (See Figure 14). Among these 5 groups only Group 7 used the addition/elimination technique. Group 7 provided this solution on their worksheet (see Figure 13, for the solution to Task 1 by Group 7).

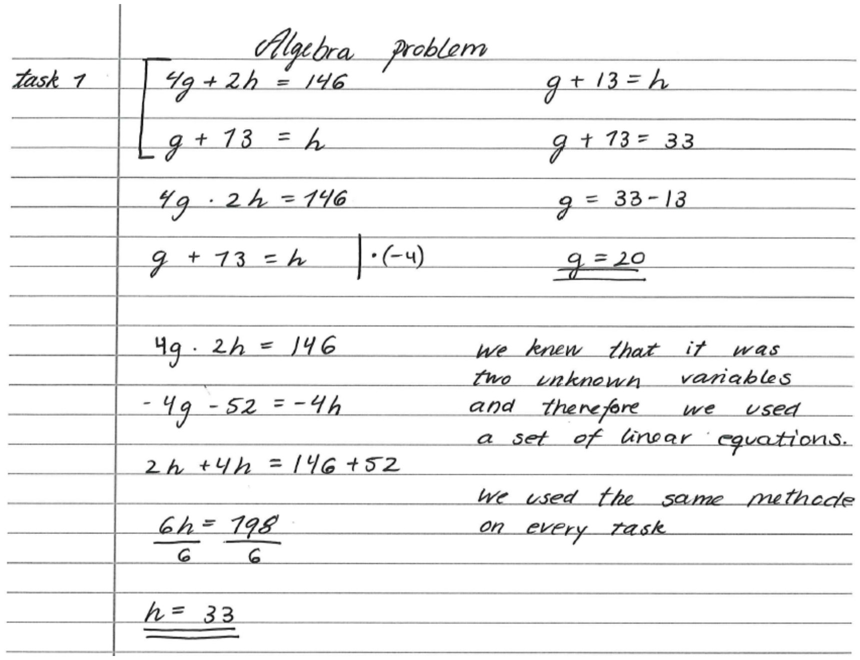



**Figure 13:** Group 7's solution to Task 1.

From Figure 13, we observe that the students translated the word problem into variables (g = goats and h = hens) and setup an algebraic equation (that is, simultaneous set of two linear equations). They used the additions/elimination technique to compute the algebraic equations. When the student representing Group 7 was asked about the processes they went through in solving Task 1, Arvid offered the following response:

671. Teacher: Can you tell me the processes you went through in solving these algebraic word problems?

673. Arvid: We tried a lot of different equations, we knew that we needed a set of linear equations because it was too many variables, so we started to sort of find the right equation, but we use a little time, because we had forgotten how to do it.

676. Teacher: So, what processes did you use in finding the right equations?

677. Arvid: We just tried different, and then we just look if some of them may give us the right answer, so yeah, we just tried a lot of different ones.

**Excerpt 5.2.3.0**

In Arvid's response, he stated that his group tried a lot of different equations since they knew they needed a set of linear equations to solve the problem. They tried to organized their data (that's the variables and information from the problem) and find the right equation per the information they had. Interestingly, Arvid noted that they spent some time in finding the right equations since they had forgotten how to go about it. Group 1 used the direct substitution technique to solve Task 1 (see Figure 14, for the solution to Task 1 by Group 1). They made an error when summing up the total number of goats and

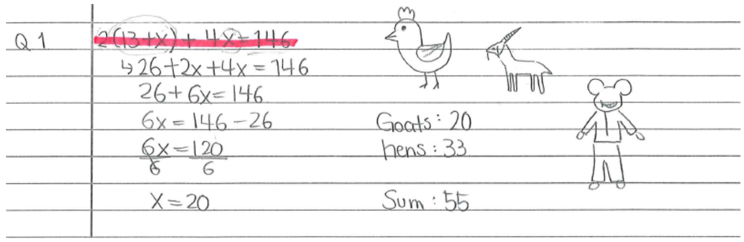



**Figure 14:** Group 1's solution to Task 1.

hens, although the they setup the right equation for the word problem. Bjørg, representing Group 1, offered the following response when prompted about the error:

41. Teacher: How many animals in all does the farmer have?
42. Bjørg: Then I got that, 20 goats in the garden. No! In the farm. Then I plus this together [Points at 20 goats and 33 hens] and I got 55 as an answer.
59. Teacher: Looking at 20 goats and 33 hens, you mean the sum is 55?
60. Bjørg: Oh!! Wait, what, 55, it supposed to be 53.

**Excerpt 5.2.3.1**

Bjørg just stated the sum of goats and hens that they calculated in their worksheet but when she was challenged about the total sum, she quickly had a second thought about the calculation of the total sum where she identified the error.

Interestingly among the seven groups, only two groups used an arithmetic technique to solve Task 1. Group 2 first tried the algebraic method but when they could not arrive at a desired solution they then used the arithmetic method. When Julie from Group 2 was asked why the first attempt failed (see Figure 15, for the first attempt solution of Group 2), she offered the following response:

197. Teacher: Ohk! How come the first attempt in question one failed?
198. Julie: Umm.. I don't know... Yeah, because when we.. These are the same [Points at $(2x+13*2)+4y = 146$ and $4y = 146-(2x+13*2)$] and this [Points at $(2x+13*2)$] is negative, like this [Points at $(2x+13*2)$] was subtracted by this [Points at $(2x+13*2)$] one, because we will sort the x's in one side and then this [Points at 146] one will go over to this [Points at 146] side, and then they will just subtracted each other and then the equation would be zero, equals zero [Laughs]. I don't know if that is right [Laughs].

**Excerpt 5.2.3.2**

In Julie's reasoning, after setting up the equation $(2x + 13 * 2) + 4y = 146$ in Figure 15, they made $4y$ the subject (that is, $4y = 146 - (2x + 13 * 2)$) and then substituted it into

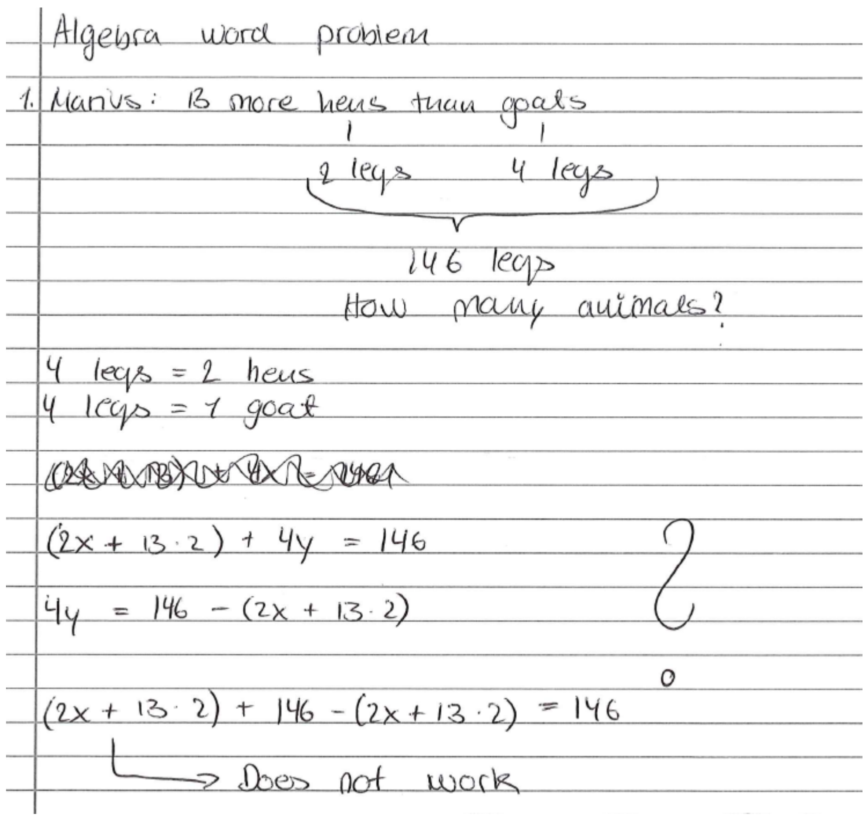



**Figure 15:** Group 2's first attempt of Task 1.

the first equation to eliminate $y$ and rather find $x$. However, in the end they found that the variables cancel each other and that they have 146 in both side of the equation. The error in the first attempt was because they wrote $4y$ instead of $4x$ in the first equation, since if the goats are $x$ then the hens should be $13 + x$ or if the goats were $y$ then the hens should also have been $13 + y$. When the algebraic method failed, Group 2 resorted to an arithmetic method (see Figure 16, for arithmetic solution to Task 1 by Group 2). Julie offered the following response when asked to explain her working process:

154. Teacher: Ohk! So, can you tell me the processes you went through in solving these algebraic word problems?

156. Julie: Umm. Well, I think we first thought about it which it which way we could solve this, if we could use umm two equations like $x$ and $y$ for the hens and goats um. But then I think we figured out that, that did not worked umm and then we tried to look at how many um animals they were or legs they were in um altogether. Umm, then we found out that since they were 13 more hens than goats we could just um subtract the hens legs from all of the legs. And then we could just part the legs in two um... Or divide them [Laughs] in two, and find out how many they were, I think.

**Excerpt 5.2.3.3**

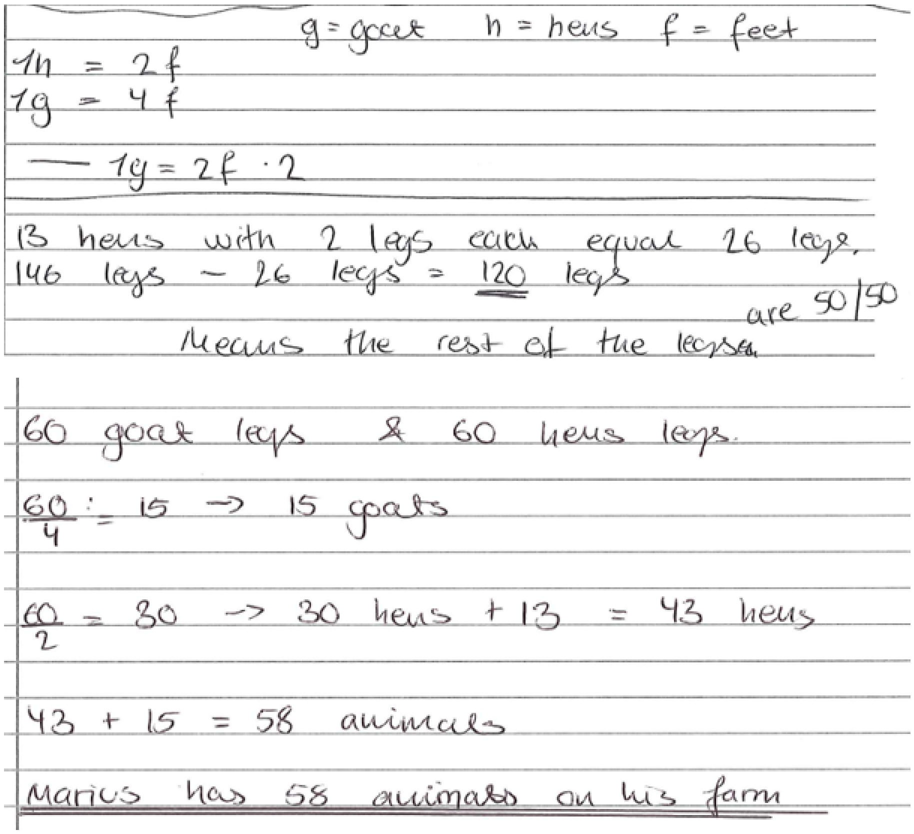
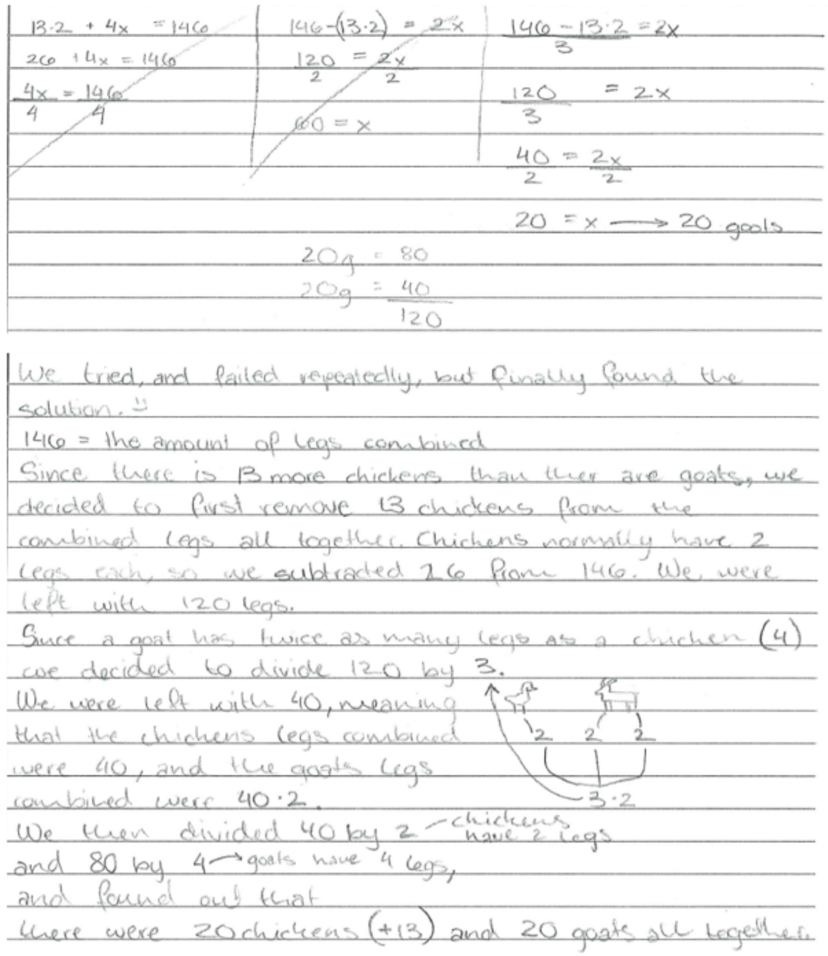

g = goat h = hens f = feet
1h = 2f
1g = 4f
— 1g = 2f · 2
13 hens with 2 legs each equal 26 legs.
146 legs − 26 legs = 120 legs
Means the rest of the legs are 50/50
60 goat legs & 60 hens legs.
$\frac{60}{4}$ = 15 → 15 goats
$\frac{60}{2}$ = 30 → 30 hens + 13 = 43 hens
43 + 15 = 58 animals
Marius has 58 animals on his farm

**Figure 16:** Group 2's arithmetic solution to Task 1.

Julie explains that since there are 13 more hens than goats, they subtracted $(13 \times 2)$ hen legs from the total legs and then equally divided the remaining legs among the goats and hens. From Figure 16, after dividing equally the remaining legs, they further divided by the number of legs of a goat and a hen respectively. And that they added the 13 to 30 making a total of 43 hens. In the end they had a total of 58 animals, which is a correct sum (that is $[15 \times 4] + [43 \times 2] = 146$).

Group 5 also used the arithmetic method but in a different way (see Figure 17, for the arithmetic solution to Task 1 by Group 5). Group 5 used the same method as Group 2 but the difference is when they noted that the goats have twice as many legs as the hens and therefore they divided the remaining 120 legs into 3.


$13 \cdot 2 + 4x = 146$
$26 + 4x = 146$
$\frac{4x}{4} = \frac{146}{4}$

$146 - (13 \cdot 2) = 2x$
$\frac{120}{2} = \frac{2x}{2}$
$60 = x$

$\frac{146 - 13 \cdot 2}{3} = 2x$
$\frac{120}{3} = 2x$
$\frac{40}{2} = \frac{2x}{2}$
$20 = x \longrightarrow 20$ goats

$20g = 80$
$20g = 40$
$120$

We tried, and failed repeatedly, but finally found the solution. :)
146 = the amount of legs combined
Since there is 13 more chickens than ther are goats, we decided to first remove 13 chickens from the combined legs all together. Chickens normally have 2 legs each, so we subtracted 26 from 146. We were left with 120 legs.
Since a goat has twice as many legs as a chicken (4) we decided to divide 120 by 3.
We were left with 40, meaning that the chickens legs combined were 40, and the goats legs combined were $40 \cdot 2$.
2 2 2
$3 \cdot 2$
We then divided 40 by 2 – chickens have 2 legs
and 80 by 4 → goats have 4 legs,
and found out that
there were 20 chickens (+13) and 20 goats all together.

**Figure 17:** Group 5's arithmetic solution to Task 1.

Below is the written text by Group 5 in Figure 17:

> We tried and failed repeatedly, but finally found the solution. 146 = the amount of legs combined. Since there is 13 more chickens than there goats, we decided to first remove 13 chickens from the combined legs all together. Chickens normally have 2 legs each, so we subtracted 26 from 146. We were left with 120 legs. Since a goat has twice as many legs as a chicken (4) we decided to divide 120 by 3. We were left with 40, meaning that the chickens legs combined were 40, and the goats legs combined were 40.2. We then divided 40 by 2 and 80 by 4 and found out that there were 20 chickens (+13) and 20 goats all together.

From Table 12 on page 78, if you look at the results of Task 2 from the students' worksheet, all the groups used the usual known techniques (SM or AEM) to mathematize the second task successfully. It is also seen from the results of Task 2 that all the groups but one used the direct substitution method (SM) setting-up a linear equation whilst only one group setup a simultaneous set of two linear equations and used the additions/elimination (AEM) technique to solve it. The same situation applies to Task 3 but in this time around one particular group misinterpreted (MT) the task which led to an incorrect expression/equation. Figure 18 shows how Group 7 misinterpreted (MT) the third task during the solution process. If you look at the solution in Figure 18, you see that there is a missing link between

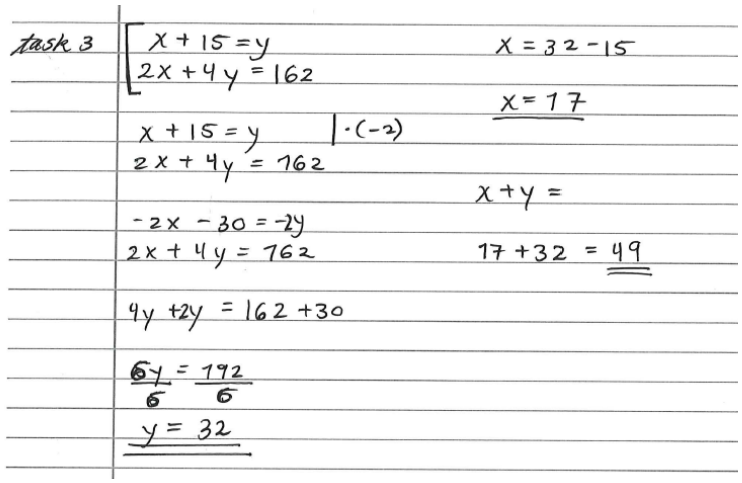



**Figure 18:** Group 7's solution to Task 3.

the first and second sentence in Task 3 (see Figure 10 on page 58). That is, in the first sentence they have the second number to be $x$ and the given number to be $y = x + 15$. When it comes to the second sentence which says 'the sum of two times the first number and four times the second number', they mistakenly took the first number to be $x$ and the second number to be $y$. The response of Arvid from Group 7 reveals that the students were not able to identify that the given number is the same as the first number. Arvid offered the following response when asked to explain how he solved the third task:

693. Teacher: Going back to the third and fourth questions. Can you resolve it again for me?

...............

698. Arvid: The value of a given number is fifteen more than the value of a second number. The sum of two times the first number and four times the second number is 162. So, we put variables $x$ and $y$, and we knew that $x$ is going, umm $y$ is going to be fifteen more than $x$.

702. Teacher: So, $y$ is the first number?

703. Arvid: $y$ is the first number.

704. Teacher: And $x$ is the second number?

705. Arvid: Umm, yes!

706. Teacher: Ohk! Then the next line=

707. Arvid: =No! No! Wait, no $x$ is the first. The second number is $y$.

...............

710. Teacher: And then we go to the next line.

711. Arvid: Umm, the sum of two times the first number, so $2x$ and four times the second number $4y$, which equals urr 162.

...............

715. Teacher: The first is $x$?

716. Arvid: Yes, and the second is $y$.

717. Teacher: So, it says the value of a given number is fifteen more than the value of the second number.

719. Arvid: Yes, so it wasn't... The first number was $x$, so the $x$ is fifteen more than the second number.

721. Teacher: So, the second number is $y$?

722. Arvid: Oh!! Ohk! Oh, so it was wrong [Laughs]. Ohk, so I don't know what to do then... It should be switch around?

724. Teacher: Yeah.

725. Arvid: Ohk!

726. Teacher: That's why I asked about that.

727. Arvid: Ohk! So, it should be $y + 15$ equals $x$.

728. Teacher: So, it like the first line and the second line of the question.

729. Arvid: Ohk! So, we just misunderstood the task then.

**Excerpt 5.2.3.4**

Arvid misunderstood the task, he could not comprehend the task well which therefore led to an incorrect expression. In Arvid's reasoning, he considered the first number as $x$ and the second as $y$. However, when the task asked that the given number is fifteen more than the second number, he used the expression $x + 15 = y$ instead of $x = y + 15$. This reveals that Arvid and his group members were not able to identify that the given number is the same as the first number.

From Table 12 on page 78, the solutions to the fourth task on students' worksheet revealed that 2 out 7 groups misinterpreted (MT) the fourth task. If you look at Table 12 under Task 4, you can see that Groups 5 and 7 misinterpreted (MT) the word problem and therefore used an incorrect expressions for the task. For Group 7, since Task 3 is equivalent to Task 4, it is not surprising that they applied the same way of solving Task 3 to solve Task 4. However, it is not the same as Group 5 since they successfully solve Task 3 but misinterpreted Task 4 although Task 3 is equivalent to Task 4. Tonje from Group 5 offered the following response when asked to resolve Task 4 (see Figure 19, for the solution to Task 4 by Group 5):

492. Teacher: Ohk! Can you please explain question four in the algebraic word problem task for me?

494. Tonje: So, um 'A given number is six more than a second number'.

495. Teacher: Yeah.

496. Tonje: Umm, we put the second number as $x$, because you don't add something to it, and then the first number is then $x + 6$ because you add something. So, then we just put in the umm the multiplications or the number.

499. Teacher: So, can you explain the multiplication for me?

500. Tonje: Yeah, so the first number you multiply with 4.

501. Teacher: So, the first number was $x$?

502. Tonje: The first number was umm, wait [Laughs]. Oh! I wrote something wrong [Laughs]. Ohk! So, I might have done something wrong here, umm. I meant to put 4 there [Points at $x + 6$ in the equation] and 2 here [Points at $x$ in the equation]. Ohk! Yeah [Laughs].

**Excerpt 5.2.3.5**

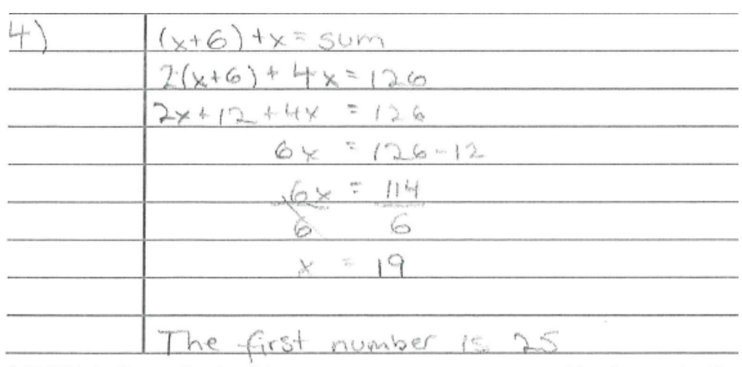

4)
$(x+6)+x = \text{sum}$
$2(x+6)+4x = 126$
$2x+12+4x = 126$
$6x = 126-12$
$\frac{6x}{6} = \frac{114}{6}$
$x = 19$
The first number is 25

**Figure 19:** Group 5's solution to Task 4.

Tonje identifies the error in the process of resolving the task, the error here was that she wasn't able to identify that the given number is the same as the first number.

The results of the third interview question (Q3a) from Table 14 on page 80, which asked students how did they know their answers were right, revealed that the students looked back (LB) or substituted their answer in the equation to double check. Six students (out of 7) verified their answers by looking back (LB), that's substituting the final answer in the equation to double check whether they are right. Among these six groups, Hilde from Group 3 explains how her group members discuss the answer among themselves and also think about the answer whether it sounds right or make sense. Hilde offered the following response:

311. Teacher: So, in the end how did you know your answer was right?
312. Hilde: ... Umm. We just talked about it and we kind of try to see if we could do it other ways or umm just use logic. So, we know that if there were 20 goats, and 20 goats have 4 legs, that's 80 legs. And if there were 33 hens and they have 2 legs, we just.. Yeah, we went round and put $x$ equals 20 into the equation and see if it equals 146.

**Excerpt 5.2.3.6**

In Hilde's reasoning, they thought about the answer, whether it sounds right or make any sense; that is, working backwards using the answer she already got. For instance, in Task 1 she had the number of goats to be 20 and the number of hens to be 33. And that if goats have 4 legs and hens have 2 legs, there will be 80 goats legs and $(33 \times 2)$ hens legs and the sum will be $80 + 66 = 146$ as found in the equation.

In summary, the results revealed that the students use the known techniques mostly to solve algebraic word problems. However, if they are not able to comprehend the word problem or translate the problem text into a suitable algebraic expression, they reread the problem and then use the arithmetic method to solve the algebraic word problem. It is also seen in Table 12 on page 78, that very few of the students have difficulties (MT) in comprehending abstract factual and abstract hypothetical word problems (Task 3 and 4). The results also shows that all the students have some prior knowledge about the tasks they solved (very few of the students were unfamiliar with some of the algebraic word problem tasks). For this reason, it was not surprising that most of the students performed well. The results again reveal that most of the students view the concrete factual word problem (Task 1) as the most difficult, for the reason that it was the first task and they have to get used to the way of solving it. By assumption, if the tasks were to be rearranged in terms of numbering, then the students might have chosen the task that comes first as the most difficult task.

The next section presents the analysis of the mathematical modelling activities of the students.

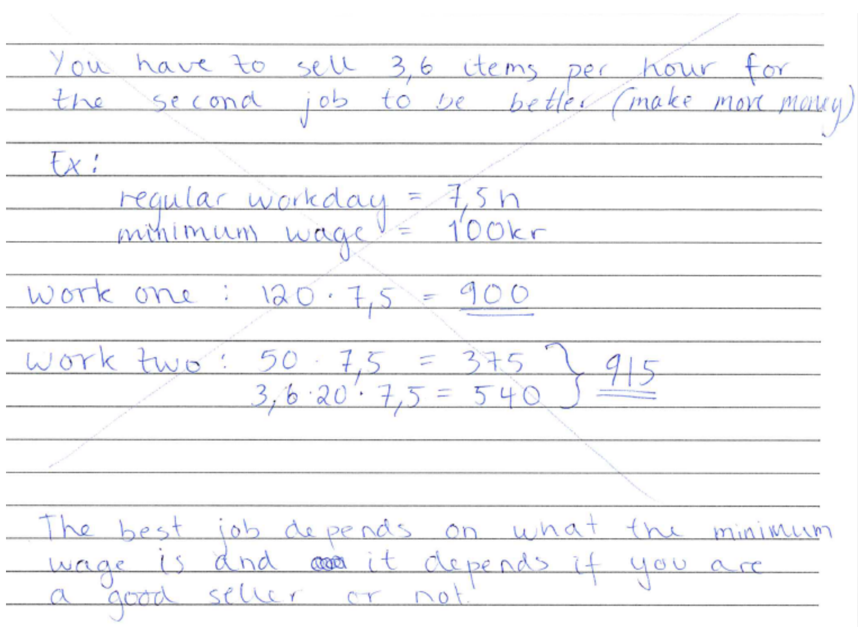
## 5.3 Mathematical Modelling

This section has five sub-sections; the first subsection starts with an analysis of the modelling process of the students. An analysis of the various mathematical representations the student used is presented followed by an analysis of how the students interpreted their model. Next to this, is the analysis of the students conception on modelling activities at school and the importance of group work. Tables 13 and 14 on pages 79 and 80 respectively presents the table of summary of results from the students' modelling worksheets and the students interviews respectively.

### 5.3.1 Modelling process/cycle

In the modelling process, there is the need to transform the real world problem to a mathematical problem by making some assumptions and also formulating an expression or equation.

The results in Table 13 revealed that, six (out of seven) groups clearly stated their assumptions (MA) whilst one group indirectly considered some assumptions (that was only seen in the equation they used, see Figure 25 on page 101 for the algebraic and graphical representation of Group 5's solution). The modelling task requires two different equations representing the two jobs described in Figure 11 on page 60. From Table 13, only one group (that is Group 4) used the arithmetic method (AC) to find an expression and also the break-even point of the two jobs, whilst the other groups formulated equations (FE) and solved them the algebraic way. Figure 20 shows how Group 4 solved the modelling task the arithmetic way. Although this particular group used the arithmetic method, they were able to find the correct break-even point. They used direct translation of the problem text alongside with numerical values in finding the desired solution.



You have to sell 3,6 items per hour for the second job to be better (make more money)

Ex:
regular workday = 7,5 h
minimum wage = 100kr

work one : 120 · 7,5 = 900

work two : 50 · 7,5 = 375
3,6 · 20 · 7,5 = 540
915

The best job depends on what the minimum wage is and it depends if you are a good seller or not

**Figure 20:** Group 4's solution to the modelling task.

Five other groups formulated expressions/equations (FE) for the two jobs but the equation for the second job was incorrect (IE) as seen in Table 13. Group 7 did not clearly formulate an expression/equation (FE) but their equations were seen through the graphical representation (GR) they used, however they also had the equation that describes the second job in the modelling task to be incorrect (IE) (see Figure 28 on page 104 for the graphical representation of Group 7's solution). Two of these five groups (that is Group 1 and 3) thought that the commission in the amount of 20kr per item she sells is what Kristin earns the whole working period but not on the hourly basis.

Group 3 tried to find the break-even point (the point at which job 1 equals job 2, or the number of items that Kristin must sell for Job 2 to be equal as Job 1) but got it wrong since they formulated an incorrect expression for job 2. Figure 21 shows how Group 3 solved the modelling task by formulating two sets of equations for the respective jobs. If you look at the second equation in Figure 21, we see that although the second equation is incorrect (the correct expression should be [$y = \frac{a}{2}x + 20bx$]), the student went further to manipulate the two equations to find the break-even point.

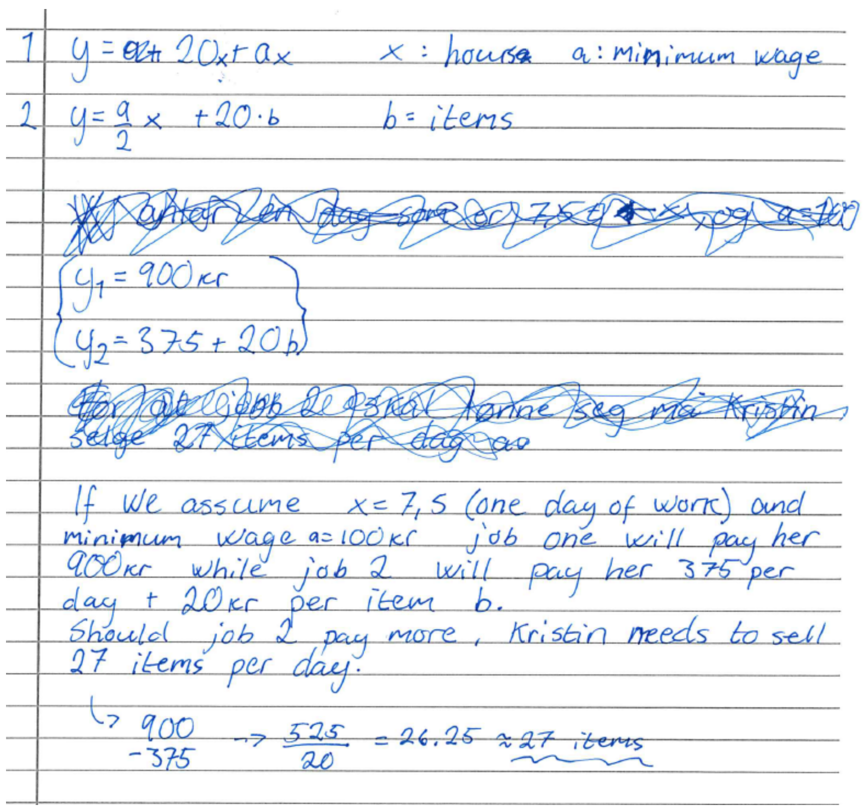


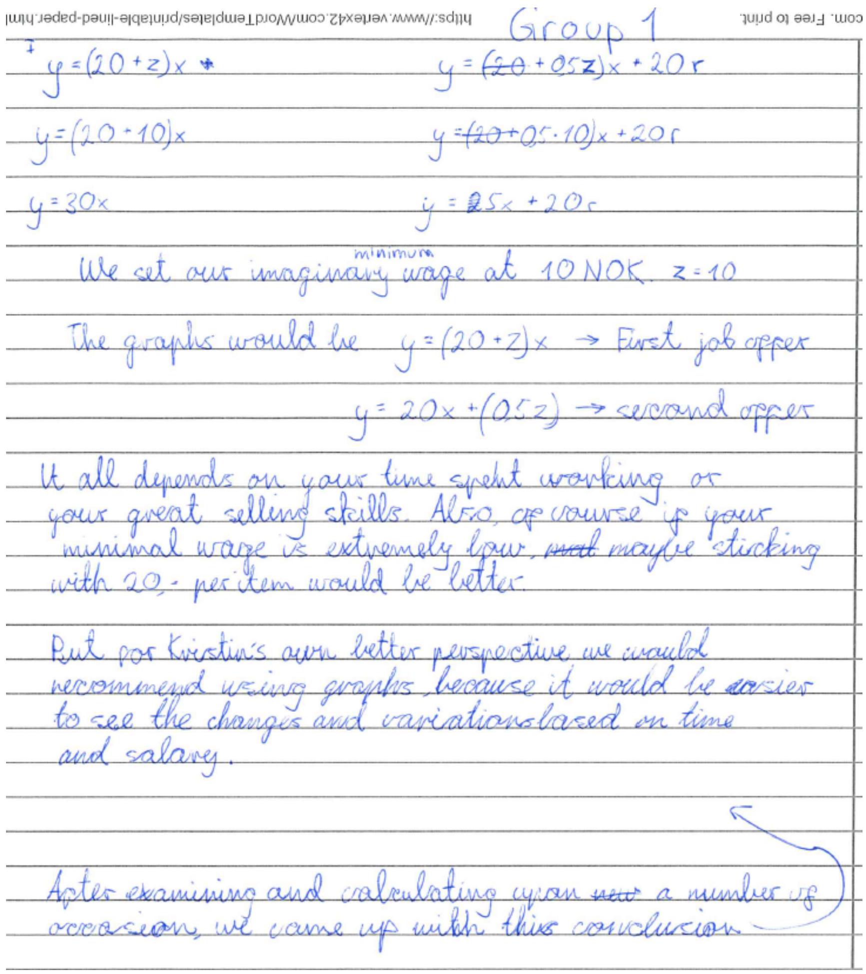

**Figure 21:** Group 3's solution to the modelling task.

On the other hand, Group 1 had similar equations as Group 3 (see Figure 22, for the solution to the modelling task by the first group) but the students were not able to manipulate the two equations to find the break-even point. This was as a results of the third variable introduced in the second equation since the students are more used to two sets of equations involving the variables $x$ and $y$. Bjørg from Group 1 offered the following response when asked for some clarifications on the second job equation:

109. Teacher: Ohk! What does the $r$ means in the equation for the second job?
110. Bjørg: Umm. I think is for the amount of the items.
111. Teacher: So, is it the amount of items for only one hour or?
112. Bjørg: We thought that per item cost 20kr. So, $x$ is the hour and $r$ is the amount of the items.
114. Teacher: Ohk! It means that you have $x$ here [Points at $5x$ in the second equation] but you don't have it here [Points at $20r$ in the second equation].
116. Bjørg: No.
117. Teacher: Why?
118. Bjørg: Umm. . . .
119. Teacher: Maybe you can have $x$ at both sides [Writes $y = 5x + 20xr$]
120. Bjørg: It could be.
121. Teacher: Because every hour you have 20kr on each of the items you sell.
122. Bjørg: Yeah.
123. Teacher: Because, this [Points at the equation $y = 5x + 20r$] sounds like only one hour.
124. Bjørg: Yeah, it makes sense. So, $x$ could be on both sides or we could just write the equation $y$ equals $(5 + 20r)x$.
126. Teacher: Yeah. Did you checked the point at which equation 1 and 2 will be the same?
127. Bjørg: We didn't, we thought that this had two different, whole different answer. So, we did not.
129. Teacher: Maybe you didn't consider this equation [Points at the equation $y = 5x+20xr$].
130. Bjørg: Yes, we didn't quiet understood the second one, the second job.

**Excerpt 5.3.1.0**

Bjørg realized that the equation should have being $y = 5x + 20xr$ instead of $y = 5x + 20r$ but when asked about the point at which both jobs will be the same, she noted that the two equations are different with different answers. This was as a results of the third variable ,$r$, introduced in the second equation. Bjørg's conclusion reveals that Group 1 did not really understood the statement for the second job which led to an incorrect expression/equation (IE).



Group 1

$y=(20+z)x$ $\qquad$ $y=(\cancel{20}+0.5z)x+20r$

$y=(20+10)x$ $\qquad$ $y=(\cancel{20}+0.5\cdot 10)x+20r$

$y=30x$ $\qquad$ $y=25x+20r$

We set our imaginary minimum wage at 10 NOK. $z=10$

The graphs would be $y=(20+z)x$ → First job opper

$y=20x+(0.5z)$ → second opper

It all depends on your time spent working or your great selling skills. Also, op course ip your minimal wage is extremely low, ~~not~~ maybe sticking with 20,- per item would be better.

But por Kristin's own better perspective we would recommend using graphs because it would be easier to see the changes and variations based on time and salary.

Apter examining and calculating upon ~~new~~ a number op occasion, we came up with this conclusion.

**Figure 22:** Group 1's solution to the modelling task.

Below is the text written in Figure 22 by Group 1:

> It all depends on your time spent working or your great selling skills. Also of course if your minimal wage is extremely low, maybe sticking with 20 per item would be better. But for Kristin's own better perspective we would recommend using graphs, because it would be easier to see the changes and variations based on time and salary. After examining and calculating upon a number of occasion, we came up with this conclusion.

Group 5, 6 and 7 had a similar equation for the second job described in the modelling task (their equation were different from Group 1, 2 and 3). These groups could not manipulate the third variable in the second job equation, and that they used the idea that in every hour Kristin earns a commission of 20kr on items she sells plus a constant half minimum wage (that is $y = 50 + 20x$, if the minimum wage is 100). This was done in order to have two equations with only $x$ and $y$ variables that they can handle since they are more used to such equations at school. The solution by Group 6 which presents the equation, numerical values and graph gives a true representation of the solution of the other two groups (Group 5 and 7). See Figure 23, for the illustration of how Group 6 solved the modelling task. In Figure 23, if you look at the two equations above and below the graph, the students substituted some real values into the equations and used the values obtained to plot the graph.

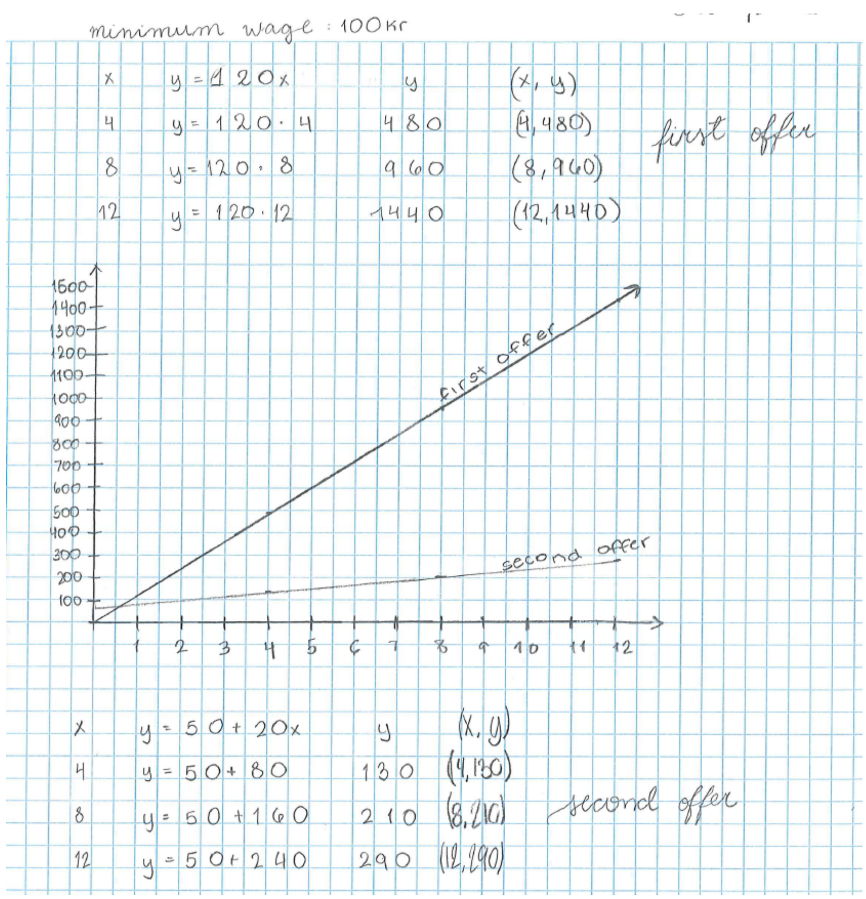
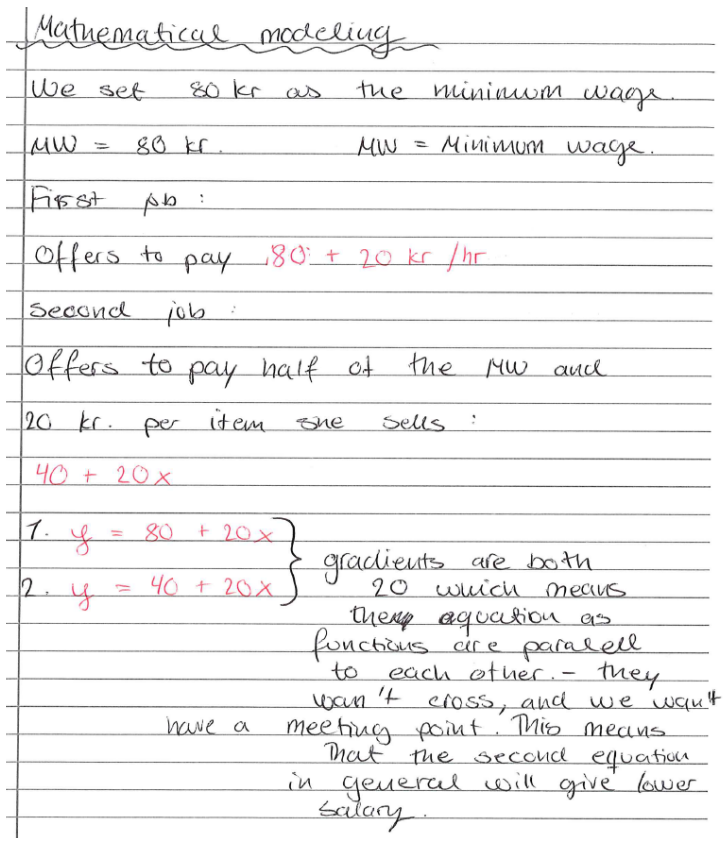



**Figure 23:** Group 6's solution to the modelling task.

One group (Group 2) out of the other six groups (the groups with incorrect (IE) second job equation) had both equations for the respective jobs incorrect. In Group 2's solution (see Figure 24, an illustration of the solution to the modelling task by Group 2), they considered that in both jobs the minimum wage and half the minimum wage will be constant whilst the number of hours only affects the additional 20kr above the minimum wage. If you look at the two equations in Figure 24, they used the variable $x$ where $x$ in the first equation



Mathematical modeling

We set 80 kr as the minimum wage.

MW = 80 kr. MW = Minimum wage.

First job:

Offers to pay 80 + 20 kr/hr

Second job:

Offers to pay half of the MW and

20 kr. per item she sells:

40 + 20x

1. y = 80 + 20x
2. y = 40 + 20x

gradients are both 20 which means the equation as functions are parallell to each other. – they won't cross, and we won't have a meeting point. This means that the second equation in general will give lower salary.

**Figure 24:** Group 2's solution to the modelling task.

represents the number of hours but also represents the number of items in the second equation. This was done to avoid the introduction of a third variable and only have the $x$ and $y$ variables that they can handle or manipulate. Julie from Group 2 offered the following response when asked to explain Group 2's solution process in Figure 24:

230. Teacher: Can you please tell me the processes you went through in solving this modelling task?

232. Julie: Umm. Well, first we read it and then we figured out that there was no specific minimum wage, so we just set one just to.. a kind of put a picture on how it looks umm. And then we put together the two equations because we wanted to use two and then put them into a diagram in Geogebra and then find the similarities or where the lines cross each other. But then we find out that the gradients were the same which means they kind of parallel to each other umm, which made the task very confusing because we thought much about it and then it just became more and more confusing. But then we tried to, like think how it would be in real life if she sells much, she would umm earn more with the um first or second job, yeah the second job which she sells items and so if she sells like minimum two or three items she would earn even more than the first one, even though the first one has like a better um salary in general when you see it at first. So, we tried to put it in a diagram but I don't know if it went right

> because the axis are different since the first equation has hours and the second one has items so, yeah. And then we just concluded with that the second job would be more umm, would better because she would mostly or umm, she is going to sell more than two items an hour or three, because yeah, if it's a summer job people go shopping all the time [Laughs].

**Excerpt 5.3.1.1**

In Julie's response, reading the problem and also making some assumptions was the first step taken in solving the modelling task. They formulated two equations for both jobs and tried to use Geogebra to find the similarities between the two equations (that is, where the two equations crosses each other). In the process of finding the similarities they realized that both equations have the same gradient (that is, the line graph of the two equations were parallel to each other) which made the task more confusing for them. To resolve this confusion, they try to think about how it would be in real life if Kristin sells more items. They then found that if Kristin sells two or three items she would earn more.

From the mathematical problem to a mathematical solution, the students would have to solve the equations and also verify their model by looking backwards to check their answer. Results from the students' worksheet in Table 13 on page 79 revealed that apart from Group 4 who successfully used the arithmetic method, two groups (Group 2 and 3) out of the six groups that had their equation to the incorrect (IE), solved (STE) their equations by finding the gradient and break-end point of the two equations respectively (see Figures 24 and 21). The remaining 4 groups did not solve the equations because they couldn't manipulate the third variable in the second equation. If you again look at Table 13, only two groups (out of the 7 groups) verified (VM) their model whereas the other five groups did not verify their model.

In summary, the results revealed that few of the students used the arithmetic method (AC) to successfully solve the modelling task whilst most of the students formulated equations (FE) to solve the model. However, not all the students who used the algebraic equations could set up the right equation (mostly they had the second job equation incorrect). These students were not able to find the correct break-even point since it was difficult to manipulate the third variable in the second equation, and that they tried to get rid of the third variable to have only $x$ and $y$ variables (since they are more used to simultaneous equations with two variables). The results also revealed that few of the students were able to solved and verified the equations they formulated but then their equations were incorrect.

The next section presents the analysis of the various mathematical representations the students used for clarity when interpreting their model.

### 5.3.2 Mathematical representations

The students used graphical (GR), numerical (NR), algebraic (AR) and verbal (VR) representations in finding a suitable model for the modelling task. The results in Table 13 on page 79 revealed that the students (in each group) used at least two of the representations above when solving the modelling task.

The students used a graphical representation (GR) to provide a clearer picture of the real valued function they modeled from the task. The results of the solution to the modelling task on students' worksheet in Table 13, revealed that five groups (out of seven) used graphical representation to explain their model. Group 5, 6 and 7 had similar graphs since their formulated equations where the same (see Figures 23 and 25, for the graphical representations of Group 6 and 5 respectively).

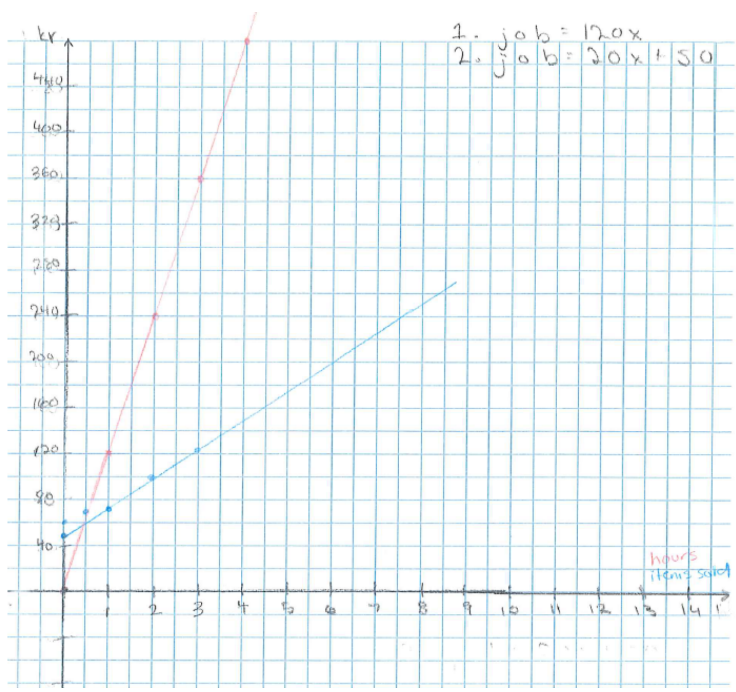



**Figure 25:** Group 5's solution to the modelling task.

In Figure 25, the students used the same $x$ variable to represent the number of hours in first equation and number of items in second equation. Although the expression/equation for the second job is incorrect, their graph gives a clear picture of the two equations ($y = 120x$ and $y = 20x + 50$).

Group 1 had the same graph as Groups 5 and 6, but the difference was that they considered the situation where no item was sold (see Figure 26, for Group 1's graphical solution to the modelling task). From Figure 26, Group 1 used the graph to find out the difference between the two jobs if no item was sold. That is, a person will earn more in job 1 in situations where there is no sales. The two equations in Figure 26 were $y = (10+20)x$ and $y = (5 + 20z)x$, where they considered $z = 0$ (that is the point where no item was sold).

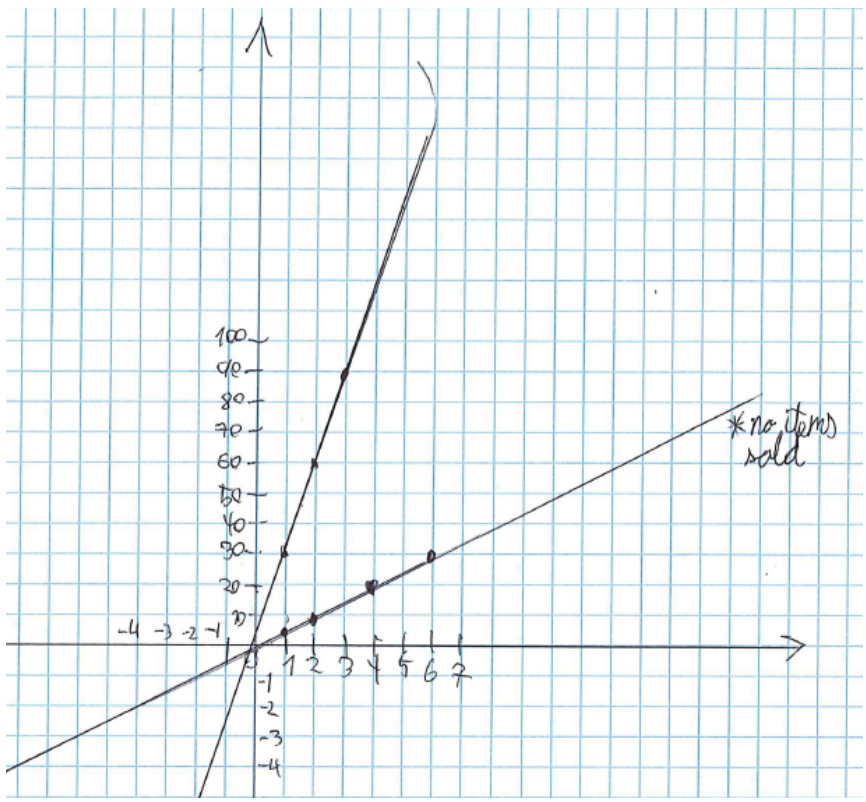



**Figure 26:** Group 1's graphical solution to the modelling task.

Group 2's graphical representation of the modelling task was quiet interesting. In Figure 27 (Group 2's graphical solution to the modelling task), the $x$ variable in both equations represents the number of hours but there is a hidden variable multiplying $20x$ in the second equation ($y = 40 + 20x$). It is seen beneath the graph in Figure 27 that, if a person sells 3 or 2 items then he/she will earn $40+60x$ or $40+40x$ respectively (the hidden variable here is $z$, that is $y = 40 + 20xz$). This was done by the students to discard the third variable in the second equation to have only $x$ and $y$ variables for easy manipulation.

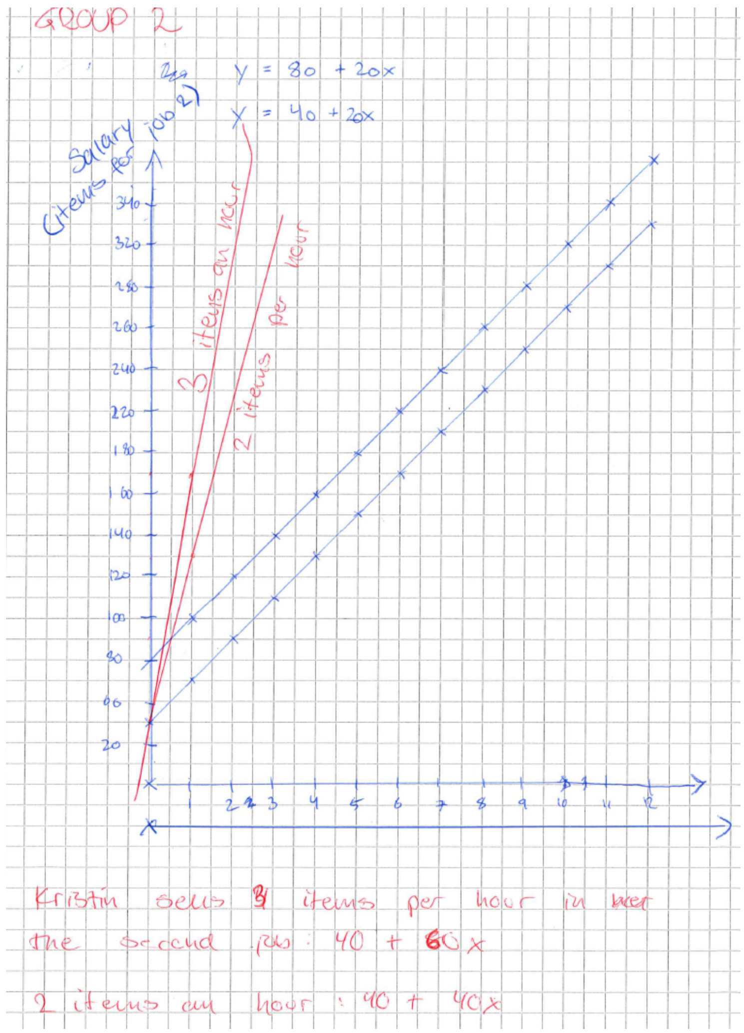



**Figure 27:** Group 2's graphical solution to the modelling task.

Below is the written text by Group 2 beneath the graph in Figure 27:

> Kristin sells 3 items per hour in the second job: $40 + 60x$
> 2 items an hour: $40 + 40x$

The students again used the numerical representation (NR) in acquiring a first understanding of the problem and also in investigating a particular case. The results of the solution to the modelling task on students' worksheet in Table 13 on page 79, revealed that two groups (Group 4 and 6) used numerical representation in their working process. It is seen in Figure 20 on page 94, that Group 4 solved the modelling task the arithmetic way. Group 4 directly translated the problem text using some numerical values, however their solution lacks generality since it does not provide a general picture of the solution (the introduction of some algebraic equations could have made the solution more general). Group 6 on the other hand also used the numerical representation in their solution. It is seen in Figure 23 on page 98, that Group 6 substituted some numerical values in the algebraic equations they formulated and also drew a graph based on the results they had. Although the second job equation they formulated was incorrect however the use of algebraic, numerical and graphical representations gives a more clearer and general picture of their solution.

The students used the algebraic representation (AR) for a concise and general picture of their solution and also the formulation of a mathematical model for their solution. The results in Table 13, revealed that five out of seven groups used the algebraic representation in their working process. These five groups formulated algebraic equations for both the first and second jobs. Although Group 7 did not write down their equations, the formulated equations was seen through the graphs they drew.

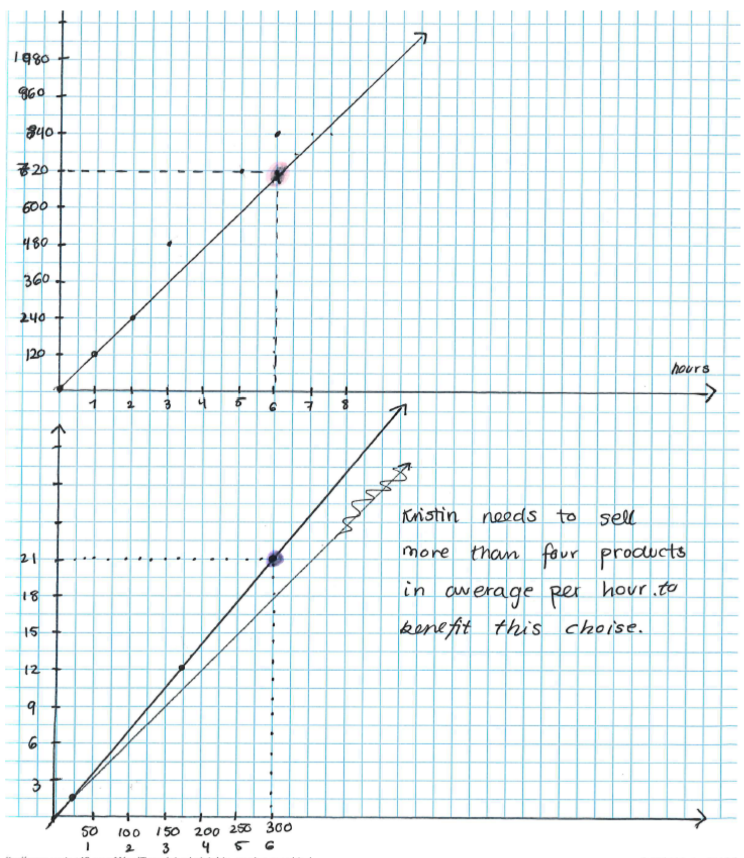



**Figure 28:** Group 7's graphical solution to the modelling task.

It is seen in Figure 28 (the graphical solution to the modelling task by Group 7), that Group 7 used the equation $y = 120x$ in drawing the first graph but the second graph was a little complicated. Arvid from Group 7 offered the following response when asked to explain the second graph in Figure 28:

776. Teacher: But the second I don't really understand.
777. Arvid: No we were.. We want as a... We didn't care [Laughs] if you can say like that, we made a few short cuts, we just come to an answer and just look like it's going to give us the right answer.
780. Teacher: But can you tell me more about the second graph?
781. Arvid: We should probably switch around the $x$ and $y$ axis, because then we get the kroners up here [Points at the $y$-axis] and the stuff she needs to sell up there [Points at the $x$-axis]. And it should start at 50, because the minimum wage is 50 in the second, so it should start at 50 and then go up by 20 per items she sells, so we work a bit not so perfectly in line with that one.
786. Teacher: Ohk! So, you made the equation 50kr plus the items that she sells=
787. Arvid: =Yeah, $y$ equals $20x + 50$ I think
788. Teacher: So, it means if she sells one item then it's going to be 70.
789. Arvid: Yeah, 70.
790. Teacher: Ohk, so when it's one then we have 70, but it's 50 on the graph.
791. Arvid: Yeah, it was not as perfect as we wanted it to be, but wasn't it also going up by 50 at the time in the hour as well. Wasn't it?
793. Teacher: Ohk! You had an idea.
794. Arvid: The idea was around that, but we didn't know how to write it down actually [Laughs].

**Excerpt 5.3.2.0**

In Arvid's response, the second job graph starts with 50 since the minimum wage is 50 and then go up by 20 per the item Kristin sells (that is, $y = 50 + 20x$). The equation Arvid gave did not match with the second graph and when he was further asked to explain, he made known that the graph was not as perfect as they wanted it to be and although they had an idea but they didn't know how to write it down. This was as a results of their inability to manipulate the third variable in the second equation since they were used to equations with two variables.

The students used verbal representation (VR) for their final interpretation of the results obtained in the solution process. The results in Table 13, revealed that six groups used verbal representation in the solution process.

The next section presents an analysis of how the students representing their respective groups interpreted their model during the interview section.

### 5.3.3 Interpretation of the model

Solving the mathematical modelling task mathematically alone was not enough for the students to answer the question. The opinions of the students' matters and it also influences the kind of answer they gave to this modelling task. If a student is a risk-averse, he/she might conclude that Kristin should take the first job since the pay is guaranteed. On the other hand, a risk-seeking student might conclude that Kristin should go for the second job for the possibility of making much more money. The students' might also argue about the kind of items sold and the personality of Kristin.

Apart from Group 4 that used arithmetic method (that is solving the modelling task numerically) to successfully solve the modelling task, all the other six groups formulated an incorrect algebraic equation for the second job. Although they had the second job equation to be incorrect yet still the kind of interpretation they gave shows how they reconcile their mathematical answer with reality, which made the mathematics more relevant and interesting. The students offered the following responses when asked to interpret their model:

102. Teacher: Let say I'm your friend. Can you interpret your modelling results to me, so that I can make a choice?

104. Bjørg: [Laughs] Ohk!... Umm.. It all depends on your time spent working or your great selling skills. Also, of course if your minimal wage is extremely low, maybe sticking with 20kr per item would be better. But umm.. Kristin's own better perspective we would recommend using graphs, because it would be easier to see the changes and variations based on time and salary.

251. Julie: Umm. I would say that if you would like to... If you are a person who works a lot and works for the money and earns.. and is a good person who manages to talk to your costumers and umm yeah, who knows stuff about what you are selling umm, then the second job would more.. would be better.

370. Hilde: Oh! That's hard umm. So, I would kind of [Laughs] ask, are you good at selling, because if you have to sell 27 items for it to be better than the first job offer. If you are good at selling then ohk, you will after a while earn more with equation number 2 or, yeah. But if you feel sort of insecure then this [Points at the equation of the first job] choice is safer.

443. Eirik: It depends on how much the minimum is and it depends on how much you are able to sell in one hour, yeah.

542. Tonje: Yes, and if you choose the other job then you have to be more, I don't know, you have to be like go further into how the people are or into what the people are going to buy, because you earn more when they buy more, which means you have to be more active, you have to um advertise the product, you have to go like, 'Hey, you wanna buy chocolate'. So, you have to work, you have to actually work to [Laughs] get money.

Yeah, the other one is like you do your job, but you still earn your money without nothing more.

643. Helge: We saw that the first offer was way better umm, because it depends on how many umm... So, you should pick the first offer um, but only if you good at the job um, so if you are a good seller umm and you at least sell 11 things per hour on average then you should pick the second job.

764. Arvid: Umm, I think it depends on the.. What was the store she was going to work in? [He reads the mathematical modelling task] I will take the second option because you have the potential to earn a lot more than you have to do with the first, but if you don't sell as much as you need to do in one of the hours you can probably sell a double amount in the second, so I will probably go with the second one even though it's a risk.

**Excerpt 5.3.3.0**

From the responses above, most of the students are risk-seeking students who believe there is a possibility of making much more money depending mostly on the kind of items sold, the personality of Kristin, and the minimum wage. Hilde, Eirik and Helge talked about the number of items sold per hour. Hilde and Helge were specific that if a person sells more than 27 and 11 items respectively then the second job would be better, although the values 27 and 11 were not correct as a results of miscalculation or formulation of incorrect equations. Few of the students went in for the first job since they thought the pay is decent and guaranteed.

In summary, it is seen from the students response that the choices they made depended on the minimum wage, the 'break-even' point (that is the number of items their friend needs to sell hourly in order for job 1 to be the same as job 2), the personality of their friend and the kind of items sold. These opinions of the students made their mathematical solution more relevant and interesting since their interpretation shows the link between reality and the mathematics they did.

The next section presents an analysis of the opinions of the students on their interest in modelling in school activities during the interview section.

### 5.3.4 Modelling in school activities

The students interviewed gave their thoughts about the connection between the mathematics they learn at school and the real world. Results of the fifth interview question (see page 66 for the fifth interview question, Q5) in Table 14 on page 80, which asked students if they see any connection between the mathematics they learn at school and the outside world, generally revealed that most of the students see some connection (SCO). Four of the students see some connection, however two out of these four students see rather a sort of imaginary connection (SIC), that is they believe there exist a connection but can not pinpoint any aspect of the mathematics they learn, where these connections can be seen. Julie from Group 2 gave a specific example when asked about the connection between the mathematics at school and the real world. Julie offered the following response:

207. Teacher: Umm. Do you see any connection between the mathematics you learn at school and the outside world?
209. Julie: Yeah, well not any of this connection.
210. Teacher: Can you tell me more about that?
211. Julie: Umm. If you want to be a person who builds houses and stuff, you can use trigonometry to like find the angles and sides and um yeah, with the goats on the farm if they [Laughs] don't want to go around and count every single one, they can [Laughs]..

**Excerpt 5.3.4.0**

In Julie's response, one can apply the trigonometry they learn at school to find the angles and sides when building a house and other architectural stuff. Two of the students do not really see any connection (NCO) between the mathematics they learn at school and the outside world. Eirik and Helge from Group 4 and 6 offered the following responses respectively:

416. Eirik: Umm, No. Most of the mathematics we go through now is more of theoretical, so you can't really use it in real life, unless something special.
607. Helge: Umm.. No, I don't think there is much of a connection.. Umm, but that's only because I don't really use um, or depends on what you do on your spare time, but I don't really use equations, I don't make like any real graphs on the spare time, but if you like to do this and you want to study mathematics when you grow older umm, when you get older you might want to be a teacher then it's kind of be helpful, but Uhh [Laughs].

**Excerpt 5.3.4.1**

From Eirik's response, most of the mathematics they study at class are more theoretical and that one can not really use it or apply it in real life unless something special. Helge on the other hand does not see any connection since he does not use the mathematics he learn on his spare time. Helge is also of the view that if a person wants to study mathematics when he/she is older or wants to become a teacher then it will be necessary if the person could work a bit more on the applications of mathematics on his/her spare time, like applying equations and sketching graphs. One of the students interviewed thought the connection depends much more on the particular mathematics subject (CDS). Arvid from Group 7 offered this response when asked about the connection between the mathematics at school and the real world:

> 732. Arvid: In sort of.. In some of the subject, like trigonometry, we have no idea what to use in the real world but some... But things like equations and some things like that, they are easy to use in the real world.

**Excerpt 5.3.4.2**

In Arvid response, it is easier to apply mathematical equations in the real world but for some subjects like trigonometry they have no idea how it can be applied in the real world.

The students in the next interview question (Q6) were asked about the number of times they solve mathematical modelling task or realistic task at school. The results of the sixth interview question (see page 67 for the sixth interview question) in Table 14 on page 80, revealed that most of the students are of the view that they only solve less mathematical modelling task (LMT) at school. Six of the students answered that they usually solve less mathematical modelling task at school. The kind of task the students solve at school are more close, where they use specific techniques to find the solution (see Appendix F on page 181 for examples of the task the students solve at school). One of these six students, Arvid from Group 7, tries to differentiate between the task they have at school and the modelling task in this research study, and also the reason why they solve less mathematical modelling task at school. Arvid offered the following response:

> 735. Teacher: Ohk! How often do you solve mathematical modelling task at school?
> 736. Arvid: Umm, not often like that. We found that task really hard actually, so we don't usually solve that.
> 738. Teacher: And what do you think might be the reason?
> 739. Arvid: I don't know, maybe is not in the things we need to learn, maybe comes further on in the year, semester, I don't know.. but we might actually get it, I don't know.

**Excerpt 5.3.4.3**

In Arvid's response, they found the modelling task really hard to solve compared to what they solve at class since the task was more open where you have to make some assumptions, find equations, and maybe draw a graph whilst the one they have at school is more close and direct, that is they follow a required procedure to arrive at a specific and unique answer. Arvid also thought that maybe they don't solve open tasks like the task in this research study because it might not be in the things they need to learn or maybe it comes further in the next semester.

The results of the seventh interview question (see page 67 for the seventh interview question, Q7) in Table 14, which asked students their opinion about the modelling task in research study, revealed that most students see the task as difficult (DT) and a realistic problem (RT). Six out of seven students interviewed have the opinion that the modelling task in this research study is a difficult task and that the task had no specific answer. Hilde from Group 3 gave some reasons why she sees the task as difficult. Hilde offered the following response:

344. Teacher: Ohk! What's your opinion about the modelling task you solved?
345. Hilde: ...Umm, my thought about it. It's abstract, so it's very hard to actually come up with an answer when you don't have all the information and it was kind of [laughs] irritating, because I always wanted to just find an answer, but I guess it's good to solve, to also just think about it.

**Excerpt 5.3.4.4**

In Hilde's response, the task is abstract and that it was difficult to come up with an answer due to less information outlined in the problem. She found this as irritating because she is used to the ways of finding a specific answer with a specific technique. In spite of Hilde's challenges with regards to the modelling task, she also acknowledge that the task is good to solve and also when thinking about it.

The eleventh question (see page 68 for the eleventh interview question) sought the conception of the students about the inclusion of mathematical modelling in school activities. Results of the eleventh interview question (Q11) in Table 14, which asked students if they want more of the modelling task at school, revealed that most students would want to have more of the modelling task (MMT) at school. Each of the students, those that want more of the modelling task, gave a unique reason as to why they want more of the modelling task at school.

Bjørg from Group 1 offered the following response:

131. Teacher: Ohk! So will you want more of this modelling task at school?
132. Bjørg: Yeah.
133. Teacher: Why?
134. Bjørg: Because I struggled with this problem, so I want to be good at this.
135. Teacher: And does it help in the learning process of mathematics?
136. Bjørg: Yeah.
137. Teacher: How?
138. Bjørg: Like in different, umm. Like you can think. To solve this problem, you have to think in many different ways and you have to like [Laughs]. You can have many different thinking ways.

**Excerpt 5.3.4.5**

In Bjørg's response, she struggled when solving the task and for that reason she wants more of the modelling task in other to be good at it. She also have the conception that the process of solving the modelling task helps the learning process since one have to think in many different ways when solving the task.

Julie from Group 2 offered the following response:

279. Teacher: So, will you want more of this modelling task at school?
280. Julie: Yeah. Well, I think it's a good task to like put several um situations together and have to use more um of the things you learn in class to solve the equation or the problem, you don't have to only use the one formula you learned in the class umm, you have to use like the formula you had last week or the one you learned a year ago, and then you have to put it all together to solve the problem, yeah.

**Excerpt 5.3.4.6**

In Julie's response, the task was a good task and that it requires the use of more of the things studied at class to solve it. That is, one does not need to use only one formula or technique learned at class but it involves putting together a formula studied a week or a year ago to solve it.

Eirik from Group 4 offered the following response:

447. Teacher: Ohk! Will you want more of this modelling task at school?
448. Eirik: Umm, yeah, it works fine.
449. Teacher: Can you tell me more about that?
450. Eirik: I'm just, you just fine with tasks that are a little bit harder and you gonna fine the tools and just use the tools to find what's there. It's nothing more than that.

**Excerpt 5.3.4.7**

Eirik has the conception that the task is a good task and fine to solve since its a little bit harder and one needs to find the tools used in solving the task and nothing more than that.

Tonje from Group 5 offered the following response:

552. Teacher: Ohk! So, will you want more of the modelling task at school?
553. Tonje: Yeah, I would say.
554. Teacher: Why?
555. Tonje: I think I learn more umm, what we do now is kind of just she gives us a paper or some explanation of what we are doing and then she just kind of let us read one sentence about what we are learning, and we are just on our own. But I feel like here we got to work together more and we got more task that um still match what we are learning, but what we are doing now is just go way out just.. very complicated um

yeah. But they kind of mention that since this is T-math or theoretical math, then we have to like make up our own theories kind of, so her giving us like one sentence is kind of what we are suppose to do but it doesn't help us. You kind of have to be good at math to be able to do stuff, so if we do this more then we can kind of interact more and we won't like slide out and change the theme while working together as often, because then we will have to focus on the certain thing.

**Excerpt 5.3.4.8**

In Tonje's response, she made it known that she had learned more during the modelling process. She also compared the current situation (the activities during the research study) to what is normally done at school where the teacher gives an explanation of what they are doing and then they are on their own. She also felt that in this modelling activity they have to work together and work on a task that match what they learn at school. Tonje made it known that one has to be good at maths to be able to solve the usual problems at school, however the modelling activity helps them to interact more whilst they don't deviate and change the theme while working together since they are more focus on the things they need to do.

Helge offered the following response:

655. Teacher: So, will you want more of the modelling task at school?
656. Helge: Umm, yeah!
657. Teacher: Why?
658. Helge: Because I think this task are fun to solve umm, but it's a little bit easier to do on a computer umm and it's takes a lot of time to like drawing the lines and the axes, so I like this kind of task, but I would like to do it on the PC.

**Excerpt 5.3.4.9**

In Helge's response, the modelling task was fun to solve but he suggested that it would have been easier using the computer since it takes longer time sketching the graph manually.

Arvid offered the following response:

796. Teacher: So, will you want more of the modelling task at school?
797. Arvid: Maybe more of the equations and yes, things we need to do or things we can use in the future as well. Like we don't need to know function analysis and things like that or a trigonometry but things like we can use umm, is good to learn things like that.
801. Teacher: Yeah, because every math that you study at school probably can be used outside there=
803. Arvid: =Yeah.
804. Teacher: Like trigonometry, they can ask you maybe you want to build or paint a house,

where do I place the ladder? So, you need to find the angles involve. Math can be used everywhere.

807. Arvid: Yes, it can, but some of the things we go through, I don't know how to use.

808. Teacher: So, the problem is how.. As teachers how, we are supposed to give task that you can relate=

810. Arvid: =Yes, or maybe if we have uhm... I think we are going to maybe say what we can use this for in the future, because some just say you will find out, you will figure it out, and then we don't know what to.

**Excerpt 5.3.4.10**

In Arvid's response, they would need more of the things they can use in the future. He gave an example that there is no need to study function analysis and trigonometry but rather things they can use outside school. When an example of how trigonometry can be used in the real world, Arvid made it known that some of the things they go through at school he does not know how to use them in the real world. He further suggests that teachers should tell them how the things they learned at school can be used in the future, since they just tell them they will find out which leaves them in a situation where they don't know what to do.

### 5.3.5 Group Work

The students worked in groups to complete both the algebraic word problem and modelling task. During the interviews the students where asked about their conception on whether working in groups was helpful (GWH) or not (GNH). The results of the third interview question (see page 66 for the third interview question, Q3c) in Table 14 (on page 80) , which asked students whether working in groups was helpful or not, revealed that most of the students have the conception that the group activities helped (GWH) them to achieve what they could not achieve on their own. Helge from Group 6 gave reasons why working in groups was helpful and also the disadvantage involve. Helge offered the following response:

591. Teacher: Ohk! Was it helpful when you solve these questions in a group?

592. Helge: Yeah, because if you didn't know how to solve it umm then when another person manages to solve it, you can (inaudible) ask them, or how did you solve this task? Umm, but if no one can really solve it then it would have been a problem but then you just have to ask the teacher um, but it is also like a downside within the group because umm the group might not be helpful, you might understand what they are doing but it is not really beneficial because um if you are in a group where everyone is working on their own um and you aren't really sharing ideas and you don't know what to do, then you will just be sitting there wasting time.. umm, but it can be really

helpful.

**Excerpt 5.3.5.0**

In Helge's response, group work helps since if you doesn't know how to solve the problem you can get help from a friend and if no one in the group knows how to solve then the group can get some hints from the teacher as they work together. However, there could be a challenge within the group if everyone works on his/her own and aren't sharing ideas and if you don't know what to do then you will just be sitting there wasting time.

In summary of this chapter, the results revealed that the students have some prior knowledge about the algebraic word problem tasks they solved. However very few of the students had difficulties in comprehending abstract factual and abstract hypothetical word problems. The students mainly used a known technique to solve the algebraic word problems, and that if they are not able to comprehend the word problem or translate the problem text into a suitable algebraic expression, they reread the problem and then use an arithmetic method to solve the problem. Few of the students again used an arithmetic method to successfully solve the mathematical modelling task whilst most of the students formulated equations to solve their model. Although their formulated equations were not correct but their equations, graphs and interpretation showed how they reconcile their mathematical answer with reality making the mathematics more relevant and interesting. The results also revealed that most students would want to have more modelling activities at school.

The next chapter presents the discussion of the study where the results are discussed in light of the literature review.

# 6 DISCUSSION

This chapter presents a discussion of the research findings. The discussion involves the link between the researcher's interpretations and the literature review. The research questions are first addressed. This is followed by a discussion of the study which entails students' prior knowledge, the types of word problems, the mathematization of the word problem, students' modelling processes, students' conceptions on both modelling in school activities and group work.

## 6.1 Addressing the Research Questions

I repeat, for the reader, the two research questions:

1. How do upper secondary students' justify their strategies for solving algebraic word problems?

2. How do these students interpret their findings after solving an algebraic word problem in a form of mathematical modelling?

The aim of the study was to investigate how upper secondary students solve algebraic word problems in the area of mathematical modelling. Data were collected through worksheets (group work) and students' interviews (individual students representing their respective groups) to answer the research questions. Specifically, data from the students' solutions to the algebraic word problem tasks aided in answering the first research question, whilst data from the students' solution to the mathematical modelling task also aided in answering the second research question.

In the first tasks (the algebraic word problem tasks) the students use a specific technique to solve the tasks in the process of mathematization (that is, justifying their strategy) whereas the second task (the mathematical modelling task) is open and does not require a unique technique in solving it. The justification of students strategies is also applicable in the second research question since justification is all about how the students mathematize the task but then the interpretation of their findings also captures their mathematization process since an interpretation of their findings may depend on the initial understanding of the problem, the construction of a mathematical model, the actual computation activities, and an evaluation of the outcome of the computation (which forms part of the mathematization process).

### 6.1.1 Addressing the first research question

The students usually use the strategy of transforming algebraic word problem tasks into a linear equation or simultaneous set of two linear equations and then use addition/elimination or substitution method to solve them at school. The justification of students' strategies is how they mathematize the problem. That is: understanding the problem (step 1); translating the problem using variables and setting up an algebraic equation (step 2); solving the equation using a known technique (step 3); verifying and presenting the final answer (step 4).

Students' worksheet responses were analyzed together with the interviews in order to find out how the students justify their strategies in solving the algebraic word problem tasks. The analysis of students' worksheet revealed that most students were successful on most of the tasks, but few were unsuccessful (had incorrect equation/expression 'EI', see Table 12 on page 78) on Task 3 and 4 whilst few of the students also used a different technique (an arithmetic method 'SPA', see Table 12) other than what they normally used at school to solve Task 1. Below is Task 1, 3 and 4:

> 1. Marius, a young farmer, has 13 more hens than goats. Since hens have two legs each, but goats have four legs each, all together the animals have 146 legs. How many animals in all does Marius have?
> 3. The value of a given number is fifteen more than the value of a second number. The sum of two times the first number and four times the second number is 162. What is the sum of the two numbers?
> 4. A given number is six more than a second number. If the first were four times and the second two times as large, their sum would be 126. What is the first number?

Two out of the seven groups of students used a different technique (arithmetic method) other than what they would normally use at school to solve Task 1, but they were successful in arriving at the desired answer. These students resorted to an arithmetic method when an attempt to set up an algebraic equation failed. The difficulty with Task 3 and 4 was about the students not able to comprehend the word problem which led to the setting up of a wrong equation.

**Task 1**: Why did the students use an arithmetic method instead of setting up an algebraic equation and solving it with a known technique? The students were familiar with the word problem they solved as revealed by the students' interviews. Most of the students view

Task 1 as the most difficult task for the reason that it was the first question and therefore they have to get used to the way of solving it. Now, if the students are familiar with the problem, why then did they use an arithmetic method instead of setting up an algebraic equation and solving it with a known technique. In the process of justifying their strategies or mathematizing Task 1, the students understood the problem but the issue was about setting up the right equation (step 2). That is, using the substitution method to formulate a linear equation with only one variable but they rather had an equation with two variables instead. These students found it difficult to solve one equation with two variables. When the algebraic method failed, they reread the problem and then used an arithmetic method to solve it.

**Task 3**: The task was familiar to the students but only one group could not comprehend the word problem. The difficulty in justifying or mathematizing Task 3 was in step 1 (understanding the problem). That is, the students misunderstood the problem and they could not find the appropriate link between the first and second sentence in Task 3 (precisely "a given number" in the first sentence and "the first number" in the second sentence), leading to an incorrect algebraic equation. Although the algebraic equation set-up by the students was incorrect, the equation can still be solved and verified by putting the final answer back into the incorrect equation. The students went on to step 3 and 4 since the incorrect equation they set-up could be solved and verified (see for example Group 7's solution to Task 3 in Figure 18 on page 90).

**Task 4**: The task was familiar to the students but two groups could not comprehend the word problem. The difficulty in justifying or mathematizing Task 4 was in step 1 (understanding the problem). That is, the students misunderstood the problem and they could not find the appropriate link between the first and second sentence in Task 4, leading to an incorrect algebraic equation. Although the algebraic equation set-up by the students was incorrect, the equation can still be solved and verified by putting the final answer back into the incorrect equation. The students went on to step 3 and 4 since the incorrect equation they set-up could be solved and verified (see for example Group 5's solution to Task 4 in Figure 19 on page 92).

Note that one group of students were not able to comprehend both Task 3 and 4. Task 3 is equivalent to Task 4, meaning they have a common story context and common solution procedure (similar equation). Apparently, the students might have applied the same solution procedure of Task 3 to Task 4 without making any adjustment.

In summary, the students were familiar with the algebraic word problem tasks they solved.

Most of the students regarded the first task (Task 1) as the most difficult task for the reason that it was the first task and that they have to get used to the way of solving it. In the process of justifying or mathematizing Task 1 to Task 4 (the algebraic word problem tasks), the students translate the problem using variables and set-up an algebraic equation. In a situation where the algebraic equation (whether correct or incorrect) can be solved and verified, they go on by solving the equation with a known technique whereas in situations where they can not solve the equation, they reread the problem and then resort to an arithmetic method.

### 6.1.2 Addressing the second research question

An interpretation of a modelling results may usually depend on: the initial understanding of the problem, the construction of a mathematical model, the actual computational activities, and the evaluation of the outcome of the computations. Students' worksheet responses were analyzed together with the interviews in order to find how the students interpret their findings after solving an algebraic word problem in a form of mathematical modelling. The analysis of students' worksheet revealed that very few students, who used an arithmetic method, were successful on the modelling task (Group 4 used an arithmetic method 'AC' and they did not get an incorrect expression 'IE', see Table 13 on page 79). Although most of the students formulated an incorrect equation, the use of a graphical representation of their model gave a general and clearer picture of the real value function they modeled.

One group of students used an arithmetic method to solve the modelling task. The students had an initial understanding of the problem text. They made some assumptions and went on to solve the problem numerically instead of formulating an algebraic equation. Although they successfully solved the modelling task, but the generality of their model could not be seen (see Group 4's solution to the modelling task in Figure 20 on page 94). That is, the numerical approach only investigate a specific or particular case and that the solution lacks generality.

The other group of students formulated an algebraic equation in the process of solving the modelling task. The students misunderstood the problem text (specifically the text that described the second job option in the modelling task). Although their second equation was wrong, the students went further to use a graphical representation in providing a clearer picture of the real valued function (algebraic equation) they modeled from the task. The formulated equation gave a concise and general picture of their solution. Why was the second equation not correct? The first equation (example: $\boldsymbol{y} = 120\boldsymbol{x}$) had two variables whilst the second (example: $\boldsymbol{y} = 50\boldsymbol{x} + 20\boldsymbol{zx}$) had three variables. The students were more familiar with two set of equations with only two variables respectively. Solving two set of equations where one of the equation has three variables was difficult for the students. The students formulated an equation which had only two variables for easy manipulation.

The students that used an arithmetic method and the students that formulated an algebraic equation (model) both evaluated the outcome of their computations. Apparently, the opinions of the students influenced the kind of answer they gave to the modelling task, and their conclusions were not solely based on the computations. That is, their conclusions depended on these factors: the minimum wage; the 'break-even point' (that is the number of items their friend needs to sell hourly in order for job 1 to be the same as job 2); the personality of their friend; and the kind of items sold. The responses of the students reveal that most of the students are risk-seeking students who believe there is a possibility of making much more money depending on the factors listed above (for example, see Excerpt **5.3.3.0**: line 764 on page 107). The opinions of the students made their mathematical solution more relevant and interesting since their interpretation shows the link between reality and the mathematics they did.

On the whole, in the process of interpreting the findings of the modelling results by the students, one out of seven groups used an arithmetic method to successfully solve the modelling task whilst the rest formulated an algebraic equation but their equation was incorrect. Although their equation was incorrect, it gave a concise and general picture of their solution. Most of the students compared their computations with reality, that made their mathematical solution more relevant and interesting since their interpretation gives a link between reality and the mathematics they did (that is, their conclusion depended on other factors and not solely on their computations).

## 6.2 Discussion of the Study

A general discussion of the results in connection with the literature review is presented in this section. The discussion entails the prior knowledge or transfer of problem solving, the types of word problems, the mathematization of the word problem, the students' modelling processes, students' conceptions on the modelling activities in school and group work.

### 6.2.1 Prior knowledge/Transfer of problem solving

The algebraic word problem tasks given to the students were isomorphic to the word problems they usually solve at school, that is the word problem is different in terms of the story context but similar in terms of the equations involve. The interview results revealed that the students were more familiar with the algebraic word problem tasks they solved in this study (For example see Excerpt **5.2.1.0**, page 82). Despite the familiarity of the word problem, two groups of students were not able to comprehend some of the word problem tasks. This is consistent with Reed (1998) which reports that the transfer of isomorphic problems is a complex and difficult process that usually takes place under certain experimental circumstances. That is after students have been involved in well-designed forms of training, it is then that they can successfully deal with isomorphic problems. The students were taught to translate the word problem using variables, set-up an algebraic equation and then compute the equation using a known technique when solving algebraic word problem tasks. Morales et al. (1985) argues that even though students have mastered the technical competencies of doing the mathematical operations involved in the word problems, they still experience considerable difficulty with simple word problems that require application of these techniques.

### 6.2.2 Types of word problems

The algebraic word problem tasks in this study are conceptual problems (non-routine problems) as described by Boaler (1997). A conceptual problem here is a problem that cannot be answered from memory alone but requires thought, that is a great deal of thought is required even if the correct rule/method had already been learned. Students mastering a technique for solving an algebraic word problem task is not enough, but they would need to understand the problem text and the links between the sentences in the problem text. Elia et al. (2009) and Boonen et al. (2016b) argue that in solving a non-routine problem, students are required to develop an adequate understanding of the situation described in the word problem text before deriving a mathematical model.

The four algebraic word problem tasks given to the students were concrete factual, concrete hypothetical, abstract factual and abstract hypothetical respectively. Most of the students have the conception that the concrete factual word problem task (Task 1) was the most difficult task. The main reason was that, since it was the first task they have to get used to the way of solving it (for example see Excerpt **5.2.2.1** on page 83). By assumption, if the tasks were rearranged in terms of numbering, the students may then have chosen the first task as the most difficult task according to the reason they gave. Although most of the students viewed Task 1 as the most difficult task, all the students still successfully solved the problem. It was only two groups that used an arithmetic method instead of the algebraic method to solve Task 1, since they did not set-up the right equation.

One group of students were unsuccessful with the abstract factual (Task 3) word problem task whilst two groups of students were also unsuccessful with the abstract hypothetical (Task 4) word problem task. This is consistent with Caldwell and Goldin (1979) which reports that concrete problems (factual and hypothetical; Task 1 and Task 2) are the least difficult, followed by abstract hypothetical and abstract factual problems. According to the results from the students' worksheets (Task 3 and 4 had some incorrect equations 'EI', see Table 12 on page 78) the abstract factual and the abstract hypothetical problems were more difficult to some of the students than the concrete problems (factual and hypothetical), however most of the students have the conception that the concrete factual problem was the most difficult task (see Q2 in Table 14 on page 80, four out of seven students regard Task 1 as the most difficult 'D' task). The students regarding the first task as the most difficult for the reason of it been the first question is not found in the literature discussed in chapter three. The literature only reports on the students' difficulty found in the type of word problems based on their performance. Apparently, the students might have the habit of completing the first task given before they move on to the next task that follows. In this case whichever task, among the four algebraic word problem tasks, that comes first would be considered difficult as the students try to find their rhythm (especially when the word problems have the same format).

### 6.2.3 Mathematizing word problems

Mathematizing an algebraic word problem is an act of finding a solution to a problem that is unknown. The four phases of problem solving described by Polya (2004) was used in this study as a process of justifying ones strategies when solving algebraic word problems. That is: Understanding the problem; Devising a plan (translating the problem using variables and setting up an algebraic equation); Carrying out the plan (solving the equation using a known technique); Looking back (verifying and presenting the final answer).

Most of the students were able to comprehend the algebraic word problem tasks in this study. Kyttälä and Björn (2014) argues that for an individual to comprehend word problems in written form, he/she needs to be able to read and understand the text that describes the tasks. Few of the students were not able to comprehend Task 3 and Task 4 due to the fact that they misunderstood the problem text (for example see Excerpt **5.2.3.4**: line 729, on page 91).

Kieran (2007) points out that the solving of word problems has two phases, that is the setting up of an equation to represent the relationships inherent in the word problem and the actual solving of the equation. It appears that most students in this study set-up an algebraic equation from the problem text in all the tasks. Few of the students used an arithmetic method to solve Task 1. Also these students who were not able to comprehend Task 3 and 4, set-up incorrect equations for the two tasks respectively. Polya (2004) argues that setting up equations comes along with some difficulties which is usually the difficulties of translation. The students had some difficulties in comprehending Task 3 and 4, which led to an incorrect translation resulting to an incorrect equation set-up. To overcome the difficulties of translation, Polya (2004) argues that one must thoroughly understand the conditions in the problem text and also be familiar with the forms of mathematical expression. Riley et al. (1983) also points out that most students use little conceptual knowledge, focusing instead primarily on how to translate the English problem statement directly into a corresponding set of equations when solving algebraic word problems.

Most of the students solve the algebraic word problem tasks with a known technique. However, a few of the students did not set-up the right equation for Task 1 hence they were not able to solve the equation. They used the substitution method to formulate a linear equation that involves only one variable but they rather had an equation with two variables instead (see Figure 15 on page 87). Mason et al. (2011) argues that one major thing that usually happens during the attack phase of problem solving is 'stuck', the point at which much can be learned when the student reflects on his/her work and tries to work all over again. The students found it difficult to solve one equation with two variables, therefore they resorted to an arithmetic method (see Figure 16 on page 88) when they got stuck. The use of an arithmetic method helped the students in this situation but it may not always be the case. Stacey and MacGregor (1999) argues that although translating word problems into an equation can be challenging but using algebraic equations to solve problems is important, that is problems relating to equations with the unknown on only one side are easy to solve without algebra but the ones with equations with the unknown on both sides require hard thinking if algebra is not used.

Most of the students verified and presented their answers to the algebraic word problem tasks, even the ones with an incorrect equation. It is easy to verify an answer whether the translated equation is correct or not. Hence the main problem is not about the computation but the translation of the word problem. Boonen et al. (2016a) argues that to effectively solve a mathematical word problem and the understanding of the text are related in such a way that developing a deeper understanding of the word problem serves as a crucial step before the correct mathematical computations can be performed.

On the whole, in the process of mathematizing an algebraic word problem, the students translate the problem using variables and set-up an algebraic equation. In a situation where the algebraic equation (whether correct or incorrect) can be solved and verified, they go on by solving the equation with a known technique whereas in situations where they can not solve the equation or get stuck along the line, they reread the problem and then resort to an arithmetic method.

### 6.2.4 Students' modelling processes

An interpretation of a mathematical model may usually depend on; the initial understanding of the problem, the construction of a mathematical model, the actual computational activities, and the evaluation of the outcome of the computations. In Figure 8 on page 46 (a simple view of the mathematical modelling process), one needs an initial understanding in order to translate the real world problem into a mathematical problem by making some assumptions and also formulating equations. The actual computational activities takes place when one moves from a mathematical problem to a mathematical solution by solving the formulated equations and also verifying one's model. The evaluation of the outcome of the computations also takes place when one moves from a mathematical solution to a real world solution by interpreting the model.

The modelling task used in this study was an open-ended task that can not be answered from memory alone but requires thought. The mastering of a particular technique in solving the modelling task is not enough but rather one needs to develop an adequate understanding of the situation described in the problem. Most of the students considered some assumptions before the formulation of a mathematical model. They spent some time researching about the minimum wage and the number of items one has to sell every hour in order to earn minimum wage. Most of the students formulated an algebraic expression or an equation after making some assumptions. One group of students used an arithmetic method to successfully solve the modelling task (only Group 4 used an arithmetic method 'AC', see Table 13 on page 79). Although they successfully solved the modelling task numerically, their solutions lacked generality since the numerical approach only investigates a specific or particular case. This is one of the disadvantages listed by Friedlander and Tabach (2001), nevertheless they also point out that the combine use of the various representations (algebraic, graphical, numerical and verbal) can cancel out the disadvantages. The choice of the numerical approach by these group of students might have been the students' thinking style or an attempt to overcome difficulties encountered during the use of another representation as suggested by Friedlander and Tabach (2001).

All the students that formulated an algebraic equation in the process of solving the modelling task had at least one incorrect equation. The students specifically misunderstood the problem text that described the second job option in the modelling task. Although they had an incorrect equation but their equation gave a concise and general picture of their solution. One reason for the incorrect formulated equation is that, the students are more familiar with two set of equations with only two variables respectively and so dealing with two equations—where one equation has two variables (example: $y = 120x$) and the other equation has three variables (example: $y = 50 + 20zx$)—was a problem. The students formulated two equations with only two variables respectively to avoid the third variable in one of the equations, for easy manipulation. This is consistent with Pedersen (2015) which reports that Norwegian upper secondary students' tend to perform weakly on items that place high demands on symbol manipulation.

All students used at least two of the mathematical representations (graphical, numerical, algebraic and verbal) in their modelling process. Most of the students that formulated equations also use the graphical representation. Although their equation was incorrect, the graphical representation provided a clearer picture of the real valued function they modeled. Duval (1999) argues that the shift between representations is important in learning, and that it helps students to avoid confusing the mathematical object with it's representation. The movement from algebraic to graphical representation was a direct translation (see the text beneath Figure 9 on page 51) as described by Janvier (1987), however one group of students used the indirect translation by moving from algebraic to table (numerical) and to a graphical representation (see Figure 23 on page 98, for Group 6's indirect movement from algebraic to graphical representation).

One interesting aspect of the students' modelling process was the movement from the mathematical solution to the real world solution, that is the evaluation of their computations. Both the students that formulated an algebraic equation (model) and the students that used an arithmetic method gave an interpretation of their modelling results. The opinions of the students influenced the kind of answer they gave, that is their interpretation shows the link between reality and the mathematics they did. According to Garfunkel and Montgomery (2016), the computations of a modelling task is just one aspect and that students will have to think about making decisions in the face of uncertainty, that is doing the mathematics and reconciling the answer with reality which makes the mathematics more relevant and interesting.

### 6.2.5 Students' conceptions on modelling in school activities

The students usually solve world problems that requires the use of a specific technique but the modelling task in this study was an open ended task. Although the students viewed the modelling task as a realistic task, they also had the conception that the task was difficult since it has no specific or direct answer. Some of the students found the task as irritating (for example see Excerpt **5.3.4.4** on page 110) as they are used to problems that have a direct answer and requires a specific technique, like the algebraic word problem tasks in this study. Boaler (1998) argues that students that follow a traditional approach (like solving problems with direct answers) develop a procedural knowledge which is of limited use to them in unfamiliar situations. Van de Walle et al. (2007) also points out that procedural understanding is task-oriented knowledge (which often devolves into rote memorization or rules without understanding) that may lead to frustration when not connected with concepts. Solving an open-ended task may be irritating for students if they only practice or solve tasks with the traditional approach.

In going through the teaching materials and the tasks the students solve at school, I found the tasks to be more close-ended tasks (tasks that requires a direct answer and with a specific technique). The students have the conception that they solve fewer modelling tasks at school and one of the reason is that open-ended tasks like the modelling task in this study might not be in the things they need to learn or maybe it comes further in the following semesters (for example see Excerpt **5.3.4.3**: line 739, on page 109). Pedersen (2015) points out that the Norwegian curriculum for upper secondary school mathematics places a greater emphasis on applying procedures and methods, and to a far lesser extent the curricular objectives describe activities such as analyzing, investigating, assessing, discussing, proving, modelling, and generalizing. Although the students solve much fewer open-ended tasks at school, some of them also have the conception that there is a connection between the mathematics they study at school and the real world (for example see Excerpt **5.3.4.0** on page 108). Others also have the conception that the connection is based on the particular mathematics subject taught at school (for example see Excerpt **5.3.4.2** on page

109), for instance one of the students interviewed did not know how trigonometry can be applied in the real world. Erling et al. (2016) points out that it is important to remind students that it is through modelling activities (solving of open-ended tasks) that a lot of mathematics is used in careers beyond school. Garfunkel and Montgomery (2016) also argues that modelling activities can be used to reinforce new concepts and to illustrate their applications. So designing a modelling task that requires the use of trigonometry might change the student's thinking that trigonometry can not be applied in the real world.

Despite students viewing the modelling task as difficult and irritating, most of them state that they would want to solve more of the modelling task at school. The modelling task may be challenging to the students since they struggled during the solution process but they stated that they wanted to work more on such tasks to be good at them (see for example Excerpt **5.3.4.5** on page 110). The students are of the view that one has to think in many different ways and also use more of the things studied at class to come up with a solution to the modelling task. Garfunkel and Montgomery (2016) argues that an extended modelling activities helps students to pull together ideas from different parts of a course and from different courses. One does not need to use one formula or technique learned at class to solve an open-ended tasks, but solving such tasks requires putting together a formula studied a week or a year ago (for example see Excerpt **5.3.4.6** on page 111). One of the students interviewed is of the view that, in the modelling activities, the students have to work together on a task that match what they learn at school and that the activities helped them to interact with each other more, whilst they don't deviate and change the theme of the subject while working together since they are more focused on the things they need to do. On the other hand, this student also has the conception that the current classroom culture, where the teacher gives them a paper or some explanation of what they are doing and then they are on their own making up their own theories, is not helpful; that is, one needs to be good at mathematics to be able to develop his/her theories (see for example Excerpt **5.3.4.8** on page 112). A well designed modelling tasks may offer the opportunity for low performing students to engage with minimal prerequisite knowledge and skills, and a high performing students can also explore more complex concepts. Another student interviewed is also of the view that he would want more of the modelling tasks that involves equations and also things that will be needed in future, but not the likes of function analysis, trigonometry, and among others that are of no use to them. And that, they don't need to know such subjects if they can not use them in the future or the real world. This student stated that, he does not know how to use some of the things they went through at school in the real world and also suggested that the teacher must be able to tell the students how they can use the things they study at school in future or the real world, since some teachers just say you will find out or figure it out but then they don't know exactly what to do (see for example Excerpt **5.3.4.10** on page 113). Kolis (2011) argues that when the mathematics taught at school has some connection with the students' lives, they might come to realize the importance of mathematics to their daily lives. Matthews (2018) also reports that mathematics teachers infrequently connect their instruction to the real world, and that the teacher's messages about the real world relevance of mathematics matters in shaping how students value mathematics. As reported by Garfunkel and Montgomery (2016), modelling activities can be used as motivation for learning new techniques and new content. The inclusion of modelling activities in the curriculum might help the students to view all the mathematics subjects as important since they can relate each subject to the real world through the teacher's guidance.

### 6.2.6 Students' conceptions on group work

The students worked together in groups in solving both the algebraic word problem tasks and the mathematical modelling task. At the end of the computations, one student representing his/her group gave his/her conception about the effectiveness of group work. All the students interviewed have the conception that the group activities helped them to achieve what they could not achieve on their own. One student interviewed is of the view that if one belongs to a group and doesn't know how to solve the problem, he/she could get help from another group member, and if no one in the group knows how to solve it then the whole group can get some hints from the teacher as they work together. Gillies (2016) argues that in cooperative learning students work together to achieve common goals or complete group tasks, that's goals and tasks that they would be unable to complete by themselves. As I went round the class whilst the students solved the tasks in groups, I noticed that when a student found a solution he/she explained it to the other group members, and when the group members found out that the solution is not accurate, they then teamed up and upgraded the solution to the point that they all agreed that it is the correct solution. Gillies (2016) reports that students feel more comfortable in a cooperative learning environment and by this motivation they turn to ask questions and also express their ideas. On the other hand, this same student interviewed is also of another view that group work is not really beneficial if one does not know what to do and also belongs to a group where everyone is working on their own and aren't really sharing ideas, then he/she will just be sitting among the other members wasting time (for example see Excerpt **5.3.5.0** on page 114). Sharan (2010) reports that the implementation of cooperative learning may be challenged with resistance and hostility from students who believe that they are being held back by their slower teammates or by students who are less confident and feel that they are being ignored or demeaned by their team.

The next chapter presents the conclusion of the study which entails the summary of results and discussion, the limitations of the study, an implication of the study for teaching and also suggestions for further research.

# 7 CONCLUSION

In this final chapter, the summary of results and discussion are presented. Afterwards the limitations of the study follows and the chapter finally ends with an implication of the study for teaching and also suggestions for further research.

## 7.1 Summary

This research study is a case study of one group of 1T Norwegian upper secondary students on the topic “how upper secondary students solve algebraic word problems in the area of mathematical modelling. The data used in this study was collected through worksheets (group work) and students’ interviews (individual students representing their respective groups) in addressing the research questions.

The students in this study were familiar with the algebraic work problem tasks they solved. The majority of the students interviewed regarded the first task as the most difficult task since it was the first task and that they must get used to the way of solving it. However, few of the students where not able to comprehend the abstract factual and abstract hypothetical (Task 3 and 4 respectively) word problems, although they regarded the concrete factual (Task 1) word problem as the most difficult task. In the process whereby the students mathematize the four algebraic word problem tasks, they first translated the problem using variables and then set-up an algebraic equation. In a situation where the algebraic equation (whether correct or incorrect) can be solved and verified, the students went on by solving the equation with a known technique whereas in situations where they can not solve the equation, they reread the problem and then resorted to an arithmetic method.

The students’ interpretation of their modelling findings mostly depends on their initial understanding of the problem, the construction of a mathematical model, the actual computational activities, and the evaluation of the outcome of their computations. Few of the students used an arithmetic method to successfully solve the mathematical modelling task, whilst most of the students formulated equations to solve the task. One of the equations formulated by the students was incorrect. Although their formulated equation was not correct, but it gave a concise and general picture of their solution. The conclusions made by the students depended on other factors and not solely on their computations. Almost all the students compared their computations with reality, which made their mathematical solution interesting since their interpretations gave a link between reality and the mathematics they did. In the end, most of the students are of the view that they would want to have more modelling activities at school, giving some reasons why. They also have the conception that group work is helpful, and they tend to achieve more when working in groups.

I repeat that, these results can not be generalized but are only meant to be suggestive and any conclusions drawn are tentative. The next section presents the limitations of this study.

## 7.2 Limitations

The findings of this study must be seen in the light of the study's limitations. The first limitation of this study concerns the sample size. The findings of this study can not be generalized because of the sample size involved (twenty-three students participated in this study). The second limitation concerns the number of tasks given to the students. The findings are not generalized to every algebraic word problem and mathematical modelling tasks. The third limitation concerns the data collection instruments used. The individual interviews did not capture the views of all the students within the group but only one person (a representative of the group).

The twenty-three participants compared to the entire Norwegian upper secondary students does not provide grounds for the findings to be generalized. The students used in this study is just a case study of one group of Norwegian upper secondary students. A large number of students will need a longer time in coding and analyzing the data that will be collected. The research questions could be answered using a survey designed for the purpose of generalization, where we find out statistically students' performance in both the algebraic word problem and the mathematical modelling tasks. However, this might not reveal in detail the unique experiences of the individuals and the kind of conceptions they have. The choice of the participants in this study was partially determined by the time frame of the master thesis. On the other hand, the time frame for the master thesis also influenced the number of algebraic word problem tasks given to the students. The time allocated by the mathematics teacher of the students for the research activities did not give much room for a lot of tasks. Using a lot of algebraic word problems in this study would have produced much more data and would require a longer time in coding and analyzing. Nonetheless, considering a lot of algebraic word problem and modelling tasks could have impact on the findings. The findings could be generalized to type of every algebraic word problem and mathematical modelling tasks if a lot of tasks were considered. The individual interviews (interview of an individual representing his/her group) used in this study did not capture the views of all the students within the group. Interviewing every member within the group could have an impact on the findings since the opinions of the students differ. Again, interviewing every member of the group would have produced a large volume of data that would have required a great deal of time in coding and analyzing.

To over these limitations in future studies I will suggest the use of multiple case studies, where one selects his/her participants from different upper secondary schools across Norway. It must not be all upper secondary schools in Norway, but some randomly selected schools will be enough to generalize the findings. If the tasks are designed to suit most of the things the teacher teaches at class, then these tasks can be used by the teacher on a regular basis. This is one way of finding more time to give more tasks. Lastly, the use of group interviews could help in getting almost all the views of the students within the group since their opinions differ. Also, the use of a video tape in collecting the data could give details about students' communications, actions and their thoughts when solving the task. The video tape will be best for just one group (between 3 to 4 students) due to the large amount of data it would produce, if the research is at the master level within a limited time.

The next section presents the implication of the study for teaching.

## 7.3 Implications for Teaching

Previous research (Caldwell and Goldin, 1979, 1987) as well as this current study reveal that students finds abstract factual and abstract hypothetical word problems more difficult than concrete factual and concrete hypothetical word problems. In going through samples of the algebraic word problem tasks (see Appendix F on page 181) given to the students at class, I found that the tasks do not include more of the abstract factual and abstract hypothetical word problems. In this regard I suggest that teachers should include more of the abstract (factual and hypothetical) word problems when designing tasks for the students, since some of the students have difficulty with such tasks. I also suggests that teachers should encourage the use of algebraic and graphic way of solving word problems. Although the arithmetic way is helpful, according to Stacey and MacGregor (1999) translating word problem into an equation can be challenging but using algebraic equations to solve problems is important, since problems relating to equations with the unknown on only one side are easy to solve without algebra but the ones with equations with the unknown on both sides require hard thinking if algebra is not used.

In comparing the algebraic word problem tasks and the modelling task used in this study, the modelling task is more open and does not require a unique technique to solve it. The findings in this study revealed that most students formulated equations which were incorrect, however their equations gave a concise and general picture of their solution. Also, the students used a graphical representation in providing a clearer picture of the real valued function (algebraic equation) they modeled from the task and the link between their computations and reality made the mathematics more interesting. The students have the opinion that they solve fewer modelling task at class. Previous research (Stillman, 2007) reveals that, at the upper secondary level and particularly in some European countries, there is a strong bias against mathematical modelling and greater attention is given to high level mathematics (theorems, proofs, formulas, and among others). Antonius (2004) argues that it is still hard to find time for modelling activities which are time consuming to a very high degree in the Nordic context. However, recently the Norwegian Ministry of Education and Research has decided on a new curricular where 'modelling and applications' is one of the core elements of mathematics (but the curricular will take into effect in the autumn 2020). For mathematical modelling to be added to the ordinary didactical system, Artaud (2007) suggests that the teaching process must be accorded extra time. The findings revealed that some students are of the view that they don't know how to use some of the things they went through at school in the real world. Some students also suggested that the teacher must be able to tell them how they can use the things they study at school in future or the real world, since teachers usually just say, you will find out or figure it out and that they don't know exactly what to do. Kolis (2011) argues that when the mathematics taught at school has some connection with the students' lives, they might come to realize the importance of mathematics to their daily lives. Matthews (2018) also reports that mathematics teachers infrequently connect their instruction to the real world, and that the teacher's messages about the real-world relevance of mathematics matters in shaping how students value mathematics. As reported by Garfunkel and Montgomery (2016), modelling activities can be used as motivation for learning new techniques and new content. The inclusion of modelling activities in the curriculum might help the students to view all the mathematics subjects as important since they may be able to relate each subject to the real world through the teacher's guidance.

## 7.4 Future Research

This study was conducted at the upper secondary school level. It would be interesting to conduct the same research with students at the lower secondary school level or students from the teacher education class at the university level in finding out if students at these levels have challenges with abstract algebraic word problems and also if they can manipulate the expressions in the modelling task. Again, since the new Norwegian curricular (which will take into effect in the autumn 2020) has modelling and applications as one of the core elements, a similar research at these levels might be necessary.

In the course of the research study an interesting issue came up. That is, one student talked about how easier it would be when sketching the graphs with a computer (see Excerpt **5.3.4.9**: line 658, on page 112), whilst another student also talked about the use of GeoGebra in making diagrams during the modelling process (see Excerpt **5.3.1.1**: line 232, on page 100). This is an interesting issue which could be studied in a more extensive and comprehensive way. I suggest research on the use of technological tools for modelling realistic problems, where there is an investigation of how students mathematize a realistic problem using a technological tool and also how these students interact with the tool as they develop a technology-based solution to a realistic problem.

Advancements in the field of digital technology are currently making immense changes in our world. This gives birth to various digital technologies which are observed to be developing gradually. The integration of digital technology offers opportunities to the education systems, although it might have some challenges. Majumdar (2015) reports that the use of digital technology in the education system promotes learner centered and collaborative learning principles. The National Council of Teachers of Mathematics (2000), also argues that technology is essential in teaching and learning mathematics and that it influences the mathematics that is taught and enhances students' learning. They also emphasized the need to develop students' abilities to successfully use technological tools in dealing with complex problem solving. Despite the integration of technological tools in the education system, teachers, educators and researchers are still confronted with many questions. Twenty-six year ago, Watson et al. (1993) in the Impact project identified some negative factors that comes along with the integration of technological tools in the education system: teachers having insufficient knowledge of the software and understanding of the principles behind its use; students being unable to cooperate effectively; and students having difficulties in learning to use the software. Although some measures have been taken over the years, yet still the challenges still exist in this field. Drijvers (2015) reports that the design of the digital technology and the corresponding tasks and activities, the role of the teacher, and the educational context are factors that emerge as decisive and crucial when it comes to the integration of technological tools in the education system.

Mathematical modelling, as reviewed in this study on page 42, is one of the appropriate media that supports the use of technological tools in dealing with complex problem solving. Modelling offers students the opportunity to draw out their own mathematical and scientific ideas in the process of finding real world solutions to real world problems. Mousoulides et al. (2007), argues that the presence of technological tools are an important factors that can interfere in students' work in modelling activities and that the availability of these tools, such as computer software or graphic calculators, might change the way student solve a problem. They further points out that the use of appropriate technological tools by students during modelling activities may improve the quality of students' work which may result in better models and solutions. Stillman (2007) reports that technology allows more authentic modelling situations. However, Strässer (2007) warns that technology should not only be considered as a means to enhance the students' modelling abilities and to enrich the students' experience of applications and modelling, since the use of technology as everyday and professional instruments deeply changes the scope and way mathematics is used in society. Siller and Greefrath (2010) describes an extended modelling cycle that involves the use of technology, in which they design a model that links the three worlds: Real world (situation, real results); Mathematical world (mathematical model, mathematical results); and Technology world (computer model, computer results). The technology world describes the world where problems are solved through the help of technology. Siller and Greefrath (2010) reports that using technology broadens the possibilities to solve certain mathematical models that would not be solved if technology would not be available. However, the development of a mathematical model at this point does not depend only on the skills in certain software tools but also demands mathematical knowledge.

In the study by Christou et al. (2004), students used dynamic geometry software to model and mathematize a realistic problem, and some features of the software enabled the students to explore the problem and make mathematical conjectures. The students also utilized the dragging features of the software to check some specific geometrical figures in verifying whether their conjecture hold. Mousoulides et al. (2007) reported that students' interactions with Potters Wheel, spatial geometry software for generating solids by revolution, assisted students in developing the necessary mathematical constructs and processes as they actively engaged and solved a realistic problem through meaningful mathematical modelling. In this study, students were able to reach models and solutions that they could not probably do without using the software. Mousoulides (2011) also reported that students' use of the GeoGebra software assisted them in broadening their explorations and visualization skills in the process of modelling real world problems and also making connections between the real world and the mathematical world. These recent research studies reveal that the incorporation of an appropriate technological tool in modelling activities could help in understanding how students approach realistic problems. Siller and Greefrath (2010) points out that the use of technology broadens the possibilities to solve certain mathematical models, but unfortunately a lot of teachers and educators prefer not to work with realistic problems with an example that teachers do not want to use technology in class since it is very costly in terms of time. Monaghan (2016) on the other hand provides a critique of ideas of the integration of technology in mathematical modelling, pointing out the complex nature of technology in this area (in reality).

I also suggests that tasks used in further research should be designed in connection with the mathematics teacher of the students based on the scheme reported by Leung and Bolite-Frant (2015). Leung and Bolite-Frant (2015) reports on four considerations in designing tasks that make use of tools, these considerations are: epistemological and mathematical considerations; tool-representational considerations; pedagogical considerations; and discursive considerations. Leung and Bolite-Frant (2015, p. 194) argues that "different epistemological approaches[42] to mathematical knowledge have different implications on task design". Nonetheless, the same tool can be used in task designs with different epistemological stances. A challenge to tool-based task design under this consideration is the determination of a possible range of epistemological orientation and the type of mathematical knowledge that a tool can afford and to choose them appropriately for pedagogical situations. For example, a tool like dynamic geometry software can be used in task design in covering a large epistemic spectrum from drawing precise robust geometrical figures to exploration of new geometric theorems and development of argumentation discourse.

Another consideration taken into account when designing tool-based task is tool representational considerations. The way a chosen tool represent mathematical knowledge is at the heart of tool-based task design. These considerations are considered: How far away from the expected symbolic representation is in the tool's potential to represent the mathematical concept?; and Is the tool capable enough of representing the targeted mathematical

[42] Sfard's (2008) participationist epistemological orientation and acquisitionist epistemological orientation. The participationist epistemological orientation would favor a tool-based design with the potential for students to participate in the construction of shared mathematical experiences or discourses, whereas the acquisitionist epistemological orientation would use tools to explore and consequently construct personal mathematical knowledge

knowledge parallel to the corresponding symbolic representation? The next consideration is the pedagogical considerations. Tool-based task design must be supported by a suitable pedagogical environment. Also, familiarity with a tool and how to use it effectively to teach and learn are important pedagogical considerations for tool-based task design. The final consideration is the discursive considerations. Practicing to use a tool to accomplish a task involves formation of appropriate tool-based vocabularies in the development of utilization routines. The designed tool-based tasks should be able to bring about discourses for mathematical knowledge mediated by tools in the mathematics classroom. Another point to note, is how does these discourses relate to mathematics knowledge (pp. 194-198). Now, to implement these considerations (when designing a tool-based task for modelling activities) more research in this field needs to be done.

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# Appendices

## A Interview Guide

I'm working on a project, and it's about how upper secondary students solve algebraic word problems in the area of mathematical modelling. In this regard, I would like to ask you some questions and hear your thoughts and opinions. You can say exactly what you want, there are no correct or wrong answers. Say all that comes to your mind.

I will be using a voice recorder. However, I will never use your name or anything else that can trace your identity. If you have questions or something you do not understand now or during the interview, just ask. We can start now, we have some time!

Questions:

1. Have you solved or encountered similar problems like these algebraic word problems at class before?
   (a) Were there any difference?
   (b) Can you tell me what the difference was?
2. Which of the questions among the algebraic word problem task, was the most difficult?
   (a) Can you tell me why?
   (b) How about the easiest? Why?
3. Can you tell me the processes you went through in solving these algebraic word problems?
   (a) How did you know your answer was right?
   (b) Do you know of any other method used in solving these algebraic word problems? Can you tell me?
   (c) Was it helpful when you solved these questions in a group? How? How about solving them individually?
4. Questions about specific errors found in their solutions to the algebraic word problem tasks.
5. Can you see any connection between the mathematics you learn at school and the outside world? Why?
6. How often do you solve mathematical modelling task at school?
   (a) Any reason?

7. What is your opinion about the modelling task you solved recently?
    (a) Does this modelling task has any connection between the mathematics at school and the outside world? Why?
8. Can you tell me the processes you went through in solving this modelling task?
    (a) How did you know your answer was right?
9. Please, interpret your modelling results to me?
10. Questions about specific errors found in their solution to the modelling task.
11. Will you want more of the modelling task at school? Why?

We will end the interview now. Is there anything more you want to say/tell?

Remember that I will process everything anonymously, and you cannot track the answers back to you. Thank you very much for your time.

# B Consent Form

***Are you interested in taking part in the research project "(How upper secondary students solve algebraic word problems in the area of mathematical modelling)"?***

This is an inquiry about participation in a research project where the main purpose is to investigate the underlying understanding of students when solving algebraic word problems in the area of mathematical modelling. In this letter we will give you information about the purpose of the project and what your participation will involve.

**Purpose of the project**

The purpose of the research project is to investigate students' understanding as they justify their strategies and also the interpretation of their findings when solving algebraic word problems in the area of mathematical modelling. The research study tends to address these main questions:

- How do upper secondary students justify their strategies for solving algebraic word problems?
- How do these students interpret their findings after solving algebraic word problems in a form of mathematical modelling?

The research project forms part of the master's thesis undertaken at the University of Agder.

**Who is responsible for the research project?**

The University of Agder (Mathematical Sciences Department) is the institution responsible for the project. The researcher responsible is Obed Opoku Afram (Master student) and the project is however supervised by Professor John David Monaghan.

**Why are you being asked to participate?**

You have been selected due to the fact that you are in the upper secondary school and currently working with algebraic word problems. On the other hand, the teacher responsible for the mathematics class and the head of department at the upper secondary school have been contacted for their consent.

**What does participation involve for you?**

The methods which will be employed for data collection are handwritten materials, interviews and classroom observation. The handwritten materials are mainly about the answer sheets provided by the students whilst the interviews will be tape recorded (All data will be completely anonymous, there will be no link between the registered signatures on the consent forms and the data collected).

- If you chose to take part in the project, this will involve that you provide a justification or explain the strategies you used in solving some algebraic word problems. You will be working in groups. It will take approximately 45 minutes.
- You will also participate in a 10-20 minutes interview, where you talk about your solutions to the modelling task. A voice recorder will be used during the interview.
- I will observe, participant and take notes during some mathematics lessons (no personal data will be registered on the individual students during the observation).
- I will also ask your teacher to provide information about the mathematics courses you have taken and the teaching activities organized by the teacher. The teacher will also be ask for reasons why he/she uses a particular method when presenting the mathematics lessons and also why he/she allow students to work in groups instead of individual or vice versa. I will record the interview with a voice recorder.

**Participation is voluntary**

Participation in the project is voluntary. If you chose to participate, you can withdraw your consent at any time without giving a reason. All information about you will then be made anonymous. There will be no negative consequences for you if you chose not to participate or later decide to withdraw. It will not affect your relationship with your school or teacher. No information or data will be recorded on students who do not participate in the research, especially during the mathematics lessons observation.

**Your personal privacy – how we will store and use your personal data.**

We will only use your personal data for the purpose(s) specified in this information letter. We will process your personal data confidentially and in accordance with data protection legislation (the General Data Protection Regulation and Personal Data Act).

- The only persons that can have access to the personal data are the researcher, supervisor and the censor at the University of Agder.
- All necessary precautions will be taken to ensure that no unauthorized persons are able to access the personal data. I will use pseudonym to ensure your identity is not revealed in any part of the research project. The personal data will be stored on an external hard drive and the University of Agder server.

The participants will not be recognizable in any form of publications.

**What will happen to your personal data at the end of the research project?**

The project is scheduled to end on the 30th of May 2019. The personal data including the voice recordings will be completely deleted within a period of six months after the research project is done.

**Your rights**

So long as you can be identified in the collected data, you have the right to:

- access the personal data that is being processed about you
- request that your personal data is deleted
- request that incorrect personal data about you is corrected/rectified
- receive a copy of your personal data (data portability), and
- send a complaint to the Data Protection Officer or The Norwegian Data Protection Authority regarding the processing of your personal data.

**What gives us the right to process your personal data?**

We will process your personal data based on your consent.

Based on an agreement with the University of Agder (Mathematical Sciences Department), NSD – The Norwegian Centre for Research Data AS has assessed that the processing of personal data in this project is in accordance with data protection legislation.

**Where can I find out more?**

If you have questions about the project, or want to exercise your rights, contact:

- The University of Agder (Department of Mathematical Sciences) via
  - Professor John David Monaghan (Supervisor), by email: john.monaghan@uia.no
  - Obed Opoku Afram (student), by email: obedoa17@student.uia.no or by telephone: +47 40384669.
- NSD - The Norwegian Centre for Research Data AS, by email: (personverntjenester@nsd.no) or by telephone: +47 55 58 21 17.

Yours sincerely,

| | |
|---|---|
| ...................... | ...................... |
| Obed Opoku Afram | Prof. John David Monaghan |
| (Researcher/Student) | (Supervisor) |

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## Consent form

I have received and understood information about the project 'How upper secondary students solve algebraic word problems in the area on mathematical modelling' and have been given the opportunity to ask questions. I give consent:

☐ to participate in the group activities.

☐ to participate in an interview.

☐ for notes to be taken about my involvement during mathematics class lessons.

☐ for my teacher to give information about me concerning the mathematics courses I have taken and the teaching activities at class to this project.

I give consent for my personal data to be processed until the end date of the project, approximately October 2019.

......................

Signed by participant, date.

(**NB! If you are under 16 years old, your superior also needs to sign**)

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# C Summary of 1T Course Content

**Course Content**

1. Geometry and Trigonometry
   1.1 Trigonometry
   1.2 Geometry
   1.3 Law of sine and law of cosine
2. Algebra
   2.1 Order of operations
   2.2 Calculations with fractions
   2.3 Calculations with variables
   2.4 Linear equations and formulas
   2.5 Inequalities
   2.6 Simultaneous sets-linear equations
   2.7 Special binomial products
   2.8 Common factors and factorization
   2.9 Rational expressions
   2.10 Power and standard form
   2.11 Roots and power
3. Functional Analysis
   3.1 Graph of linear functions - graphical solution of linear equation sets
   3.2 Graphs of linear functions in Geogebra
   3.3 Linear adjustment - linear model for the relationship between two sizes
   3.4 Introduction to second-degree functions
   3.5 Second degree formula
   3.6 Nonlinear equation sets
   3.7 Second degree inequalities
   3.8 Function concept
   3.9 Exponential functions
   3.10 Introduction to exponential equations
   3.11 Logarithmic equations
   3.12 Rational functions
4. Probability
   4.1 probability calculation - introduction of basic concepts
   4.2 product set for independent and dependent events
   4.3 Venn diagram. Cross-table. Addition rule
   4.4 exercises in probability count
5. Derivative and functional discussion
   5.1 Average growth rate
   5.2 Instantaneous growth
   5.3 Introduction to the derivative rules
   5.4 The definition of the derivative
   5.5 Function discussion

# D Transcripts of Students' Interviews

## D.1 Interview Transcript, Student A

Teacher: Have you solved or encountered similar problems like these [Points at the algebraic word problem task] algebraic word problems at class before?=[43]

Bjørg: =Yeah.

Teacher: Were there any difference?

Bjørg: Umm..[44] I don't think so.

Teacher: So there wasn't any difference?=

Bjørg: =No.

Teacher: So how similar was it?

Bjørg: .. Very close.

Teacher: Very close?

Bjørg: Yeah, I guess.

Teacher: Ohk!.. So, when you went through the questions, which of the questions among the algebraic word problem task was the most difficult?

Bjørg: Umm.. None of them actually. It was like same form.. It was kind of same questions. It had similar, I don't know, equations, but um I kind of forgot to read through whole text, like whole problems, so I forgot to sum up in the end, so I got failed at the first time, but when I went through it like, I read the problem, then I got the right answer as the others in my group.

Teacher: Ohk! So, can you tell me the processes you went through in solving these algebraic word problems? ... What did you do?

Bjørg: What did I do.. Umm.. Can I take this [Pointing at the first question in the algebraic word problem task] as an example?=

Teacher: =Yeah, sure.

Bjørg: Umm.. I start like, in this [question]=

Teacher: [Yeah]

Bjørg: =(inaudible)[45] Umm, you know the amount of hens, there are 13 more hens than goats. I set goats as $x$ and... I don't know how to explain it like.. Can I write?

Teacher: Yes, you can.

Bjørg: So, this is the value of goats [writes $x$], because 13 hens more than goats, maybe

[43] When one speaker immediately follows the speaker before, without any pause between them.

[44] A brief pause in speech, .. means one second pause, ... three second pause, ..... more than five second pause.

[45] Inaudible or missing words.

not, let me think. . . .

Teacher: You can use the solution you already have.

Bjørg: Yeah, so $13 + x$ means goats and hens has two legs, so this [Points at $2(13 + x)$ in the equation] is the amount of the legs of hens and the goats, which is x, has four legs, so if I.. Umm plus these two amounts [Points at $2(13 + x)$ and $4x$] then I get 146 legs and yeah. How many animals in all does Marius have? Then I have just to like sort $x$'s to the one side and then the numbers to the other side, then I get $6x$ is 120 and I divide into 6 because I want $x$ to be alone and I get 20. And this is not the answer or maybe. . .

Teacher: Yeah.

Bjørg: Yeah, this is not the answer because the problem is..

Teacher: How many animals in all does the farmer have?

Bjørg: Then I got that, 20 goats in the garden. No! In the farm. Then I plus this together [Points at 20 goats and 33 hens] and I got 55 as an answer.

Teacher: Ohk! So that is the process you went through?

Bjørg: Yeah.

Teacher: So, you first of all understood the question=

Bjørg: =Yeah.

Teacher: And then you tried to make an equation=

Bjørg: =Yeah.

Teacher: And then you solve the equation. So how did you know your answer was right?

Bjørg: [Laughs] Umm I talked with my friends in my group and they had same answer as me, so we thought it was right.

Teacher: So, do you know of any other method used in solving these problems, apart from the one you know already?

Bjørg: Umm.. Not really. . . Umm, I don't know.

Teacher: Ohk! Was it helpful when you solve these questions in a group?

Bjørg: Not really, because I solve it alone, like every.. All of them by myself and then I talked with my friends in my group.

Teacher: Looking at 20 goats and 33 hens, you mean the sum is 55?

Bjørg: Oh!! Wait, what, 55, it supposed to be 53.

Teacher: Ohk.

Bjørg: Yeah [Laughs]

Teacher: Can you see any connection between the mathematics you learn at school and the outside world?

Bjørg: Yeah, it might be=

Teacher: =Can you tell me more about that.

Bjørg: [Laughs] Umm, like I don't know, um... maybe in super market when you have to like see the price in some. I don't actually know.

Teacher: Just say what you are thinking.

Bjørg: Like math is in daily life, actually like you can find them every day [Laughs]. But I don't have exact example.

Teacher: So.. How often do you solve mathematical modelling task at school?

Bjørg: In math classes?

Teacher: Yeah.

Bjørg: Umm.. 10 percent maybe.

Teacher: Any reason?

Bjørg: Umm.. We get problems, they are like um, they are most like these questions [Points at the algebraic word problem task], not this one [Points at the modelling task].

Teacher: Ohk! So, what is your opinion about the modelling task you solved?

Bjørg: It was difficult for me.

Teacher: Ohk!

Bjørg: So, I have to talk with my group.

Teacher: And does this modelling task has any connection between the mathematics at school and the outside world.

Bjørg: Umm, yeah it might be.

Teacher: So, what is the connection here?

Bjørg: Umm, like in this problem they ask about job and you can, like if you in real life and you want to find a job and you can like choose one of the two jobs, and if you want to know which is more, which is better than the other job, then you can like compare.

Teacher: Can you please tell me the processes you went through in solving the modelling task?

Bjørg: Umm. We made equations I guess... Yeah first we made two equations, one for one job and second for the other job, and we drew, and we made a graph looking like this [Points at their solution graph] umm. And we found out that our first graph which crosses the line here [Points at the line $y = 30x$ on the graph] is better than the other in this line [Points at the line $y = 5 + 20x$ where $x$ is taken as zero on the graph].

Teacher: So, you set up the equation and then you try to make a graph.
Bjørg: Yeah.
Teacher: Let say I'm your friend. Can you interpret your modelling results to me, that I can make a choice?
Bjørg: [Laughs] Ohk!... Umm.. It all depends on your time spent working or your great selling skills. Also, of course if your minimal wage is extremely low, maybe sticking with 20kr per item would be better. But umm.. Kristin's own better perspective we would recommend using graphs, because it would be easier to see the changes and variations based on time and salary.
Teacher: Ohk! What does the $r$ means in the equation for the second job?
Bjørg: Umm. I think is for the amount of the items.
Teacher: So, is it the amount of items for only one hour or?
Bjørg: We thought that per item cost 20kr. So, $x$ is the hour and $r$ is the amount of the items.
Teacher: Ohk! It means that you have $x$ here [Points at $5x$ in the second equation] but you don't have it here [Points at $20r$ in the second equation].
Bjørg: No.
Teacher: Why?
Bjørg: Umm....
Teacher: Maybe you can have $x$ at both sides [Writes $y = 5x + 20xr$]
Bjørg: It could be.
Teacher: Because every hour you have 20kr on each of the items you sell.
Bjørg: Yeah.
Teacher: Because, this [Points at the equation $y = 5x + 20r$] sounds like only one hour.
Bjørg: Yeah, it makes sense. So, $x$ could be on both sides or we could just write the equation $y$ equals $(5 + 20r)x$.
Teacher: Yeah. Did you checked the point at which equation 1 and 2 will be the same?
Bjørg: We didn't, we thought that this had two different, whole different answer. So, we did not.
Teacher: Maybe you didn't consider this equation [Points at the equation $y = 5x + 20xr$].
Bjørg: Yes, we didn't quiet understood the second one, the second job.
Teacher: Ohk! So will you want more of this modelling task at school?
Bjørg: Yeah.
Teacher: Why?
Bjørg: Because I struggled with this problem, so I want to be good at this.

Teacher: And does it help in the learning process of mathematics?
Bjørg: Yeah.
Teacher: How?
Bjørg: Like in different, umm. Like you can think. To solve this problem, you have to think in many different ways and you have to like [Laughs]. You can have many different thinking ways.

## D.2 Interview Transcript, Student B

Teacher: Have you solved or encountered similar problems like these [Points at the algebraic word problem task] algebraic word problems at class before?=
Julie: =Yeah. We did some in high school. Oh, not high school but the previous school I went to, in 10$^{\text{th}}$ grade and 9$^{\text{th}}$ grade and just also some other years in math.
Teacher: Ohk! And recently, have you been working with these problems?
Julie: Well, not that much. Not that much.
Teacher: Were there any difference or they were similar?
Julie: Yes, they were quiet similar.
Teacher: Ohk! So, when you went through the questions, which of them was the most difficult?
Julie: Umm.. I think we straggled the most with the first one [Laughs]. Umm but then when we got the first one we figured out. I think we got the other ones too. But Umm.. also the one with the um.. Oh no, that one [Laughs].
Teacher: Ohk! So, can you tell me the processes you went through in solving these algebraic word problems?
Julie: Umm. Well, I think we first thought about it which it which way we could solve this, if we could use umm two equations like $x$ and $y$ for the hens and goats um. But then I think we figured out that, that did not worked umm and then we tried to look at how many um animals they were or legs they were in um altogether. Umm, then we found out that since they were 13 more hens than goats we could just um subtract the hens legs from all of the legs. And then we could just part the legs in two um... Or divide them [Laughs] in two, and find out how many they were, I think.
Teacher: Ohk! So, the same applies to the second one?
Julie: Umm... Yeah.
Teacher: The same process?
Julie: Umm.. Maybe. Here only we um... I think we used two equations to find firstly how many students they were, like this [Points at $13g + 2b = 346$] and we said $g$ for

the girls and $b$ for the boys. Ummm.. And then um yeah, since there were 7 more girls than boys, we put that. . . .

Teacher: So, it was more like a simultaneous equation.

Julie: Yeah, I think so [laughs], I don't really know.

Teacher: So, the issue here is that you try to understand the word problem, and then you tried to put them into variables=

Julie: =Yeah.

Teacher: Then you form equations and then you solve the equations.

Julie: Yeah, yeah.

Teacher: Ohk! So, how did you know your answer was right?

Julie: . . . Well, I think we did. I don't know, we thought it was right or not but. . .

Teacher: When you arrived at the answer, you did nothing else?

Julie: Yeah, maybe, but I think we were a little confuse but, in the end, we got an answer that we were satisfied with. But.. and then also, make sense um because there were 13 more girls than boys. And I think we tried to put altogether in the end and that made 346.

Teacher: Ohk! So, do you know of any other method used in solving these problems, apart from what you just did?

Julie: Ummm.. I [Laughs]. Not that I know of now.

Teacher: And was it helpful when you solve the problems in a group?

Julie: Yeah, it was.

Teacher: How helpful was it?

Julie: Umm. I think discussing with others and like getting more ideas will just.. make the process [Laughs] more smooth. Yeah, it makes it quicker.

Teacher: Ohk! How about solving them individually?

Julie: Well, I think it will work the same way, but it will maybe take some more time, some more thinking. But if it would be a task on a test it would maybe be the same because you work it as you know that this is a test and you have to solve it, and you only have this much time so you like under pressure and then, it also works [Laughs].

Teacher: Ohk! How come the first attempt in question one failed?

Julie: Umm.. I don't know. . . Yeah, because when we.. These are the same [Points at $(2x + 13 * 2) + 4y = 146$ and $4y = 146 - (2x + 13 * 2)$] and this [Points at $(2x + 13 * 2)$] is negative, like this [Points at $(2x + 13 * 2)$] was subtracted by this [Points at $(2x + 13 * 2)$] one, because we will sort the x's in one side and then this [Points at 146] one will go over to this [Points at 146] side, and then they will just subtracted each other and then the equation would be zero, equals zero [Laughs]. I don't know if that is right [Laughs].

Teacher: Is it because, maybe you didn't set up the equation right?

Julie: Yeah, maybe.

Teacher: Umm. Can you see any connection between the mathematics you learn at school and the outside world?

Julie: Yeah, well not any of this connection.

Teacher: Can you tell me more about that?

Julie: Umm. If you want to be a person who builds houses and stuff, you can use trigonometry to like find the angles and sides and um yeah, with the goats on the farm if they [Laughs] don't want to go around and count every single one, they can [Laughs]..

Teacher: How often do you solve mathematical modelling task at school?

Julie: Well, I think it's often when we have about this [Points at the modelling task] in algebra, functions and stuff umm and it also a common question on (inaudible) the logic test, we have a five hour test um, so we have sometimes.

Teacher: Ohk! What is your opinion about the modelling task you solved?

Julie: Umm. Well.. it is a good question kind of, because it relates to real life and it could have happen in real life. Only maybe we would not solve it this way [Laughs] in real life, umm.

Teacher: You will not solve it this way in real life?=

Julie: =No! [Laughs]. I think we will just like think about it and it would be like ohk , like this one will be more, but maybe it will be fun just to solve it since it in real life, but.. Yeah, I think it a good question.

Teacher: So how about the mathematics you have at school?

Julie: Yeah, I think so, maybe um I don't see it as well but I think that there is one. There has to be a reason why we learn all of it [Laughs].

Teacher: Can you please tell me the processes you went through in solving this modelling task?

Julie: Umm. Well, first we read it and then we figured out that there was no specific minimum wage, so we just set one just to.. a kind of put a picture on how it looks umm. And then we put together the two equations because we wanted to use two and then put them into a diagram in Geogebra and then find the similarities or where the lines cross each other. But then we find out that the gradients were the same which means they kind of parallel to each other umm, which made the task very confusing because we thought much about it and then it just became more and more confusing. But then we tried to, like think how it would be in real life if she sells much, she would umm earn more with the um first or second job, yeah the second job which she sells items and so if she sells like minimum two or three items she would earn even more than the first one, even though the first one has like a better um salary in general when you see it at first. So, we tried to put it in a diagram but I don't know if it went right because the axis are different since the first equation has hours and the second one has items so, yeah. And then we just concluded with that the second job would be more umm, would better because she would mostly or umm, she is going to sell more than two items an hour or three, because yeah, if it's a summer job people go shopping all the time [Laughs] .

Teacher: Ohk! So, let's say I'm your friend, how do you interpret your results to me for me to understand and make a choice?

Julie: Umm. I would say that if you would like to... If you are a person who works a lot and works for the money and earns.. and is a good person who manages to talk to your costumers and umm yeah, who knows stuff about what you are selling umm, then the second job would more.. would be better.

Teacher: Looking at your equations, does the $x$ affect this one [Points at 80 in the first equation] or only that one [Points at 20 in the first equation]?

Julie: Only that one [Points at 20 in the first equation].

Teacher: So, which means in 2 hours, she will still get 80?

Julie: I don't know actually [Laughs].

Teacher: Because it reads, every hour you have the minimum wage plus 20kr.

Julie: Ahh! Ohk! It wrong to put it that way.

Teacher: Can you give me the new equation?

Julie: How we would put it?

Teacher: Yeah.

Julie: So, it is [Writes $(80 + 20)x = y$].

Teacher: Same with the second equation.

Julie: Ohk!! Yeah.

Teacher: But you said if somebody sells 2 or 3 items, then it is better than the first job. How do you explain that?

Julie: I guess it was because we did write the equations like this [Points at $y = 80+20x$] and not like this [Points at $(80 + 20)x = y$].

Teacher: Ohk!

Julie: Because we thought that this was only the thing you got one time, and if that will be the situation umm, you will get more there [Points at the equation for the second job]. Because 20 times 3 would be 60 and then you sell items for 60kr per hour um, you would get more than here [Points at the equation for the first job] because you would just get 20 per hour. So, that makes sense. But with this equation [Points at $(80 + 20)x = y$], that will not be the situation I guess.

Teacher: So, will you want more of this modelling task at school?

Julie: Yeah. Well, I think it's a good task to like put several um situations together and have to use more um of the things you learn in class to solve the equation or the problem, you don't have to only use the one formula you learned in the class umm, you have to use like the formula you had last week or the one you learned a year ago, and then you have to put it all together to solve the problem, yeah.

## D.3 Interview Transcript, Student C

Teacher: Have you solved or encountered similar problems like these [Points at the algebraic word problem task] algebraic word problems at class before?

Hilde: Arr, yeah.

Teacher: Were there any difference?

Hilde: Umm.. It was a bit similar [Laughs], yeah I think.

Teacher: Ohk! So, when you went through the questions, which of them was the most difficult?

Hilde: Umm. I think it was maybe the first because, we just had to get used to the way of thinking to solve it but after we knew how to do it, we just applied the same method to do the rest of them. But umm I noticed some of my fellow students in my group, they got confuse with example, 'A given number is six more than a second number', so this is the same as the first number, and they got confuse, so the first number is 4 times as large and they took the second number instead of the first number. So, it was kind of the most difficult to actually understand what they were saying and the information we got. And to apply them correctly in the equation.

Teacher: Ohk! So, can you tell me the processes you went through in solving these algebraic word problems?

Hilde: Umm. We started by um collecting information, so writing down what we knew and we had $x$ of course for unknowns, so we decided what is $x$? In the first task we decided $x$ is number of goats, total number of goats because then we can put $x + 13$ which equals total number of hens, and then we had total number of legs which is

146. And then because we know goats have legs, we put 4 multiplied by $x$ plus 2 multiplied by, and then blankets $x + 13$, which equals 146. So, the two and four or, yeah it's the number of legs.. Yeah, and then we just solve the equation, and we saw that $x$ equals 20, which is number of goats, and we put 20 plus 13, which is number of hens. Yeah!

Teacher: So, in the end how did you know your answer was right?

Hilde: ...Umm. We just talked about it and we kind of try to see if we could do it other ways or umm just use logic. So, we know that if there were 20 goats, and 20 goats have 4 legs, that's 80 legs. And if there were 33 hens and they have 2 legs, we just.. Yeah, we went round and put $x$ equals 20 into the equation and see if it equals 146.

Teacher: Ohk! So, do you know other methods used in solving these questions?

Hilde: From school?

Teacher: Yeah, other methods apart from what you did here.

Hilde: Umm. Not sure of, never learned a specific method to use about. Always just the kind of thought about it myself and figured a way around it.

Teacher: So, was it helpful when you solve the problems in a group?

Hilde: Umm. Our group actually started by interpreting or kind of understanding the question and working by ourselves first, and then just join forces last to see if we got the same answer and if we didn't, why? Go over what we did then, yeah.

Teacher: So, how about solving them individually?

Hilde: I think it's good, I work better when I work alone because at least, in first or at first, because then I can just think for myself and I can do it of my own tempo and I don't want to rush if the others are faster than me or stay behind and explain to them if I'm the one that's faster. So, it's just good to do that, but it's also good to compare answers afterwards to see if I did some yeah, if there is kind of plus or minus wrong.

Teacher: Can you see any connection between the mathematics you learn at school and the outside worlds?

Hilde: Yeah, if I think about it.. But I mean it's not like I go round and then look for it all the time, sometimes I do it because of, you know, it a bit fun [Laughs].

Teacher: How often do you solve mathematical modelling task at school?

Hilde: Umm. We do it when we are working on subject, but this was a bit different because there were so many variables, there wasn't a concrete answer I think at least, because we got, the first one we had two variables $x$ and $a$, we didn't know minimum wage and the second one we add three because items were added, and we named it $b$, so it would have being hard to drew it in, yeah, this graph [Points at their solution graph] because we didn't have all the information that is required to use those from us.

Teacher: Ohk! What's your opinion about the modelling task you solved?

Hilde: ...Umm, my thought about it. It's abstract, so it's very hard to actually come up with an answer when you don't have all the information and it was kind of [laughs] irritating, because I always wanted to just find an answer, but I guess it's good to solve, to also just think about it.

Teacher: Yeah, do you see any connection between this modelling task and the real world?

Hilde: Yeah, I mean um like which jobs, we can see that. I mean you can always compare two jobs to see ohk which one will be the best for me, or like this and this situation, there are several of those situations.

Teacher: Ohk! Umm.. Can you tell me the processes you went through in solving the modelling task?

Hilde: Yeah, it a bit similar, we decided what was $x$ and in this case, we took $x$ as hours, number of hours, and $y$ of course is the total money she earns and then we had 20 because it says 20kr per hour, so $20x$ and then we didn't know minimum wage so we put a as minimum wage, so plus $ax$. And the same here [Points at the equation for the second job]. So, because it's in the same graph we again had $x$ as hours so $\frac{a}{2}$ multiplied by hours plus $20b$ because she earns 20kr per item.

Teacher: Ohk! So, did you check whether your answer was right?

Hilde: That was very hard, because of course we did it by um just giving a number for each job, we said minimum wage is 100 and then we tried this out and said ohk, because of so and so, and here we kind of stuck at the.. So, we saw. We found how many items she has to sell for this [Points at the equation for the second job] to better than that one [Points at the equation for the first job], and we got the answer, approximately 27 items.

Teacher: Ohk! So, let's say I'm your friend, interpret your results for me to understand and make a choice?

Hilde: Oh! That's hard umm. So, I would kind of [Laughs] ask, are you good at selling, because if you have to sell 27 items for it to be better than the first job offer. If you are good at selling then ohk, you will after a while earn more with equation number 2 or, yeah. But if you feel sort of insecure then this [Points at the equation of the first job] choice is safer.

Teacher: Why don't we have $x$, the number of hours here [Points at $20b$ in the second equation] in the equation?

Hilde: Because it's kind of constant, it's not if you sell so many items per hour you get this for one day of work. You have number of items multiplied by 20.

Teacher: If you have $x$ here [Points at $20b$ in the second equation] then the number of items she sells will decrease, it won't be 27 items but far less. Let's say 3 items. Do you get it?

Hilde: [Laughs]. I think I got it.

Teacher: Ohk! So, will you want more of the modelling task at school?

Hilde: Umm. I don't know [Laughs]. Maybe, but we are yet to cover a lot of other stuffs, so it kind of depends.

Teacher: So, if you want this type of questions, why will you want them?

Hilde: Umm.. Just to get more used to the way of working and thinking about it and which method we have to use to solve it, because we don't use it, all of it to solve.

## D.4 Interview Transcript, Student D

Teacher: Have you solved or encountered similar problems like these [Points at the algebraic word problem task] algebraic word problems at class before?

Eirik: Umm, yes.

Teacher: Was it very similar?

Eirik: Yes, it was very similar.

Teacher: Ohk! So, which of the questions among the algebraic word problem task was the most difficult?

Eirik: Umm... None of them was very complicated.

Teacher: Like it was kind of the same?

Eirik: Yeah.

Teacher: So, can you please tell me the processes you went through in solving these algebraic word problems?

Eirik: Umm. As I said, I did many of those tasks before, and I know you got to solve them with an equation, so I just try to put all the information I get in equation to solve them the mathematical way.

Teacher: Ohk! So, when you solved them, how did you know your answer was right?

Eirik: Ummm, you can check in equation.

Teacher: So, you put it back into the equation and checked?

Eirik: Mmm!

Teacher: Do you know of any other method used in solving those questions, apart from the method you used?

Eirik: Umm, no I don't, or you can try to think about it in your head but it's almost the same as having an equation.

Teacher: Ohk! Was it helpful when you solve these questions in a group?

Eirik: No, I really solve it myself.

Teacher: Can you see any connection between the mathematics you learn at school and the outside world?

Eirik: Umm, No. Most of the mathematics we go through now is more of theoretical, so you can't really use it in real life, unless something special.

Teacher: Ohk! So, how often do you solve mathematical modelling task at school?

Eirik: Umm, often we solve those ones in… No, just one um, some questions in every topic, just to try if you understood the theme.

Teacher: Yeah! Was it as open as this one [Pointing to the modelling task]?

Eirik: No, they are more related to the theme that we have in class, but you can't be sure (inaudible) either solve them.

Teacher: What is your opinion about this modelling task?

Eirik: Ummm, it's irritating, there is no actual answer because there are too many um variables, yeah.

Teacher: Does the modelling task has any connection between the mathematics at school and the real word?

Eirik: Umm. We don't do things like that in our class anymore. They do the thing B Math, if you know what I'm talking about, we don't, because it looks, it's too easy as in mathematics task, so we did a lot of that in urr lower secondary school.

Teacher: So, can you tell me the processes you went through in solving this modelling task?

Eirik: Umm, one of the mates of the group tried to set $x$ as, if you think about different variables as a… how much you work, how much you sell, and the umm.. that was third variable um and minimum wage umm, if the minimum wage is 100kr, then he tried to find out what it is, but it's not an answer because minimum wage affects how much she earns, and it doesn't make an answer and the best job. Yeah, you can really find an answer and too many variables you can take umm, you can shorten it to two variables as you get minimum wage and the umm… how much she sells but.. I don't have enough information to solve this task.

Teacher: So, how will you interpret your modelling results to me?

Eirik: It depends on how much the minimum is and it depends on how much you are able to sell in one hour, yeah.

Teacher: Where from the 3.6 in your calculations?

Eirik: 3.6 is how much you need to sell to earn more than a…

Teacher: Ohk! Will you want more of this modelling task at school?

Eirik: Umm, yeah, it works fine.

Teacher: Can you tell me more about that?

Eirik: I'm just, you just fine with tasks that are a little bit harder and you gonna fine the tools and just use the tools to fine what's there. It's nothing more than that.

## D.5 Interview Transcript, Student E

Teacher: Have you solved or encountered similar problems like these [Points at the algebraic word problem task] algebraic word problems at class before?

Tonje: Yes, the three last ones, I would say but not the first one. But they are all the same though, like kind of.. we have been through this in pre-school or high school [Laughs]. No, middle school.

Teacher: Middle school?

Tonje: Yes, we've been through this, but it's kind of hard to like recall what we were taught or in mine opinion, that was my case.

Teacher: So, you've been through the last three=

Tonje: =Yes, I will say that.

Teacher: Ohk! Was it very similar?

Tonje: Yeah.

Teacher: Ohk! So, which of the questions was the most difficult?

Tonje: The first one, because we had to like get into the way of calculating the other task as well. So, when we first solve the first one, we were kind of in the game, so we knew how to solve the others [Laughs]. The first one is like a very difficult warm up whilst the three last ones were kind of like, ohk we've done this, it's ohk now.

Teacher: Can you please tell me the processes you went through in solving these algebraic word problems?

Tonje: We tried and we failed [Laughs] several times on the first one.. Yeah, so we just kind of.. We decided to try and solve them using algebraic methods, however we kind of failed couple of times before we first or finally figured how to solve the task.

Teacher: Ohk!

Tonje: And then, after that the three last ones just went like a breeze [Laughs].

Teacher: Ohk! So, how did you know your answer was right?

Tonje: We tested it out. Yeah, we checked it on the calculator, we put in um what we thought was $x$ into the equation and calculate on the calculator.

Teacher: So, do you know of any other method used in solving these algebraic word problems?

Tonje: Umm, you could kind of just put a variable in there um by yourself and instead of the $x$, instead of doing what the um question kind of ask you to do, but that's kind of [Laughs] takes a longer time.

Teacher: Ohk! Was it helpful when you solve these questions in a group?

Tonje: Yes, indeed [Laughs].

Teacher: How helpful was it?

Tonje: Well, because when you are by yourself and you are like trying your best, you only have your own mind like you don't get help from anyone, but when you are two people you both share your thoughts and if one has one opinion then the other person might have another, so you can compare and combine maybe, it works things out.

Teacher: Ohk! Can you please explain question four in the algebraic word problem task for me?

Tonje: So, um 'A given number is six more than a second number'.

Teacher: Yeah.

Tonje: Umm, we put the second number as $x$, because you don't add something to it, and then the first number is then $x + 6$ because you add something. So, then we just put in the umm the multiplications or the number.

Teacher: So, can you explain the multiplication for me?

Tonje: Yeah, so the first number you multiply with 4.

Teacher: So, the first number was $x$?

Tonje: The first number was umm, wait [Laughs]. Oh! I wrote something wrong [Laughs]. Ohk! So, I might have done something wrong here, umm. I meant to put 4 there [Points at $x + 6$ in the equation] and 2 here [Points at $x$ in the equation]. Ohk! Yeah [Laughs].

Teacher: So, can you see any connection between the mathematics you learn at school and the outside world?

Tonje: Yes.

Teacher: What is the connection here?

Tonje: Ohk! I just answered that on impulse, because we've had that question several times and we've always being proven wrong because it is in the real world, like kind of the examples put much easier. I guess like, if you are working and you earn a certain amount of money for one hour then you can use for example algebraic methods of calculating how much you will earn. Yeah, for several hours [Laughs].

Teacher: Ohk! So, how often do you solve mathematical modelling task at school?

Tonje: Like realistic problems?

Teacher: Yeah.

Tonje: Ummm, only this year I think I have done it um, in the $10^{\text{th}}$ grade it wasn't really that realistic. Yeah, like twenty thousand million against one apple [Laughs]. Is not gonna happen but.. Ohk! Well, it now that actually using real life problems or real-life situations to teach us math which I really appreciate.

Teacher: So, can you tell me the processes you went through in solving this modelling task?

Tonje: We kind of struggled because we thought we had to know what the minimum wage was, so we kind of like wasted a lot of time just trying to find out the minimum wage. However, we decided to continue without it and just made our own minimum wage, like put in our own number and then after that we just made a graph which is a little incorrect but [Laughs] I realize that as soon as I started, but yeah, then we just compared this two [Points at the two equations for first and second job], saw where they meet each other and which one will benefit Kristin in the long run and, yeah!

Teacher: So, this line [Points at the red line on the graph] is what?

Tonje: This is the first job and the second job is the blue one.

Teacher: So, how do you interpret your results to me?

Tonje: Well, if our calculations are correct, then we advise you to [Laughs] choose the first job, because you will earn the most in the fasters time period.

Teacher: Ohk!

Tonje: Because as you can see um, it pays 20kr an hour above the minimum wage and we put the minimum wage as 100 um, and then we start there and it rises by 20kr an hour, I think.

Teacher: Ohk!

Tonje: Yes, and if you choose the other job then you have to be more, I don't know, you have to be like go further into how the people are or into what the people are going to buy, because you earn more when they buy more, which means you have to be more active, you have to um advertise the product, you have to go like, 'Hey, you wanna buy chocolate'. So, you have to work, you have to actually work to [Laughs] get money. Yeah, the other one is like you do your job, but you still earn your money without nothing more.

Teacher: You should have had $x$ in both sides and a new variable representing the number of items like y equals $50x + 20bx$.

Tonje: Yes, but we kind of struggled, I mean we.. Our opinions didn't match each other.

Teacher: Ohk! So, will you want more of the modelling task at school?

Tonje: Yeah, I would say.

Teacher: Why?

Tonje: I think I learn more umm, what we do new is kind of just she gives us a paper or some explanation of what we are doing and then she just kind of let us read one sentence about what we are learning, and we are just on our own. But I feel like here we got to work together more and we got more task that um still match what we are learning, but what we are doing now is just go way out just.. very complicated um yeah. But they kind of mention that since this is T-math or theoretical math, then we have to like make up our own theories kind of, so her giving us like one sentence is kind of what we are suppose to do but it doesn't help us. You kind of have to be good at math to be able to do stuff, so if we do this more then we can kind of interact more and we won't like slide out and change the theme while working together as often, because then we will have the focus on the certain thing.

## D.6 Interview Transcript, Student F

Teacher: Have you solved or encountered similar problems like these [Points at the algebraic word problem task] algebraic word problems at class before?

Helge: Umm.. Yes, in algebra. So, we have encounter several of these.. umm, number one um number two, I was not as familiar with the third one um and I was not as familiar with the fourth one, but the first and second one...

Teacher: So, when you went through the questions, which of the questions among the algebraic word problem task was the most difficult?

Helge: Umm, definitely these two, number three and four, because I didn't really know how to solve them as um, I had solved some tasks that were kind of some link, but I didn't really solve the exact some ones, umm and so those were the most difficult ones, definitely.

Teacher: Ohk! Can you tell me the processes you went through in solving these algebraic word problems?

Helge: Ohk! So, first we read through the question umm like together aloud and then we umm, we just explain what we thought about the task, and then we started working on our own. So, having shared our ideas, we started working on our own and when someone thought that they had come to an answer, they started explaining to the other group members um and then we say like, AHHH Ohk so you finish it, or we also saw that maybe it wasn't correct, so then we started working on our own again, but then we used that person's method um to get to the right answer, yeah.

Teacher: Umm. Apart from the method you used, do you know of any other method used in solving these algebraic word problems?

Helge: Umm, not really, um especially for one and two we used simultaneous equation. I don't really know how to solve them any other way. You can probably make a graph umm but I didn't know.

Teacher: Ohk! Was it helpful when you solve these questions in a group?

Helge: Yeah, because if you didn't know how to solve it umm then when another person manages to solve it, you can (inaudible) ask them, or how did you solve this task? Umm, but if no one can really solve it then it would have been a problem but then you just have to ask the teacher um, but it is also like a downside within the group because umm the group might not be helpful, you might understand what they are doing but it is not really beneficial because um if you are in a group where everyone is working on their own um and you aren't really sharing ideas and you don't know what to do, then you will just be sitting there wasting time.. umm, but it can be really helpful.

Teacher: Ohk! So, how did you know your answer was right?

Helge: Umm, we like back checked them, so we for example in the equation we got, like if $x$ is 20, then we put 20 into the first equation, into like $x$ and then we will solve it and see like, Oh this is actually correct.

Teacher: Ohk! Can you see any connection between the mathematics you learn at school and the outside world?

Helge: Umm.. No, I don't think there is much of a connection.. Umm, but that's only because I don't really use um, or depends on what you do on your spare time, but I don't really use equations, I don't make like any real graphs on the spare time, but if you like to do this and you want to study mathematics when you grow older umm, when you get older you might want to be a teacher then it's kind of be helpful, but Uhh [Laughs].

Teacher: So, you don't see any connection between the math at school and the real world?=

Helge: =Not really, also it depends on what we are learning about, if it is like only, is like right in the math like plus minus, yeah then it can be helpful I guess.

Teacher: How often do you solve mathematical modelling task at school?

Helge: Umm.. that, like I'm more use to like doing this, but I also think it depends on what school you go to.

Teacher: I mean like what you now..

Helge: Oh!

Teacher: Do you do more of this [Points at the modelling task].

Helge: Yeah, I think we just finish a chapter on like, like linear graph and stuffs, umm we have worked quiet a lot actually with this kind of task.

Teacher: Ohk! What is your opinion about the modelling task you solved recently?

Helge: I mean like, like overall situation around um the task and how it's related to like holidays and gifts umm, stuff like that I think I can more like relate to the task and it might make it easier to solve it, so I think it's a nice task.

Teacher: So, it means there is a connection between this modelling task and the real world?

Helge: Yeah, there is a connection like between this and the real life, and so I can understand the task better.

Teacher: Can you tell me the processes you went through in solving this modelling task?

Helge: Umm, yeah.. First, because what we always do is that, when we get like a text, we have to like sort out the most important part, so we did that first and then we made um like different kind of $x$ and $y$ variables umm, and then we solve them, and then we put them into the graph um, it was a little bit more difficult to make the graph on like paper, because we are used to using the computer, umm but I think we made it like pretty good um, yeah, in my opinion, yeah

Teacher: Ohk! So, did you verify your answer?

Helge: Umm, not sure, I don't think we double checked, because I don't think we knew how to double checked. I guess we could have done it on like the PC afterwards to see it was right, but I don't think we double checked.

Teacher: Ohk! Please interpret your modelling results to me?

Helge: We saw that the first offer was way better umm, because it depends on how many umm... So, you should pick the first offer um, but only if you good at the job um, so if you are a good seller umm and you at least sell 11 things per hour on average then you should pick the second job.

Teacher: Why 11?

Helge: Umm.. I'm not sure, we just wrote it [Laughs], it don't make sense though, that's true umm, but.. you should pick the first offer though, because you will make the most amount of money, yeah.

Teacher: The first equation is ohk. Now the second equation is half the minimum wage plus 20kr on every item you sell. So, there should be an $x$ on both sides like $y = 50x+20ax$ and a new variable $a$ which stands for the number of items.

Helge: Ohk! [Laughs].

Teacher: So, will you want more of the modelling task at school?

Helge: Umm, yeah!

Teacher: Why?

Helge: Because I think this task are fun to solve umm, but it's a little bit easier to do on a computer umm and it's takes a lot of time to like drawing the lines and the axes, so I like this kind of task, but I would like to do it on the PC.

## D.7 Interview Transcript, Student G

Teacher: Have you solved or encountered similar problems like these [Points at the algebraic word problem task] algebraic word problems at class before?

Arvid: Yes, we have looked at similar task.

Teacher: Was it very similar?

Arvid: Umm, quiet but it wasn't like in text form, it was like only umm.. equation.

Teacher: So, which of the questions among the algebraic word problem task was the most difficult?

Arvid: Umm, the first one was the hardest because we didn't know exactly how to solve it. So, we use a little time on getting the right equation, after that it was kind of easy when we knew how to do it.

Teacher: Can you tell me the processes you went through in solving these algebraic word problems?

Arvid: We tried a lot of different equations, we knew that we needed a set of linear equations because it was too many variables, so we started to sort of find the right equation, but we use a little time, because we had forgotten how to do it.

Teacher: So, what processes did you use in finding the right equations?

Arvid: We just tried different, and then we just look if some of them may give us the right answer, so yeah, we just tried a lot of different ones.

Teacher: So, how did you know your answer was right?

Arvid: Umm, we tested them afterwards if we got the answer, let say g equals 20 and we put them in the formulae afterwards and look if it will give us the right things as the task said.

Teacher: Ohk! Do you know of any method used in solving these algebraic word problems?

Arvid: It was another different method inside the theme linear equations, but my group thought that this one was the most easiest to solve,

Teacher: Ohk! Was it helpful when you solved these questions in a group?

Arvid: Yea! We were sort of dividing the task, so if two of us solved it, we discuss how to solve them and then the two others sort of fill it in and wrote what we did so we sort of find a way to cooperate.

Teacher: Ohk! How about solving them individually?

Arvid: I work best individually because I know usually what I need to do, but when we are a group, like it's nice to hear others opinion as well and we got different opinions.

Teacher: Going back to the third and fourth question. Can you resolve it again for me?

Arvid: Yeah.

Teacher: Step by step.

Arvid: Ohk! So, what we did, we knew that.. I gonna read the task=

Teacher: =Ohk! Go ahead.

Arvid: The value of a given number is fifteen more than the value of a second number. The sum of two times the first number and four times the second number is 162. So, we put variables $x$ and $y$, and we knew that $x$ is going, umm $y$ is going to be fifteen more than $x$.

Teacher: So, $y$ is the first number?

Arvid: $y$ is the first number.

Teacher: And $x$ is the second number?

Arvid: Umm, yes!

Teacher: Ohk! Then the next line=

Arvid: =No! No! Wait, no $x$ is the first. The second number is $y$.

Teacher: The second number is $y$ and the first number is $x$?

Arvid: Yes.

Teacher: And then we go to the next line.

Arvid: Umm, the sum of two times the first number, so $2x$ and four times the second number $4y$, which equals urr 162.

Teacher: Oh! The first number is $y$ and the second number=

Arvid: =No, no, the first is $x$.

Teacher: The first is $x$?

Arvid: Yes, and the second is $y$.

Teacher: So, it says the value of a given number is fifteen more than the value of the second number.

Arvid: Yes, so it wasn't... The first number was $x$, so the $x$ is fifteen more than the second number.

Teacher: So, the second number is $y$?

Arvid: Oh!! Ohk! Oh, so it was wrong [Laughs]. Ohk, so I don't know what to do then... It should be switch around?

Teacher: Yeah.

Arvid: Ohk!

Teacher: That's why I asked about that.

Arvid: Ohk! So, it should be $y + 15$ equals $x$.

Teacher: So, it like the first line and the second line of the question.

Arvid: Ohk! So, we just misunderstood the task then.

Teacher: It was same with the fourth question.. So, can you see any connection between the mathematics you learn at school and the outside world?

Arvid: In sort of.. In some of the subject, like trigonometry, we have no idea what to use in the real world but some... But things like equations and some things like that, they are easy to use in the real world.

Teacher: Ohk! How often do you solve mathematical modelling task at school?

Arvid: Umm, not often like that. We found that task really hard actually, so we don't usually solve that.

Teacher: And what do you think might be the reason?

Arvid: I don't know, maybe is not in the things we need to learn, maybe comes further on in the year, semester, I don't know.. but we might actually get it, I don't know.

Teacher: Ohk! What is your opinion about the modelling task you solved recently?

Arvid: It was easier to solve in the head than it was to write in paper, because we struggled a bit with the (inaudible), it was a bit hard to write in paper, and what we did, make lines go as we wanted to go, so it was much easier to just think about the answer in the head than it was to put it down on paper.

Teacher: But in all, do you see any connection between this modelling task and the outside world?=

Arvid: =Yes, that one is really easy to combine with the real world, because it's some problem we might actually stumble upon in future.

Teacher: Can you tell me the processes you went through in solving this modelling task?

Arvid: Umm, we sort of knew what we have to do. We need to write a graph and look at the... What sort of wage was the most exclusive, for saying like that, but we struggled with finding right numbers and just putting it together, but uh we found a solution but we don't know if it's the smartest one.

Teacher: Ohk! But did you double check as you did with the algebraic word problem?

Arvid: Umm no, we didn't. We hadn't got enough time, we were a bit late with that one.

Teacher: So, please interpret your modelling results to me?

Arvid: Umm, that's a hard question. I will probably give you what you would earn the most. But.. I will probably do some same things as I did here, but maybe I will taken in head, and maybe just solve it in head or I just give you my thought on that, I think.

Teacher: So, the first job or the second job? What do you think?

Arvid: Umm, I think it depends on the.. What was the store she was going to work in? [He reads the mathematical modelling task] I will take the second option because you have the potential to earn a lot more than you have to do with the first, but if you don't sell as much as you need to do in one of the hours you can probably sell a double amount in the second, so I will probably go with the second one even though it's a risk.

Teacher: So, do you have an equation for the graphs?

Arvid: No [Laughs]. We didn't come that far either but..

Teacher: You just made=

Arvid: =We just made a line [Laughs] and hope for the best, to be honest.

Teacher: Because, the first graph looks like $y$ equals $120x$.

Arvid: Yes, actually it's does.

Teacher: But the second I don't really understand.

Arvid: No we were.. We want as a… We didn't care [Laughs] if you can say like that, we made a few short cuts, we just come to an answer and just look like it's going to give us the right answer.

Teacher: But can you tell me more about the second graph?

Arvid: We should probably switch around the $x$ and $y$ axis, because then we get the kroners up here [Points at the $y$-axis] and the stuff she needs to sell up there [Points at the $x$-axis]. And it should start at 50, because the minimum wage is 50 in the second, so it should start at 50 and then go up by 20 per items she sells, so we work a bit not so perfectly in line with that one.

Teacher: Ohk! So, you made the equation 50kr plus the items that she sells=

Arvid: =Yeah, $y$ equals $20x + 50$ I think

Teacher: So, it means if she sells one item then it's going to be 70.

Arvid: Yeah, 70.

Teacher: Ohk, so when it's one then we have 70, but it's 50 on the graph.

Arvid: Yeah, it was not as perfect as we wanted it to be, but wasn't it also going up by 50 at the time in the hour as well. Wasn't it?

Teacher: Ohk! You had an idea.

Arvid: The idea was around that, but we didn't know how to write it down actually [Laughs].

Teacher: So, will you want more of the modelling task at school?

Arvid: Maybe more of the equations and yes, things we need to do or things we can use in the future as well. Like we don't need to know function analysis and things like that or a trigonometry but things like we can use umm, is good to learn things like that.

Teacher: Yeah, because every math that you study at school probably can be used outside there=

Arvid: =Yeah.

Teacher: Like trigonometry, they can ask you maybe you want to build or paint a house, where do I place the ladder? So, you need to find the angles involve. Math can be used everywhere.

Arvid: Yes, it can, but some of the things we go through, I don't know how to use.

Teacher: So, the problem is how.. As teachers how, we are supposed to give task that you can relate=

Arvid: =Yes, or maybe if we have uhm… I think we are going to maybe say what we can use this for in the future, because some just say you will find out, you will figure it out, and then we don't know what to.

# E Mathematizing Task 2, 3 and 4

## E.1 Task 2

A Polyaian way of mathematizing Task 2 in Figure 10 (page 55) is illustrated below:

Linear equation:

> *Step 1*: Summarize the information in a table. That is, translate the problem using variables.

| | number | Total number of pupils making 346 |
|---|---|---|
| boys | $x$ | $2x$ |
| girls | $7+x$ | $13(7+x)$ |

*Step 2*: Set up an equation

Total number of pupils making 346 = $13(7+x)+2x=346$

Total number of pupils in the mathematics class = $(7+x)+x=7+2x$

*Step 3*: Solve the equation

$$\begin{aligned}13(7+x)+2x&=346\\91+13x+2x&=346\\15x&=255\\x&=17\end{aligned}$$

$7+2x=7+2(17)=41$

*Step 4*: Present the final answer

There are 17 boys and 24 girls. Altogether there are 41 pupils in the mathematics class.

## E.2 Task 3

The Polyaian way of mathematizing Task 3 in Figure 10 (page 55) is illustrated below:

Linear equation:

*Step 1*: Summarize the information in a table. That is, translate the problem using variables.

| | value | The sum when both numbers are increased is 162 |
|---|---|---|
| $1^{st}$ number | $15+x$ | $2(15+x)$ |
| $2^{nd}$ number | $x$ | $4x$ |

*Step 2*: Set up an equation

The sum of the two numbers after the increment $= 2(15 + x) + 4x = 162$

The sum of the two numbers $= (15 + x) + x = 15 + 2x$

*Step 3*: Solve the equation

$$\begin{aligned} 2(15 + x) + 4x &= 162 \\ 30 + 2x + 4x &= 162 \\ 6x &= 132 \\ x &= 22 \end{aligned}$$

$15 + 2x = 15 + 2(22) = 59$

*Step 4*: Present the final answer

The $1^{\text{st}}$ number is 37 and the $2^{\text{nd}}$ number is 22. Their sum is 59.

## E.3 Task 4

The Polyaian way of mathematizing Task 4 in Figure 10 (page 55) is illustrated below:

Linear equation:

*Step 1*: Summarize the information in a table. That is, translate the problem using variables.

| | value | The sum when both numbers are hypothetically increased is 126 |
|---|---|---|
| $1^{\text{st}}$ number | $6 + x$ | $4(6 + x)$ |
| $2^{\text{nd}}$ number | $x$ | $2x$ |

*Step 2*: Set up an equation

The sum of the two numbers after the hypothetical increment $= 4(6+x)+2x = 126$

The value of the 1st number $= 6 + x$

*Step 3*: Solve the equation

$$\begin{aligned} 4(6+x)+2x &= 126 \\ 24+4x+2x &= 126 \\ 6x &= 102 \\ x &= 17 \end{aligned}$$

$6 + x = 6 + 17 = 23$

*Step 4*: Present the final answer

The value of the 1st number is 23 whilst the 2nd number is 17.

# F 1T Mathematics Class Exercises

## 2.6 Simultaneous Sets of two Linear equations / Linear Systems with Two Variables

### Introduction – visualisation

ROW 1

ROW 2


**Intro problem 1**

Find out, without calculating the price for each item, which item cost the most, the cap or the umbrella?

**Intro problem 2**

Use the two rows to make a new combination consisting of umbrellas and caps, and calculate the combined price for one umbrella and one cap.

**Intro problem 3 - Additions method (Elimination method)**

a) Use row 1 to make a second that shows the price for 4 umbrellas and 2 caps.
b) Combine the row in a) and row 2 to make a row consisting only of umbrellas.
c) Use the row in b) to decide the price for an umbrella, and then to decide the price for a cap.


**Intro problem 4 - Substitution method**

a) The price for an umbrella is $p$ and the price on a cap is $c$, set up an equation for the price of 2 umbrellas and 1 cap.
b) In the same way set up an equation for the price of 1 umbrella and 2 caps.
c) Find a formula for $c$ using the equation in a).
d) Substitute $c$ in equation b) with the formula in c). Use this new equation to find the price $p$ for an umbrella.
e) Lastly, find the price $c$ for a cap.

## 2.6 Problems

**Problem 1.** Solve the linear systems using addition method.

a) $\begin{bmatrix} 2x+y=8 \\ x-y=-2 \end{bmatrix}$

b) $\begin{bmatrix} 3x-2y=5 \\ 5x+4y=1 \end{bmatrix}$

c) $\begin{bmatrix} 2x-5y=4 \\ x-4y=5 \end{bmatrix}$

**Problem 2.** Solve the linear systems using the substitution method.

a) $\begin{bmatrix} x=y+1 \\ 2x+3y=-3 \end{bmatrix}$

b) $\begin{bmatrix} y=2x-1 \\ x-2y=-1 \end{bmatrix}$

c) $\begin{bmatrix} 4x+2y=14 \\ 2x+3y=15 \end{bmatrix}$

**Problem 3.** Solve the linear systems.

a) $\begin{bmatrix} x+2y=3 \\ 2x+3y=4 \end{bmatrix}$

b) $\begin{bmatrix} 3x+4y=5 \\ 4x+5y=6 \end{bmatrix}$

c) $\begin{bmatrix} x-2y=4 \\ -y+3x=-3 \end{bmatrix}$

**Problem 4**

Solveig buys in total 13 kg of potatoes and carrots, and pays 97 kr. The potatoes cost 6.50 kr per kg, and the carrots costs 9 kr per kg. How many kilograms of each vegetable did she buy?

**Problem 5.**

The sum of two numbers $x$ and $y$ equals 14. The difference between the two numbers is 40. Find the two numbers, $x$ and $y$.

**Problem 6**

Altogether, it costs 56 kr for 3 kids and 2 adults for bus tickets. One adult ticket cost twice as much as a kid ticket. How much does an adult ticket cost, and how much does a kid ticket cost?

**Problem 7**

It costs 68 kr for 2 kg of apples and 3 kg of oranges. Furthermore, it costs 60 kr for 3 kg of apples and 1 kg of oranges. How much does 1 kg apples costs, and how much does 1 kg of oranges costs?

**Problem 8**

Find two numbers where the sum equals 42 and the difference is 12.

**Problem 9**

It was sold 800 tickets to a pop concert. Normal tickets cost 90 kr and VIP-tickets cost 120 kr. There were sold tickets for 85500 kr. How many normal tickets were sold?

**Problem 10**

One day on a local café, I could not stop noticing the guests sitting by two other tables.

The people on table 1 had eaten six pieces of cake and three coffees and had to pay 435 kr.

On table 2, the guests had eaten eleven pieces of cake and had seven coffees, and they had to pay 880 kr.

My family had five pieces of cake and six coffees, what did we have to pay?

**Problem 11**

Use a digital program (CAS in geogebra) to solve problem 3b), 1a), 2b) and 3c).

## 2.6 Solutions

1. a) $x=2$ , $y=4$ b) $x=1$ , $y=-1$ c) $x=-3$ , $y=-2$
2. a) $x=0$ , $y=-1$ b) $x=1$ , $y=1$ c) $x=1{,}5$ , $y=4$
3. a) $x=-1$ , $y=2$ b) $x=-1$ , $y=2$ c) $x=-2$ , $y=-3$
4. 8 kg potatoes and 5 kg carrots
5. $x=27$ and $y=-13$
6. Kids ticket: 8 kr, Adult ticket: 16 kr
7. Apples: 16 kr, Oranges: 12 kr
8. 27 and 15
9. 350
10. 555 kr

## 3.3 Linear regression – simulate linear models

Regression - predictive model of the relationships between variables (eg. points on a graph).

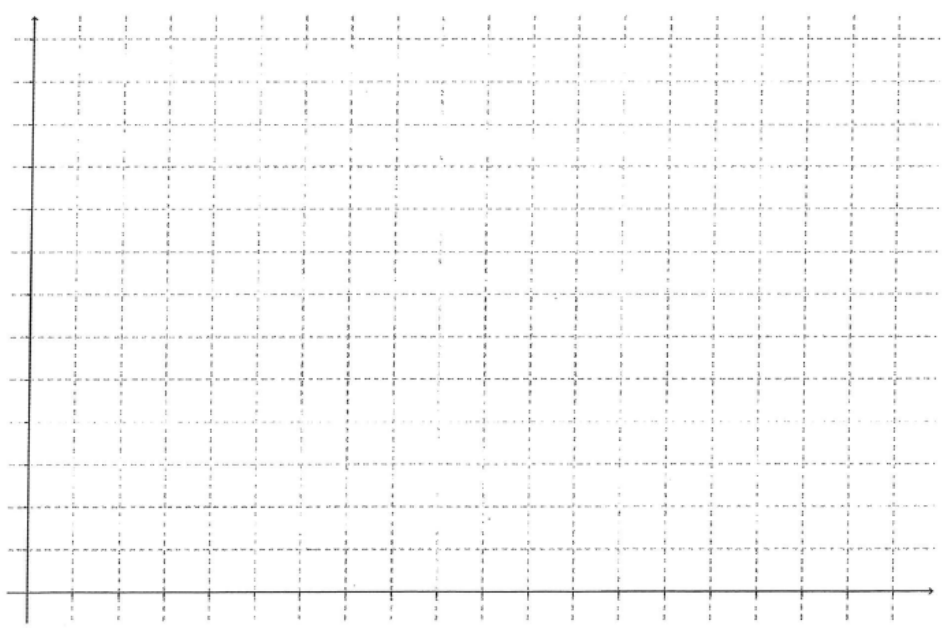
**Introduksjonsoppgave 1 Lineær modell for kraft og lengde ved strekking av Laban**

I dette forsøket skal dere strekke en seigmann ved hjelp av en kraftmåler. Legg seigmannen på en linjal og hold hodet fast. Stikk en ståltråd/binders gjennom nederste del av beina. Strekk seigmannen ved å dra med kraftmåleren i ståltråden.

a) Mål sammenhørende verdier av kraft og lengde og før resultatene inn i tabellen under.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Kraft (N) | | | | | | | | |
| Lengde (cm) | | | | | | | | |

b) Tegn måleresultatene inn som punkter i koordinatsystemet nedenfor. La x-aksen være kraften og y-aksen være lengden. Husk å skrive navn og enheter på aksene.



c) Tegn en rett linje som passer best mulig med punktene og finn en formel for hvordan lengden henger sammen med kraften (en formel for lengden; L = ?). **Skriv likningen for linjen her;**

Dette er et eksempel på en *lineær modell.* Dere har nå funnet en matematisk modell for sammenhengen mellom lengden og kraften ved å utføre en *lineær regresjon*.

d) Forklar hva konstantene (tallene) i formelen forteller.

### 3.3 Oppgaver med lineær regresjon i geogebra

**Oppgave 1**

Kari er på biltur sammen med kjæresten Ivar. Bilen har en digital bensinmåler der de kan lese av nøyaktig hvor som er igjen på tanken. Ivar fylte bensin da turen startet. Kari nullstilte da tripptelleren. Hun leser med ujevne mellomrom av kjørelengden x (km) og bensinmengden y (l) som er igjen på tanken. Her er tallene;

| x (km) | 0 | 120 | 250 | 380 |
|---|---|---|---|---|
| y (l) | 60 | 51,5 | 42,1 | 33,8 |

a) Finn ved regresjon likningen for den rette linja som passer best.
b) Hvor mye bensin bruker bilen per mil i gjennomsnitt på turen?
c) Bruk modellen du fant i a) til å finne ut hvor mye bensin er igjen på tanken etter 32 mil?
d) Bruk modellen du fant i a) til å finne ut hvor langt bilen har kjørt når det er 10 liter igjen på tanken.

**Oppgave 2**

Tabellen under viser folkemengden i Norge for noen utvalgte år i perioden fra 1950 til 2000.

| **Årstall** | 1950 | 1960 | 1970 | 1980 | 1990 | 2000 |
|---|---|---|---|---|---|---|
| **Folkemengde** | 3 249 954 | 3 567 707 | 3 683 221 | 4 078 900 | 4 233 116 | 4 478 497 |

a) Finn en lineær modell som beskriver sammenhengen mellom år og folkemengde. La $x$ være antall år etter 1950 og $y$ folkemengden i millioner.

b) Hvor mye øker folkemengden per år ut fra uttrykket du fant i a)?

c) Dersom denne utviklingen fortsetter. Hva vil folkemengden i Norge være i år 2050?

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**3.3 Fasit**

1 a) $y = -0,069x + 59,8$ b) 0,69 l/mil c) 38 l d) 72 mil

2 a) $y = 0,024x + 3,315$ b) 24 000 per år c) 5 715 000