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Global Non-monotonicity of Solutions to Nonlinear Second-Order Differential Equations

Mervan Pašić and Yuriy V. Rogovchenko

Abstract. We study behavior of solutions to two classes of nonlinear second-order differential equations with a damping term. Sufficient conditions for the first derivative of a solution $x(t)$ to change sign at least once in a given interval (in a given infinite sequence of intervals) are provided. These conditions imply global non-monotone behavior of solutions.

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1. Introduction

1.1. Differential Equations and Main Assumptions

We are concerned with the following two classes of nonlinear second-order differential equations:

$$(r(t)x')' + p(t)x' + q(t)x + f(t,x) = e(t), \qquad t \geq t_0, \tag{1.1}$$

and

$$(r(t)x')' + p(t)x' + q(t)g(x) = e(t), \qquad t \geq t_0, \tag{1.2}$$

where $e(t)$ is a continuous non-homogeneous term, and nonlinear terms $f(t,x)$ and $g(x)$ satisfy, respectively, conditions

$$f(t,u)u \geq 0 \qquad \text{and} \qquad f(t,u) = -f(t,-u) \qquad \text{for all } t \geq t_0 \text{ and } u \in \mathbb{R}, \tag{1.3}$$

and

$$g'(u) \geq K > 0 \qquad \text{and} \qquad g(u) = -g(-u) \qquad \text{for all } u \in \mathbb{R}. \tag{1.4}$$

Assumptions (1.3) and (1.4) are satisfied, for instance, for Emden–Fowler differential equations where $f(t,u) = g(u) = |u|^{\nu}\text{sgn}(u)$, $\nu > 0$.

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Our main condition on the coefficients $r(t)$ and $q(t)$ in this paper is that there exists a sequence of intervals $[a_n, b_n]$, $t_0 < a_n < b_n \le a_{n+1}$, $n \in \mathbb{N}$, $a_n \to \infty$ as $n \to \infty$, such that

$$r(t) > 0 \qquad \text{and} \qquad q(t) > 0 \qquad \text{on } [a_n, b_n]. \tag{1.5}$$

Whenever necessary, this assumption is used on a single interval $[a, b]$, that is,

$$r(t) > 0 \qquad \text{and} \qquad q(t) > 0 \qquad \text{on } [a, b]. \tag{1.6}$$

1.2. Monotone and Non-monotone Solutions of Differential Equations

Monotonicity (or lack of it) is one of fundamental properties of solutions to differential equations. Exponential functions e^t and e^{-t} have been traditionally viewed as prototypes for monotone solutions due to the fact that they are fundamental solutions of linear differential equations with constant coefficients. Since e^{-t}, in addition to monotonicity, possesses other important characteristics (boundedness, positivity, decay to zero at infinity), these properties have been often studied alongside with the existence of monotone solutions.

Among pioneering results on monotone solutions of linear differential equation we would like to mention papers by Hartman and Wintner [11,12]; related theorems can be also found in Hartman's monograph [10, Chapter XIV, Part I]. In addition to monotonicity of solutions, monotonicity of certain functions of solutions has also been studied with the goal to represent some solutions in the form of Laplace–Stiltjes transforms of monotone functions. As demonstrated by Liberto Jannelli [19], extensions of monotonicity theorems from linear to nonlinear differential equations are essentially nontrivial. Several important results on monotone solutions to different classes of nonlinear differential equations and systems of differential equations have been obtained by Elias and Kreith [8], Kreith [14], Marini [20], Švec [35] and, more recently, by Cecchi et al. [3], Evtukhov and Klopot [9], Li and Fan [18], Tanigawa [36], Wang [40]; see also the bibliography in the cited papers.

Existence of non-monotone solutions to different classes of differential equations has been usually associated with their oscillatory nature since oscillating solutions are obviously non-monotone. Although lack of monotonicity has received much less attention in the literature, a number of interesting results in this direction have been reported by Cecchi et al. [3], Detki [5,6], Lepin [16,17] and other authors. It should be noted that non-monotone behavior is observed in many phenomena in applied sciences; see, for instance, the papers by Pašić [26] or Zhang [38] and the references cited therein. Complementing the aforementioned studies, in this paper we obtain sufficient conditions which ensure non-monotonicity of all extendable solutions to Eqs. (1.1) and (1.2).

1.3. Non-monotonicity of Smooth Functions

In what follows, we deal with real-valued functions $h = h(t)$ of the real variable t defined on (t_0, ∞) :

Definition 1.1. *A function* $h \in C^1(t_0, \infty)$ *is called non-monotone on* (t_0, ∞) *(a weakly oscillatory function on* (t_0, ∞)*) if there exists a sequence of points* $\{s_n\}_{n\in\mathbb{N}} \in (t_0, \infty)$, $s_n \to \infty$ *as* $n \to \infty$, *such that* $h'(t)$ *changes sign at* $t = s_n$ *for all* $n \in \mathbb{N}$.

Weakly oscillatory in the sense of Definition 1.1 solutions have been studied for several classes of differential and functional differential equations, often in relation to oscillation and monotonicity of solutions. In particular, Cecchi et al. [4] demonstrated that a linear second-order differential equation

$$(r(t)x'(t))' + q(t)x(t) = 0 \tag{1.7}$$

with positive coefficients possesses either oscillatory solutions or solutions which are eventually strictly monotone, but (1.7) does not posses weakly oscillatory solutions. Cecchi and Marini [2] presented sufficient conditions for every solution $x(t)$ of a nonlinear functional differential equation

$$(r(t)h(x)x'(t))' + q(t)f(x(g(t)) = 0$$

to be either oscillatory or weakly oscillatory, whereas similar results for a second-order nonlinear neutral differential equation

$$(a(t)(x(t) + bx(t-\tau_1) + cx(t+\tau_2))')' + p\,(t)\,x^{\alpha}(t-\sigma_1) + q(t)x^{\beta}(t+\sigma_2) = 0$$

were reported by Thandapani et al. [37]. Nonexistence of weakly oscillatory solutions for a third-order nonlinear functional differential equation

$$x'''(t) + q(t)x'(t) + r(t)f(x(t)) = 0$$

and for a nonlinear differential equation with the p-Laplacian

$$\left(|x'(t)|^{p-1}\,x'(t)\right)' + f\,(t, x(t), x'(t)) = 0, \qquad p \geq 1,$$

have been established, respectively, by Bartušek et al. [1] and Pekarkova [28].

However, in the research literature one can come across different interpretations of the term "weakly oscillatory solution". For instance, Mihalikova [21] calls a solution $(x_1(t), x_2(t))$ of a system of ordinary differential equations weakly oscillatory if at least one component is oscillatory, whereas Parhi and Padhi [24] use the term "weakly oscillatory differential equations" for differential equations which posses both oscillatory and non-oscillatory solutions. To avoid possible confusion, we prefer not to use the term "weakly oscillatory solutions" and refer to monotonicity properties instead.

We distinguish the following two types of non-monotonicity of a function $h\,(t)$ on an infinite interval (t_0, ∞) :

(i) $h(t)$ is non-monotone on (t_0, ∞) in the sense of Definition 1.1,

or

(ii) $\liminf_{t\to\infty} h(t) < \limsup_{t\to\infty} h(t)$.

It is not difficult to check that (ii) implies (i) because condition (ii) ensures existence of two sequences $\{t_n\}_{n\in\mathbb{N}}$ and $\{s_n\}_{n\in\mathbb{N}}$, $t_n < s_n < t_{n+1}$, $t_n \to \infty$ as $n \to \infty$, such that

$$h(t_n) < h(s_n) \qquad \text{and} \qquad h(s_n) > h(t_{n+1}), \qquad \text{for } n \in \mathbb{N}.$$

However, the converse statement is not true. Indeed, the first derivative of a positive function $h(t) = t^{-1}(2 + \sin t)$,

$$h'(t) = t^{-2} \left(t \cos t - \sin t - 2 \right),$$

changes sign infinitely many times on any interval (t_0, ∞), $t_0 \in \mathbb{R}$, and thus $h'(t)$ is a non-monotone on (t_0, ∞) function, but

$$\liminf_{t\to\infty} h(t) = \limsup_{t\to\infty} h(t) = 0.$$

Non-monotone solutions satisfying condition (ii) for Eqs. (1.1) and (1.2) have been studied recently by Pašić and Tanaka [27] for the linear case where $f(t, u) \equiv 0$ and $g(u) \equiv u$ respectively. In particular, the following important result has been proved:

Theorem 1.1 ([27, Theorem 2.5, part (ii)]). *Suppose that* $r(t) > 0$ *and* $q(t) \not\equiv 0$ *on* $[t_0, \infty)$ *and*

$$\int_{t_0}^{\infty} \frac{1}{r(s)} \int_{t_0}^{s} q(r)\, \mathrm{d}r\, \mathrm{d}s < \infty.$$

If there exists a solution $x_0(t)$ *of a linear differential equation*

$$(r(t)x')' + p(t)x' + q(t)x = e(t), \qquad t \geq t_0, \tag{1.8}$$

satisfying

$$\liminf_{t\to\infty} x(t) < \limsup_{t\to\infty} x(t), \tag{1.9}$$

then every positive bounded solution $x(t)$ *of Eq.* (1.8) *satisfies* (1.9) *and thus such solution* $x(t)$ *is non-monotone on* (t_0, ∞).

Unlike recent studies by Pašić [25] and Pašić and Tanaka [27], in this paper we do not restrict our discussion only to positive solutions $x(t)$ of Eqs. (1.1) and (1.2). Furthermore, our approach here differs from that in the latter paper because non-monotonicity of the type (ii) is deduced in [27] from the asymptotic analysis of a linear differential equation

$$(r(t)y'(t))' = e(t).$$

1.4. Reciprocal (Dual) Differential Equation

A differential equation associated with Eqs. (1.1) and (1.2),

$$\left(\frac{1}{q(t)} y'\right)' - \frac{p(t)}{q(t)r(t)} y' + \frac{K}{r(t)} y = 0, \qquad t \in (a, b), \tag{1.10}$$

where K is a positive constant defined in (1.4), is called a *reciprocal equation* (also known in the literature as *dual equation*). In particular, for Eq. (1.1), $K = 1$. Reciprocal equation (1.10) is a linear differential equation obtained from Eqs. (1.1) and (1.2) by replacing the coefficients $r(t)$ and $q(t)$ with the reciprocal to $q(t)$ and $r(t)$ functions $1/q(t)$ and $1/r(t)$, respectively; its solution $y \in \mathrm{C}^2(a, b)$.

Remark 1.1. In the literature, the reciprocal (dual) equation is usually defined on the whole half-line (t_0, ∞) because $r(t) \neq 0$ and $q(t) \neq 0$ for all $t \geq t_0$, see [4, the dual equation (E_2), p. 386] and [7, the reciprocal equation (1.2.11), p. 22]. This is not, however, possible in our case since condition (1.5) does not exclude the possibility for either $r(t)$, or $q(t)$, or both to have zeros at points which are not located in (a, b). Therefore, coefficients $1/q(t)$ and $1/r(t)$ may have singularities outside (a, b), as illustrated in the example that follows. For further discussion on the reciprocal (dual) equations on (t_0, ∞) in the case where $p(t) \equiv 0$, we refer the reader to the papers by Cecchi et al. [4] and Potter [29] and to the monograph by Došlý and Řehák [7].

Example 1.1. Consider a Hill differential equation

$$x'' + (\sin t)x' + (\cos t)x = 0. \tag{1.11}$$

Writing Eq. (1.11) in the self-adjoint form,

$$\left(\mathrm{e}^{-\cos t}x'\right)' + (\cos t)\,\mathrm{e}^{-\cos t}x = 0, \qquad t \in \mathbb{R}, \tag{1.12}$$

we observe that $x(t) = \mathrm{e}^{\cos t}$ is a positive non-oscillatory solution of Eq. (1.12). Hence, by Sturm separation theorem, every solution $x(t)$ of the linear differential equation (1.12) is also non-oscillatory. The reciprocal equation (1.10) associated with Eq. (1.12) has the following form:

$$\left(\frac{\mathrm{e}^{\cos t}}{\cos t}y'(t)\right)' + \mathrm{e}^{\cos t}y(t) = 0, \qquad t \in (a, b). \tag{1.13}$$

Observe that $r(t) = \mathrm{e}^{-\cos t} > 0$ and $q(t) = (\cos t)\,\mathrm{e}^{-\cos t} > 0$ on the interval $(a, b) = (-\pi/4 + 2n\pi, \pi/4 + 2n\pi)$, $n \in \mathbb{N}$, but the coefficient $1/q(t) = (\cos t)^{-1}\,\mathrm{e}^{\cos t}$ has infinitely many singular points $t_n = \pi/2 + n\pi$ located outside (a, b). Despite this, $y(t) = C \sin t$, $C \in \mathbb{R}$, is a one-parameter family of smooth solutions of the reciprocal equation (1.13).

2. Auxiliary Results

2.1. Critical Points and Monotonicity of Solutions

The following lemma demonstrates that if, under some additional conditions, the derivative $x'(t)$ of a solution $x(t)$ vanishes at a certain point $t_* \in [a, b]$, then t_* should not be an inflection point of $x(t)$.

Lemma 2.1. *Suppose that* (1.3) (*respectively,* (1.4)) *and* (1.6) *hold. Let* $x(t)$ *be a solution of* (1.1) (*respectively,* (1.2)) *such that*

$$|x(t)| + |e(t)| \neq 0 \qquad \text{and} \qquad x(t)e(t) \leq 0 \qquad \textit{on } [a, b]. \tag{2.1}$$

If $x'(t)$ *has a zero in* $[a, b]$, *then* $x'(t)$ *changes sign on* $(a - \varepsilon, b + \varepsilon)$ *for all sufficiently small* $\varepsilon > 0$.

Proof. By the assumptions of the lemma, there exists a $t_* \in [a, b]$ such that

$$x'(t_*) = 0. \tag{2.2}$$

Suppose, contrary to the claim of the lemma, that $x(t)$ is monotone on $(a-\delta, b+\delta)$ for some small $\delta>0$ and thus $x'(t)$ does not change sign on $(a-\delta, b+\delta)$, that is, t_* should be an inflection point, $x''(t_*)=0$. This implies that

$$(r(t)x'(t))'|_{t=t_*}=0, \tag{2.3}$$

and it follows from (1.1) (respectively, (1.2)), and (2.3) that

$$F(t_*, x(t_*))=e(t_*), \tag{2.4}$$

where

$$F(t,u)=\begin{cases} q(t)u+f(t,u), & \text{in the case of (1.1)},\\ q(t)g(u), & \text{in the case of (1.2)}.\end{cases}$$

Note that assumption $|x(t)|+|e(t)|\neq 0$ on $[a,b]$ excludes the possibility for $x(t)$ and $e(t)$ to vanish simultaneously at $t=t_*$. Hence, if $x(t_*)=0$ and $e(t_*)\neq 0$ (respectively, $e(t_*)=0$ and $x(t_*)\neq 0$), it follows from (1.3) (respectively, (1.4)) and (2.4) that $0=e(t_*)\neq 0$ (respectively, $0\neq F(t_*, x(t_*))=e(t_*)=0$), a contradiction. If both $x(t_*)\neq 0$ and $e(t_*)\neq 0$, then assumptions (1.3) (respectively, (1.4)), $x(t)e(t)\leq 0$, and $q(t)>0$ on $[a,b]$, imply that the left- and right-hand sides in (2.4) have different signs, which is also not possible. Therefore, $x'(t)$ has to change the sign on $(a-\varepsilon, b+\varepsilon)$ for all small enough $\varepsilon>0$. □

As an immediate consequence of Lemma 2.1, we derive the following result used in the sequel:

Corollary 2.1. *Let assumptions of Lemma 2.1 be satisfied. If a solution $x(t)$ of (1.1) (respectively, (1.2)) is monotone on $(a-\varepsilon, b+\varepsilon)$ for some $\varepsilon>0$, then $x'(t)\neq 0$ for all $t\in[a,b]$.*

2.2. Relation Between Zeros of $x'(t)$ and $y(t)$

In order to illustrate relationship between zeros of the solution $y(t)$ of Eq. (1.10) and the derivative $x'(t)$ of the solution $x(t)$ of Eq. (1.1) (respectively, (1.2)), consider once again Hill differential equation (1.12) along with the reciprocal equation (1.13). Observe that $y(t)=\sin t$ is an exact solution of (1.13) possessing zeros in $[a_n, b_n]=[-\pi/4+2n\pi, \pi/4+2n\pi]$, $n\in\mathbb{N}$. On the other hand, the first derivative $x'(t)$ of the exact solution $x(t)=\mathrm{e}^{\cos t}$ of (1.12) also has zeros in the intervals $[a_n, b_n]$ because $x'(t)=-(\sin t)\,\mathrm{e}^{\cos t}$ (in this particular case, exactly the same zeros $t_n=2\pi n$, $n\in\mathbb{N}$). The following lemma establishes that, in general, if $y(t)$ has a zero in (a_n, b_n), so does $x'(t)$, but zeros may not necessarily coincide.

Lemma 2.2 ([25, Theorem 3.1]). *Suppose that assumptions (1.3), (1.5) hold and*

$$(-1)^n e(t)\leq 0 \qquad \text{on } (a_n, b_n),\ n\in\mathbb{N}.$$

Assume further that

$$p^2(t)\leq 4r(t)q(t) \qquad \text{on } (a_n, b_n),\ n\in\mathbb{N}. \tag{2.5}$$

If every solution $y(t)$ of the reciprocal equation (1.10) *on* (a_n, b_n), $n \in \mathbb{N}$, *has a zero in* (a_n, b_n), $n \in \mathbb{N}$, *then the derivative* $x'(t)$ *of every positive solution* $x(t)$ *of* (1.1) (*respectively, of Eq.* (1.2)) *also has a zero in* (a_n, b_n), $n \in \mathbb{N}$.

Remark 2.1. If the reciprocal (dual) equation (1.10) is defined on the infinite interval (t_0, ∞) rather than on a bounded interval (a, b), the so-called reciprocity (duality) principle states that the main equation

$$(r(t)x')' + p(t)x' + q(t)x = 0, \qquad t \ge t_0,$$

is non-oscillatory if and only if Eq. (1.10) is non-oscillatory, see, for instance, Cecchi et al. [4, p. 386] or Potter [29, p. 474]. Since in our case the reciprocal equation (1.10) is defined only on a sequence of intervals (a_n, b_n), $n \in \mathbb{N}$, a similar reciprocity principle cannot be applied and a different result is required instead. Unlike standard reciprocity principles, Lemma 2.2 relates zeros of solutions $y(t)$ of (1.10) and zeros of the derivatives $x'(t)$ of solutions $x(t)$ of (1.1) rather than zeros of solutions.

Using a local result which holds on (a, b) rather than on the whole half-axis (t_0, ∞), we observe that condition (2.5) in Lemma 2.2 is satisfied automatically. Introducing a new function

$$\Theta(t) = \exp\left(\int_a^t \frac{p(\tau)}{r(\tau)}\,\mathrm{d}\tau\right), \qquad t \in (a, b), \tag{2.6}$$

we eliminate damping terms in Eqs. (1.1), (1.2), and (1.10):

$$(\Theta(t)r(t)x')' + \Theta(t)q(t)x + \Theta(t)f(t, x) = \Theta(t)e(t), \qquad t \in (a, b), \tag{2.7}$$

$$(\Theta(t)r(t)x')' + \Theta(t)q(t)g(x) = \Theta(t)e(t), \qquad t \in (a, b), \tag{2.8}$$

$$\left(\frac{1}{\Theta(t)q(t)}y'\right)' + \frac{K}{\Theta(t)r(t)}y = 0, \qquad t \in (a, b). \tag{2.9}$$

Note that every solution $x(t)$ of (1.1) on (a, b) (respectively, solution $y(t)$ of (1.10)) also satisfies Eq. (2.7) (respectively, Eq. (2.9)) and vice versa; this is also valid for solutions $x(t)$ of (1.2) and solutions $y(t)$ of (1.10). The same relation holds for Eqs. (1.2) and (2.8). Furthermore, (2.9) is a reciprocal equation to (2.7) as well; coefficients of (2.7) satisfy (2.5) since the damping term in (2.7) is equal to zero and other coefficients are positive functions. Thus, applying Lemma 2.2 directly to Eqs. (2.7) and (2.9) instead of Eqs. (1.1) and (1.10), we do not need anymore assumption (2.5). This trick, however, does not work for oscillation criteria where damping terms play important role, see, for instance, Mustafa et al. [23], Rogovchenko and Rogovchenko [30,31], Rogovchenko [32], or Rogovchenko and Tuncay [34].

Lemma 2.3. *Suppose that conditions* (1.3) (*respectively,* (1.4)) *and* (1.6) *hold. If every solution $y(t)$ of the reciprocal equation* (1.10) *has a zero in* $[a, b]$, *then for every solution $x(t)$ of* (1.1) (*respectively,* (1.2)) *satisfying condition* (2.1) *on* $[a, b]$ *the derivative $x'(t)$ changes sign on* $(a-\varepsilon, b+\varepsilon)$ *for all small enough* $\varepsilon = \varepsilon(x) > 0$.

Proof. Let $x(t)$ be a solution of Eq. (1.1) (respectively, (1.2)) satisfying condition (2.1) on $[a,b]$. Suppose that the conclusion of lemma is false, then there exists a $\delta > 0$ such that $x(t)$ is a monotone function on $(a-\delta, b+\delta)$. Let

$$w_a \stackrel{\text{def}}{=} \frac{G(x(a))}{\Theta(a)r(a)x'(a)} \in \mathbb{R},$$

where

$$G(u) \stackrel{\text{def}}{=} \begin{cases} u, & \text{for Eq. (1.1)}, \\ g(u), & \text{for Eq. (1.2)}. \end{cases}$$

By Corollary 2.1, $x'(t) \neq 0$ for all $t \in [a,b]$ and the functions

$$u(t) = w_a \qquad \text{and} \qquad v(t) = \frac{G(x(t))}{\Theta(t)r(t)x'(t)}$$

are well-defined and continuously differentiable for all $t \in [a,b]$. Taking into account that

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 0, \qquad \Theta(t)q(t) > 0, \qquad \text{and} \qquad \frac{1}{\Theta(t)r(t)} > 0, \qquad t \in [a,b],$$

we conclude that, for all $t \in [a,b]$,

$$\frac{\mathrm{d}u}{\mathrm{d}t} < \Theta(t)q(t)u^2 + \frac{K}{\Theta(t)r(t)} \qquad \text{and} \qquad u(a) = w_a, \tag{2.10}$$

where K is a positive constant defined in (1.4). On the other hand, since $\Theta(t) > 0$, $r(t) > 0$, $q(t) > 0$ and $G(x(t))e(t) \leq 0$ on $[a,b]$, by virtue of (2.7) and (2.8), we deduce that, for all $t \in [a,b]$,

$$\frac{\mathrm{d}v}{\mathrm{d}t} \geq \Theta(t)q(t)v^2 + \frac{K}{\Theta(t)r(t)} \qquad \text{and} \qquad v(a) = w_a. \tag{2.11}$$

Since the right-hand side of the inequality (2.11) is strictly positive, the function $v(t)$ is increasing on $[a,b]$ and thus,

$$u(t) = w_a = v(a) < v(t), \qquad t \in [a,b]. \tag{2.12}$$

It follows from (2.10) to (2.12) and [15, Theorem 1.2.1] (see also [22]) that there exists a function $w \in C^1([a,b])$ such that, for all $t \in [a,b]$,

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \Theta(t)q(t)w^2 + \frac{K}{\Theta(t)r(t)} \qquad \text{and} \qquad w(a) = w_a.$$

It is not difficult to check that the function

$$y(t) = \exp\left(-\int_a^t \Theta(s)q(s)w(s)\,\mathrm{d}s\right)$$

satisfies the reciprocal equation (2.9) for all $t \in [a,b]$ and thus it satisfies Eq. (1.10) as well. Since $y(t) > 0$ for all $t \in [a,b]$, this contradicts the assumption of the lemma stating that every solution $y(t)$ of the reciprocal equation (1.10) has a zero in $[a,b]$. Thus, $x(t)$ should be non-monotone on $(a-\varepsilon, b+\varepsilon)$ for all small enough $\varepsilon > 0$. The proof is complete now. $\square$

Example 2.1. To illustrate Lemma 2.3, consider a linear second-order homogeneous differential equation

$$x'' + \frac{\sin t}{\sin t + 2}\, x = 0, \qquad t \in \mathbb{R}, \tag{2.13}$$

and the associated reciprocal equation

$$\left(\frac{\sin t + 2}{\sin t}\, y' \right)' + y = 0, \qquad t \in (a, b), \tag{2.14}$$

where $0 < a < \pi/2 < b < \pi$. Clearly,

$$q(t) = \frac{\sin t}{\sin t + 2} > 0$$

on $[a, b]$. Since $e(t) \equiv 0$ in Eq. (2.13), condition (2.1) reads as $x(t) \neq 0$, $t \in [a, b]$. It is easy to see that $x(t) = \sin t + 2$ is a solution of (2.13) such that $x(t) > 0$ on $[a, b]$ and $x'(t_*) = 0$ for $t_* = \pi/2 \in [a, b]$. Observe that $y(t) = \cos t$ is a solution of the reciprocal equation (2.14) that vanishes at the same point $t_* = \pi/2 \in [a, b]$.

With a similar argument as in Lemma 2.3, one can also derive the following result:

Lemma 2.4. *Let conditions* (1.3) (*respectively,* (1.4)) *and* (1.5) *be satisfied, and* $[a_n, b_n]$, $n \in \mathbb{N}$, *be a sequence of intervals defined in* (1.5). *If every solution* $y(t)$ *of the reciprocal equation* (1.10) *has a zero in each of the intervals* $[a_n, b_n]$, *then every solution* $x(t)$ *of* (1.1) (*respectively, of Eq.* (1.2)) *satisfying* (2.1) *on* $[a_n, b_n]$ *is non-monotone on* (t_0, ∞).

Note that in order to satisfy the assumption of Lemma 2.4 that every solution $y(t)$ of (1.10) on (a_n, b_n) has a zero in $[a_n, b_n]$, one can apply to the reciprocal equation (1.10) any appropriate interval oscillation criteria which guarantee location of zeros in a given sequence of intervals, see Kong and Pašić [13] or Rogovchenko and Tuncay [33] and the references cited therein.

2.3. Location of Zeros of $x(t)$ and $y(t)$ in a Given Interval

As pointed out in Sect. 2.2, in this paper we are interested in sufficient conditions on $r(t)$, $p(t)$, and $q(t)$ which ensure that solutions $y(t)$ of (1.10) and the derivatives $x'(t)$ of solutions $x(t)$ of (1.1) (respectively, of Eq. (2.9)) have zeros located in a sequence of intervals $[a_n, b_n]$, $n \in \mathbb{N}$, whereas solutions $x\,(t)$ may not vanish in these intervals. In fact, referring again to Eq. (1.12), recall that every member of a one-parameter family of solutions $y(t) = C \sin(t)$, $C \neq 0$ of reciprocal equation (1.13) has a zero in $[a_n, b_n]$ for all $n \in \mathbb{N}$, yet all solutions $x(t)$ of (1.12) are non-oscillatory.

However, in this section, we prove a criterion which guarantees that both solutions $y(t)$ of (1.10) and $x(t)$ of (1.1) vanish at least once in a given interval.

Theorem 2.1. *Let* $\Theta(t)$ *be defined as in* (2.6). *Assume that conditions* (1.3) *and* (1.6) *hold. If there exists a positive number* γ *such that*

$$\frac{1}{\pi} \int_a^b \min\left\{ \frac{\Theta(t) q(t)}{\gamma}, \frac{\gamma}{\Theta(t) r(t)} \right\} \mathrm{d}t \geq 1, \tag{2.15}$$

then all solutions $x(t)$ of (1.1) *satisfying* (2.1) *on* $[a,b]$ *and all solutions* $y(t)$ *of the associated reciprocal equation* (1.10) *have at least one zero in* $[a,b]$.

In the proof of Theorem 2.1, we exploit a Riccati transformation and a comparison principle. We also use properties of the integral

$$I(t)=\int_a^t \min\left\{\frac{\Theta(s)q(s)}{\gamma},\frac{\gamma}{\Theta(s)r(s)}\right\}\mathrm{d}s, \qquad t\in[a,b], \tag{2.16}$$

and of the function

$$\varphi(t)=\tan\left(I(t)+\arctan\omega_0\right), \qquad t\in[a,T^*), \tag{2.17}$$

where ω_0, $T^*\in\mathbb{R}$ and $T^*>a$. We start by proving the following auxiliary result:

Lemma 2.5. *Suppose that* (1.6) *is satisfied and let $\gamma>0$ be a real number such that* (2.15) *holds. Then, for every $\omega_0\in\mathbb{R}$, there exists a $T^*\in(a,b)$ such that the function $\varphi(t)$ defined in* (2.17) *has the following properties:*

$$\begin{cases} \frac{\mathrm{d}\varphi}{\mathrm{d}t}\le\frac{\gamma}{\Theta(t)r(t)}\varphi^2+\frac{\Theta(t)q(t)}{\gamma}, & t\in(a,T^*),\\ \frac{\mathrm{d}\varphi}{\mathrm{d}t}\le\frac{\Theta(t)q(t)}{\gamma}\varphi^2+\frac{\gamma}{\Theta(t)r(t)}, & t\in(a,T^*),\\ \varphi(a)=\omega_0, & \lim_{t\to T^*}\varphi(t)=\infty. \end{cases} \tag{2.18}$$

Proof. It follows from (2.15) and (2.16) that the function $I(t)$ is continuous on $[a,b]$, $I(a)=0$, and $I(b)\ge\pi$. Therefore,

$$I(a)+\arctan\omega_0=\arctan\omega_0<\frac{\pi}{2}<\pi+\arctan\omega_0\le I(b)+\arctan\omega_0.$$

Since $I(t)$ is continuous on $[a,b]$, there exists a $T^*\in(a,b)$ such that

$$I(T^*)+\arctan\omega_0=\frac{\pi}{2}$$

and thus,

$$\lim_{t\to T^*}\varphi(t)=\infty. \tag{2.19}$$

An elementary calculation yields

$$\begin{aligned}\frac{\mathrm{d}\varphi}{\mathrm{d}t}=\frac{1}{\cos^2(I(t)+\arctan\omega_0)}\frac{\mathrm{d}I}{\mathrm{d}t} &= \left[1+\tan^2\left(I(t)+\arctan\omega_0\right)\right]\frac{\mathrm{d}I}{\mathrm{d}t}\\ &=\left[1+\varphi^2(t)\right]\frac{\mathrm{d}I}{\mathrm{d}t}\\ &=\left[1+\varphi^2(t)\right]\min\left\{\frac{\Theta(s)q(s)}{\gamma},\frac{\gamma}{\Theta(s)r(s)}\right\}.\end{aligned} \tag{2.20}$$

The validity of both differential inequalities in (2.18) follows immediately from (2.20) and assumption (2.15). The proof of the lemma is complete.

□

Proof of Theorem 2.1. Let $x(t)$ and $y(t)$ be solutions of Eqs. (1.1) and (1.10). Suppose, contrary to the claim of the theorem, that $x(t) \neq 0$ and $y(t) \neq 0$ on $[a,b]$, in which case the functions

$$\xi(t) = -\frac{\gamma\Theta(t)r(t)x'(t)}{x(t)} \qquad \text{and} \qquad \psi(t) = -\frac{\gamma y'(t)}{\Theta(t)q(t)y(t)}$$

are well defined on $[a,b]$. Now, using assumptions (2.7) and (2.9), we conclude that $\xi, \psi \in C^1([a,b])$ and, for all $t \in [a,b]$,

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} \geq \frac{\gamma}{\Theta(t)r(t)}\xi^2 + \frac{\Theta(t)q(t)}{\gamma} \qquad \text{and} \qquad \frac{\mathrm{d}\psi}{\mathrm{d}t} \leq \frac{\Theta(t)q(t)}{\gamma}\psi^2 + \frac{\gamma}{\Theta(t)r(t)}. \tag{2.21}$$

Choosing first $\omega_0 = \xi(a)$ and then $\omega_0 = \psi(a)$, using (2.18), (2.21), and the comparison principle [39], we deduce that $\varphi(t) \leq \xi(t)$ and $\varphi(t) \leq \psi(t)$ for $t \in [a,b]$. In particular, it follows that $\varphi \in C^1([a,b])$, which contradicts condition (2.19). Therefore, our assumption is wrong and both $x(t)$ and $y(t)$ should vanish at least once in $[a,b]$. The proof for Eq. (2.9) follows the same lines. □

3. Main Results

We provide first a test for a non-monotone behavior of solutions of Eqs. (1.1) and (1.2). Let the intervals $[a_n, b_n]$, $n \in \mathbb{N}$, and the function $\Theta(t)$ be defined in (1.5) and (2.6), respectively.

Theorem 3.1. *Let $t_0 > 0$, and assume that (1.3) (respectively, (1.4)) and (1.5) hold. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be two sequences of positive numbers such that, for all $n \in \mathbb{N}$ and for all $t \in [a_n, b_n]$,*

$$\beta_n \geq \alpha_n \left(\frac{1}{4} + \frac{\pi^2}{\ln^2(b_n/a_n)} \right) \tag{3.1}$$

and

$$\frac{\beta_n}{q(t)} \leq \Theta(t) \leq \frac{\alpha_n K t^2}{r(t)}. \tag{3.2}$$

Then every solution $x(t)$ of Eq. (1.1) (respectively, of Eq. (1.2)) satisfying for $t \in [a_n, b_n]$ condition (2.1) is non-monotone on (t_0, ∞).

Proof. For $t > 0$ and $n \in \mathbb{N}$, consider the Euler differential equation

$$\frac{1}{\beta_n} z'' + \frac{1}{\alpha_n t^2} z = 0. \tag{3.3}$$

If $M_n = \beta_n/\alpha_n$, assumption (3.1) yields

$$M_n \geq \frac{1}{4} + \frac{\pi^2}{\ln^2(b_n/a_n)}, \qquad n \in \mathbb{N}.$$

Since $M_n > 1/4$, the numbers $\rho_n = \sqrt{M_n - 1/4}$ are well-defined and

$$\rho_n \geq \frac{\pi}{\ln(b_n/a_n)} > 0, \qquad n \in \mathbb{N}. \tag{3.4}$$

Observe that, by virtue of (3.4), the function

$$z(t) = \sqrt{t}\sin(\rho_n \ln(t/a_n)) \tag{3.5}$$

is a solution of Eq. (3.3) satisfying

$$z(a_n) = z(t_n) = 0, \tag{3.6}$$

where $t_n = a_n \exp(\pi/\rho_n) \in (a_n, b_n]$. Furthermore, a_n and t_n are two consecutive zeros of $z(t)$ in $[a_n, b_n]$ for all $n \in \mathbb{N}$, since for every $t \in (a_n, b_n)$ there exists an $s \in (0, \pi)$ such that $t = a_n \exp(s/\rho_n)$ and hence, $z(t) > 0$.

Next, it follows from the assumption (3.2) that for $t \in [a_n, b_n]$,

$$\frac{1}{\Theta(t)q(t)} \le \frac{1}{\beta_n} \qquad \text{and} \qquad \frac{K}{\Theta(t)r(t)} \ge \frac{1}{\alpha_n t^2}. \tag{3.7}$$

Using (3.7) and applying Sturm comparison theorem to the reciprocal equation (2.9) and Euler equation (3.3), we deduce that every solution $y(t)$ of equation (2.9) has at least one zero between two consecutive zeros $t = a_n$ and $t = t_n$ of a solution $z(t)$ of Eq. (3.3) in each interval $[a_n, b_n]$, $n \in \mathbb{N}$. An application of Lemma 2.4 completes the proof of the theorem. □

Example 3.1. For $t \ge 1$, consider a nonlinear differential equation

$$\left(\alpha_0 t^k \sin(\ln t) x'\right)' + \alpha_0 t^{k-1} \sin(\ln t)\, x' + t^l x + g(t)|x|^m \operatorname{sgn}(x) = e(t), \tag{3.8}$$

where $\alpha_0 \in (0, 2/5]$, $k \le 1$, $l \ge 0$, $m > 0$, and $g(t) \ge 0$ on $[1, \infty)$. We claim that all assumptions of Theorem 3.1 are satisfied with

$$\alpha_n = \alpha_0, \qquad \beta_n = \frac{5}{2}\alpha_0, \qquad a_n = e^{2n\pi + \pi/6}, \qquad \text{and} \qquad b_n = e^{2n\pi + 5\pi/6}.$$

In fact, note that $t_0 = 1$, $r(t) = \alpha_0 t^k \sin(\ln t) > 0$, $p(t) = \alpha_0 t^{k-1} \sin(\ln t)$, $q(t) = t^l > 0$, and $\Theta(t) = t/a_n$ on $[a_n, b_n]$. Since

$$\beta_n = 5\alpha_0/2 \le 1 \le e^{(2n\pi + \frac{\pi}{6})l} = a_n^l,$$

we have $a_n \beta_n \le a_n^{l+1} \le t^{l+1}$ for $t \in [a_n, b_n]$. This establishes the estimate from below for $\Theta(t)$ in (3.2), $\beta_n/q(t) \le \Theta(t)$, $t \in [a_n, b_n]$.

Next, since $k - 1 \le 0$ and $a_n > 1$, we have

$$0 < \alpha_0 t^{k-1} \sin(\ln t) \le \alpha_0 a_n^{k-1} \le \alpha_n a_n,\ t \in [a_n, b_n],$$

that is, $r(t) \le \alpha_n a_n t$ for $t \in [a_n, b_n]$, so the estimate from above for $\Theta(t)$ in (3.2) also holds, $\Theta(t) \le \alpha_n K t^2 / r(t)$, where $K = 1$.

Finally, observe that

$$\beta_n = \frac{5}{2}\alpha_0 = \frac{5}{2}\alpha_n = \alpha_n \left(\frac{1}{4} + \frac{\pi^2}{\ln^2(b_n/a_n)}\right),$$

and thus (3.1) holds. Therefore, by Theorem 3.1, every solution $x(t)$ of equation (3.8) satisfying (2.1) on $[a_n, b_n]$ is non-monotone on (t_0, ∞).

Remark 3.1. Recall that we have used Euler differential equation (3.3) in the proof of Theorem 3.1. For the solution (3.5) of Eq. (3.3) satisfying (3.6), condition $t_n \le b_n$ is equivalent to the inequality (3.1). Letting $\alpha_n = \alpha_0 > 0$ in (3.1), we conclude that

$$\liminf_{n\to\infty} \beta_n \geq \frac{\alpha_0}{4} > 0. \tag{3.9}$$

Application of Sturm comparison theorem to Eqs. (1.1) and (2.9) requires that

$$\frac{1}{\alpha_n t^2 q(t)} \leq \Theta(t) \leq \frac{1}{\beta_n r(t)}. \tag{3.10}$$

However, if $\Theta(t)r(t)$ becomes unbounded as $t \to \infty$, inequalities (3.10) yield

$$\liminf_{n\to\infty} \beta_n = 0,$$

which contradicts (3.9). For instance, condition (3.9) holds in Example 3.1, and, on the other hand,

$$\begin{aligned}\limsup_{t\to\infty}(\Theta(t)r(t)) &= \alpha_0 \limsup_{t\to\infty}(t^{k+1}\sin(\ln t)) \\ &= \alpha_0 \limsup_{n\to\infty} e^{(k+1)(2n+1/2)} = \infty,\end{aligned}$$

that is, $\Theta(t)r(t)$ is unbounded as $t \to \infty$. Therefore, even for linear equation (1.8), application of Sturm comparison theorem to Euler differential equation (3.3) does not lead to the situation described in Sect. 2.3.

The following result demonstrates how an oscillation criterion like Theorem 2.1 can be turned into a non-monotonicity test.

Theorem 3.2. *Suppose that conditions* (1.3) *and* (1.5) *hold, and let assumption* (2.15) *be satisfied on a sequence of intervals* $[a_n, b_n]$, $n \in \mathbb{N}$, *defined in* (1.5). *Then every solution* $x(t)$ *of Eq.* (1.1) *satisfying for* $t \in [a_n, b_n]$ *condition* (2.1) *is non-monotone on* (t_0, ∞).

Proof. Observe that the first derivative $x'(t)$ of any nontrivial oscillatory function $x(t)$ changes sign infinitely many times on (t_0, ∞) and thus, $x(t)$ is non-monotone on (t_0, ∞). On the other hand, applying Theorem 2.1 on the intervals $[a_n, b_n]$, we conclude that every nontrivial solution $x(t)$ of equation (1.1) has at least one zero in each of the intervals $[a_n, b_n]$. This implies that $x(t)$ is a nontrivial oscillatory function and thus $x(t)$ is non-monotone on (t_0, ∞). Finally, note that every solution $x(t)$ of Eq. (1.1) satisfying for $t \in [a_n, b_n]$ condition (2.1) is a nontrivial function. Otherwise Eq. (1.1) would yield $e(t) \equiv 0$, which is not possible due to condition (2.1). This completes the proof of the theorem. □

Example 3.2. Let $t_0 \in \mathbb{R}$ and let ω satisfy $0 < 3\omega \leq (3/4)^{1/4}$. Consider a nonlinear differential equation

$$x'' + \sin(\omega t)x + f(t,x) = e(t), \qquad t \geq t_0, \tag{3.11}$$

where the function $f(t,x)$ satisfies (1.3). We claim that for γ such that

$$1 \leq \gamma \leq \frac{1}{\pi\omega} \qquad \text{or} \qquad 3\omega \leq \gamma \leq \left(\frac{3}{4}\right)^{1/4},$$

all assumptions of Theorem 3.2 are met with

$$a_n = \frac{1}{\omega}\left(\frac{\pi}{3} + 2n\pi\right) \qquad \text{and} \qquad b_n = \frac{1}{\omega}\left(\frac{2\pi}{3} + 2n\pi\right).$$

In fact, for both choices of γ, we conclude that inequalities $0 < 3\omega \leq (3/4)^{1/4} < 3/\pi$ imply $1 \leq 1/(\pi\omega)$ and $3\omega \leq (3/4)^{1/4}$. Furthermore, since $r(t) \equiv 1$ and $p(t) \equiv 0$, we have $\Theta(t) \equiv 1$ and the left-hand side of (2.15) assumes the form

$$\frac{1}{\pi}\int_{a_n}^{b_n} \min\left\{\frac{\Theta(t)q(t)}{\gamma}, \frac{\gamma}{\Theta(t)r(t)}\right\} \mathrm{d}t = \frac{1}{\pi}\int_{a_n}^{b_n} \min\left\{\frac{\sin(\omega t)}{\gamma}, \gamma\right\} \mathrm{d}t.$$

We need to verify condition (2.15). To this end, let first $1 \leq \gamma \leq 1/(\pi\omega)$. Then $1 \leq \gamma^2$ and $\sin(\omega t)/\gamma \leq \gamma$. This yields

$$\begin{aligned}\frac{1}{\pi}\int_{a_n}^{b_n} \min\left\{\frac{\Theta(t)q(t)}{\gamma}, \frac{\gamma}{\Theta(t)r(t)}\right\} \mathrm{d}t &= \frac{1}{\pi}\int_{a_n}^{b_n} \frac{\sin(\omega t)}{\gamma}\,\mathrm{d}t \\ &= \frac{1}{\pi\gamma\omega} \geq 1, \qquad n \in \mathbb{N}.\end{aligned}$$

Suppose now that $0 < 3\omega \leq \gamma \leq (3/4)^{1/4}$. In this case,

$$\sin(\omega t) \geq \min_{[a_n,b_n]} \sin(\omega t) = \frac{\sqrt{3}}{2} \geq \gamma^2 \qquad \text{on} \quad (a_n, b_n),$$

that is, $\sin(\omega t)/\gamma \geq \gamma$ on (a_n, b_n), and hence,

$$\begin{aligned}\frac{1}{\pi}\int_{a_n}^{b_n} \min\left\{\frac{\Theta(t)q(t)}{\gamma}, \frac{\gamma}{\Theta(t)r(t)}\right\} \mathrm{d}t &= \frac{1}{\pi}\int_{a_n}^{b_n} \gamma\,\mathrm{d}t \\ &= \frac{\gamma}{3\omega} \geq 1, \qquad n \in \mathbb{N}.\end{aligned}$$

Thus, in both cases condition (2.15) is satisfied. Consequently, it follows from Theorem 3.2 that every solution $x(t)$ of Eq. (3.11) satisfying (2.1) on $[a_n, b_n]$ is non-monotone on (t_0, ∞).

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Mervan Pašić
Department of Applied Mathematics, Faculty of Electrical Engineering
and Computing
University of Zagreb
10000 Zagreb
Croatia
e-mail: mervan.pasic@gmail.com

Yuriy V. Rogovchenko
Department of Mathematical Sciences, Faculty of Technology and Science
University of Agder
Post Box 422, 4604 Kristiansand
Norway
e-mail: yuriy.rogovchenko@uia.no

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