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On the existence of solutions to the higher order Korteweg de Vries-Burgers equation

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Master thesis in mathematics

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Sammendrag

Vi studerer eksistensen av løsninger til den høyere ordens Korteweg de Vries-Burgers likningen $u_t + 2uu_x - u_{xx} + u_{xxxxx} = 0$. Ved å jobbe med estimater i Sobolev rom finner vi at det eksisterer unik løsning i Sobolev rommet H^s for $s > 1/2$. Ved å bruke det relativt modderne Bourgain rommet klarer vi å redusere denne verdien betraktelig, og vi viser at det eksisterer løsninger i H^s for $s > -2$.

Abstract

We study the existence of solutions to the higher order Korteweg de Vries-Burgers equation $u_t + 2uu_x - u_{xx} + u_{xxxxx} = 0$. By working out a priori estimates in Sobolev spaces we find that there exists unique solution in the Sobolev space H^s for $s > 1/2$. Through the relatively modern Bourgain spaces, we manage to reduce this value significantly and find that there exists unique solution in H^s for $s > -2$.

Acknowledgement

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Contents

1 Notation

$\mathcal{F}(u)$ denotes the Fourier transform of u

$\mathcal{F}^{-1}(u)$ denotes the inverse Fourier transform of u

$\mathcal{F}_x(u)$ denotes the Fourier transformation in x of u

$A \lesssim B$: there exists a constant $c > 0$ such that $|A| \leq c|B|$

$A \sim B$: there exists constants $c, c' > 0$ such that $|A|c \leq |B| \leq |A|c'$

$A \gg B$: there exists a constant $c \geq 100$ such that $|A| \geq c|B|$

$\langle \cdot \rangle^s = (1 + |\cdot|^2)^{\frac{s}{2}}$

iff: if and only if

$\mathbb{N}_0$: $\mathbb{N} \cup \{0\}$

2 Introduction

The generalized Korteweg de Vries equation describes the propagation of waves in shallow water, and was proposed by Diederik Johannes Korteweg and Gustav deVries in 1895 [12] and is given by

$$u_t + u_{xxx} + uu_x = 0 \tag{2.1}$$

It is classified as a nonlinear dispersive partial differential equation with dispersive term u_{xxx}

The generalized Korteweg de Vries-Burgers equation, KdV-Burgers for short, also has a dissipative term u_{xx} and is given by

$$u_t + \alpha uu_x + \beta u_{xx} + \gamma u_{xxx} = 0 \tag{2.2}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $\alpha\beta\gamma \neq 0$, and arises in several physical contexts, e.g flow of liquids containing gas bubbles [23]. With coefficients $\alpha = 1, \gamma = 1, \beta = -1$, Molinet and Ribaud has shown existence of unique solutions in the Sobolev space H^s for $s > -1$ [14]. The equation we will be studying is the higher order KdV-Burgers equation

$$\begin{cases} u_t + 2uu_x - u_{xx} + u_{xxxxx} = 0, & t > 0, x \in \mathbb{R} \\ u(0,x) = u_0(x) \end{cases} \tag{2.3}$$

Different variants of the KdV-Burgers equation are heavily studied, and there are countless papers publsihed on the subject. Well-posedness of this specific equation is to the writers knowledge not known to have been studied before.

We will approach our equation in two different function spaces, namely Sobolev spaces and Bourgain spaces. This thesis is structured as follows: In chapter 3 we review some important preliminary subjects with essential theorems and properties that will be usefull for our further discussion. We will take a brief look into functional analysis and the most usefull norm inequalities and theorems for our need. We also give some important results regarding integral calculus, and introduce the reader to distributions and operations on distributions. We conclude chapter 3 with some results on convolutions. Furthermore, in chapter 4 we give insights on the Fourier transformation and its properties, both in L^1, L^2 and for tempered distributions. In chapter 5 we review the basics of Sobolev spaces.

In chapter 6 we look for solutions in the Sobolev space H^s and work to get the lowest value of s we can set for our a priori estimates to hold. Then we find a suitable function space in which we can apply the contraction principle, which is our main tool for getting a unique solution.

In chapter 7 we look for solutions in the Bourgain space $X^{s,b}$ and hope to lower our value for s. The work in this section is somewhat technical, but explained in quite detail.

From chapter 3 throughout chapter 5 we omit all proofs. the reader can resort to the sources given for detailed proofs and further results on the subject. The results will be stated without specific reference, the sources are rather given at the start of each section. When spesific results are gathered from a source not stated in the beginning of the section, it will be specified. Examples calculated in these sections will be used for work in chapters 6 and 8, and are calculated by the writer. In chapters 6 and 7 there are several theorems and propositions with proofs. These are done by the writer, but inspired by the work of others. Some more so than others. Specific references will be given when due. Consequently, proofs and results in these two sections without sources are done by the writer.

3 Preliminaries

3.0.1 Functional analysis

For the part of normed vector spaces and Hilbert spaces, we mainly follow the works of Weaver [24, Chapters 3,5], but also Bowers & Kalton [4, Chapters 2,6] and Rudin [17, Chapters 1-3]. For results on real analysis, we use Ross [16, Chapters 2-4].

Normed vector spaces

Recall the definition of a norm

Definition 3.1. *A norm on a vector space E over $\mathbb{F} = (\mathbb{R}, \mathbb{C})$ is a mapping $\|\cdot\| : E \to [0, \infty)$ which satisfies the conditions*

$$\begin{aligned} (i) :& \|x\| = 0 \iff x = 0 \\ (ii) :& \|ax\| = |a|\|x\| \\ (iii) :& \|x + y\| \leq \|x\| + \|y\| \end{aligned}$$

for all $x, y \in E$ and for all $a \in \mathbb{F}$.

A vector space E equipped with a norm is called a normed vector space, and a normed vector space where the associated metric $d(x, y) = \|x - y\|$ is complete, is called a Banach space. Typical examples of Banach spaces are the L^p spaces. The fact that the space is complete is of great significance for us. Since every cauchy sequence converges, it gives us the greatest freedom to take limits. In normed vector spaces, continuity and boundedness are equivalent:

Proposition 3.1. *Let E, F be normed vector spaces, and let $T : E \to F$ be a linear mapping. Then T is continuous iff $\exists C \geq 0$ s.t $\|T(x)\| \leq C\|x\|$ for all $x \in E$*

We define the notion of a mapping being an isometry

Definition 3.2. *Let E, F be normed spaces and $T : E \to F$ be a linear mapping. If*

$$\|T(x)\| = \|x\|$$

for all $x \in E$ then T is an isometry

In other words, the mapping T preserves the norm. We have another definition regarding norm-preserving mappings:

Definition 3.3. *Let E, F be normed spaces. A bijective linear map $T : E \to F$ is called and isomorphism if both T and T^{-1} are continuous. We then say that E and F are isomorphic. If T is also a isometry on E, we say that T is an isometric isomorphism*

An important result that will help us when extending the Fourier transform beyond L^1 is this weaker version of the Hahn-Banach theorem.

Proposition 3.2. *Let E_0 be a dense subspace of a normed vector space E and let F be a Banach space. Then any bounded linear map T from E_0 to F extends uniquely to a bounded linear map, $\tilde{T}$ from E to F. Furthermore, the norm of T is equal to the norm of $\tilde{T}$.*

Whereas this proposition holds only for dense subspaces, the Hahn-Banach theorem states that this is also true for all subspaces if the target space is $\mathbb{F}$. These kinds of mappings are important for studying Banach spaces

Definition 3.4. *Let E be a normed vector space. If the mapping $T : E \to \mathbb{F}$ is linear we call T a linear functional.*

The space of all linear functionals is called a dual space

Definition 3.5. *The dual space of a normed vector space E is the Banach space E' consisting of all bounded linear functionals on E*

The study of dual spaces can tell us much about Banach spaces, especially when working in Hilbert spaces.

Hilbert spaces

Recall the definition of an inner product

Definition 3.6. *An inner product on a vector space E over $\mathbb{F}$ is a mapping $\langle\cdot,\cdot\rangle : E \times E \to \mathbb{F}$ which satisfies*

$$\begin{aligned} i) :& \langle v, v\rangle \geq 0 \quad \textit{with equality iff} \quad v = 0 \\ ii) :& \langle w, v\rangle = \overline{\langle w, v\rangle} \\ iii) :& \langle av + bu, w\rangle = a\langle v, w\rangle + b\langle u, w\rangle \end{aligned}$$

for all $u, v, w \in E$ and all $a, b \in \mathbb{F}$.

The norm associated with the inner product is given by

$$\|v\| = \langle v, v\rangle^{1/2}$$

An important inequality we will use often is the Cauchy-Schwartz inequality

Theorem 3.1 (Cauchy-Schwartz inequality)**.** *Let E be a vector space equipped with an inner product. Then we have*

$$|\langle u, v\rangle| \leq \|u\|\|v\|$$

for all $u, v \in E$.

Using the properties of the inner product, it is not hard to get the triangle inequality satisfied

Lemma 3.1. *Let E be a vector space and let $u, v \in E$. Then we have*

$$\|u + v\| \leq \|u\| + \|v\|$$

We are now ready to define a Hilbert space

Definition 3.7 (Hilbert space)**.** *A Hilbert space is a normed vector space equipped with an inner product for which the the associated norm is complete.*

Classical examples of Hilbert spaces are both the l^2 space of all scalar sequences that are square summable and the L^2 space of square integrable functions with their respective norms $\|a\|_{l^2} = \Big(\sum |a_n|^2\Big)^{1/2}$ and $\|f\|_{L^2} = \Big(\int |f(x)|^2 dx\Big)^{1/2}$. It can be shown that these two are the two only examples of Hilbert spaces, up to an isometric isomorphism.

The key concept in Hilbert spaces which differs it from standard Banach spaces is the orthogonality property

Definition 3.8. *Let H be a Hilbert space. We say that $u, w \in H$ are orthogonal if $\langle u, w \rangle = 0$.*

There are subspaces in which every element might be orthogonal to some other subspace, this is defined by

Definition 3.9. *Let E, F be two closed subspaces of a Hilbert space H. We say that E is orthogonal to F if for all $u \in E$ and for all $v \in F$, $\langle u, v \rangle = 0$.*

If we have vectors $e_n \in H$ with the property that e_n and e_m are orthogonal for $n \neq m$ and $\|e_n\| = 1$ for all n, we call these vectors orthonormal. An orthonormal basis for a Hilbert space is an orthonormal set of vectors whose span is dense in H, and it can be shown that all seperable Hilbert spaces have a countable orthonormal basis. Orthogonality leads to several important results.

Theorem 3.2 (Pythagoras)**.** *If u, v are elements of an inner product space and u is orthogonal to v we have that*

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

The first key result is the closest point lemma

Lemma 3.2 (Closest point lemma)**.** *Let H be a Hilbert space and suppose that E is a nonempty closed convex subset of H. Given $x \in H$ there exists a unique point $y \in E$ such that*

$$\|x - y\| = d(x, E) = \inf_{z \in E} \|x - z\|$$

Next, we have the following duality result

Theorem 3.3 (Riesz representation theorem)**.** *Let H be a Hilbert space. Then every $u \in H$ gives rise to a bounded linear functional $v \to \langle v, u \rangle$ with norm equal to $\|u\|$, and every bounded linear functional on H has this form.*

In some cases, it can be easier to work with the norm of the inner product instead of the norm of $u \in H$. We will see this theorem flourishing in the nonlinear estimates in chapter 7.

Arrising from orthonormality, we have Bessels inequality

Theorem 3.4 (Bessels inequality). *Suppose $(e_j)_{j=1}^N$ is a countable orthonormal set, where $N \in \mathbb{N} \cup \infty$. For $x \in H$ we have*

$$\sum_{j=1}^{N} |\langle x, e_j \rangle|^2 \leq \|x\|^2$$

and, if we have equality in the above equation we have

$$\sum_{j=1}^{N} \langle x, e_j \rangle e_j = x$$

When we have equality in the first expression, the property is called Parsevals identity. But under what conditions do we have Parsevals identity? It can be shown that the following three conditions are equivalent

Theorem 3.5. *Let H be a Hilbert space and suppose $(e_n)_{n=1}^{\infty}$ is an orthonormal sequence in H. Then the following are equivalent*

1: *The sequence $(e_n)_{n=1}^{\infty}$ is an orthonormal basis for H*

2: *If $x \in H$ then we have Parsevals identity satisfied*

3: *$(e_n)_{n=1}^{\infty}$ is a maximal orthonormal sequence*

The most important result in this section tells us that any Hilbert space is Hilbert isomorphic to l^2. We use the notation $l^2_\infty = l^2$

Theorem 3.6. *Let H be a Hilbert space. Then H is Hilbert isomorphic to l^2_n, where $n = \dim H$. That is, there exists a linear bijection between H and l^2_n which preserves the inner product.*

Operators on normed vector spaces

Recall that a linear operator T is a mapping $T : E \to E$ for a vector space E such that $T(ax + by) = aTx + bTy$ for any $a, b \in \mathbb{F}$. The utmost important result of this section is the Banach contraction principle:

Theorem 3.7. ***Banach contraction-mapping principle***
Let $I : E \to E$ be a linear operator and E a non-empty complete metric space. If there exists a constant $0 < k < 1$ such that $d(Ix, Iy) \leq kd(x, y)$, for all $x, y \in E$ then I is called a contraction mapping and I has a unique fixed point.

All norms give rise to a distance function in the form of $d(x, y) = \|x - y\|_X$. Thus, the statement is equivalent to

$$\|I(u) - I(v)\|_X \leq k\|u - v\|_X \tag{3.1}$$

This is our main tool for showing existence of unique solutions. For operators between Hilbert spaces, we have this helpfull result

Lemma 3.3. *Let H be a Hilbert space. If $T : H \to H$ is a bounded linear operator, there exists a unique linear operator $T^\star : H \to H$ such that*

$$\langle Tx, y \rangle = \langle x, T^\star y \rangle$$

for all $x, y \in H$, and such that

$$\|T\| = \|T^\star\|$$

This operator $T^\star$ is called the Hilbert space adjoint of T.

3.0.2 Integral calculus and L^p spaces

In this section we will review some essential results on integral theory. The sources used in this section are Weaver [24, Chapters 2,4] and Taylor [22, Chapters 4,9]. For simplicity, the limits of integration and the measure we are integrating with respect to is always the measure space (X, μ) noted in the start of the results or definitions.

Definition 3.10. *Let (X, μ) be a measure space. A measurable function $f : X \to \mathbb{F}$ is called integrable if*

$$\int |f| < \infty$$

We state some trivial but usefull results for integrable functions

Lemma 3.4. *Let (X, μ) be a measure space and let $f, g : X \to \mathbb{F}$ be integrable and $a \in \mathbb{F}$. Then we have*

$$(i) : \int (f + g) = \int f + \int g$$
$$(ii) : \int af = a \int f$$
$$(iii) : |\int f| \leq \int |f|$$

For integrable functions it follows from the linearity of the integral that we can move sums in and out of the integral aswell, as long as the sum is integrable (i.e finite). Next we review some important results which gives us the tools we need to interchange limits and integration, the most important being the Dominated convergence theorem (DCT).

Theorem 3.8 (Monotone convergence theorem). *Let* (X, μ)*be a measure space and let* (f_n) *be a pointwise increasing sequence of measurable functions* $(f_n) : X \to [0, \infty)$. *Then*

$$\int \lim f_n = \lim \int f_n$$

Theorem 3.9 (Fatous' lemma). *Let* (X, μ) *be a measure space and let* (f_n) *be any sequence of measurable functions* $f_n : X \to [0, \infty)$. *Then we have*

$$\int (\liminf f_n) \leq \liminf \int f_n$$

Theorem 3.10 (Dominated convergence theorem). *Let* (X, μ) *be a measure space and let* (f_n) *be a sequence of integrable functions which converges pointwise to a function* f. *Suppose there is an integrable function* $g \geq 0$ *such that* $|f_n| \leq g$ *for all* n. *Then* f *is also integrable, and we have*

$$\int f_n \to \int f \tag{3.2}$$

The next theorems allows us to interchange the order of integration, under given conditions.

Theorem 3.11 (Tonelli's theorem). *Let* (X, μ) *and* (Y, ν) *be complete* σ*-finite measure spaces and let* $f : X \times Y \to [0, \infty)$. *Then we have that both* f_x *and* f_y *are measurable functions for almost every* x *and* y *and the functions*

$$x \rightarrowtail \int f_x d\nu \quad \textit{and}$$
$$y \rightarrowtail \int f_y d\mu$$

are also measurable, and

$$\int f d(\mu \times \nu) = \int \Big(\int f(x, y) d\nu \Big) d\mu = \int \Big(\int f(x, y) d\mu \Big) d\nu$$

Tonelli's theorem has restrictions on the range of f, but we have a more general result which allows the range to be $\mathbb{F}$:

Theorem 3.12 (Fubini's theorem). *Let (X, μ) and (Y, ν) be complete σ-finite measure spaces and let $f : X \times Y \to \mathbb{F}$. Then we have that both*

$$x \rightarrowtail \int f_x d\nu \quad \text{and}$$
$$y \rightarrowtail \int f_y d\mu$$

are integrable for almost every x and y. Moreover,

$$\int f d(\mu \times \nu) = \int \Big(\int f(x,y) d\nu \Big) d\mu = \int \Big(\int f(x,y) d\mu \Big) d\nu$$

Combining these two theorems gives us the Fubini-Tonelli theorem [25], commonly refered to as just Fubini's theorem

Theorem 3.13 (Fubini-Tonelli theorem). *Let (X, μ) and (Y, ν) be complete σ-finite measure spaces and f a measurable function. Then we have that*

$$\int_X \Big(\int_Y |f(x,y)| d\nu \Big) d\mu = \int_Y \Big(\int_X |f(x,y)| d\mu \Big) d\nu = \int_{X \times Y} |f(x,y)| d(\mu \times \nu)$$

Furthermore, if any of the three integrals

$$\int_X \Big(\int_Y |f(x,y)| d\nu \Big) d\mu$$
$$\int_Y \Big(\int_X |f(x,y)| d\mu \Big) d\nu$$
$$\int_{X \times Y} |f(x,y)| d(\mu \times \nu)$$

is finite, we have that

$$\int_X \Big(\int_Y f(x,y) d\nu \Big) d\mu = \int_Y \Big(\int_X f(x,y) d\mu \Big) d\nu = \int_{X \times Y} f(x,y) d(\mu \times \nu)$$

As we will see later on, we mostly study integrals with absolute value on the integrand. This makes the Fubini-Tonelli theorem essential. The next result gives us even more freedom to change the order of integration [26]

Lemma 3.5 (Minkowski's integral inequality). *Let (X,μ) and (Y,ν) be measure spaces and $f : X \times Y \to \mathbb{F}$ be measurable. For $1 \leq p \leq \infty$ we have*

$$\Big(\int_Y |\int_X f(x,y)d\mu|^p d\nu\Big)^{1/p} \leq \int_X \Big(\int_Y |f(x,y)|^p d\nu\Big)^{1/p} d\mu \tag{3.3}$$

We move on to the L^p spaces. We start with the special cases $p=1$ and $p=\infty$. For simplicity, we always let (X,μ) be a measure space and know that if we say $f \in L^1$ it is implied that $f \in L^1(X,\mu)$.

Definition 3.11. *L^1 is the set of integrable functions $f : X \to \mathbb{F}$, identifying functions which agree almost everywhere, equipped with the norm*

$$\|f\|_{L^1} = \int |f|$$

Definition 3.12. *L^∞ is the set of essentially bounded measurable functions $f : X \to \mathbb{F}$, identifying functions which agree almost everywhere, equipped with the essential supremum norm*

$$\|f\|_{L^\infty} = \sup\{|z| : z \in \quad \textit{ess ran}(f)\}$$

and the essential range of f is the range of f outside of sets of measure zero.

This definition of $\|\cdot\|_{L^\infty}$ is similar to the straight forward supremum, just with modifications that takes care of the sets of measure zero.

Definition 3.13. *Let $1 < p < \infty$. L^p is the set of measurable functions $f : X \to \mathbb{F}$ such that*

$$\int |f|^p < \infty$$

identifying functions which agree almost everywhere, equipped with the norm

$$\|f\|_{L^p} = \Big(\int |f|^p\Big)^{1/p}$$

The most important case of L^p spaces is when $p=2$ since L^2 is a Hilbert space. It can be shown that all L^p spaces are Banach spaces, though showing that they are complete can be a bit tedious. Showing that the triangle inequality holds is not trivial either. In fact, it is a consequense of Hölders inequality which we will use often.

Lemma 3.6 (Hölders inequality)**.** *Let $1 < p < \infty$ and $f, g : X \to \mathbb{F}$ be measurable. For $\frac{1}{p} + \frac{1}{q} = 1$ we have that*

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}$$

We see that for $p = q = 2$ this coincides with the Cauchy-Schwartz inequality. The triangle inequality for L^p spaces is named the Minkowski inequality.

Lemma 3.7 (Minkowskis inequality)**.** *Let $1 < p < \infty$ and $f, g : X \to \mathbb{F}$ be measurable. Then we have*

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

Our last result combined with the fact that all cauchy sequences in L^p converge will help us when expanding the Fourier transform.

Lemma 3.8. *Let (X, μ) be a measure space and $1 \leq p < \infty$. Then we have*

$i)$: *The simple functions in L^1 are dense in L^p*

$ii)$: *The compactly supported continuous functions are dense in L^p.*

3.0.3 Distributions

The theory and results of this section is gathered from Mitrea [13, Chapters 1-3] and Rudin [17, Chapter 6]. The calculated examples are motivated by their appearence in Chapter 7.

First, we denote the space of continuously differentiable functions

Definition 3.14. *The space of continuously differentiable functions is defined as*

$$C^k(\Omega) = \{f(x) \in \Omega : \quad f \text{ is } k \text{ times differentiable and } f^{(k)} \text{ is continuous}\}$$

The space of continuously differentiable functions vanishing outside of a compact interval is denoted C^k_0. $f \in C^k_\infty$ means that f vanishes at infinity.

A distribution is a much larger class of objects than that of functions. It is an extension which includes more exotic elements, like the Dirac-Delta distribution. Before we give the formal definition of a distribution, we review some new concepts and spaces. We first introduce the notion of a multi-index α, which is usefull when working with functions of n variables.

Definition 3.15 (Multi-index)**.** *A multi-index α is an ordered n-tuple*

$$\alpha =(\alpha_1, \alpha_2, \dots, \alpha_n)$$

With the properties

$$\begin{aligned}|\alpha| :=&\alpha_1 + \alpha_2 + \cdots + \alpha_n\\ \partial^\alpha :=&\partial_1^{\alpha_1} \cdot \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}, \quad \partial_j := \frac{\partial}{\partial x_j}, \quad j = 1, 2 \dots, n\\ \alpha! :=&\alpha_1! \cdot \alpha_2! \cdots \alpha_n!\end{aligned}$$

We let Ω be an open subset of $\mathbb{R}^n$ if not specified otherwise. The support of a function f is

$$\text{supp } f =\overline{\{x : f(x) \neq 0\}}$$

In other words: The support of f is a compact set K on which f is not zero, and f is zero outside of K. We let K be the standard notation for a compact set in $\mathbb{R}^n$.

Definition 3.16. *$\mathcal{D}(\Omega)$ is the vector space equipped with the topology κ*

The topology κ is defined as follows

Definition 3.17. *A sequence $(\phi_j)_{j\in\mathbb{N}} \subset C_0^\infty(\Omega)$ converges in $\mathcal{D}(\Omega)$ iff both the following conditions are satisfied:*
i): There exists a compact set $K \subset \Omega$ such that supp $\phi_j \subset K$ for all $j \in \mathbb{N}$ and supp$\phi \subset K$.
ii): For any $\alpha \in \mathbb{N}_0^n$ we have $\lim_{j\to\infty} \sup_{x\in K} |\partial^\alpha(\phi_j - \phi)(x)| \to 0$

The second condition will for convenience be denoted as $\phi_j \to \phi$ in $\mathcal{D}(\Omega)$. We are ready to formalize the definition of a distribution

Definition 3.18. *$u : \mathcal{D}(\Omega) \to \mathbf{C}$ is a distribution if u is linear and continuous*

In other words, a distribution is a complex-valued linear functional which is also continuous. The space of all distributions in Ω is denoted $\mathcal{D}'(\Omega)$. For a functional $u : \mathcal{D}(\Omega) \to \mathbb{C}$ working on a test-function $\phi \in C_0^\infty$ we use the notation $\langle u, \phi\rangle$. There are several equivalent definitions of continuity in the topology we stated:

Lemma 3.9. *Let $u : \mathcal{D}(\Omega) \to \mathbb{C}$ be a linear map. Then u is a distribution iff for each compact set $K \subset \Omega$ there exists $k \in \mathbb{N}_0$ and $C \in (0, \infty)$ such that*

$$|\langle u, \phi \rangle| \leq C \sup_{x \in K, |\alpha| \leq k} |\partial^\alpha \phi(x)|$$

for all $\phi \in C_0^\infty$ with compact support in K.

A distribution can be seen as a generalization of a function. So every operation we do on a distribution schould also hold for functions. Several of the operations on distributions are in fact generalized through studying first what the operation schould be defined as when working with locally integrable functions. We state the most important operations we need for our further study

Proposition 3.3 (Multiplication with a C^∞ function)**.** *If $u \in \mathcal{D}'(\Omega)$ and $a \in C^\infty(\Omega)$ then au is a distribution, and*

$$\langle au, \phi \rangle = \langle u, a\phi \rangle$$

Definition 3.19 (Differentiation)**.** *If $u \in \mathcal{D}'(\Omega)$ we have for each $\alpha \in \mathbb{N}_0$ defined the distributional derivative, or derivative in the distributional sense, of order α of u to be the mapping*

$$\langle \partial^\alpha u, \phi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \phi \rangle$$

for all $\phi \in C_0^\infty(\Omega)$

Differentiating a distribution also yields a distribution

Proposition 3.4. *For each $\alpha \in \mathbb{N}_0$ and each $u \in \mathcal{D}'$ we have $\partial^\alpha \in \mathcal{D}'$*

It can also be shown that the multiplication rule for derivatives in the classical sense also holds for the distributional derivative. In chapter 7 there is a need for two different derivatives in the distributional sense, which is what motivates these two examples. First, note the Heaviside function $H(x)$ and the sign function $\mathrm{sgn}(x)$.

$$H(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

$$\mathrm{sgn}(x) = \begin{cases} 1 \quad \text{if} & x > 0 \\ 0 \quad \text{if} & x = 0 \\ -1 \quad \text{if} & x < 0 \end{cases} \tag{3.5}$$

Example 3.1. *For every $c \in \mathbb{R}$ we have*

$$(e^{-c|x|})' = -sgn(x)ce^{-c|x|} \quad in \quad \mathcal{D}'(\mathbb{R}) \tag{3.6}$$

Proof. Writing out the definition of the distributional derivative, we have that for any $\phi \in C_0^\infty$ with compact support

$$\begin{aligned}
\langle (e^{-c|x|})', \phi \rangle &= -\langle e^{-c|x|}, \phi' \rangle \\
&= -\int_{-\infty}^{\infty} e^{-c|x|}\phi'(x)dx \\
&= -\int_{0}^{\infty} e^{-cx}\phi'(x)dx - \int_{-\infty}^{0} e^{cx}\phi'(x)dx \\
&= \phi(0) - \int_{0}^{\infty} ce^{-cx}\phi(x)dx - \phi(0) + \int_{-\infty}^{0} ce^{cx}\phi(x)dx \\
&= -\int_{-\infty}^{\infty} H(x)ce^{-c|x|}\phi(x)dx + \int_{-\infty}^{\infty} H(-x)ce^{-c|x|}\phi(x)dx \\
&= \int_{-\infty}^{\infty} ce^{-c|x|}\phi(x)(H(-x) - H(x))dx \\
&= \int_{-\infty}^{\infty} -\text{sgn}(x)ce^{-c|x|}\phi(x) \\
&= \langle -\text{sgn}(x)ce^{-c|x|}, \phi(x) \rangle
\end{aligned}$$

And thus

$$(e^{-c|x|})' = -\text{sgn}(x)ce^{-c|x|} \tag{3.7}$$

In the distributional sense.

□

Another example which will also be needed for calculations in chapter 7 is

Example 3.2.

$$(|t|^n)' = sgn(t)nt^{n-1} \quad in \quad \mathcal{D}'(\mathbb{R}) \tag{3.8}$$

Proof. Writing out the definition yields

$$\begin{aligned}
(|x|^n)' &= -\int_{-\infty}^{\infty} |t|^n \phi'(t)dt \\
&= -\int_{0}^{\infty} t^n \phi'(t)dt - \int_{-\infty}^{0} (-t)^n \phi'(t)dt \\
&= \int_{0}^{\infty} nt^{n-1}\phi(t)dt + \int_{-\infty}^{0} (-n)(-t)^{n-1}\phi(t)dt \\
&= \int_{0}^{\infty} nt^{n-1}\phi(t)dt + \int_{-\infty}^{0} (-n)(-t)^{n-1}\phi(t)dt \\
&= \int_{-\infty}^{\infty} n|t|^{n-1}\phi(t)H(t)dt - \int_{-\infty}^{\infty} n|t|^{n-1}\phi(t)H(-t)dt \\
&= \int_{-\infty}^{\infty} n|t|^{n-1}\mathrm{sgn}(t)\phi(t)dt
\end{aligned}$$

And thus,

$$(|t|^n)' = \mathrm{sgn}(t)nt^{n-1} \tag{3.9}$$

In the distributional sense.

□

There are some key concepts in distribution theory not yet touched. Both convolutions of distributions and taking the Fourier transform of distributions will be dealt with in the sections to come. Before we move on to convolutions, we briefly note the definition of the support of a distribution.

Definition 3.20. *The support of a distribution $u \in \mathcal{D}'(\Omega)$ is defined as supp $u := \{x \in \Omega :$ there is no ω open such that $\ x \in \omega \subseteq \Omega \ $ and $\ u|_\omega = 0\}$*

An important class of distributions when we consider convolutions is the set of compactly supported distributions

Definition 3.21. *The set of compactly supported distributions is the set*

$$\mathcal{D}'_c(\Omega) := \{u \in \mathcal{D}'(\Omega) : \textit{supp } u \textit{ is a compact subset of } \Omega\}$$

3.0.4 Convolutions

The theory from this section follows the works of Gasquet & Witomski [9, Lessons 20,21,32], Mitrea [13, Chapter 2.8] and Rudin [17, Chapter 6].

The domain we are working on is always $\mathbb{R}^n$ for convolutions. The convolution of two functions f and g is a function defined as

$$(f * g)(x) = \int f(x-t)g(t)dt = \int f(y)g(h-y)dy$$

We obviously need some assumptions on f and g for this integral to even exist. We start first with functions in L^1

Proposition 3.5. *If $f, g \in L^1$ we have that*

$$f * g \in L^1$$

and the convolution is defined almost everywhere. The convolution is also a continuous bilinear operator from $L^1 \times L^1 \to L^1$ satisfying

$$\|f * g\|_{L^1} \leq \|f\|_{L^1}\|g\|_{L^1}$$

Next, we look at what assumptions we need for the existence of convolutions of L^p functions.

Proposition 3.6. *If $f \in L^p$ and $g \in L^q$ where $\frac{1}{p} + \frac{1}{q} = 1$ we have that*

$$\|f * g\|_{L^\infty} \leq \|f\|_{L^p}\|g\|_{L^q}$$

and the convolution is defined everywhere and is both continuous and bounded on $\mathbb{R}$

We have an important inequality that shows under what conditions the convolution in L^r exists

Proposition 3.7 (Young's inequality)**.** *For $f \in L^p$ and $g \in L^q$ we have the inequality*

$$\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}$$

whenever $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$

Setting $p = 1, q = 2, r = 2$ we see that for $f \in L^1$ and $g \in L^2$ we have that

$$\|f * g\|_{L^2} \leq \|f\|_{L^1}\|g\|_{L^2}$$

which is a special case appearing frequently. We are also interested in where the convolutions of functions in L^1 and L^2 lie

Proposition 3.8. *i) If $f, g \in L^2$ we have*

$$(f * g)(x) \subset L^\infty \cap C^0$$

ii) If $f \in L^2$ and $g \in L^1$ we have

$$(f * g)(x) \subset L^2$$

iii) If $f \in L^1$ and $g \in L^\infty$ we have

$$(f * g)(x) \subset L^\infty \cap C^0$$

Until now we have only looked at convolutions of functions in L^p. We can also have convolutions of functions both in L^p and C^n.

Proposition 3.9. *Let $f \in L^1$ and $g \in C^n(\mathbb{R})$. If $g^{(k)}$ is bounded for all $k = 0, 1, \ldots, n$ we have*

$$\begin{aligned} i) :& f * g \in C^n(\mathbb{R}) \\ ii) :& (f * g)^{(k)} = f * g^{(k)}, k = 1, 2, \ldots, n \end{aligned}$$

The convolution of distributions is highly related to where the distribution is supported. We first consider the convolution of a test function with a distribution

Definition 3.22. *Let $u \in \mathcal{D}'$ and $\psi \in \mathcal{D}$. Then*

$$\langle u * \psi(x), \phi(x) \rangle = \langle u, \psi(-x) * \phi(x) \rangle$$

for all $\phi \in \mathcal{D}$

Taking the convolution with test functions we the following properties

Proposition 3.10. *If $u \in \mathcal{D}'$ and $\psi, \phi \in \mathcal{D}$ we have*

$$\begin{aligned} i) : &\quad u * \psi \in C^\infty \\ ii) : &\quad \partial^\alpha (u * \psi) = \partial^\alpha u * \psi = u * \partial^\alpha \psi \\ iii) : &\quad u * (\psi * \phi) = (u * \psi) * \phi \end{aligned}$$

for every multi-index α

For a distribution $u \in \mathcal{D}'_c$ we have the following proposition

Proposition 3.11. *Let u be a distribution with compact support and let $\phi \in C^\infty$ and $\psi \in \mathcal{D}$. Then we have*

$$\begin{aligned} i): &\quad u * \phi \in C^\infty \\ ii): &\quad \partial^\alpha(u * \phi) = \partial^\alpha u * \phi = u * \partial^\alpha \phi \\ iii): &\quad u * \psi \in \mathcal{D} \\ iv): &\quad u * (\psi * \phi) = (u * \psi) * \phi \end{aligned}$$

It is also possible to take the convolution of two distributions $u, v \in \mathcal{D}'$, provided at least one of them has compact support.

Proposition 3.12. *Let $u, v, w \in \mathcal{D}'$. If atleast one of them has compact support, we have that*

$$u * v = v * u$$

If atleast two of them has compact support, we have that

$$u * (v * w) = (u * v) * w$$

Next, we have the support of the convolution of two distributions to be

Proposition 3.13. *Let $u, v \in \mathcal{D}$ and let at least one of them have compact support. Then*
supp $(u * v) \subset$ supp u +supp v

Lastly, we have the following important result

Proposition 3.14. *Let $u, v \in \mathcal{D}'$. If one of them has compact support, $u * v$ is a distribution. Moreover, for $u \in \mathcal{D}'$ and $v \in \mathcal{D}$, $u * v$ is a tempered distribution.*

Tempered distributions are yet to be defined, but we look into the subject in the next chapter.

4 Fourier Transform

Theory and results on the Fourier transform is gathered from Gasquet & Witomski [9, Lessons 17-23], Gonzàles-Velasco [10, Chapter 7] and Stein & Shakarchi[18, Chapter 5]. For the preface of this chapter the information is gathered from [8] and Gonzàles-Velasco [10, Chapter 1]. We start the chapter off with a short introduction before we move on to defining the Fourier transform for functions in L^1 and properties of the transform. Then we move on to defining the transform for functions in the Schwartz space, for functions in L^2 and lastly for tempered distributions. Results will only be in $\mathbb{R}$, but expanding to $\mathbb{R}^n$ is not difficult.

Integral transformations were invented by Leonhard Euler (1707-1783) and are quite usefull when solving different kinds of differential equations. The french mathmatician Jean Baptise Joseph Fourier (1768-1830) formally introduced the idea of Fourier series and Fourier transforms back in 1807 and 1811 when he was working on the propagation of heat. These works were later on expanded and gathered and then published in his famous book, Théorie analytique de la chaleur - or as we would call it, The analytical theory of heat, in 1822. The Fourier transform is to this day extremely usefull, e.g when working with signal processing, image processing, and of course for solving partial differential equations.

4.1 The Fourier transform in L^1, $\mathcal{S}$ and L^2.

We start by defining the Fourier transform for functions in $L^1(\mathbb{R})$. For simplicity we will only write L^1 and know that it implicitly means $L^1(\mathbb{R})$ unless stated otherwise.

Definition 4.1. *For $f \in L^1$ we denote*

$$\mathcal{F}[f] = \widehat{f}(\xi) = \int f(x)e^{-i2\pi x\xi}dx \tag{4.1}$$

to be the Fourier transformation of $f(x)$.

We see that this definition makes sense only when $f \in L^1$ since

$$|\widehat{f}(\xi)| \leq \int |f(x)||e^{-i2\pi x\xi}|dx = \int |f(x)|dx < \infty$$

The following theorem tells us about the behaviour of $\widehat{f}$ and the operator $\mathcal{F}[\cdot]$.

Theorem 4.1 (Riemann-Lebesgue)**.** *If $f \in L^1$ then $\widehat{f}$ satisfies*

$$i): \quad \mathcal{F}[f] \quad \textit{is continuous and bounded on} \quad \mathbb{R}$$

$$ii): \quad \mathcal{F} \quad \textit{is a continuous linear operator from } L^1 \textit{ to } L^\infty \textit{ and} \quad \|\widehat{f}\|_{L^\infty} \leq \|f\|_{L^1}$$

$$iii): \quad \lim_{|\xi|\to+\infty} |\widehat{f}(\xi)| = 0$$

So the operator $\mathcal{F}$ has nice and continuous properties, and $\widehat{f}(\xi)$ vanishes at infinity which makes it more pleasant to work with. What makes the Fourier transform so applicable in the setting of partial differential equations is that when taking the transform of a differentiated function it gives us the transform of the function it self, multiplied with a monomial. The same goes for translated functions and even the convolution of functions; taking the transform will interchange different operations and yield more pleasant expressions to work with. The next proposition shows that we can even switch the transform on the product of two functions under integration.

Proposition 4.1. *If $f, g \in L^1$ then $\widehat{f}g \in L^1$, $f\widehat{g} \in L^1$ and*

$$\int f(t)\widehat{g}(t)dt = \int \widehat{f}(x)g(x)dx$$

We state some usefull properties of the Fourier transform

Proposition 4.2 (Differentiation)**.** *1: If $x^k f(x) \in L^1$ for $k = 0, 1, \ldots, n$ then $\widehat{f}$ is n times differentiable and*

$$\mathcal{F}[(-i2\pi x)^k f(x)] = \widehat{f}^{(k)}(\xi)$$

2: If $f \in L^1$ and f is n times differentiable and all the derivatives $f^{(k)} \in L^1$ for $k = 0, 1, \ldots, n$ then

$$\mathcal{F}[f^{(k)}(x)] = (i2\pi\xi)^k \widehat{f}(\xi)$$

As we can see, the Fourier transform simply interchanges differentiation with multiplication by ξ and some multiple of $i2\pi$.

Proposition 4.3 (Translation). *If $f \in L^1$ we have for any $a \in \mathbb{R}$ that*

$$i): \mathcal{F}[f(x+a)] = \widehat{f}(\xi)e^{i2\pi a\xi} \tag{4.2}$$

$$ii): \mathcal{F}[f(x)e^{-i2\pi ax}] = \widehat{f}(\xi + a) \tag{4.3}$$

Proposition 4.4 (Dilation). *If $f \in L^1$ and $a \neq 0$ we have that*

$$\mathcal{F}[f(ax)] = \frac{1}{|a|}\widehat{f}(\frac{\xi}{a}) \tag{4.4}$$

Now we are equipped with a lot of usefull tools we can use to compute Fourier transformations when calculating it directly from the definition does not seem to lead us anywhere. But what we have done thus far is without use if we cannot go back to the domain we came from, at least in the sense of solving differential equations. Luckily, the inverse Fourier transform comes from the transform it self, which is simply $\mathcal{F}^{-1}[\widehat{f}(\xi)] = \int \widehat{f}(\xi)e^{i2\pi x\xi}d\xi$. However, $f \in L^1$ does not imply $\widehat{f} \in L^1$, so we need some additional assumptions on f for the inverse Fourier transform to exist.

Theorem 4.2 (Fourier inverse transform). *If f and $\widehat{f}$ both are in L^1, then*

$$\mathcal{F}^{-1}[\widehat{f}(\tau)](t) = f(t) \tag{4.5}$$

at all points where f is continuous.

Next, we have a sufficient condition for $\widehat{f}$ to be in L^1

Proposition 4.5. *If $f \in C^2$ and if f, f', f'' are all in L^1 then $\widehat{f} \in L^1$ aswell.*

Lastly, we have a result that helps us greatly when computing the inverse transform

Proposition 4.6. *If f is continuous and integrable and if $\widehat{f} \in L^1$ then for all $x \in \mathbb{R}$ we have that*

$$\mathcal{F}[\widehat{f}(\xi)](x) = f(-x) \tag{4.6}$$

Now we have a grasp of what the Fourier transform is all about, and we move on to the Fourier transformation of functions in the Schwartz space $\mathcal{S}$. The Schwartz space is the space of rapidly decreasing functions.

Definition 4.2 (Rapid decay)**.** *A function* $f : \mathbb{R} \to C$ *is said to decay rapidly if, for all* $p \in \mathbb{N}$*,*

$$\lim_{|x| \to \infty} |x^p f(x)| = 0$$

We denote the Schwartz space formally by

Definition 4.3 (Schwartz space, $\mathcal{S}$)**.** $\mathcal{S}$ *denotes the vector space of functions* $f \to \mathbb{C}$ *that satisfy the following properties:*
i): f *is infinitely differentiable*
ii): f *and all of its derivatives decay rapidly.*

This space has several pleasant properties which we will state in a theorem

Theorem 4.3. *The Schwartz space* $\mathcal{S}$ *satisfy the following properties:*
i) $\mathcal{S}$ *is invariant under multiplication by a polynomial*
ii) $\mathcal{S}$ *is invariant under differentiation.*
iii) $\mathcal{S}$ *is invariant under the Fourier transform*
iv) $\mathcal{S}$ *is invariant under convolutions*
v) $\mathcal{S} \subset L^1$

I.e, if $f \in \mathcal{S}$ then for any polynomial $P(x)$ we have that $P(x)f(x) \in \mathcal{S}$. Likewise, $f \in \mathcal{S} \implies f^{(k)} \in \mathcal{S}$, the convolution of two Schwartz function is again a Schwartz function, and most importantly $f \in \mathcal{S} \implies \widehat{f} \in \mathcal{S}$. Since $\mathcal{S} \subset L^1$, all the properties we showed for the Fourier transform in L^1 also works in $\mathcal{S}$, due to the invariance properties of $\mathcal{S}$. The last result of this section will be the famous Plancherel theorem which states that the Fourier transform is an isometry in the L^2 norm

Theorem 4.4 (Plancherel's theorem)**.** *If* $f \in \mathcal{S}$ *we have that*

$$\|\widehat{f}\|_{L^2} = \|f\|_{L^2}$$

Since L^2 is complete, we know that all cauchy sequenses converge. It can be shown that the Schwartz space is in fact a dense linear subspace of L^2, which tells us that every function in L^2 can be approximated by a Schwartz function. From the Plancherel theorem, we know that the Fourier transform is an isometry on $\mathcal{S}$ in the L^2 norm. Now, by setting $\mathcal{S}$ to be E_0 and E and F to be L^2 in Proposition 3.2, we get the following result

Theorem 4.5. *The Fourier transform and its inverse on the Schwartz space* $\mathcal{S}$ *extend uniquely to isometries on* L^2*. Thus, we have the following results for all* $f, g \in L^2$*:*

1: $\mathcal{F}[\mathcal{F}^{-1}[f]] = \mathcal{F}^{-1}[\mathcal{F}[f]] = f$ *almost everywhere.*

2: $\|f\|_{L^2} = \|\widehat{f}\|_{L^2}$

We now move on to the Fourier transform of functions in L^2. The Fourier transform is extended to L^2 by taking limits:

Definition 4.4. *If $f \in L^2$, the Fourier transform is defined as*

$$\widehat{f}(\xi) = \lim_{N\to\infty} \int_{-N}^{N} f(x)e^{-i2\pi x\xi}dx \tag{4.7}$$

where the limit is taken in L^2, i.e

$$\lim_{N\to\infty} \|\widehat{f} - \int_{-N}^{N} f(x)e^{-i2\pi x\xi}dx\|_{L^2} = 0$$

Likewise, the inverse Fourier transform is defined as

Definition 4.5. *If $\widehat{f}(\xi) \in L^2$, the inverse Fourier transform is defined as*

$$f(x) = \lim_{N\to\infty} \int_{-N}^{N} \widehat{f}(\xi)e^{i2\pi x\xi}d\xi \tag{4.8}$$

where the limit is taken in L^2, i.e

$$\lim_{N\to\infty} \|f(x) - \int_{-N}^{N} \widehat{f}(\xi)e^{i2\pi x\xi}d\xi\|_{L^2} = 0$$

For functions in $L^1 \cap L^2$, the definitions of the Fourier transform we have defined in L^1 and L^2 respectively, coincide. It can be shown through taking limits that the properties of the Fourier transform we have from the Schwartz space also hold in L^2. Having the Plancherel theorem also count for $f \in L^2$, we immediately see that this implies $\widehat{f}$ is also in L^2. We will no longer use the different definitions for the transform in L^1 and L^2, but know that it is indeed a limit if $f \in L^2$. We also see that $\widehat{f}$ is no longer defined pointwise if $f \in L^2$. We have yet to see what the Fourier transform does on convolutions or products. This is because we have seen that it is not trivial to see in which space the convolution of two functions from two different spaces is. Since we now also have the transform for functions in L^2 and its relation with the transform in $\mathcal{S}$ and L^1 we can take a closer look. We will see that taking the transform of a product interchanges multiplication with convolution, and likewise it interchanges (under given conditions) convolution with multiplication.

Proposition 4.7. *For $f, g \in L^1$ we have that*

$$\widehat{f * g}(\xi) = \widehat{f}(\xi) \cdot \widehat{g}(\xi)$$

If also $\widehat{f}, \widehat{g} \in L^1$ we have that

$$\widehat{f \cdot g}(\xi) = \widehat{f} * \widehat{g}(\xi)$$

Proposition 4.8. *For $f, g \in L^2$ we have that*

$$f * g(x) = \overline{\mathcal{F}}[\widehat{f} \cdot \widehat{g}](x)$$

and

$$\widehat{f \cdot g}(\xi) = \widehat{f} * \widehat{g}(\xi)$$

Since $f * g \in L^\infty \cap C^0$, we cannot readily define its Fourier transform just yet. However, it can be shown that this is indeed a tempered distribution so there is still hope for this convolution. We will see this in the next section

4.2 The Fourier transform on tempered distributions

We will also define the Fourier transform of distributions, but first we need some concepts not introduced yet.

Definition 4.6 (Space of tempered distributions)**.** *The space of tempered distributions is the dual space of the Schwartz space:*
$\{u : \mathcal{S} \to \mathbb{C} : \quad u \text{ is linear and continuous}\}$

If a functional belongs to the dual of the Schwartz space we call it a tempered distribution. We state a result regarding when the linear functional from the Schwartz space is continuous

Proposition 4.9. *If the functional $u : \mathcal{S} \to \mathbb{C}$ is linear, it is also continuous iff* $\exists m, k \in \mathbb{N}_0$ *and $C > 0$ s.t*

$$|u(\phi)| \leq C \sup_{|\alpha| \leq m, |\beta| \leq n} \sup_{x \in \mathbb{R}} |x^\beta \partial^\alpha \phi(x)|$$

for multi-indices α, β, and $\forall \phi \in \mathcal{S}$

What kind of distributions are tempered, one might ask. There are several propositions classifying these.

Proposition 4.10. *If $u \in \mathcal{S}'$ we have*

1: *For each $n \in \mathbb{N}$, $x^n u \in \mathcal{S}'$*

2: *For each $n \in \mathbb{N}$, the derivative $u^{(n)} \in \mathcal{S}'$*

3: *The mappings $u \to x^n u$ and $u \to u^{(n)}$ are continuous from $\mathcal{S}'$ to $\mathcal{S}'$*

It can be shown that indeed all functions in L^p are tempered distributions:

Proposition 4.11. *If $f \in L^p$ for $p \geq 1$, then $f \in \mathcal{S}'$.*

We are ready to define the Fourier transform on a tempered distribution.

Proposition 4.12. *If $u \in \mathcal{S}'$, the mapping*

$$\widehat{u} : \mathcal{S} \to \mathbb{C}, \quad \widehat{u}(\phi) := \langle u, \widehat{\phi} \rangle \tag{4.9}$$

is well defined, linear and continuous for all $\phi \in \mathcal{S}$. Thus, $\widehat{u} \in \mathcal{S}'$.

This extends the Fourier transform from L^1 or L^2 to tempered distributions. Since all functions in L^p spaces are tempered distributions, we have the Fourier transform defined for all $f \in L^p$. The question that arises next is, what properties does the transform on tempered distributions satisfy? Luckily, the most important ones from the preceding sections are still applicable, with some modifications.

Proposition 4.13. *If $u \in \mathcal{S}'$, we have for all multi-indices $\alpha \in \mathbb{N}_0$ that*

1: $\widehat{\partial^\alpha u} = \xi^\alpha u$

2: $\widehat{x^\alpha u} = (-1)^\alpha \partial^\alpha u$

Theorem 4.6. *The Fourier transform is a linear, 1-to-1, bicontinuous mapping from $\mathcal{S}'$ to $\mathcal{S}'$. The inverse mapping, $\mathcal{F}^{-1} = \overline{\mathcal{F}}$, is defined $\forall \phi \in \mathcal{S}$ by*

$$\langle \mathcal{F}^{-1}[u], \phi \rangle = \langle u, \mathcal{F}^{-1}[\phi] \rangle \tag{4.10}$$

And, for all $u \in \mathcal{S}'$ we have

$$\mathcal{F}[\mathcal{F}^{-1}[u]] = \mathcal{F}^{-1}[\mathcal{F}[u]] = u \tag{4.11}$$

We established the basics of convolutions of both test functions and distributions and several combinations of the two in the last chapter. We conclude this section with the study of the Fourier transform on these convolutions. We want to find out under what conditions the essential property $\widehat{u * v} = \widehat{u} \cdot \widehat{v}$ is satisfied

We begin with convolutions of functions in $\mathcal{S}$

Proposition 4.14. *If $\phi \in \mathcal{S}$ and $u \in \mathcal{S}'$ then*

$$\widehat{u * \phi} = \widehat{u} \cdot \widehat{\phi}$$

and

$$\widehat{u \cdot \phi} = \widehat{u} * \widehat{\phi}$$

For a compactly supported distribution we have the following result

Proposition 4.15. *If $u \in \mathcal{D}'_c$ and $v \in \mathcal{S}'$ we have*

$$\widehat{u * v} = \widehat{u} \cdot \widehat{v}$$

As noted in the convolution of L^2 functions, we can now restate Proposition 4.8. We know that for $f, g \in L^2$, $f * g \in L^\infty \cap C^0$. In the theory we established for the Fourier transform in the preceding section we know that it is not possible to take the transform of such a function, but now we also know that $f * g \in \mathcal{S}'$. Thus, Proposition 4.8 can be restated by simply taking the Fourier transform [9, p. 313]

Proposition 4.16. *For $f, g \in L^2$, we have*

$$\widehat{f * g} = \widehat{f} \cdot \widehat{g}$$

and

$$\widehat{fg} = \widehat{f} * \widehat{g}$$

5 Sobolev spaces

In this section we follow the works of Linares & Ponce [12, Chapter 3], Brezis [5, Chapters 8,9], Bahouri & Chemin & Danchin[1, Chapter 1] and Tao [21]

5.1 Motivation

The given norm of a function can be interpreted as a way to quantify the properties of the function. For example, the height and width of a function can be quantified through the $L^p(X, \mu)$ norm [21]. Taking the example from Tao [21] we see that when considering a step function $f = A1_E$ where $A \in \mathbb{R}$ and E is some bounded interval,

$$\|f\|_{L^p} = |A|\mu(E)^{1/p}$$

we get a combination of the height and width. There are other function spaces which can also take account for the smoothness of the function, or in other words, how many times we can differentiate it and it still be a continuous function. Sobolev spaces combine all of these properties, which makes it suitable as a tool for partial differential equations.
Sobolev spaces are named after russian mathematician Sergei Sobolev (1908-1989) [27] [28] and are critical for the study of partial differential equations. Sobolev spaces combine weak derivatives of functions along with their integrability in L^p spaces. Looking back at the distributional derivative, the weak derivative is quite similar. We can find weak derivatives of functions not being differentiable in the classical sense, as long as they are integrable.

Definition 5.1. *Let I be a compact interval and $u \in L^1(I)$. We call $v \in L^1(I)$ the weak derivative of u, if*

$$\int_I u(x)\phi'(x)dx = -\int_I v(x)\phi(x)dx$$

for all $\phi \in C_I^\infty$

What differs from the distributional derivative is that here we want the v to be a locally integrable function. When taking the distributional derivative, we might end up with a distribution such as the dirac delta which is not integrable[20]. To quickly see what motivates the need for Sobolev spaces, we take a brief look at the classical example from Brezis [5, p. 201]: Given $f \in C([a,b])$, find a function u such that

$$-u'' + u = f, \quad u(a) = u(b) = 0$$

is satisfied. We see that a classical (or strong) solution has to be a C^2 function on $[a,b]$. If we multiply the equation with $\phi \in C^1([a,b])$ and integrate by parts we get

$$\int_a^b u'\phi' dx + \int_a^b u\phi = \int_a^b f\phi$$

for all $\phi \in C^1([a,b])$ and $\phi(a) = \phi(b) = 0$. This equation now makes sense for all $u \in C^1([a,b])$, and $u, u' \in L^1([a,b])$, where u' is the weak derivative. The solution of this equation denotes the weak solution of the original equation. It can be shown that the weak solution is equal to the classical solution. Thus, we conclude that working in a function space that quantifies the weak derivatives is a clever place to start looking for solutions.
We define the classical Sobolev spaces [21]

Definition 5.2. *Let $1 \leq p \leq \infty$ and $k \geq 0$ a natural number. A function f is in the classical Sobolev space $W^{k,p}(\mathbb{R}^d)$ if its weak derivatives $\frac{\partial}{\partial_{x_j}} f$ exists and lie in $L^p(\mathbb{R}^d)$ for all $j = 0, 1, \ldots, k$. If f is in $W^{k,p}(\mathbb{R}^d)$, the norm is defined by*

$$\|f\|_{W^{k,p}(\mathbb{R}^d)} := \sum_{j=0}^{k} \|\frac{\partial f}{\partial_{x_j}}\|_{L^p(\mathbb{R}^d)} \tag{5.1}$$

This Sobolev space has a lot of nice properties and embeddings, but we are limited to k being an integer. The space we are interested in for our use is the L^2 based Sobolev space $W^{k,2}(\mathbb{R}^d)$, where we can use Plancherels theorem and thus work with the Fourier transformated function instead. Following the introduction made by Tao [21] and limiting ourselves to the case of $\mathbb{R}$, we can get an equivalent formulation of $W^{k,2}(\mathbb{R})$ in terms of the Fourier transform. Using Plancherel and that the Fourier transform interchanges differentiation with mulitplication by ξ and a factor of π we get

$$\int |f'(x)|^2 dx = \int (2\pi\xi)^2 |\widehat{f}(\xi)|^2 d\xi$$

for all $f \in \mathcal{S}$. Summing all derivates up to k we get

$$\begin{aligned} \int \sum_{j=0}^{k} |f^{(j)}|^2 dx &= \int \sum_{j=0}^{k} (2\pi\xi)^{2j} |\widehat{f}(\xi)|^2 d\xi \\ \|f\|_{W^{k,2}(\mathbb{R})} &= \int \sum_{j=0}^{k} (2\pi\xi)^{2j} |\widehat{f}(\xi)|^2 d\xi \end{aligned}$$

For all integers $k \geq 0$. And even though we assumed only that $f \in \mathcal{S}$, it can be shown that this holds for all $f \in W^{k,2}(\mathbb{R})$ [21, Tao]. Note that the sum in the above formula can be compared to

$$\begin{aligned} \sum_{j=0}^{k} (2\pi\xi)^{2j} &= (1 + (2\pi\xi)^2 + \ldots + (2\pi\xi)^{2k}) \\ &\leq C(1 + \xi^{2k}) \leq C(1 + |\xi|^2)^k = \langle \xi \rangle^{2k} \end{aligned}$$

Thus,

$$\sum_{j=0}^{k} (2\pi\xi)^{2j} \sim \langle \xi \rangle^{2k}$$

And we can compare the $W^{k,2}$ norm to

$$\|f\|_{W^{k,2}(\mathbb{R})} \sim \|\langle \xi \rangle^k \widehat{f}(\xi)\|_{L^2}$$

Now, for all $s \in \mathbb{R}$, we denote the space H^s to be the space of all tempered distributions $\langle \xi \rangle^s \widehat{f}(\xi)$ to be in L^2 with norm

$$\|f\|_{H^s} := \|\langle \xi \rangle^s \widehat{f}(\xi)\|_{L^2}$$

It can be shown that the spaces H^s and $W^{k,2}$ are in fact equal [21, Tao].

5.2 The space H^s

The space H^s will be our main tool in chapter 6. This space has several properties which makes it easy to work with. We state a formal definition

Definition 5.3. *For all $s \in \mathbb{R}$ we define the Sobolev space H^s of order s as*

$$H^s(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \langle\xi\rangle^k \widehat{f}(\xi) \in L^2(\mathbb{R}^n)\} \tag{5.2}$$

equipped with the norm

$$\|f\|_{H^s} = \|\langle\xi\rangle^s \widehat{f}(\xi)\|_{L^2} \tag{5.3}$$

There is an equivalent definition when s is a positive integer which will be usefull, which is basically what we stated in the preceding subsection but now with distributional derivatives. We formalize it as follows

Proposition 5.1. *If k is a positive integer, then $H^k(\mathbb{R}^n)$ coincides with the space of functions $f \in L^2(\mathbb{R}^n)$ whose derivatives in the distributional sense $\partial_x^\alpha f$ also lie in $L^2(\mathbb{R}^n)$ for every multi-index $|\alpha| \leq k$. If this is the case, the norm is*

$$\|f\|_{H^k} = \sum_{|\alpha|\leq k} \|\partial_x^\alpha f\|_{L^2} \tag{5.4}$$

From the definition of the space H^s we can derive several properties.

Lemma 5.1. *If $s > s'$, then $H^s(\mathbb{R}^n) \subset H^{s'}(\mathbb{R}^n)$*

Thus, lowering the value of s gives us a larger class of functions.

Lemma 5.2. *$H^s(\mathbb{R}^n)$ is a Hilbert space with respect to the inner product $\langle\cdot,\cdot\rangle_s$ defined as*

$$\langle f, g\rangle_s = \int_{(\mathbb{R}^n)} \langle\xi\rangle^s f(\xi)\overline{g(\xi)} d\xi \tag{5.5}$$

for $f, g \in H^s(\mathbb{R}^n)$.

We also have the following density result

Lemma 5.3. *The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$*

When we have comparable orders of s we can also have an estimate between them under given conditions.

Lemma 5.4. *If $s_1 \leq s \leq s_2$ with $s = as_1 + (1-a)s_2, 0 \leq a \leq 1$ then*

$$\|f\|_{H^s(\mathbb{R}^n)} \leq \|f\|^a_{H^{s_1}(\mathbb{R}^n)} \|f\|^{1-a}_{H^{s_2}(\mathbb{R}^n)} \tag{5.6}$$

The results we will state next lets us relate the derivative in the distributional sense to the classical derivatives. These are called embedding theorems.

Theorem 5.1. *If $s > n/2 + k$, then $H^s(\mathbb{R}^n)$ is continuously embedded in $C^k_\infty(\mathbb{R}^n)$. Thus, with a possible modification in sets of measure zero,*

$$\|f\|_{C^k} \lesssim \|f\|_{H^s} \tag{5.7}$$

In chapter 6 we only work in $\mathbb{R}$, so in our case the theorem states that if $f \in H^s$ when $s > 1/2 + k$ then we also have that $f \in C^k_\infty$

Theorem 5.2. *If $s \in (0, n/2)$ then $H^s(\mathbb{R}^n)$ is continuously embedded in $L^p(\mathbb{R}^n)$ with $p = \frac{2n}{n-2s}$ or equivalently with $s = n(1/2 - 1/p)$. So*

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{H^s(\mathbb{R}^n)} \tag{5.8}$$

When $s > n/2$ the Sobolev space is an algebra

Theorem 5.3. *For $s > n/2$, $H^s(\mathbb{R}^n)$ is an algebra with respect to the product of functions. That is, if $f, g \in H^s(\mathbb{R}^n)$ then $fg \in H^s(\mathbb{R}^n)$ and*

$$\|fg\|_{H^s(\mathbb{R}^n)} \lesssim \|f\|_{H^s(\mathbb{R}^n)} \|g\|_{H^s(\mathbb{R}^n)} \tag{5.9}$$

Our last result is regarding the duality between H^s and H^{-s}

Theorem 5.4. *For any $s \in \mathbb{R}$, the bilinear functional $\mathcal{B} : \{\mathcal{S} \times \mathcal{S} \to \mathbb{C} : (\phi, \psi) \rightarrowtail \int_{\mathbb{R}^d} \phi(x)\psi(x)dx\}$ can be extended to a continuous bilinear functional on $H^{-s} \times H^s$. Moreover, if L is a continuous linear functional on H^s, a unique tempered distribution u exists in H^{-s} such that*

$$\forall \phi \in \mathcal{S}, \langle L, \phi \rangle = \mathcal{B}(u, \phi)$$

6 Solutions in H^s

We have used Linares & Ponce [12, Chapter 5] for inspiration in this chapter, though most calculations done there are not simliar to those made here. This chapter will be divided in three parts. To make it easier to work with, we start by turning (2.3) into its integral formulation. Once we have a managable equation, we continue with estimating its Sobolev norm. The goal of these estimates is to get the norm bounded by some function of t which tends to 0 as $t \to 0$, multiplied with some constant $C > 0$. When we have the estimate we look to find a suitable function space in which we can use the contraction principle to get solutions for $t \in [0, T]$, $T > 0$.

We start by turning the equation into its integral formulation, through Fourier transformation

$$\begin{aligned}
\widehat{u}_t - \widehat{u_{xx}} + \widehat{u_{xxxxx}} &= -2\widehat{uu_x} \\
\widehat{u}_t - \widehat{u}(i2\pi\xi)^2 + \widehat{u}(i2\pi\xi)^5 &= -2\widehat{uu_x} \\
\widehat{u}_t + \widehat{u}(i2^5\pi^5\xi^5 + 4\pi^2\xi^2) &= -2\widehat{uu_x}
\end{aligned}$$

Denote $\eta = i2^5\pi^5\xi^5 + 4\pi^2\xi^2$, and solve this for $\widehat{u}$ as an ordinary differential equation

$$\begin{aligned}
\int_0^t (\widehat{u}e^{r\eta})' dr &= -2\int_0^t \widehat{uu_x} e^{r\eta} dr \\
\widehat{u}e^{t\eta} - \widehat{u_0} &= -2\int_0^t \widehat{uu_x} e^{r\eta} dr \\
\widehat{u} &= \widehat{u_0}e^{-t\eta} - 2e^{-t\eta}\int_0^t \widehat{uu_x} e^{r\eta} dr \\
\widehat{u}(\xi, t) &= \widehat{u_0}(\xi)e^{-t\eta} - 2\int_0^t \widehat{uu_x}(\xi, r)e^{-(t-r)\eta} dr \qquad (6.1)
\end{aligned}$$

This is the formulation we will use the most. There is no urgent need to solve for $u(x, t)$ explicitly, since the Fourier transform of u is what we use in the sobolev norms. To make use of the contraction mapping, we do however need to define $I(u)$, the integral formulation of our equation (2.3)

$$I(u) = \mathcal{F}_x^{-1}[\widehat{u_0} e^{-t\eta} - 2\int_0^t e^{-(t-r)\eta} \widehat{uu_x} dr](x,t) \tag{6.2}$$

We state our main result of this chapter

Theorem 6.1. *For all $u_0 \in H^s$ with $s > 1/2$ there exists $T = T(\|u_0\|_{H^s}, s) > 0$ and a unique solution u of* (2.3) *in* $[0,T]$ *with*

$$u \in C([0,T], H^s(\mathbb{R})) \cap L^\infty([0,T], H^s(\mathbb{R}))$$

6.1 A priori estimates

Proposition 6.1. *Let $u_0 \in H^s$ with $s > 1/2$. Then there exists $C > 0$ such that*

$$\|u\|_{H^s} \leq \|u_0\|_{H^s} + C\sqrt{t} \sup_{t\in[0,r]} \|u(r,x)\|_{H^s}^2 \tag{6.3}$$

Proof. We start by directly inserting (6.1) in the sobolev norm and work our way forward

$$\begin{aligned}
\|u\|_{H^s} &= \|\langle\xi\rangle^s(\widehat{u_0} e^{-t\eta} - 2\int_0^t \widehat{uu_x} e^{-(t-r)\eta} dr)\|_{L^2_\xi} \\
&\leq \|\langle\xi\rangle^s \widehat{u_0} e^{-t\eta}\|_{L^2_\xi} + \|2\langle\xi\rangle^s \int_0^t \widehat{uu_x} e^{-(t-r)\eta} dr\|_{L^2_\xi} \\
&= N_1 + N_2
\end{aligned} \tag{6.4}$$

Consider first N_1:

$$\begin{aligned}
N_1 &= \|\langle\xi\rangle^s \widehat{u_0} e^{-t\eta}\|_{L^2_\xi} \\
&= \|\langle\xi\rangle^s \widehat{u_0} e^{-4\pi^2\xi^2 t}\|_{L^2_\xi} \\
&\overset{\mathrm{t}\geq 0}{\leq} \|\langle\xi\rangle^s \widehat{u_0}\|_{L^2_\xi} \\
&= \|u_0\|_{H^s}
\end{aligned} \tag{6.5}$$

Consider now N_2:

$$\begin{aligned}
N_2 &= \|2\langle\xi\rangle^s \int_0^t \widehat{uu_x} e^{-(t-r)\eta} dr\|_{L^2_\xi} \qquad (6.6)\\
&= \|4\int_0^t \langle\xi\rangle^s \cdot \widehat{u^2} \cdot i\pi\xi e^{-(t-r)\eta} dr\|_{L^2_\xi}
\end{aligned}$$

Using Hölders inequality in time we get

$$\leq C\|\sqrt{\int_0^t |\langle\xi\rangle^s \cdot \widehat{u^2}|^2 dr} \cdot \sqrt{\int_0^t |i\pi\xi e^{-(t-r)\eta}|^2 dr}\|_{L^2_\xi} \tag{6.7}$$

Note that

$$\int_0^t |i\pi\xi e^{-(t-r)\eta}|^2 dr = \pi^2\xi^2 e^{-8\pi^2\xi^2 t} \cdot \frac{e^{8\pi^2\xi^2 t} - 1}{8\pi^2\xi^2} \leq 1, \forall\xi$$

Using this estimate in (6.7) we get

$$\leq C\sqrt{\int_{\xi\in\mathbb{R}} \int_0^t |\langle\xi\rangle^s \cdot \widehat{u^2}|^2 dr d\xi} \tag{6.8}$$

By Fubini, we interchange the integrals

$$\leq C\sqrt{\int_0^t \int_{\xi\in\mathbb{R}} |\langle\xi\rangle^s \cdot \widehat{u^2}|^2 d\xi dr}$$

$$\leq C\sqrt{\int_0^t \|u^2\|_{H^s}^2 dr}$$

And lastly, by Proposition 5.3 we have that $s > 1/2$ implies H^s is an algebra and thus

$$\leq \sqrt{\int_0^t \|u\|_{H^s}^4 c_s dr}$$

$$\leq \ C_s\sqrt{t} \cdot \sup_{r\in[0,t]} \|u(r,x)\|_{H^s}^2 \tag{6.9}$$

Inserting for the estimates we found for N_1 in (6.5) and N_2 in (6.9) we get our desired result

$$\|u\|_{H^s} = N_1 + N_2 \leq \|u_0\|_{H^s} + C\sqrt{t} \sup_{r\in[0,t]} \|u(r,x)\|_{H^s}^2 \tag{6.10}$$

□

6.2 Contraction argument

The work in this section follows the same ideas done by Cazenave [6, Chapter 4.10]. We look for a solution in a space where the supremum is present, since we also have that in our a priori estimates. We define the space $E = \{u \in L^\infty([0,T], H^s(\mathbb{R})) : \|u\|_{L^\infty([0,T],H^s(\mathbb{R}))} \leq 2\|u_0\|_{H^s}\}$. We will show that I is a contraction mapping on E.

Proposition 6.2. *Let $u, v \in E$. For all $u_0 \in H^s$ with $s > 1/2$ there exists $C > 0$ such that for $T \leq \frac{1}{25C^2\|u_0\|^2_{H^s}}$, the mapping $I : E \to E$ is contractive.*

Proof. First we show that $I : E \to E$. Using the estimates proven in Proposition 6.1 we know there exists $C > 0$ such that

$$\begin{aligned}\|I(u)\|_{L^\infty([0,T],H^s(\mathbb{R}))} &\leq \sup_{t\in[0,T]} \left(\|u_0\|_{H^s} + C\sqrt{t} \sup_{r\in[0,t]} (\|u(r,x)\|^2_{H^s})\right) \\ &\leq \|u_0\|_{H^s} + C\sqrt{T}(\sup_{t\in[0,T]} \|u\|_{H^s})^2\end{aligned}$$

Inserting for T we get

$$\leq \|u_0\|_{H^s} + C\frac{1}{\sqrt{25C^2\|u_0\|^2_{H^s}}}\|u\|^2_{L^\infty([0,T],H^s(\mathbb{R}))}$$

Using the restriction on the last norm from the definition of E we get

$$\begin{aligned}&\leq \|u_0\|_{H^s} + C\frac{1}{\sqrt{25C^2\|u_0\|^2_{H^s}}}4\|u_0\|^2_{H^s} \\ &< 2\|u_0\|_{H^s}\end{aligned}$$

So, for the restriction $\|u\|^2_{L^\infty([0,T],H^s(\mathbb{R}))} \leq 2\|u_0\|_{H^s}$ we stay within the closed ball of radius $r = 2\|u_0\|_{H^s}$. For the contractive argument, we start by inserting directly in the definition of the norm

$$\begin{aligned}&\|Iu - Iv\|^2_{L^\infty([0,T],H^s(\mathbb{R}))} \\ &\leq C^2 \sup_{t\in[0,T]} \| \int_0^t \langle\xi\rangle^s \xi\pi e^{-(t-r)\eta}(\widehat{u^2} - \widehat{v^2})dr\|^2_{L^2_\xi} \\ &\leq C^2 \sup_{t\in[0,T]} \| \int_0^t \langle\xi\rangle^s \xi\pi e^{-(t-r)\eta}\widehat{u^2 - v^2}dr\|^2_{L^2_\xi}\end{aligned}$$

Following the steps that lead to (6.8) in the previous section, we use Hölder in time and then Fubini to get

$$
\begin{aligned}
&\le C^2 \sup_{t\in[0,T]} \int_0^t \int_{\mathbb{R}} |\langle\xi\rangle^s \widehat{u^2 - v^2}|^2 d\xi dr \\
&= C^2 \sup_{t\in[0,T]} \int_0^t \|u^2 - v^2\|_{H^s}^2 dr \\
&= C^2 \sup_{t\in[0,T]} \int_0^t \|(u-v)(u+v)\|_{H^s}^2 dr
\end{aligned}
$$

Again, by Proposition 5.3

$$
\le C^2 \sup_{t\in[0,T]} \int_0^t \|u+v\|_{H^s}^2 \|u-v\|_{H^s}^2 dr
$$

Using the triangle inequality we get

$$
\begin{aligned}
&\le C^2 \sup_{t\in[0,T]} \int_0^t (\|u\|_{H^s} + \|v\|_{H^s})^2 \|u-v\|_{H^s}^2 dr \\
&\le C^2 T \sup_{t\in[0,T]} \big((\|u\|_{H^s} + \|v\|_{H^s})^2 \|u-v\|_{H^s}^2 \big) \qquad (6.11) \\
&\le C^2 T (\sup_{t\in[0,T]} \|u\|_{H^s} + \sup_{t\in[0,T]} \|v\|_{H^s})^2 \|u-v\|_{L^\infty([0,T],H^s(\mathbb{R}))}^2 \\
&\le C^2 T (\|u\|_{L^\infty([0,T],H^s(\mathbb{R}))} + \|v\|_{L^\infty([0,T],H^s(\mathbb{R}))})^2 \|u-v\|_{L^\infty([0,T],H^s(\mathbb{R}))}^2
\end{aligned}
$$

Using the restriction we have from the definition of E we get

$$
\le 16 C^2 T \|u_0\|_{H^s}^2 \|u-v\|_{L^\infty([0,T],H^s(\mathbb{R}))}^2
$$

Taking roots yields

$$
\|Iu - Iv\|_{L^\infty([0,T],H^s(\mathbb{R}))} \le \sqrt{T} 4C \|u_0\|_{H^s} \|u-v\|_{L^\infty([0,T],H^s(\mathbb{R}))} \qquad (6.12)
$$

Setting $T \le (25C^2\|u_0\|_{H^s}^2)^{-1}$ we get a strictly contracative mapping, and thus we have a unique solution of (2.3) in E for t in $[0,T]$ where $T = T(\|u_0\|_{H^s}, s) > 0$ when $s > 1/2$. This concludes the proof of this proposition $\square$

Proof of Theorem 6.1 By the preceding proposition, we know that our solution u lies in $L^\infty([0,T], H^s(\mathbb{R}))$ for $s > 1/2$. To show that it is also in $C([0,T], H^s)$, we use the embedding theorems for Sobolev spaces. By Proposition 5.1, u is continuously embedded in C^0 since $k = 0$ and $n = 1$. $\square$

7 Solutions in $X^{s,b}$

What is now commonly refered to as Bourgain type spaces was formally introduced by Jean Bourgain in 1993 [2][3] and are shown to be suitable for some types of dispersive and dissipative nonlinear partial differential equations. Our main goal for this chapter is to lower the value of s we found in the preceding chapter, and it will be split in four parts. We start by defining the space $X^{s,b}$ along with some important properties. Then we work our way through the linear part before we do a bit of technical work on the estimate of the nonlinear part. Lastly, we do the contraction argument. For this chapter, we have mainly followed the works done by Molinet & Ribaud [15], Han & Peng [11] and Chen & Li [7].

7.1 $X^{s,b}$ spaces

For the general theory on $X^{s,b}$ spaces we have followed the works of Tao [19, p. 97-107]. The results we state here can be found with proofs and further results in this book.

If we take the spacetime Fourier transformation of a linear constant-coefficient dispersive equation, it can be shown that its solutions are supported on the hypersurface $\tau = h(\xi)$ where h is some polynomial depending on the equation. If we consider the equation with the nonlinearity one would think that this destorts the support of solutions significantly. However, if we localize enough in time with a Schwartz cut-off function $\psi(t)$, the localized Fourier transform of ψu still concentrates around $\tau = h(\xi)$. To grip this dispersive effect, $X^{s,b}$ spaces are suitable. We use the notation $X^{s,b}_{\tau=h(\xi)} = X^{s,b}$, knowing that the space always depends on the equation we study.

Definition 7.1. *Let $h : \mathbb{R}^d \to \mathbb{R}$ be a continuos function and let $s, b \in \mathbb{R}$. The Bourgain space $X^{s,b}$ is defined to be the closure of the Schwartz functions $\mathcal{S}_{x,t}(\mathbb{R} \times \mathbb{R}^d)$ under the norm*

$$\|u\|_{X^{s,b}(\mathbb{R}\times\mathbb{R}^d)} = \|\langle\xi\rangle^s\langle\tau - h(\xi)\rangle^b\widehat{u}(\xi,\tau)\|_{L^2_\xi L^2_\tau(\mathbb{R}\times\mathbb{R}^d)} \tag{7.1}$$

A suitable example is our own equation. To get $h(\xi)$, disregard the nonlinearity in (2.3) and consider the Fourier transform in both time and space.

$$
\begin{aligned}
i\tau\widehat{u} + i\xi^5\widehat{u} + \widehat{u}\xi^2 &= 0 \\
\widehat{u}(i(\tau + \xi^5) + \xi^2) &= 0
\end{aligned}
$$

our $X^{s,b}$ norm then becomes

$$
\|u\|_{X^{s,b}} = \|\langle\xi\rangle^s \langle i(\tau + \xi^5) + \xi^2\rangle \widehat{u}(\xi,\tau)\|_{L^2_{\xi,\tau}(\mathbb{R}^2)} \tag{7.2}
$$

The $X^{s,b}$ spaces are Banach spaces and have the nesting

Proposition 7.1 (Nesting)**.** *For $s < s'$ and $b < b'$ we have*

$$
X^{s',b'} \subset X^{s,b}
$$

We also have the following duality relation, which will be essential when estimating the nonlinearity.

Proposition 7.2 (Duality)**.**

$$
\left(X^{s,b}\right)^\star = X^{-s,-b} \tag{7.3}
$$

The Bourgain spaces also have some helpfull invariance properties

Proposition 7.3 (Invariance properties)**.** *The $X^{s,b}$ spaces are invariant under translations. If $h(\xi)$ is odd, we have that $X^{s,b}$ is invariant under complex conjugation so that*

$$
\|\overline{u}\|_{X^{s,b}} = \|u\|_{X^{s,b}} \tag{7.4}
$$

The complex conjugate of u in $X^{s,b}$ is actually given by

$$
\|\overline{u}\|_{X^{s,b}_{\tau=-h(-\xi)}} = \|u\|_{X^{s,b}}
$$

but for h odd the left hand side does not change. To establish solutions with continuity in time, we have the following result

Proposition 7.4. *Let $b > 1/2$, $s \in \mathbb{R}$ and $h : \mathbb{R}^d \to \mathbb{R}$ be continuous. Then, for any $u \in X^{s,b}(\mathbb{R}^d \times \mathbb{R})$, we have*

$$
\|u\|_{C^0_t H^s_x(\mathbb{R}^d \times \mathbb{R})} \lesssim \|u\|_{X^{s,b}(\mathbb{R}^d \times \mathbb{R})} \tag{7.5}
$$

Lastly, we have this usefull result for our contraction argument

Proposition 7.5. *The $X^{s,b}$ spaces are stable with resepct to time localization. That is, for any Schwartz function $\psi(t)$ in time we have*

$$\|\psi(t)u\|_{X^{s,b}} \lesssim \|u\|_{X^{s,b}}$$

We already have the $\widehat{u}(\xi, t)$ representation of u which we will continue to use, since the transform in x doesnt change when adding a time cut-off. We will consider the time truncated integral formulation

$$F(u) = \psi(t)\mathcal{F}_x^{-1}[\widehat{u_0}\widehat{W}(\xi, t) - 1_{\mathbb{R}^+}(t)\int_0^t \widehat{W}(\xi, (t-r))(\psi_T^2(r)\widehat{\partial_x(u^2)}(\xi, r))dr] \tag{7.6}$$

Where ψ is a Schwartz cut-off function satisfying $\psi(t) \in C_0^\infty$, supp$\psi \subset [-1, 1]$ and $\psi \equiv 1$ for $t \in [-1/2, 1/2]$. The localized Bourgain space we will study is for $T > 0$ defined by

$$\|f\|_{X_T^{s,b}} = \inf_{g \in X^{s,b}} \{g(t) = f(t) \quad \text{on } [0, T]\}$$

We state the main result of this section. We only need to consider the integral formulation of (2.3), and we will use the contraction principle on the time truncated form (7.6). If we prove that u is a solution to (7.6) then it is clear that u is also a solution of (6.2) on $[0, T]$ for $T < 1/2$.

Theorem 7.1 (Localized solution)**.** *Let $u_0 \in H^s$ with $s > -2$. There exists $b > 1/2$ and $T = T(\|u_0\|_{H^s}) > 0$ such that (2.3) has a unique solution u with*

$$u \in C([0, T], H^s) \cap X_T^{s-2b+1,b}$$

Similar to chapter 6, for finding solutions we make an intuitive approach and insert directly in the $X^{s,b}$ norm and work our way forward. It quickly becomes clear that we need two different linear estimates. Using the terminology from [15], we refer to these as the free term and forcing term

7.2 Linear estimates

In this section we estimate the linear parts of the problem.

First, we set $\widehat{W}(\xi, t) = e^{-|t|\xi^2 - it\xi^5}$. For convenience, we drop the multiples of $i2\pi$. We start by proving the linear estimates of the free term.

7.2.1 Linear estimates of the free term

Proposition 7.6. *Let $s \in \mathbb{R}$, and $1/2 \leq b \leq 1$. Then for all $u_0 \in H^s$ we have $\|\psi u_0 W\|_{X^{s+(1-2b),b}} \lesssim \|u_0\|_{H^s}$.*

Proof. We start by inserting directly in $\|\cdot\|_{X^{s,b}}$:

$$\begin{aligned}\|\psi u_0 W\|_{X^{s,b}} &= \|\langle i(\tau+\xi^5)+\xi^2\rangle^b \langle\xi\rangle^s \mathcal{F}_t[\widehat{u_0}(\xi)\psi\widehat{W}](\tau)\|_{L^2_{\xi,\tau}} \\ &= \|\langle\xi\rangle^s \widehat{u_0}\|\langle i(\tau+\xi^5)+\xi^2\rangle^b \mathcal{F}_t[\psi e^{-|t|\xi^2} e^{-it\xi^5}](\tau)\|_{L^2_\tau}\|_{L^2_\xi}\end{aligned}$$

By the translation property of the Fourier transform we get

$$= \|\langle\xi\rangle^s \widehat{u_0}\|\langle i(\tau+\xi^5)+\xi^2\rangle^b \mathcal{F}_t[\psi e^{-|t|\xi^2}](\tau+\xi^5)\|_{L^2_\tau}\|_{L^2_\xi}$$

A change of variables yields

$$= \|\langle\xi\rangle^s \widehat{u_0}\|\langle i\tau+\xi^2\rangle^b \mathcal{F}_t[\psi e^{-|t|\xi^2}](\tau)\|_{L^2_\tau}\|_{L^2_\xi}$$

By the triangle inequality we get

$$\begin{aligned}&\lesssim \|\langle\xi\rangle^s \widehat{u_0}\|\langle\tau\rangle^b \mathcal{F}_t[\psi e^{-|t|\xi^2}](\tau)\|_{L^2_\tau}\|_{L^2_\xi} \\ &+ \|\langle\xi\rangle^{s+2b} \widehat{u_0}\|\mathcal{F}_t[\psi e^{-|t|\xi^2}](\tau)\|_{L^2_\tau}\|_{L^2_\xi} \\ &:= B_1 + B_2 \end{aligned} \tag{7.7}$$

We start with B_1. Let $g_\xi = \psi e^{-|t|\xi^2}$. We want to show that $\|g_\xi\|_{H^b} \lesssim \langle\xi\rangle^{2b-1}$. We first assume $|\xi| \geq 1$.

$$\|g_\xi\|_{H^b} = \|\langle\tau\rangle^b \mathcal{F}_t[\psi e^{-|t|\xi^2}]\|_{L^2_\tau} \tag{7.8}$$

Rewriting this expression and using Hölders inequality, setting $p = \frac{1}{b}$ and $q = \frac{1}{1-b}$ yields

$$\begin{aligned}\|g_\xi\|_{H^b} &= \|(\langle\tau\rangle\mathcal{F}_t[\psi e^{-|t|\xi^2}])^b \mathcal{F}_t[\psi e^{-|t\xi^2}]^{1-b}\|_{L^2_\tau} &(7.9)\\ &\lesssim \|\psi e^{-|t|\xi^2}\|^b_{H^1}\|\mathcal{F}_t[\psi e^{-|t|\xi^2}]\|^{1-b}_{L^2} &(7.10)\\ &\lesssim \langle\xi\rangle^{2b-1} &(7.11)\end{aligned}$$

The last inequality follows from the estimates of both norms in (7.10). Considering the L^2 norm first, by Plancherel we get

$$\begin{aligned}\|\mathcal{F}_t[\psi e^{-|t|\xi^2}]\|_{L^2_\tau}^{1-b} &= \|\psi e^{-|t|\xi^2}\|_{L^2_t}^{1-b} && (7.12)\\ &\leq (2\int_0^1 e^{-2t\xi^2}dt)^{\frac{1-b}{2}}\\ &= (-\frac{2(e^{-2\xi^2}-1)}{2\xi^2})^{\frac{1-b}{2}}\\ &\lesssim |\xi|^{-2\frac{1-b}{2}}\\ &\sim \langle\xi\rangle^{b-1} && (7.13)\end{aligned}$$

Continuing with the H^1 norm of (7.10) we write out the equivalent definition of the H^b norm when $b \in \mathbb{Z}^+$. By Proposition 5.1 we get

$$\|\psi e^{-|t|\xi^2}\|_{H^1}^b = (\|\psi e^{-|t|\xi^2}\|_{L^2_\tau} + \|(\psi e^{-|t|\xi^2})'\|_{L^2_\tau})^b$$

The derivative is in the distributional sense. Using what we calculated in Example 1, inserting for (3.7) we get

$$\lesssim (\|\psi e^{-|t|\xi^2}\|_{L^2_t} + \|\psi' e^{-|t|\xi^2}\|_{L^2_t} + \xi^2\| - \mathrm{sgn}(t)\psi e^{-|t|\xi^2}\|_{L^2_t})^b$$

Since $\psi \in C_0^\infty$ implies that $\psi' \in C_0^\infty$ and the compact support is also in $[-1,1]$ for ψ'. It is easy to see that all these L^2 norms are bounded by $|\xi|^{-1}$. Thus we get

$$(7.14)$$

$$\begin{aligned}&\lesssim (\frac{1}{|\xi|} + \frac{1}{|\xi|} + |\xi|)^b\\ &\lesssim \langle\xi\rangle^b && (7.15)\end{aligned}$$

Combining (7.13) and (7.15) we get

$$\|g_\xi\|_{H^b} \lesssim \langle\xi\rangle^b\langle\xi\rangle^{b-1} = \langle\xi\rangle^{2b-1} \tag{7.16}$$

Consider now for $\|g_\xi\|_{H^b}$ for $|\xi| \leq 1$:

$$\|g_\xi\|_{H^b} = \|\psi e^{-|t|\xi^2}\|_{H^b}$$

Using Taylors formula and the triangle inequality we get

$$\lesssim \|\sum_{n\geq 0}\frac{\psi(|t|\xi^2)^n}{n!}\|_{H^b}$$

Since $|\xi| \leq 1$ we have that $\xi^{2n} \leq 1$ for all n. Thus, we get

$$\lesssim \sum_{n\geq 0} \frac{1}{n!} \|\psi|t|^n\|_{H^b} \tag{7.17}$$

We need to take a closer look at the H^b norm. As earlier, by using Hölder we get

$$\|\psi|t|^n\|_{H^b} \leq \|\psi|t|^n\|_{H^1}^b \|\psi\|_{L_t^2}^{1-b}$$

Writing out the definition of the H^1 norm again, Proposition 5.1 yields

$$\begin{aligned} &= (\|\psi|t|^n\|_{L^2} + \|\psi'|t|^n\|_{L^2} + \|\psi(|t|^n)'\|_{L^2})^b \|\psi\|_{L_t^2}^{1-b} \\ &\lesssim n \end{aligned} \tag{7.18}$$

The bounds of all these terms follows from the properties of ψ, but we will show the first and the third norm. The first:

$$\|\psi|t|^n\|_{L^2}^2 = \int_{\mathbb{R}} |\psi|t|^n|^2 dt$$

Since $-1 \leq t \leq 1$ we have that $|t|^{2n} \leq 1$ for all n and thus

$$\lesssim \sup|\psi| \int_{-1}^{1} |t|^{2n} dt \lesssim 1$$

Now consider the third norm, with the derivative of $|t|$. From the definition of the norm, the derivative is in the distributional sense. Recalling the calculations from Example 3.2, we insert for (3.9):

$$\begin{aligned} \|\psi(|t|^n)'\|_{L^2}^2 &= \int_{\mathbb{R}} |\psi \mathrm{sgn}(t) n |t|^{n-1}|^2 dt \\ &\lesssim \sup|\psi| \int_{-1}^{1} n^2 |t|^{2n-2} dt \\ &\lesssim n^2 \end{aligned} \tag{7.19}$$

For $n = 0$ we have $\|\psi|t|^0\|_{H^b} = \|\psi\|_{H^b}$, which is of course also bounded. To conclude, inserting for (7.18) in (7.17) we get that for $|\xi| \leq 1$

$$\begin{aligned}
\|g_\xi\|_{H^b} &\lesssim \|\psi\|_{H^b} + \sum_{n\geq 1} \frac{n}{n!}\|\psi|t|^n\|_{H^b} \\
&\lesssim 1 + \sum_{n\geq 1} \frac{1}{(n-1)!} \\
&\lesssim 1 \\
&\sim \langle\xi\rangle^{2b-1}
\end{aligned} \tag{7.20}$$

Since $|\xi| \leq 1$. Thus, combining (7.16) and (7.20) gives us the desired result:

$$\|g_\xi\|_{H^b} \lesssim \langle\xi\rangle^{2b-1} \tag{7.21}$$

Which concludes the part of B_1. Consider now B_2, first for $|\xi| \leq 1$:

$$B_2 = \|\langle\xi\rangle^s\langle\xi\rangle^{2b}\widehat{u_0}\|\mathcal{F}_t[\psi e^{-|t|\xi^2}]\|_{L^2_\tau}\|_{L^2_\xi} \tag{7.22}$$

Since $|\xi| \leq 1$, we can remove $\langle\xi\rangle^{2b}$ entirely. We then add $\langle\tau\rangle^b$ in the L^2_τ norm, so that we have exactly $\|g_\xi\|_{H^b}$ which we have just shown to be bounded by $\langle\xi\rangle^{2b-1}$ $\forall\xi \in \mathbb{R}$. Consider now B_2 for $|\xi| \geq 1$: By Plancherel, we get

$$\begin{aligned}
\|\mathcal{F}_t[\psi e^{-|t|\xi^2}]\|_{L^2_\tau} &= \|\psi e^{-|t|\xi^2}\|_{L^2_t} \\
&\leq (2\sup|\psi| \int_0^1 e^{-2t\xi^2}dt)^{1/2} \\
&\sim \langle\xi\rangle^{-1}
\end{aligned} \tag{7.23}$$

Inserting this estimate in (7.22) we get that

$$B_2 \lesssim \|\langle\xi\rangle^s\langle\xi\rangle^{2b-1}\widehat{u_0}\|_{L^2_\xi} \tag{7.24}$$

for all $\xi \in \mathbb{R}$. Thus, inserting for $\|g_\xi\|_{H^b}$ and (7.24) in (7.7) we get that

$$\begin{aligned}
\|\psi u_0 W\|_{X^{s,b}} &\lesssim \|\langle\xi\rangle^s\widehat{u_0}\|\langle\tau\rangle^b\mathcal{F}_t[\psi e^{-|t|\xi^2}]\|_{L^2_\tau}\|_{L^2_\xi} \\
&\quad + \|\langle\xi\rangle^{s+2b}\widehat{u_0}\|\mathcal{F}_t[\psi e^{-|t|\xi^2}]\|_{L^2_\tau}\|_{L^2_\xi} \\
&\lesssim \|\langle\xi\rangle^{s-(1-2b)}\widehat{u_0}\|_{L^2_\xi}
\end{aligned} \tag{7.25}$$

Replacing s with $s + (1 - 2b)$ concludes the proof. ∎

It is now evident that we will in fact be considering the $X^{s-2b+1,b}$ norm and not the $X^{s,b}$ norm. As we want a continuous solution in time, we are going to choose our b to be stricly larger than $1/2$ at some point. This will however not be convenient until we reach the nonlinear estimate. We move on to proving the linear estimates of the forcing term.

7.2.2 Linear estimates of the forcing term

Some of the work done here is similar to that of Han & Peng [11, Prop 2] and Molinet & Ribaud [15, Prop 2.2], but is calculated in more detail and with some own ideas.

Proposition 7.7. *Let $s \in \mathbb{R}$ and $f \in X^{s,-b'}$. For $1/2 \leq b$, $b' < 1/2$ and $b+b' \leq 1$ we have $\|\langle i\tau+\xi^2\rangle \mathcal{F}_t[K_\xi]\|_{L^2_\tau} \lesssim \langle\xi\rangle^{2(b+b'-1)}\|\frac{\widehat{f}(\tau)}{\langle i\tau+\xi^2\rangle^{b'}}\|_{L^2_\tau}$, where K_ξ is defined as $K_\xi = \psi\int_0^t e^{-|t-t'|\xi^2} f(t')dt'$*

Proof. For convenience we denote $\|\frac{\widehat{f}(\tau)}{\langle i\tau+\xi^2\rangle^{b'}}\|_{L^2_\tau}$ by $\|f\|_Z$. We assume $t \geq 0$ and start by rewriting K_ξ, following Molinet & Ribaud [15]

$$K_\xi = \psi\int_0^t e^{-|t-t'|\xi^2} f(t')dt'$$

Rewriting $f(t')$ in the form of a Fourier transform gives

$$= \psi e^{-|t|\xi^2}\int_0^t e^{t'\xi^2}\int_{\mathbb{R}} e^{it'\tau_1}\widehat{f}(\tau_1)d\tau_1 dt'$$

Using Fubini yields

$$= \psi e^{-|t|\xi^2}\int_{\mathbb{R}} \widehat{f}(\tau_1)\int_0^t e^{t'(\xi^2+i\tau_1)}dt'd\tau_1$$

Calculating the integral we get

$$\begin{aligned}
&= \psi\int_{\mathbb{R}} \widehat{f}(\tau_1)\frac{e^{it\tau_1} - e^{-|t|\xi^2} + 1 - 1}{i\tau_1+\xi^2}d\tau_1 \\
&= \psi\int_{\mathbb{R}} \frac{1-e^{-|t|\xi^2}}{i\tau_1+\xi^2}\widehat{f}(\tau_1)d\tau_1 + \psi\int_{\mathbb{R}}\frac{e^{it\tau_1}-1}{i\tau_1+\xi^2}\widehat{f}(\tau_1)d\tau_1 \\
&:= J_1 + J_2 \qquad (7.26)
\end{aligned}$$

Having K_ξ on this form is convenient for applying Taylors formula. As we move forward we will need to divide these integrals even further. Our goal for each integral is getting it bounded by either $\|f\|_Z\langle\xi\rangle^{2(b+b'-1)}$ directly, or by $\|f\|_Z$ when $|\xi|\leq 1$ since that implies $1\sim\langle\xi\rangle^{2(b+b'-1)}$.
Consider now $\|\langle i\tau+\xi^2\rangle^b\widehat{J}_1(\tau)\|_{L^2_\tau}$ for $|\xi|\leq 1$:

$$\|\langle i\tau+\xi^2\rangle^b\widehat{J}_1(\tau)\|_{L^2_\tau}=\|\langle i\tau+\xi^2\rangle^b\mathcal{F}_t[\psi\int_{\mathbb{R}}\frac{1-e^{-|t|\xi^2}}{i\tau_1+\xi^2}\widehat{f}(\tau_1)d\tau_1]\|_{L^2_\tau} \tag{7.27}$$

We move out the expression in the numerator since this does not depend on τ_1. Then we can remove out the entire integral, due to Minkowskis integral inequality since this is a function of (ξ,τ_1). Thus, we get

$$\lesssim\|\langle i\tau+\xi^2\rangle^b\mathcal{F}_t[\psi(1-e^{-|t|\xi^2})]\|_{L^2_\tau}\int_{\mathbb{R}}|\frac{\widehat{f}(\tau_1)}{i\tau_1+\xi^2}|d\tau_1$$

Adding $\frac{\langle i\tau_1+\xi^2\rangle^{b'}}{\langle i\tau_1+\xi^2\rangle^{b'}}$ to the integral and using Cauchy-Schwartz we get

$$\lesssim\|\langle i\tau+\xi^2\rangle^b\mathcal{F}_t[\psi(1-e^{-|t|\xi^2})]\|_{L^2_\tau}\Big(\int_{\mathbb{R}}\frac{\langle i\tau_1+\xi^2\rangle^{2b'}}{|i\tau_1+\xi^2|^2}d\tau_1\Big)^{1/2}\|\frac{\widehat{f}(\tau_1)}{\langle i\tau_1+\xi^2\rangle^{b'}}\|_{L^2_\tau}$$

Using the triangle inequality yields

$$\lesssim\|f\|_Z\Big(\|\psi(1-e^{-|t|\xi^2})\|_{H^b}+|\xi|^{2b}\|\mathcal{F}_t[\psi(1-e^{-|t|\xi^2})]\|_{L^2_\tau}\Big)$$
$$\cdot\Big(\int_{\mathbb{R}}\frac{\langle i\tau+\xi^2\rangle^{2b'}}{|i\tau+\xi^2|^2}d\tau\Big)^{1/2} \tag{7.28}$$

All the terms in (7.28) must be evauluated. We consider them first for $|\tau|\leq 1$. We start with evaluating the first norm in (7.28): By Taylors formula and the triangle inequality we have

$$\|\psi(1-e^{-|t|\xi^2})\|_{H^b}\lesssim\sum_{n\geq 1}\frac{\xi^{2n}\|\psi|t|^n\|_{H^b}}{n!}$$

Since $|\xi|\leq 1$ we have that $\xi^{2n}\leq\xi^2$ for $n\geq 1$. Using this, aswell as the calulations from (7.18) we get

$$\lesssim\xi^2\sum_{n\geq 1}\frac{1}{n!}\|\psi|t|^n\|_{H^b}\lesssim\xi^2 \tag{7.29}$$

Continuing with the second norm in (7.28), we add $\langle\tau\rangle^b$ to get the H^b norm

$$\begin{aligned}
&|\xi|^{2b}\|\mathcal{F}_t[\psi(1-e^{-|t|\xi^2})]\|_{L^2_\tau}\\
&\lesssim |\xi|^{2b}\|\psi(1-e^{-|t|\xi^2})\|_{H^b}\\
&\lesssim |\xi|^{2b+1}
\end{aligned}\tag{7.30}$$

This last inequality follows from using the same estimate in (7.29), but using the fact that since $|\xi|\leq 1$ it is also true that $|\xi|^2\leq|\xi|$

What remains is the integral in (7.28). Recalling that $|\tau|\leq 1$ we get

$$\Big(\int_{|\tau|\leq 1}\frac{\langle i\tau+\xi^2\rangle^{2b'}}{|i\tau+\xi^2|^2}d\tau\Big)^{1/2}\lesssim\Big(\int_{|\tau|\leq 1}\frac{\langle\tau\rangle^{2b'}}{\xi^4}d\tau\Big)^{1/2}\sim\Big(\frac{1}{\xi^4}\Big)^{1/2}=\frac{1}{\xi^2}\tag{7.31}$$

for $\xi\neq 0$

Combining the estimates (7.29)-(7.31) gives

$$\begin{aligned}
\|\langle i\tau+\xi^2\rangle^b\widehat{J}_1(\tau)\|_{L^2_\tau}&\lesssim\|f\|_Z(|\xi|^2+|\xi|^{2b+1})\frac{1}{|\xi|^2}\\
&\sim\|f\|_Z\langle\xi\rangle^{2b-1}
\end{aligned}$$

Since $|\xi|\leq 1$ we can add $\langle\xi\rangle^{2b'-1}\sim 1$ and thus

$$\begin{aligned}
&\sim\|f\|_Z\langle\xi\rangle^{2b-1}\langle\xi\rangle^{2b'-1}\\
&=\|f\|_Z\langle\xi\rangle^{2(b+b'-1)}
\end{aligned}\tag{7.32}$$

Consider now (7.27) for $|\tau|\geq 1$. We evaluate the integral in (7.28):

$$\Big(\int_{|\tau|\geq 1}\frac{\langle i\tau+\xi^2\rangle^{2b'}}{|i\tau+\xi^2|^2}d\tau\Big)^{1/2}\lesssim\Big(\int_{|\tau|\geq 1}\frac{\tau^{2b'}}{\tau^2}d\tau\Big)^{1/2}\lesssim 1\tag{7.33}$$

since $b'<1/2$. We have shown that the other norms in (7.28) are bounded by $(|\xi|^2+|\xi|^{2b+1})$ regardless of what τ is. Using that $|\xi|\leq 1$ we just replace them with 1, thus getting the bound for (7.27) when $|\tau|\geq 1$ and $|\xi|\leq 1$ to be

$$\begin{aligned}
\|\langle i\tau+\xi^2\rangle^b\widehat{J}_1(\tau)\|_{L^2_\tau}&\lesssim\|f\|_Z\\
&\lesssim\|f\|_Z\langle\xi\rangle^{2(b+b'-1)}
\end{aligned}\tag{7.34}$$

Since $\langle\xi\rangle^{2(b+b'-1)} \sim 1$, when $|\xi| \leq 1$.
Consider now J_1 for $|\xi| \geq 1$. We start by using the triangle inequality:

$$\begin{aligned}\|\langle i\tau+\xi^2\rangle^b \widehat{J}_1(\tau)\| &\lesssim \|\langle i\tau+\xi^2\rangle^b \mathcal{F}_t[\psi \int_{\mathbb{R}} \frac{\widehat{f}(\tau_1)}{i\tau_1+\xi^2} d\tau_1]\|_{L^2_\tau} \\ &+ \|\langle i\tau+\xi^2\rangle^b \mathcal{F}_t[\psi \int_{\mathbb{R}} \frac{e^{-|t|\xi^2}}{i\tau_1+\xi^2} \widehat{f}(\tau_1) d\tau_1]\|_{L^2_\tau} \\ &:= J_{1,1} + J_{1,2} \qquad (7.35)\end{aligned}$$

We consider $J_{1,1}$ first. Following the idea by Han & Peng [11, p. 176] we use the following inequality for $0 \leq b \leq 1$

$$\langle i\tau+\xi^2\rangle^b \lesssim \langle i\tau - i\tau_1\rangle^b + |i\tau_1+\xi^2|^b \qquad (7.36)$$

First, we mulitply with $e^{-it\tau_1}e^{it\tau_1}$ inside the integral.

$$J_{1,1} = \|\langle i\tau+\xi^2\rangle^b \mathcal{F}_t[\psi \int_{\mathbb{R}} \frac{\widehat{f}(\tau_1)e^{it\tau_1}e^{-it\tau_1}}{(i\tau_1+\xi^2)} d\tau_1\|_{L^2_\tau}$$

We can now rewrite the integral as a Fourier transform

$$= \|\mathcal{F}_t[\langle i\tau+\xi^2\rangle^b \psi \mathcal{F}_t[\frac{\widehat{f}(\tau_1)e^{-it\tau_1}}{(i\tau_1+\xi^2)}]]\|_{L^2_\tau}$$

Inserting for (7.36) and using the Cauchy-Schwartz along with interchanging multiplication with convolution when taking the transform of a product we get

$$\lesssim \|(\langle\cdot\rangle^b \widehat{\psi}) * \frac{\widehat{f}(\cdot)e^{-it\cdot}}{|i\cdot+\xi^2|}\|_{L^2_\tau} + \|\widehat{\psi} * \frac{\widehat{f}(\cdot)e^{-it\cdot}}{|i\cdot+\xi^2|^{1-b}}\|_{L^2_\tau}$$

By Youngs inequality we get

$$= \|\langle\tau\rangle^b \widehat{\psi}\|_{L^1_\tau} \|\frac{\widehat{f}(\tau)e^{-it\tau}}{|i\tau+\xi^2|}\|_{L^2_\tau} + \|\widehat{\psi}\|_{L^1_\tau} \|\frac{\widehat{f}(\tau)e^{-it\tau}}{|i\tau+\xi^2|^{1-b}}\|_{L^2_\tau} \qquad (7.37)$$

$$\lesssim \|\frac{\widehat{f}(\tau)}{\langle i\tau+\xi^2\rangle^{b'}} \cdot \frac{\langle i\tau+\xi^2\rangle^{b'} e^{-it\tau}}{|i\tau+\xi^2|}\|_{L^2_\tau} + \|\frac{\widehat{f}(\tau)}{\langle i\tau+\xi^2\rangle^{b'}} \cdot \frac{\langle i\tau+\xi^2\rangle^{b'} e^{-it\tau}}{|i\tau+\xi^2|^{1-b}}\|_{L^2_\tau} \qquad (7.38)$$

Taking supremum over all $\tau \in \mathbb{R}$ on everything that doesnt contribute to getting the $\|f\|_Z$ norm gives us

$$\lesssim \|f\|_Z \Big(\big(\sup_\tau \frac{|e^{-it\tau}|^2(\tau^2+\xi^4)^{b'}}{(\tau^2+\xi^4)} \big)^{1/2} + \big(\sup_\tau \frac{|e^{-it\tau}|^2(\tau^2+\xi^4)^{b'}}{(\tau^2+\xi^4)^{1-b}} \big)^{1/2} \Big) \tag{7.39}$$

Recalling that $|\xi| \geq 1$, we can divide by ξ^4 and get

$$\lesssim \|f\|_Z \Big(\big(\sup_\tau \frac{(\frac{\tau^2}{\xi^4}+1)^{b'}|\xi|^{4b'}}{(\frac{\tau^2}{\xi^4}+1)\xi^4} \big)^{1/2} + \big(\sup_\tau \frac{(\frac{\tau^2}{\xi^4}+1)^{b'}|\xi|^{4b'}}{(\frac{\tau^2}{\xi^4}+1)^{1-b}|\xi|^{4(1-b)}} \big)^{1/2} \Big)$$

Since we have the restriction $b+b' \leq 1$ and $b' < 1/2$ both these supremums are bounded $\forall \tau \in \mathbb{R}$ and $|\xi| \geq 1$. Thus, we get

$$\begin{aligned} &\lesssim \|f\|_Z (1+|\xi|^{2b'+2b-2}) \\ &\lesssim \|f\|_Z \langle \xi \rangle^{2(b+b'-1)} \end{aligned} \tag{7.40}$$

Consider now $J_{1,2}$:

$$\begin{aligned} J_{1,2} &\lesssim \|\langle i\tau+\xi^2\rangle^b \mathcal{F}_t[\psi e^{-|t|\xi^2}]\|_{L^2_\tau} \int_{\mathbb{R}} |\frac{\widehat{f}(\tau_1)}{i\tau_1+\xi^2}| d\tau_1 \\ &\lesssim \Big(\|\langle \tau\rangle^b \mathcal{F}_t[\psi e^{-|t|\xi^2}]\|_{L^2_\tau} + |\xi|^{2b} \|\mathcal{F}_t[\psi e^{-|t|\xi^2}]\|_{L^2_\tau} \Big) \|f\|_Z \\ &\cdot \Big(\int_{\mathbb{R}} \frac{\langle i\tau_1+\xi^2\rangle^{2b'}}{\tau_1^2+\xi^4} d\tau_1 \Big)^{1/2} \end{aligned}$$

By inserting for g_ξ in the first norm and the estimate we calculated in (7.16) and using Plancherel in the second, we get

$$\begin{aligned} &\lesssim \Big(\|g_\xi\|_{H^b} + |\xi|^{2b} \|\psi e^{-|t|\xi^2}\|_{L^2_t} \Big) \|f\|_Z \Big(\int_{\mathbb{R}} \frac{\langle i\tau_1+\xi^2\rangle^{2b'}}{\tau_1^2+\xi^4} d\tau_1 \Big)^{1/2} \\ &\lesssim \|f\|_Z \langle\xi\rangle^{2b-1} \Big(\int_{\mathbb{R}} \frac{\langle i\tau_1+\xi^2\rangle^{2b'}}{\tau_1^2+\xi^4} d\tau_1 \Big)^{1/2} \end{aligned} \tag{7.41}$$

The evalutation of the integral gives us the desired result:

$$\int_{\mathbb{R}} \frac{\langle i\tau+\xi^2\rangle^{2b'}}{\tau^2+\xi^4} d\tau \lesssim \int_{\mathbb{R}} \frac{(1+\frac{\tau^2}{\xi^4})^{b'}|\xi|^{4b'}}{(1+\frac{\tau^2}{\xi^4})\xi^4} d\tau$$

Making a change of variables yields

$$= \int_{\mathbb{R}} \frac{\langle \tau \rangle^{2b'}}{\langle \tau \rangle^2} d\tau \langle \xi \rangle^{4b'-2}$$

Which is bounded for $b' < 1/2$, thus we get

$$\lesssim \langle \xi \rangle^{2(2b'-1)} \tag{7.42}$$

Taking roots and inserting in (7.41) we get

$$\lesssim \|f\|_Z \langle \xi \rangle^{2(b+b'-1)} \tag{7.43}$$

This concludes the part with J_1.
Let us now consider J_2:

$$\begin{aligned}
\|\langle i\tau + \xi^2 \rangle^b \widehat{J}_2(\tau)\|_{L^2_\tau} &= \|\langle i\tau + \xi^2 \rangle^b \mathcal{F}_t[\psi \int_{\mathbb{R}} \frac{e^{it\tau_1} - 1}{i\tau_1 + \xi^2} \widehat{f}(\tau_1) d\tau_1]\|_{L^2_\tau} \\
&\lesssim \|\langle i\tau + \xi^2 \rangle^b \mathcal{F}_t[\psi \int_{|\tau_1| \leq 1} \frac{e^{it\tau_1} - 1}{i\tau_1 + \xi^2} \widehat{f}(\tau_1) d\tau_1]\|_{L^2_\tau} \\
&+ \|\langle i\tau + \xi^2 \rangle^b \mathcal{F}_t[\psi \int_{|\tau_1| \geq 1} \frac{e^{it\tau_1} - 1}{i\tau_1 + \xi^2} \widehat{f}(\tau_1) d\tau_1]\|_{L^2_\tau} \\
&:= J_{2,1} + J_{2,2}
\end{aligned} \tag{7.44}$$

We start by considering $J_{2,1}$ for $|\xi| \geq 1$.

$$J_{2,1} \lesssim \|\langle i\tau + \xi^2 \rangle \mathcal{F}_t[\psi \int_{|\tau_1| \leq 1} \frac{e^{it\tau_1} - 1}{i\tau_1 + \xi^2} \widehat{f}(\tau_1) d\tau_1]\|_{L^2_\tau}$$

By Taylors formula we get

$$\begin{aligned}
&\lesssim \|\langle i\tau + \xi^2 \rangle \mathcal{F}_t[\int_{|\tau_1| \leq 1} \sum_{n \geq 1} \frac{\psi |it\tau_1|^n}{n!} \frac{\widehat{f}(\tau_1)}{i\tau_1 + \xi^2} d\tau_1]\|_{L^2_\tau} \\
&\lesssim \sum_{n \geq 1} \frac{\|\langle i\tau + \xi^2 \rangle \mathcal{F}_t[\psi |t|^n]\|_{L^2_\tau}}{n!} \int_{|\tau_1| \leq 1} |\frac{\widehat{f}(\tau_1)}{i\tau_1 + \xi^2}| d\tau \\
&\lesssim \sum_{n \geq 1} \frac{\|\langle \tau \rangle^b \mathcal{F}_t[\psi |t|^n]\|_{L^2_\tau} + |\xi|^{2b} \|\mathcal{F}_t[\psi |t|^n]\|_{L^2_\tau}}{n!} \int_{|\tau_1| \leq 1} |\frac{\widehat{f}(\tau_1)}{i\tau_1 + \xi^2}| d\tau_1 \\
&\lesssim |\xi|^{2b} \sum_{n \geq 1} \frac{\|\psi |t|^n\|_{H^b}}{n!} \int_{|\tau_1| \leq 1} |\frac{\widehat{f}(\tau_1)}{i\tau_1 + \xi^2}| d\tau_1
\end{aligned}$$

Using Cauchy-Schwartz on the integral yields

$$
\begin{aligned}
&\lesssim \langle\xi\rangle^{2b}\sum_{n\geq 1}\frac{1}{(n-1)!}\|f\|_Z\Big(\int_{|\tau_1|\leq 1}\frac{\langle i\tau_1+\xi^2\rangle^{2b'}}{\tau_1^2+\xi^4}d\tau_1\Big)^{1/2}\\
&\lesssim \|f\|_Z\langle\xi\rangle^{2(b+b'-1)} \qquad (7.45)
\end{aligned}
$$

The explanation for the last inequality comes from the evaluation of the integral

$$
\int_{|\tau|\leq 1}\frac{\langle i\tau+\xi^2\rangle^{2b'}}{\tau^2+\xi^4}d\tau \lesssim \int_{|\tau|\leq 1}\frac{(1+\frac{\tau^2}{\xi^4})^{b'}|\xi|^{4b'}}{(1+\frac{\tau^2}{\xi^4})\xi^4}d\tau
$$

Making a change of variables gives

$$
\begin{aligned}
&= \int_{|\tau|\leq \xi^{-2}}\frac{1}{\langle\tau\rangle^{2-2b'}}d\tau\langle\xi\rangle^{2(2b'-1)}\\
&\lesssim \int_{-\xi^{-2}}^{\xi^{-2}}d\tau\cdot\langle\xi\rangle^{2(2b'-1)}\\
&\sim \langle\xi\rangle^{2(2b'-2)}
\end{aligned}
$$

Consider now $J_{2,1}$ for $|\xi|\leq 1$. In the preceding estimates, we dropped τ^n completely since it is bounded by 1. The only difference now is that we take it with us in the integral. So again, by Taylors formula and using that $|\tau_1|^n\leq|\tau_1|$ for $n\geq 1$ we get

$$
\sum_{n\geq 1}\frac{\|\langle\tau\rangle^b\mathcal{F}_t[\psi|t|^n]\|_{L^2_\tau}+|\xi|^{2b}\|\mathcal{F}_t[\psi|t|^n]\|_{L^2_\tau}}{n!}\int_{|\tau_1|\leq 1}|\frac{|\tau_1|\widehat{f}(\tau_1)}{i\tau_1+\xi^2}|d\tau_1
$$

By Cauchy Schwartz again, we get

$$
\begin{aligned}
&\lesssim \sum_{n\geq 1}\frac{\|\psi|t|^n\|_{H^b}+\|\psi|t|^n\|_{L^2_t}}{n!}\|f\|_Z\Big(\int_{|\tau_1|\leq 1}\frac{|\tau_1|^2\langle i\tau_1+\xi^2\rangle^{2b'}}{\tau_1^2+\xi^4}d\tau_1\Big)^{1/2}\\
&\lesssim \|f\|_Z\\
&\sim \|f\|_Z\langle\xi\rangle^{2(b+b'-1)} \qquad (7.46)
\end{aligned}
$$

The sum is already shown to be bounded earlier, so we only need to evaluate the integral to see that this last inequality is indeed true

$$
\int_{|\tau|\leq 1}\frac{|\tau|^2\langle i\tau+\xi^2\rangle^{2b'}}{\tau^2+\xi^4}d\tau \lesssim \int_{|\tau|\leq 1}\frac{\tau^2\langle\tau\rangle^{2b'}}{\tau^2}\lesssim 1
$$

What remains is to evaluate $J_{2,2}$ By the triangle inequality we get

$$\begin{aligned}
J_{2,2} &\lesssim \|\langle i\tau+\xi^2\rangle^b \mathcal{F}_t[\psi \int_{|\tau_1|\geq 1} \frac{e^{it\tau_1}}{i\tau_1+\xi^2}\widehat{f}(\tau_1)d\tau_1]\|_{L^2_\tau} \\
&+ \|\langle i\tau+\xi^2\rangle^b \mathcal{F}_t[\psi \int_{|\tau_1|\geq 1} \frac{\widehat{f}(\tau_1)}{i\tau_1+\xi^2}d\tau_1]\|_{L^2_\tau} \\
&:= J_3 + J_4
\end{aligned}$$

We evaluate J_4 first for $|\xi| \leq 1$

$$\begin{aligned}
J_4 &\lesssim \|\langle i\tau+\xi^2\rangle^b \widehat{\psi}\|_{L^2_\tau} \int_{|\tau_1|\geq 1} |\frac{\widehat{f}(\tau_1)}{i\tau_1+\xi^2}| d\tau_1 \\
&\lesssim \Big(\|\langle\tau\rangle^b\widehat{\psi}\|_{L^2_\tau} + |\xi|^{2b}\|\widehat{\psi}\|_{L^2_\tau}\Big)\|f\|_Z \Big(\int_{|\tau_1|\geq 1} \frac{\langle i\tau_1+\xi^2\rangle^{2b'}}{\tau_1^2+\xi^4}d\tau_1\Big)^{1/2} \\
&\lesssim \|f\|_Z \Big(\int_{|\tau_1|\geq 1} \frac{\langle\tau_1\rangle^{2b'}}{\tau_1^2}d\tau_1\Big)^{1/2} \\
&\lesssim \|f\|_Z \\
&\sim \langle\xi\rangle^{2(b+b'-1)}\|f\|_Z
\end{aligned} \tag{7.47}$$

For $|\xi| \geq 1$, J_4 is exactly the same as $J_{1,1}$ which we have already shown is true for all τ_1, so setting the restriction $|\tau_1| \geq 1$ obviously does not change the result. This concludes the part of J_4. The case of J_3 is also quite similar. In $J_{1,1}$ we multiplied with $e^{-it\tau_1}e^{it\tau_1}$ to get it on the form of a Fourier transform. Here, this is not necessary since we already have $e^{it\tau_1}$ present. Except from this, following the same steps as in $J_{1,1}$, the only thing we need to check is the supremum in (7.39) for $|\tau_1| \geq 1$ and $|\xi| \leq 1$:

$$\begin{aligned}
&\big(\sup_{|\tau_1|\geq 1} \frac{(\tau_1^2+\xi^4)^{b'}}{(\tau_1^2+\xi^4)}\big)^{1/2} + \big(\sup_{|\tau_1|\geq 1} \frac{(\tau_1^2+\xi^4)^{b'}}{(\tau_1^2+\xi^4)^{1-b}}\big)^{1/2} \\
&\lesssim \big(\sup_{|\tau_1|\geq 1} \frac{\tau_1^{2b'}}{\tau_1^2}\big)^{1/2} + \big(\sup_{|\tau_1|\geq 1} \frac{\tau_1^{2b'}}{\tau_1^{2(1-b)}}\big)^{1/2} \\
&\lesssim 1
\end{aligned} \tag{7.48}$$

Since $b' < 1/2$ and $b + b' \leq 1$, which concludes the part of J_3 as well as our proof. □

The last proposition in this section shows some smoothing properties between Bourgain spaces. We study the linear operator defined by

$$L : f \to \psi(t)1_{\mathbb{R}^+}(t)\int_0^t W(t-t')f(t')dt' \tag{7.49}$$

Proposition 7.8. *For $s \in \mathbb{R}$, $b \geq 1/2$, $b' < 1/2$, $b + b' \leq 1$ and $f \in X^{s+2b'-1,-b'}$ we have*

$$\|\psi(t)1_{\mathbb{R}^+}(t)\int_0^t W(t-t')f(t')dt'\|_{X^{s-2b+1,b}} \lesssim \|f\|_{X^{s+2b'-1,-b'}} \tag{7.50}$$

Proof. Following a standard argument that can be found in nearly every paper on the subject, for example Han & Peng [11, Prop 3], we insert in the $X^{s,b}$ norm

$$\|\psi(t)1_{\mathbb{R}^+}(t)\int_0^t W(t-t')f(t')dt'\|_{X^{s,b}}$$

$$= \|\langle i(\tau+\xi^5)+\xi^2\rangle^b\langle\xi\rangle^s\mathcal{F}_{x,t}[\psi(t)1_{\mathbb{R}^+}(t)\int_0^t W(t-t')f(t')dt'](\xi,\tau)\|_{L^2_{\xi,\tau}}$$

$$= \|\langle i(\tau+\xi^5)+\xi^2\rangle^b\langle\xi\rangle^s\mathcal{F}_t[\psi(t)1_{\mathbb{R}^+}(t)\int_0^t e^{-|t-t'|\xi^2}e^{-i(t-t')\xi^5}\mathcal{F}_x[f(t')]dt'](\tau)\|_{L^2_{\xi,\tau}}$$

$$= \|\langle i(\tau+\xi^5)+\xi^2\rangle^b\langle\xi\rangle^s\mathcal{F}_t[\psi(t)1_{\mathbb{R}^+}(t)\int_0^t e^{-|t-t'|\xi^2}e^{-i\xi^5 t'}\mathcal{F}_x[f(t')]dt'](\tau+\xi^5)\|_{L^2_{\xi,\tau}}$$

Making a change of variable we get

$$= \|\langle i\tau+\xi^2\rangle^b\langle\xi\rangle^s\mathcal{F}_t[\psi(t)1_{\mathbb{R}^+}(t)\int_0^t e^{-|t-t'|\xi^2}e^{-i\xi^5 t'}\mathcal{F}_x[f(t')]dt'](\tau)\|_{L^2_{\xi,\tau}}$$

Setting $\mathcal{F}_x[e^{-it'\xi^5}f](t',\xi) = f_1$ we get

$$= \|\langle i\tau+\xi^2\rangle^b\langle\xi\rangle^s\mathcal{F}_t[\psi(t)1_{\mathbb{R}^+}(t)e^{-|t|\xi^2}\int_0^t e^{t'\xi^2}f_1(t',\xi)dt'](\tau)\|_{L^2_{\xi,\tau}}$$

$$= \|\langle i\tau+\xi^2\rangle^b\langle\xi\rangle^s\mathcal{F}_t[\psi(t)1_{\mathbb{R}^+}(t)e^{-|t|\xi^2}\int_0^t e^{t'\xi^2}\int_{\mathbb{R}} e^{it'\tau_1}\widehat{f}_1(\tau_1,\xi)d\tau_1 dt'](\tau)\|_{L^2_{\xi,\tau}}$$

And by the same calculations we used to arrive at K_ξ we get

$$\lesssim \|\langle\xi\rangle^s\|\langle i\tau+\xi^2\rangle^b\mathcal{F}_t[1_{\mathbb{R}^+}K_\xi]\|_{L^2_\tau}\|_{L^2_\xi}$$

Using Proposition 7.7 on $f = f_1$ yields

$$\lesssim \|\langle\xi\rangle^{s+2(b+b'-1)}\|f\|_Z\|_{L^2_\xi} \tag{7.51}$$

Given that $\|\mathcal{F}_t[1_{\mathbb{R}^+}K_\xi]\|_{H^b} \lesssim \|\mathcal{F}_t[K_\xi]\|_{H^b}$, but this is not hard to get. See for example [14]. Replacing s with $s-2b+1$ finishes the proof.

$$\tag{7.52}$$

□

7.3 Nonlinear estimate

We have up until now not set any restriction on s, other than it has to be a real number. This is where we get our limitations on s. We now also set $b = 1/2 + \delta/2$ and $b' = 1/2 - \delta/2$ for any $\delta > 0$ small. This guarantees a continuous solution in time, and our estimates done in the linear part still holds since $b > 1/2$, $b' < 1/2$ and $b + b' \leq 1$.

Proposition 7.9. *For every $s > -2 + 25\delta/8$ and every $\delta > 0$, there exists $C, \Gamma > 0$ such that for any $(u, v) \in X^{s-\delta,1/2+\delta/2}$ with compact support in $[0, T]$,*

$$\|\partial_x(uv)\|_{X^{s-\delta,-1/2+\delta/2}} \leq CT^{\Gamma}\|u\|_{X^{s-\delta,1/2+\delta/2}}\|v\|_{X^{s-\delta,1/2+\delta/2}} \tag{7.53}$$

Using Riesz representation theorem (Proposition 3.3), we can use a duality argument to get an equivalent formulation of (7.53). From Proposition 7.2 we know that the dual space of $X^{s,b}$ is $X^{-s,-b}$. Thus, proving (7.53) is equivalent to showing that for any function $w \in X^{-s+\delta,1/2-\delta/2}$ with compact support in $[0, T]$,

$$|\langle\partial_x(uv), w\rangle| \leq CT^{\Gamma}\|u\|_{X^{s-\delta,1/2+\delta/2}}\|v\|_{X^{s-\delta,1/2+\delta/2}}\|w\|_{X^{-s+\delta,1/2-\delta/2}} \tag{7.54}$$

Writing out the inner product yields

$$\begin{aligned}
|\langle\partial_x(uv), w\rangle| &\sim \left|\int_{\mathbb{R}^2} \mathcal{F}_{x,t}[\partial_x(uv)]\mathcal{F}_{x,t}[w]d\xi d\tau\right| \\
&= \int_{\mathbb{R}^2} |\widehat{w}(\xi,\tau)d\xi d\tau \int_{\mathbb{R}^2} i\xi\widehat{u}(\xi_1,\tau_1)\widehat{v}(\xi-\xi_1,\tau-\tau_1)d\xi_1 d\tau_1| \\
&= \int_{\mathbb{R}^4} |\xi\widehat{w}(\xi,\tau)\widehat{u}(\xi_1,\tau_1)\widehat{v}(\xi_2,\tau_2)|d\xi d\xi_1 d\tau d\tau_1 \qquad (7.55)
\end{aligned}$$

Where

$$\begin{aligned}
\tau &= \tau_1 + \tau_2 \\
\xi &= \xi_1 + \xi_2
\end{aligned}$$

Set now

$$\begin{aligned}
\widehat{f}(\xi,\tau) &= \langle i(\tau+\xi^5) + \xi^2\rangle^{1/2+\delta/2}\langle\xi\rangle^{s-\delta}\widehat{u}(\xi,\tau) \\
\widehat{g}(\xi,\tau) &= \langle i(\tau+\xi^5) + \xi^2\rangle^{1/2+\delta/2}\langle\xi\rangle^{s-\delta}\widehat{v}(\xi,\tau) \\
\widehat{h}(\xi,\tau) &= \langle i(\tau+\xi^5) + \xi^2\rangle^{1/2-\delta/2}\langle\xi\rangle^{-s+\delta}\widehat{w}(\xi,\tau)
\end{aligned}$$

Rearranging and inserting for $\widehat{u}, \widehat{v}, \widehat{w}$ we see that (7.54) is implied by

$$I \lesssim T^{\Gamma} \|f\|_{L^2(\mathbb{R}^2)} \|g\|_{L^2(\mathbb{R}^2)} \|h\|_{L^2(\mathbb{R}^2)} \tag{7.56}$$

Where

$$I = \int_{\mathbb{R}^4} \frac{K(\xi, \xi_1, \tau, \tau_1) \widehat{h}(\xi, \tau) \widehat{f}(\xi_1, \tau_1) \widehat{g}(\xi_2, \tau_2)}{\langle \sigma \rangle^{\delta/8}} d\xi d\xi_1 d\tau d\tau_1 \tag{7.57}$$

and

$$K(\xi, \xi_1, \tau, \tau_1) = \frac{|\xi| \langle \xi \rangle^{s-\delta} \langle \xi_1 \rangle^{-s+\delta} \langle \xi_2 \rangle^{-s+\delta}}{\langle \sigma \rangle^{1/2 - 5\delta/8} \langle \sigma_1 \rangle^{1/2+\delta/2} \langle \sigma_2 \rangle^{1/2+\delta/2}} \tag{7.58}$$

and

$$\begin{aligned} \sigma &= i(\tau + \xi^5) + \xi^2 \\ \sigma_k &= i(\tau_k + \xi_k^5) + \xi_k^2 \end{aligned}$$

Before we proceed further on the proof of Proposition 7.9, we need some results regarding the integrals that will arrise from modyfing (7.57) further. There are three separate integrals we must evaluate, and to ease our way of dealing with the four-dimensional integral we need some basic estimates on σ:

$$\max(|\sigma|, |\sigma_1|, |\sigma_2| \gtrsim |\sigma - \sigma_1 - \sigma_2| \tag{7.59}$$

We know that

$$|\sigma - \sigma_1 - \sigma_2| \geq |\operatorname{Re} \sigma - \sigma_1 - \sigma_2|$$

and

$$|\sigma - \sigma_1 - \sigma_2| \geq |\operatorname{Im} \sigma - \sigma_1 - \sigma_2|$$

To get an estimate for the maximum (7.59) we insert for σ get:

$$\begin{aligned} |\operatorname{Im} \sigma - \sigma_1 - \sigma_2| &= |\tau + \xi^5 - \tau_1 - \xi_1^5 - \tau_2 - \xi_2^5| \\ &= |\tau - (\tau_1 + \tau_2) + (\xi_1 + \xi_2)^5 - \xi_1^5 - \xi_2^5| \\ &= |(\xi_1 + \xi_2)^5 - \xi_1^5 - \xi_2^5| \\ &= |\xi||\xi_1||\xi_2||3\xi_1^2 - 3\xi_1 \xi + \xi^2 + 2\xi_1 \xi_2 \xi| && (7.60) \\ &= |\xi||\xi_1||\xi_2||\xi_1^2 + \xi \xi_2| && (7.61) \\ &= |\xi||\xi_1||\xi_2||\xi^2 - \xi_1 \xi_2| && (7.62) \end{aligned}$$

For the real part we have

$$|\operatorname{Re}\sigma - \sigma_1 - \sigma_2| = |\xi^2 - \xi_1^2 - \xi_2^2| \tag{7.63}$$
$$= 2|\xi_1||\xi_2| \tag{7.64}$$

Note also that

$$\begin{aligned}\langle\sigma\rangle &= (1+|\sigma|^2)^{1/2}\\ &= (1+|\tau+\xi^5|^2+|\xi^2|^2)^{1/2}\\ &\geq (1+|\xi|^4)^{1/2}\\ &\sim \langle\xi\rangle^2\end{aligned} \tag{7.65}$$

Thus, we have the two estimates on the maximum

$$\max(|\sigma|,|\sigma_1|,|\sigma_2|) \gtrsim 2|\xi_1||\xi_2| \tag{7.66}$$
$$\max(|\sigma|,|\sigma_1|,|\sigma_2|) \gtrsim |\xi||\xi_1||\xi_2||\xi_1^2+\xi\xi_2| \tag{7.67}$$

Any of the three equalities (7.60), (7.61) and (7.62) can of course be used as needed in (7.67). For the integrals to come, we use the same partition of $\mathbb{R}^4$ as Chen & Li [7, p. 1139] and divide $\mathbb{R}^4$ in mainly three parts A, B and D.

Lemma 7.1. *For every $s > -2 + 25\delta/8$ and every $\delta > 0$ and any fixed $(\xi_1, \tau_1) \in \mathbb{R}^2$ with $|\xi_1| \gtrsim 1$ there exists $C > 0$ such that*

$$I_A = \int_A \frac{|\xi|^2\langle\xi\rangle^{2s-2\delta}\langle\xi_1\rangle^{-2s+2\delta}\langle\xi_2\rangle^{-2s+2\delta}}{\langle\sigma\rangle^{1-5\delta/4}\langle\sigma_1\rangle^{1+\delta}\langle\sigma_2\rangle^{1+\delta}} d\xi d\tau \leq C \tag{7.68}$$

where $A = A(\xi_1,\tau_1) = \{(\xi,\tau) \in \mathbb{R}^2 : |\xi| \leq 2|\xi_1|, |\xi_2| \gtrsim 1, |\sigma_1| \geq |\sigma_2|\}$.

Proof. Since $|\xi_1| \gtrsim |\xi|$ we have two cases to consider:
Either **1:** $|\xi_1| \gg |\xi_2|$ or **2:** $|\xi_1| \sim |\xi_2|$.
1. Here we have to consider two different cases, either $|\sigma_1| \geq |\sigma|$ or $|\sigma| \geq |\sigma_1|$
1.1 Consider first $|\sigma_1| \geq |\sigma|$. Then σ_1 is the maximum. Using that $|\xi_1| \gtrsim |\xi|$ we get from (7.67) that the maximum is bounded by

$$\begin{aligned}|\sigma_1| &\gtrsim |\xi||\xi_1||\xi_2||\xi_1^2+\xi\xi_2|\\ &\gtrsim |\xi||\xi||\xi_2||\xi_1^2+(\xi_1+\xi_2)\xi_2|\\ &\gtrsim |\xi||\xi||\xi_2||\xi_1^2+\xi_1\xi_2+\xi_2^2|\\ &\gtrsim |\xi||\xi||\xi_2||\xi_1^2+\xi_1\xi_2|\\ &\gtrsim |\xi||\xi||\xi_2||\xi_1(\xi_1+\xi_2)|\\ &\gtrsim |\xi||\xi||\xi_2||\xi_1\xi|\\ &\gtrsim |\xi|^4|\xi_2|\end{aligned} \tag{7.69}$$

And thus

$$\langle\sigma_1\rangle \gtrsim \langle\xi\rangle^4\langle\xi_2\rangle \tag{7.70}$$

The estimates for the maximum in the other lemmas has similar calculations as those for (7.69). We show it in full detail just once. Note that since $|\xi_1| \gg |\xi_2|$ and $\xi_1 + \xi_2 = \xi$ we must have $|\xi_1| \sim |\xi| \gg 1$ Using this, aswell as (7.70) and (7.65) we can estimate our integral:

$$\begin{aligned} I_A &= \int_A \frac{|\xi|^2\langle\xi\rangle^{2s-2\delta}\langle\xi_1\rangle^{-2s+2\delta}\langle\xi_2\rangle^{-2s+2\delta}}{\langle\sigma\rangle^{1-5\delta/4}\langle\sigma_1\rangle^{1+\delta}\langle\sigma_2\rangle^{1+\delta}} d\xi d\tau \\ &\lesssim \int_A \frac{\langle\xi\rangle^2\langle\xi\rangle^{2s-2\delta}\langle\xi_1\rangle^{-2s+2\delta}\langle\xi_2\rangle^{-2s+2\delta}}{\langle\xi\rangle^{2(1-5\delta/4)}\langle\xi\rangle^{4(1+\delta)}\langle\xi_2\rangle^{1+\delta}\langle\sigma_2\rangle^{1+\delta}} d\xi d\tau \end{aligned}$$

Since $|\xi_1| \sim |\xi|$ it follows that $\langle\xi\rangle^2 \sim \langle\xi_1\rangle^2$, and likewise $\langle\xi_1\rangle^{4(1-\delta)} \sim \langle\xi\rangle^{4(1-\delta)}$. Thus,

$$\lesssim \int_A \frac{\langle\xi_1\rangle^2\langle\xi_1\rangle^{2s-2\delta}\langle\xi_1\rangle^{-2s+2\delta}\langle\xi_2\rangle^{-2s+2\delta}}{\langle\xi_1\rangle^{2(1-5\delta/4)}\langle\xi_1\rangle^{4(1+\delta)}\langle\xi_2\rangle^{1+\delta}\langle\sigma_2\rangle^{1+\delta}} d\xi d\tau$$

Gathering these terms lead to

$$= \int_A \frac{1}{\langle\xi_1\rangle^{4+3\delta/2}\langle\xi_2\rangle^{2s+1-\delta}\langle\sigma_2\rangle^{1+\delta}} d\xi d\tau$$

Since $|\xi_1| \gg |\xi_2|$ implies $\langle\xi_1\rangle \gtrsim \langle\xi_2\rangle$, we replace $\langle\xi_1\rangle$ with $\langle\xi_2\rangle$ and gather the terms, which yields

$$\begin{aligned} &\lesssim \int_A \frac{1}{\langle\xi_2\rangle^{2s+5+\delta/2}} d\xi \int_{\mathbb{R}} \frac{1}{\langle\sigma_2\rangle^{1+\delta}} d\tau \\ &\leq C \end{aligned} \tag{7.71}$$

For $s > -2 - \delta/4$.

1.2 Consider now the case of $|\sigma| \geq |\sigma_1|$. Then $|\sigma|$ is the maximum, and we get the estimate from (7.70).

$$\langle\sigma\rangle \gtrsim \langle\xi\rangle^4\langle\xi_2\rangle$$

We have from (7.65) that

$$\langle\sigma_1\rangle \gtrsim \langle\xi_1\rangle^2$$

Using these in I_A yields

$$\begin{aligned} I_A &= \int_A \frac{|\xi|^2\langle\xi\rangle^{2s-2\delta}\langle\xi_1\rangle^{-2s+2\delta}\langle\xi_2\rangle^{-2s+2\delta}}{\langle\sigma\rangle^{1-5\delta/4}\langle\sigma_1\rangle^{1+\delta}\langle\sigma_2\rangle^{1+\delta}} d\xi d\tau \\ &\lesssim \int_A \frac{\langle\xi\rangle^2\langle\xi\rangle^{2s-2\delta}\langle\xi_1\rangle^{-2s+2\delta}\langle\xi_2\rangle^{-2s+2\delta}}{\langle\xi\rangle^{4(1-5\delta/4)}\langle\xi_2\rangle^{1-5\delta/4}\langle\xi_1\rangle^{2(1+\delta)}\langle\sigma_2\rangle^{1+\delta}} d\xi d\tau \end{aligned}$$

By the same arguments used in **1.1** and gathering the terms we get

$$\lesssim \int_A \frac{1}{\langle\xi_2\rangle^{2s+5-24\delta/4}} d\xi \int_{\mathbb{R}} \frac{1}{\langle\sigma_2\rangle^{1+\delta}} d\tau \tag{7.72}$$

when $s > -2 + 25\delta/8$. This concludes **1.**

2. $|\xi_1| \sim |\xi_2|$:

2.1 Assume first $|\sigma_1| \geq |\sigma|$. Then $|\sigma_1|$ is maximum. Since $2|\xi_1| \geq |\xi|$ and $|\xi_1| \sim |\xi_2|$ it follows that $|\xi_2| \sim |\xi_1| \sim |\xi|$. Thus we can estimate the maximum by

$$\begin{aligned} |\sigma_1| = \max(|\sigma|, |\sigma_1|, |\sigma_2|) &\gtrsim |\xi||\xi_1||\xi_2||\xi_1^2 + \xi\xi_2| \\ &\gtrsim |\xi_2|^5 \end{aligned}$$

which implies

$$\langle\sigma_1\rangle \gtrsim \langle\xi_2\rangle^5$$

We then have

$$\begin{aligned} I_A &= \int_A \frac{|\xi|^2\langle\xi\rangle^{2s-2\delta}\langle\xi_1\rangle^{-2s+2\delta}\langle\xi_2\rangle^{-2s+2\delta}}{\langle\sigma\rangle^{1-5\delta/4}\langle\sigma_1\rangle^{1+\delta}\langle\sigma_2\rangle^{1+\delta}} d\xi d\tau \\ &\lesssim \int_A \frac{\langle\xi\rangle^2\langle\xi\rangle^{2s-2\delta}\langle\xi_1\rangle^{-2s+2\delta}\langle\xi_2\rangle^{-2s+2\delta}}{\langle\xi\rangle^{2(1-5\delta/4)}\langle\xi_2\rangle^{5(1+\delta)}\langle\sigma_2\rangle^{1+\delta}} d\xi d\tau \\ &\sim \int_A \frac{\langle\xi_2\rangle^2\langle\xi_2\rangle^{2s-2\delta}\langle\xi_2\rangle^{-2s+2\delta}\langle\xi_2\rangle^{-2s+2\delta}}{\langle\xi_2\rangle^{2(1-5\delta/4)}\langle\xi_2\rangle^{5(1+\delta)}\langle\sigma_2\rangle^{1+\delta}} d\xi d\tau \\ &\sim \int_A \frac{1}{\langle\xi_2\rangle^{2s+5+\delta/2}} d\xi \int_{\mathbb{R}} \frac{1}{\langle\sigma_2\rangle^{1+\delta}} d\tau \\ &\leq C \end{aligned} \tag{7.73}$$

for $s > -2 - \delta/4$.
2.2 Assume now $|\sigma| \geq |\sigma_1|$. By the exact same argumentation we now get

$$\begin{aligned} I_A &\lesssim \int_A \frac{\langle \xi_2 \rangle^2 \langle \xi_2 \rangle^{2s-2\delta} \langle \xi_2 \rangle^{-2s+2\delta} \langle \xi_2 \rangle^{-2s+2\delta}}{\langle \xi_2 \rangle^{5(1-5\delta/4)} \langle \xi_2 \rangle^{2(1+\delta)} \langle \sigma_2 \rangle^{1+\delta}} d\xi d\tau \\ &\lesssim \int_A \frac{1}{\langle \xi_2 \rangle^{2s+5-25\delta/4}} d\xi \int_{\mathbb{R}} \frac{1}{\langle \sigma_2 \rangle^{1+\delta}} d\tau \\ &\leq C \end{aligned} \tag{7.74}$$

when $s > -2 + 25\delta/8$. This concludes the proof of Lemma 7.1 □

Lemma 7.2. *For every $s > -2 + 25\delta/8$ and every $\delta > 0$ and any fixed $(\xi, \tau) \in \mathbb{R}^2$ with $|\xi| \gtrsim 1$ there exists $C > 0$ such that*

$$I_B = \int_B \frac{|\xi|^2 \langle \xi \rangle^{2s-2\delta} \langle \xi_1 \rangle^{-2s+2\delta} \langle \xi_2 \rangle^{-2s+2\delta}}{\langle \sigma \rangle^{1-5\delta/4} \langle \sigma_1 \rangle^{1+\delta} \langle \sigma_2 \rangle^{1+\delta}} d\xi_1 d\tau_1 \leq C \tag{7.75}$$

where $B = B(\xi, \tau) = \{(\xi, \tau) \in \mathbb{R}^2 : |\xi| \geq 2|\xi_1|, |\xi_2| \gtrsim 1, |\xi_1| \gtrsim 1, |\sigma_1| \geq |\sigma_2|\}$.

Proof. Similarly, we have to estimate two different cases. Either **1.** $|\xi_2| \gg |\xi_1|$ or **2.** $|\xi_2| \sim |\xi_1|$. First case first:
1.1 We assume first that $|\sigma| \geq |\sigma_1|$. Since $|\xi_2| \gg |\xi_1|$ and $\xi = \xi_1 + \xi_2$ it follows that $|\xi_2| \sim |\xi|$. The estimate for the maximum then becomes

$$\begin{aligned} |\sigma| &\gtrsim |\xi||\xi_1||\xi_2||\xi_1^2 + \xi\xi_2| \\ &\gtrsim |\xi|^4 |\xi_1| \end{aligned}$$

and thus

$$\langle \sigma \rangle \gtrsim \langle \xi \rangle^4 \langle \xi_1 \rangle \tag{7.76}$$

We then get

$$\begin{aligned} I_B &= \int_B \frac{|\xi|^2 \langle \xi \rangle^{2s-2\delta} \langle \xi_1 \rangle^{-2s+2\delta} \langle \xi_2 \rangle^{-2s+2\delta}}{\langle \sigma \rangle^{1-5\delta/4} \langle \sigma_1 \rangle^{1+\delta} \langle \sigma_2 \rangle^{1+\delta}} d\xi_1 d\tau_1 \\ &\lesssim \int_B \frac{\langle \xi \rangle^2 \langle \xi_2 \rangle^{2s-2\delta} \langle \xi_1 \rangle^{-2s+2\delta} \langle \xi_2 \rangle^{-2s+2\delta}}{\langle \sigma \rangle^{1-5\delta/4} \langle \sigma_1 \rangle^{1+\delta} \langle \sigma_2 \rangle^{1+\delta}} d\xi_1 d\tau_1 \\ &\lesssim \int_B \frac{\langle \xi \rangle^2 \langle \xi_1 \rangle^{-2s+2\delta}}{\langle \xi \rangle^{4(1-5\delta/4)} \langle \xi_1 \rangle^{1-5\delta/4} \langle \xi_1 \rangle^{2(1+\delta)} \langle \sigma_2 \rangle^{1+\delta}} d\xi_1 d\tau_1 \\ &\lesssim \int_B \frac{1}{\langle \xi_1 \rangle^{2s+5-25\delta/4}} d\xi_1 \int_{\mathbb{R}} \frac{1}{\langle \sigma_2 \rangle^{1+\delta}} d\tau_1 \\ &\leq C \end{aligned} \tag{7.77}$$

for $s > -2 + 25\delta/8$.
1.2 Consider now $|\sigma_1| \geq |\sigma|$. Then $|\sigma_1|$ is maximum, and we use (7.76), aswell as the fact that $\langle\sigma\rangle \gtrsim \langle\xi\rangle^2$. So now we get

$$
\begin{aligned}
I_B &= \int_B \frac{|\xi|^2\langle\xi\rangle^{2s-2\delta}\langle\xi_1\rangle^{-2s+2\delta}\langle\xi_2\rangle^{-2s+2\delta}}{\langle\sigma\rangle^{1-5\delta/4}\langle\sigma_1\rangle^{1+\delta}\langle\sigma_2\rangle^{1+\delta}} d\xi_1 d\tau_1 \\
&\lesssim \int_B \frac{\langle\xi\rangle^2\langle\xi_2\rangle^{2s-2\delta}\langle\xi_1\rangle^{-2s+2\delta}\langle\xi_2\rangle^{-2s+2\delta}}{\langle\sigma\rangle^{1-5\delta/4}\langle\sigma_1\rangle^{1+\delta}\langle\sigma_2\rangle^{1+\delta}} d\xi_1 d\tau_1 \\
&\lesssim \int_B \frac{\langle\xi\rangle^2\langle\xi_1\rangle^{-2s+2\delta}}{\langle\xi\rangle^{2(1-5\delta/4)}\langle\xi\rangle^{4(1+\delta)}\langle\xi_1\rangle^{1+\delta}\langle\sigma_2\rangle^{1+\delta}} d\xi_1 d\tau_1 \\
&\lesssim \int_B \frac{1}{\langle\xi\rangle^{4+3\delta/2}\langle\xi_1\rangle^{2s+1-\delta}} d\xi_1 \int_{\mathbb{R}} \frac{1}{\langle\sigma_2\rangle^{1+\delta}} d\tau_1 \\
&\lesssim \int_B \frac{1}{\langle\xi_1\rangle^{2s+5+\delta/2}} d\xi_1 \\
&\leq C
\end{aligned} \tag{7.78}
$$

When $s > -2 - \delta/4$. This conlcudes case **1**.
2. Consider now $|\xi_2| \sim |\xi_1|$. Since we also have that $|\xi| \geq 2|\xi_1|$ it follows that $|\xi_2| \sim |\xi_1| \sim |\xi|$. The estimate for the maximum now becomes

$$
\begin{aligned}
\max(|\sigma|, |\sigma_1|, |\sigma_2|) &\gtrsim |\xi||\xi_1||\xi_2||\xi_1^2 + \xi\xi_2| \\
&\gtrsim |\xi_1|^5
\end{aligned} \tag{7.79}
$$

Assume first that $|\sigma_1| \geq |\sigma|$. Then we have $\langle\sigma_1\rangle \gtrsim \langle\xi\rangle^5$. As earlier, we arrive at

$$
\begin{aligned}
I_B &\lesssim \int_B \frac{\langle\xi\rangle^2\langle\xi_1\rangle^{-2s+2\delta}}{\langle\xi\rangle^{2(1-5\delta/4)}\langle\xi\rangle^{5(1+\delta)}\langle\sigma_2\rangle^{1+\delta}} d\xi_1 d\tau_1 \\
&\lesssim \int_B \frac{1}{\langle\xi_1\rangle^{2s+5+\delta/2}} d\xi_1 \int_{\mathbb{R}} \frac{1}{\langle\sigma_2\rangle^{1+\delta}} d\tau_1 \\
&\leq C
\end{aligned} \tag{7.80}
$$

when $s > -2 - \delta/4$. Assume now $|\sigma| \geq |\sigma_1|$. This case is again quite similar, as we now get

$$
\begin{aligned}
I_B &\lesssim \int_B \frac{\langle\xi\rangle^2\langle\xi_1\rangle^{-2s+2\delta}}{\langle\xi\rangle^{5(1-5\delta/4)}\langle\xi_1\rangle^{2(1+\delta)}\langle\sigma_2\rangle^{1+\delta}} d\xi_1 d\tau_1 \\
&\lesssim \int_B \frac{1}{\langle\xi_1\rangle^{2s+5-25\delta/4}} d\xi_1 \int_{\mathbb{R}} \frac{1}{\langle\sigma_2\rangle^{1+\delta}} d\tau_1 \\
&\leq C
\end{aligned} \tag{7.81}
$$

For $s > -2 + 25\delta/8$. This concludes the proof of lemma 7.2. □

We proceed to the last integral we need to evaluate

Lemma 7.3. *For every $s > -2 + 25\delta/8$ and every $\delta > 0$ and any fixed $(\xi, \tau) \in \mathbb{R}^2$, there exists $C > 0$ such that*

$$I_D = \int_D \frac{|\xi|^2 \langle \xi \rangle^{2s-2\delta} \langle \xi_1 \rangle^{-2s+2\delta} \langle \xi_2 \rangle^{-2s+2\delta}}{\langle \sigma \rangle^{1-5\delta/4} \langle \sigma_1 \rangle^{1+\delta} \langle \sigma_2 \rangle^{1+\delta}} d\xi_1 d\tau_1 \leq C \tag{7.82}$$

where $D = D(\xi, \tau) = \{(\xi_1, \tau_1) \in \mathbb{R}^2 : |\xi_1| \lesssim 1\}$.

Proof. We have two cases to consider. Contrary to our two other proofs, we do not need any estimates on the maximum. We only need the estimate from (7.65). Since $|\xi_1| \lesssim 1$ we have that $|\xi| \sim |\xi_2|$ and thus $\langle \xi \rangle \sim \langle \xi_2 \rangle$. Then we get

$$\begin{aligned}
I_D &\lesssim \int_D \frac{|\xi|^2 \langle \xi \rangle^{2s-2\delta} \langle \xi_1 \rangle^{-2s+2\delta} \langle \xi \rangle^{-2s+2\delta}}{\langle \xi \rangle^{2(1-5\delta/4)} \langle \xi_2 \rangle^{2(1+\delta)} \langle \sigma_1 \rangle^{1+\delta}} d\xi_1 d\tau_1 \\
&\lesssim \int_D \frac{|\xi|^2 \langle \xi_1 \rangle^{-2s+2\delta}}{\langle \xi \rangle^{2-5\delta/2)} \langle \xi_2 \rangle^{2+2\delta)}} d\xi_1 \int_{\mathbb{R}} \frac{1}{\langle \sigma_1 \rangle^{1+\delta}} d\tau_1 \\
&\lesssim \int_{|\xi_1| \lesssim 1} \frac{|\xi|^2}{\langle \xi \rangle^{4-\delta/2}} \cdot \frac{1}{\langle \xi_1 \rangle^{2s-2\delta}} d\xi_1 \\
&\lesssim \int_{|\xi_1| \lesssim 1} \frac{1}{\langle \xi_1 \rangle^{2s-2\delta}} d\xi_1 \\
&\leq C
\end{aligned} \tag{7.83}$$

Since we are integrating over $|\xi_1| \lesssim 1$ this integral is finite for any finite s, which concludes the proof of this lemma. □

Remark 7.1. *The same procedure could have been followed in Lemma 7.3 if $|\xi_2| \lesssim 1$, since they play the same part in the integral. We also covered the case of both $|\xi_1| \sim |\xi_2| \lesssim 1$, since that still implies $|\xi| \lesssim 1$ and thus $|\xi| \sim |\xi_1| \sim |\xi_2|$. Using the fact that $|\xi_2| \sim |\xi|$ still holds, we can do the exact same computation. The same goes for our assumptions in Lemma 7.1 and Lemma 7.2 aswell; the assumption that $|\sigma_1| \geq |\sigma_2|$ could have been $|\sigma_2| \geq |\sigma_1|$. Either way, the proof is similar and yields the same result.*

Lastly, we have this lemma which gives us the contraction factor.

Lemma 7.4. *Suppose* $0 < \beta < 1/2$, $\gamma \in \mathbb{R}$ *and* $\omega \in X^{\gamma,\beta}$ *with compact support in* $[0,T]$. *Define* $h(t,x)$ *by* $\widehat{h}(\xi,\tau) = \langle\sigma\rangle^{\beta}\langle\xi\rangle^{\gamma}\widehat{\omega}(\xi,\tau)$. *For any* $0 < \theta < \beta$ *there exists* $\Gamma > 0$ *such that*

$$\|\frac{\widehat{h}(\xi,\tau)}{\langle\sigma\rangle^{\theta}}\|_{L^2_{\xi,\tau}} \lesssim T^{\Gamma}\|h\|_{L^2_{x,t}} \tag{7.84}$$

The proof is quite straight forward, using interpolation and Sobolev embedding. We refer the reader to Han & Peng [11, Lemma 2] for full proof.

Continuing the proof of Proposition 7.9:

Recall that what we need to show is

$$I \le CT^{\Gamma}\|f\|_{L^2(\mathbb{R}^2)}\|g\|_{L^2(\mathbb{R}^2)}\|h\|_{L^2(\mathbb{R}^2)}$$

Where

$$I = \int_{\mathbb{R}^4} \frac{K(\xi,\xi_1,\tau,\tau_1)\widehat{h}(\xi,\tau)\widehat{f}(\xi_1,\tau_1)\widehat{g}(\xi_2,\tau_2)}{\langle\sigma\rangle^{\delta/8}} d\xi d\xi_1 d\tau d\tau_1$$

and

$$K(\xi,\xi_1,\tau,\tau_1) = \frac{|\xi|\langle\xi\rangle^{s-\delta}\langle\xi_1\rangle^{-s+\delta}\langle\xi_2\rangle^{-s+\delta}}{\langle\sigma\rangle^{1/2-5\delta/4}\langle\sigma_1\rangle^{1/2+\delta/2}\langle\sigma_2\rangle^{1/2+\delta/2}}$$

We divide $\mathbb{R}^4$ in six parts by $\mathbb{R}^4 = A \cup B \cup D \cup A_0 \cup B_0 \cup D_0$ where

$$\begin{aligned}
A &= \{(\xi,\xi_1,\tau,\tau_1) \in \mathbb{R}^4 : |\xi| \lesssim |\xi_1|, |\xi_1| \gtrsim 1, |\xi_2| \gtrsim 1, |\sigma_1| \ge |\sigma_2|\}\\
B &= \{(\xi,\xi_1,\tau,\tau_1) \in \mathbb{R}^4 : |\xi| \gtrsim |\xi_1|, |\xi_1| \gtrsim 1, |\xi_2| \gtrsim 1, |\sigma_1| \ge |\sigma_2|\}\\
D &= \{(\xi,\xi_1,\tau,\tau_1) \in \mathbb{R}^4 : |\xi_1| \lesssim 1 \quad \text{or} \quad |\xi_2| \lesssim 1 \quad \text{or both}, |\sigma_1| \ge |\sigma_2|\}\\
A_0 &= \{(\xi,\xi_1,\tau,\tau_1) \in \mathbb{R}^4 : |\xi| \lesssim |\xi_1|, |\xi_1| \gtrsim 1, |\xi_2| \gtrsim 1, |\sigma_2| \ge |\sigma_1|\}\\
B_0 &= \{(\xi,\xi_1,\tau,\tau_1) \in \mathbb{R}^4 : |\xi| \gtrsim |\xi_1|, |\xi_1| \gtrsim 1, |\xi_2| \gtrsim 1, |\sigma_2| \ge |\sigma_1|\}\\
D_0 &= \{(\xi,\xi_1,\tau,\tau_1) \in \mathbb{R}^4 : |\xi_1| \lesssim 1 \quad \text{or} \quad |\xi_2| \lesssim 1 \quad \text{or both}, |\sigma_2| \ge |\sigma_1|\}
\end{aligned}$$

Considering first the integral in A:
By Cauchy-Schwartz we get

$$\begin{aligned}
I_A \lesssim &\Big(\int_A |K(\xi,\xi_1,\tau,\tau_1)\widehat{f}(\xi_1,\tau_1)|^2 d\xi d\xi_1 d\tau d\tau_1\Big)^{1/2}\\
&\cdot \Big(\int_A |\frac{\widehat{h}(\xi,\tau)}{\langle\sigma\rangle^{\delta/8}}\widehat{g}(\xi_2,\tau_2)|^2 d\xi d\xi_1 d\tau d\tau_1\Big)^{1/2}
\end{aligned}$$

Taking supremum over all (ξ_1, τ_1) in the integral with $K(\xi, \xi_1, \tau, \tau_1)$ and by using Fubini in both integrals, we get

$$\lesssim \Big(\sup_{(\xi_1,\tau_1)} \int_A |K(\xi,\xi_1,\tau,\tau_1)|^2 d\xi d\tau \int_{\mathbb{R}^2} |\widehat{f}(\xi_1,\tau_1)|^2 d\xi_1 d\tau_1 \Big)^{1/2}$$
$$\cdot \Big(\int_{\mathbb{R}^2} |\frac{\widehat{h}(\xi,\tau)}{\langle\sigma\rangle^{\delta/8}}|^2 d\xi d\tau \int_{\mathbb{R}^2} |\widehat{g}(\xi-\xi_1,\tau-\tau_1)|^2 d\xi_1 d\tau_1 \Big)^{1/2}$$

By Plancherel we get

$$\lesssim \Big(\sup_{(\xi_1,\tau_1)} \big(\int_A |K(\xi,\xi_1,\tau,\tau_1)|^2 d\xi d\tau \big) \|f\|_{L^2}^2 \Big)^{1/2}$$
$$\cdot \Big(\int_{\mathbb{R}^2} |\frac{\widehat{h}(\xi,\tau)}{\langle\sigma\rangle^{\delta/8}}|^2 d\xi d\tau \|g\|_{L^2}^2 \Big)^{1/2}$$

Since we now have fixed (ξ_1, τ_1) we can use Lemma 7.1 on the first integral, and Lemma 7.4 on the second integral to conclude

$$\lesssim CT^{\Gamma} \|f\|_{L^2} \|h\|_{L^2} \|g\|_{L^2} \tag{7.85}$$

which completes the proof in A. By remark 7.1, proving the estimate of the integral in A_0 follows from the symmetry of $|\sigma_1|$ and $|\sigma_2|$. The integral in B is quite similar:
Since the order of the convolution does not matter, we interchange them. Using Cauchy Schwartz we again arrive at

$$I_B \lesssim \Big(\int_B |K(\xi,\xi_1,\tau,\tau_1)\widehat{f}(\xi_2,\tau_2)|^2 d\xi d\xi_1 d\tau d\tau_1 \Big)^{1/2}$$
$$\cdot \Big(\int_B |\frac{\widehat{h}(\xi,\tau)}{\langle\sigma\rangle^{\delta/8}} \widehat{g}(\xi_1,\tau_1)|^2 d\xi d\xi_1 d\tau d\tau_1 \Big)^{1/2}$$

By the same arguments as in the part with A we now get

$$\lesssim \Big(\sup_{(\xi,\tau)} \int_B |K(\xi,\xi_1,\tau,\tau_1)|^2 d\xi_1 d\tau_1 \int_{\mathbb{R}^2} |\widehat{f}(\xi-\xi_1,\tau-\tau_1)|^2 d\xi d\tau \Big)^{1/2}$$
$$\cdot \Big(\int_B |\frac{\widehat{h}(\xi,\tau)}{\langle\sigma\rangle^{\delta/8}}| d\xi d\tau \int_{\mathbb{R}^2} |\widehat{g}(\xi_1,\tau_1)|^2 d\xi_1 d\tau_1 \Big)^{1/2}$$

By Plancherel, Lemma 7.2 and Lemma 7.4 we again get

$$\lesssim CT^{\Gamma} \|f\|_{L^2} \|h\|_{L^2} \|g\|_{L^2} \tag{7.86}$$

The integral in B_0 is proven by symmetry. The integral in D follows by the same procedure as A and B. Thus, the proposition is proven. □

7.4 Contraction argument

Now we are ready to prove Theorem 7.1. As stated earlier, we consider the time-truncated version of the integral formulation of (2.3)

$$F(u) = \psi(t)\mathcal{F}_x^{-1}[\widehat{u_0}\widehat{W}(\xi,t) - 1_{\mathbb{R}^+}(t)\int_0^t \widehat{W}(\xi,(t-r))(\psi_T^2(r)\widehat{\partial_x(u^2)}(\xi,r))dr] \tag{7.87}$$

It is now also clear why we need to consider $\psi_T^2 u^2$ inside the integral, since Proposition 7.9 only holds for functions with compact support in time.
We do a standard contraction argument, similar to that of chapter 6. We will show that the mapping F is contractive in the closed ball of radius r, defined as

$$B(0,r) = \{u \in X^{s-\delta,1/2+\delta} : \|u\|_{X^{s-\delta,1/2+\delta/2}} \leq 2C\|u_0\|_{H^s} = r\}$$

First we show that the mapping F is from the closed ball to itself, i.e $F : B(0,r) \to B(0,r)$: By Propositions 7.8 and 7.9 there exists $\delta, C > 0$ such that

$$\|F(u)\|_{X^{s-\delta,1/2+\delta/2}} \leq C(\|u_0\|_{H^s} + \|\partial_x(\psi_T^2 u^2)\|^2_{X^{s-\delta,-1/2+\delta/2}}) \tag{7.88}$$

Setting $w = \psi_T u$ and using Proposition 7.9 on w^2 we know that there exists $T, \Gamma > 0$ such that

$$\leq C(\|u_0\|_{H^s} + T^\Gamma\|\psi_T u\|^2_{X^{s-\delta,1/2+\delta/2}})$$

From Proposition 7.5 we know what the $X^{s,b}$ spaces are stable with respect to time localization by a Schwartz function. Thus,

$$\leq C(\|u_0\|_{H^s} + T^\Gamma\|u\|^2_{X^{s-\delta,1/2+\delta/2}})$$

Setting $T = (8C^2\|u_0\|_{H^s})^{-\Gamma}$ we get

$$\leq C\Big(\|u_0\|_{H^s} + \frac{\|u\|^2_{X^{s-\delta,1/2+\delta/2}}}{8C^2\|u_0\|_{H^s}}\Big)$$

Using the restriction $\|u\|_{X^{s-\delta,1/2+\delta/2}} \leq 2C\|u_0\|_{H^s}$ we get

$$\leq 3/2C\|u_0\|_{H^s} \tag{7.89}$$

So the mapping F is indeed from the closed ball of radius r into it self. Thus, we can calculate the contraction argument. Using the same arguments as above, and the fact that $\partial_x(u^2) - \partial_x(v^2) = \partial_x\big((u-v)(u+v)\big)$ we get

$$\begin{aligned}&\|F(u) - F(v)\|_{X^{s-\delta,1/2+\delta/2}} \le CT^{\Gamma}\|u-v\|_{X^{s-\delta,1/2+\delta/2}}\|u+v\|_{X^{s-\delta,1/2+\delta/2}}\\ &\le CT^{\Gamma}\big(\|u\|_{X^{s-\delta,1/2+\delta/2}} + \|v\|_{X^{s-\delta,1/2+\delta/2}}\big)\|u-v\|_{X^{s-\delta,1/2+\delta/2}}\end{aligned}$$

Inserting for T we get

$$= C(8C^2\|u_0\|_{H^s})^{-1}\big(\|u\|_{X^{s-\delta,1/2+\delta/2}} + \|v\|_{X^{s-\delta,1/2+\delta/2}}\big)\|u-v\|_{X^{s-\delta,1/2+\delta/2}}$$

Using the restriction on the $X^{s,b}$ norms on u and v we get

$$\begin{aligned}&\le \frac{4C^2\|u_0\|_{H^s}}{8C^2\|u_0\|_{H^s}}\|u-v\|_{X^{s-\delta,1/2+\delta/2}}\\ &\le 1/2\|u-v\|_{X^{s-\delta,1/2+\delta/2}}\end{aligned}$$

So we have a contraction mapping in the ball $B = (0,r)$ for $r = 2C\|u_0\|_{H^s}$ and $T = (8C^2\|u_0\|_{H^s})^{-\Gamma}$, and thus we have a unique solution on the interval $[0,T]$ for $T = T(\|u_0\|_{H^s}) > 0$. $\square$

Proof of Theorem 7.1: We have from the preceding proposition that our solution is in $X^{s-2b+1,b}$. To show that it is also in $C([0,T], H^s)$, we need only refer to Proposition 7.4 which guarantees us that this embedding holds since we have $b > 1/2$.

8 Conclusion

The value for s in chapter 6 was found through an intuitive approach in Sobolev spaces and we managed to prove that there exists a unique solution of (2.3) in H^s for $s > 1/2$. By using the relatively modern (and quite techinical) Bourgain space we managed to lower this value significantly, and proved that we have a unique solution of (2.3) in H^s for $s > -2$.
We conclude this thesis with a small remark on the globality of the solution. We have only shown local existence, but it is probable that the solution can be extended globally by using a similar arguments as Han & Peng [11, Prop 5].

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