

Studying pupils' mathematical thinking
through problem solving and
view of mathematics

Case studies of Finnish comprehensive school pupils

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Preface

My journey as a PhD student began at University of Helsinki already a decade ago. After some year and a half of studying and looking for funding, I accepted a three-year position as a research fellow at University of Agder in 2009. That is when the adventure really began! The courses in mathematics didactics were demanding and challenging, but also so fun and rewarding. The people in mathematics department were warm and welcoming. I got to participate in NoGSME, YESS and Nordic summer and winter schools, attend a number of conferences, and meet so many amazing people!

I want to thank University of Agder for funding my studies and making my adventure possible. Thank you NoGSME for funding the summer and winter school participations, I learnt so much! I am also grateful for the opportunity to participate in a NOMAD workshop. The discussions about my paper during the workshop helped me to improve my paper into an actual journal article that was published in NOMAD in 2017.

I would not be finalising my PhD studies without my supervisors at University of Agder, Professor emerita Barbro Grevholm and Professor Olav Dovland. Barbro and Olav, thank you for your endless encouragement, guidance and support in my endeavour of finding my own way as a researcher. Thank you for your warmth, kindness and understanding, especially when I felt lost. I have had a bumpy road but you have never stopped supporting me, giving me the strength to continue on this path. I have always been able to count on your help.

I would not be here today also without Professor emeritus Erkki Pehkonen. He is the one who encouraged me to apply for a PhD position first in Helsinki and then in Kristiansand. He has stood by me during the entire journey. Erkki, thank you for your dedicated support on my way to become the person I am today, both academically and personally. Thank you for supervising me, I always knew you got my back. And thank you for reminding me to take care of ‘me’ and my family.

In addition to my three supervisors, I also want to thank Professor Markku Hannula from University of Helsinki. He has supported and guided me throughout the years, starting from the time when he actually was my supervisor. Markku, thank you for the countless times you have listened to me talking about my work, for reading my texts, and for commenting my ideas in such sharp and productive ways. I am grateful that you welcomed me to the mathematics didactics research community at University of Helsinki, giving me a place where I belong also in the Finnish context. Thank you for being so kind, patient, honest and inspirational!

I also want to thank my 90% opponent Professor Lieven Verschaffel from KU Leuven. Your precise and thoughtful comments and suggestions guided me to reflect upon my own work, which helped me to improve my thesis. And the teachers and pupils in the study: Thank you for sharing your time and thoughts with me. It was very inspirational to get to know you all! And thank you Enrique for helping me to improve the text and English language in my Kappa.

It has been a privilege to be a PhD student at University of Agder. I have been surrounded by incredible people, not only in Kristiansand and Helsinki but also around the world. I want to thank all the research fellows and friends around the world for welcoming and accepting me to be part of this amazing community. I sincerely hope that we will continue working together also in the future!

And then, last but definitely not least, I want to thank my family. Mom and dad, you have always been so proud of me. You have supported me whatever choices I have made in life, even if the decision has taken me to live in another country. Thank you for believing in me. And mom, thank you for using almost all of your holidays for babysitting while I was travelling for work. Without you I would not have been able to become part of the research community that I so much enjoy being.

Toni, my dear husband, thank you for always being there for me. Thank you for listening to the endless pondering of – whatever is on my mind. Thank you for taking care of all the kids while I have to work, again. Thank you for loving me. Thank you for being you. And Ennilotta, Iina-Matilda and Mea-Martta, the best daughters in the whole world, thank you for your love and understanding, your patience and kindness. Thank you for the laughter and joy you bring into my life! Rakastan teitä.

Abstract

Developing pupils' mathematical thinking is in the heart of mathematics education, also according to the Finnish curriculum. Finnish pupils' mathematical performance has been assessed in many national and international studies (e.g. PISA and TIMSS). However, we know very little about Finnish pupils' mathematical thinking at the end of comprehensive school that go beyond paper tests.

In this study, a theoretical framework is built for studying mathematical thinking through problem solving and view of mathematics. Problem solving represents the fluctuating state data that is studied through problem-solving processes, metacognition, affect and meta-affect. View of mathematics represents the more stable trait data. These temporally distinct aspects have seldom been studied together as one entity aiming to show the dynamic processes of pupils' mathematical thinking.

The theoretical framework is used as an analytic tool to study four Finnish high-achieving pupils' mathematical thinking. On theoretical level the results show the dynamic and complex processes of mathematical thinking, as well as the intertwined relationship between cognition and affect and the state and trait aspects in mathematical thinking. On practical level the results reveal strengths and weaknesses in pupils' mathematical thinking, which directs attention further towards the question of how pupils' mathematical thinking could be developed.

The four high-achieving pupils also represent interesting cases of Finnish pupils' mathematical thinking at the end of comprehensive school. The results show that the pupils are similar only on the surface level: they all like and enjoy doing mathematics, they are successful problem solvers, and they are motivated to learn mathematics. However, a closer look into their problem-solving processes and view of mathematics reveal their very different skills and competences in mathematics.

The aim in building up the theoretical framework was to create an analytical tool that can give rich data about comprehensive school pupils' mathematical thinking. However, there was an opportunity to interview one of the pupils also at the beginning of his university studies. As a consequence, the theoretical framework was developed so that it can be used at different educational levels. The new results also validate some findings from comprehensive school and thus, increase the reliability of the original results.

The analytical tool is one of the main implications of the study. The tool could be used by researchers for instance to powerfully evaluate pupils in interventional studies giving rich data about the development of pupils' mathematical thinking over time. Also after some modifications,

the tool could be used by mathematics teachers to support pupil evaluation that would actively involve the pupil.

Other implications are the rich descriptions of the individual pupils' mathematical thinking. The pupils amaze with their different personal characteristics in mathematics: Alex is a very conscious and justifying thinker and learner in mathematics who could benefit from recognising mathematics more in his own life. Daniel is extremely confident in mathematics but needs support in becoming aware of his own learning and problem-solving processes. Emma is very thorough in problem solving and learning of mathematics but needs further emotional support to learn mathematics. Nora is fluent in expressing her thoughts and connecting mathematics to real life but needs help in looking back and checking in problem solving.

There are seven papers that constitute this thesis. In the first six papers, the theoretical framework is built up step-by-step. These first papers also contain more detailed discussions of the four individual pupils' mathematical thinking. In the final paper, the four pupils' results are brought together to study the similarities and differences in their mathematical thinking and to discuss what characterises these four pupils' mathematical thinking. The development of one of the pupils' mathematical thinking from comprehensive school to university is discussed in the sixth paper.

Contents

1	Introduction	13
2	Theoretical framework	15
2.1	Mathematical thinking	15
2.2	Studying mathematical thinking	16
2.3	Hannula's three-dimensional model of affect	17
2.4	State: Problem solving	19
2.4.1	Problem-solving process	20
2.4.2	Metacognition: metacognitive knowledge and skills	21
2.4.3	Affective state	24
2.4.4	Meta-affect	24
2.5	Trait: View of mathematics	25
2.5.1	Pupil profile	26
2.6	Summary of the framework	26
3	Purpose of the study	29
3.1	Research questions	29
3.2	List of the papers	30
4	Methodology	31
4.1	Research paradigm	31
4.2	Research design	31
4.3	Participants	32
4.4	Data collections	33
4.4.1	Classroom observations	33
4.4.2	Problem solving	34
4.4.3	View of mathematics	38
4.4.4	Pilot study	41
4.5	Data analysis	41
4.5.1	Problem solving	42

4.5.2	View of mathematics	42
4.5.3	Mathematical thinking: Combining state and trait results	43
4.6	Ethical issues	43
4.6.1	The participants	43
4.6.2	Validity, reliability and trustworthiness	44
4.6.3	The research community	45
5	Results	47
5.1	Constructing a tool for studying mathematical thinking (Papers 1-7)	47
5.1.1	Paper 1: Mathematical thinking	47
5.1.2	Paper 2: View of mathematics	48
5.1.3	Paper 3: Problem solving and view of mathematics	49
5.1.4	Paper 4: The interrelationship between problem solving and view of mathematics	50
5.1.5	Paper 5: A tool for studying mathematical thinking	50
5.1.6	Papers 6 & 7: Mathematical thinking at different ages	52
5.1.7	Summary	53
5.2	Comprehensive school pupils' mathematical thinking (Papers 2-7)	54
5.2.1	Alex (Papers 2, 6 & 7)	54
5.2.2	Nora (Papers 3 & 7)	56
5.2.3	Emma (Papers 3, 4 & 7)	57
5.2.4	Daniel (Papers 5 & 7)	59
5.2.5	Summary (Paper 7)	60
5.3	The development of Alex's mathematical thinking (Paper 6)	61
6	Limitations of the study	63
7	Discussion on results and theory	67

7.1	Developing mathematical thinking through formative and summative assessments	67
7.2	Connections to Finnish curriculum development	69
7.3	Gender differences	71
7.4	Connections to mathematical modelling	73
8	Conclusion and implications	75
8.1	Theoretical considerations (RQ 1)	75
8.2	Practical considerations (RQ 2)	76
8.3	Concluding words	77
9	References	79
10	Appendices	93

1 Introduction

Developing mathematical thinking is one of the key tasks for mathematics instruction in the Finnish curriculum (the Finnish National Board of Education [FNBE], 2004, 2014). Mathematical thinking is a dynamic process that can be developed and studied through problem solving (see examples in Felmer, Pehkonen, & Kilpatrick, 2016). In addition, affective components have a crucial role in problem solving and learning mathematics (Hannula, 2011, 2012; Op't Eynde, de Corte, & Verschaffel, 2002). Both of these aspects, problem solving and affective components, have also been recognised as influencing the development of mathematical thinking in the Finnish curriculum (see FNBE, 2014).

In the past years, the Finnish National Board of Education has evaluated pupils'¹ mathematics learning outcomes at the end of comprehensive school twice, in 2011 and 2012. These are the first large scale studies assessing pupils' performance related to the 2004 curriculum. Even though FNBE reported that the average mathematics performance is at a satisfactory level, the results are alarming: pupils' mathematics performance has dropped from previous assessments (Hirvonen, 2012; Rautopuro, 2013). For instance, problem-solving performance as well as performance in all branches of mathematics has dropped consistently (Hirvonen, 2012).

Similar results showing the descending trend of Finnish pupils' mathematics performance have also been found in international studies (Väljjarvi, 2014; Kupari & Nissinen, 2015). For instance the Programme for International Student Assessment (PISA) organised by the Organisation for Economic Cooperation and Development (OECD) has shown that the Finnish national mean in mathematics literacy has diminished by 33 points between 2003 and 2015 (Vettenranta et al., 2016). This correlates with a difference of more than half a year of schooling (Kupari & Nissinen, 2015). A significant decline in the Finnish performance was also visible between years 1999 and 2011 in the Trends in International Mathematics and Science Study (TIMSS) that was established by the International Association for the Evaluation of Educational Achievement (IES; Mullis, Martin, Foy, & Arora, 2012). Even though the trend in Finnish pupils' performance in mathematics is descending, the results remain nationally 'at a satisfactory level' and internationally above OECD average (Hirvonen, 2011; Vettenranta et al., 2016).

In addition, studies on Finnish pupils' mathematics-related affect do not show encouraging results. The large-scale national studies show that

¹ In the Nordic tradition, the word 'pupil' refers to a person who is studying in comprehensive school, also outside a classroom setting (cf. the tradition in English language).

pupils' who are finishing comprehensive school in Finland consider mathematics as useful but they express fairly low self-competence and do not like mathematics (e.g. Metsämuuronen, 2013; Hirvonen, 2012). Similarly, international studies show that Finnish pupils have low level of positive attitudes towards mathematics (Mullis et al., 2012) including liking and valuing mathematics (Kupari, Vettenranta, & Nissinen, 2012). Interestingly, Finnish pupils' self-concept, such as expressing confidence in learning mathematics, was the most significant predictor for their performance in the TIMSS Study (Kupari & Nissinen, 2013).

While these large scale studies show the descending trend of Finnish pupils' mathematics performance, we have very little qualitative knowledge about the mathematical thinking that pupils have at the end of comprehensive school. What characterises their mathematical thinking? How could their mathematical thinking be developed (cf. the task for mathematics instruction in the Finnish curriculum; FNBE, 2004, 2014)? Furthermore, situational and contextual problem-solving data (what I call state) and more stable affective data (what I call trait) have seldom been studied together as one entity (Hannula, 2011). So, is it possible to build a framework that is used to study explicitly both fluctuating state data and quite stable trait data, while still showing the dynamic processes of pupils' mathematical thinking? The purpose of my study is to find answers to these questions.

In the seven papers that constitute this thesis, the theoretical framework that is used to analyse pupils' mathematical thinking is built up step-by-step. Mathematical thinking is studied through problem solving and view of mathematics. Problem solving represents the fluctuating state data that is studied through problem-solving processes, metacognition, affect and meta-affect. View of mathematics represents the more stable trait data. Even though problem solving and view of mathematics are studied separately, the results complement each other revealing the dynamic process of mathematical thinking and aspects that influence it.

While building a rich analysing tool for mathematical thinking, individual cases of Finnish pupils at the final year of comprehensive school are introduced in the papers for two purposes: to illustrate how theory is used to analyse the data, and to introduce interesting cases of Finnish pupils and their mathematical thinking. The results show how the analysing tool can reveal different strengths and weaknesses in different pupils. These different characterisations of pupils' mathematical thinking give us hints of what high achievers' mathematical thinking can be like at the end of Finnish comprehensive school. Also, the results of the pupils' mathematical thinking motivated us to reflect upon how this tool could be used in classrooms to support the development of individual pupils' mathematical thinking.

2 Theoretical framework

2.1 Mathematical thinking

Despite the wide use of the term ‘mathematical thinking’ in mathematics education, and perhaps because of it, there is no common understanding of the meaning of mathematical thinking or even a consensus on the abilities or predispositions that underlie mathematical thinking (e.g. Sternberg, 1996). Many researchers seem to think of the concept as thinking about mathematics, others might think of it as combination of complicated processes, something that makes use of mathematical operations, processes, or dynamics (Burton, 1984); and others might look at mathematical thinking through different worlds of mathematics (see Tall, 2013).

In addition to these more general assumptions of mathematical thinking, studies are influenced, for instance, by the specific mathematical domain in which the study is conducted, or a special viewpoint about how mathematical thinking could be studied. The focus of a study can be for instance different thinking skills or styles (e.g. McGregor, 2007; Burton, 1999), or problem solving (e.g. Polya, 1957; Mason, Burton, & Stacey, 1982/2010; Schoenfeld, 1985). Another focus might be the study of issues that have an effect on mathematical thinking such as research on metacognition (e.g. Stillman & Mevarech, 2010; Schoenfeld, 1987; Flavell, 1979) and mathematics related affect (e.g. Pepin & Rösken-Winter, 2015; Hannula, 2012). Or yet, a study can focus on mathematical thinking in different mathematical domains (e.g. Hähkiöniemi, 2006; Joutsenlahti, 2005; Merenluoto, 2001), or how mathematical thinking can be improved through teaching (e.g. Lester & Cai, 2016; Doerr, 2006; Sfard, 2001).

In the Finnish curriculum (the 2004 curriculum was implemented at the time of data collection for this project), mathematical thinking is described through a list of thinking skills and methods that pupils should learn during the comprehensive school. This learning objective is presented as parallel to the other learning objectives (numbers and calculations, algebra, functions, geometry, and probability and statistics). The list of mathematical thinking skills and methods that pupils should learn between grades 6-9 is presented below (FNBE, 2004, p.164):

- Functions that demand logical thinking, such as classification, comparison, organization, measurement, constructing, modelling, and looking for and presenting rules and correlations
- Interpretation and use of concepts needed in drawing comparisons and correlations
- Interpretation and production of mathematical texts

- Introduction to proof: justified conjectures and experiments, systematic trial-and-error method, demonstrating incorrectness, direct proof
- Solving combinatorial problems by different methods
- Use of tools and drawings that assist thinking
- History of mathematics

Introducing these processes in mathematical thinking skills and methods, and introducing thinking skills and methods as its own learning objective in parallel to five other mathematics domains, follow the definition of mathematical thinking by Mason et al. (1982/2010). In their definition, mathematical thinking is about mathematical processes, rather than about any particular branch of mathematics. The process view is also adopted in my study, where mathematical thinking is considered to be an individual activity (cf. Sfard, 2007), and ‘pupils’ activities, actions and explanations during problem solving are interpreted as visible signs or expressions of their mathematical thinking’ (Viitala, 2015, p. 138). Following this definition of mathematical thinking, the next question is, how can these mathematical processes be studied?

2.2 Studying mathematical thinking

After a thorough literature review, Schoenfeld (1992) recognised five aspects that are important in a study on mathematical thinking. These are the knowledge base, problem-solving strategies, monitoring and control, beliefs and affects, and practices. Similar findings have been found in connection to literature on problem-solving performance (Lester, 1994), and are also listed as part of final-assessment criteria in the current Finnish curriculum (see FNBE, 2014, pp. 433-434).

Even though the list still remains relevant to current research in mathematics education (see Schoenfeld, 2015), the terms used today are somewhat different. The development in terminology has also influenced my study and matured step-by-step throughout the PhD project, in which the theoretical framework initiated from Schoenfeld’s (1992) paper and the Finnish 2004 curriculum. I will follow Schoenfeld’s list of aspects that are important in a study on mathematical thinking and explain how they are present in my study. After this short introduction to the terms used in this study, more thorough explanations of them are presented in separate sections of this chapter.

First of all, the knowledge base and problem-solving strategies are called resources and heuristics in my study (cf. Carlson & Bloom, 2005). Resources refer to ‘the conceptual understandings, knowledge, facts, and procedures used during problem solving’ (ibid., p. 50) and heuristics describe the specific procedures and approaches in problem solving (for instance subdividing the problem; see ibid.). Resources and heuristics are an inherent part of problem-solving processes that cannot be studied

thoroughly with paper-and-pencil mathematics tests. Since problem-solving processes are also listed as final-assessment criteria for mathematical thinking skills and methods in the curriculum (see FNBE, 2004, p. 166), mathematical thinking is studied through problem-solving processes.

In close connection to problem-solving processes is metacognition. Schoenfeld (1992) used the terms monitoring and control to refer to the part of metacognition and metacognitive skills that include active monitoring and consequent regulation of problem-solving processes (e.g. Schoenfeld, 2015, 1987; Veenman, Elshout, & Meijer, 1997). Definitions and explanations for metacognition and metacognitive skills follow the section on problem-solving processes.

Perhaps the most significant change in research that has occurred relates to beliefs and affects. Definitions have moved from beliefs towards dynamic affective systems (see Pepin & Rösken-Winter, 2015). In this study, affect is seen as a psychological domain with state and trait aspects (Hannula, 2011). The affective state is situational and contextual, and can be studied together with pupils' problem-solving processes. The affective trait, on the other hand, is relatively stable and directs pupils' engagement and success in problem solving. Affective trait will be studied through pupils' view of mathematics, from which a pupil profile can be derived to obtain background information about the pupils.

Since the study focuses on individual pupils, the role of practices is relatively small in this study. It is part of the social dimension in Hannula's model (2011, 2012; see section 2.3) and is included in the study mainly through the pupils' view of mathematics.

Missing from Schoenfeld's list is meta-affect. Meta-affect can be seen as 'standing in relation to affect much as metacognition stands in relation to cognition, and powerfully transforming individuals' emotional feelings' (DeBellis & Goldin, 2006, p.132). Meta-affect will be introduced in section 2.4.4 following the section discussing affective state.

The division of affect into two temporally distinct aspects, state and trait, inspired me to divide the entire study into two parts: problem solving (state) and view of mathematics (trait). This division is visible throughout the study. More about Hannula's (2011, 2012) three-dimensional theoretical model of affect can be found in the next section before introducing the theoretical considerations for problem solving and view of mathematics.

2.3 Hannula's three-dimensional model of affect

The theory about affect, its concepts, and their connections has been used in very diverse ways both in Finnish and international research (see e.g. Hannula, 2007; Zan, Brown, Evans, & Hannula, 2006; Furinghetti &

Pehkonen, 2002; Pepin & Rösken-Winter, 2015). While early research surveying mathematics anxiety and attitudes in mathematics lacked a proper theoretical foundation (Zan et al., 2006), the subsequent research on mathematical problem solving clarified and categorised affective concepts in mathematics education (e.g. into beliefs, attitudes and emotions by McLeod, 1992; or beliefs, attitudes, emotions and values by DeBellis & Goldin, 1997). The most current theorising of affect aims to provide more dynamic representations or systems of affect in mathematics education (Pepin & Rösken-Winter, 2015; also for a review of previous theorising of affect, see Hannula, 2011, 2012).

Perhaps the most discussed model of affect today is Hannula's model dating to 2011/2012. In that model, the term affect is used as 'an umbrella concept for those aspects of human thought which are other than cold cognition, such as emotions, beliefs, attitudes, motivation, values, moods, norms, feelings and goals' (Hannula, 2012, p. 138). Hannula's model has three distinct dimensions. First we have the division of affect into cognitive, motivational and emotional aspects of affect. The cognitive domain includes mental representations that have a truth value of some kind to the individual, for instance knowledge, beliefs and memories (e.g. Goldin, 2002). Motivation reflects personal preferences and explains choices; emotions contain different feelings, moods and emotional reactions (Hannula, 2011).

The second dimension in Hannula's model is the division of affect into two aspects, one of rapidly changing affective states and another one of relatively stable affective traits. Contrary to some of the previous theories of affect, such as McLeod's (1992) model, all the above-mentioned components of affect (cognition, emotion and motivation) can be found in both state and trait aspects. For instance in a problem-solving situation, a pupil can have a belief trait, 'I cannot solve word tasks,' but after reading a word task description he/she can have a belief state, 'this task isn't so difficult, I can solve this problem,' and start working on the word task (cognition). Similar examples can be given for emotions and motivation: state confidence (emotion) can be guiding a problem-solving process for instance through motivation to work on the task, whereas an overall confidence in school mathematics (emotion) can determine how ready the pupil might be to make an effort to learn mathematics (motivation).

The third dimension in the model comprises the physiological, psychological and social nature of affect. The physiological and psychological levels are considered to be individual phenomena that interact with the social context. Studies on the physiological dimension can focus for instance on the neural activations (state) or neural connections (trait) in the brain during mathematics problem solving. Studies about the social

aspect of affect can investigate social interaction (state) or classroom norms (trait) that influence problem-solving processes (cf. the social turn, e.g. Lerman, 2000). This study follows the psychological tradition of looking explicitly at individuals' cognition, motivation and emotion as both a state and as a trait. Examples of the psychological aspects of Hannula's model can be seen in Table 2.1.

Table 2.1 Examples of the psychological aspects of Hannula's model (cf. Hannula, 2011, 2012).

	Cognitive	Emotional	Motivational
State	Thoughts, belief states	Feelings, state confidence	Active goals
Trait	Beliefs, conceptions	Confidence, emotional dispositions	Desires, goal orientations

2.4 State: Problem solving

Problem solving has had a central role in mathematics education and curriculum development nationally and internationally for decades (for national development, see e.g. Pehkonen, 2009). During those years, however, problem solving has had multiple meanings from doing routine calculations to doing mathematics as a professional, and the definition might still differ from country to country (e.g. Törner, Schoenfeld, & Reiss, 2007; Schoenfeld, 1992).

In the 2004 curriculum in Finland (implemented at the time of data collection), learning problem solving is one of the three key tasks for mathematics instruction together with developing mathematical thinking and learning of mathematical concepts (FNBE, 2004). Problem solving is listed in general learning objectives ('[t]he pupils will learn to [...] solve mathematical problems'; *ibid.*, p. 164) that will be evaluated through a number of final assessment criteria (e.g. '[t]he pupils will know how to estimate a possible result and prepare a plan for solving a problem; they will have dependable basic calculation skills'; *ibid.*, p. 166). Especially, in connection to mathematical thinking, the problem-solving process is listed as part of final-assessment criteria for mathematical thinking skills and methods (*ibid.*, p. 166):

The pupils will [...] know how to transform a simple problem in text form to a mathematical form of presentation, make a plan to solve the problem, solve it, and check the correctness of the result

These problem-solving processes are at the core of this study where I uncover some of the complex cognitive processes of mathematical thinking. Problems that enable rich non-linear problem-solving processes are of the kind in which the solver has to combine previously known infor-

mation in a new way to him/her. This definition separates ‘a problem’ from a routine task or an exercise (e.g. Kantowski, 1980; cf. individual experiences of tasks/problems in Lester, 1994). But what is the role of metacognition, affect and meta-affect in these problem-solving processes?

As an example, Carlson and Bloom (2005) introduced a multidimensional problem-solving framework for individual problem solvers. They studied professional mathematicians and made detailed observations on how resources and heuristics interact with problem-solving behaviour, and how monitoring and affect were expressed during four problem solving phases (orienting, planning, executing and checking). Their analysis showed how all of the attributes (resources, heuristics, affect and monitoring) are present in every behavioural phase of problem solving. What Carlson and Bloom call ‘affect’, is divided into affective states and meta-affect in my study.

With the broader knowledge about problem-solving behaviour of professional mathematicians (mainly states; Carlson & Bloom, 2005), together with other reports (e.g. Schoenfeld, 1992, 2015; Lester, 1994; FNBE, 2014), the theoretical framework for studying comprehensive school pupils’ mathematical thinking through situational problem-solving processes will be discussed next.

2.4.1 Problem-solving process

In the Finnish curriculum, the problem-solving process is described through four steps: knowing how to transform a simple problem in text form to its mathematical form, making a plan to solve the problem, solve it, and check the correctness of the result (FNBE, 2004, p. 166). These steps are very similar to Polya’s (1957) classical problem solving phases: understanding the problem, devising a plan, carrying out the plan, and looking back (see table 2.2). While making a plan to solve a problem and solving it are part of both process descriptions for mathematical problem solving, the notions of transforming a simple problem in text to its mathematical form and checking the result are only two occurrences of understanding the problem and looking back.

Polya’s (1957) model for problem solving has been criticised for being too simple or not adequate for students’ learning (e.g. Schoenfeld, 1992). Thus, the model has been further developed by many researchers over the years (e.g. Mason et al., 1982/2010; Schoenfeld, 1985). In these subsequent models, the nonlinearity or cyclic movements of the different problem solving phases have been emphasised. Problem solving has also been divided into more detailed phases, some being more general for mathematical problem-solving (e.g. *ibid.*), and some referring to specific aspects of problem solving (e.g. solving open-ended problems; Hähkiöniemi, Leppäaho, & Fransisco, 2013).

In 2005, Carlson and Bloom introduced a multidimensional problem-solving framework that also has four phases: orienting, planning, executing and checking (see Table 2.2). In orienting, the solver makes sense of the problem as well as organises and constructs knowledge to solve it. When planning the problem solution, the solver makes decisions about the strategies and approaches he/she will make when executing the plan. After solving the problem, the solver checks the solution and verifies the answer. The model emphasises the cyclic nature of problem solving, especially between planning, executing and checking.

Table 2.2 Problem-solving phases described in the Finnish curriculum (FNBE, 2004), Polya’s model (1957), and Carlson and Bloom’s (2005) multidimensional problem-solving framework.

Finnish curriculum	Polya’s model	Carlson & Bloom’s model
Transforming a problem into a mathematical presentation	Understanding the problem	Orienting
Making a plan to solve the problem	Devising a plan	Planning
Solving the problem	Carrying out the plan	Executing
Checking the correctness of the result	Looking back	Checking

In the planning phase of professional mathematicians, Carlson and Bloom (2005) found a sub-cycle of conjecturing, testing, and evaluating. In this sub-cycle, the ideas for solving a problem are initially tested before executing the selected plan. However, in my project with comprehensive school students, the conjecture cycle has not been visible (see Viitala 2015, 2016, 2017a). Thus, the sub-cycle of conjecture seems to be a quality of expert problem solvers (cf. metacognitive activities of novice and expert problem-solvers e.g. in Stillman & Galbraith, 1998; Schoenfeld, 1992).

When problem-solving processes are studied in this study, the processes are understood to be cyclic in nature and the focus is on the four main phases of problem-solving behaviour (not on the sub-cycle that Carlson and Bloom found from mathematicians; Carlson & Bloom, 2005). In the following sections I explain how metacognition, affective state and meta-affect are present in these processes.

2.4.2 Metacognition: metacognitive knowledge and skills

Even though a pupil might have the knowledge and skills for solving a problem, inefficient control mechanisms can be a major obstacle in solv-

ing problems (Carlson, 1999). These control mechanisms refer to metacognition. Similarly as for problem solving, metacognition also has many different meanings in educational research (e.g. Tarricone, 2011). After decades of research on metacognition, a majority of the researchers have returned to Flavell's early definition of it (Stillman & Mevarech, 2010):

Metacognition refers to one's knowledge concerning one's own cognitive processes and products or anything related to them [...] Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects or data on which they bear, usually in service of some concrete goal or objective. (Flavell, 1976, p. 232)

Metacognition has been commonly categorised into metacognitive knowledge and metacognitive skills. Flavell (1979) describes metacognitive knowledge as the interplay between person-characteristics, task-characteristics, and strategy. Person-characteristics refer to beliefs about the individual and others as cognitive processors, task-characteristics refer to task management and confidence for achieving the goal, and strategy refers to evaluations of the effectiveness of chosen strategies to achieve the goal. Of these, the two aspects referring to beliefs and estimation of confidence will be discussed as part of affect. Hence, only the evaluation of the effectiveness of the chosen strategy in problem solving is considered to be a metacognitive activity in my study.

Metacognitive skills refer to control and self-regulation (Schoenfeld, 1987; Veenman, Elshout, & Meijer, 1997; Veenman, van Hout-Wolters, & Afflerbach, 2006). Metacognitive control during problem solving includes monitoring problem-solving progress, deciding on the next step, and directing resources (Schoenfeld, 1987). These metacognitive skills seem to increase with age, and at the age of 14-15, metacognitive skills seem to predict performance in mathematics even more than intelligence (van der Stel, Veenman, Deelen, & Haenen, 2010; Alexander, Carr, & Schwanenflugel, 1995).

In my study, metacognitive skills are studied through four mathematics specific metacognitive activities that can be studied through pupils' overt behaviour. These activities are orientation, planning, evaluation, and elaboration (van der Stel et al., 2010; see Table 2.3 for simplified connections between metacognitive activities and Carlson and Bloom's (2005) problem-solving phases). These metacognitive activities can be studied in parallel with cognitive actions in problem-solving processes. For instance, making a graph for representing a geometrical problem is a cognitive action preceded by a metacognitive decision to make sense of the problem at hand with a picture (metacognitive skilfulness).

Table 2.3 A simplistic way of parallelising Carlson and Bloom’s (2005) problem solving phases with van der Stel et al. (2010) mathematics-specific metacognitive activities. Examples are taken from van der Stel et al. (2010, p. 220).

Problem solving phases	Metacognitive activities	Examples of mathematics-specific metacognitive activities
Orienting	Orientation	Estimating the answer Making a sketch of the problem to represent the problem
Planning	Planning	Designing a step-by-step action plan, instead of working by trial and error
Executing		Writing down calculations step-by-step
Checking	Evaluation	Monitoring action plan Checking an answer by recalculating
	Elaboration	Paraphrasing the problem Drawing conclusions while referring to the problem statement

As can be seen from Table 2.3, problem-solving and metacognitive activities are closely related. Orientation as a problem-solving activity as well as metacognitive activity happens prior to the task performance. During that phase, the solver organises and makes sense of the given data (e.g. draws a picture, makes an estimation of the answer, or writes down all the relevant information from the task description). Planning and evaluation as metacognitive activities happen during the actual task performance (that is, planning, executing and checking as problem-solving (PS) phases). Planning as a metacognitive activity happens during planning and execution as problem-solving activity. An example of this activity might be a decision to write down calculations step-by-step: it might happen during planning of the problem solution, or while executing that plan (PS phases; cf. Viitala, 2017a).

Similarly as in planning, evaluation can occur as a metacognitive activity also during execution and checking as problem-solving activity. As an example, the active monitoring of an action plan (made during planning as a PS phase) occurs during execution of the plan. On the other hand, checking an answer by recalculating might happen during execution or checking of the given plan (PS activities). Elaboration as a metacognitive activity happens after the task performance in mathematics. These activities are not necessarily part of problem-solving activities (e.g. paraphrasing the problem). However, they might provide an oppor-

tunity for learning in future occasions (cf. mathematical modelling in section 7.4).

Furthermore, similarly as for the cyclic nature of problem-solving processes, metacognitive processes are understood to be cyclic in this study. For instance, monitoring actions might result in returning to the task description and/or devising the plan again (problem-solving activities). These actions are influenced by the solver's metacognitive skilfulness.

(Metacognition and self-regulation are often being used as closely linked concepts in educational research. For a review of their properties and differences, see e.g. Dinsmore, Alexander, & Loughlin, 2008.)

2.4.3 Affective state

Affective states influence pupils' problem-solving processes through situational and contextual emotions, cognitions and motivations (cf. Hannula, 2011; see section 2.3). Affective states as psychological phenomena are traditionally studied in connection to problem-solving processes. Instead of categorising further the affective states that might emerge in problem-solving processes (cf. the definition for affect; Hannula, 2012; see section 2.3), all affective states emerging in a pupil's problem-solving situation are considered to be part of the individual pupil's problem-solving processes. However, special attention in my study is given to the task related beliefs (cognition), changing emotions, feeling of confidence, and task motivation that emerge in pupils' problem-solving and metacognitive processes (cf. task characteristics of metacognition in Flavell, 1979).

2.4.4 Meta-affect

How affective states emerging in problem-solving situations influence the actual problem-solving process is a result of meta-affective skilfulness. Meta-affect can be seen as 'standing in relation to affect much as metacognition stands in relation to cognition, [...] powerfully transforming individuals' emotional feelings' (DeBellis & Goldin, 2006, p.132). For instance, in their study on professional mathematicians' problem solving, Carlson and Bloom (2005) emphasised the role of effective management of frustration and anxiety in problem solving. This was shown to be an important factor in mathematicians' persistent pursuit of solutions to complex problems. Similar management of affect can also guide a comprehensive school pupil's problem-solving behaviour (see Viitala, 2016).

Carlson and Bloom (2005) used the terms 'mathematical intimacy' and 'mathematical integrity' in relation to what in my study I call meta-affect (cf. DeBellis & Goldin, 2006). These constructs express the bond between the problem solver and the problem (or the problem-solving situation). Mathematical intimacy refers to the deep and vulnerable emo-

tional engagement an individual may have with the problem. Mathematical integrity, on the other hand, is ‘the individual’s fundamental commitment to mathematical truth, search for mathematical understanding, or moral character guiding mathematical study’ (ibid., p. 132). These constructs are not considered when studying pupils’ mathematical thinking in comprehensive school. However, I used these constructs when I had an opportunity to analyse one of the pupils’ mathematical thinking also as a university student (see paper 6; about the additional longitudinal data, see sections 5.1.6 and 5.3).

2.5 Trait: View of mathematics

In addition to affective states, the relatively stable affective traits have been shown to have an influence on individuals’ problem-solving processes and mathematical thinking (e.g. Zan et al., 2006; DeBellis & Goldin, 2006; Vinner, 2004; Schoenfeld, 1992). Affective traits direct pupil’s engagement and success in mathematics. This has also been recognised in the current curriculum, according to which mathematics teaching should support pupils’ positive attitude towards mathematics and strengthen motivation, positive self-image and self-confidence as a mathematics learner (FNBE, 2014, p. 374).

The study on affective trait is structured in the same way as affective state: It is a psychological phenomenon and a mixture of cognitive, motivational and emotional processes (Hannula, 2011, 2012; see section 2.3). Following the tradition, affective traits are studied through pupils’ view of mathematics (see e.g. Op’t Eynde et al., 2002). However, even though frameworks for studying pupils’ view of mathematics are often adopted from belief-research, all the affective components are included in the selected framework, including emotions and motivation (thus the word ‘view’, see Rösken, Hannula, & Pehkonen, 2011).

In my study, pupils’ view of mathematics is studied through four components: mathematics (as science and as a school subject), oneself as a learner and user of mathematics, learning mathematics, and teaching mathematics (Pehkonen, 1995). Similar categories have been found in many other studies (see Op’t Eynde et al., 2002). Pehkonen’s (1995) categorisation has been criticised for not considering social aspects of pupil’s view of mathematics (social and socio-mathematical norms in mathematics classroom; Op’t Eynde et al., 2002). These aspects belong to the social component of Hannula’s (2011, 2012) model that is not studied explicitly in this study. However, social aspects are present in pupils’ problem-solving activities and view of teaching mathematics that might reflect social norms in the classroom, and thus are not excluded from the study either.

In addition to the influence on mathematical thinking on a trait level (e.g. motivation to learn mathematics and confidence in school mathematics), pupil's view of mathematics influences mathematical thinking also on a state level, in this case through problem solving. A classic example is the belief that a mathematics problem should be solved in five minutes, which might limit a pupil's effort to work on a task. These traits in states (problem-solving situations) are important to recognise to help pupils develop their problem-solving behaviour.

Pupils' answers to questions about their view of mathematics might also raise metacognitive and meta-affective issues. These are considered to be traits when the answers are based on memories of earlier experiences, for instance explanations about self-regulation in mathematics learning (metacognition), a belief about oneself as a cognitive processor (person characteristics in Flavell's (1979) model for metacognitive knowledge), or how the feeling of anxiety towards word problems is handled (meta-affect).

2.5.1 Pupil profile

In connection to pupil's view of mathematics, a pupil profile is made to have as background information of the pupil (cf. Pehkonen, 1995). The pupil profile is a short description of the pupil that is constructed using the information that is considered to form the core of the pupil's view of him/herself as a learner of mathematics. These components are ability, difficulty of mathematics, success, and enjoyment of mathematics (Hannula & Laakso, 2011; Rösken et al., 2011). Ability and success relate to personal beliefs and contain statements such as 'math is hard for me' (ability) and 'I am sure I can learn math' (cf. beliefs about oneself as a learner and a user of mathematics in Pehkonen, 1995). Difficulty of mathematics refers to mathematics as a subject (cf. beliefs about mathematics in Pehkonen, 1995) and enjoyment of mathematics to emotions.

Even though motivation did not end up as component of its own, in the study by Rösken et al. (2011) it is considered as an important factor directing pupils' problem solving and mathematics learning. It is also one of the key components of affect in Hannula's model (2011, 2012). Hence, motivation to learn mathematics is added to the pupil profile.

2.6 Summary of the framework

The initial purpose of this thesis was to examine mathematical thinking through problem solving and view of mathematics. The aim was to go beyond ordinary mathematics tests to get a deeper understanding of the mathematical thinking Finnish pupils might have at the end of compulsory school. In order to achieve this goal, I started forming the theoretical framework following the Finnish 2004 curriculum (which the pupils had followed during the main part of their compulsory school; FNBE, 2004)

and Schoenfeld's (1992) list of important aspects for a study on mathematical thinking. In the end, most of my time in the development of my study went into building up the theoretical (and analytical) framework presented in this chapter.

Multiple approaches are used to study the dynamic processes of mathematical thinking. First, problem solving is studied through problem-solving processes (orienting, planning, executing and checking; Carlson & Bloom, 2005) that are influenced by metacognitive skills (orientation, planning, evaluation, and elaboration; van der Stel et al., 2010), affective states (cognitive, emotional and motivational components as a psychological phenomenon; Hannula, 2011, 2012) and meta-affect (transforming the affective states in problem solving; DeBellis & Goldin, 2006). Secondly, the affective trait is studied through pupils' view of mathematics (mathematics, oneself as a learner and user of mathematics, learning mathematics and teaching mathematics; Pehkonen, 1995) and the pupil profile which is formed as background information about the pupil (following the core of pupil's view of him/herself as a learner of mathematics: ability, difficulty of mathematics, success, and enjoyment of mathematics; Rösken et al., 2011).

As was seen in the sections of this chapter, the different components of my study are highly connected and sometimes even theoretically overlapping (e.g. problem-solving and metacognitive activities, or metacognitive knowledge and affect, respectively). Hence, choices were made to categorise the overlapping aspects under one component. However, since mathematical thinking is understood to be a dynamic process, all these components will be brought together in the end to describe pupils' mathematical thinking through the interrelated components of problem solving and view of mathematics. Furthermore, the framework is built to gain deep and informed knowledge about pupils' mathematical thinking. Hence, this interpretive study is open also to other aspects arising from the data (e.g. the influence of social aspect on mathematical thinking; Hannula, 2011, 2012).

A very simplistic structure of the theoretical framework is presented in Figure 2.1. The structure is not meant to be exhaustive with respect to components influencing mathematical thinking, nor does it show the connections between the different components. Simply, the purpose of the figure is to show the tools through which mathematical thinking is studied in the present work.

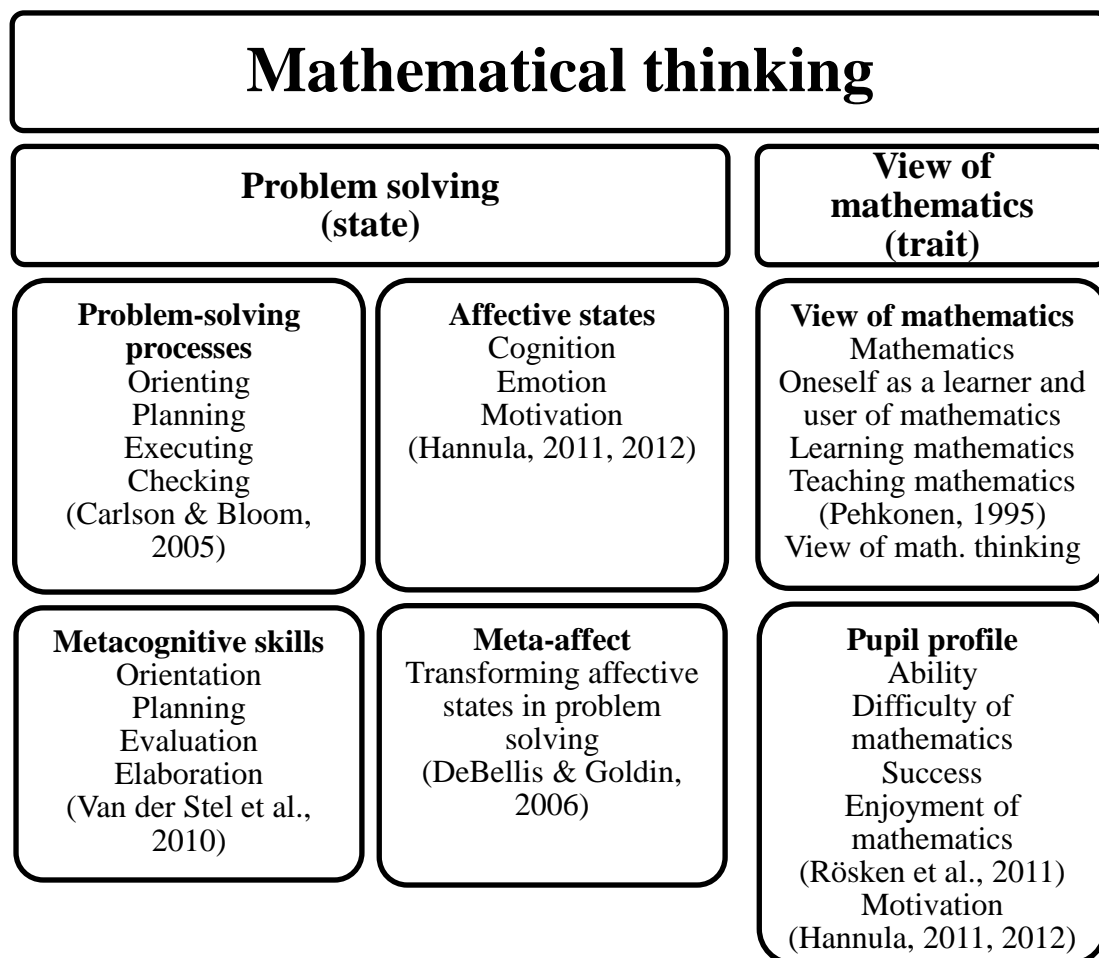


Figure 2.1 A simplistic structure of the components studied in relation to mathematical thinking.

A note about the terms used in this study. In academic literature, there are many parallel terms used for the same or similar phenomena. As explained before, one example is metacognition and self-regulation (see Dinsmore et al., 2008). Another example is the connection between self-concept and self-efficacy, and how they relate, for instance, to self-confidence (see e.g. Lee, 2009; Ferla, Valcke, & Cai, 2009). Similarly as the terms used in this study, these ‘self-terms’ can overlap and differ in stability (e.g. self-concept is a more stable trait and self-efficacy relate to state affects, cf. *ibid*; see also McLeod, 1992). However, the ‘self-terms’ are often studied with similar test items in questionnaires as the terms used in this study (e.g. mathematical self-efficacy as task-related confidence (state), cf. Lee, 2009). To enable the reader to relate the research results, it is important to explain the terms used in a study, as well as to provide examples of the test items used to study a phenomenon. Hence, the interview protocols for this study can be found in appendix 2.

3 Purpose of the study

3.1 Research questions

In the large-scale assessments, whether they are national or international, we can see the descending trend of the Finnish pupils' performance in mathematics (Kupari & Nissinen, 2015; Välijärvi, 2014; Rautopuro, 2013; Hirvonen, 2012; Mullis et al., 2012). Since the beginning of PISA and TIMSS studies as well as the preceding national large scale studies, the Finnish curriculum has been renewed twice, in 2004 and 2014. Thus, it is justifiable to ask, how much of the development in Finnish pupils' mathematics achievement is due to the curriculum?

Some Finnish mathematicians have criticised curriculum development that is moving from exact definitions and proofs to a more descriptive mathematics curriculum where, for instance, geometry is neglected (e.g. Martio, 2009). Also the modest performance in TIMSS compared to PISA have raised the question of why Finnish pupils perform better in problem solving and applying mathematics than with more conventional curriculum material (see e.g. Andrews, Ryve, Hemmi, & Sayers, 2014). In addition, the new curriculum (see FNBE, 2014) seems to emphasise applications and multi-disciplinary learning even more than the previous curriculum (see FNBE, 2004).

Nevertheless, despite the recent development of Finnish students' mathematics achievement, already earlier results suggest that Finnish students' level of conceptual understanding and mathematical thinking have been at an alarmingly low level (Kupiainen & Pehkonen, 2008). Similar results have also been found in other Finnish studies (e.g. Kaasila, Pehkonen, & Hellinen, 2010; Merenluoto, 2001; Huhtala, 2000). These studies have been carried out at different levels of education. What we are lacking is more recent qualitative knowledge about pupils' mathematical thinking, especially at the end of comprehensive school.

Standard and standardised tests have been criticised for testing pupils with short answer questions containing only low-level facts and skills (Lesh & Clarke, 2000) that do not provide insight into pupils' abilities (Iversen & Larson, 2006; Niss, 1999). Hence, the present qualitative study is largely based on interviews, aiming to get closer to and deeper into pupils' mathematical thinking. However, the question arises, how do we get in-depth knowledge about pupils' dynamic mathematical thinking processes that considers various aspects that influence it? With these reflections and the theoretical framework presented in Chapter 2, the study at hand aims to answer the following questions:

1. What can be revealed when mathematical thinking is studied through two different data sets: problem solving (state) and view of mathematics (trait)?
2. What characterises the mathematical thinking of four Finnish high-achieving pupils at the end of comprehensive school?

To answer the first question, a theoretical and analytical framework was developed step-by-step and individual pupils' mathematical thinking was studied using the tool created (Papers 1-6). For the second question, the results from the individual pupils (Papers 2-6) were brought together to see the characteristics, and the similarities and differences the pupils might have in their mathematical thinking (Paper 7).

3.2 List of the papers

1. Viitala, H., Grevholm, B., & Nygaard, O. (2011). About affect in five Finnish dissertations on mathematical thinking. In K. Kislenko (Ed.), *Current state of research on mathematical beliefs XVI: Proceedings of the MAVI-16 Conference, June 26-29, 2010, Tallinn, Estonia* (pp. 313-330). Tallinn, Estonia: University of Tallinn, The institute of Mathematics and Natural Sciences.
2. Viitala, H. (2013). Alex's world of mathematics. In M. S. Hannula, P. Portaankorva-Koivisto, A. Laine, & L. Näveri (Eds.), *Current state of research on mathematical beliefs XVIII: Proceedings of the MAVI-18 Conference, September 12-15, 2012, Helsinki, Finland* (pp. 71-81). Helsinki, Finland: The Finnish Research Association for Subject Didactics, Publications in Subject Didactics 6.
3. Viitala, H. (2015). Two Finnish girls and mathematics: Similar achievement level, same core curriculum, different competences. *LUMAT*, 3(1), 137–150.
4. Viitala, H. (2016). Emma's mathematical thinking, problem solving and affect. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 1294–1300). Prague, Czech Republic: Charles University and ERME.
5. Viitala, H. (2017a). A tool for understanding pupils' mathematical thinking. *Nordic Studies in Mathematics Education*, 22(2), 5–31.
6. Viitala, H. (2017b). *A framework for studying students' mathematical thinking at different ages: A longitudinal case study of Alex*. Manuscript in preparation.
7. Viitala, H. (2017c). A case study on Finnish pupils' mathematical thinking: problem solving and view of mathematics. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education* (pp. 1218-1225). Dublin, Ireland: DCU Institute of Education and ERME.

4 Methodology

The key methodological question in this study is: How can we study mathematical thinking so that we get in-depth knowledge about pupils' mathematical thinking that also shows the dynamic nature of mathematical thinking? The first step is to form a theoretical framework that can be used to guide the data collection and which can also be used as an analytic framework. Such theoretical framework was introduced in Chapter 2. In this part of the thesis, I will present the key methodological choices that were made in connection to data collection for the empirical part of the study. The tools for analysing the data are briefly introduced (as they are based on the theoretical framework already presented in Chapter 2). Also, different ethical considerations about the study are discussed briefly.

4.1 Research paradigm

Paradigms represent belief systems that attach users to particular worldviews and guide their actions (e.g. Denzin & Lincoln, 2000; Guba, 1990). These basic beliefs concern ontological, epistemological, and methodological issues. In this study, the ontological and epistemological assumptions for the object of study, individual pupils' mathematical thinking, belong to social constructivism that is considered to be an instance of interpretive, naturalistic paradigms (cf. Cohen, Manion, & Morrison, 2007; Lincoln & Guba, 2000).

The constructivist paradigm is based on relativism where realities are locally and specifically constructed, so that multiple realities exist. Findings are co-created by knower and respondent (Lincoln & Guba, 2000) and ask for personal involvement of the researcher (Cohen et al., 2007). The aim of the research is to understand and interpret. The constructivist paradigm assumes a naturalistic set of methodological procedures such as hermeneutic or dialectic methodology (Lincoln & Guba, 2000).

4.2 Research design

Following the constructivist paradigm, this study relies on individual cases (see Denzin & Lincoln, 2000). The purpose is not to describe the pupils as part of a group, which is a kind of ethnography, but the pupils are considered as individuals, explaining and interpreting their own thinking.

Case studies are not defined by the methods of inquiry that are used (Stake, 2000; Yin, 2009). This study aims to 'investigate and report the complex dynamic and unfolding interactions of events, human relationships and other factor in a unique instance' (Cohen et al., 2007, p. 253, citing Sturman, 1999, p. 103). In this case, the individual pupils' mathe-

mathematical thinking is studied with the intention of developing a theoretical framework which can help researchers understand pupils' mathematical thinking and its development (cf. Cohen et al., 2007).

More particularly, this is a case study of eight pupils. Following the constructivist-interpretive paradigm, a naturalistic approach is taken and the study is implemented in the pupils' own school environment. Furthermore, Stake (2000) would call this particular study a collective case study as it studies a number of cases in order to investigate mathematical thinking as a phenomenon, and similarities and differences in pupils' mathematical thinking as a general condition.

4.3 Participants

There are eight participants in this study. In order to find similarities and differences between genders, achievement level, and the possible effect of classroom practices, two pupils were selected from four mathematics classes. From each mathematics class, one boy and one girl were chosen, one of whom should be a high achiever and the other a low achiever. Each class were from a different school. Because this is a case study, there is no need for the classes to be randomly selected, so the schools were chosen from areas in Finland that were geographically convenient for the researcher to attend.

Before the actual data collection, the principals of the schools were contacted in person, and they decided whether the school was willing to participate in the study. This was done in the spring 2010. In the participating schools, the principal chose the mathematics classes for the study according to the teachers' willingness to participate. In the beginning of autumn 2010 and just before the data collection began, the teachers were contacted in person. During this time, the teachers were able to ask questions about the study and give an (oral) informed consent for participation (cf. Sowder, 1997).

The selection of the pupils was made with the teacher before the researcher met the pupils. The teachers were told to think of pupils who might be able to verbalise their thinking, and to select one girl and one boy, one of whom is a high achiever and the other one a low achiever. This plan worked well with three teachers. However, one of the teachers chose the participants before meeting the researcher and had already asked the pupils about their interest to participate in the study. These pupils also took part in the study. Even though we had a strategy for choosing the pupils, in all cases participation in the study was voluntary.

The selection of participants was made in the mathematics classroom except in the one case where the teacher had asked the pupils before meeting the researcher. The pupils were introduced to the research project and volunteers were asked to raise their hands. Pupils were told that

one boy and one girl would be selected. In the case when there were volunteers, the teacher helped in choosing the participants among them. In other cases, the teacher asked specific pupils if they wanted to be part of the study. No one who was asked refused. Thereafter, the chosen participants had an opportunity to ask questions to the researcher and the form for informed consent was given to the pupils and their parents. (More ethical issues are discussed later in section 4.6. The letters for principals, teachers, pupils and their parents, as well as the form of informed consent for target pupils and their parents are in appendix 3.)

Due to the selection process that was based on the pupils' willingness to participate, not all of the original criteria for participant selection were met. In the end, four girls and four boys participated in the study, of whom six were high achievers and two low achievers. All participants were in ninth grade (the final grade of comprehensive school in Finland). All participating schools, teachers and pupils remained in the project from the beginning to the end.

4.4 Data collections

Case studies rely on interviewing, observing, and document analysis (Denzin & Lincoln, 2000). These are all part of the data collection in this study. For instance, the researcher observed around 20 mathematics lessons in each of the four classes, collected problem-solving solutions on paper from each target pupil (the solutions were used to support data analysis and stimulated-recall interviews), and interviewed each target pupil three times (1-2 hours per interview).

The data was collected in the autumn of 2010 in three cycles over the course of three and a half months. During each of the cycles, all four classes were visited for 1-2 weeks, classroom observations were made, problems were solved in mathematics classrooms, and individual interviews were held with each of the target pupils. More detailed methods of data collection will be discussed in what follows.

4.4.1 Classroom observations

All the classes were observed for four weeks over the course of three months: two weeks in the first round of data collection, and one week in the second and third round of data collection. In the first round, the purpose was to let the teacher and the pupils get used to the researcher's presence in the classroom and to get background information on the working methods normally used in the lessons. This 'background information' can be used to explain some of the possible differences observed in pupils' problem solving and mathematical thinking.

In the first round of data collection, only the mathematics teaching was observed. When the pupils worked on problem-solving tasks in the classroom, the researcher followed the teacher. In the second and third

round of data collection, the target pupils' work was observed as well. During these occasions, the researcher sat next to the target pupil (one at a time) and made notes about the pupil's work and mathematics related interactions in the lesson.

To sum up, during the observations, free notes were taken regarding the mathematics teaching, mathematics related questions asked and answers given in the lessons, and working methods used.

4.4.2 Problem solving

The data for problem solving (problem-solving processes, metacognition, affective states and meta-affect) were collected from mathematics lessons and pupil interviews. In the lessons, the data collection on problem solving focused on simultaneous 'on-line methods' of following pupils' problem-solving processes (observations and videotapes). In interviews, the data collection followed the retrospective 'off-line methods' of stimulated-recall interviews (from metacognitive research cf. e.g. Veenman et al., 2006; van der Stel & Veenman, 2014; Akturk & Sahin, 2011).

In each of the data collection cycles, the classes were given one or two mathematics tasks to solve. The teacher introduced and distributed the task(s) to the whole class. The target pupils' desks were videotaped while they worked on the given tasks, so that the interactions among individuals (teacher and other pupils) could be heard and the solution process observed. The target pupils were advised to think aloud during problem solving, but in quiet classes this invitation was not extended. Thus, in addition to the problems solved in the lessons, more problems were solved in the interviews.

A natural classroom setting was used to give the pupils an opportunity to work in ways familiar to them (a naturalistic approach; cf. Cohen et al., 2007). During problem solving, the pupils were able to ask for help from their teacher and peers. If the pupils talked about the problem with someone, the researcher was able to follow their reasoning. The naturalistic approach and interacting with others also enabled the researcher to find out what kind of difficulties the pupils faced in solving the given problems, and how they would typically try to solve them. After working on their tasks in the classroom, the pupils' work on paper was collected.

The target pupils were interviewed, each individually, on the same or the day following task work in the classroom. The interviews took between one and two hours and consisted of two parts, one about problem solving and one about the pupil's view of mathematics. The problem solving part of the interview was semi-structured and focused (Kvale & Brinkmann, 2009). The stimulated-recall method was used to get a deeper and more elaborate understanding of the pupil's mathematical thinking. Both, the classroom video and the solution on paper were used as

stimuli in the interviews. Before looking at the videos, the pupil was asked about:

- emotions during problem solving (affective states)
- planning the problem solving (resources / heuristics / problem-solving activities / metacognition)
- how did he/she start solving the problems posed in the task (problem-solving activities / metacognition)
- could he/she have proceeded in another way with the problem (resources / heuristics / metacognition)

In the stimulated recall part, the pupil was asked to explain as much as possible of what could be seen in the video: what he/she did and why, if he/she felt something special at any point, or anything that came into the pupil's mind. The researcher also asked questions that came to her mind from watching the video (questions about problem-solving phases, resources, metacognitive activities, affective states or meta-affective activities). Furthermore, observation data was used in the interviews if it was relevant to the problem solution under discussion (e.g. if the voices were not recorded properly in the video, I could ask 'I saw you talking to two friends in the lesson at this point of the problem solution. What did you talk about with them, do you remember? Did it help you to solve the problem? How did it help?'). During the stimulated-recall process, the researcher and the pupil engaged in constructing a shared view of the pupils' mathematical thinking (cf. constructivist world view; e.g. Lincoln & Guba, 2000).

After the stimulated recall part of the interview, the pupil was asked the following questions about the problem-solving process:

- Did your emotions change during problem solving? (At which point? How? ...) (affective states)
- Did you think about your solution while solving, or after solving the problem in the task? (At which point? How? ...) (problem-solving phases / metacognition)
- Did the problem remind you of a similar problem solved earlier? (A method? A concept? A feeling?) If so, how did it affect your solution process? (resources / heuristics / metacognition / affective states)
- What motivated you to solve the problem? (affective state)

Additionally, at this point the pupil was asked to assess his/her confidence in solving the problems in the task. The pupil was asked to use a 10-centimetre-long line to assess his/her confidence prior, during and after solving the problem(s), as well as current confidence in school mathematics. Similar estimations of certainty have been done in previous Finnish studies (e.g. Merenluoto, 2001; Hannula, Pehkonen, Maijala, &

Soro, 2006). An example of the confidence line is presented below in Figure 4.1 (published earlier in Viitala, 2015, p. 141):

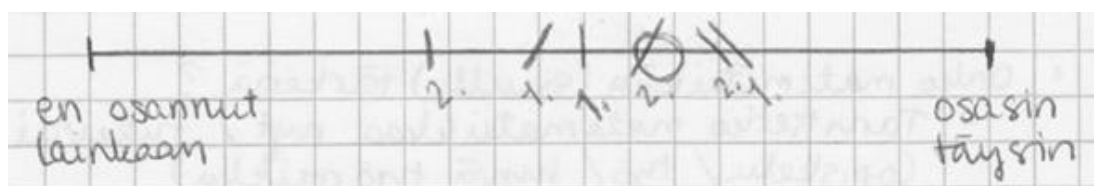


Figure 4.1 An example of the confidence line (2 tasks). The line was 10 cm long with a scale from ‘I couldn’t do it at all’ (left) to ‘I could do it perfectly’ (right). Symbols: Confidence after reading the task |, while solving the problem /, after solving the problem \, and confidence in school mathematics o.

All interviews were videotaped. As in the classrooms, the videos were directed so that the desk with the solution papers and a computer screen were visible, instead of filming the target pupil.

The pupils solved also some additional problems in the interviews. In the second interview, the pupils solved one problem in connection to the discussion about the possible number of answers to a mathematics problem. Also, because the target pupils did not think aloud during problem solving in classroom, the third interview contained a problem-solving section. In that section, problems were solved by the pupil, perhaps with the help of the researcher. The pupil was asked to think aloud when solving the problems and the researcher could ask questions about the solution at any time of the process. This allowed the researcher to get closer to the pupil’s mathematical thinking during problem solving, even though outside of the natural setting. However, data collected from this part of the interview only complemented the results from the actual data collection setting in classrooms and stimulated-recall interviews.

A note about the confidence line. In a study on pre-service teachers’ affective pathways in problem solving, Morselli and Sabena (2015) found that writing down emotions and feelings while solving a problem is a demanding task asking for meta-affective skilfulness, while also interrupting the problem-solving process. Hence, many of the pre-service teachers did not evaluate their confidence during problem solving. Morselli and Sabena concluded that these evaluations should be done only after completing the task.

The present study asked for evaluations and explanations of the pupils’ confidence before, during and after solving the problem. As Morselli and Sabena’s (2015) example showed, this would have been difficult to accomplish in the classroom situation. In addition to the young age of the pupils and the fact that the situation was not authentic anymore in the interview, similar problems would probably have occurred in the stimulated-recall part. Hence, it was justifiable to do the whole evaluation of confidence only after the stimulated-recall part when the solu-

tion process was vivid in the pupils' minds and explanations to the evaluations were easy to give.

4.4.2.1 Tasks

The tasks used in this study are released PISA-items. PISA-tasks were selected because they are well-tested, designed for 15-year-olds, based on real-life situations (problems related to real-life are emphasised in the Finnish curriculum; see FNBE, 2004), and translations are available in different languages (making discussion easier). The selected PISA-tasks represent various mathematical domains (e.g. algebra or statistics), mathematical content (e.g. change and relationship, or uncertainty), mathematical processes (mathematizing) and different types of tasks (open- and closed-constructed response items; see task characteristics e.g. in OECD, 2006). Furthermore, the tasks were selected so that all 15-year-old students should be able to solve the problem tasks introduced in lessons, regardless of the level of their mathematical performance.

Most of the PISA-tasks used in this study were tested for a pilot study in the spring of 2010. One high-achieving 14-year-old pupil solved a number of released PISA tasks from the PISA 2006 framework (see OECD, 2006). Thus, in addition to the abovementioned reasons for the task selection, the tasks were chosen so that they would contain some elements that might cause obstacles in solving the problem and force the pupils out of their 'comfort zone' (e.g. the given data, difficult task description, or multiple choices for presenting or interpreting the results).

A list of chosen tasks and some of the reasons for choosing them are introduced in Table 4.1. The tasks that all pupils worked on in the mathematics lessons and which the target pupils discussed in the stimulated-recall interviews are Holiday, School excursion and Indonesia. Indonesia also included some questions in the lesson about reading the given table. However, the number of problems solved in the second lesson and in the third interview was dependent on the time it took the pupil to solve the given problems. Thus, not all of the problems were actually solved by all pupils. The problem descriptions can be found in appendix 1.

Table 4.1 The PISA tasks used in the study (Sources: 1) OECD, 2006; 2) OECD, 2009; 3) Finnish Institute for Educational Research)

		Task	Some reasons for selection	Source
Round 1	Lesson	Holiday	Problem solving Multiple ways to solve Unusual table of data Long task description	1) pp. 77-78
Round 2	Lesson	School Excursion	Change and relationship Uncertainty Functions	1) p. 87
		Carpenter	Geometry	2) p. 111
	Interview	Distance (modified to 3 and 5 km)	Problem solving Multiple ways to solve Multiple answers Uncertainty	1) p. 102
Round 3	Lesson	Indonesia	Produce a graph Statistics	1) p. 111
	Interview	Growing up	Interpreting a graph Combining data from different sources (task description and a graph)	2) p. 106
		Braking	Unusual graph to present data Using graph data to form a function Combining data from two sets of graphs	2) p. 128-129; 3)

4.4.3 View of mathematics

The data about pupils' view of mathematics was collected in the interviews. For this part of the interviews, the outline for data collection was taken from Pehkonen (1995). The questions asked from the pupils were categorised into mathematics (as a science and as a school subject), oneself as a learner and user of mathematics, learning mathematics, and teaching mathematics. Also background information of the pupil and his/her family (e.g. parents' education and occupation) was collected and questions about the pupils' view of mathematical thinking were asked. The questions in the interviews included cognitive (beliefs and concep-

tions), emotional (feelings) and motivational aspects of affect (cf. Hanula, 2011).

The interview questions were inspired by categorisations, definitions and statements of, for instance, pupils' view of mathematics (Pehkonen, 1995), mathematics related attitudes (Fennema & Sherman, 1976) and affect instrument used in Norwegian KIM-study (see e.g. Kislenko, 2009). The interview protocols are listed in appendix 2 and example questions are seen in Table 4.2.

Table 4.2 Interview themes and example questions.

Number of the interview	Theme	Example questions
1	Background	Tell me about your family. What kind of hobbies do you have? What are your parents' occupations?
1	Mathematics	What is mathematics as a school subject? What is mathematics as a science? Does mathematics exist outside of school? (How? Where?)
1	Oneself and mathematics	Is mathematics important to you? Does it help you think logically? (How?) Has your view of mathematics changed in comprehensive school? (How? When?)
2	Learning mathematics	How do you learn mathematics? Does learning mathematics take time? What motivates you to learn mathematics? How confident are you in learning mathematics? Is it most important to get a correct answer?
3	Teaching mathematics	Does teaching matter to your learning? (How?) What is good teaching? If it depends on teaching, can everybody learn mathematics?
3	Mathematical thinking	What does mathematical thinking mean? How do you recognise it? Where does it exist?

Similarly as in the stimulated-recall part, also in this part of the interviews the observation data was used to ask more specific questions about the pupil's view of mathematics. For example, if the pupil often asks help from a friend in the observed lessons, I could ask 'I saw you asking help from friends quite often in the lessons. Do you usually rather ask help from a friend or the teacher? Why? Does it help you learn mathematics? How does it help? ...'

4.4.4 Pilot study

The data collection methods were tested in a pilot study in the spring of 2010. One teacher, a colleague of the researcher, agreed to make a pilot study. The principal of the school was reached by phone and consent to come to the school was granted. The pupils were chosen by the teacher, and one girl and one boy took part in the pilot study. In the pilot study, the pupils and one of their parents gave informed consent to participate.

The purpose of the pilot study was to test the data collection methods. The researcher observed the lessons for one week and made notes similarly as described before. All the pupils in the class solved the problem tasks from Holiday (OECD, 2006, p. 77-78) in one lesson. The classroom setting was similar to the setting introduced above. The problem solutions were discussed in a stimulated-recall interview with the target pupils in which issues arising from the solutions were asked and discussed. The pilot interviews did not specifically address pupils' view of mathematics, even though some issues were discussed in the interviews. Also, the teacher gave a description of the target pupils' mathematical thinking.

After the pilot study, the pilot class worked on a task called Rising Crimes (OECD, 2006, pp. 94-95) and the solutions were given to the researcher. However, the pupils in the pilot test did not find this problem surprising or problematic at all, so this problem task was not included in the main study.

The pilot study confirmed that the data collection methods are purposeful and thorough for studying pupils' mathematical thinking. The main development concerns measuring confidence: Instead of asking pupils to evaluate their overall confidence in relation to the task, the pupils are asked to evaluate their confidence at different phases of problem solving. Additionally, the researcher gained experience and confidence in observing lessons, making notes and, most importantly, carrying out interviews.

4.5 Data analysis

Yin (2009) argues that analysing case study evidence is 'one of the least developed and most difficult aspects of doing case studies' (p. 127). In this study, the data analysis includes both theory and data driven analyses. How the analysis was made, is described next.

A note about data analysis in the papers. Due to the step-by-step construction of the theoretical (and analytical) framework, the results described in the individual papers are not fully analysed with the methods described in this section of the thesis. The analytical frame was fully used only in papers 6 and 7.

4.5.1 Problem solving

In problem solving, the analysis was mostly theory driven and codes were used to categorise and then further analyse the data (see Kvale & Brinkmann, 2009). First, problem solving was analysed by following the problem-solving phases of Carlson and Bloom (2005) in order to structure and time-frame the problem-solving processes. Then, tasks were reviewed one-by-one to analyse the resources and heuristics (following Carlson & Bloom, 2005), metacognitive actions (following van der Stel et al., 2010), affective states (see Hannula, 2011) and meta-affective actions (see DeBellis & Goldin, 2006) that emerged in the problem-solving processes.

In the analytical steps listed above, the categories and definitions given in the theoretical framework (in Chapter 2) guided the data analysis. However, the actual analysis of resources, heuristics, metacognitive actions, affective states and meta-affective actions was data driven. The purpose of studying pupils' problem solving was to analyse the emergent processes, and thus the correctness of answers was not evaluated.

In paper 6, the development of one pupil's mathematical thinking was reported from comprehensive school to university. In that paper, pupil's mathematical intimacy and integrity (DeBellis & Goldin, 2006; see section 2.4.4) were studied as part of meta-affect in problem solving.

4.5.2 View of mathematics

The pupil's view of mathematics was first analysed by interpreting and describing the issues that the pupil highlighted the most in every category of view of mathematics (see Pehkonen, 1995) and mathematical thinking. In this process, meaning condensation that moves from the more general to the more detailed analysis was used. This method for interview analysis has five steps (Kvale & Brinkmann, 2009):

First, the complete interview is read through to get a sense of the world. Then, the natural "meaning units" of the text, as they are expressed by the subjects, are determined by the researcher. Third, the theme that dominates a natural meaning unit is restated by the researcher as simply as possible, thematizing the statements from the subject's viewpoint as understood by the researcher. [...] The fourth step consists of interrogating the meaning units in terms of the specific purpose of the study. In the fifth step, the essential, nonredundant themes of the entire interview are tied together into a descriptive statement. (ibid., p. 205, 207)

The first three steps help to condense the data, whereas the two remaining steps can include a theoretical analysis which rises from the related literature and is reflected upon along with it (the interview themes following Pehkonen, 1995; see also Yin, 2009). Meaning condensation is based on phenomenological approach: the subject's own perspectives and his descriptions of his own world experiences are preserved while condensing data. Finally, the pupil's confidence in mathematics was assessed based on the confidence line introduced in Section 4.4.2.

At the end, the pupil profile (a short description of the pupil) was constructed following the categorisation and example statements introduced by Rösken et al. (2011; ability, success, difficulty of mathematics, and enjoyment of mathematics; cf. ‘Oneself as a learner and user of mathematics’ in Pehkonen, 1995). The pupil profile also contains the achievement level of the pupil, his/her motivation to study mathematics, and possibly some key element of the pupil’s view of mathematics (emphasised in the pupil’s view of mathematics).

4.5.3 Mathematical thinking: Combining state and trait results

After analysing the problem solving (state) and view of mathematics (trait) data, the results were combined to see the possible trends in the pupil’s mathematical thinking. Particularly, the comparison showed if the explanations of the pupil’s view of mathematics (trait) are consistent with the actual problem-solving behaviour (state). Results from the state data show the fluctuating mathematical thinking skills the pupil have, whereas the combination of the state and trait results display the more stable competences influencing the pupil’s mathematical thinking.

4.6 Ethical issues

Ethical issues are considered from three perspectives: the participants, methodological issues (data collection, analysis and reporting), and the research community in mathematics education. These issues are explained separately in the following sections.

4.6.1 The participants

Informed consent stems from recognizing the individual’s personal dignity and autonomy. Informed consent means that the participant is fully informed, is competent to give consent, fully comprehends the condition of the consent, and gives it voluntarily (Sowder, 1997). For the participants to be fully informed, the aims, methods and possible risks of the study have to be explained so that the participants are able to understand them (Tuomi & Sarajärvi, 2003). As for giving the informed consent voluntarily, the participant also has to have the right to refuse to participate in the study, withdraw from the study at any phase (for any or no reason), the right to forbid the use of material concerning them in the study afterwards, and the right to know these rights (ibid.). These issues are told to the participants in the study in writing or orally.

When asking for the consent to do the study, there were three different letters, one for the principals and mathematics teachers, one for all pupils and their parents, and one for the target pupils and their parents. The letters included information about the researcher, her employer (University of Agder in collaboration with University of Helsinki), what is to be studied, how the study is conducted, how much time the data collection will take, what will be the main concern of the study, and what

methods will be used in data collection. Anonymity and the possibility to withdraw from the study are explained. The participants are promised that the data will not be presented to anyone outside the study without the participant's permission. Also, the researcher's and her supervisors' contact information are provided in case the participants have further questions. All letters can be found in appendix 3.

The study was explained shortly to the pupils during the first day in the classroom, describing the researcher's role in there, giving them an opportunity to ask questions about the study, and encouraging them to ask questions throughout the data collection. The consent from the schools (principals), teachers and pupils were requested orally. The parents gave their consent for the target pupils to participate through email or on paper.

The process of getting informed consent from the participants was the first step in building trust and respect between the researcher and the participants. Because the research objects are only 15-year-olds, it was particularly important to be very clear in explaining what the research is about. The pupils needed to feel safe, and trust between them and the researcher was very important. To enhance trust, the pupils and the researcher discussed the study together and went through its phases. Mathematical thinking might be too abstract for the students to understand, so being as concrete as possible in the explanation of the study was important. I told the participants that I was interested in their thinking. I asked them to think everything aloud, and the fact that they were not being evaluated in any way was emphasised: their intelligence or performance was not studied; they were not given grades based on the study; nor would their teachers hear what we discussed in the interviews. It was my hope that doing all this would result in the participants feeling more confident in expressing their thoughts in the study.

The pupils were explained why and what was videotaped (namely, the desk, so that the problem-solving process can be seen and discussions heard during the stimulated-recall interview and for the data analysis). They were told that the camera would be behind them (so they could not be seen or recognised in the video), and they had the opportunity to see beforehand, in the video camera, what would be visible in the recording. Additionally, before the first problem solving situation in the class, the video cameras were brought to the classroom so that the pupils could see them in advance and get used to having them in class.

Permission to collect the data was requested from the Norwegian data protection agency NSD. All the names in the study are pseudonyms.

4.6.2 Validity, reliability and trustworthiness

The validity of the study was improved with triangulation. Triangulation 'reflect[s] an attempt to secure an in-depth understanding of the phe-

nomenon in question' (Denzin & Lincoln, 2000, p. 5). For triangulation, data about individual pupils' mathematical thinking was collected on paper (the solutions of the problems), observations on problem solving were made, and pupils' verbal descriptions of their thinking were collected (cf. interviews, observations and document analysis in triangulation; Stake, 2000). Problem solving was studied through different perspectives: cognition, emotion and motivation, as well as problem-solving processes, metacognition, affect and meta-affect. Further, the results were compared and enriched with pupil's view of mathematics.

Another validity question is crystallization where 'the writer tells the same tale from different points of view' (Denzin & Lincoln, 2000). In the case of mathematical thinking, the (multiple) problem-solving processes and pupil's view of mathematics both shed light on the pupil's mathematical thinking from different aspects (state and trait data). Especially the problem-solving processes and the pupil's view of learning mathematics seemed to tell the same story about the pupil's thinking (cf. Viitala, 2016).

Case studies have been criticised (among other things) of being subjective (internal validity) and not being generalizable (external validity; e.g. Diefenbach, 2009). To answer this critique, the various words and phrases that the participants used have been preserved in different steps of the data analysis and reporting. The criteria of choosing to report specific pupils' mathematical thinking have been explained in the individual papers that constitute this thesis. Special attention has also been paid not to interpret the results more than is suggested by the data, as the reported data have already been filtered through the theoretical positions (theoretical framework) and biases (Merriam, 1988, cited in Sowder, 1997).

The key in analysing and reporting the data is to be open, honest and precise. If the findings are reported clearly, the misinterpretations of the readers of the results can be decreased. The questions above also measure the reliability and trustworthiness of interpretations.

4.6.3 The research community

The main task for a study is to take knowledge and understanding further. The study has to refer to foregoing research and the results should be applicable and beneficial either directly or indirectly (cf. Mäkelä, 1998). These ethical questions are about being ethical to the education research community. This study is strongly connected to the related literature from the starting point of the project, through data collection and analysis, to the results presented in the papers. The result (especially the formed theoretical and analytical framework for mathematical thinking) is directly applicable to further research on mathematical thinking. In addition, the results from using the framework (especially the develop-

ment of mathematical thinking from comprehensive school to university)
have novelty value.

5 Results

This study has two research questions, the first one referring to the theoretical construct of the framework and its potential to shed light on the nature of pupils' mathematical thinking, and the second one addressing the empirical results that were found by using the theoretical framework as an analytic tool (see Chapter 3). Answers to both research questions can be found from the seven papers that constitute this thesis. Next, the answers to the research questions are presented by walking through the papers one-by-one, starting with the first research question (Section 5.1) and continuing towards the second research question (Section 5.2). In Section 5.3, the development of one pupil's mathematical thinking from comprehensive school to university is described. All the individual papers can be found in appendix 5.

5.1 Constructing a tool for studying mathematical thinking (Papers 1-7)

Even though not all of the seven papers are necessarily needed for the dissertation at hand, I wanted to keep all of them for one specific reason: together they show the path I took while constructing the theoretical and analytical framework for this study. The answer to the first research question 'what can be revealed when mathematical thinking is studied through two different data sets: problem solving (state) and view of mathematics (trait)?' is constructed next by walking through the papers 1-7, one-by-one.

5.1.1 Paper 1: Mathematical thinking

The first step in an attempt to study mathematical thinking is to define mathematical thinking and try to frame it to meet the purposes of the study. From the very beginning it was clear that all the papers in mathematics education are somehow connected to mathematical thinking or its development. However, even when mathematical thinking was specifically mentioned in a study, the term 'mathematical thinking' was often left undefined.

Hence, during the journey of defining mathematical thinking, together with two of my supervisors we decided to write a conference paper about mathematical thinking in Finnish dissertations to grasp how mathematical thinking had been defined earlier in some Finnish studies. Learning on my background in belief-research, the research question for the first paper was 'what have Finnish researchers said about mathematical thinking, with special emphasis on affective factors?'

All the five dissertations studied for the paper were about Finnish secondary school pupils' mathematical thinking published after the year

2000. One of the dissertations aimed at clarifying and refining the definition of affect (Hannula, 2004) and one built a new model for mathematical thinking resting on numerous previous definitions for mathematical thinking (e.g. information processes; Joutsenlahti, 2005). The remaining three did not define mathematical thinking explicitly (Hähkiöniemi, 2006; Hihnala, 2005; Merenluoto, 2001) but looked at mathematical thinking as thinking in or about mathematics (e.g. calculating or explaining understanding).

Affective factors were studied in three dissertations, more explicitly in two of them (Hannula, 2004; Joutsenlahti, 2005) and in one only through the feeling of certainty (Merenluoto, 2001). Hannula's dissertation can be considered as preliminary work towards the model used in this study. Furthermore, there were notions of metacognition in four of the dissertations (Hannula, 2004; Joutsenlahti, 2005; Hihnala, 2005; Merenluoto, 2001), most explicitly in Hannula's dissertation where metacognition was considered to be a central part of the meta-level of mind.

Writing this paper proved to me that it is very complicated and difficult to define mathematical thinking as a phenomenon. Instead, we decided to take a more practical viewpoint to it, defining mathematical thinking in the paper as 'thinking about, on or in mathematics, and in most cases it is thinking that occurs when mathematical tasks or problems are solved or discussed' (Viitala, Grevholm, & Nygaard, 2011, p. 315). This definition has come along with me throughout the PhD process.

The papers also raised my awareness on the construct of affective factors and especially on metacognition as part of the meta-level of mind. Metacognition is reported explicitly as part of the problem-solving process and mathematical thinking for the first time in Paper 3.

5.1.2 Paper 2: View of mathematics

The second paper was about pupil's view of mathematics that at this point was called 'affect in mathematics'. In addition to describing one pupil's view of mathematics, the paper was set to be 'an initial step in the research project to understand how [the] 'interrelationship between affect and cognition' works' (Viitala, 2013, p. 72; referring to Zan et al., 2006).

View of mathematics was analysed by categorising the data following Pehkonen's (1995) categorisation and then through data condensation (Kvale & Brinkmann, 2009; see Section 4.5.2), one pupil's view of mathematics was described. Hence, in addition to discovering the relation of cognition and affect in mathematical thinking, the researcher improved her skills to analyse, condense, and report a large amount of data. This way of analysing pupils' view of mathematics followed me throughout the research project.

The short description of Alex's mathematical thinking at the end of the paper (see the quote below) is illustrative at least in two ways: First, it gives hints of the interrelationship between cognition and affect in Alex's mathematical thinking. The quote includes notions of Alex's mathematical knowledge base (resources), metacognitive skills, affective traits and real-life connection of mathematics (emphasised in the curriculum, see FNBE, 2004). Second, it can be considered to be the first unintentional version of the pupil profile with short descriptions of ability, success, and enjoyment of mathematics (cf. Rösken et al., 2011).

What makes Alex interesting is his high ability to explain his own thinking and the awareness of his own learning. He enjoys doing mathematics but it is not enough to carry the interest outside the classroom. He seems to be very down to earth with his abilities in mathematics and he recognizes that his mastery of mathematics is limited to school mathematics. It seems that it is possible to have highly positive affect in mathematics in school without being that interested in it in everyday life. (Viitala, 2013, p. 81)

5.1.3 Paper 3: Problem solving and view of mathematics

For the third paper, pupils' problem-solving behaviour was analysed for the first time, and the paper represents the first attempt to connect problem-solving data with view of mathematics. Also metacognition was studied narrowly as part of problem-solving behaviour. Pupils' mathematical thinking was studied mainly through criteria set in the curriculum for problem-solving activities, metacognition, and affect (see FNBE, 2004). At this point, view of mathematics was called 'affect related to mathematics'.

For this paper, only the first cycle of classroom and interview data was analysed. The research question for the paper was 'what characterizes the problem solving of two Finnish girls solving a PISA task?' with a more theoretical sub-question 'how do the results reflect the learning objectives, core content and final-assessment criteria of 'thinking skills and methods' described in the Finnish curriculum?' (Viitala, 2015, p. 138) Also the emergent affective states were studied together with task motivation and confidence to solve the problem task. The analysis for pupils' view of mathematics (trait) followed the procedures tested for Paper 2.

The results from problem solving and affect related to mathematics showed very different results for the two pupils: While Emma seemed more competent in problem solving, Nora was more confident, could express her thinking more unambiguously, her view of mathematics was wider, she could connect mathematics to real world more easily, and seemed to take a bigger responsibility of her own learning than Emma. All of these features are listed in the curriculum as learning objectives in mathematics. However, only problem solving is described as final-assessment criteria.

Even though these two girls seemed quite similar on the surface level, they showed very different competences in mathematics. This is when the overarching aim for the PhD project changed from describing the similarities and differences in pupils' mathematical thinking and for instance dividing their thinking into different levels, into creating a tool that can help recognising what kind of support different pupils might need in learning mathematics and developing their mathematical thinking.

5.1.4 Paper 4: The interrelationship between problem solving and view of mathematics

In the fourth paper all the pupil data for a single pupil was analysed for the first time (all three rounds of data collection). Hence, multiple problem-solving processes were compared together and the results were combined with the pupil's 'affect related to mathematics' (which was later in the paper also called 'pupil's view of mathematics'). The aim of the paper was to understand the interrelationship between cognition and affect in mathematical thinking.

The analysis on problem-solving phases followed Polya's (1957) description (rather than the description in the curriculum, as in previous papers). Metacognition was still studied through criteria set in the curriculum (FNBE, 2004) but the definition was widened to include also other metacognitive behaviour described by Schoenfeld (1987). Affective states and traits were analysed similarly as in earlier papers.

Results from Emma's problem solving and view of mathematics seemed to give a well-matching picture of Emma's mathematical thinking. The problem-solving behaviour (state) correlated with Emma's explanations about her learning of mathematics (trait) on cognitive, emotional and motivational levels (cf. Hannula, 2011). Furthermore, it seemed to be Emma's uncertainty that made her successful in mathematics (the impact of emotions on cognition through actions in mathematical problem-solving and learning).

A closer analysis on Emma's mathematical thinking truly showed the interrelationship between cognition and affect, as well as state and trait aspects of mathematical thinking. Nonetheless, the theoretical and analytical tools needed much more work to be complete.

5.1.5 Paper 5: A tool for studying mathematical thinking

The first four papers were initial steps on a journey to create an analytic tool that could be used to analyse pupils' mathematical thinking, a tool that would show the dynamic nature of mathematical thinking both between cognition and affect, and between state and trait aspects of mathematical thinking. The first results presented in papers 1-4 showed a promising start for this aim. Hence, the purpose of the fifth article was to further develop the theoretical framework and analytic tool for studying

mathematical thinking. The research question for the paper was a theoretical one: Is it possible to construct a tool for understanding pupils' mathematical thinking that shows the dynamic process of problem solving, metacognition and affect in their thinking? (Viitala, 2017a)

After further work with related literature, the theoretical framework for this paper started to look a lot like the framework presented in Chapter 2 (see Figure 5.1, cf. Figure 2.1). The same framework was used as an analytical tool in the paper. Problem solving was studied through problem-solving phases (Polya, 1957), mathematical metacognitive activities (van der Stel et al., 2010) and affective states (Hannula, 2011, 2012). As part of affective states also meta-affect (DeBellis & Goldin, 2006) was recognised. View of mathematics still followed the definition from Paper 2 (following Pehkonen, 1995) but a pupil profile was added to the framework (Rösken et al., 2011). A simple structure of the framework is presented in Figure 5.1.

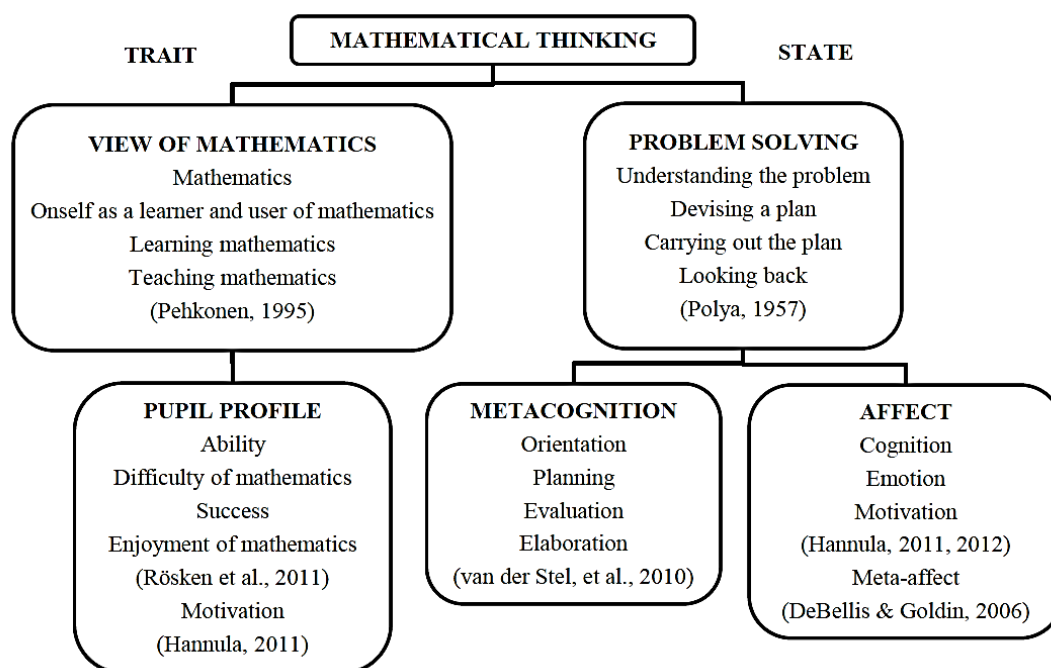


Figure 5.1 A simple structure of the theoretical and analytical framework presented in Paper 5 (Viitala, 2017a; cf. Figure 2.1).

After analysing Daniel's mathematical thinking with this tool, the results of Daniel's problem-solving processes and view of mathematics did show the interrelated and dynamic construct of his mathematical thinking. In fact, problem-solving activities, metacognitive skills and affect were shown to be an inseparable part of Daniel's thinking process that was also supported by his view of mathematics. However, a comparison with Emma's results (Paper 4) showed that the tool can give different results for different pupils. The results also revealed the weaknesses that

pupils have in thinking mathematically. Hence, in addition to assessing pupils' mathematical thinking skills, the framework can be used to help pupils to develop their mathematical thinking (cf. tasks for instruction in FNBE, 2004).

Additionally to successfully construct a tool that shows the dynamic process of problem solving, metacognition and affect (state and trait) in pupils' thinking, the tool allows individuals to be individuals. Hence, slowly I started to become closer to the answer for the first research question in the PhD project (what can be revealed when mathematical thinking is studied through two different data sets: problem solving (state) and view of mathematics (trait)?). However, some modifications still needed to be done before the tool was 'ready'.

5.1.6 Papers 6 & 7: Mathematical thinking at different ages

After writing Paper 5, I got an opportunity to interview Alex (Paper 2) again as a university student. Hence, the aim of the sixth paper was to restructure the framework for studying mathematical thinking so that it could be used at different ages. For this purpose, the framework used in Paper 5 was combined with a similar framework that was formed after studying professional mathematicians' problem-solving behaviour (Carlson & Bloom, 2005) in a hope that it could contribute a new insight into how to study mathematical thinking and its development at different educational levels.

There were three modifications done to the previous version of the framework (presented in Paper 5). First, problem solving was studied through the phases introduced by Carlson and Bloom (2005). In addition to the similar problem-solving phases that Polya (1957) had, Carlson and Bloom emphasised the cyclic nature of problem solving and introduced a sub-cycle of conjecturing in planning. While conjecturing, the mathematicians imagined the problem solution process through before starting to solve the problem. Another change was done with the categorisation of meta-affect. In this version of the paper, meta-affect is 'lifted' from a sub-category of affect into its own category that is parallel to metacognition. Also, a sub-category to meta-affect was formed for this paper following Carlson and Bloom, and mathematical intimacy and integrity were studied as part of meta-affect.

For the sixth paper, Alex's data was analysed both from the comprehensive school and from university points of view. In addition to the development of mathematical thinking that Alex had gone through in the four years between comprehensive school and university, the results showed which elements are more useful in studying young adult's mathematical thinking as opposed to studying a 15-year-old. First, the conjecturing cycle was not visible in Alex's problem solving in 9th grade, nor was it found from any other pupils studied for this project. Additionally,

mathematical intimacy and integrity were not that prominent for a comprehensive school pupil, nor was it that important as a university student either (perhaps due to the task selection for problem solving at university). Hence, the conjecture cycle, and mathematical intimacy and integrity were not included in the theoretical framework used in Paper 7 and presented in Chapter 2 for studying comprehensive school pupils' mathematical thinking. However, I wanted to use Carlson and Bloom's (2005) model for problem solving rather than Polya's (1957) model, since it emphasises the cyclic nature of problem solving, and it arises attention to the possible emerging of the conjecture cycle.

5.1.7 Summary

The purpose of Section 5.1 was to show the path I went through while constructing the theoretical and analytical framework for studying comprehensive school pupils' mathematical thinking. This was done keeping in mind the first research question for the PhD project: What can be revealed when mathematical thinking is studied through two different data sets: problem solving (state) and view of mathematics (trait)?

From the very beginning it was clear to me that I wanted to study mathematical thinking from two perspectives: problem solving and view of mathematics. What these components can reveal depends on the lenses we choose to look through while collecting and analysing my data. The initial focus on problem solving came from the Finnish curriculum. After searching through literature and testing approaches used in earlier studies, I ended up looking at problem solving through problem-solving activities, metacognitive skills, affective states and meta-affect. Even though pupils' view of mathematics was studied separately, only after combining the results with problem-solving results a thorough view of pupils' mathematical thinking was formed.

On one hand, the results show that looking at mathematical thinking through problem solving and view of mathematics can reveal insights into the dynamic and complex processes pupils go through when thinking mathematically. The interaction between state and trait aspects of thinking as well as the intertwined relationship between cognition and affect can both be revealed when using the analytic tool created for this study.

Additionally, the tool reveals individual pupils' strengths and weaknesses in solving problems and learning mathematics. This knowledge can be further used to support the growth of individual pupils' mathematical thinking in school. An example of how this can be achieved is given in Paper 5 (Viitala, 2017a). In the next section, the results of individual pupils' mathematical thinking will be introduced through the results reported in papers 2-7.

5.2 Comprehensive school pupils' mathematical thinking (Papers 2-7)

The second research question is the one this PhD project actually started out with: What characterises the mathematical thinking of four Finnish high-achieving pupils at the end of comprehensive school? The purpose was to get in-depth knowledge of the pupils' mathematical thinking and perhaps reveal some trends in high and low achievers' mathematical thinking. However, this aim changed after writing the third article. Some aspects of the possible trends can be seen in large scale studies on pupils' mathematical thinking and problem solving. What cannot be seen from these studies is how these pupils' can be supported individually in developing their mathematical thinking.

In Section 5.2, the purpose is to describe shortly some aspects of pupils' view of mathematics and problem-solving processes without repeating too much of what have already been said in the seven papers constituting this thesis. The descriptions start with the pupil profile that emphasises the strengths of the pupil's mathematical thinking. This is followed by some examples of the pupil's view of mathematics (following the categories of mathematics as a school subject or as a science, oneself as a learner and user of mathematics, learning mathematics and teaching mathematics; Pehkonen, 1995) and results from problem-solving processes. At the end, these results are brought together and a recommendation is given on how the development of the pupil's mathematical thinking could be supported.

5.2.1 Alex (Papers 2, 6 & 7)

The case of Alex was discussed in three papers. In Paper 2, Alex's view of mathematics was described quite thoroughly. In Paper 6, the results of Alex's view of mathematics was compared and combined with his problem-solving processes. In paper 7, Alex's results were compared with other pupils'. Alex was chosen to be the first case analysed from the group of participants because he seemed exceptionally competent as a mathematics learner and as a problem solver. In this section of the paper, Alex's mathematical thinking is described as it was in comprehensive school. The development of Alex's mathematical thinking from comprehensive school to university will be discussed in Section 5.3.

Pupil profile: Alex is a high achiever in mathematics. He feels able to do mathematics and enjoys learning it. He is motivated, self-confident and trusts his own reasoning more than his calculations. He is aware of his own mathematical thinking and learning processes, and seems to have a very clear and organised net of knowledge that has a strong tool-value for him.

The pupil profile says it all: Alex has a very positive view of mathematics and picture of himself as a learner and user of mathematics. Further, Alex explains that mathematics as a science is ‘explaining different problems or natural phenomena, or such, with the assistance of calculations’ (Viitala 2013, p. 75). He thinks mathematics is important but he is not that interested in it outside school. He explains that his own learning process includes understanding the goal in learning something, calculating tasks from easier to more difficult and finding routine, and seeking similarities between the new thing and things learnt before. The teaching in his mathematics lessons seems to support his way of learning.

Alex is fluent and thorough also as a problem solver. For instance, he gets orientated by looking for a simpler problem, making sense of the problem, organising information and estimating the answer. All this gives him confidence for the rest of the problem-solving process. Furthermore, he can move naturally between different phases of problem solving, he is aware of his own thinking, and he can fluently explain and justify his cognitive and metacognitive actions. Below in Figure 5.2 is a typical confidence line showing Alex’s confidence in problem-solving process, as well as in mathematics.

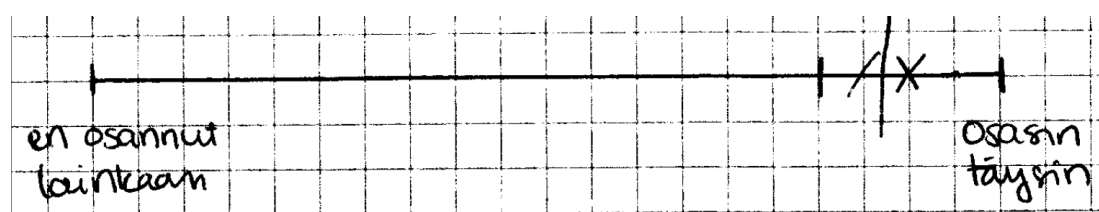


Figure 5.2 Alex’s estimation of confidence with Holiday, task 1 (OECD 2006, p.77). The line is 10 cm long with a scale from ‘I couldn’t do it at all’ (left) to ‘I could do it perfectly’ (right). Alex used the following symbols: confidence after reading the task, ‘shorter’ |; while solving the problem, /; after solving the problem, X; and confidence in school mathematics, ‘longer’ |.

Alex’s view of mathematics (trait) and problem solving (state) gives a well matching picture of Alex’s mathematical thinking. He is aware of his thinking and fluent in explaining and justifying his cognitive and metacognitive actions both in learning mathematics and problem solving. He is confident and has a positive attitude towards learning mathematics and solving problems. However, even though Alex’s mathematical thinking is thorough and flexible, there still is something that might help him to develop his mathematical thinking: Alex’s view of mathematics in his own life is quite limited. He also does not relate the problems in the project to real life. Hence, recognising mathematics more in his own life could enrich Alex’s view of mathematics, and through that, also his understanding of school mathematics might develop.

5.2.2 Nora (Papers 3 & 7)

Nora was selected for the third paper together with Emma because they seemed to have similar skills and achievement in mathematics. In Paper 3, Nora's view of mathematics as a school subject and as a science was described together with the problem-solving processes she went through in the first data collection cycle. In Paper 7, the overall results from Nora's mathematical thinking was described shortly and compared with other pupils in the study.

Pupil profile: Nora is a high achiever in mathematics. She likes mathematics very much and is motivated to learn it. She is quite confident in mathematics, and even though learning mathematics takes time and effort for her, she enjoys learning it. She can explain her thinking processes fluently and connects mathematics easily to her everyday life.

For Nora, mathematics offers confidence in her life. She sees mathematics everywhere: in her own life, other school subjects and in science. For Nora, learning mathematics is interesting and produces feelings of success. Even though Nora's family members cannot help her with homework (on the contrary, Nora helps her mother in mathematics), they value learning mathematics as one of the basic things that should be learnt in school. Hence, Nora thinks mathematics is important, even though her friends and classmates do not share this view. Furthermore, in a similar way as the way in which teaching proceeds in her class, Nora starts learning new content in mathematics with something that feels easy and that is connected to earlier content (the latter does not always happen), and applying comes later on.

In problem-solving situations, Nora is flexible in directing her actions based on the affective states occurring in the thinking processes (meta-affective skills). Her confidence in problem solving might fluctuate anywhere between 0 and 10 in the confidence line (see an example in Figure 5.3). Furthermore, she is fluent in moving between orienting, planning and executing in problem solving. However, her planning and executing often goes in quick cycles: she might simply start solving something, and make the plan step-by-step as she goes along. Also, she is often happy with the first answer she gets, and does not check her results.

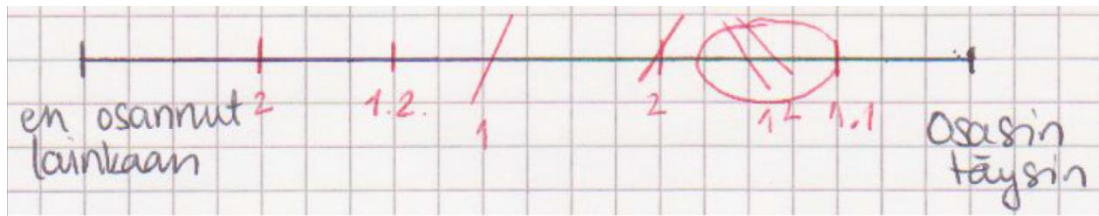


Figure 5.3 Nora's estimation of confidence with Holiday, tasks 1 and 2 (OECD 2006, p.77). The line is 10 cm long with a scale from 'I couldn't do it at all' (left) to 'I could do it perfectly' (right). Nora used the following symbols: confidence after reading the task, |; while solving the problem, /; after solving the problem, \; and confidence in school mathematics, o. Confidence after reading the first task are marked twice, 1.1 is confidence after reading the text, and 1.2 confidence after reading the table in the task.

What characterises Nora's mathematical thinking best is her fluency in expressing her thinking both about learning mathematics and in problem solving; and 'seeing' mathematics all around her in her own life, in other school subjects and in science. She could benefit from (both in state- and trait-wise) directing more attention to the checking phase in the problem-solving processes. For Nora, a mathematics problem always has one exact answer. That, together with her very straightforward way of solving problems and her tendency to make careless errors, directs attention towards supporting her to look back, and perhaps exposing her more to open-ended problems, for instance. These might help her to become a more reflective user and learner of mathematics, and through them, to further develop her mathematical thinking.

5.2.3 Emma (Papers 3, 4 & 7)

Emma's mathematical thinking was discussed in three papers. For Paper 3, the first round of data collection was analysed in order to compare Emma's mathematical thinking with Nora's thinking. Even though Emma was a high achiever and a successful problem solver, her lack of confidence made her an interesting case for further studies. Thus, for Paper 4, the data from all three cycles of data collection were analysed and Emma's mathematical thinking was described more thoroughly. In Paper 7, Emma's results were compared with other pupils' mathematical thinking.

Pupil profile: Emma is a high achiever in mathematics. She values mathematics, thinks it is important, a bit easy and can be quite fun to learn. She is not very confident in mathematics but her thoroughness in problem solving and learning mathematics makes her successful. Yet, succeeding in mathematics and being proud of herself keeps her motivated to learn mathematics.

In addition to the positive view of herself as a mathematics learner, Emma also links many less positive affects to it. For instance, learning can be irritating and tiring, it takes time and effort, and she believes her grade in mathematics is the highest she can achieve. Emma sees mathe-

matics as calculations both in and outside school. She knows well what is needed for her to learn mathematics, and good teaching supports these needs: for instance, Emma needs time for studying and searching for routine, and above all, she needs opportunities to ask questions. Emma does not actively link new knowledge to prior knowledge. She takes responsibility of her own learning but still at some level she connects her success in mathematics to her teacher.

Due to her uncertainty in mathematics, Emma uses a considerable time for getting oriented in problem solving. She explains that she wants to understand every aspect of a problem before starting to solve it. Consequently, she plans the solution quickly and starts solving the problem. On the other hand, not understanding the task completely might also hinder her ability to solve the problem. She moves fluently between problem-solving phases, in which affective states and meta-affect seem to play a key role. She is very careful in her thinking and working. Her affective states fluctuate much more than her confidence in mathematics in general, which was very stable throughout the study (between 5.5 and 6.25). An example of Emma's confidence line is presented in Figure 5.4 (already presented as Figure 4.1; Viitala, 2015, p. 141).

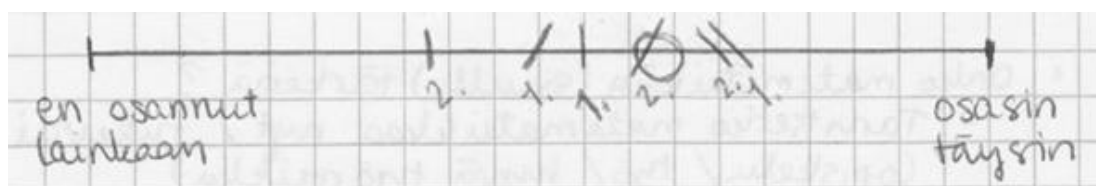


Figure 5.4 Emma's estimation of confidence with Holiday, tasks 1 and 2 (OECD 2006, p.77). The line is 10 cm long with a scale from 'I couldn't do it at all' (left) to 'I could do it perfectly' (right). Emma used the following symbols: confidence after reading the task, |; while solving the problem, /; after solving the problem, \; and confidence in school mathematics, ○.

What is prominent in Emma's learning mathematics and solving problems is her uncertainty and subsequent thoroughness. In both activities she asks a lot of questions from her teacher and friends, and learning seems to happen in a social communication with others. Emma has already turned her uncertainty in mathematics into success in problem solving and mathematics learning. However, she still learns every topic in mathematics as its own entity and does not connect new knowledge to prior knowledge. This might hinder her learning. Hence, supporting Emma emotionally could open doors to a more thorough learning and understanding of mathematics that would develop also her mathematical thinking.

5.2.4 Daniel (Papers 5 & 7)

Daniel's mathematical thinking was discussed in Papers 5 and 7. For Paper 5, all the data about Daniel's mathematical thinking was analysed and reported. Daniel was selected for the paper because his performance in mathematics seemed to be very similar to Alex's. It would be interesting to compare their mathematical thinking later on and find out if their mathematical thinking was similar when studied through the tool constructed for this thesis. This was done in Paper 7, in which Daniel's mathematical thinking was compared with all four high achievers' thinking reported in earlier papers.

Pupil profile: 'Daniel is very confident and successful in mathematics. He has the highest grade in mathematics and he is very aware of his success. He likes mathematics, it is easy for him and he is motivated to learn it. He values mathematics and it is one of his favourite subjects.' (Viitala, 2017a, p. 17)

Daniel thinks that mathematics is the most important school subject, and that mathematics is needed everywhere through life. Mathematical thinking appears through calculations and models of thinking. Daniel has difficulties in explaining how he learns mathematics, things just seem to 'click together' or 'become familiar'. He does most of his learning in mathematics lessons, which might indicate that he learns through the steps the teacher goes through while teaching.

Daniel is confident, successful and thorough problem solver who moves naturally between different problem-solving phases. Orienting and checking are mostly guided by the affective states and Daniel's meta-affective skills: In both phases, he looks for a feeling of confidence before he is ready to move on to the next phase in his problem-solving process. Even though confidence plays a key role in planning and executing, it is Daniel's metacognitive skills that are prominent in these phases of problem solving. As an example, a surprising answer might cause him to return to the task description or devise a new plan for the problem solution.

Daniel is extremely confident in learning mathematics and in problem solving. He becomes more and more confident in problem solving every step he takes reaching 95-100% confidence at the end of the problem-solving process. He expresses the same confidence also in school mathematics, which was estimated three times during the research project. An example of his confidence line can be seen in Figure 5.5. This example has the most variance in the estimations of confidence (for instance in one case, all but the first estimations were utter 100%).

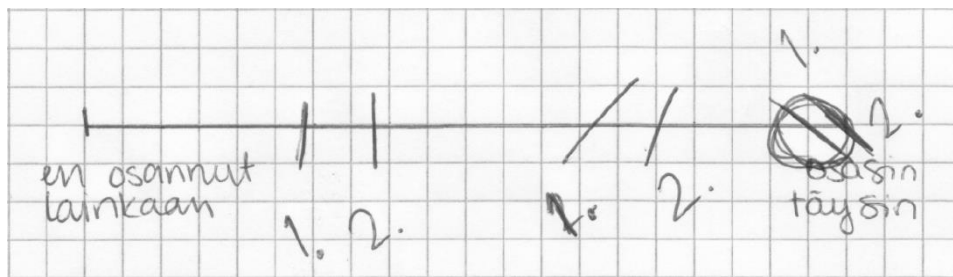


Figure 5.5 Daniel's estimation of confidence with Holiday, tasks 1 and 2 (OECD 2006, p.77). The line is 10 cm long with a scale from 'I couldn't do it at all' (left) to 'I could do it perfectly' (right). Daniel used the following symbols: confidence after reading the task, |; while solving the problem, /; after solving the problem, \; and confidence in school mathematics, ○.

Even though Daniel's thinking and problem-solving processes are successful, his work on paper can be quite disorganised. His notes are messy, they often do not proceed chronologically, and one written expression might be used to calculate many calculations. He also has difficulties in explaining his mathematics learning and problem solving post-situation. These together direct my attention towards a point where Daniel could be helped to become a more successful mathematical thinker: Daniel might benefit from paying more attention to his problem-solving and learning processes. Furthermore, practicing more on clear and precise pencil-and-paper work could also develop his problem-solving and learning skills, and through them, his mathematical thinking.

5.2.5 Summary (Paper 7)

Paper 7 was written for the purpose of bringing together the results of the four pupils' mathematical thinking and answer the question 'what similarities and differences related to mathematical thinking can be found between these [four high achieving] pupils?' After going shortly through Alex's, Nora's, Emma's and Daniel's results on their view of mathematics and problem solving, the answer to the research question set for the paper was as follows (Viitala, 2017c, p. 1223):

The results showed that the similarities between the pupils were found to be mainly on a surface level: all the pupils liked mathematics, were motivated to learn it, enjoyed doing mathematics and were successful problem solvers. However, after a deeper look into their problem-solving processes and view of mathematics, the study revealed a great deal of differences between the pupils, and showed different competences: Alex is a very conscious thinker and learner of mathematics, and excellent in justifying his thinking and actions in mathematics. Daniel is extremely confident and metacognitive skills are prominent in his problem solving. Emma is an unsure but very thorough problem solver and learner of mathematics. Nora is fluent in expressing her thoughts and connecting mathematics to real life.

Hence, the answer to the second research question 'what characterises the mathematical thinking of four Finnish high-achieving pupils at the end of comprehensive school?' depends on the characteristics of the in-

dividual. Even though all the pupils were successful problem solvers and enjoyed learning mathematics, what characterised their thinking best were their personal skills as problem solvers and competences in mathematics. Hence, the similarities found in their mathematical thinking can be said to be on a surface level and the differences were shown to be deeply personal characteristics.

5.3 The development of Alex's mathematical thinking (Paper 6)

Even though the development of Alex's thinking does not fit into the scope of this thesis, there are two specific reasons I want to report on it: First, it further validates the results gained from comprehensive school. Secondly, it shows the growth in Alex's mathematical thinking and expresses how recognising mathematics more in his own life (in- and outside school) has enriched his view of mathematics, and his understanding of school mathematics (cf. Section 5.2.1).

As a comprehensive school pupil, Alex seemed to like explaining his thinking to the researcher. In mathematics class, he measured his competences by comparing himself to his classmates. Hence, he valued this opportunity to get feedback about his mathematical thinking from an outsider. At the end of the data collection he asked for an honest feedback about his mathematical thinking. Hence, after Paper 2 about Alex's world of mathematics was published, I sent it to Alex. Surprisingly, he commented on it and agreed to a follow-up interview that was agreed to happen later that year. (A translation of the reflective email Alex sent as a comment to Paper 2 is in appendix 4.) At the time of the follow-up interview, Alex had just started his university studies.

For Paper 6, the theoretical framework was developed so that it can be used to study mathematical thinking at different ages, or at different educational levels (see Section 5.1.6 or Paper 6 for further details). The theoretical framework was then used as an analytic tool to study the development of Alex's mathematical thinking from comprehensive school to university. Some of the key results of the development are introduced next.

In upper secondary school, Alex's interest moved from mathematics towards physics and chemistry. His motivation to learn mathematics decreased and he needed more time and effort to learn mathematics. However, he still considered mathematics important and interesting, enjoyed learning it, believed in his abilities and success in it, was confident and got excellent grades.

Student profile (upper secondary school / university): 'Alex is a successful student who thinks that mathematics is exciting and challenging in an interesting way. He is self-motivated and diligent in learning

mathematics but motivated mostly by a good grade. He sees mathematics as a tool and an inseparable part of physics and chemistry.’ (Viitala, 2017b)

As a comprehensive school pupil, Alex’s weakness in mathematics related to the lack of linking mathematical problems to real life (problem solving) and seeing mathematics quite narrowly in his own life (view of mathematics). Perhaps the biggest change in Alex’s view of mathematics from comprehensive school to university concerned this exact observation. In upper secondary school Alex became more aware of the role of mathematics in his life (a tool) and he became more interested in mathematics in real life. Also the development of more general study skills opened new perspectives in seeing mathematics as ways of thinking and learning. As a result of this development, Alex sees mathematics everywhere now: for instance in architecture, arts, philosophy, traffic control and medicine. Mathematics has become ‘a buttress’ that he can trust and lean on, bringing confidence and order to a hectic world. He sees mathematical thinking at the core of all learning. Furthermore, he says that

It is exciting how mathematics can be used to describe the world, with numbers invented by humans, and still it feels like there is some truth behind it. Somehow, nature is based on that even though it is invented by humans. It is exciting!

As a problem solver, Alex’s development can be best characterised by the growth of mathematical knowledge. The processes he goes through when solving problems as a university student are the same as in comprehensive school (for a thorough description of one problem-solving process as a comprehensive school pupil, see Paper 6). However, after upper secondary school Alex started to use the conjecture cycle in his problem-solving processes: he started to think problems through as thoroughly as possible to avoid careless errors. This resulted into an increase of time used for orienting and planning. Simultaneously it moved him closer to the activities of experienced problem-solvers (cf. Carlson & Bloom, 2005). As a university student, Alex thinks that the ability to solve problems is essential in a world where reasonable results should be reached quickly (cf. the connection between problem solving and real world).

After the first version of the manuscript of Alex’s second paper (Paper 6) was ready, it was sent to Alex for commenting. In addition to the supplementary data about Alex’s thinking, the comments also increase the validity of the study. Alex agreed with the descriptions I had written about his mathematical thinking in comprehensive school and how it had developed over the years. For more detailed comments, see the translation of the reflective email Alex sent as a comment to Paper 6 in appendix 4.

6 Limitations of the study

As all studies, this study has limitations. In this chapter, I will shortly reflect on some of the limitations I have come across during the PhD project. I will explain how these limitations affected the study and/or how the challenges were dealt with.

First of all, there were eight pupils that participated in the study. However, only four pupils' mathematical thinking has been reported. This is due to the limitation of the study time. When I started the PhD project, I planned to study pupils' mathematical thinking at the end of comprehensive school through problem solving and view of mathematics. These both aspects revealing and influencing mathematical thinking have been studied extensively over the years. However, what came as a surprise for me was that these two aspects have rarely been studied deeply together. The construction of the theoretical and analytical framework was time consuming, and testing the tools needed thorough work with the data collected from individual pupils. Thus, only four pupils' mathematical thinking ended up being part of the results reported in the PhD project.

This brings me to another limitation of the study: All the reported pupils were high achievers in mathematics, even though also low achievers were participating in the study. This causes an imbalance to the results. Hence, the second research question was altered from 'What characterises the mathematical thinking of Finnish pupils at the end of comprehensive school?' into 'What characterises the mathematical thinking of four Finnish high-achieving pupils at the end of comprehensive school?' The next reasonable step would be to analyse the low achieving pupils' mathematical thinking and report them to the research community. This would strengthen the applicability of the analytic tool and bring trustworthiness to the results emerging from the project.

The third limitation of the study concerns the absence of thinking-aloud protocol as an on-line data collection method. Thinking aloud would help the researcher to follow the pupils' authentic in-situ thinking in problem-solving processes (e.g. Ericsson & Simon, 1998). The pupils were instructed to think aloud while working on the tasks in mathematics lessons. At the same time, the problems were solved as part of ordinary mathematics lessons following the teachers' instruction for the problem-solving situation (a naturalistic approach; cf. Cohen et al., 2007; Lincoln & Guba, 2000). In many cases, this meant that the pupils worked on the tasks in silence and did not talk with each other unless they confronted some obstacles in their problem-solving processes.

The lack of thinking aloud in mathematics lessons and supporting solely on stimulated-recall methods in the interviews might cause a loss

of causality in pupil's explanations of mathematical thinking (Ericsson & Simon, 1998). In the two other data collection cycles after noticing the absence of thinking aloud in problem-solving situations, there were two attempts to correct the situation. First of all, the problem solution on paper was compared with the video showing the in-situ problem-solving process before the stimulated-recall interview. This helped the researcher to keep track of the problem-solving phases the pupil went through while solving the problem. Often the pupils also talked about the problem with a friend at some point of the process, which made it easier for the researcher to follow their thinking.

Additionally, to make up for the loss of data from the absence of thinking-aloud protocol, some problems were solved also in the third interview. In those situations, the researcher kept on reminding the pupils to think aloud. These problems were used to complement the results gained from problem-solving processes in the mathematics lessons, so that the naturalistic approach would not be compromised. Also, the pupils' view of mathematics, and especially their explanations about their mathematics learning often supported the findings from the problem-solving processes, which increases the reliability of the results in the study.

The fourth limitation of the study considers the social aspect of studying pupils' mathematical thinking (cf. Hannula, 2011, 2012). Many recent studies also in Finland have moved from studying individuals as individuals to studying individuals as part of a group. An example of this is a recent dissertation on mathematics related to affect as a social and cultural phenomenon (Tuohilampi, 2016). Originally, in my study, the teaching was observed and teachers were interviewed to get a clear picture of the teaching and learning culture in the participating classes. The purpose was to reflect about the individual pupils' problem-solving activities and mathematical thinking with the practices in the mathematics classrooms (social traits in Hannula, 2012). However, due to the limitation of time and the extensive amount of data collected in this study, only the observation data was used to support questioning about individual pupils' problem-solving activities and view of mathematics in the interviews (on a state level e.g. metacognitive decision to draw a picture of a problem if it is usually done in mathematics lessons; or on a trait level e.g. explanations about teaching mathematics). A further examination of the pupils' mathematical thinking in light of this data would enrich the results found from this study.

The fifth limitation concerns additional validation of the results. In the teacher interviews (that were not analysed for the PhD project), the teachers' views of the individual pupils' mathematical thinking were collected. This was done for crystallising the research results through an

analysis that would involve the pupil, his/her teacher, and the researcher explaining the same pupil's mathematical thinking (see crystallisation e.g. in Denzin & Lincoln, 2000). However, combining problem-solving data (state) and pupil's view of mathematics (trait) already offers two different data sets that aim to describe the same phenomenon (mathematical thinking). Hence, I decided that the teacher's view of the pupil's mathematical thinking would not bring extra value to the already existing description of the mathematical thinking (that had been constructed together with the pupil; cf. co-created findings in interpretive paradigm; Lincoln & Guba, 2000).

The final limitation is connected to the interpretation of the results. In addition to designing the study, I collected, analysed and reported the data alone. I have dealt with this matter in the project in various ways. Here are some examples: First of all, the validation process started from the planning stage of the data collection. The individuals' results are based on multiple problem-solving processes (repetition) and three separate interviews, in which I, for instance, repeated or rephrased pupils' answers to make sure I understood them correctly. During the analysis, I pursued to find corresponding actions and explanations to support initial findings. In reports, I used excerpts from the data to support the interpretations of the results. Moreover, I have discussed my research project and results in various seminars and conferences. On these occasions, colleagues and members of the research community have had opportunities to comment and ask questions about my research project and results. Following these discussions, I have been able to improve my writing, simultaneously improving the reliability and trustworthiness of my work and interpretations (more ethical issues in section 4.6).

7 Discussion on results and theory

7.1 Developing mathematical thinking through formative and summative assessments

In the past decade or so, Finnish comprehensive school pupils' mathematical performance has been evaluated both nationally and internationally (see e.g. Metsämuuronen, 2013; Vettenranta et al., 2016; Mullis et al., 2012). Even though these large scale studies show how Finnish pupils' mathematics performance is descending, the results remain at a satisfactory level or above OECD average (e.g. Hirvonen, 2011; Vettenranta et al., 2016). But what entails this 'satisfactory level'? What kind of mathematical thinking is involved in Finnish pupils' mathematical activities? What we lack in Finland is deeper knowledge about individual pupils' mathematical thinking, especially at the end of comprehensive school. Moreover, we need more information on how these pupils' thinking could be developed at an individual level so that the descending trend could be decelerated or even reversed.

This study is a small initial step towards answering the questions above. The mathematical thinking of four high-achieving pupils at the end of comprehensive school is described using the theoretical and analytical framework developed for this study, and suggestions are given on how these pupils' mathematical thinking could be developed. The suggestions can involve cognitive, metacognitive, affective (state and/or trait) and meta-affective aspects depending on the individual's strengths and weaknesses in mathematical thinking. The central idea in developing pupils' mathematical thinking is in highlighting and preserving pupils' strengths in mathematics and directing their attention towards their weaknesses. The actual changes happen slowly and recognising them is just a beginning for the pupils. However, with the help from their mathematics teachers, this development is possible.

Even though not studied explicitly, this study offers one indication of the impact of noticing the weaknesses in someone's mathematical thinking. The suggestion to develop Alex's mathematical thinking in comprehensive school involved paying more attention to the real-life connections both in problem solving (state) and in his own life (trait). This observation turned out to be one of the key factors in the development of Alex's mathematical thinking from comprehensive school to university. Paying conscious attention to the real-world connections made Alex even more reflective citizen and moved him a step closer to the thinking of professional mathematicians described by Carlson and Bloom (2005; see Paper 6).

The new curriculum (FNBE, 2014) describes pupils as active learners who, during the comprehensive school, should learn to solve problems and set goals to their learning. While learning new knowledge and skills, they should learn how to reflect their learning, experiences and feelings (ibid., p. 17). Teachers can help their pupils with these learning goals through formative assessment. Formative assessment happens in everyday activities by observing pupils' learning processes and communicating with the pupils. Pupil feedback should be qualitative and descriptive and should help pupils to perceive and understand what they are supposed to learn, what they have learnt already and how they could advance their own learning and improve their performance (ibid., pp. 50-51). On the other hand, summative assessment can also include verbal evaluation. It allows the teacher to describe the level of a pupil's performance, and his/her strengths, progressions, and targets of development (ibid.).

The descriptions of formative and summative assessments agree well with this study's research methods and handling of the results. First of all, by observing the pupils' learning and problem-solving processes, the teachers gain knowledge about their pupils' knowledge and skills in mathematics, as well as indications of their mathematics related metacognitive skills and affects. This knowledge can then be used as part of summative evaluations. These summative evaluations are expected to happen in learning discussions with the pupils.

The learning discussions do not have to be time-consuming. The teacher can use his/her observations as a starting point to the discussions. The teacher and the pupil can start by discussing the pupil's strengths in mathematics and continue towards the issues that are difficult for the pupil. The learning discussion is also an opportunity for the teacher to ask questions about the pupil's view of mathematics. Following this part of the discussion, the teacher can set long-term goals for learning together with the pupil. The first learning discussion is a start of a learning cycle in which these goals are evaluated in mathematics lessons as part of formative assessment and the goals are evaluated and perhaps altered in the next learning discussion (that might happen two or three months after the previous discussion). For a more detailed example, see Paper 5.

To be able to use the framework constructed in this study, teachers need to recognise the cognitive, metacognitive, affective and meta-affective aspects in mathematical thinking. For this purpose, the teachers can be given short descriptions of them with examples. Furthermore, the teachers should be given a few key questions to support the learning discussions. With these tools, the highly educated teachers in Finland can start using the framework to support their evaluation and to make it their own.

7.2 Connections to Finnish curriculum development

Some Finnish mathematicians have been criticising the curriculum development for moving too much towards applications and more descriptive mathematics (e.g. Martio, 2009). This view could be supported by the descending trend in Finnish pupils' mathematics performance in large scale studies (e.g. Hirvonen, 2011; Vettenranta et al., 2016) or the modest performance in TIMSS compared to PISA (Andrews et al., 2014). At the same time, the recent curriculum development has moved even more towards applications and multi-disciplinary learning (see FNBE, 2014). Even though it is difficult to determine the reasons behind the development in pupils' mathematics performance, it is justifiable to ask: How much of the development in Finnish pupils' mathematics achievement is caused by the curriculum development?

This study offers a counter-example to the critique towards applications. In comprehensive school, Alex was a fluent mathematical thinker with a well-organised net of knowledge. He mastered his problem-solving processes but did not relate them to real life. Actually, from the four pupils discussed as part of this thesis, only Daniel actively imagined the problem situations in real life. Further, what described the development of Alex's mathematical thinking best was the growth of mathematical knowledge and starting to see the connections between mathematics and his life. Only after seeing mathematics as a toolkit and ways of thinking, his intrinsic interest towards learning mathematics increased and the power of applying carried him to learn mathematics in a more meaningful way.

In a recent paper, Lester and Cai (2016) searched through 30 years of research on teaching through or with mathematical problem solving. After looking at teaching problem solving from multiple perspectives, they concluded that 'Focusing on problem solving in the classroom not only impacts the development of students' higher-order thinking skills but also reinforces positive attitudes. Finally, there is little evidence that we should worry that students sacrifice their basic skills if we teachers focus on developing their mathematical problem-solving skills.' (p. 130). What remained unresolved, however, was how problem solving should be taught in a most effective way.

While the new curriculum moves more towards applications and multi-disciplinary learning, also the assessment criteria changes from the previous curriculum. For instance, where applications were described as final assessment criteria mainly through problem solving in the 2004 curriculum (FNBE, 2004), applying mathematics in other school subjects and surrounding society is written separately as final-assessment criteria in the new curriculum (FNBE, 2014). How can this aim of applying mathematics in diverse environments be evaluated? Furthermore, in the

new curriculum taking responsibility of own learning, participating constructively in the activities of the mathematics group, and expressing mathematical thinking are also listed as part of final-assessment criteria (see FNBE, 2014). Even though these are desirable features for pupils, their evaluation challenges Finnish mathematics teachers due to their subjective nature.

In addition to the international trend of teachers being resistant to use other than ordinary mathematics tests as assessment methods due to their subjective nature (Watt, 2005; Watson, 2000), Finnish pupils at the end of comprehensive school are already evaluated unequally (a systematic difference in the same performance level can be up to two marks; Rautopuro, 2013). Finnish teachers seem to evaluate their pupils comparing them with other pupils they teach (*ibid.*), and even though girls had higher school grades in mathematics, boys performed better in recent national assessments (*ibid.*; Hirvonen, 2012). While the assessment method based on this study (introduced in Section 7.1) might help teachers to assess their pupils' development of mathematical thinking also through the abovementioned assessment criteria, it does not take away the subjective nature of the assessment.

There are hints of the unequal assessment methods also in this study. Emma was reported to be more competent in problem solving than Nora. On the other hand, Nora was more confident in mathematics, she could express her thinking more unambiguously, her view on mathematics was broader, she could connect mathematics to real world more easily, and she seemed to take a bigger responsibility of her own learning than Emma. All these features are listed in the 2004 curriculum as part of learning objectives but only problem solving is listed as final-assessment criteria (see FNBE, 2004). Nonetheless, Emma's mathematics grade was 9 and Nora's 10 on a whole number scale from 4 to 10. For another perspective, could the difference be due to their mathematics groups? Nora's class could be described as low achieving whereas Emma's class had pupils from all achievement levels (based on the researcher's observations).

As can be seen above, observations about the participating pupils' mathematical thinking and their relations to the Finnish curriculums (both 2004 and 2014) raise more questions than it gives answers. As a summary, more research is needed to see the potential of the new curriculum to increase pupils' mathematical knowledge, develop their thinking skills and affects, and help them to apply mathematics in a broader and more meaningful way. Additionally, more attention should be paid to pupil assessment in order to keep it diverse but more equal to all pupils in Finland.

7.3 Gender differences

All the four pupils whose results were discussed in this thesis were high achievers. They felt able to do mathematics, enjoyed learning it, succeeded in it and did not find it that difficult (cf. Hannula et al., 2011). Furthermore, they thought that mathematics is useful and liked mathematics as a school subject. The latter point contradicts with the previous results (Metsämuuronen, 2013; Hirvonen, 2012) but coincides with PISA results where the top pupils in Finland seem to be interested in and enjoy mathematics (e.g. Törnroos, Ingemansson, Pettersson & Kupari, 2006). (Whether the four pupils are part of the top two PISA groups have not been studied.)

Despite the high achievers' positive view of mathematics, research shows that Finnish pupils end their comprehensive school with unnecessarily negative affect towards mathematics (Tuohilampi & Hannula, 2013; Hirvonen, 2012). This trend is visible also in many other countries (e.g. Lee, 2009). Furthermore, girls have shown to have a lower self-confidence in mathematics than boys (e.g. Metsämuuronen, 2017; Hannula, Maijala, Pehkonen, & Nurmi, 2005). The trend in gender differences is visible also in this study, especially when all the eight pupils' confidence in mathematics was compared (see Table 7.1). The estimations of confidence were supported by the pupils' explanations about their view of mathematics (cf. confidence factors (I can get a good grade in mathematics, I can succeed in mathematics, I could handle more difficult mathematics, and I can learn mathematics) in Giaconi, Varas, Tuohilampi, & Hannula, 2016).

Table 7.1 Pupils' estimations of their confidence in mathematics. The scale is 0-10, 0 being the negative and 10 the positive end of the scale. Error margin is ± 0.25 . (H) is for high achieving pupil, (L) is for low achieving pupil.

		Round 1	Round 2	Round 3	Mean
Daniel	(H)	9.5	10	10	10
Oliver	(H)	10	9.5	9	9.5
Alex	(H)	9	8	8.5	8.5
Lucas	(L)	5.5	4.5	5.5	5

Boys' mean 8.5

Nora	(H)	8	8.5	7	8
Olivia	(H)	7	7.5	7	7
Emma	(H)	6.5	5.5	5.5	6
Maya	(L)	4	4	2.5	3.5

Girls' mean 6

A recent intervention study in Finland aimed at improving elementary school pupils' problem-solving skills and their mathematics-related affect from 3rd to 5th grade (Tuohilampi, Näveri, & Laine, 2015). In the study, pupils were involved with a mathematical problem monthly. The most significant result from the three-year intervention was that the girls' affect in mathematics decreased less than in the control group (ibid). This is a significant result connected to for example girls' opportunities for further studies and career choices (for further information about gender differences in Finnish mathematics education, see Metsämuuronen, 2017). There is also something positive in the development of gender differences in Finland. In the 2012 PISA assessment, girls performed better than boys in problem solving. This is significant because girls outperformed boys only in five of the participating countries, and from these countries, all the other four countries succeeded weakly in the assessment (Kyllönen & Nissinen, 2014).

Because of, and partly despite of the results presented above, the next step would be to do yet another problem-solving intervention in Finnish lower secondary schools (from 7th to 9th grade). The intervention should support strongly on the most recent curricular aims for mathematics learning (cf. FNBE, 2014), perhaps through mathematical modelling (see Section 7.4; for an international review on the effects of teaching mathematical problem solving, see Lester & Cai, 2016).

7.4 Connections to mathematical modelling

During my post-graduate studies I was asked why I studied mathematical thinking through applied problem solving, why not through applied mathematical modelling. Below, I will introduce (applied) mathematical modelling shortly and present some reflections to this question.

Mathematical modelling is a central concept connected to solving problems. It is seen as a future direction for problem-solving research (English & Sriraman, 2010), and it is also used for instance in PISA frameworks (see e.g. OECD, 2016). In the Finnish 2004 curriculum, modelling is mentioned once as a task for mathematics instruction: ‘The core task of mathematics instruction in the sixth through ninth grades is to [...] furnish adequate basic capabilities encompassing the modelling of everyday mathematical problems’ (FNBE, 2004, p. 163). Even though the connection between Finnish curriculum and mathematical modelling is relatively loose, mathematical modelling can be seen as a rich way to prepare pupils for future occupations (see English, 2008).

One of the three prevailing general descriptions for mathematical modelling connects the extra-mathematical (real world) and mathematical worlds through modelling cycles (English, Ärlebäck, & Mousoulides, 2016). This perspective of applied modelling is the one adopted in the PISA frameworks (see e.g. OECD, 2016). In the (applied) mathematical modelling cycle, the solver first structures real-world facts and data into mathematical form through mathematizing, and then after mathematical work, interprets and validates the results in terms of the original real-world problem (Niss, Blum, & Galbraith, 2007). To complete the modelling cycle, the solver has to evaluate the proposed solution both mathematically and in context, and present recommendations that are argued in terms of the modelling effort (*ibid.*).

When the modelling cycle is compared with the problem-solving process presented in this study (see Table 2.3), there does not seem to be many differences. So why talk about problem-solving processes instead of applied mathematical modelling? The first argument deals with the goal for the problem-solving activities. In mathematical modelling, the core activity is the transfer process between the extra-mathematical and the mathematical world through different (real and mathematical) models, emphasising also the modelling-loops for validation (Krug & Schukajlow-Wasjutinski, 2013; see also Lesh, Doerr, Carmona, & Hjalmarson, 2003). In problem solving, the focus is not in the models pupils (might) construct but in the processes they go through while solving problems (that might resemble the mathematical modelling cycle) and in the explanations and justifications for these actions.

Another reason for not studying pupils’ mathematical modelling is the difficulty of evaluating the modelling activities, especially because

mathematical modelling is often done as group work (e.g. Niss et al., 2007). In PISA frameworks (e.g. OECD, 2016) the problematic situation is solved by simplifying the modelling cycle: ‘it is often not necessary to engage in every stage of the modelling cycle, especially in the context of an assessment (Niss et al., 2007). The problem solver frequently carries out some steps of the modelling cycle but not all of them [...], or goes around the cycle several times to modify earlier decisions and assumptions’ (OECD, 2016, p. 65). If the modelling cycle is not studied as a whole, then why not study the activities through problem-solving processes discussed in the framework presented in this study?

The third, and the most important reason for studying pupils’ mathematical thinking through problem-solving processes instead of mathematical modelling, lays in the curriculum. Unlike mathematical modelling and modelling activities, problem solving and problem-solving abilities are frequently referred to both as learning objectives and as final-assessment criteria in the 2004 curriculum, and concrete problem-solving abilities are listed to support the evaluation of pupils’ problem-solving skills. These objectives and assessment criteria have served a starting point for studying pupils’ mathematical thinking in this study, following the instructions for teaching and learning that pupils have been encountered throughout their nine years of schooling.

After the study at hand was implemented, the Finnish curriculum has been renewed. In the process, the current curriculum (FNBE, 2014) emphasise applications and multi-disciplinary learning even more than the previous curriculum that was effective during the data collection for this study (FNBE, 2004). Mathematical modelling might offer a rich way for developing pupils’ mathematical thinking and expose them to multi-disciplinary learning in mathematics (cf. English et al., 2016). These skills can develop pupils’ mathematics learning but also prepare them for future careers where the key competence might be analysing and comprehending rich situations in a short period of time (English, 2008; cf. Alex’s explanations about problem solving in Paper 6).

8 Conclusion and implications

8.1 Theoretical considerations (RQ 1)

The first research question for this study is ‘what can be revealed when mathematical thinking is studied through two different data sets: problem solving (state) and view of mathematics (trait)? To answer this question, a theoretical and analytical framework was developed based on previous research done in the field of mathematics education. The novelty in the framework rests upon a couple of facts: first, the framework is constructed explicitly around the state and trait aspects related to mathematical thinking. These two phenomena have seldom been studied explicitly as one dynamic construct (Hannula, 2011). Secondly, additional novelty brings the aim of constructing a tool for studying pupils’ mathematical thinking that can be used at different educational levels.

In addition to connecting state and trait aspects, the framework is multifaceted because it studies the state and trait aspects from multiple different angles: Problem solving (state) is studied through problem-solving processes, mathematics related metacognitive skills, affective states and traits, and meta-affective skills, not forgetting the influence of mathematical knowledge base and heuristics (cf. e.g. Schoenfeld, 1992; Carlson & Bloom, 2005). Pupils’ view of mathematics (trait) is studied through cognitive, emotional and motivational aspects related to mathematics as a science and as a school subject, oneself as a learner and user of mathematics, learning mathematics and teaching mathematics, not forgetting pupils’ view of mathematical thinking (Hannula, 2011, 2012; Pehkonen, 1995; cf. also Op’t Eynde et al., 2002). Also a pupil profile is used to shortly describe a pupil (following Rösken et al., 2011). (For an overview, see Figure 2.1.) These components together form the deep and holistic view of a pupil’s mathematical thinking.

On theoretical level, the results of the study show the dynamic processes of mathematical thinking and the intertwined relationship between cognition and affect as well as the state and trait aspects in mathematical thinking. On practical level, it reveals the strengths and weaknesses in pupils’ mathematical thinking, and through them directs attention towards the ways in which pupils’ mathematical thinking could be developed on an individual level. (See Papers 2-7)

The thus formed theoretical framework and analytical tool was tested with Finnish pupils. However, since the construction of the tool is based on a wide range of international studies, its applicability is not restricted to any specific country. The framework could be used by researchers to study, evaluate and further develop pupils’ mathematical thinking. Further, it can be used for cross-sectional studies as well as for longitudinal

studies aiming at studying the development of pupils' mathematical thinking at different ages (see Paper 6).

At its best, the analytic tool could be used to very powerfully evaluate pupils in interventional studies giving rich data about the development of their mathematical thinking over time. Another way to utilise the tool, would be to evaluate and develop pupils' mathematical thinking in schools. After some modifications, the tool could be used by mathematics teachers to support pupil evaluation that would also actively involve the pupil (cf. evaluation criteria described in FNBE, 2014; for an example, see Paper 5). If the tool is actually helpful to develop pupils' mathematical thinking as part of ordinary school activities, is a matter of another continuation study.

The purpose of the analytical tool is to study individual pupils' mathematical thinking. However, I believe that the framework can be adapted to different viewpoints in mathematics education research. For instance, if the research team is interested to study individuals as part of a group (a social aspect; cf. Hannula, 2012), the pupils' activities can be studied through collaborative problem-solving processes, and the view of mathematics can be studied by emphasising pupils' roles in these activities. In this example study, the results might show how the group-work influences individual's mathematical thinking, or how the individual influences another pupil's thinking, and illustrate the roles that the pupils' have in such problem-solving situations. How the theoretical framework could be adapted to different situations has not been tested yet, but it would be another interesting continuation to this study.

8.2 Practical considerations (RQ 2)

The second research question was based more on the empirical part of the study: 'what characterises the mathematical thinking of four Finnish high-achieving pupils at the end of comprehensive school?' This question was answered by analysing individual pupils' mathematical thinking first individually with the analytic tool created for this study (Papers 2-6) and then by bringing those results together in the final paper (Paper 7).

On individual level, pupils' view of mathematics supported and complemented the results gained from problem-solving processes giving clear and unambiguous descriptions of the individuals' strengths and weaknesses in mathematical thinking. Especially pupils' descriptions of learning mathematics entailed similar strengths and weaknesses as their problem-solving processes (e.g. in metacognitive or meta-affective skills). Even so, all the categories of mathematical thinking were important to study; For example, without the interviews on view of mathematics we would lack the information about Alex's well organised net of

knowledge, his reflective thinking while learning mathematics, or his limitations to connect mathematics to his own life.

On a group level, the comparison between individuals' mathematical thinking indicated that the similarities remained on a surface-level: All the pupils liked mathematics, were motivated to learn it and were successful problem solvers. However, a deeper analysis into their mathematical thinking revealed a great deal of differences between the pupils. These differences are due to personal characteristics that might not necessarily be connected only to mathematics or found only from Finnish pupils.

If these different pupil characteristics turn into more general categories of mathematics pupils showing similar skills (state) and competences (trait) in mathematical thinking, is a matter of a continuation study. In such a study, the tool should be used to analyse a large amount of pupils' mathematical thinking. This analysis could be part of an intervention study aiming to develop mathematical thinking and mathematics related affect for instance through mathematical problem solving or modelling (for an example of such study, see e.g. Tuohilampi et al., 2015). However, the results as they are in this study are by no means generalizable.

8.3 Concluding words

Constructing the theoretical and analytical framework is hopefully just a beginning of a long journey. It is not constructed just to study individuals' mathematical thinking per se, but keeping in mind its applicability in other, larger studies on the development of mathematical thinking. Also its applicability as part of school evaluation is an intriguing idea that is strongly connected to the current Finnish curriculum (see FNBE, 2014) and that, at this point, is tested loosely and unofficially only by the researcher herself. And the four cases, even though selected quite randomly, amazed with their differences to think mathematically. To be able to find these differences was of course due to the power of the analytical tool. And finally, it was a privilege to study the development of Alex's mathematical thinking, a privilege that does not need to end along with this PhD study.

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10 Appendices

APPENDIX 1: PISA-tasks used in the main study	95
Holiday	97
School excursion	99
Indonesia	100
Carpenter	101
Distance	102
Growing up	103
Braking	104
APPENDIX 2: Interview protocols	109
Interview 1	111
Interview 2	114
Interview 3	117
APPENDIX 3: Documents connected to ethics	121
Letter to the principals and mathematics teachers	123
Letter to all pupils and their families	125
Letter to the target pupils and their families: An informed consent	126
APPENDIX 4: Alex's emails	129
Reaction to Paper 2: Alex's view of mathematics	131
Reaction to the manuscript of Paper 6: The development of Alex's mathematical thinking from comprehensive school to university	133
APPENDIX 5: The papers	137
List of the papers	139

APPENDIX 1:
PISA-tasks used in the main study

Holiday

Mathematics Example 2: HOLIDAY

This problem is about planning the best route for a holiday.

Figures A and B show a map of the area and the distances between towns.

Figure A. Map of roads between towns

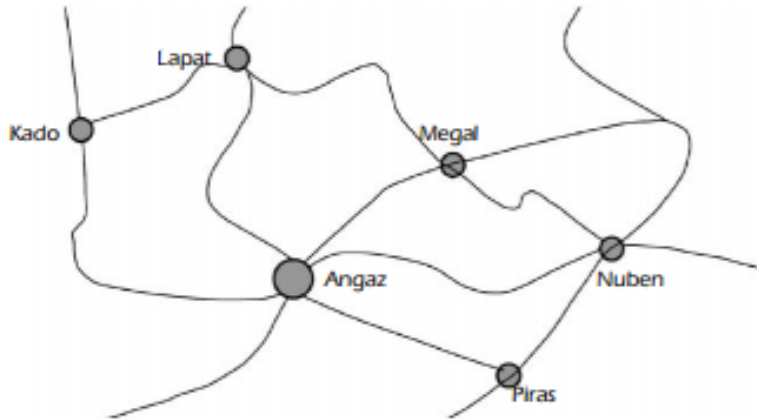


Figure B. Shortest road distance of towns from each other in kilometres

Angaz						
Kado	550					
Lapat	500	300				
Megal	300	850	550			
Nuben	500		1300	450		
Piras	300	850	800	600	250	
	Angaz	Kado	Lapat	Megal	Nuben	Piras

Question 1: HOLIDAY

Calculate the shortest distance by road between Nuben and Kado.

Distance: kilometres.

Question 2: HOLIDAY

Zoe lives in Angaz. She wants to visit Kado and Lapat. She can only travel *up to 300 kilometres* in any one day, but can break her journey by camping overnight anywhere between towns.

Zoe will stay for two nights in each town, so that she can spend one whole day sightseeing in each town.

Show Zoe's itinerary by completing the following table to indicate where she stays each night.

Day	Overnight Stay
1	Camp-site between Angaz and Kado
2	
3	
4	
5	
6	
7	Angaz

(OECD, 2006, pp. 77-78)

School excursion

Mathematics Example 5: SCHOOL EXCURSION

A school class wants to rent a coach for an excursion, and three companies are contacted for information about prices.

Company A charges an initial rate of 375 zed plus 0.5 zed per kilometre driven. Company B charges an initial rate of 250 zed plus 0.75 zed per kilometre driven. Company C charges a flat rate of 350 zed up to 200 kilometres, plus 1.02 zed per kilometre beyond 200 km.

Question 1: SCHOOL EXCURSION

Which company should the class choose, if the excursion involves a total travel distance of somewhere between 400 and 600 km?

(OECD, 2006, p. 87)

Indonesia

Mathematics Example 25: INDONESIA

Indonesia lies between Malaysia and Australia. Some data of the population of Indonesia and its distribution over the islands is shown in the following table:

Region	Surface area (Km ²)	Percentage of total area	Population in 1980 (millions)	Percentage of total population
Java/Madura	132 187	6.95	91 281	61.87
Sumatra	473 606	24.86	27 981	18.99
Kalimantan (Borneo)	539 460	28.32	6 721	4.56
Sulawesi (Celebes)	189 216	9.93	10 377	7.04
Bali	5 561	0.30	2 470	1.68
Irian Jaya	421 981	22.16	1 145	5.02
TOTAL	1 905 569	100.00	147 384	100.00

One of the main challenges for Indonesia is the uneven distribution of the population over the islands. From the table we can see that Java, which has less than 7% of the total area, has almost 62% of the population.

Source: de Lange and Verhage (1992). Used with permission

Question 1: INDONESIA

Design a graph (or graphs) that shows the uneven distribution of the Indonesian population.

(OECD, 2006, p. 111)

Tasks for the pupils:

Task 1. What percentage of population lives on the Indonesian island biggest in surface area?

Task 2. What is the percentage of population living on the Indonesian islands that are biggest and smallest in surface area?

Task 3. The PISA task above (Question 1).

Task 4. How much bigger is the Indonesian island biggest in surface area, than the Indonesian island smallest in surface area?

Task 5. Which islands form one third of the population?

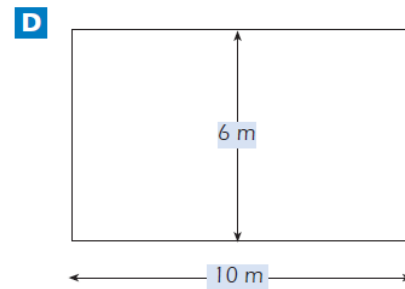
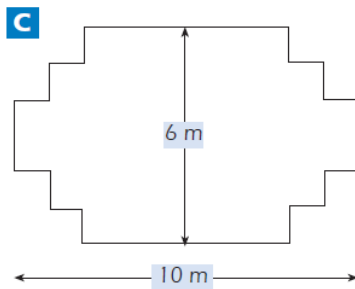
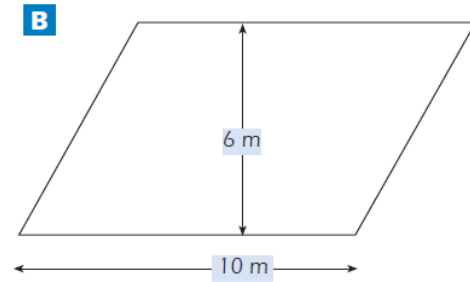
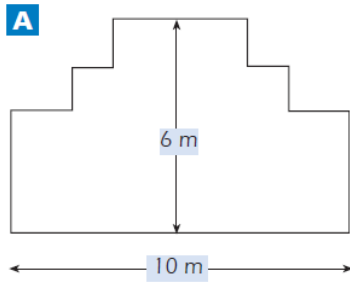
Task 6. What other interesting things could you calculate from the table? What could you do with the information?

Carpenter

MATHEMATICS UNIT 10: CARPENTER

QUESTION 10.1

A carpenter has 32 metres of timber and wants to make a border around a garden bed. He is considering the following designs for the garden bed.



Circle either "Yes" or "No" for each design to indicate whether the garden bed can be made with 32 metres of timber.

Garden bed design	Using this design, can the garden bed be made with 32 metres of timber?
Design A	Yes / No
Design B	Yes / No
Design C	Yes / No
Design D	Yes / No

(OECD, 2009, p. 111)

Distance

Mathematics Example 18: DISTANCE

Mary lives two kilometres from school, Martin five.

Question 1: DISTANCE

How far do Mary and Martin live from each other?

(OECD, 2006, p. 102)

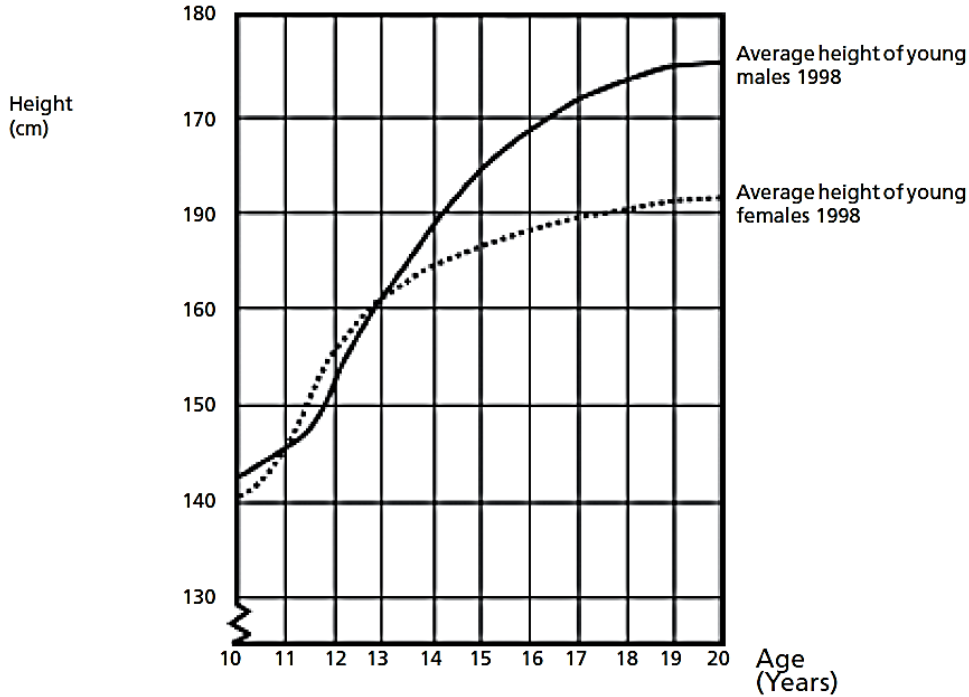
The pupils solved this problem with 3 and 5 km.

Growing up

MATHEMATICS UNIT 6: GROWING UP

Youth grows taller

In 1998 the average height of both young males and young females in the Netherlands is represented in this graph.



QUESTION 6.1

Since 1980 the average height of 20-year-old females has increased by 2.3 cm, to 170.6 cm. What was the average height of a 20-year-old female in 1980?

Answer: _____ cm

QUESTION 6.2

Explain how the graph shows that on average the growth rate for girls slows down after 12 years of age.

.....
.....

QUESTION 6.3

According to this graph, on average, during which period in their life are females taller than males of the same age?

.....
.....

(OECD, 2009, p. 106)

In connection to questions 6.1 above, some pupils were also asked to calculate how many percentages the female average height increased.

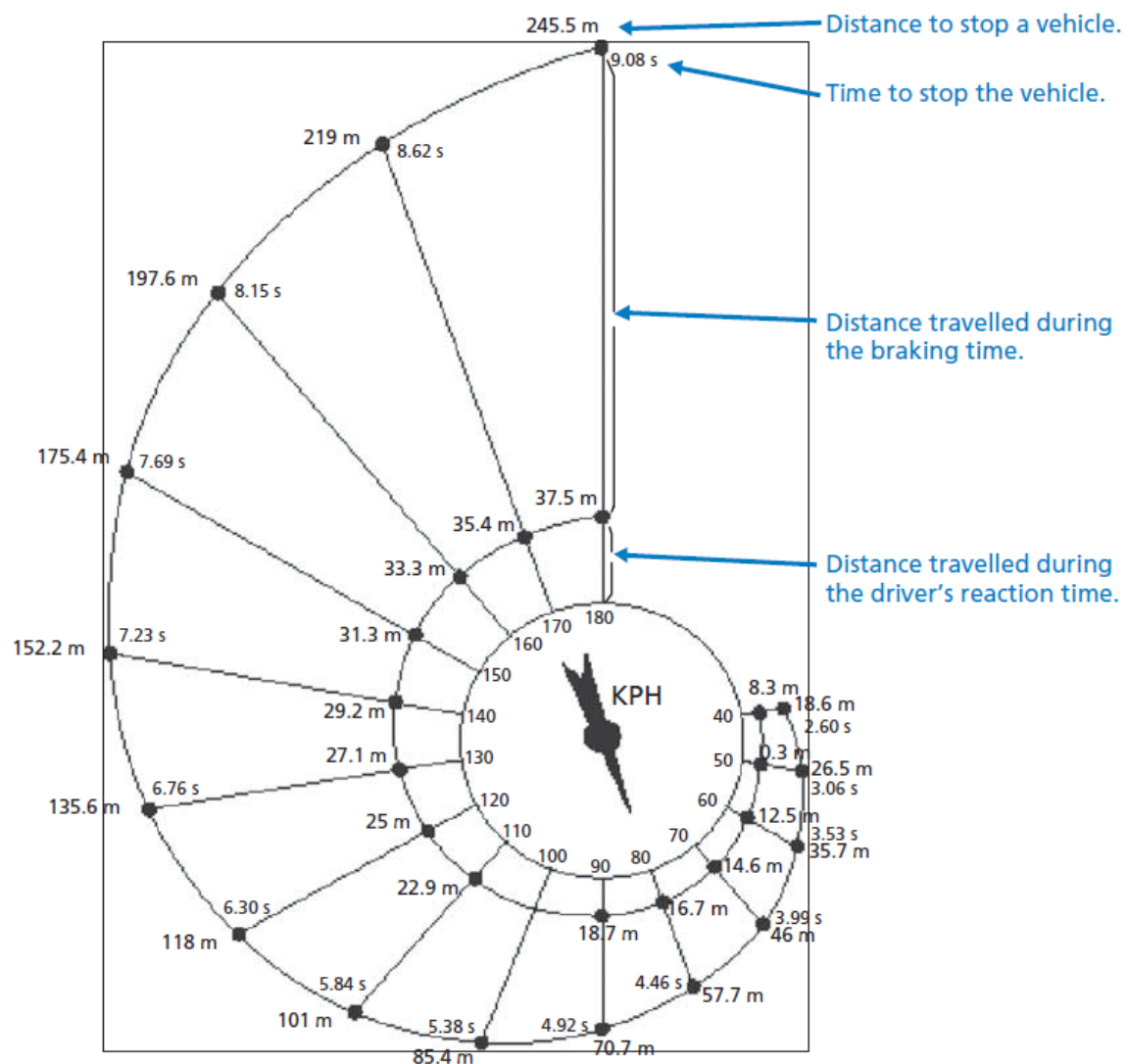
Braking

MATHEMATICS UNIT 31 : BRAKING

The approximate distance to stop a moving vehicle is the sum of:

- the distance covered during the time the driver takes to begin to apply the brakes (reaction-time distance)
- the distance travelled while the brakes are applied (braking distance)

The 'snail' diagram below gives the theoretical stopping distance for a vehicle in good braking conditions (a particularly alert driver, brakes and tyres in perfect condition, a dry road with a good surface) and how much the stopping distance depends on speed.



Source: La Prévention Routière, Ministère de l'Éducation nationale, de la Recherche et de la Technologie, France.

QUESTION 31.1

If a vehicle is travelling at 110 kph, what distance does the vehicle travel during the driver's reaction time?

.....
.....

QUESTION 31.2

If a vehicle is travelling at 110 kph, what is the total distance travelled before the vehicle stops?

.....
.....

QUESTION 31.3

If a vehicle is travelling at 110 kph, how long does it take to stop the vehicle completely?

.....
.....

QUESTION 31.4

If a vehicle is travelling at 110 kph, what is the distance travelled while the brakes are being applied?

.....
.....

QUESTION 31.5

A second driver, travelling in good conditions, stops her vehicle in a total distance of 70.7 metres. At what speed was the vehicle travelling before the brakes were applied?

.....
.....

(OECD, 2009, pp. 128-129)

Tehtävä 49: JARRUTUS

M215Q06

Kostealla tiellä, muiden olosuhteiden pysyessä ennallaan, kasvaa jarrutusmatka 40%:a (ei reaktioaikana edetty matka).

Tiedetään, että hyvissä olosuhteissa 80 kilometriä tunnissa etenevän ajoneuvon pysäyttämiseen kuluu 57,7 m.

Millä seuraavista tavoista voidaan laskea ajoneuvon pysähtymiseen kuluva kokonaismatka märissä olosuhteissa?

- A $57,7 \cdot 1,4$
- B $(57,7 - 16,7) \cdot 1,4$
- C $16,7 + (57,7 \cdot 1,4)$
- D $16,7 + (57,7 - 16,7) \cdot 1,4$

[On a damp road, other conditions remaining the same, the braking-time increases by 40% (not the distance travelled during reaction-time). We know that in good conditions, to completely stop a vehicle travelling 80 kph takes 57.7 m.

Which of the following ways can be used to calculate the total distance for the vehicle to stop in damp conditions?]

Tehtävä 50: JARRUTUS

M215Q07

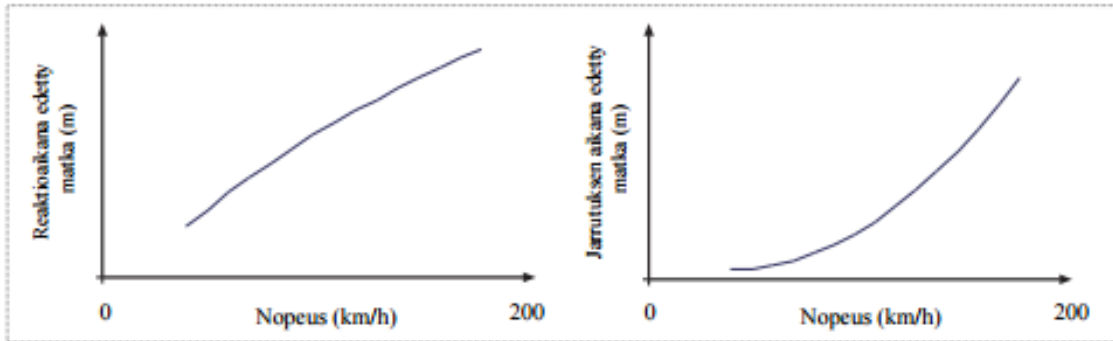
Seuraavalla sivulla on neljä kuvaajaparia, jotka esittävät kuljettajan reaktioaikana edetyn matkan ja jarruttamisen aikana edetyn matkan. Vaaka-akselilla on kuvattu auton nopeus kilometreinä tunnissa ja pystyakselilla edetty matka metreinä.

[On the next page, there are four pairs of graphs showing the distance that the driver has travelled during reaction-time and braking-time. The horizontal axis represents the speed of the car in kph and the vertical axis the distance travelled in metres.]

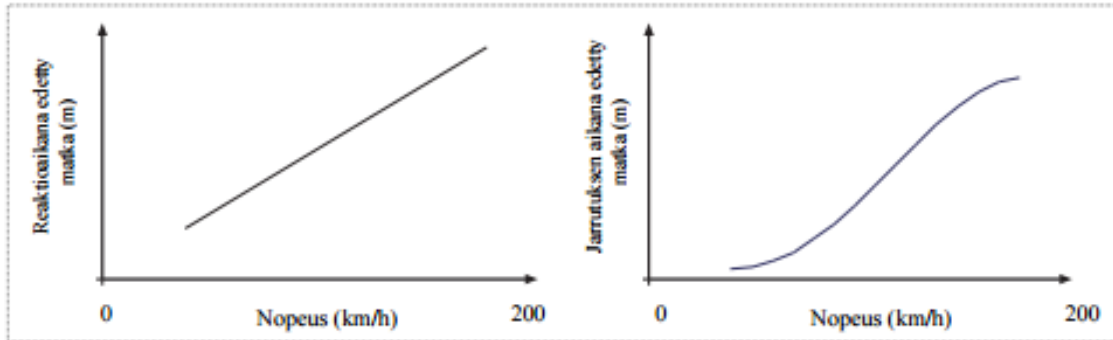
[In which of the pair of graphs the information is in-line with the ‘snail’ diagram?]

Missä kuvaajaparissa annetut tiedot ovat yhdenmukaisia "simpukka"-diagrammin kanssa?

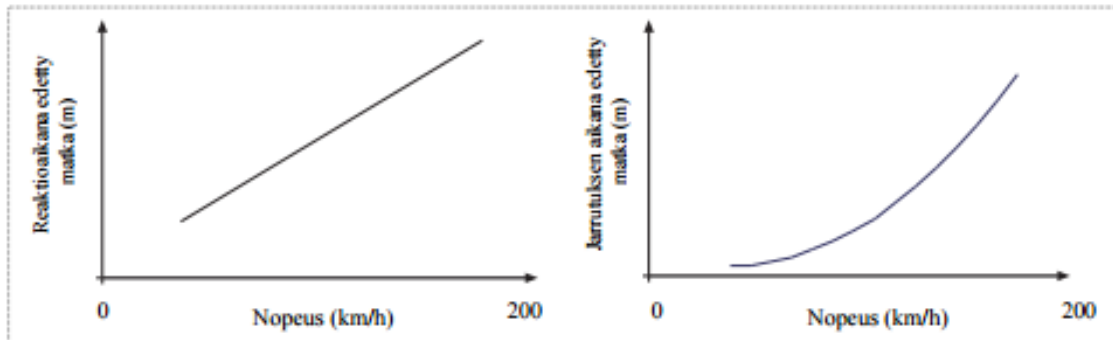
A.



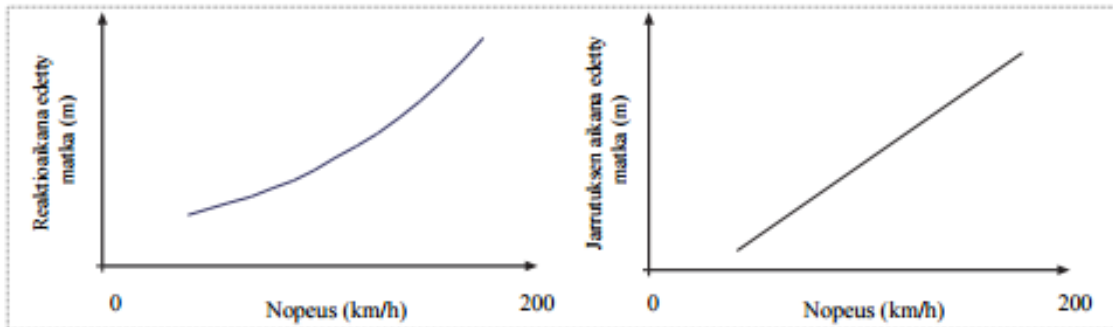
B.



C.



D.



(FIER)

APPENDIX 2:

Interview protocols

Interview 1

The discussion followed these themes and questions freely, depending on the pupil's explanations at a given moment (cf. semi-structured interview). If a question that was not originally in the interview protocol was discussed more with one pupil, it was asked also from other pupils in the following interviews. An example is the last question in 'mathematics and me': What do you do if you get stuck with a difficult problem?

Background

- Tell me about yourself.
- What do you do as a hobby?
- Tell me about your family.
- What do your parents do for a job?

Mathematics as a school subject and as a science

- What is mathematics as a school subject?
- What is mathematics as a science?
- Does mathematics exist outside school?
 - Can you give examples?
- Do you need mathematics outside school?
 - What mathematics?
 - Where?
 - How?
 - Can you give examples?
- How is mathematical knowledge gained?
- How do you think that the mathematical knowledge is correct?
- Do you need mathematics now / in future?
 - Education / job / a good working place?
- Does mathematics help you in other school subjects?
 - Where?
 - Can you give examples?

Mathematics and me (oneself as a user and learner of mathematics)

- What mathematics is to you?
- What mathematics do you need?
- What mathematics mean to you?
- Is mathematics important to you?

- Is your view of mathematics changed during the nine years of schooling?
 - How?
 - Why?
 - When?
 - Did teaching change?
- Does mathematics help you think logically?
 - How?
 - Can you give examples?
- What do you do if you get stuck with a difficult problem?
 - Do you quit or continue?
 - How?
 - Why?

Before solving the problem (after reading the task description)

- How did you feel?
 - How strong was that feeling?
- Did you plan the problem solution at this point?
 - What was the plan?
 - Why?
- Could you have proceeded differently?
- Did you read only the first task in the beginning, or both?

Before stimulated-recall interview

The purpose is that you tell me everything that is happening in the video: What did you do, how, why, did you feel something, did you talk with someone, whatever comments come to mind. Either one can stop the video to tell or ask something at any time.

After solving the problem

- Did you set any sub-goals?
 - What?
 - Why?
- Did you think about the solution during or after problem solving?
 - Do you usually do so?
- Did you think about your own thinking during or after problem solving?
 - Do you usually do so?

- Did you think of a similar problem you have solved before (mathematics / method / concept / feeling)?
 - What?
 - Did it help you in some way?
 - How?
- What did you feel when you solved the problem?
 - Did the feelings change during the problem-solving process?
 - How?
 - Why?
- What motivated you to solve the problem?
- What if the distance for a day was 500 km? Would you have solved the problem differently?
 - Nights?
 - First camping area?

Confidence line

You see a 10-centimetre line that has ‘I couldn’t do it at all’ in one end, and ‘I could do it perfectly’ in the other end. Please mark on that line the following things:

- Where were you after reading the task?
 - Why?
- Where were you while solving the problem?
 - Why?
- Where were after you solved the problem?
 - Why?
- So you felt more/less confident than in ...?
 - Why?
- Where are you now in school mathematics?

Interview 2

The discussion followed these themes and questions freely, depending on the pupil's explanations at a given moment (cf. semi-structured interview). If a question that was not originally in the interview protocol was discussed more with one pupil, it was asked also from other pupils in the following interviews. An example is including the task Distance into the second interview.

Learning mathematics

- What is learning mathematics to you?
- How do you learn mathematics?
 - What happens when you learn mathematics?
 - Do you actively connect new knowledge to the old?
 - How?
 - Can you give examples?
- What feelings does learning mathematics bring to you?
- What adjectives could you use to describe what learning mathematics is to you?
 - Is it interesting / easy / difficult / exciting?
- What motivates you to learn mathematics?
- Does learning mathematics take time?
- Do you want to use time for learning mathematics?
- Does learning mathematics require a lot of work?
- How much time do you use for mathematics homework?
 - Is it enough?
 - Is it useful?
- Is learning mathematics memorising / learning to memorise?
 - Why / why not?
- Is mathematics rules?
 - Is it something else?
 - What?
- Do you always need a rule or a formula in a mathematics problem?
- Is it the most important to get a correct answer?
 - Why / why not?
- Can there be many ways to solve a problem?
 - Can you give examples?
- Can there be many answers to a problem?
 - Can you give examples?
 - Matt lives 3 km from school and Mary 5 km. How far do they live from each other? (Distance)
 - Can there be another answer?

- What about another? ...
- Are you afraid to do mistakes in mathematics?
 - Why / why not?
- What does it mean to make a mistake?
- Do girls and boys differ in learning mathematics?
 - If so, how?
 - Are others more gifted / need math more / more diligent / have better grades / more interested?
- What is your grade in mathematics?
 - Is it fair?
 - Does it represent your abilities?
- Could you get a better grade?
 - How?
- Do you want to learn mathematics?
- How certain (confident) are you in yourself / of your skills in mathematics?

Other questions

- What is pi?
- How do you check your tasks concretely, e.g. in tests?

Before solving the problem (after reading the task description)

- How did you feel?
 - How strong was that feeling?
- Did you plan the problem solution at this point?
 - What was the plan?
 - Why?
 - Did you have sub goals?
 - What?
- Could you have proceeded differently?
- How did you start solving the problem?

Stimulated-recall interview

Issues rising from the stimulated-recall interview and discussed with the pupils:

- Units in the calculations, how do they use them
- How do you do mental calculations?
- Can you show me how would do you calculate (e.g.) $200 \cdot 1.02$?
- Could you draw graphs for the prices of the bus companies?
- If not, why did not use a ruler for measurements in the task Carpenter.

After solving the problem

- Did you set any sub-goals?
 - What?
 - Why?
- Did you think about the solution during or after problem solving?
- Did you think about your own thinking during or after problem solving?
- Did you think of a similar problem you have solved before (mathematics / method / concept / feeling)?
 - What?
 - Did it help you in some way?
 - How?
- What did you feel when you solved the problem?
- Did the feelings change during the problem-solving process?
 - How?
 - Why?
- What motivated you to solve the problem?

Confidence line

You see a 10-centimetre line that has 'I couldn't do it at all' in one end, and 'I could do it perfectly' in the other end. Please mark on that line the following things:

- Where were you after reading the task?
 - Why?
- Where were you while solving the problem?
 - Why?
- Where were after you solved the problem?
 - Why?
- So you felt more/less confident than in ...?
 - Why?
- Where are you now in school mathematics?

Interview 3

The discussion followed these themes and questions freely, depending on the pupil's explanations at a given moment (cf. semi-structured interview). If a question that was not originally in the interview protocol was discussed more with one pupil, it was asked in this interview, at the latest.

Before solving the problem (after reading the task description)

- How did you feel?
 - How strong was that feeling?
- Did you think of a similar problem you have solved before (mathematics / method / concept / feeling)?
- Did you plan the problem solution at this point?
 - What was the plan?
 - Why?
 - Did you have sub goals?
 - What?
- How did you start solving the problem?
- Could you have proceeded differently?

After solving the problem

- Did the feelings change during the problem-solving process?
 - How?
 - Why?
- Did you think about the solution during or after problem solving?
- What motivated you to solve the problem?

Confidence line

You see a 10-centimetre line that has 'I couldn't do it at all' in one end, and 'I could do it perfectly' in the other end. Please mark on that line the following things:

- Where were you after reading the task?
 - Why?
- Where were you while solving the problem?
 - Why?
- Where were after you solved the problem?
 - Why?
- So you felt more/less confident than in ...?
 - Why?
- Where are you now in school mathematics?

Problems solved in the interview

While the tasks Growing and Braking were solved, the pupils were asked to think aloud and explain their thought and actions.

From Growing, the tasks 6.3 and 6.1 were done by all pupils (in this order).

From Braking, the tasks 31.1-31.5 were done by all pupils.

Teaching mathematics

- Does teaching matter?
 - How?
- If it depended on the teaching, can everybody learn mathematics?
 - What if studying had to done alone?
- Could you compare the good and bad things in teaching that your mathematics teachers have had?
- How would you want that you were taught in mathematics?
- What is good teaching?
 - How is teaching done / working habits / ...?

Mathematical thinking

- What does mathematical thinking mean?
- Where does it exist?
- How do you recognise mathematical thinking?

Other questions

These questions were asked from all pupils:

- Had you thought of mathematics as a science before the first interview?
- What is your family's / friends' / mathematics class's attitude towards mathematics?
- What is your favourite subject?
 - What is your least favourite subject?
 - Where is mathematics on this line?
 - Are your feelings / confidence same in these subjects?
- What do tests mean to you?
- How important mathematics tests are for you?
- What feelings do mathematics tests / test situation bring to you?
 - Are you anxious?
 - What about other school subjects?

- What is the reason for your success in a mathematics test?
 - Are you usually successful?
 - What is the reason for the possible lack of success?
 - Are you usually unsuccessful?
 - What is the reason for the possible success?
- Who is responsible of your mathematics learning?
- Can I show one of the problem-solving videos from the lesson to the teacher? Still nothing from the interviews or what we have discussed will be revealed to the teacher.

Due to the time limit for the interviews these questions were asked only from some of the pupils:

- If you knew all the rules in mathematics, would you be successful in it?
- Do you sometimes get tired when solving mathematics problems?
- Is mathematics boring?
- Does mathematics 'fit' to you?
- Do your mathematical abilities grow during a learning process?
 - How?

APPENDIX 3:
Documents connected to ethics

Letter to the principals and mathematics teachers

Dear rector / teacher,

I approach you because of my doctoral studies. First I will tell you shortly about the subject of my study, and then I will introduce my plans and timetable for collecting data in which I hope your school will be part of. In the end I will introduce myself and my background.

The object of my study is the mathematical thinking of 9th grade pupils in Finnish compulsory school. The Finnish curriculum for compulsory school, where mathematical thinking and its development are set to be one of the aims for teaching in mathematics, and PISA-assessments, where pupils' ability to answer problems connected to everyday life and future demands, have inspired me to do this study. This research is a case study and I will study pupils' mathematical thinking when they solve PISA-tasks. The aim is to get qualitative in-depth knowledge about the mathematical thinking of individuals at the end of their compulsory schooling.

There will be altogether four classes taking part in the study, all from different schools. From each class the main target is the mathematical thinking of two pupils, but I will observe also the teaching which influences the development of mathematical thinking. The study will be carried out during next fall semester in three cycles. In each cycle the pupils will solve one PISA-task as part of their lesson. Solving the task takes about 15-20 minutes. During each cycle, the two "target pupils" will be videotaped while they solve the task. With the video, an interview will be kept about their mathematical thinking somewhere in the school building (a familiar environment to the pupils) at the same or the next day, after the pupils' own school day. Every interview is one-to-one interview between the researcher and the pupils individually and it takes about one hour. During the study, also the teacher will be interviewed once. The teacher interview will take about 1-2 hours.

The timetable for your school goes as follows. In August/September I will get to know the class by observing it for about two weeks. First the pupils will be informed about the study and they will have an opportunity to ask questions. Additionally the pupils will have an opportunity to get acquainted for me being there in the classroom. I will not interfere the teaching. During the second week the first PISA-problem will be solved and the pupil interviews kept. After these two weeks I will visit the class twice one week at the time every fourth weeks. During each of these weeks the pupils will solve one PISA-task. The final timetable will be decided after each of the schools knows their own timetable. All the

pupils and the teachers will be kept anonymous in the research, and also the school names will be changed in research reports.

Then something about me. My name is Hanna Viitala, I am a young PhD student in mathematics didactics from University of Agder, Norway. Originally I am from Etelä-Pohjanmaa and I graduated as mathematics teacher from University of Helsinki in 2007 where I also started my PhD studies in didactics in 2008. In September 2009 I started my studies in Kristiansand, Norway, where the position of research fellow enabled me to do research full time. The work in a new environment has offered me a unique chance to take part in the activities of an active multicultural group of researchers, and the planning of my research has moved rapidly forward. My aim is to get my PhD qualification from University of Agder in the fall 2012, and the following parts of the study will be concluded in 2013.

If you have any questions, you can contact me or my supervisors. I have been lucky; I have three supervisors, two in Norway and one in Finland. The language of my work is English. Below you will find all email addresses to my supervisors. To Barbro and Olav you can write in Swedish or English and to Erkki in Finnish. If you have any practical issues, please contact me primarily.

If needed, this letter can be copied to the mathematics teachers in your school.

Thank you for your attention!
Sincerely,

Hanna Viitala
hanna.l.viitala@uia.no
050-5336860

My supervisors:

Professor Barbro Grevholm, University of Agder, Norway (barbro.grevholm@uia.no)

Professor Olav Nygaard, University of Agder, Norway (olav.nygaard@uia.no)

Professor emeritus Erkki Pehkonen, University of Helsinki (erkki.pehkonen@helsinki.fi)

Letter to all pupils and their families

Dear pupil and parent,

I am a PhD student from University of Agder (Norway) and I am studying 9th graders' mathematical thinking in a project of University of Agder and University of Helsinki. My aim is to get qualitative in-depth knowledge of pupils' mathematical thinking at the end of their compulsory schooling. Mathematical thinking and its development is one aim for mathematics in curriculum for basic education. My goal is to get my PhD qualification from University of Agder in the fall 2012, and the following parts of the study will be concluded by the end of 2013.

There will be four classes taking part in the study, all from different schools. My main object in the study is pupils, but I will also observe teaching in the mathematics classrooms as well as interview the mathematics teachers as background information to the study. The study will be carried out in three cycles. In each cycle the whole class will solve one PISA-task as part of a lesson. It will take about 15 minutes. I am coming to collect data for my study in your class in fall 2010 (all three cycles).

From all classes two pupils will be asked to be the main object of my study. While the pupils solve the given tasks, the two 'target' pupils will be videotaped so that their voice can be heard and their paper visible in the video. After the lesson, on the same or the next day, one-to-one interviews with the researcher and the pupil will be conducted in the school building. One interview will take about one hour. The time for the interview will be agreed together with the pupil. The two pupils in question, as well as their parents, will be provided with additional information and a request to participate in the study. Participation will be voluntary.

All the personal information collected in the study will be anonymous. The knowledge gained will not be used to anything else than to my study. The data will be stored properly. The participation of the pupil will be voluntary and he/she has a right to refuse to take part any stage of the research. The data collection will be concluded by the end of 2010, and the study will be completed by the end of 2013.

I will gladly answer your questions concerning the study, my contact information is below. Thank you for your collaboration!

Sincerely,

Hanna Viitala
hanna.l.viitala@uia.no

Letter to the target pupils and their families: An informed consent

Dear pupil and parent,

I refer to my earlier letter to you and ask your permission for the pupil to participate to the closer study. This means that the pupil will be videotaped in the classroom while the whole class solves PISA-tasks. In the video, the pupil's voice will be heard and his/her table and task-paper visible (no faces). We will talk this situation through with the pupil before we videotape.

After the lesson, in the same or the next day, I will interview the pupil and we will go through the task solution. This will also be videotaped, and again, the pupil's face won't be visible in the video, but his/her voice will be heard and table and task solution visible. The interview will take about one hour and it will be held somewhere in the school building. The time for the interview will be agreed together with the pupil. As the data collection proceeds in three cycles, all this will be done three times during the fall (about once in four weeks).

During the interview most of the questions will be connected to the pupil's solving of mathematical tasks. However, in order to analyse the observations, I will also ask some few more personal questions. The pupil will be asked about his/her grade in mathematics. I will also ask the pupil to tell me what their parents' occupation and education are and who the pupil asks if he/she needs help or wants to discuss mathematics at home. These questions come somewhere during the interviews and the pupil can answer according to his/her own knowledge and experiences. They do not have to prepare themselves to the interviews in any way.

Anonymity will be guaranteed in data collection and in the research reports. The pupil has a right to stop his/her participation at any time during the study. All the data will be destroyed after the research is finished by the end 2013.

I will happily answer to your questions concerning the study, my contact information is below. Thank you for your collaboration!

Sincerely,

Hanna Viitala
hanna.l.viitala@uia.no

-- Cut this and give the lower part back to Hanna --

I agree to take part in the closer part of the study done by Hanna Viitala.

(The name of the pupil)

My child has a permit to take part in the closer part of the study done by Hanna Viitala.

(The name of the pupil's parent)

APPENDIX 4:

Alex's emails

Reaction to Paper 2: Alex's view of mathematics

Selected parts from Alex's reaction to Paper 2 (email)

Dear Hanna,

[...]

Your article about 'Alex' describes well me and my relationship with mathematics and learning. During upper secondary school years, I have recognised the nature of my mathematical thinking even more, and you have discovered that well already from a boy in lower secondary school. On the other hand, my favourite subjects in school have changed a bit, and it was fun to read things about me that I don't recall anymore (like was English really my favourite subject after sports). It was also a bit amusing that I didn't like Swedish and Germany due to their illogicality but English was still one of my favourite subjects. Nowadays, I am very interested in natural sciences, especially physics and chemistry because they are exact mathematical sciences. Also, my interest in mathematics itself has increased, but it is still most importantly a tool in other sciences and life in general. I don't like learning by memorising and I always pursue for understanding instead of memorising (just like before). For example, even though in physics matriculation exams it was allowed to use [a book with a collection of formulas and information about physical conditions], I don't think that the formulas are much of use unless you don't understand how they represent nature and how they are formulated.

[...] During the past year I have noticed a great deal of development in my studying. [...] In matriculation exams] unfortunately I got one point under [laudatur, the highest grade in the exams] in mathematics. However, I emphasised physics, chemistry and Finnish language more keeping in mind my postgraduate studies. So, laudatur from mathematics was not really my goal either. In the upper secondary school diploma, my mathematics grade was 9 (mean 9.4), physics was 10 and chemistry 9 [on a whole number scale from 4 to 10].

[...] Based on the previous [university entrance exams to medical school] and cram school, I expected a very difficult entrance exam that emphasises calculations in physics and chemistry. This would suit me well because I think I'm better in calculating than many other applicants. I also hoped for a broad material and tasks related to it based on previous exams, because I'm fast in perceiving text entities and finding relevant information from them. However, there were no materials in the exam at all and a lot of theoretical biology tasks, and not those tasks that require

complicated logics that I so much hoped for. The two most difficult physics and chemistry calculations were easy for me and I would have hoped for more of them.

[...]

If you want further information or clarifications, I'm happy to answer.

Best wishes,

[Alex]

Researcher's note: Alex did get into the medical school at the first attempt.

Reaction to the manuscript of Paper 6: The development of Alex's mathematical thinking from comprehensive school to university

First reaction

Dear Hanna,

Today, I read through your article concentrating especially to the chapters related to me. You have succeeded well in condensing our long discussion in the article. I managed quite well to go through the field-specific English terms in your article but could you clarify for me a bit what the terms 'mathematical integrity and intimacy' mean in this specific case?

Especially the direct quotes described my thinking well. The quote at the end of chapter 6.2 I think best synthesises my abstract perception of mathematics. It synthesises my interest in deeper mathematics (mathematics outside everyday mathematics). The quote also shows why mathematics creates positive or sometimes confusing feelings. Signs developed by humans can be used to describe the world and develop for example new technology.

The comparison of the development in my mathematical thinking over the four years is very similar to how I have understood it myself. However, I don't think that I would have stopped to actively think about the development of my mathematical thinking without our discussions. Indeed, my mathematical thinking has not just grown like branches in a tree but it has gotten more dimensions. The model developed in comprehensive school has served as a good basis to build on.

When describing the development of my problem-solving skills you say that orienting takes longer in upper secondary school and it is more precise than in lower secondary school. This is completely true and maybe the only aspect in my problem solving abilities that I have consciously developed over the years, and in my opinion, a lot of development has happened. The more precise orienting helps me to focus, avoid careless errors and unnecessary rushing. It also helps me to concentrate (which has probably been the most important factor leading to success in matriculation exams and in university entrance exams. In other words, [it helps] in situations in which right choices need to be done quickly. The idea could be parallelised with urgent situations in the ER as a doctor in fu-

ture when the concentration is also effected by the responsibility of another person's health.)

Your writing about self-confidence in problem solving and problem-solving phases and strategies describes me well and I don't know how to comment them further. Mainly I paused to think that planning at least the quite easy problems have not changed much. I can't say would there be a change in planning with more difficult tasks either. In the future, in problem solving (probably more in diagnosing patients than with numbers), it would be good to develop in this regard. I have always solved problems with the first way that I come up with and I have not paused to think of alternative/easier ways of solving a problem.

I hope that I have opened up my thoughts in an understandable way and I hope that my comments are useful for your article. If you want clarifications or more comments about some specific point, just ask. [...]

'Alex'

Researchers note: Further comments are on the next page.

Second reaction (after clarification of 'mathematical integrity and intimacy')

Dear Hanna,

I try to comment the analysis on [mathematical integrity and intimacy] even though internalising them is not that easy.

If I understood correctly, [mathematical integrity] on my account means that I trust mathematics and I don't question the possibilities of mathematics leading me towards the correct answer and the mathematical understanding of a task. In any case, this is exactly how I think. I'm not that motivated to think about mathematics as science and as truth, that I would have a need to question the approaches and their usefulness while solving tasks after I have understood the logics in the approach.

What comes to [mathematical intimacy], I consider more difficult tasks as intellectual challenges per se and solving them is even exciting. Easy and pattern-like tasks don't produce big emotions (except boredom if a lot of similar tasks have to be solved at once). If I don't immediately succeed in solving a task, my interest towards it grows. Making a mistake or having an error in my logics might frustrate a bit, but usually I can utilise this emotion to solve the task again. In other words, the emotional states caused by the tasks mainly increase the commitment to solve the task.

I remember that in upper secondary school proofing (and vectors) caused frustration in me. Could it be called 'proofing by induction', that I didn't understand at all. I tried to go through an example task repeatedly, in which something was proofed with mathematical induction, but at any point I just didn't understand the logic of it (How does this proof anything?!). This is perhaps a point where my [mathematical integrity] has staggered.

I hope that my vague thoughts concerning these terms that are difficult to understand, can be exploited.

[Alex]

APPENDIX 5:

The papers

List of the papers

1. Viitala, H., Grevholm, B., & Nygaard, O. (2011). About affect in five Finnish dissertations on mathematical thinking. In K. Kislenko (Ed.), *Current state of research on mathematical beliefs XVI: Proceedings of the MAVI-16 Conference, June 26-29, 2010, Tallinn, Estonia* (pp. 313-330). Tallinn, Estonia: University of Tallinn, The institute of Mathematics and Natural Sciences.
2. Viitala, H. (2013). Alex's world of mathematics. In M. S. Hannula, P. Portaankorva-Koivisto, A. Laine, & L. Näveri (Eds.), *Current state of research on mathematical beliefs XVIII: Proceedings of the MAVI-18 Conference, September 12-15, 2012, Helsinki, Finland* (pp. 71-82). Helsinki, Finland: The Finnish Research Association for Subject Didactics, Publications in Subject Didactics 6.
3. Viitala, H. (2015). Two Finnish girls and mathematics: Similar achievement level, same core curriculum, different competences. *LUMAT*, 3(1), 137–150.
4. Viitala, H. (2016). Emma's mathematical thinking, problem solving and affect. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 1294–1300). Prague, Czech Republic: Charles University and ERME.
5. Viitala, H. (2017a). A tool for understanding pupils' mathematical thinking. *Nordic Studies in Mathematics Education*, 22(2), 5–31.
6. Viitala, H. (2017b). *A framework for studying students' mathematical thinking at different ages: A longitudinal case study of Alex*. Manuscript in preparation.
7. Viitala, H. (2017c). A case study on Finnish pupils' mathematical thinking: problem solving and view of mathematics. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education* (pp. 1218-1225). Dublin, Ireland: DCU Institute of Education and ERME.

Article 1

ABOUT AFFECT IN FIVE FINNISH DISSERTATIONS ON MATHEMATICAL THINKING

Hanna Viitala, Barbro Grevholm, Olav Nygaard
University of Agder, Norway

In this paper we study five Finnish dissertations on mathematical thinking from the last 10 years. We intend to answer the question 'what have Finnish researchers said about mathematical thinking, with special emphasis on affective factors.' In the studies presented, mathematical thinking is mostly seen as a cognitive function and only two of the five dissertations take affective factors profoundly into account in their theoretical framework. In the empirical part of the dissertations, mathematical thinking is viewed through mathematical or information processes, conceptual change, or different representations. The paper discusses the approaches to mathematical thinking and affect which appear in these studies.

INTRODUCTION

In Finnish comprehensive school “[t]he task of instruction in mathematics is to offer opportunities for the development of mathematical thinking” (National Core Curriculum for Basic Education 2004, p.157). This aim to develop pupils’ mathematical thinking is emphasized in every level of mathematics education. What mathematical thinking is, however, is left undefined in the curriculum and the reader is expected to have an intuitive sense of its meaning.

'Mathematical thinking' is a term that is used widely in research articles in mathematics education. Many authors describe how mathematical thinking can be improved through teaching (e.g. Doerr 2006, Sfard 2001) or by using some specific problems (e.g. McGregor 2007), how mathematical thinking can be measured in school (e.g. Baker 1993; Bisanz, Watchorn, Piatt & Sherman 2009), or what kind of mathematical thinking pupils have (e.g. Joutsenlahti 2005, Merenluoto 2001, 2004).

Despite the wide use of the concept, there is no consensus of what is meant by mathematical thinking (Sternberg 1996). Many researchers seem to think of the concept as thinking about mathematics when others might think of it as combination of complicated processes, something that makes use of mathematical operations, processes, or dynamics (Burton 1984). Additionally, as seen also from these definitions, mathematical thinking is often connected to cognition. But what is the role of affect in mathematical thinking? It would be helpful to explore the affective factors of relevance for mathematical thinking before one begins to study mathematical thinking more deeply.

In her doctoral studies, Viitala is trying to describe what characterises Finnish 15-year-old pupils' mathematical thinking when they are about to end their 9-year compulsory schooling. Basic school tests can only give hints of the underlying thinking. To get deeper knowledge about mathematical thinking, more in-depth research needs to be done. For this reason Viitala intends to study pupils' mathematical activity and actions during problem solving and interprets this activity as visible signs or expressions of mathematical thinking. Finding the role of affect in mathematical thinking is one important step on the way to understand mathematical thinking in the study, and this paper serves as the start on that road, by investigating what has earlier been said about this issue in selected Finnish doctoral dissertations.

RESEARCH QUESTION AND CENTRAL CONCEPTS

In this paper we intend to answer the following question: What have Finnish researchers said about mathematical thinking, with special emphasis on affective factors? Mathematical thinking is considered as thinking about, on or in mathematics, and in most cases it is thinking that occurs when mathematical tasks or problems are solved or discussed.

Affective components for us follow the work of McLeod (1994) who classified affective components into emotions, beliefs and attitudes, and DeBellis and Goldin (1997) who developed this classification further by adding values to it. Emotions are mostly affective and the least stable of these, whereas beliefs are mostly cognitive and the most stable. Attitudes and values belong somewhere in between these two. The four components are different from each other, but they are interacting so that the study of one component cannot be completely separated from the three other components.

METHOD AND METHODOLOGY

Mathematical thinking is often considered as a purely cognitive function. However, we claim that affective factors are closely connected to this cognitive side of mathematical thinking. There are vast amounts of research done in both research areas: mathematical thinking and affect. Our goal is to explore how these two areas of research are connected in reports. It is clear that careful selection was needed in exploring these reports. In this chapter we explain our selection process.

In her doctoral work, Viitala is studying Finnish pupils' mathematical thinking in the end of their compulsory schooling, at the age of 15. This encouraged us to concentrate on the research

done on mathematical thinking in Finland. There exists high level research on affect in Finland, and a review concerning it has been published by Hannula (2007). Thus, our focus is on reports about mathematical thinking and the aim is to find connections between mathematical thinking and affective components in various Finnish studies.

We started our search by exploring larger studies on mathematical thinking, and Finnish doctoral dissertations served as a good starting point. When we investigated the dissertations, the focus was not on the results but on how mathematical thinking and affect are presented in the theory. Nonetheless, there were too many doctoral studies on mathematical thinking to report on in this paper, so we limited the exploration to research that have been done in secondary school within the last 10 years.

Finally, we managed to limit the discussion to five dissertations on mathematical thinking. In many cases there are further reports and development on theory published, however, because of the limitation of pages for this paper we concentrate only on the five 'original' reports from Hannula (2004) and Hihnala (2005) from lower secondary school, and Joutsenlahti (2005), Merenluoto (2001), and Hähkiöniemi (2006) from upper secondary school.

MATHEMATICAL THINKING AND AFFECT IN FIVE DISSERTATIONS

We will now describe what has been said about mathematical thinking and affect in the five dissertations, one by one. Two of the dissertations are based on data from lower secondary school and three of them from upper secondary. We start with the two from lower secondary school, and then proceed to the remaining three.

The study by Markku Hannula

The aim of the study by Markku S. Hannula (2004) is to “increase the coherence of the theoretical foundation for the role of affect in mathematical thinking and learning” (p. 4). The dissertation includes theoretical and empirical work, and three research tasks are set: to make an analysis of the concepts used for describing affect in mathematics education and, if necessary, refine the definitions, to describe the role of affect in mathematical thinking and learning, and to describe how experiences influence the development of affect (*ibid*, pp. 36-37). From these, some results of the first two questions are presented below.

Hannula studies affect in mathematics education research from the same starting point as we do. He starts with McLeod’s (1994) classification of affect dividing it into beliefs, attitudes and emotions, and adds values to this categorization following DeBellis and Goldin (1997). However, as Hannula notes, these four concepts do not cover the whole field of affect, and from other concepts used in literature, he adds motivation into the discussion. Hannula shows how affective components are viewed from different theoretical frameworks (e.g. from cognitive or social dimensions). Hence, in pursuing to construct a holistic framework of the human mind, he includes physiological, psychological, and social views into his search.

After reviewing the literature and re-evaluating the concepts used in them, Hannula ends up with cognition, motivation, and emotion which all belong to the individual’s self-regulative system. In this system, cognition and emotion are viewed as representational systems which require motivation as an energizing system. Cognition codes information about self and environment, and emotions about progress towards personal goals. Motivation originates from human needs. They all are deeply intertwined and each regulates the others to some extent.

From the original four concepts only emotion fits into the framework Hannula introduces, when the other three (beliefs, attitudes, and values) are seen as mixtures of motivational, emotional and cognitive processes. This framework is built to clarify the role of affect in processes of the human mind. Emotion, cognition and motivation are described as “fundamentally different kinds of processes that together constitute the human mind” (ibid, p. 20). Attitudes, beliefs, values, and even emotions are defined in ways that include motivational, emotional, and cognitive processes. The model of human mind is presented in figure 1.

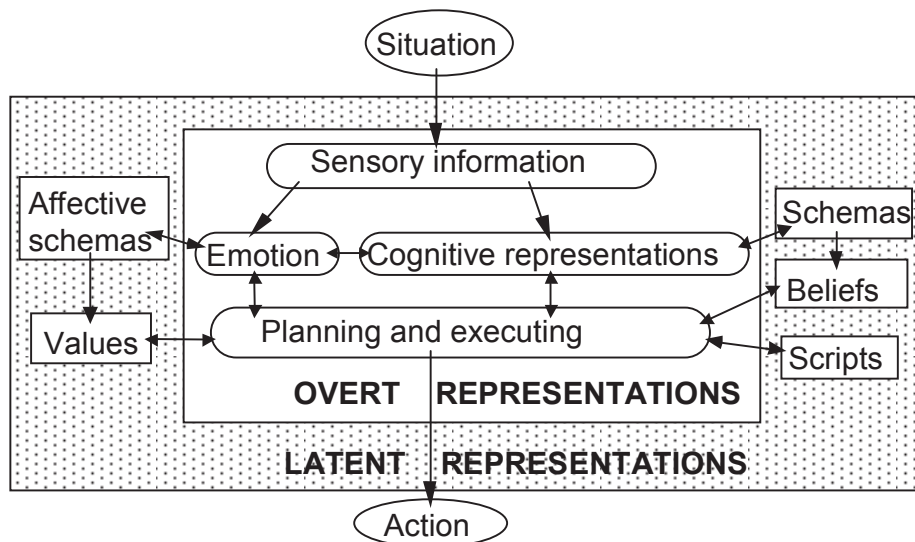


Figure 1. A model of human mind (Hannula 2004, p. 51)

In addition to the model of human mind, Hannula complements the theory with ‘the meta-level of mind.’ Meta-level of mind stems from cognition-emotion interaction and consists of four aspects: metacognition (cognitions about cognitions), emotional cognition (cognitions about emotions), cognitive emotions (emotions about cognitions), and meta-emotions (emotions about emotions). These clusters are considered qualitatively different

and they can be conscious as well as unconscious. People can only tell about things they are aware of, and thus, research based on self-report can only reach metacognition and emotional cognition (Hannula 2001, one of the articles in Hannula's dissertation). Emotions can be reached directly for example observing facial expressions (*ibid.*).

As a conclusion, there is a relationship between affect and mathematical thinking, as Hannula (2004) describes it:

In mathematical thinking, the motivational aspect determinates goals in a situation. [...] Emotions are an evaluation of the subjective progress towards goals and obstacles on the way. [...] Cognition is a non-evaluative information process that interprets the situation, explores possible actions, estimates expected consequences, and controls actions. (p.55)

The study by Kauko Hihnala

The second dissertation on mathematical thinking, to which the empirical data was collected in lower secondary school, is from Kauko Hihnala (2005). The aim for his research is to describe the development of mathematical thinking when shifting from arithmetic to algebra. Hihnala (2005) approaches mathematical thinking through van Hiele's (1986) theory. Hihnala does acknowledge that van Hiele's five level theory for mathematical thinking was developed to describe the levels of geometrical thinking, however, he adapts parts of the theory to describe the levels of algebraic thinking. In the study, mathematical thinking is studied through algebraic thinking.

When doing a literature review on mathematical thinking, Hihnala identifies four lines of research that are often connected to mathematical thinking: problem solving (when pupils' metacognitive skills are emphasized), reasoning, conceptual change in knowledge inquiry, and understanding (where processes are important). In categorizing previous research, he

bases mostly on Finnish studies. In his own study he claims to examine mathematical thinking through knowledge processing. The data is collected mainly by analysing solutions of tasks that pupils gave on paper.

Knowledge is examined through problem solving and it is divided into procedural knowledge and conceptual knowledge. Here Hihnala refers to the work of Hiebert and Lefevre (1986), Kieran (1992), and Sfard (1991). In his study Hihnala analysed the procedural knowledge used in tasks but acknowledges that it is the procedures that change the conceptual knowledge into perceivable form (Hiebert & Lefevre 1986).

When Hihnala is constructing the theoretical framework for his study, he does not take affective factors into consideration. He is investigating the tasks and what tools he needs in analysing them. In the discussion of the results, however, he talks about motivation when he considers reasons for possible changes in pupils' grades as they move forward in their studies. Also teachers' task in motivating pupils to study is recognised.

The study by Jorma Joutsenlahti

In the remaining three dissertations, the empirical data were collected in upper secondary school. The first dissertation we are exploring is from Jorma Joutsenlahti (2005). While Hannula (2004) did his most significant work in clarifying and refining the definitions of concepts in the affective domain, Joutsenlahti (2005) does profound work in the domain of mathematical thinking.

Joutsenlahti's dissertation includes also theoretical and empirical work. He examines different approaches to mathematical thinking and makes his own model for the concepts. Although his main problem in the study is to describe features of the pupils' test-oriented mathematical thinking, as before, we are

concentrating on the theoretical framework he is constructing and the role of affect in his theory.

In Finnish curriculum one aim is to develop pupils' mathematical thinking. This was the starting point for Hihnala's (2005) work (in lower secondary school), as it is for Joutsenlahti (in upper secondary school). However, as Joutsenlahti highlights, mathematical thinking is something that cannot be observed directly. He introduces five central starting points for studying pupils'¹ mathematical thinking that can impact essentially on the thinking process, or by which mathematical thinking can be understood or described. These starting points are beliefs, culture, mathematical abilities, information processing, and problem solving.

Joutsenlahti places these starting points into five different approaches to mathematical thinking following Sternberg (1996). These approaches are *the psychometric approach* (mathematical abilities), *the anthropological approach* (culture, beliefs), *the pedagogical approach* (beliefs, problem solving), *the mathematics as science approach* (problem solving, information processes), and *the information process approach* (information processes, problem solving).

From the listed approaches, Joutsenlahti uses the information process approach. Here, the concept of knowledge is emphasized instead of viewing thinking as computer-like manipulation of symbols. As problem solving is part of that approach, also pupils' metacognitions, beliefs, attitudes, and emotions (as part of the belief system that is directed to mathematics and learning mathematics) play an essential role in Joutsenlahti's research.

¹ Joutsenlahti uses the word 'student' to refer to secondary school pupils. In this paper, however, we follow the Scandinavian way, and we call them 'pupils.'

These all are connected to strategies that are part of strategic knowledge.

Knowledge in Joutsenlahti's study is divided into procedural knowledge (includes mastery of skills), conceptual knowledge (includes also knowledge that is understood), and strategic knowledge. These categories of knowledge are all connected to problem solving. Knowledge is linked to mathematical proficiency (Kilpatrick, Swafford & Findell 2001) through versatile control of mathematics. Understanding leads to 'conceptual understanding', problem solving to 'strategic competence' and 'adaptive reasoning', skills to 'procedural fluency', and beliefs to view of mathematics which Kilpatrick et al. (2001) calls 'productive disposition'.

All in all, Joutsenlahti (2005) defines pupils' mathematical thinking to be his or her processing of mathematical knowledge (procedural, conceptual, strategic knowledge) that is guided by his or her metacognition, where the individual reorganizes his or her web of knowledge. From affective components, beliefs, attitudes and emotions are central part of his analysis through the study of pupils' view of mathematics. The aim for mathematical thinking is considered as getting deeper understanding of concepts or sets of concepts, or getting through problem solving processes.

The study by Kaarina Merenluoto

The fourth dissertation is from Kaarina Merenluoto (2001). She looks into upper secondary school pupils' understanding of real numbers, and the aim is to "describe the conceptual change, which is needed when the number concept is enlarged from the natural numbers to the domains of integers, rational and real numbers" (ibid, p. 6). In addition to exploring theory in Merenluoto's dissertation, we also look into results where she

introduces a framework for levels of mathematical thinking based on the theory and the empirical results of her study.

Merenluoto approaches abstract mathematical concepts, or 'creatures' as she calls them, through representations. These concepts have a dual nature, and following the work of Sfard (1991), Merenluoto interprets mathematical concepts operationally as processes, or structurally as objects. These classifications are seen as complementary (Sfard 1991). In connecting the new knowledge that pupils are trying to learn and their prior thinking, Merenluoto (2001) uses the theories of conceptual change that have been more in use in research on learning physics, as she points out.

From the data of 640 pupils, Merenluoto constructs a five level classification for mathematical thinking based on theoretical starting points. In different levels she combines the levels of structurality of a concept (Sfard 1991), and the thinking in theories of conceptual change (e.g. Vosniadou 1999, diSessa 1993). Also the theories of Goodson-Espy (1998), and Cifarelli (1988) are used. In the following, the levels of mathematical thinking are described and their connection to Sfard's theory is indicated.

The first and lowest level of mathematical thinking is the elementary level. Here the pupil's answer is based on logic of natural numbers and/or everyday experiences. The second level is recognition where the pupil recognises some essential characteristic of a concept, but his or her prior thinking is in control. The third level, which together with the second level is comparable with Sfard's (1991) interiorization, is called reproduction. Here the pupil's justification is based on mental operations. The fourth level is structural abstraction, where the pupil recognizes some structure of a concept. This level together with the fifth is comparable with Sfard's (1991) condensation. The fifth level is called structural awareness where the pupil pays attention to the structure of the number concept and shows

ability to compare them. Sfard's (1991) reification can be seen as the area of experts' thinking (Merenluoto 2001).

Affective components in Merenluoto's study (2001) are left in the background. Affect is visible, however, when Merenluoto talks about certainty judgements of mathematical solutions or answers. These certainty judgements are considered as emotional (e.g. feeling of confidence or difficulty). Prior conceptions are mentioned to be central in conceptual change, however, they are not studied in Merenluoto's work. Also feeling of control and certainty in mathematical task performance, and experiences are discussed in the final chapter, and for example beliefs and self-regulation are mentioned in the discussion as being something that has guided prior research done in the area of mathematical performance. All in all, very little of affective factors are included in the study, but different parts of these factors are identified to be connected to the subject at hand.

The study by Markus Hähkiöniemi

The fifth and final dissertation we explore is written by Markus Hähkiöniemi (2006). He studied the role of representations in learning the derivative, and the aim for his research is "to find out how students may use different kinds of representations for thinking about the derivative in a specific approach" (ibid, p. 3). The ideas of student centeredness (Davis & Maher 1997) and open problem solving (e.g. Pehkonen 1997) inspired Hähkiöniemi in his study aiming to acquire information on how pupils think.

Representation is one of the central concepts in Hähkiöniemi's research (2006). He considers representations not only as tools for expressing our thoughts, but also as tools to think with. Further, representations are not seen only as symbolic, but also as graphical and kinesthetic, and there are invisible internal sides in them as well as visible external sides (e.g. gestures). These sides are inseparable. Different representations can enrich pupils'

mathematical thinking, and “[t]he object of thinking is constructed through using different representations” (ibid, p. 15).

For a more general framework, Hähkiöniemi (2006) uses Tall’s theory of three worlds of mathematics (e.g. Tall 2004). These three worlds are the symbolic world where symbols act dually as processes and concepts, the embodied world of visuo-spatial images, and the formal world of properties. From these, Hähkiöniemi studies pupils’ use of different representations in the embodied and symbolic worlds. He concentrates on what kind of procedural and conceptual knowledge pupils are using and how they consider derivative as a process and as an object.

In the theoretical framework, or in the results, we could not find any mentioning of affective factors, only the pupils’ cognitive activity was studied. Affective factors and how they influence the learning is not discussed even in the discussion chapter.

SUMMARY AND DISCUSSION

All the five dissertations discussed above were strongly connected to (secondary school) pupils’ mathematical thinking. In the work of Hannula (2004), the focus was more on affect, and on the role of affective factors in mathematical thinking. Both affect and mathematical thinking are concepts that are widely used in mathematics education research, and with either of the concepts there is no common agreement on their definitions. Where Hannula (2004) aimed at clarifying and refining the definition of affect, Joutsenlahti (2005) built a new model for mathematical thinking resting on numerous previous definitions. The remaining three, Hihnala (2005), Merenluoto (2001), and Hähkiöniemi (2006) did not define mathematical thinking explicitly. It seems like they refer to mathematical thinking as thinking in mathematics, where mathematical thinking appears when the pupils calculate something, explain their understanding

of a mathematical situation, or thinking is interpreted from the pupils' written solutions to different tasks.

In the studies presented, mathematical thinking is viewed through mathematical or information processes, conceptual change, or different representations. Also problem solving among many other approaches to mathematical thinking is mentioned often and can be interpreted as being part of some of the approaches taken. This illustrates how describing mathematical thinking is complex, and that it can be viewed from many different starting points. This is also the case with affect, as Hannula (2004) shows in his theory review. Only Hannula (2004) and Joutsenlahti (2005) clearly deal with affect in their studies. Hannula defines affect through self-regulation where emotion, cognition and motivation are central concepts. Beliefs, values, and attitudes are seen as mixtures of motivational, emotional and cognitive processes. Joutsenlahti (2005) consider pupils' view of mathematics as influential to mathematical thinking, and beliefs, attitudes, and emotions are studied. This view can be connected to Hannula's model in future studies.

In Merenluoto's (2001) research many affective components are recognised to have an influence in learning and performing in mathematics, and from such components for instance concepts of emotions in certainty judgements, beliefs that influence our search of knowledge, and prior experiences are mentioned in the dissertation. From these, only emotions as feelings of certainty are actually studied. Hihnala (2005) and Hähkiöniemi (2006) do not mention clearly any affective factors in their theoretical framework. Hihnala (2005), however, mentions motivation in his discussion in the conclusion part of his dissertation, as it might explain some of the variation of pupils' grades through time. Hähkiöniemi (2006) continues to interpret the theory and results strictly through cognition. Thus the conclusion is that several of these researchers study mathematical thinking without giving

any emphasis to affective factors, although these factors must be considered important. It seems that this confirms the view that beliefs and other affective factors constitute a hidden variable in the classroom (Leder, Pehkonen & Törner 2002).

One interesting thing from the dissertations is the notions of metacognition. When Hihnala (2005) talks about problem solving as one way to look into pupils' mathematical thinking he mentions how the pupils' metacognitive skills are important in problem solving. This is argued also in Joutsenlahti (2005), where metacognitions are additionally considered as part of information processes. In the work of Merenluoto (2001), more closely in her theory review, metacognitive awareness is mentioned as something that is missing from novice's explanations, in comparison to experts' explanations.

For Hannula (2004) metacognition is a central part of the meta-level of mind. He also clearly demonstrates how different parts of meta-level of mind can be recognised from data (Hannula 2001). However, in research based on pupils' own explanations about task solving, only metacognition and emotional cognition can be detected from the data (ibid.). Aspects based on emotions (meta-emotions and cognitive emotions) cannot be studied directly from what someone says. However, even considering these aspects of meta-level of mind, what the pupils can express in interviews can enrich data and will be part of Viitala's work. Especially metacognitions play a central role in the interviews as it is the metacognitive awareness of pupils that might differentiate some thinking to be on higher level than other.

Finally, even though there is a strong line of research on affect in Finland (Hannula 2007), many times it does not reach the research on mathematical thinking. Mathematical thinking is seen as a cognitive function, and the definitions of knowledge are important in these studies. From the dissertations we studied, only Joutsenlahti (2005) and Hannula (2004) took affective

components into account in their studies explicitly and they both utilized McLeod's classification of affect (emotions, beliefs and attitudes).

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Article 2

Alex's world of mathematics

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Abstract

This is a story of a high achieving 15-year-old boy called Alex. The story is not about the achievement in class per se, but rather a story of what kind of role mathematics plays in his everyday life. High performance in school mathematics does not automatically mean thinking is flexible and performance reaches the same level outside school. However, this story shows how a boy, who does not value mathematics higher than any other school subject, can transfer the knowledge he has in mathematics outside the classroom quite naturally and spontaneously seek for valid examples from school mathematics when talking about mathematics in general.

Keywords

affect, mathematical thinking, case study

Introduction

This paper derives from a research project on Finnish 15-year-pupils' mathematical thinking which aims to describe pupil's mathematical thinking in two perspectives; On one hand the aim is to find out if it is possible to combine results on mathematical thinking in different domains in mathematics (e.g. problem solving, algebra, and statistics), and on other hand combine results from cognitive and affective data.

When studying pupils' mathematical thinking, research has usually concentrated purely on the cognitive aspect. However, it has become clear that if we really want to describe mathematical thinking, we should also relate to affective factors (Vinner 2004). Nowadays "[a]rguably the most important problem for research on affect in mathematics is the understanding of the interrelationship between affect and cognition" (Zan, Brown, Evans & Hannula 2006).

The present paper concentrates on the affective data but not forgetting the cognitive aspects of learning mathematics. It discusses one case, a Finnish 15-year-old Alex, and aims to see what his own explanations reveal about his affect in mathematics, what role mathematics plays in his everyday life, and what he can say about his own mathematical thinking. This is an initial step in the research project to understand how Alex's 'interrelationship between affect and cognition' works.

Theoretical framework

There are many studies on affect in mathematics in Finland. Some reviews have already been published about the subject (e.g. Hannula 2007; Viitala, Grevholm & Nygaard 2011). The short literature review below about studies on affect concentrates on the findings from Finnish lower secondary school. This is the level where Alex is at the time of the data collection.

The core of pupils' view of mathematics in grade 8 has been found to be constituted by four components: ability, difficulty of mathematics, success, and enjoyment of mathematics (Hannula & Laakso 2011). Similar results have also been found among different age groups (e.g. Rösken, Hannula & Pehkonen 2011). Positive dimensions correlated positively to other positive correlations, and negative dimensions correlated to negative views. The grade 8 pupils are "more clearly divided into those with a positive view of mathematics and to those who hold a negative view of mathematics" (Hannula et al. 2011, p.13).

The most recent national report on the Finnish learning results at the end on comprehensive school wonders if the calculation skills in Finland are declining (Hirvonen 2012). Together with mathematics assignments, a background survey including information about attitudes towards mathematics was collected. The results show how pupils "considered mathematics to be useful, but they did not like it at all that much" (ibid., p.12). Pupils' perceptions of their own skills were slightly positive. Gender differences were found to be minor.

Some affective data has also been collected in PISA assessments. The results show that Finnish pupils lack interest and enjoyment in mathematics. Only the pupils on the two highest proficiency levels seemed to be interested in and enjoy mathematics. Anxiety in mathematics was below OECD average and boys had more positive attitudes towards mathematics than girls. (Törnroos, Ingemansson, Pettersson & Kupari 2006) Finnish pupils were also characterized by

“below average self-efficacy and low level of control strategies used. [...] In Finland affect was an important predictor of achievement. Mathematical self-concept was the strongest predictor of mathematics performance, and this correlation was strongest among countries in the study.” (Hannula 2007, p. 201)

Theoretical framework around affect, its concepts and their connections have been used in very diverse way both in Finnish and international research (see e.g. Hannula 2007, Zan et al. 2006, Furinghetti & Pehkonen 2002, and MAVI proceedings throughout the years). Thus, some clarification is needed here also.

In the present paper affect and its different components such as beliefs, attitudes (McLeod 1994) and values (DeBellis & Goldin 1997) are not separated from each other. Instead, affective factors are seen as mixtures of motivational, emotional and cognitive processes (Hannula 2004). Moreover, affect is viewed through a model of the individual's self-regulative system, where cognition and emotion are viewed as representational systems which require motivation as an energizing system (ibid.).

When talking about affective data collected as part of a project on mathematical thinking it is also important to explain the tight connection between mathematical thinking and affect. This connection is well articulated by Hannula (ibid., p. 55):

In mathematical thinking, the motivational aspect determinates goals in a situation. [...] Emotions are an evaluation of the subjective progress towards goals and obstacles on the way. [...] Cognition is a non-evaluative information process that interprets the situation, explores possible actions, estimates expected consequences, and controls actions.

Methods

The aim in this paper is to discuss one case, a Finnish 15-year-old Alex, and see what his own explanations reveal about his affect in mathematics, what role mathematics plays in his everyday life, and what he can say about his own mathematical thinking. This aim is reached by analysing video data from three semi-structured and focused interviews (Kvale & Brinkmann 2009) I had with Alex in the autumn 2010.

The interviews followed six themes. Four of them followed Pehkonen's categorization of mathematics related beliefs on 1) mathematics, 2) mathematics learning, 3) mathematics teaching and 4) oneself within mathematics (Pehkonen

1995, discussed also in Op't Eynde et al. 2002). Pupil's background and mathematical thinking were the two remaining themes. In the interviews pupil's own lines of thoughts were emphasized and followed whenever possible. A more elaborated structure of the interviews together with some example questions can be found in Table 1.

Table 1. Interview themes and example questions.

<i>Interview</i>	<i>Theme</i>	<i>Example questions</i>
1	Background	Tell me about your family.
	Mathematics	What is mathematics as science? Does it exist outside school? (How? Where?)
	Oneself within mathematics	Is mathematics important to you? Does it help you think logically? (How?)
2	Mathematics learning	How do you learn mathematics? Is it most important to get a correct answer?
3	Mathematics teaching	Does teaching matter to your learning? (How?) What is good teaching?
	Mathematical thinking	What does mathematical thinking mean? How do you recognize it?

The themes of the interviews also guided the data analysis and reporting of the results. In addition, data about learning mathematics was further analysed using Hannula's (2004) self-regulation system introduced above. The videotapes were first transcribed and categorized roughly into the six themes (in Finnish). In this process also some data reductions were done (shortening of sentences and leaving some parts of longer examples outside the transcription). Then the data was translated into English.

After having the original transcriptions in both languages, more data reduction was done following strictly the six themes introduced above. Throughout the analysis the words used by the interviewee were preserved. Only in the very end the key findings were put together and interpreted as is seen in this report, still offering some original data from the interviews to support the interpretations.

Results

The categorization of the results follows the structure of the interviews (see Table 1) emphasizing the part of mathematics learning (which most reveals the relationship of affect and mathematical thinking). The chapter about mathematics learning also follows (loosely) Hannula's model of self-regulation (2004). The categorization is not exclusive; many of the findings could belong to different categories.

Background

Alex is in his final grade in Finnish comprehensive school starting his 9th year of schooling. He is the only child with parents who both have higher level university degrees. Alex spends a lot of time doing sports, and mathematics is his third favourite subject in school after sports and English language. His mathematics grade¹ is 10 and it describes his skills in mathematics well because, in his own words, "I usually know the mathematics taught in school quite thoroughly." After comprehensive school he will go to upper secondary school.

Mathematics

Alex's view of mathematics is rather dynamic: For instance, he emphasizes that, rather than changed, mathematics has expanded during the 9 years in school. As an example of this expansion he explains how "many different calculations can be calculated in different ways still getting a correct answer." He also sees mathematics strongly as a tool: as a science mathematics is "explaining different problems or natural phenomena, or such, with the assistance of calculations." Mathematics is important as a school subject because "it is very useful in school subjects such as physics, chemistry, and other natural sciences."

When asked about his use of mathematics outside school, he finds situations (dealing with money: gas for the moped, other expenses, earnings) where "simpler" mathematics is needed, and he finds examples of the mathematics he needs (calculations with percentages) or does not need "much" (geometry) or apply "yet" (systems of two equations, subject they were learning in class at the time of the interview). He also recognizes that in working life mathematics is needed "in quite many jobs."

¹ The scale in Finland is 4...10 where 10 is the best possible grade.

Oneself within mathematics

Alex's affect in mathematics is very positive. He has a lot of self-confidence and he trusts himself "pretty much" in mathematics. He values mathematics and thinks that mathematics is important "as a school subject," and he sees that this view is also shared by his family and friends. The majority of the feelings he connects with mathematics are positive, he enjoys challenges in mathematics and he is persistence to find answers to his questions.

"When you learn, [learning mathematics] is fun and interesting" whereas "calculating basic calculations, that are being calculated a hundred times, is a bit boring. However, then the routine is found so it [learnt mathematics] can be done also later on." Learning mathematics "might be exiting if it has something to do with oneself."

"With those [tasks] that I really have to think and I discover something [mathematics] is definitely not boring, they [the tasks] are very interesting."

Mathematics "is usually quite easy but challenges can, of course, be found and [mathematics] can be hard if it goes far enough." If mathematics feels hard "I think about it quite much [...] why [something is done, ..., and] it keeps bothering me. [..., I do want to find answers because] then it would not bother me anymore."

Alex's motivation to study mathematics is twofold: he studies mathematics "for a good grade which also benefits future studies, and also for learning and understanding" mathematics. From these, the first (external) motivation seems to be dominating over the second (internal) one: Despite the very positive affect in mathematics, Alex sees that mathematics "is not more special than other [school subjects]" and he would not study mathematics (at home) if it were not compulsory. Nonetheless, it is clear that he recognizes the value of learning mathematics.

Mathematics learning

Alex is very aware of his learning in mathematics and he can explain it in two levels: the overall process of learning and connecting new knowledge to prior knowledge. The overall learning process ("understanding what is being pursued" and "calculating tasks from easier to more difficult") is important to Alex because "without learning process one cannot discover everything" (that needs

to be learnt). Routine (even though boring) is also important so the calculations wouldn't feel difficult.

After the more general discussion of learning the discussion moved to learning new things and making connections in particular. This small part of one interview presented below gives a good example about Alex's awareness of his learning and net of knowledge, and how he does not always even realize making the connections:

Int: (When you learn new things) do you for example search for connections to mathematics that has been learnt before?

Alex: Yes, I seek for connections to mathematics learnt before, I look for similarities. For example last year we had polynomial calculations, and now drawing lines and solving equations. They have quite a lot of the same things.

Int: So you remember similar things and you connect them to each other when you learn new things?

Alex: Yes, I don't necessarily always realise them if they are in different places, sometimes I do realize them, and sometimes they are self-evident and I don't think about them.

Learning mathematics for Alex is more understanding than remembering and memorizing. Understanding means two things: First one has to understand why something is done (e.g. in polynomial calculations "understanding for example why the terms are moved to another side"). Secondly, one has to know another way to verify the solution than the one used in the task.

The emotions connected to learning mathematics are mostly positive as described before. Alex thinks learning mathematics is "fun and interesting," whereas rote learning is boring. He has a lot of self-confidence and trusts more his own reasoning than his calculations. Making mistakes does not frighten him, but when he does them, they disturb his thinking (he thinks it is hard to find the error). Mathematics "is usually quite easy but challenges can, of course, be found and it can be hard if it goes far enough."

Alex is motivated to learn mathematics and he aims for understanding. He also recognizes that he is responsible of his own learning. To know if mathematical knowledge is correct "one has to calculate or discover it oneself." Having a good grade in mathematics is the most important motivation for Alex to study

mathematics. Hence, he prepares for mathematics tests carefully and usually knows what kind of tasks there should be in the test. In addition, it tests, he checks his answers carefully: First he checks the units, next estimates if the answer is reasonable (if the magnitude of the answer is correct), then he rethinks the expression or equation and how he got it, and finally checks if the answer is correct.

Mathematics teaching

Teaching is central to Alex's learning. "[He would] not study mathematics alone at home if it was not compulsory. [He learns] in school when [mathematics] is taught. Usually it is enough and [he does] not have to study it separately for tests at home." Teaching mathematics in school proceeds from details to wider connections. First calculating and solving equations is learnt, and then it is expanded and applied.

Good mathematics teaching is "illustrative: [learnt mathematics] is connected to 'somewhere it is really needed', and [explanations are also given on] what kind of phenomena can be transformed into calculations being learnt. [However, making connections] are many times hard in the beginning when calculating is rehearsed mechanically."

Mathematical thinking

When explaining mathematical thinking, Alex brings up the same tool aspect as when describing mathematics as a science: For Alex mathematical thinking means "transforming different attributes, and for example weather conditions and natural phenomena into some form of calculations," or vice versa, he recognizes his mathematical thinking when "some form of calculating, or using or applying rules of natural sciences, applying things" exists.

Alex likes things to be logical: He likes Swedish and German languages least as school subjects as they are "not that logical and have a lot of exceptions." Mathematics helps Alex to think logically "as things can be made to numbers."

As an example of Alex's clear and prompt mathematical thinking here is a problem I gave him to solve in connection to discussion about having many answers to one problem. It is a modified PISA-task (originally 2 and 5 km): *Mary lives 3 kilometres from school, Martin 5. How far do Mary and Martin live from each other? (OECD 2009, p. 111) This is how Alex solved it without using any concrete tools to help him:*

Alex: When thought quickly, 2 if you calculate the difference, but of course it can be 8 kilometres or something in between.

Int: Can it be anything in between?

Alex: [Pause] Almost, yes.

Int: Why?

Alex: Because they go by radius' from school. And apparently it forms circles for both and they can be to any ratio to each other. So, it becomes anything in between.

Conclusion and Discussion

This paper aims to discuss one case, a Finnish 15-year-old Alex, and see what his own explanations reveal about his affect in mathematics, what role mathematics plays in his everyday life, and what he can say about his own mathematical thinking. In this part of the paper the results presented above are discussed. Also some results from other data are brought up as part of the discussion.

Alex's affect in mathematics is very positive. He enjoys learning mathematics and is motivated to study it. Even though he considers mathematics important as one of the school subjects and seems not that interested in mathematics outside school, he understands the value of learning mathematics and works towards learning it in a very thorough way. Mathematics has strongly a tool value for Alex both as a science and as part of his everyday life.

Alex is very aware of his own mathematical thinking. This emerges most when discussing about his learning of mathematics. He is aware of his own learning process (understanding the goal in learning something, calculating tasks from easier to more difficult and finding routine). He can also explain well more detailed parts of learning new things, (e.g.) seeking similarities between the new thing and things learnt before. At the same time he explains teaching to be central to his learning and it seems (from the results presented here and the observations on the teaching in Alex's class) the teaching is supporting his way of learning new things and developing his mathematical thinking.

Alex seems to have a clear and organized (mathematical) thinking and net of knowledge. He can express himself in a very clear way when answering questions, is able to give spontaneous examples from school mathematics to

explain his thinking, and needs (at least in some occasions) just one stimuli to connect different topics in mathematics (in this paper polynomial calculations, equations and drawing lines) to each other. Also Alex's view on mathematical thinking seems broader than just thinking mathematics as calculations within mathematics; he also connects mathematical thinking to natural phenomena and natural sciences.

In connection to previous results on affect in Finland among Alex's age group, Alex is not a very exceptional pupil. He feels able to do mathematics, he enjoys it, succeeds in it and does not find it that difficult (cf. Hannula et al. 2011). In addition, he clearly thinks mathematics is useful, at least within natural sciences, and he likes mathematics as a school subject. The latter point contradicts previous results (Hirvonen 2012) but coincides with PISA results where the top pupils in Finland seem to be interested in and enjoy mathematics (Törnroos et al. 2006). (Whether Alex actually is part of the top two PISA groups, has not been studied.)

What makes Alex interesting is his high ability to explain his own thinking and the awareness of his own learning. He enjoys doing mathematics but it is not enough to carry the interest outside the classroom. He seems to be very down to earth with his abilities in mathematics and he recognizes that his mastery of mathematics is limited to school mathematics. It seems that it is possible to have highly positive affect in mathematics in school without being that interested in it in everyday life.

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Article 3

TWO FINNISH GIRLS AND MATHEMATICS: SIMILAR ACHIEVEMENT LEVEL, SAME CORE CURRICULUM, DIFFERENT COMPETENCES

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Abstract Mathematical thinking and problem solving are essential parts of learning mathematics described in the Finnish National Core Curriculum for Basic Education. Evaluations on both have been done at national and international level. However, in a request for deeper understanding of pupils' mathematical thinking we need to move beyond paper tests. This paper is a first look into the mathematical thinking of two Finnish girls, Emma and Nora, in their final year of Finnish comprehensive school. After solving a real-life situated problem in a classroom, the girls talk about mathematics and problem solving in an interview. The focus of the analysis is on the learning objectives, core content and final-assessment criteria related to thinking skills and methods in the Finnish curriculum. Also some results on metacognition and affect will be reported. The results suggest that while both pupils have similar achievement level in mathematics, their competences are different: Emma is more competent in problem solving and Nora is more self-confident and self-guided in learning mathematics and can more easily recognize mathematics outside school.

1 Introduction

The Finnish National Core Curriculum for Basic Education has three tasks for mathematics instruction: developing mathematical thinking, learning mathematical concepts, and learning most widely used problem solving methods (FNBE, 2004). All these instructional tasks are evaluated near the end of comprehensive school at local, national (e.g. Rautopuro, 2013; Hirvonen, 2012) and international (e.g. OECD, 2014; Mullis, Martin, Foy & Arora, 2012) level. While Finland continuously stays among top countries in PISA assessments, most recent studies show how the level of mathematical skills is declining (Väljjarvi, 2014; Rautopuro, 2013; Hirvonen, 2012). To better understand the situation with mathematical thinking, more research going beyond paper tests is needed. For this aim, in this research project we move closer to the pupils and ask: What characterises mathematical thinking of Finnish pupils at the end of comprehensive school?

Part of the general mission of basic education is to offer pupils opportunities to obtain 'the knowledge and skills they need in life, [and] become capable of further study' (FNBE, 2004, p. 12). Real-life connections are highlighted also in mathematics learning objectives, and the national curriculum emphasizes that instruction should utilize effectively problems that come up in day-to-day situations (ibid.). To study mathematical thinking of 15-year-olds, we need concrete tools (tasks or problems) to be able to talk about mathematics and mathematical thinking. PISA assessment offers well-tested mathematical problems designed for 15-year-olds that are based on real-life situations. Choosing tasks that move beyond the kinds of situations and problems that are typically encountered in school

classrooms (OECD, 2009) we aim to interpret what kind of mathematical thinking pupils enter the world outside school with.

This paper is a step towards understanding Finnish pupils' mathematical thinking better. Problem solving is an important part of mathematical thinking and serves as a starting point for the analysis. The focus in this paper is on the learning objectives, core content and final-assessment criteria related to (mathematical) thinking skills and methods in the Finnish Core Curriculum for Basic Education (FNBE, 2004). In addition to cognitive and metacognitive aspects of the curriculum, also some affective aspects will be discussed.

The paper aims to answer the following questions:

What characterizes the problem solving of two Finnish girls solving a PISA task?

- a. What similarities and differences can be found in their problem solving?
- b. How do the results reflect the learning objectives, core content and final-assessment criteria of 'thinking skills and methods' described in the Finnish curriculum?

2 Theoretical framework

The key concept in the research project is mathematical thinking. Despite its wide use in the literature, there is no common understanding of what is meant by mathematical thinking (e.g. Sternberg, 1996; Burton, 1984). With difficulties in defining the term most studies adopt a practical view, without framing the concept, focusing on questions like how mathematical thinking can be measured or improved in school (e.g. McGregor, 2007; Doerr, 2006), or what kind of mathematical thinking do students have (in Finland e.g. Hähkiöniemi, 2006; Joutsenlahti, 2005). Here, thinking is considered being mathematical when it relies on operations that are mathematical in separation of thinking about the subject matter of mathematics (Burton, 1984). Furthermore, pupils' activities, actions and explanations during problem solving are interpreted as visible signs or expressions of their mathematical thinking. In the following, the three aspects of mathematical thinking that are discussed in this paper will be introduced. These are problem solving, metacognition and affect.

2.1 Problem solving

Problem solving is an essential part of, and thus an important tool in understanding pupils' mathematical thinking. Since both terms 'problem' and 'problem solving' have many meanings in mathematics education (Törner, Schoenfeld & Reiss, 2007), they need clarification. Here, mathematical task is called a problem if the solver has to combine previously known data in a new way to her to solve a task (e.g. Kantowski, 1980). Furthermore, with problem solving we refer to the activities and actions pupils perform while solving a given mathematical task or a problem.

In the present paper we want to emphasize the problem solving phases that the Finnish curriculum lists in its final-assessment criteria of thinking skills and methods for a grade of

8 (FNBE 2004). These phases are transforming a text problem to a mathematical form of presentation, making a plan to solve a problem, solving it, and checking the correctness of the result. These phases follow the problem solving principles described by Polya (1957). Transforming text to mathematical presentations requires *understanding the problem*, and further, checking the results is a part of *looking back* (cf. *ibid.*).

2.2 Metacognition

When studying pupils' mathematical thinking, especially through problem solving, also their metacognitive skills should be recognized. Similarly as terms 'problem' and 'problem solving', also 'metacognition' has many different meanings in educational research (Stillman & Mevarech, 2010). In 1987 Schoenfeld (1987, p.190) listed three aspects of research on metacognition: 'your knowledge about your own thought processes', 'control or self-regulation', and 'beliefs and intuitions'. Even though theories on metacognition has been developed since (Stillman et al., 2010), the three aspects of metacognitive research (Schoenfeld, 1987) give a useful starting point for studying of pupils' metacognition in problem solving.

Also the Finnish curriculum (FNBE, 2004) lists some metacognitive factors in learning objectives for sixth to ninth graders. According to the curriculum, pupils should 'learn to trust themselves, and to take responsibility for their own learning in mathematics' and 'learn to work in a sustained, focused manner, and to function in a group' (*ibid.*, p. 164). Excluding group work, findings of the above mentioned learning objectives will be discussed.

2.3 Affect

When studying pupils' mathematical thinking, research has usually concentrated purely on the cognitive aspect (here, problem solving and metacognition). However, it has become clear that if we really want to describe mathematical thinking, we should also relate to affective factors (e.g. Vinner, 2004). One aim of the research project is to understand the interrelationship between affect and cognition (Zan, Brown, Evans & Hannula, 2006) in mathematical thinking.

Affect is seen as mixture of cognitive, motivational, emotional processes and can be expressed as follows (Hannula, 2012, p. 144):

Cognition deals with information (self and the environment), while motivation directs behaviour (goals and choices). Success or failure in goal-directed behaviour is reflected in emotions (e.g., shame). These emotions, in turn, act as a feedback system to cognitive and motivational processes.

In addition to looking at affect through cognitive, motivational and emotional processes, it also has physiological, psychological and social domains as well as trait and state aspects (Hannula, 2011). The present paper deals affect as a psychological domain and looks at it from both trait and state aspect. Connected to problem solving, rapidly changing affective

state is in focus. In discussion, also some aspects of more stable affective trait will be discussed.

3 Methods and methodology

3.1 Participants

The aim of the research project is to study Finnish pupils' mathematical thinking at the end of comprehensive school. Thus, data was collected in the first semester of 9th grade when pupils are 15 years of age. High achieving girls Emma (mathematics grade 9¹) and Nora (mathematics grade 10) are from different schools. Similar achievement level is the reason why they were selected for this paper. Additionally they both worked mainly individually with the tasks in the classroom.

3.2 Data collection

The data for this paper was collected both from a classroom and from an interview. Emma and Nora solved one PISA task in an ordinary classroom situation. The teacher chose the way she² introduced the task to the whole class and the researcher acted as an observer. Emma and Nora were video recorded when they solved the task and their solutions on paper were collected. Natural classroom setting was used to give the pupils an opportunity to work in a familiar way to them. They were able to ask help from their teacher and peers. It also enabled the researcher to find out what kind of difficulties the pupils faced and how they were accustomed to solve them. Additionally, if the girls talked about the task with someone, the researcher was able to follow their reasoning.

Emma and Nora were interviewed on the same or the following day after solving the PISA task in classroom. The interviews contained two parts. The first part had three themes (see Table 1): pupil's background, mathematics and oneself within mathematics (following Pehkonen, 1995; more about interview themes in the project see Viitala, 2013). This part of the interview was semi-structured and focused (Kvale & Brinkmann, 2009), focusing on affective components within and towards mathematics.

Table 1 Interview themes and example questions

Theme	Example questions
Background	Tell me about your family.
Mathematics	What is mathematics as science? Does mathematics exist outside school? (How? Where?)
Oneself within mathematics	Is mathematics important to you? Does mathematics help you think logically? (How?)

¹ On a scale 4-10.

² Due to the small amount of participants in the study, all teachers are treated as females to preserve anonymity.

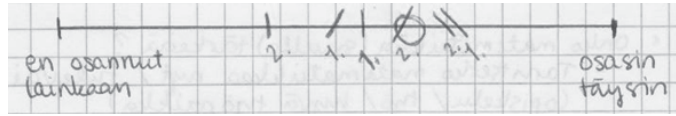


Figure 1 Emma's confidence line from the interview (2 tasks). The line is 10 cm long with a scale from 'I couldn't do it at all' (left) to 'I could do it perfectly' (right). Symbols: Confidence after reading the task |, while solving the task /, after solving the task \, and confidence in school mathematics o.

The second part of the interview was about problem solving. The classroom video was used as stimuli when the pupil's problem solving phases were discussed. Also the solution paper was used to support the discussion. After the stimulated recall part, some metacognitive (thinking about own thinking) and affective (feelings, motivation) questions concerning the problem solving situation were asked and a 10 cm long confidence line was introduced (see Figure 1). The pupils used the confidence line to assess their confidence prior, during and after solving the problems as well as their current confidence in school mathematics. Similar estimations of certainty was used e.g. in Merenluoto (2001).

Emma and Nora were interviewed individually by the researcher. The interviews were video recorded. Video camera in classroom was directed towards the pupil's desk showing her work on paper. In the interview, the camera pointed also to the computer from which the classroom video was watched. These settings were chosen to ease the analysis and to preserve anonymity.

3.3 The PISA task

The PISA task discussed in this paper is called 'Holiday'. Holiday was chosen from PISA 2003 problem solving survey (OECD, 2006, pp. 77-78). It only requires elementary arithmetical content knowledge so all pupils should master the pure mathematics in the task. Additionally, it has all five aspects of mathematizing present: the problem is situated in reality, the problem solver has to identify the relevant mathematics and reorganise the problem and gradually trim away the reality, solve the mathematical problem and reflect on the mathematical solution in terms of the real situation (ibid., pp. 74-75, 78).

Holiday consist of two tasks both which Emma and Nora solved. The first task is to calculate the shortest route between two towns and the second is to plan where to stay overnight on a holiday trip. A simplified map of the area and a table of distances between towns³ are given within the task.

3.4 Analysis

The analysis was divided into two sections: Problem solving, and Affect related to mathematics. These titles are somewhat misleading and need elaboration. Principally, both of these sections contain cognitive, metacognitive and affective aspects.

³ In the pupils' version of 'Figure B' all grey cells were white (blank). Additionally there is an error in 'Figure B' (corrected in OECD, 2009): The distance between Nuben and Lapat should be 1000, instead of 1300. Neither of the pupils used this information.

In problem solving, the main focus is on the cognitive problem solving process written in the curriculum as final-assessment criteria. After analysing problem solving processes, also some other cognitive aspects from core content and final-assessment criteria of thinking skills and methods will be discussed (e.g. interpreting and producing mathematical texts, and presenting possible alternative solutions systematically). In problem solving, thinking about own thinking as well as control and self-regulation (e.g. keeping track what is being done during problem solving) will be discussed as part of metacognition. Pupil's motivation to solve the tasks as well as feelings and confidence during problem solving will be reported as part of psychological affective state.

In affect related to mathematics, pupils' view on mathematics and connections between mathematics and real life will be reported. Discussion on metacognition concentrates on the metacognitive aspects listed in learning objectives in the curriculum (e.g. trusting oneself). From affect, some aspects of relatively stable psychological trait will be discussed (e.g. feelings and beliefs towards and within mathematics).

4 Results

This chapter starts with describing the problem solving phases of Emma and Nora. In chapter 4.1, classroom and interview data are combined and some metacognitive and affective data is included. When summarizing problem solving results in chapter 4.1.3, some interpretations of the problem solving results, and other aspects of thinking skills and methods from the curriculum will be examined and reported. Finally, aspects of affect related to mathematics will be discussed in chapter 4.2. These results are based on the interviews.

4.1 Problem solving

Emma

Task 1

Emma read through and thought about the first task for almost six minutes before starting to solve it. The task felt easy for her in the beginning but the atypical table (Figure B in OECD, 2006, p. 77) made her feel nervous. Because she had difficulties in understanding the table in the first task, she also read the second task before deciding to do the 'easier' first task first.

During the six first minutes, Emma says she used most of her time on reflecting the table. She struggled with how to read the table and wondering 'what is the distance'. Also the structure of the table disturbed her: she didn't understand 'the steps', why there is just a line on the bottom right corner, and why Piraz is at the bottom twice. In Emma's words, she panicked and it took her almost five and a half minutes (including a minute long announcement through speakers) to 'really starting to concentrate'. After realizing it is the empty spot in the table she needed to 'do', she made a plan and started to solve the task.

$$\begin{array}{l}
 \text{Kado - Angaz } 550 \text{ km} \\
 \text{Angaz - Nuben } 500 \text{ km} \\
 550 \text{ km} + 500 \text{ km} = 1050 \text{ km} \\
 \\
 \text{Kado - Lapat } 300 \text{ km} \\
 \text{Lapat - Megal } 550 \text{ km} \\
 \text{Megal - Nuben } 450 \text{ km} \\
 300 \text{ km} + 550 \text{ km} = 850 \text{ km} \rightarrow \\
 850 \text{ km} + 450 \text{ km} = 1300 \text{ km}
 \end{array}$$

Figure 2 Emma's calculations of task 1

Emma decided which routes to calculate by estimating distances from the map. In addition to the route that seemed shortest to her (Kado-Angaz-Nuben), she decided to calculate another (in her view the second shortest) route (Kado-Lapat-Megal-Nuben) to confirm the result. She did not calculate more routes because they seemed longer than the chosen two. She calculated the route distances proceeding city-by-city, starting with the one that seemed the shortest. Throughout the task solution process, Emma was quick in looking at the distances from the table and moved her pen near the table only twice.

In her calculations Emma was very thorough and wrote everything down neatly step-by-step (see Figure 2). All the calculations she did mentally. Realizing that the task is about doing elementary mathematics made her feel uneasy. Only once she made a mistake in her calculations and/or writing (writes '15') which she immediately corrects (erases '5' and corrects it to '1050'). After calculating the second route, she drew a line over the calculations for that route and wrote the answer ('1050 km') to the answer line. Emma was done with the first task after ten and a half minutes (including the announcement that took a minute from the task).

Emma checked the calculations for the shortest route, not for the longer one. However, it is not clear when she did so (probably during the last minute of working with the tasks when she moves her papers around and also completes her answer to the second task). In addition, she asked her friends if they also got 1050 as an answer to the first task as a confirmation.

Task 2

Emma started the second task by reading it three times. The task felt hard because 'it had a lot of text'. First she read the task quickly through. On second reading, she marked all the given information to the map. Then she continued thinking how to 'calculate the 300' and realized she needs to use the table (Figure B in OECD, 2006, p. 77). She wrote all the needed distances to the map. Then she read the task for the third time.

After almost three minutes Emma started to solve the task by calculating the overall distance of the trip ('550 km + 500 km + 300 km = 1350 km'). In the interview, she could not explain why she calculated it even after thinking about it for quite a while. While she was doing the calculation, the teacher encouraged the class to talk about the tasks with

friends. So far very silent class got noisy quickly and Emma started talking about the second task with her friends.

First, Emma's classmates talked about how the task should be done. Emma listened the discussion and wrote '550 km - 300 km = 250 km' on paper under the calculated overall distance. Then she asked the girls if she has proceeded correctly and got a confirmation. The discussion continued by one of the girls starting to explain the beginning of the task to Emma. Very quickly Emma took over and started explaining the task to the girl and asking confirmative questions (e.g. 'So I put two nights here in Kado?'). Emma went through the whole task and wrote down the answer as she went on.

While explaining, Emma did not have to go to the task description again; she remembered all the needed details from reading the task. After starting to explain the solution to her friend, it took Emma only a bit more than a minute to finish solving the task. After Emma was done, the teacher walked by and Emma asked her how much she has to justify her thinking on paper. The teacher says it is important to justify so the thinking becomes visible. Emma is concerned about the time, so after asking the researcher they agree that Emma can explain (justify) her thinking in the interview.

In the end, the problem felt easy to Emma. She checked the answer of the second task after working with both tasks and completed it: 'Kado' became 'In Kado' (rows 2 and 3) and 'Lapat' became 'In Lapat' (rows 4 and 5). Then she took the paper to the teacher. The overall time she worked with the two tasks was over eighteen minutes.

Nora

Task 1

After reading the first task description Nora felt that the task is 'very easy', until she went through the table (Figure B in OECD, 2006, p. 77). She thought that the table did not have all the distances and she felt 'a bit like but not frustrated'. After looking at the table for a while she realized that if she cannot find the distance starting from left, she needs to start from down. Then she made a plan to solve the task. All this took Nora less than one and a half minutes.

Nora decided to calculate one route that seemed to be the shortest (Kado-Lapat-Megal-Nuben) based on an evaluation from looking at the map. She started to look for the distances city-by-city from the table and write the expression for the calculation as she went on (first distance: '300 +'). After getting almost to the end, she faced a problem with finding the last distance from the table. She looked Megal-Nuben from left to down, then Nuben-Megal from down to left. After that she thought this distance was not in the table and made a new plan for solving the task.

On the second try, Nora went back one distance and decided proceed from there to another direction (aiming to calculate route Kado-Lapat-Angaz-Nuben). She checked the first distance from the table and proceeded with the chosen route. She had her finger on the table pointing at Megal (down) but in the interview she could not remember why. She did

not use any information from that column. After continuing with the second route Nora faced the same problem again: she could not find the last distance from the table. She had to do yet another plan for solving the task.

Nora erased everything she had written so far and decided to start from the other end of the route (Nuben) and calculate the route she can find the distances to. At this point Nora felt frustrated. She had failed completing two routes already which made her thinking if the task was a trick. Now, she had been working with the task for less than three minutes.

Nora proceeded by looking at the distances she can find from Nuben starting from the bottom row of the table. She found only one distance (Nuben-Piraz), so she started with that. Nora managed to find all distances to the new route (Nuben-Piraz-Angaz-Kado) and made a miscalculation when calculating the overall distance (' $250 + 300 + 550 = 11\ 000$ '). The magnitude of the answer did not seem right, so she went back to the calculation and corrected the answer (see Figure 3). Then she wrote her answer ('1100') to the answer line. At this point Nora had used less than four and a half minutes of her time.

Throughout the task, Nora mostly used her fingers as support when she was looking at the distances from the table. She did not check her answer. However, after Nora had done both tasks, a classmate asked if she had the same answer to the first task as he had. They both had the same answer and Nora got a confirmation.

$$250 + 300 + 550 = 1100$$

Image 3 Nora's calculation of task 1.

Task 2

Nora did not understand the second task from first reading and she had to read it through 'at least three times'. She began to go through the task step-by-step combining the given information with the map. However, she did not write anything down on the map. She used her finger to point the first town (Angaz) and then the road between the first and the second town (Angaz-Kado). She spent half a minute pointing at the map before starting to solve the task. At this point, Nora had worked with the second task for a bit more than a minute.

Nora began to write something to the answer area (another table, OECD, 2006, p.78) and erased it. The answer area disturbed her. It seemed to her that there is a mismatch with the columns, first one indicating a day and the second one indicating a night. Soon she realized that it has to be the same day after which they spend the night. Even in the interview Nora thought that the mismatch should have been corrected so that the time and the place match.

After a half a minute confusion concerning the answer area, Nora wrote answers to the first two rows ('In Kado') and, after a short while, to the next two rows ('Lapat'). Then she was interrupted by a classmate asking her what he should do in the second task. Nora started to help him.

Nora explained to the classmate the steps needed to get from the first town to the second one (how the towns form 'a ring', there is 500 km between Angaz and Kado, first camping area is between them, she (girl from the task) can go maximum of 300 km, and so on). After Nora reached the second town in her explanations, the boy repeated what he had learnt and the teacher interrupted them by saying something to the whole class. The discussion with the boy ended and Nora returned to her paper. She started to write down the last step to the answer area (row 6). After writing 'Lapat and Angaz's', the boy interrupted her again asking for help. They went through the rest of the task together similarly as before and Nora did not return to the task again.

Nora did not check her result (which in fact was incomplete). Excluding the time used to help the boy, it took Nora only less than three minutes to solve the second task. Including the given help, Nora used five and a half minutes for the second task and ten minutes for the two tasks together.

Summing up problem solving results

Emma is a reflective problem solver. She agrees that it is important for her to understand the given information before starting to plan and solve problems. She explains how she can return to the task description even in the middle of calculations to confirm herself that she is doing the right thing. If the problem feels hard, she might try different ways of solving it and choose the one that seems correct (in tests). In the first task she also looked for alternative solution to confirm her result. She also checks her results to the given tasks.

Even though Emma seems fluent in problem solving (following all the steps from the curriculum) and self-regulation, word problems make her feel unease. She originates the dislike of word problems to elementary school where she often failed to solve them. Emma 'always remembers' her father saying: 'Remember to read word problems properly, as many times as needed, think what is asked'. This has helped her face word problems.

Nora is more direct in problem solving. She also wants to understand the problem and the given information before starting to plan and solve problems. If the problem is complicated (as the second task here), she cuts it in smaller pieces to better understand what is given. She explains how she stops to think about the solution only if there is a problem. In the first task, Nora handled problems flexibly and made new plans quickly. She says that she is happy with the first result she gets. Here, she did not check her results (in the first task she miscalculated the answer and corrected it on the spot).

Both Emma and Nora were able to interpret and produce mathematical texts. They both had difficulties with reading the table (Nora had difficulties until the end) and understanding the second task (for which Emma needed help from friends). Eventually, they understood what was given and asked. Formulating and solving calculations were easy to them. They had similar confidence in different parts of solving the tasks (see Table 2). Nora's feelings changed on a larger scale, though, ending up feeling slightly more confident than Emma.

Table 2 Emma's and Nora's confidence with the tasks. The values are given on a scale 0-10, 0 indicating the negative and 10 the positive end of the scale (see Figure 1)

	Where were you <i>after reading</i> the task?	Where were you <i>while solving</i> the task?	Where were you <i>after solving</i> the task?
Task 1			
Emma	5,5	5,0	7,25
Nora	8,5 / 3,5 (table)	4,5	7,5
Task 2			
Emma	3,75	6,25	7,0
Nora	2,0	6,5	7,75

Most of the time in their interviews, Emma and Nora were able to express their thoughts unambiguously and justify their actions. At some point, however, Emma had problems in explaining her thinking understandably and justifying her decision to calculate the overall distance of the trip in the second task. Neither of the girls was aware of their own thinking during problem solving processes. The video camera motivated them to solve the problems (extrinsic motivation). However, being able to do the tasks also motivated Emma (intrinsic motivation). Whereas Emma's answers were correct, Nora's answers were incorrect (task 1) or incomplete (task 2).

4.2 Affect related to mathematics

Emma sees mathematics as something that is very much tight to school subject: Mathematics is calculations, mathematical knowledge is gained through calculating, correctness of mathematical knowledge can be verified by asking the teacher, mathematics outside school is doing homework and reading for tests, and so forth. She recognizes that mathematics is useful and needed for instance to get a good job but she does not know how it is useful, just that it is. From school subjects, mathematics is needed in civics (stocks), chemistry and physics. All in all, it seems hard for Emma to see connections between mathematics and the real world.

For Nora, it is easy to see connections between mathematics and real world. First of all, mathematics plays a big role in science (philosophy, physics and chemistry). In addition to a tool view (doing investigations, calculations and demonstrations), Nora has also an idea of how mathematics as science develops (developing formulas, getting more accurate results and making new formulas). Secondly, Nora finds connections between mathematics and *her* world (baking and shopping). Finding mathematics in other school subjects is quite easy for her as well (physics, chemistry, geometry (maps) and history (eras)). For Nora, the meaning of mathematics is offering confidence; 'If you can calculate something is true, you can believe it'.

Both girls worked in a sustained and focused manner with the tasks. Nora is more confident in mathematics (Table 3) and takes a bigger responsibility of her own learning than Emma. Emma relates her success in mathematics to her teachers, their teaching styles and how they made her feel towards mathematics from first to ninth grade. Her grades in mathematics tests have varied substantially (6-9) over the years.

Table 3 Emma's and Nora's confidence in school mathematics. The values are given on a scale 0-10, 0 indicating the negative and 10 the positive end of the scale (see Figure 1)

	Where are you now in mathematics?		
	<i>Interview 1</i>	<i>(Interview 2)</i>	<i>(Interview 3)</i>
Emma	6,25	5,5	5,5
Nora	7,75	8,25	7,25

Emma and Nora are quite emotional when it comes to mathematics. Emma likes mathematics and is motivated to learn it. The feeling of success and being proud of herself motivates Emma to learn more mathematics. Nora starts the interview by saying that she likes mathematics very much and is the only one in her class who is really looking forward to math classes. She is also motivated to learn mathematics but her main reason for it is more traditional: She wants to get a good grade.

5 Discussion

Emma and Nora were selected for this paper because they have similar achievement level. However, the results introduce pupils that have different competences in mathematics. While Emma is more competent in problem solving, Nora is more confident, she can express her thinking better (more unambiguously), her view on mathematics is broader (application, or a tool view), she can connect mathematics to real world more easily and seems to take a bigger responsibility of her own learning than Emma.

All the above mentioned features are part of the curriculum and important aspects of mathematical thinking. Nevertheless, only problem solving is part of pupils' final-assessment criteria described in the Finnish curriculum. Based on the findings that Emma seems more fluent in problem solving and Nora is more confident and has a better grade in mathematics, a question arises: What is the role of non-measurable aspects of the curriculum in pupil evaluations? A draft version of new curriculum in Finland (will be implemented as of autumn 2016) suggests that even though pupil's motivation, positive self-image and self-confidence will not influence pupil evaluation, taking responsibility of own learning, expressing mathematical thinking and applying mathematics in different environments will be part of final-assessment criteria (FNBE, 2014). Thus, it seems that mathematical thinking will be evaluated in a more diverse way in the future.

This brings us to another question: How will the new criteria be evaluated in a fair way to all pupils? Recent research on pupils' learning results revealed how Finnish teachers seem evaluate their pupils comparing them with other pupils they teach (Rautopuro, 2013). This might also partly explain why Nora has a better grade in mathematics even though Emma seems more fluent in problem solving: Nora's class could be described as low achieving whereas Emma's class had pupils from all achievement levels (based on the researcher's observations). Whatever the reason is to the (small) difference in Emma and Nora's grades, this issue deserves more attention. Especially since the new curriculum seems to appreciate mathematical thinking in a broader way and is adding aspects to the evaluation criteria that are not based only on pure mathematics.

The results presented in this paper serve as a starting point for studying pupils' mathematical thinking at the end of comprehensive school in Finland. In future, we will continue studying Emma and Nora to see if their results remain similar when their problem solving and affect is studied with more PISA tasks and interview themes (see more about interview themes in Viitala, 2013). Additionally, we will compare the results of all 8 pupils who participated in this research project and ask what characterises the mathematical thinking of these pupils near the end of comprehensive school. As we go further, we also hope to find indications on the relationship of cognition and affect in mathematical thinking (cf. Vinner, 2004; Zan et al., 2006).

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Article 4

Emma's mathematical thinking, problem solving and affect

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This paper aims to understand one pupil's mathematical thinking through problem solving and mathematics related affect. The results reveal a successful, though quite unsure, problem solver whose affective state (connected to problem solving) seems to tell the same story as her affective trait (view of mathematics). The differences between results on affective state and trait seem to be connected mostly to emotions.

Keywords: Mathematical thinking, problem solving, affect.

THEORETICAL FRAMEWORK AND RESEARCH QUESTION

Developing mathematical thinking is one of the three tasks of instruction listed in the Finnish National Core Curriculum for Basic Education (Finnish National Board of Education [FNBE], 2004, p. 158). Some aspects of mathematical thinking are evaluated through tests at school, at the national (e.g., Rautopuro, 2013; Hirvonen, 2012) and international (e.g., OECD, 2014; Mullis, Martin, Foy, & Arora, 2012) levels. However, we lack a deeper understanding of the mathematical thinking pupils' take into their lives and possible further studies after comprehensive school.

Mathematical thinking is not defined in the curriculum, but a list of thinking and working skills is provided as evaluation criteria for every age group. These lists include learning objectives such as pupils' use of logical elements in their speech, judging truth of simple propositions and noticing parallels and regularities between different events (FNBE, 2004, p. 166). For sixth to ninth graders, 'thinking skills and methods' is also introduced as its own entity in the learning objectives parallel to core contents such as algebra and geometry (Mathematics curriculum, *ibid*, pp. 158–167).

When thinking skills and methods are listed in the curriculum, problem solving is repeatedly referred to. The term 'problem solving' is not defined. However, in final-assessment criteria for ninth graders, four problem-solving phases are introduced. These phases are similar to Polya's (1957) problem-solving phases. The process view on problem solving also guides this study where (in line with the curriculum) pupils' activities, actions and explanations during problem solving are interpreted as visible signs or expressions of their mathematical thinking.

In this study, thinking is considered mathematical when it relies on operations that are mathematical (Burton, 1984). Furthermore, a mathematical task is called a problem if the solver has to combine previously known data in a new way to her to solve the task (e.g., Kantowski, 1980).

When mathematical thinking is described, in addition to cognitive aspects, we should also explore affective factors (e.g., DeBellis & Golding, 2006; Vinner, 2004) and seek to understand the interrelationship between affect and cognition (e.g., Hannula, 2011; Zan, Brown, Evans, & Hannula, 2006). Instead of categorizing affective factors for instance as beliefs, attitudes or values, affect is seen as a mixture of cognitive, motivational and emotional processes:

Cognition deals with information (self and the environment), while motivation directs behaviour (goals and choices). Success or failure in goal-directed behaviour is reflected in emotions (e.g., shame). These emotions, in turn, act as a feedback system to cognitive and motivational processes. (Hannula, 2012, p. 144)

Affect is seen as a psychological domain with its state and trait aspects (Hannula, 2011). In connection to problem solving, we focus on rapidly changing af-

fective states. The more stable affective traits follow the categorization of pupil's view of mathematics introduced by Pehkonen (1995; discussed also in Op't Eynde, de Corte & Verschaffel, 2002). These categories are mathematics, oneself as a learner and user of mathematics, learning mathematics, and teaching mathematics.

In an aim to understand the interrelationship between cognition and affect in mathematical thinking, we look at one pupil's, Emma's, problem solving and explanations on affect related to mathematics. The information from this exemplary case can later be combined with other cases to form a more informative view on mathematical thinking at the end of Finnish comprehensive school. So, with the question 'What characterizes Emma's mathematical thinking?' we try to understand the mathematical thinking Emma takes from comprehensive school into her life and further studies (cf. mission for basic education in FNBE, 2004, p. 12).

METHODS

The data was collected in three cycles. Emma's results from the first cycle are discussed in Viitala (2015). This paper adds both problem solving and affective results and reports on findings from all three cycles of data collection.

Participant

Emma is a high achieving girl (mathematics grade 9 on a whole number scale of 4 to 10) who was selected for this paper based on a previous report (Viitala, 2015). The data collection was organized in the first semester of 9th grade when Emma was 15 years old.

Data collection

The data was collected from mathematics lessons and interviews over the course of three months. In each

of the three cycles, one mathematical task was solved in an ordinary classroom situation as a 'main task'. In Emma's case this meant that the pupils solved the tasks individually but they were allowed to talk about the tasks with a friend or ask for help from the teacher. In each of the three cycles, Emma was video recorded while she solved the task(s) in class and her solution on paper was collected.

The interviews took place either on the same day, or on the next day after solving the task in the classroom. The interviews contained two parts. The first part concentrated on affective traits and treated the following themes: pupil's background, mathematical thinking, and pupil's view of mathematics (following the categorization of Pehkonen, 1995; see example questions in Table 1). This part of the interview was semi-structured and focused (Kvale & Brinkmann, 2009).

The second part of the interview was about problem solving. The classroom data was used as stimuli when Emma's problem solving was discussed. Emma explained her actions and thinking and responded to questions such as, 'What are you thinking now?' and 'Why are you doing so?' Also, some affective and metacognitive questions were asked, for instance, 'What did you feel when you read the task?' and 'Did you think about your own thinking when solving the task?'

Finally, Emma was asked to assess her confidence before, during and after solving the problem(s), as well as her confidence in school mathematics using a 10 cm line segment (scale from 'I couldn't do it at all' to 'I could do it perfectly'; cf. estimation of certainty, e.g., in Merenluoto, 2001). All interviews were video recorded.

The tasks used in this paper are released PISA items. PISA tasks are well tested and based on real-life situ-

Theme	Example questions
Background	Tell me about your family.
Mathematical thinking	What does mathematical thinking mean? / How do you recognise it?
Mathematics	What is mathematics as a science? / Does it exist outside of school? (How? Where?)
Oneself and mathematics	Is mathematics important to you? / Does it help you think logically? (How?)
Learning mathematics	How do you learn mathematics? / Is it most important to get a correct answer?
Teaching mathematics	Does teaching matter to your learning? (How?) / What is good teaching?

Table 1: Interview themes and example questions

	Task	Given information	Why chosen?
Holiday, Q1	Calculate the shortest distance by road between Nuben and Kado.	Map of the area, Table of distances, Answer given in kilometres.	Complex situation, Combining different data
School excursion	Which (bus) company should the class choose, if the excursion involves a total travel distance of somewhere between 400 and 600 km?	Written explanation of the situation and rates that the bus companies charge.	Uncertainty, Decision making
Indonesia, Q3	Design a graph (or graphs) that shows the uneven distribution of the Indonesian population.	Table of the population of Indonesia and its distribution over the islands.	Open task

Table 2: Descriptions of some of the tasks used in the project

ations (e.g., OECD, 2006, p. 108). The 'main' PISA tasks, Holiday, School Excursion and Indonesia (ibid, pp. 77–78, 87, 111, respectively), were solved in the classroom. These tasks are open with respect to solution strategy (see task examples in Table 2). Indonesia also included two additional questions in which Emma's skill to read a table was tested (referred to as Indonesia Q1 and Q2).

In class, Emma also started to solve Carpenter (OECD, 2009, p. 111) which was added in case of extra time. In the interviews, Emma answered questions from the tasks Distance (OECD, 2006, p. 102; modified to 3 and 5 km), Growing up (OECD, 2009, p. 106; Q's 3 and 1) and Braking (ibid., pp. 128–129; also Q49 from a Web page of the Finnish Institute for Educational Research).

Analysis

The analysis was divided into two sections: Problem solving, and Affect related to mathematics. In problem solving, the main focus is on the cognitive problem solving process written in the curriculum for grades 6–9 as core content or final assessment criteria of thinking skills and methods. These phases follow the problem solving principles described by Polya (1957) and will be reported accordingly.

In connection to problem solving, pupil's thinking about their own thinking as well as control and self-regulation (e.g. keeping track of what is being done during problem solving) will be discussed as a part of metacognition (Schoenfeld, 1987). The pupil's motivation to solve the tasks as well as confidence and feelings during problem solving is reported as part of the psychological affective state.

In affect related to mathematics, pupil's view on mathematics is reported (see Table 1). These results can be referred to as affective traits. The discussion on metacognition concentrates on the aspects listed as

learning objectives in the curriculum (e.g., trusting oneself; FNBE, 2004).

The results are descriptive. More information related to methods and methodology can be found in earlier publications (Viitala, 2013, 2015).

RESULTS

Problem solving

Understanding the problem. Emma uses much of her time for understanding problems and the given information (in class 2–5 minutes which is some 30–55 % of total solution time). She seems very thorough and she says that she wants to understand every aspect of a task before starting to plan and solve it. On one hand, this seems to be a key element in her success as a problem solver. On the other hand, this might hinder her to solve a problem (not understanding all the mathematical expressions in Braking Q49) or to give a correct answer (she was prone to give an answer which she can completely understand in Braking Q49).

Emma uses graphs (maps and diagrams from the tasks) to assist her thinking when putting the given information together (e.g., marking routes and distances to the map in Holiday), understanding a problem (e.g., the graph in Braking for Q49) making a plan to solve a problem (e.g., routes in Holiday) and reflecting on the task when solving it (e.g., connecting the question to the graph in Growing up). The use of tools and drawings that assist thinking is part of mathematics learning objectives.

Making a plan. After taking the time to understand a problem and given information, Emma does not need much time to make a plan. Making a plan seems to happen on the third reading of the question, after reading the question quickly through on the first

reading and putting together the given information on second reading (according to her explanations, e.g., for Holiday Q2 and School excursion).

If a task feels hard, Emma says she thinks of alternative ways to solve it (in Holiday she calculated a second route to confirm her result). Additionally, if the task feels too simple, she might try to calculate the task further after getting the answer (Indonesia Q2, this calculation was later erased because 'it felt stupid').

Carrying out a plan. Emma seems careful and thorough in solving mathematical tasks. After understanding the task, Emma is fluent in transforming a text to a mathematical form (mathematical expressions e.g. in Holiday and School Excursion). She says she can return to the task description as a confirmation also in the middle of solving a task (e.g. in School excursion). She proceeds step-by-step with the tasks.

If Emma has different options to solve a task, she says that she chooses the one that feels more 'probable' (e.g., routes in Holiday) or has less doubt (e.g., choosing a point where to calculate School excursion). For a task that had more than one answer (Distance), she spontaneously found two answers and a third one after being probed. In most cases Emma was able to justify her actions and conclusions.

Looking back. Emma says she checks tasks only in tests. In the research project, the first class situation felt like a test situation for Emma and it was the only time she checked her answers (Holiday). Other answers to PISA tasks she checked from a friend (Holiday and School excursion) or left it until the interview. In addition, Emma feels that she does not need to check her calculations when a calculator is used.

If Emma is not sure whether she has understood the task or given information correctly, she chooses the interpretation that feels most reasonable and proceeds with that (e.g. table in Holiday, and Q1 and Q2

in Indonesia). This aspect of 'looking back' is done during the process of solving the task.

Affect related to problem solving. Emma feels unsure when faced a word problem. She might 'panic' if the task has a lot of text (Holiday Q2) or numbers (Indonesia), or she cannot understand all the given information (e.g. table in Holiday and mathematical expressions in Braking Q49). When she gets stuck with the task description, she seems to lack efficient tools to overcome the situation (Holiday Q2, Braking Q49, also visible in Emma's explanations about doing homework and learning mathematics; on getting stuck, see, e.g., Mason, 2015). In these cases, asking questions helps her to overcome the difficulties and proceed with the task.

Getting help (Holiday Q2, before solving the task) or asking the correct answer from a friend after solving the task (Holiday and School excursion) seems to have a direct influence on Emma's confidence. Similarly, not checking her answer (Indonesia) seems to make her feel very uncertain and anxious even in the interview (until the results were given). See Emma's confidence related to problem solving in Table 3.

Emma might experience many different feelings when facing, planning and solving a problem (e.g., in School excursion: nervous, unsure, doubtful, 'normal' and relieved chronologically, cf. Table 3) but she agrees that her feelings do not necessarily affect her more stable feeling of confidence. The main motivation for Emma to solve the given tasks was the video camera. However, when a mathematical obstacle was encountered, Emma was motivated to learn from it (e.g., not understanding some mathematical part of the discussion in an interview, such as graphs for companies in School excursion, or percentages when discussing Indonesia or Braking).

	Confidence <i>after reading</i> the task	Confidence <i>while solving</i> the task	Confidence <i>after solving</i> the task
Holiday, Q1	5.5	5	7.25
Holiday, Q2	3.75	6.25	7
School excursion	5	6.25	7.25
Indonesia, Q1-Q3	3.25	4	4

Table 3: Emma's confidence (0–10, ±0.25 mm) for the tasks solved in the classroom

Affect related to mathematics

Mathematics. Emma's view on mathematics is very much tied to a school subject. For her, mathematics is calculating, both as a school subject and as a science. Mathematical knowledge is gained by calculating and correctness of mathematical knowledge can be verified by asking the teacher. Emma thinks mathematics is useful and needed for instance in other school subjects (e.g. civics, physics and chemistry). When asked, it is hard for her to see connections between mathematics and real life. Emma uses mathematics outside of school when she is shopping.

Oneself and mathematics. Emma is motivated to learn mathematics. The feeling of success drives her forward and succeeding with a difficult task makes her feel proud of herself (intrinsic motivation). She values the opportunity to show her skills to her teacher and classmates by going to the black board to calculate tasks. It feels rewarding and it motivates her to learn 'the next thing' (extrinsic motivation).

Emma likes mathematics and thinks it is 'quite fun' and interesting. However, in her own words, she does not feel 'very confident' in mathematics (cf. Table 4). However, she thinks that this might be a good thing: If you are too confident, you might not use that much time for thinking or check the calculations of a task. Emma thinks that confidence and mathematics grades are two separate things. Her grade (9) is the best she thinks she can achieve.

Interview 1	Interview 2	Interview 3
6.25	5.5	5.5

Table 4: Emma's confidence (0–10, ±0.25 mm) in mathematics

Learning mathematics. Emma, together with her friends and family, values learning mathematics and thinks that mathematics is useful. She also agrees that the atmosphere with regards to mathematics in her class is positive. Emma seems to trust that if she studies mathematics she can succeed and get better in it. The feeling of success, the belief that mathematics is worthwhile and useful, and future studies motivate her to learn mathematics.

For Emma, learning mathematics is 'understanding', in addition to 'memorizing' and 'reasoning'. Understanding means you are 'able to use a method'. Learning as well as understanding takes time for

Emma. Understanding comes from calculating, asking questions, and proceeding little-by-little from easier tasks to more difficult ones. Rote learning is important. Emma learns new things as independent issues and does not actively seek for connections to previous knowledge. She 'forgets things quite quickly' and an indication of this was seen also in the interviews (calculations with percentages).

Emma's feelings in learning mathematics are also closely connected to understanding. Learning mathematics is fun when she understands or succeeds in mathematics. Not being able to understand is irritating. Sometimes learning mathematics is also tiring. Mathematics is easy for her 'but only a little' (cf. Table 4). Emma seems to take the responsibility of her own mathematics learning. She says that if she succeeds with a test, it is because she has studied for it and learned in class. Failure, on the other hand, means that she has not done enough work.

Teaching mathematics. Mathematics teaching methods and the mathematics teacher play a great role in Emma's learning of mathematics. She believes that without teaching she could not learn mathematics. A good mathematics teacher offers opportunities to ask questions, gives time for (rote) learning, does not proceed too quickly to the next thing, and proceeds from easier tasks to more difficult ones. These all are features that Emma's current mathematics teacher seems to possess (according to Emma's explanations and the researcher's observations from the classroom).

In some respect, Emma seems to connect her feelings and success in mathematics to her teacher. In elementary school she did not get along with her teacher. She was an average pupil (grade 7–8) who was not interested in mathematics, did not succeed in it and mathematics felt like torment for her. In lower secondary school Emma got a new mathematics teacher who she liked, and whose teaching she liked. Since then, Emma says she has liked mathematics and been a high achiever.

SUMMARY AND DISCUSSION

Results from Emma's problem solving and affect related to mathematics seem to give a well-matching picture of Emma's mathematical thinking. Emma is a reflective learner and problem solver who needs time for understanding. Her thoroughness and tendency to ask questions (both from friends and the teacher)

seem to be the key to her success in both respects. Emma is not very confident in mathematics or problem solving (though slightly positive, cf. Hirvonen, 2012) but, as a consequence, she seems to be very careful with her thinking and working. Moreover, her uncertainty might be a reason for her success both in problem solving and mathematics.

The results showing differences between problem solving and affect related to mathematics seem to be connected to less stable affective traits. As an example, Emma's confidence has more variance in problem solving than in mathematics and her feelings experienced during problem solving have more tendencies to negative feelings (e.g. unsureness) than her feelings in learning mathematics. What is notable regarding Emma's affect in mathematics is that, contrary to previous research results (e.g., Tuohilampi, Hannula, Laine, & Metsämuuronen, 2014), Emma's feelings towards mathematics have become more positive since elementary school.

Throughout the study, Emma worked in a sustained and focused manner with the problems. Even though she is not very confident in mathematics, she seems to trust herself as a mathematics learner (e.g., aiming to learn more mathematics so she can succeed in advanced mathematics in upper secondary school). She also seems to take responsibility for her own learning (e.g. reasons for succeeding or failing in tests). All these aspects (listed in the curriculum; FNBE, 2004) together with her problem solving skills seem to offer her a solid foundation for future studies.

In addition to preparing pupils for further studies, basic education must also provide opportunities to obtain the knowledge and skills pupils need in life (FNBE, 2004, p. 12) and mathematics teaching should help pupils to see the connection between mathematics and real life (ibid, p. 158). Nonetheless, even though the PISA tasks that were used were situated in the real world, Emma saw them purely as mathematics tasks. Additionally, she struggled to see where she uses mathematics in her own life outside of school (homework, shopping). After analysing all the cases in the project, we can see if this might be a possible trend among Finnish pupils.

The upcoming curriculum (FNBE, 2014; will be implemented in 2016) draws more attention to mathematical thinking and real-life connections. For

instance noticing connections between learned concepts and applying mathematics in other school subjects and surrounding society are written as individual learning objectives and as final-assessment criteria (ibid, p. 433–434). It is hoped that this will direct pupils' attention more towards their thinking and connections between mathematics and real life, at the same time making mathematics more worthwhile and enjoyable.

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Article 5

A tool for understanding pupils' mathematical thinking

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This article provides a tool for studying pupils' mathematical thinking. Mathematical thinking is seen as a cognitive function that is highly influenced by affect and meta-level of mind. The situational problem solving behaviour is studied together with metacognition and affect which together with pupils' view of mathematics form a dynamic construct that reveals pupils' mathematical thinking. The case of Daniel is introduced to illustrate the dynamic nature of the framework.

Understanding and developing pupils' mathematical thinking are key issues in mathematics education. The new Finnish curriculum states that the task for mathematics instruction is to develop pupils' logical, precise and creative mathematical thinking which creates a basis for understanding mathematical concepts and constructs and develops pupils' ability to handle information and solve problems (FNBE, 2014, p. 429). The development of mathematical thinking has been evaluated with school tests at local (class), national (e.g. Rautopuro, 2013; Hirvonen, 2012) and international levels (e.g. OECD, 2014; Mullis, Martin, Foy & Arora, 2012). The focus of local and national tests is on evaluating how well the learning objectives written in the curriculum are reached (e.g. Hirvonen, 2012) whereas international assessments such as PISA aim to assess education systems worldwide irrespective of national curriculums (e.g. OECD, 2013). In Finland, the results seem to be similar in all assessments: pupils' performance in mathematics is declining (Väljjarvi, 2014; Rautopuro, 2013; Hirvonen, 2012).

Standard and standardised tests have been criticised for testing pupils with short answer questions on low-level facts and skills (Lesh & Clarke, 2000) that don't provide insight into pupils' abilities (Iversen & Larson,

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2006; Niss, 1999). Nevertheless, teachers are resistant to other kinds of (formal or informal) assessment due to their subjective nature (Watt, 2005; Watson, 2000). Mathematical thinking is a cognitive process that teachers ought to be able to evaluate, and paper tests reveal only the end-product of the thinking process. Without a closer look at pupils' mathematical thinking and aspects that influence it (e.g. metacognition), the teacher has fewer tools to help pupils to develop their mathematical thinking.

The purpose of this paper is to answer the following research question: *Is it possible to construct a tool for understanding pupils' mathematical thinking that shows the dynamic process of problem solving, metacognition and affect in their thinking?* To answer this question, a theoretical framework for studying pupils' mathematical thinking is formed. Problem solving is studied with metacognition and affect as a situational process (state) that is influenced and guided by pupils' view of mathematics (trait; cf. Hannula, 2011). After forming the theoretical framework, it is tested with an example case to see if it can be used as a tool for studying pupils' mathematical thinking, and more importantly, if it actually shows the dynamic process of problem solving, metacognition and affect in mathematical thinking. Even though this study was initially built on the Finnish curriculum (see Viitala, 2015a), the present framework is adaptable to research in different countries since its theoretical building is based on international research.

The interpretation of the results is tightly connected to the example case. Thus, while discussing and summarising the results for the research question, the mathematical thinking of Daniel will also be summarised by answering the question: *What characterises Daniel's mathematical thinking and the opportunities to develop it when studied with this tool?*

Finally, one purpose of the research study on pupils' mathematical thinking was to find a tool that not only researchers, but also mathematics teachers can use during their ordinary classroom activities or as part of pupil assessment. Hence, before summarising and concluding the article, an example of how teachers can use the tool in the *Finnish context* is presented.

Theoretical framework

In spite of the wide use of the term "mathematical thinking" in mathematics education, there is no common understanding of the meaning of mathematical thinking or even a consensus on the abilities or predispositions that underlie it (see e.g. Sternberg, 1996). Studies are influenced by the underlying learning theory, the specific mathematical

domain in which the study is conducted, the special viewpoint to the issue and related literature around these issues. The focus of the study can be for instance different thinking skills or styles (e.g. creative and critical thinking, McGregor, 2007; visual, analytic and conceptual thinking, Burton, 1999), problem solving (e.g. Mason, Burton & Stacey 1982; Polya, 1957; Schoenfeld, 1985), or issues that has an effect on mathematical thinking such as research on metacognition (e.g. Stillman & Mevarech, 2010; Schoenfeld, 1987; Flavell, 1979) and mathematics related affect (e.g. Pepin & Rösken-Winter, 2015; Hannula 2012).

In 1992, after a literature review, Schoenfeld recognised five aspects that are important in a study on mathematical thinking. These are the knowledge base, problem solving strategies, monitoring and control, beliefs and affects, and practices. Similar findings have also been found in connection to literature on problem-solving performance (Lester, 1994), and are also listed as part of final-assessment criteria in the upcoming Finnish curriculum (see FNBE, 2014, pp. 433–434).

There have been some attempts to connect the abovementioned attributes in problem solving. One example is Carlson and Bloom's (2005) multidimensional problem-solving framework for individual problem solvers. They studied professional mathematicians and detailed observations were done on how resources and heuristics interact with problem solving behaviour as well as how monitoring and affect were expressed during four problem solving phases (orienting, planning, executing and checking). Their analysis showed how all of the attributes (resources, heuristics, affect and monitoring) are present in every behavioural phase of problem solving.

Together with many other studies on problem solving, the multidimensional framework of Carlson and Bloom (2005) studies problem solving from a situational and contextual viewpoint. These situational and contextual processes of problem solving, metacognition and affect are called the *states* (cf. Hannula, 2011). The affective *trait* directs pupil's engagement and success in mathematics. Affective trait is a stable pattern of "how an individual feels and thinks in these different contexts and situations" (ibid., p. 44). For instance, pupils' belief systems (traits) have been found to have an influence on their problem-solving approaches (e.g. Callejo & Vila, 2009). The two different temporal aspects reveal different competencies in pupils: the state guiding pupils' thinking and actions in a contextual problem-solving situation, whereas the trait explains pupils' learning in mathematics (cf. Bailey, Watts, Littlefield & Geary, 2014).

In the following, I will draw on existing literature and form a framework for studying pupils' mathematical thinking. The framework is new in the sense that it asks explicitly for both trait and state data. This has

seldom been the case in affect research (Hannula, 2011). The affective trait is studied for two reasons: First, it is used as background information for describing a pupil (cf. Pehkonen, 1995). Similar explanations are given by teachers describing their pupils. Second, it might have an explanatory value for direct aspects uncovered from problem solving (e.g. uncertainty in problem solving, see Viitala, 2015b). From the state aspect, problem solving, metacognition and affect are considered to form a dynamic construct that (together with the knowledge base and problem solving strategies, or resources and heuristics) reveal pupils' mathematical thinking.

The knowledge base and problem solving strategies are not given emphasis in this framework since these are aspects that, according to the Finnish curriculum (FNBE, 2004), should be evaluated with ordinary school tests. The purpose of the framework is to go beyond the information gained with ordinary school tests and offer a tool that can help teachers and researchers to evaluate pupils' mathematical thinking, and more importantly, to recognize the aspects that can help pupils to develop their mathematical thinking. In the following, the different concepts of the study are introduced following the trait (pupil profile and view of mathematics) and state (mathematical thinking, problem solving, metacognition and affect) aspects of the study.

Trait – Pupil profile and view of mathematics

The role of affect in mathematical thinking is largely recognised (e.g. Zan, Brown, Evans & Hannula, 2006; DeBellis & Goldin, 2006; Vinner, 2004; Schoenfeld, 1992; also FNBE, 2014, pp.15, 429). However, theory around affect, its concepts and their connections have been used in very diverse ways both in Finnish and international research (see e.g. Hannula, 2007; Furinghetti & Pehkonen, 2002; Pepin & Rösken-Winter, 2015). The most current theorising of affect aims to dynamic representations or systems of affect in mathematics education (see Hannula, 2011; Hannula, 2012; Pepin & Rösken-Winter, 2015). Following this line of study, the psychological phenomenon of affect is seen here as a mixture of cognitive, motivational and emotional processes (Hannula, 2011).

The term affect is used as "an umbrella concept for those aspects of human thought which are other than cold cognition, such as emotions, beliefs, attitudes, motivation, values, moods, norms, feelings and goals" (Hannula, 2012, p. 138). The cognitive domain includes mental representations that have a truth value of some kind to the individual, for instance knowledge, beliefs and memories (e.g. Goldin, 2002). Motivation reflects personal preferences and explains choices, and emotions are different feelings, moods and emotional reactions (Hannula, 2011). How these components are studied in connection to affective trait is explained below.

The affective trait is studied through pupils' view of mathematics. Unlike its origin in beliefs-research, pupils' view of mathematics is considered to include all the affective processes (cognitive, motivational and emotional processes; thus the word "view", see Rösken et al., 2011). It has four components: mathematics (as science and as a school subject), oneself as a learner and user of mathematics, learning mathematics, and teaching mathematics (Pehkonen, 1995). Similar categories have also been found in many other studies (see e.g. Op't Eynde, de Corte & Verschaffel, 2002).

Pupils' view of mathematics is a stable construct that influences the development of mathematical thinking both on a trait and a state level. On trait level, it influences the learning of mathematics (e.g. through motivation to learn mathematics, or confidence in school mathematics). On a state level, it can influence, for instance, how a pupil approaches new mathematical content or a problem (e.g. through a belief that a mathematics task should be solved in five minutes, which might limit pupil's effort to solve a task). The categorisation of the components in view of mathematics helps a researcher, or a teacher, to direct attention to the different aspects of view of mathematics.

Pehkonen's (1995) model of pupils' view of mathematics can be criticised from not considering social aspects of pupil's view of mathematics (social and socio-mathematical norms in mathematics classroom, Op't Eynde et al., 2002). Even though social aspects are not studied explicitly, they play an important role in this study. For instance, the problem solving processes in this study are influenced by the classroom culture and norms since the tasks were solved in an ordinary classroom situation. However, from a researcher's or a teacher's point of view, social aspects arise only if they are taken forward by the pupil.

Pupils' answers to questions about his/her view of mathematics might also raise metacognitive and meta-affective issues. These are considered as traits when the answers are based on memories of experiences from mathematics classes, for instance explanations about self-regulation in mathematics learning (metacognition) or how the feeling of anxiety towards a word problem is handled (meta-affect). These terms are defined later in connection to state aspects of the study.

The pupil profile is formed for background information (cf. Pehkonen, 1995). It is a short description of the pupil that is constructed using the information arising from his/her view of mathematics. A teacher forms a pupil profile while he/she is describing the pupil as a mathematics learner. Ability, difficulty of mathematics, success, and enjoyment of mathematics has been shown to constitute the core of pupil's view of him/herself as a learner of mathematics in different age groups (Hannula & Laakso, 2011; Rösken, Hannula & Pehkonen, 2011). Ability and success relate to personal beliefs and contain statements such as "math is hard for me"

(ability) and "I am sure I can learn math" (cf. beliefs about oneself as a learner and a user of mathematics, Pehkonen, 1995). Difficulty of mathematics refers to mathematics as a subject (cf. beliefs about mathematics, *ibid.*) and enjoyment of mathematics to emotions. Even though motivation did not result as its own component in Rösken et al.'s study (2011), it is one of the main aspects of affect (Hannula, 2011) and considered as an important factor directing pupils' problem solving and mathematics learning. Thus, pupil profile contains descriptions of ability, difficulty of mathematics, success, enjoyment of mathematics and motivation to learn mathematics.

State – problem solving, metacognition and affect

The purpose of the framework is to help researchers to understand and evaluate pupils' mathematical thinking and to develop it further. Problem solving is used as a tool to reach this aim. Thinking is situational, a state, and pupil's activities, actions and explanations during problem solving are interpreted as visible signs or expressions of his/her mathematical thinking. Thinking is considered being mathematical when it relies on operations that are mathematical in separation of thinking about the subject matter of mathematics (Burton, 1984). In problem solving, the cognitive and affective processes are intertwined (see e.g. Hannula, 2011; Zan et al., 2006; DeBellis & Goldin, 2006; Vinner, 2004) and directed by metacognition (e.g. Schoenfeld, 1992, 1987). Also meta-affect is seen to direct pupils' problem solving (DeBellis & Goldin, 2006). These issues are discussed next.

Problem solving. In the current curriculum in Finland, learning problem solving is one of the three tasks for mathematics instruction together with developing mathematical thinking and learning of mathematical concepts (FNBE, 2004). According to the final-assessment criteria, teachers should evaluate problem solving from two perspectives: problem-solving heuristics (e.g. "formulat[ing] a simple equation concerning a problem connected to day-to-day life and solve it either algebraically or by deduction", *ibid.*, p.166), and problem-solving phases as a thinking method. In the latter category, pupils are expected to "know how to transform a simple problem in text form to a mathematical form of presentation, make a plan to solve the problem, solve it, and check the correctness of the result" (*ibid.*, p.166).

The abovementioned four phases of problem solving are very similar to Polya's problem solving phases: understanding the problem, devising a plan, carrying out the plan and looking back (Polya, 1957; see table 1). Transforming a problem to a mathematical presentation requires

understanding the problem. The second and third phases are the same in both descriptions. Checking the result is part of looking back. These behavioural steps in problem solving offer a framework for looking at pupils' cognitive processes, that is, mathematical thinking in problem solving. The steps are not understood to happen linearly (see e.g. Schoenfeld, 1985; Mason et al., 1982; from metacognitive research e.g. Stillman & Galbraith, 1998) and going back and forth between the steps is a natural part of problem solving processes (see e.g. Mason et al. 1982; Viitala, 2015a).

Table 1. *Problem solving (PS) phases of Polya (1957) and Finnish curriculum (FNBE, 2004)*

Polya's PS model	Finnish curriculum
Understanding the problem	Transforming a problem to a mathematical presentation
Devising a plan	Making a plan to solve the problem
Carrying out the plan	Solving the problem
Looking back	Checking the correctness of the result

In this study, a mathematical task is called a problem if the solver has to combine previously known data in a new way to her to solve a task (e.g. Kantowski, 1980). Given this definition for a "problem" we need to recognise that a task can be a routine task for one pupil and a problem to another (cf. Lester, 1994; Schoenfeld, 1992). Thus, with problem solving we refer to the activities and actions pupils perform while solving a given mathematical task *or* a problem. Problem solving is directed cognitive processing that requires mathematical reasoning (Mayer, 2003).

When pupils' cognitive processes are studied, their activities, actions and explanations during problem solving are interpreted as visible signs or expressions of their mathematical thinking. These explanations and the researcher's interpretations of the problem solving process are then complemented with explanations and interpretations of metacognitive and affective processes.

Metacognition. Metacognition is an inseparable part of mathematical thinking and problem solving. Even though a pupil might have the knowledge and skills for solving a problem, inefficient control mechanisms can be a major obstacle in solving problems (Carlson, 1999). Also metacognition has many different meanings in educational research, however, a majority of the researchers have returned to Flavell's early definition (Stillman & Mevarech, 2010).

Metacognition refers to one's knowledge concerning one's own cognitive processes and products or anything related to them [...] Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects or data on which they bear, usually in service of some concrete goal or objective. (Flavell, 1976, p. 232)

Metacognition can be categorised as metacognitive knowledge and metacognitive skills (Flavell, 1979). In Flavell's model, metacognitive knowledge refers to the interplay between person characteristics, task characteristics and strategy. Person characteristics refer to beliefs about individual and others as cognitive processors, task characteristics refer to task management and confidence for achieving the goal, and strategy refers to evaluations of the effectiveness of chosen strategies to achieve the goal. Evaluation of the effectiveness of the chosen strategy in problem solving is studied as part of metacognition in this paper. However, the two other aspects referring to beliefs and estimation of confidence will be discussed as part of affect (cf. view of mathematics, Pehkonen, 1995).

Metacognitive skills refer to control and self-regulation (Schoenfeld, 1987; Veenman, Elshout & Meijer, 1997). Metacognitive control during problem solving includes monitoring problem solving progress, deciding on the next step, and directing resources (Schoenfeld, 1987). Van der Stel, Veenman, Deelen & Haenen (2010) studied metacognitive skills in problem solving through four mathematics specific metacognitive activities that can be studied from pupil's overt behaviour. These activities are orientation, planning, evaluation, and elaboration (see example questions and a simplified connection between these activities and Polya's (1957) problem solving phases in table 2). In this study, the focus is on the quality

Table 2. *Polya's (1957) problem solving (PS) phases and metacognitive activities in PS with examples (van der Stel et al., 2010, p. 220)*

Polya's PS model	Metacognitive activities	Examples of metacognitive activities
Understanding the problem	Orientation	Estimating the answer / Making a sketch of the problem to represent the problem
Devising a plan	Planning	Designing a step-by-step action plan, instead of working by trial and error / Writing down calculations step-by-step
Carrying out the plan	Evaluation	Monitoring action plan / Checking an answer by recalculating
Looking back	Elaboration	Paraphrasing the problem / Drawing conclusions while referring to the problem statement

of metacognitive skilfulness with only a little attention to the *quantity* of these skills (cf. van der Stel et al., 2010).

Affect. The affective state follows the same structure as the affective trait: affect is seen as a mixture of cognitive, motivational and emotional processes. Affective state is situational and contextual and the task related beliefs, changing emotions, feeling of confidence and task motivation are studied together with pupils problem solving and metacognitive processes (cf. task characteristics of metacognition, Flavell, 1979).

One aspect closely connected to affect is meta-affect. Meta-affect can be seen as "standing in relation to affect much as metacognition stands in relation to cognition, and powerfully transforming individuals' emotional feelings" (DeBellis & Goldin, 2006, p.132). Carlson and Bloom (2005) emphasized the role of effective management of frustration and anxiety in problem solving that were shown to be an important factor in their participants' persistent pursuit of solutions to complex problems. Recognising the different feelings in problem solving might help teachers to enhance their pupils' problem solving behaviour, especially in the case of negative emotions. In this study, meta-affect is studied together with the emotional states.

Summary of the framework

In order to understand and develop pupils' mathematical thinking, teachers and researchers need a tool that goes beyond ordinary mathematics tests. The present framework recognises all the five aspects influencing mathematical thinking that were found to be important in studies on mathematical thinking: the knowledge base, problem solving strategies, monitoring and control, beliefs and affects, and practices (Schoenfeld, 1992). The knowledge base and problem solving strategies are already tested with ordinary mathematics tests. Hence, they are not the main focus of the present study. Metacognition (monitoring and control) is influencing pupils' problem solving and present in pupils' explanations about learning mathematics. Affect (beliefs and affects) is guiding the problem solving process both from state and trait (view of mathematics) levels. Practices are not studied explicitly but they are present both in state (e.g. metacognitive decision to draw a picture of a problem in an aspiration to understand it, if it is usually done in mathematics lessons) and in trait (e.g. explanations about teaching mathematics).

Problem solving, metacognition and affect are highly connected. As the literature review showed, it is often difficult to differentiate between knowledge and metacognition, or metacognition and affect. Since problem solving is studied as a dynamic process, the somewhat unclear

categorising does not limit the study on mathematical thinking. On the contrary, seeing problem solving, metacognition and affect as highly interrelated can give us a more informed picture of pupil's mathematical thinking than studying these aspects separately in problem solving. The framework is built to direct our attention to different aspects that influence pupil's mathematical thinking. However, this interpretive study is open to all results arising from the data (such as social aspects of view of mathematics).

The structure of the framework is shown in figure 1. The structure is not meant to be exhaustive in respect to different aspects influencing mathematical thinking and their connections. It is a simplistic representation of the framework that shows the tools with which mathematical thinking is studied, and how trait and state are present in the study.

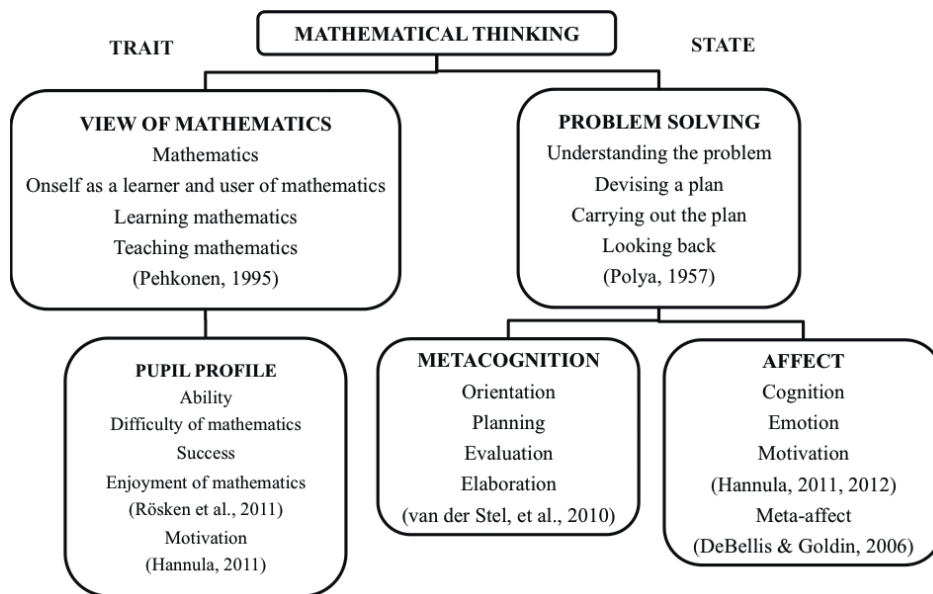


Figure 1. *A simple structure of the framework*

Methods

In this section, the phases of data collection and data analysis are explained. How teachers can use the framework in their work will be explained later in this article.

The purpose of the case of Daniel is to test the framework and to illustrate its dynamic nature. Daniel participated in a research study on pupils' mathematical thinking, and his thinking was analysed using the tool presented in this article (for previous results from the project see Viitala, 2013, 2015a, 2015b). At the time of the data collection, Daniel was at the final grade of comprehensive school (age 15).

The data used to analyse Daniel's mathematical thinking was collected from mathematics lessons and interviews in three cycles over the course of three months. In each cycle, both trait and state data were collected. The trait was about Daniel's view of mathematics and the state about problem solving. The data analysis followed the state and trait structure introduced in figure 1. Further elaborations on data analysis are given below.

Trait – pupil profile and view of mathematics

The trait data were collected through interviews. The questions treated the cognitive, emotional and motivational (Hannula, 2011) aspects of affect and followed the following themes: Daniel's background, mathematical thinking, and Daniel's view of mathematics (Pehkonen, 1995). The interviews were semi-structured and focused (Kvale & Brinkmann, 2009) and included both open and closed questions. The closed questions were either taken from large-scale studies on mathematics related beliefs (e.g. KIM-study), and/or they were asked as a follow-up question to another question (see example questions in table 3).

Table 3. *Interview themes and example questions*

Theme	Example questions
Background	Tell me about your family.
Mathematical thinking	What does mathematical thinking mean? / How do you recognise it?
Mathematics	What is mathematics as a science? / Does it exist outside of school? (How? Where?)
Oneself and mathematics	Is mathematics important to you? / Does it help you think logically? (How?)
Learning mathematics	How do you learn mathematics? / Is it most important to get a correct answer?
Teaching mathematics	Does teaching matter to your learning? (How?) / What is good teaching?

The analysis of Daniel's view of mathematics followed the same categorisation as the data collection, emphasising the connection between school mathematics and real life (also emphasised in the Finnish curriculum, FNBE, 2004) and the emergent issues from the abovementioned categories. The analysis was done one theme at a time (mathematical thinking, mathematics, oneself and mathematics, learning mathematics and teaching mathematics). After the first description about the theme issue, data reduction was executed allowing emergent and repeatedly referred

issues to be highlighted. The final and condensed description was an interpretation of these results.

The pupil profile was also derived from the interview data following the descriptions of Rösken et al. (2011) for ability, success, difficulty of mathematics, and enjoyment of mathematics (cf. "Oneself as a learner and user of mathematics", Pehkonen, 1995). Pupil profile also contained Daniel's most recent mathematics grade and his motivation to learn mathematics.

State – problem solving, metacognition and affect

The state data was collected from mathematics lessons and interviews. In each cycle, Daniel solved a real-life based mathematics task in an ordinary classroom situation (see an example task below, School Excursion, OECD, 2006, p. 87; cf. real-life connections in mathematics education in the Finnish curriculum, FNBE, 2004).

A school class wants to rent a coach for an excursion, and three companies are contacted for information about prices.

Company A charges an initial rate of 375 zed plus 0.5 zed per kilometre driven. Company B charges an initial rate of 250 zed plus 0.75 zed per kilometre driven. Company C charges a flat rate of 350 zed up to 200 kilometres, plus 1.02 zed per kilometre beyond 200 km.

Which company should the class choose, if the excursion involves a total travel distance of somewhere between 400 and 600 km?

The task solution was then further discussed in a stimulated-recall interview in which Daniel also assessed his confidence before, during and after solving the problem, as well as his confidence in school mathematics using a 10 cm line segment (scale from "I couldn't do it at all" to "I could do it perfectly"; see about estimation of certainty e.g. in Hannula, Maijala, Pehkonen & Soro, 2002; considerations on when to estimate confidence in problem solving, see e.g. Morselli & Sabena, 2015). Also some additional tasks were solved in the interviews. All interviews were video recorded.

The state data was analysed first by going through the problem solving phases (Polya, 1957) for all the tasks. Then, metacognitive decisions (van der Stel et al., 2010) and affective states (cognition, emotion, motivation, Hannula, 2011; meta-affect, DeBellis & Goldin, 2006) emerging in problem solving processes were investigated and descriptions of them were given. Finally, connections between problem solving (state) and view of mathematics (trait) were studied. The descriptive results are introduced in the following section of the paper.

Analysis and results: the case of Daniel

The purpose of this part of the article is to answer the first research question: *Is it possible to construct a tool for understanding pupils' mathematical thinking that shows the dynamic process of problem solving, metacognition and affect in their thinking?* The thinking is studied with one representative problem solving process, and only selected parts of the view of mathematics. The state and trait results are presented first separately and then brought together in discussion and summary of the results.

The results are supported with excerpts taken from the interview data. In these excerpts, the question Daniel is answering to or words completing the sentences are written in parentheses. The translation has been done by the researcher and critical words have been checked by an experienced researcher in mathematics education.

Trait – pupil profile and view of mathematics

Pupil profile. The pupil profile is a short description of Daniel's mathematics grade, motivation to learn mathematics, and view of himself as a learner of mathematics (Rösken et al., 2011; cf. oneself as a learner and user of mathematics, Pehkonen, 1995):

Daniel is very confident and successful in mathematics. He has the highest grade in mathematics and he is very aware of his success. He likes mathematics, it is easy for him and he is motivated to learn it. He values mathematics and it is one of his favourite subjects.

The following excerpts support this description:

Ability and success (personal beliefs)

(Does your grade describe your know-how?) Yes. (How would you justify your grade in mathematics?) Well, activity during lessons, test grades, the eagerness to study, how much I study and, then, how well I comprehend matters.

(How confident are you about you and your skills in mathematics?) Very confident. 100 %.

Learning (mathematics) is easy [...] if you know (something) from the beginning, then new things are easy to understand, and then it's easy.

(If you should learn mathematics on your own, would you learn it?) Yes. (Without teaching?) Yes.

Difficulty of mathematics

(Does learning mathematics require a lot of work?) Not necessarily, if you have been listening well in lessons.

(Does learning mathematics take time?) It takes some time. It might even take a day or a week to comprehend it well. But for me, it has never taken a week. Usually it takes one lesson to learn. Or then it might be a day or so: after learning something for one lesson, and then on the next day there is another lesson on the same topic, you can comprehend it just then.

Enjoyment of mathematics

(When I learn mathematics) the kind of like good feeling comes. When you learn something, or for instance if you don't get it at first and finally you do understand it, then you get a nice feeling.

At times, (learning mathematics) it is also quite fun. [...] It is not that serious [...] even though of course it is important. (For you it is laid-back?) Yes, it is. [...]

View of mathematics. Daniel thinks that mathematics is the most important school subject and it is needed everywhere through life (cf. real-life connections in the curriculum, FNBE, 2004). For these reasons he claims to be motivated to learn mathematics. Like mathematics, also mathematical thinking can exist anywhere. For Daniel, mathematical thinking is "thinking mathematically about some calculations or matters". It is not just calculating something, but also models for thinking. Daniel has a perception that if you are good in mathematics, you are able to think faster.

It is difficult for Daniel to describe how he learns mathematics: he learns by listening in mathematics lessons and doing homework. He understands the cumulative nature of mathematics but he seems to connect new knowledge to the old one actively only when it is evident. For him, mathematics is "kind of becoming familiar maybe, somehow". Listening, focusing and thinking leads to the point where "pieces click together".

When Daniel talks about teaching mathematics, he says that good teaching contains describing mathematical things in detail and teaching in a very easy way so that you understand "it" for sure. He agrees that this means that the teaching starts "with the easiest" and teaching proceeds step-by-step. He thinks that his teacher is a good mathematics teacher, and most of Daniel's mathematics learning happens in the mathematics lessons, thus, this also might be understood to be the way how Daniel builds his knowledge and skills in mathematics.

State – problem solving, metacognition and affect

The state is discussed through one task, School Excursion (OECD, 2006, p.87, see task description in Methods). The reporting follows Daniel's problem solving phases and includes metacognitive and affective (both state and trait) considerations to show the dynamic nature of the different processes in problem solving.

Researcher: How many times did you read the task?

Daniel: 3 times, I guess. [...] First I just read it and looked what has to be done.

Even though understanding the problem does not take long for Daniel, the phase is coloured with affect. Daniel estimates his confidence to be 4.7 after reading the task (on a scale 0–10, 10 being the positive end).

Daniel: Somehow, it felt at first a little like obscure because those zeds were there. Then, after I started thinking that it is probably the currency, or that it is definitely the currency, so it, the whole time, started to develop there (to a more confident direction).

Daniel feels unsure about the task description and his meta-affect directs his attention towards zeds. He then works towards understanding the meaning of zeds to become more confident to solve the task. Daniel also remembers facing a bit similar task earlier, in elementary school. Even though he does not remember any task in particular, the feeling of familiarity gives him confidence.

Daniel does not report on doing any metacognitive activities while he tries to understand the problem (e.g. estimating the answer) but at some point of solving the task, he pictured it in real life: he thought of a bus, a motorway, museums and an amusement park (cf. real-life connections in the curriculum, FNBE, 2004). Daniel *plans the task* for about 4 minutes before starting to solve it.

Daniel: [On second reading] I started looking at the numbers. [...]

Daniel: I start there, I write down those 400 and 600 km there first, for the fun of it. Then, I kind of take the intermediate result, or I mean the 500 km [...] because it's there, in between, so conveniently. So with that I try to calculate.

The interval of the travel distance disturbs Daniel "a little" but he feels confident that he can solve the task. This confidence can be traced to an affective trait: Daniel has always been able to solve the given tasks.

Researcher: Do you face challenging tasks that you don't understand right away?

Daniel: Not really because the teachers explain them well so you understand them immediately.

Researcher: There haven't been insurmountable tasks for you?

Daniel: No.

Daniel plans to write all the expressions to the prices step-by-step, for every company in the order they are written in the task description (metacognitive decision).

For the next almost 4.5 minutes Daniel *carries out his plan*: he writes down the expressions for the bus companies and then calculates all the prices with a calculator. At this point, he feels 100% confident (10, on a scale of 0–10). While writing down the expressions, he occasionally adds units after numbers. He makes a decision to add units to all numbers (metacognitive planning).

After calculating the prices for all the companies, Daniel realises that two of the companies (A and B) are equally as cheap.

Daniel: Well, I had an initial plan already, how I, that I look at all the [...] prices (for all companies at 500 km). [...] After performing that, it came to mind there that (the prices might be different with other distances).

This realisation drives Daniel to *devise a new plan* for solving the task (metacognitive activity). He calculates the prizes in 600 km for the two remaining bus companies (A and B, *carrying out the plan*).

Daniel: [...] And then in the end, I read (the task) through one more time and I made sure I have used all the numbers from there.

Daniel's justification for reading the task description through one more time refers to a belief that all the numbers from a task have to be used. Hence, his belief (trait) guided his problem solving (state). The subsequent discussion showed, however, that even though this belief guides Daniel's problem solving, it does not necessarily determine it:

Researcher: Is it usually important to use all the numbers from a task?

Daniel: In most cases all the numbers have to be used, but in some (tasks) there can be trick numbers that you don't necessarily have to use.

Researcher: The extra numbers don't disturb you too much?

Daniel: No, not too much.

When writing down the answer, Daniel is both *looking back* to his solution (problem solving behaviour) and drawing conclusions (metacognitive activity). He recapitulates his work, relates the answer to the problem and draws conclusions while referring to the problem statement:

The class should choose company A, because 600 km costs 675 z but otherwise A and B cost about the same amount.

After writing down his answer, Daniel starts to solve another task in the lesson. However, he quickly returns to the School Excursion task. This way of working with a task while doing something else was expressed also in the interviews when solving difficult tasks at home was discussed:

Daniel: I start to think about it first on my own, and then if I cannot do it, but usually it is that I get it at some point of the day.

Researcher: So you "let it brew"?

Daniel: Yes. If I cannot do it, I still think about it, maybe like quite a bit. If I don't kind of do it, for instance if I go and do something completely different, it can still circle in my head, I still think about it a little. [...]

Researcher: You have the desire to continue with a task?

Daniel: Yes.

Researcher: You don't quit?

Daniel: I never quit. I have to solve it.

Working with the task for a long time shows persistence, also with the task discussed here. On the other hand, returning to the task shows some uncertainty. Daniel decides to calculate the prices for companies A and B in 400 km (*devising a plan*), just to be sure. After solving them with a calculator (*carrying out the plan*) and not writing anything down (metacognitive decision), Daniel "feels good" (emotion) and completes his written answer (*looking back* and elaborating, metacognitive action):

But, if the distance is 400 km, B is better, because it costs 550 z, whereas A costs 575 z. So: 400 km -> B; 500 km -> A, B; 600 km -> A.

As in mathematics, Daniel is 100% confident about his work at the end (10, on a scale 0–10). It took Daniel 16 minutes to complete the task.

Discussion and summary of results

The purpose of this part is to summarise what was found in relation to the research question about the dynamic process of problem solving, metacognition and affect, and to answer the question: *What characterises Daniel's mathematical thinking and the opportunities to develop it?*

As shown above, metacognition, affective state together with meta-affect and affective trait all have an important role in Daniel's problem solving. The relationships between problem-solving, metacognitive and affective processes *are found to be dynamic*: they have an effect on each other and Daniel moves naturally between these different processes. While metacognition and affect (both state and trait, also through

meta-affect) has an effect on Daniel's problem solving behaviour, his meta-cognitive decisions and problem solving behaviour (and success) has an effect on his affective state (emotions). The answer to the question about Daniel's mathematical thinking is described below.

Daniel can be characterised as a confident, successful and thorough problem solver. Many times he does not go into details while solving problems (e.g. the units were not all correct in School Excursion which might have reduced his scores in school tests) but he ends up with correct answers. In case of problems, Daniel asks for help from friends or the teacher. Daniel liked, perhaps even enjoyed, solving the given tasks.

When *understanding a problem* or *looking back*, the role of feelings, more precisely confidence, and the way Daniel handles these feelings by directing his problem solving (meta-affect, DeBellis & Goldin, 2006) are highlighted. For instance, with another task, Daniel sat quietly after solving a task. He explained this by saying:

[...] Well, somehow I searched for the kind of confident feeling, like completely 100 % feeling of confidence, that those (calculations) are correct.

Daniel's metacognitive skills were highlighted in *planning* and *carrying out a plan*. For instance, with another task, Daniel went to an incorrect direction while solving the task but he had the metacognitive skills to monitor his work and direct his attention to a more productive direction. Additionally, while Daniel moves easily between different problem-solving phases, he might also move between different metacognitive phases within one problem solving phase.

[...] At the same time (while solving a problem) I started thinking how it would be reasonable to continue and do them, or write them down [...].

When solving problems, Daniel says in the interview that he is "quite aware" of his own thinking all the time. However, this is not visible in the stimulated-recall data. In the interviews, when Daniel was asked to explain what he was thinking in the video, he could not recall his thoughts, only actions. Similarly, when explaining his learning of mathematics, Daniel refers to behavioural actions he goes through (through teaching), as well as refers to learning as feelings (becoming familiar with something). This might mean that thinking mathematically and learning mathematics are very automatic for Daniel. On the other hand, when tasks were solved in the interviews and why-questions were asked on the spot, Daniel was more able to answer them.

One reason for not being able to explain his thinking afterwards might be that, Daniel seems to be a bit unorganised as a problem solver.

He is jumping back and forth between different phases of his calculations (and between tasks) and it is hard even for Daniel to interpret what he is doing in the video. Additionally, his notes are messy (e.g. calculations are not necessarily written chronologically and one written expression might be used to calculate many calculations). Being able to return to different problem-solving or learning situations, and practising precise and focused problem solving could develop his problem-solving and learning skills, and consequently, mathematical thinking. Thus, Daniel might benefit from paying more conscious attention to his problem-solving and learning processes.

An example on how teachers can use the tool in the Finnish context
One purpose of forming the framework for studying pupils' mathematical thinking is that teachers can use it in their mathematics lessons and as part of pupil assessment. This is particularly relevant now when Finland is under curriculum reform.

According to the new curriculum (FNBE, 2014), the main part of pupil evaluation is formative assessment that happens as part of everyday teaching and working. It asks for observing pupils' learning processes and communicating with them. Feedback that advances learning is said to be qualitative and descriptive, and should help pupils to perceive and understand what they are supposed to learn, what they have learnt already and how they could advance their own learning and improve their performance (pp. 50–51).

The summative assessment can also include verbal evaluation. The verbal evaluation allows teachers to describe the level of a pupil's performance, but also to describe the pupil's strengths, progressions, and targets of development (ibid.). Below, there is an example of how the tool can be used to develop and evaluate pupils' mathematical thinking as part of mathematics teaching in the Finnish context.

Trait – pupil profile and view of mathematics

When a teacher is asked to give a short description of a pupil in his/her mathematics class, he/she quickly forms a first version of a *pupil profile*, for instance: "Sofia is an average pupil but does not bother to study mathematics and then underachieves in it". This can be used as a starting point for learning discussion many teachers in Finland are expected to have with their pupils as part of qualitative pupil assessment.

In a learning discussion, the teacher can talk with the pupil about the teacher's observations in connection to the pupil profile and ask possible reasons for the observed issues. This discussion can be short but

informative enough to be used to set long-term goals for learning that both the teacher and the pupil agree, for instance: Sofia has a belief that she cannot do well in mathematics, and hence, does not study it. So, Sofia is asked to pay attention to the achievements that she did not believe she could accomplish (and perhaps write them down).

These learning goals can be supported by the teacher in everyday classroom situations when appropriate, and they will be taken forward in the following learning discussion (that might happen in a month or two). In the learning discussions, the pupil profile can be altered if there is a reason to do so and the long-term learning goal can be changed. If there is no obvious target for the long-term learning goal, the teacher can follow the themes behind the core of pupil's view of mathematics (ability, success, difficulty of mathematics, enjoyment of mathematics, and motivation to learn mathematics).

These observations and discussions about pupil profile open doors to pupil's *view of mathematics*. The teacher should recognise that pupil's view of mathematics can influence the development of mathematical thinking through cognitive, motivational and emotional processes. In Sofia's case, the cognitive belief that she is not good in mathematics affects her emotional and motivational bond to mathematics. Through positive experiences and supporting feedback this might change.

As a summary, pupil's view of mathematics and the pupil profile can offer a way to describe and evaluate pupil's development through lower secondary school offering documentation from the pupil's development as a mathematics learner and thinker in a long-term sense. In this manner, the pupil profile can also be used as a part of summative evaluation as well as a starting point for pupil's self-evaluation.

State – problem solving, metacognition and affect

The state offers teachers information about the situational and contextual thinking processes. It can also reveal issues connected to pupil's view of mathematics.

Information about pupil's thinking processes are given in everyday classroom situations. The key is to observe pupil's problem solving, ask questions about it, and most importantly, listen to the answers. If the purpose is to learn about pupil's problem solving, metacognition or affect, the problem should be one that the pupil is competent enough to solve. Otherwise the focus might turn more towards mathematical knowledge and heuristics.

As an example, Sofia got stuck after reading the problem and performing a first calculation. As before, she asks help and repeats that she

is not good with word-problems. In Sofia's case, there might be a problem with *affective trait* (a belief that "I am not good in word-problems") and meta-affective skills. On the other hand, if she is able to proceed with the problem after suggesting to draw a picture about the situation, the reason might also be in *metacognitive skills*. Furthermore, if the calculation does not make any sense to Sofia or the teacher in connection to the problem at hand (e.g. summing up all the numbers in the problem), the problem might be connected to *problem-solving behaviour* and insufficient planning of the problem.

The teacher gets sense of pupils' mathematical thinking while working with them in ordinary mathematics lessons. The purpose is not to understand pupils' mathematical thinking all at once, but to take small steps towards getting to know their thinking. Also with state, the discussion can continue in the learning discussions. The key for the teacher is to focus on one issue at a time (problem-solving behaviour, metacognition, or affect as in Sofia's case) so that the learning discussions can be kept short and include both short- and long-term goals for the pupils (short term goals being mathematical in most cases).

If the teacher has problems to interpret pupils' skills, the framework can offer him/her concrete tools to categorise pupils' answers so that the weak points could be recognised and the development of mathematical thinking supported. The key elements of the framework can also be developed into key questions that a teacher can use as an actual tool in his/her work. In connection to mathematical thinking, it is also important to remember that (unlike traits) the states are contextual, and in different situations the same pupil might need very different kind of help.

Summary and conclusion

This article endeavoured to answer the research question: *Is it possible to construct a tool for understanding pupils' mathematical thinking that shows the dynamic process of problem solving, metacognition and affect in their thinking?* To answer this question, a theoretical framework for studying pupils' mathematical thinking was formed based on research literature around mathematical thinking. After forming the theoretical framework, it was tested with an example case of Daniel to see if it can be used as a tool for studying pupils' mathematical thinking, and more importantly, if it actually shows the dynamic process of problem solving, metacognition and affect in mathematical thinking.

As a result, the tool for understanding pupils' mathematical thinking was found to successfully expose the dynamic processes of problem solving, metacognition and affect in Daniel's thinking. In fact, all of

these aspects were an inseparable part of Daniel's thinking process. In addition, Daniel's view of mathematics (trait) supported the findings from problem solving (state). In spite of the similar results, the trait and state perspectives are important to study separately as they report from different competencies that influence pupils' mathematical thinking: the trait revealing more stable competencies affecting pupils' mathematics learning, and the state revealing the contextual and situational competencies influencing pupils' problem solving processes.

The question about the case of Daniel concerned the use of the created tool: *What characterises pupils' mathematical thinking and the opportunities to develop it when studied with this tool?* This question was answered by interpreting the results of Daniel's mathematical thinking revealed while answering the first research question. The results showed that Daniel's metacognitive skills in problem solving as well as the natural moving between different problem solving and metacognitive phases can be characterised to be the key in his success as a mathematical thinker. His metacognitive skills outpaced affect in planning and carrying out the plan, and he was fully confident throughout the study. On the other hand, his lack of ability to return to the thinking processes after solving a task or learning in mathematics, directs our attention towards a point where Daniel could be helped to become a more successful mathematical thinker: Daniel could benefit from paying more conscious attention to his processes of problem solving and learning mathematics.

One aim of the research study was also to present a tool for studying pupils' mathematical thinking that not only researchers, but also mathematics teachers can use during their ordinary classroom activities or as part of pupil assessment. In the latter part of the paper, an example is given on how this tool could be used during ordinary classroom situations and as part of pupil assessment in the Finnish context. The first task for the teacher is to recognise if a phenomenon is connected to a state or a trait. States are less stable and can be influenced more easily. Traits, on the other hand, are more difficult to change. Thus, instead of aiming to change pupils' (affective) traits directly, teachers should aim to recognize the aspects that might hinder pupils' learning and concentrate on helping them to work through these different feelings, attitudes or beliefs in a fruitful way. Some of these obstacles that influence mathematical thinking might also be uncovered in a problem solving situation.

In an earlier publication (Viitala, 2015b), another pupil's mathematical thinking was reported using an earlier version of the framework. Unlike Daniel, this pupil, Emma, was not very confident problem solver and her affect determined many activities and actions in her problem solving processes. Learning mathematics took time for her, she asked a lot of

questions and she was quite aware of the processes needed for her to learn something. Emma was found to benefit from support to overcome her feelings of uncertainty. Thus, while both Daniel and Emma were successful problem solvers, they were found to need different support for learning mathematics and developing mathematical thinking.

Based on these two example cases, the tool can be said to successfully reveal different aspects that influence the development of individual pupil's mathematical thinking in different pupils. However, these two pupils are high achievers with a positive view of mathematics. The next step would be to adapt this framework to data from low achievers with a negative view of mathematics to see if the framework is fruitful also for understanding these pupils' mathematical thinking and for evaluating how they could be best assisted towards developing their mathematical thinking.

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Article 6

A framework for studying students' mathematical thinking at different ages: A longitudinal case study of Alex

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Understanding students' mathematical thinking is essential for educators in helping their students to develop mathematical thinking. Observing students' problem solving is a way of getting closer to their mathematical thinking as it reveals the dynamic thinking processes they undergo when solving problems. Also, students' view of mathematics influence the problem-solving and learning processes and, through them, the development of mathematical thinking. This article presents a framework for studying students' mathematical thinking at different educational levels. Mathematical thinking is studied through situational problem-solving processes and the more stable view of mathematics with the help of resources, heuristics, metacognition and affect. Also, the development of mathematical thinking of one student, Alex, is studied. The data is collected over the span of four years, during the last year of comprehensive school and during the first year of university studies. The results show that the development of mathematical knowledge and the growing interest in mathematics as a science are the most influential aspects for the development of Alex's mathematical thinking and how he sees mathematics in his everyday life.

Keywords: mathematical thinking, problem solving, view of mathematics, metacognition, affect

I guess [mathematical thinking] means changing different attributes, and for instance weather conditions and natural phenomena, to a form of calculations. (Alex, comprehensive school)

Four years later:

[Mathematical thinking is] all kinds of chains of logical deductions, whether there are numbers or not. ... I have difficulties to think where it does not exist at all. (Alex, university)

1. Introduction

Developing mathematical thinking is an important aim for school mathematics (e.g. FNBE 2015) that carries over into students' every-day life and future studies. Students develop their mathematical thinking in different ways and multiple aspects influence the development. While there is no common understanding of the meaning of mathematical thinking (Sternberg 1996), Schoenfeld recognised five aspects that are important: the knowledge base, problem-solving strategies, monitoring and control, beliefs and affects, and practices (Schoenfeld 1992). Similar findings have also been presented in connection to literature on problem-solving performance (cf. Lester 1994; Stillman and Galbraith 1998). Even though the importance of these aspects is widely recognised, there are only a few studies aiming to understand their intertwined and dynamic relationships and how they are present in mathematical thinking or problem-solving processes of individual students

(e.g. Schoenfeld 1992; Carlson and Bloom 2005). Furthermore, there is a shortage of longitudinal studies on the development of mathematical thinking that consider the variety of aspects influencing it.

Building on the existing body of research, the purpose of this article is to *form a framework for studying students' mathematical thinking at different ages* (close to secondary school) *where resources, heuristics, metacognition and affect are represented as dynamic processes influencing student's mathematical thinking*. Furthermore, a longitudinal case study on the development of mathematical thinking is introduced. The research question for the case study is: *what characterises the development of Alex's mathematical thinking when his problem solving and view of mathematics are studied before and after upper secondary school*, that is, at the end of comprehensive school and at the beginning of university studies. The data is collected in mathematics classrooms (comprehensive school) and in interviews (comprehensive school, university). The data collection, analysis and reporting are divided into two main streams affecting the development of students' mathematical thinking: the situational problem-solving process and the more stable student's view of mathematics.

2. Background

2.1 Problem solving

Research studies in mathematics education have generally accepted problem solving as a way of getting closer to students' mathematical thinking. However, the terms 'problem' and 'problem solving' have had different meanings in different contexts and in different countries (e.g. Schoenfeld 1992; Törner, Schoenfeld and Reiss 2007). For instance, with 'problem solving', researchers might mean anything from doing routine calculations to 'doing mathematics as a professional' (Schoenfeld 1992). In this study, a 'problem' is defined as follows:

A problem is only a Problem (as mathematicians use the term) if you don't know how to go about solving it. A problem that holds no "surprises" in store, and that can be solved comfortably by routine or familiar procedures (no matter how difficult!) is an exercise. (Schoenfeld 1983, p. 41)

Whereas earlier the task difficulty was considered to be a quality of a task, nowadays the difficulty of a task is seen to depend on the characteristics of the problem solver (e.g. Lester 1994). Hence, a task can be an exercise to one individual and a problem to another. Furthermore, in a study where students solve many tasks, some of them might be referred to as problems, and some as exercises. In this study, problem solving is referred to as the behaviour connected to solving an exercise *or* a problem.

Polya (1957) introduced four phases for problem solving: understanding the problem, devising a plan, carrying out the plan and looking back. This model has been criticised for being too simple or not adequate for students' learning and the model has been further developed by many researchers over the years (e.g. Mason, Burton and Stacey 1982; Schoenfeld 1985). In 2005, Carlson and Bloom introduced a multidimensional problem-solving framework that has four phases (orienting, planning, executing and checking; cf. Polya 1957) and emphasises the cyclic nature of problem solving. Additionally, in their study with mathematicians, a sub-cycle of conjecturing, testing, and evaluating emerged from a planning phase where the ideas for solving a problem are initially tested before executing the selected plan (Carlson and Bloom 2005). In a research project with comprehensive

school students, the conjecture cycle has not been visible (see Viitala 2015a; 2015b; in press). Thus, the sub-cycle of conjecture seems to be a quality of expert problem solvers (cf. metacognitive activities of novice and expert problem-solvers e.g. in Stillman and Galbraith 1998; Schoenfeld 1992).

2.2 Resources, heuristics, metacognition and affect in problem solving

Whereas resources and problem-solving heuristics have been considered to be a natural part of problem solving from the beginning (e.g. Polya 1957), the subsequent studies have also shown the importance of metacognition and affect in problem solving (e.g. Schoenfeld 1992; Stillman and Mevarech 2010; Pepin and Rösken-Winter 2015). Early studies defined metacognition as ‘one’s knowledge concerning one’s own cognitive processes and products or anything related to them’ (Flavell 1976, p. 232). One way of looking at metacognition is to see it as metacognitive skills. Metacognitive skills refer to control and self-regulation and include active monitoring and consequent regulation of problem solving processes (Schoenfeld 1987; Flavell 1979; Veenman, Elshout and Meijer 1997). In 2010, van der Stel, Veenman, Deelen and Haenen introduced a framework for studying metacognitive skills through four mathematics-specific metacognitive activities that can be studied from a student’s overt behaviour: orientation (e.g. estimating the answer), planning (e.g. writing calculations step-by-step), evaluating (e.g. monitoring action plan) and elaboration (e.g. drawing conclusions while referring to the problem statement). Even though some of these activities may be considered as cognitive, the successful application of such activities at the correct moment is a result of metacognitive skilfulness (ibid.; cf. Schoenfeld 1992).

Affect can also be seen as a dynamic part of problem solving (see e.g. Schoenfeld 1992; Viitala 2015b). Affect is ‘an umbrella concept for those aspects of human thought which are other than cold cognition, such as emotions, beliefs, attitudes, motivation, values, moods, norms, feelings and goals’ (Hannula 2012, p. 138). It is seen as a mixture of cognitive, motivational and emotional processes with a rapidly changing state and a more stable trait aspect (Hannula 2011). Cognitive processes refer to the mental representations that have a truth value of some kind to the individual (Goldin 2002); motivational processes reflect personal preferences and explain choices; and emotional processes refer to feelings, moods and emotional reactions (Hannula 2011). The affective state is easier to observe in problem solving (e.g. situational emotions and task motivation) than the affective trait. However, the trait can also have a great impact on problem solving, for instance through beliefs such as ‘all numbers from a task description have to be used’ (see e.g. Viitala 2015b), or through feelings of confidence towards mathematical problem solving (e.g. unsureness; see Viitala, in press).

In connection to affect, meta-affect has also been shown to be important in a study on problem solving (e.g. Carlson and Bloom 2005) because it manages and transforms the changing affect (e.g. emotions) in a problem-solving situation (DeBellis and Goldin 2006). Meta-affect, together with mathematical intimacy and integrity, also express the bond between the problem solver and the problem (or the problem solving situation; see e.g. Carlson and Bloom 2005). Mathematical intimacy refers to the deep and vulnerable emotional engagement an individual may have with the problem. Mathematical integrity is ‘the individual’s fundamental commitment to mathematical truth, search for mathematical understanding, or moral character guiding mathematical study’ (DeBellis and Goldin 2006, p. 132).

2.3 View of mathematics

The role of affect in mathematical thinking is largely recognised (e.g. Schoenfeld 1992; Zan, Brown, Evans and Hannula 2006; DeBellis and Goldin 2006; Vinner 2004; FNBE 2015). In addition to the problem-solving situation, affect directs students' engagement and success in all mathematics-related activities and, thus, influences the development of mathematical thinking. The affective trait has often been studied through students' view of mathematics (see Op't Eynde, de Corte and Verschaffel 2002). It has been characterised through four components: beliefs about mathematics (as a science and as a school subject), oneself as a learner and user of mathematics, learning mathematics, and teaching mathematics (Pehkonen 1995). Following the most recent development of defining affect (Hannula 2011; 2012), students' view of mathematics is seen as entailing all (cognitive, emotional and motivational) aspects of affect. Furthermore, the core of a student's view of himself as a learner of mathematics at different educational levels has been shown to consist of ability and success (personal beliefs), difficulty of mathematics (beliefs about mathematics) and enjoyment of mathematics (emotions) (Rösken, Hannula and Pehkonen 2011; Hannula and Laakso 2011).

3. Theoretical framework

The purpose of the article is to create a framework for analysing students' mathematical thinking at different ages (close to secondary school) where resources, heuristics, metacognition and affect are represented as dynamic processes influencing student's problem solving and mathematical thinking. The framework is based on two such studies. In 2005, Carlson and Bloom studied mathematicians' problem-solving behaviour with the aim to learn more about the problem-solving process and interactions of various problem-solving attributes. They used grounded approach and found that the participants' resources, heuristics, affect, and monitoring all play an important role in problem solving. They highlighted especially the relationship of metacognition and well-connected conceptual knowledge and the role of affective pathways in the problem-solving process. Similarly, in an ongoing research project, Viitala (2015b) showed how metacognition, affective state and trait, and meta-affect influence lower secondary school students' problem-solving behaviour; and these, together with the knowledge base and heuristics, form a dynamic process that reveal student's mathematical thinking. Furthermore, students' view of learning mathematics has especially revealed findings consistent with their problem-solving behaviour (see e.g. Viitala, in press), providing deeper knowledge about students' mathematical thinking.

In their framework, Carlson and Bloom (2005) wanted to bring improved clarity and coherence to the body of problem-solving literature. In many parts, their definitions are adopted also in this article. However, Carlson and Bloom focus purely on problem solving. In Viitala's study (2015b), the student's view of mathematics also greatly influences mathematical thinking (e.g. through a belief of one's capability to learn mathematics). Thus, in addition to the terms used in the Carlson and Bloom study on *problem solving* (problem solving, resources, heuristics, monitoring and affect), view of mathematics is added to the present framework on *mathematical thinking* in order to emphasise the role of the affect through cognition (e.g. beliefs), emotions and motivation (Hannula, 2011) in the development of mathematical thinking.

3.1 Resources and heuristics

Definitions for resources and heuristics (cf. the knowledge base and problem-solving strategies, Schoenfeld 1992) are adopted from the multidimensional problem-solving framework (Carlson and

Bloom 2005). Resources refer to ‘the conceptual understandings, knowledge, facts, and procedures used during problem solving’, and heuristics describe the specific procedures and approaches in problem solving (e.g. subdividing the problem; *ibid.* p. 50).

3.2 Metacognitive skills and monitoring

Carlson and Bloom (2005) found that when studying mathematicians (expert problem solvers), monitoring best characterised the metacognitive acts within each problem-solving phase. They refer to monitoring as ‘the mental actions involved in reflecting on the effectiveness of the problem-solving process and products’ (*ibid.* p. 48). However, in a research project with comprehensive school students (novice problem solvers), not all, or none of the problem-solving phases involved monitoring (see e.g. Viitala 2015a). These skills develop over time (e.g. Schoenfeld 1992; Stillman and Mevarech 2010). In many cases (see Viitala, *in press*; 2015a; 2015b) it was the small, local decisions that directed the problem-solving behaviour, such as refining a plan as the result of a solution process (see Viitala 2015b; cf. metacognitive behaviours in Carlson and Bloom 2005, p. 51). As the longitudinal study also entails data from comprehensive school, these local metacognitive acts are included in the framework in the form of metacognitive skills (following van der Stel, Veenman, Deelen and Haenen 2010).

3.3 Affect in problem solving

Following the most recent theories around affect, the cognitive, motivational and emotional processes in problem solving are studied through the state and trait aspects of affect (Hannula 2011). Also meta-affect is considered as an important mechanism managing and transforming affect in both frameworks (Carlson and Bloom 2005; Viitala 2015b). Furthermore, mathematical intimacy and mathematical integrity (DeBellis and Goldin 2006), which were an important part of Carlson and Bloom’s study, will be studied as part of problem-solving behaviour.

3.4 View of mathematics

Following Viitala (2015b), the affective trait is studied through students’ view of mathematics. It entails cognitive, emotional and motivational aspects (Hannula 2011) and has four components: mathematics (as science and as a school subject), oneself as a learner and user of mathematics, learning mathematics, and teaching mathematics (Pehkonen 1995). Since the study is about mathematical thinking, the student’s view of his mathematical thinking is also added to the view of mathematics. Additionally, a ‘student profile’ is created for background information (cf. Pehkonen 1995). The profile is a short description of the student that is based on the student’s mathematics grade, motivation to learn mathematics, and the core of his view of himself as a learner of mathematics (following Rösken, Hannula and Pehkonen 2011).

4. Methods

4.1 The participant

At the time of the first round of data collection Alex (pseudonym) was a 15-year-old student in the last (9th) grade of comprehensive school in Finland. He participated in a qualitative research project in which Finnish students’ mathematical thinking was studied. He seemed like an exceptional student among the participants and the results from Alex’s view of mathematics in comprehensive school were published in the MAVI-18 conference proceedings (Viitala 2013). In that article, it is concluded that:

What makes Alex interesting is his high ability to explain his own thinking and the awareness of his own learning. He enjoys doing mathematics but it is not enough to carry the interest outside the classroom. He seems to be very down to earth with his abilities in mathematics and he recognizes that his mastery of mathematics is limited to school mathematics. It seems that it is possible to have highly positive affect in mathematics in school without being that interested in it in everyday life. (p. 80)

After the article was published it was sent to Alex, he commented on it and agreed to a follow-up interview. The second round of data was collected four years after the data collection in comprehensive school. At that time, Alex had finished his studies in upper secondary school and started his university studies. The problem-solving data from comprehensive school and all of the university data are discussed here for the first time.

4.2 Data collection

The data from grade 9, the last year of comprehensive school, was collected in a school setting in three cycles of data collection over the course of three months. The data collection was divided into two parts: problem solving and view of mathematics. First, tasks (problems) were solved in mathematics lessons and discussed in stimulated-recall interviews. In the interviews, Alex also evaluated his confidence in connection to the tasks solved in the lessons and mathematics in general, and solved some more tasks in order to help the researcher get closer to the authentic problem-solving situation and to Alex's thoughts. Second, Alex's view of mathematics and mathematical thinking were discussed. All data was video-recorded and transcribed (around 1 hour from lessons and 3.25 hours from the interviews). More information about the data collection from comprehensive school can be found in earlier publications (see e.g. Viitala 2015a).

The data from university was collected in one interview (around 3.5 hours of transcribed video-recordings). As before, the data was collected in two parts: problem solving and view of mathematics. Some of the tasks were the same as in the first round of data collection and some of them were more closely connected to Alex's future life as an adult (taking a loan, being a medical doctor). In addition to the questions asked in comprehensive school about Alex's view of mathematics, Alex was asked to evaluate the development of his views from comprehensive school to the present time. Also, the role of problem solving in everyday life was added as a separate theme to the discussion.

All of the tasks solved in the study are released PISA-items. PISA-tasks were selected because they are well-tested, they are based on real-life situations (cf. real-life connections in mathematics curriculum in Finland; FNBE 2004), translations are available in different languages (eases the discussion), and most importantly, they are designed for 15-year-olds. The selected PISA-tasks represent various mathematical domains (e.g. algebra, statistics), mathematical content (e.g. change and relationship, uncertainty), mathematical processes (mathematising) and different types of tasks (open- and closed-constructed response items; see task characteristics, e.g. in OECD 2006), and were selected so that all 15-year-old students should be able to solve the 'main tasks' that were introduced in lessons, irrespective of the level of their mathematical performance.

The tasks solved in comprehensive school are from PISA-items *Holiday, School Excursion, Distance* (modified to 3 and 5 km), *Indonesia* (OECD 2006, pp. 77-78, 87, 102, 111, respectively), *Growing up, Carpenter, and Braking* (OECD 2009, pp. 106, 111, 128-129, respectively; Braking also included

questions from a Web page of the Finnish Institute for Educational Research, FIER). The ‘main tasks’ from the lessons are italicised.

As a university student, Alex solved tasks from the following PISA-items which were also solved in comprehensive school: Distance, Indonesia, Growing up, and Braking. Additionally, he solved tasks from PISA-items New Offer (presented in English, researcher helped with the translation; OECD 2013, p. 154) and Say No to Pain (translated by the researcher; OECD 2003, pp. 161-163).

4.3 Data analysis

In the first round of problem solving analysis, the problems were categorised following the problem-solving phases of Carlson and Bloom (2005) in order to structure and time-frame the problem-solving processes. Then, tasks were gone through one-by-one to analyse the resources and heuristics (following Carlson and Bloom 2005), metacognitive decisions (van der Stel et al. 2010), affective states (Hannula 2011) and meta-affect (DeBellis and Goldin 2006) that emerged in the problem-solving processes. Finally, connections between the problem-solving behaviour (state) and view of mathematics (trait; Pehkonen 1995) were investigated together with mathematical intimacy and integrity (DeBellis and Goldin 2006) and the results from the problem-solving processes were combined. The purpose was to analyse the emergent processes, and thus the correctness of the answer was not evaluated.

Alex’s view of mathematics was analysed first by interpreting and describing the issues that he highlighted the most in every category of view of mathematics (Pehkonen 1995) and mathematical thinking. Then, the results were compared with the problem-solving results to see if the explanations (trait) are consistent with the actual problem-solving behaviour (state). Finally, the student profile was constructed (Rösken et al. 2011).

The same data analysis was done separately to both data sets (comprehensive school and university) and compared to see the possible development. The coding in the excerpts was verified by an experienced researcher in mathematics education.

5. Results: Problem solving

5.1 Comprehensive school

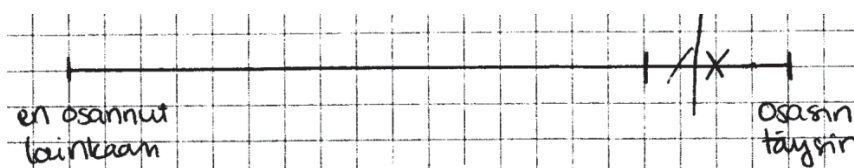
Most of the given tasks in comprehensive school were somehow familiar to Alex. Also, most of them can be referred to as exercises (cf. Schoenfeld 1983). Alex solved the tasks because he was asked to do so. Only the more unfamiliar problems (e.g. Braking) or a problem where his intuition failed (School Excursion) engaged him more. Nevertheless, his mathematical intimacy towards the tasks remained moderate.

While orienting, Alex read the task description, made sense of the given information and organised data. He employed all the attributes listed as important in a study on problem solving (see an example of the use of resources, heuristics, affect and meta-affect in Table 1). Monitoring was evident when he solved problems in the interviews and utilised self-talking, for instance when he confirmed that he had understood the situation (‘So there are many pairs of graphs. Four pairs of graphs. Ok.’), and then directed his attention to the goal of the task (‘So then I have to...’). Alex was confident in facing problems (average of 8 on a scale of 0-10, see an example of an estimation of confidence in Fig. 1) and his mathematical integrity was high.

Table 1

Excerpt on orienting from a stimulated-recall interview (Holiday, task 1; OECD 2006, p. 77)

Excerpt	Behaviour
(1) I quite often start with something simpler...	Heuristics – look for a simpler problem
(2) For me it is much harder to read here [table with distances], I don't see them directly. So, I marked them [distances] directly here [on the map]...	Sense making Affect – feeling of difficulty Meta-affect – affect directing behaviour Organising information
(3) ...the distances can't be seen directly from here [map] because that is not necessarily to scale, that drawing.	Mathematical knowledge
(4) So I marked all the needed measures [on the map] and it was easy to look at them.	Organising information Affect – satisfaction

**Fig. 1**

Alex's estimation of confidence with Holiday, task 1 (OECD 2006, p.77). The line is 10 cm long, scale from 'I couldn't do it at all' (left) to 'I could do it perfectly' (right), and symbols: confidence after reading the task 'shorter' |, while solving the task /, after solving the task X, and confidence in school mathematics 'longer' |.

Alex often started planning by estimating the answer using his conceptual knowledge and heuristics. Monitoring was not evident in the planning phase, for instance he did not evaluate the effectiveness of the selected strategies before actually starting to solve the tasks. Even in the case of re-entering the planning phase, he did not express strong emotional responses (cf. moderate intimacy towards the tasks) but his mathematical integrity pushed him forward in the problem-solving process. An example of the use of different attributes in planning phase is presented in Table 2.

Table 2

Excerpt on planning from a stimulated-recall interview (School Excursion; OECD 2006, p. 87)

Excerpt	Behaviour
(1) First I looked, thinking quickly, that C [is the cheapest]...	Conjecture
(2) It felt like the best solution so I started [with C]...	Affect – confidence
(3) I did calculate them all [prices to all companies], what they are. But that's why I started with C.	Heuristics – calculating prices for all companies
(4) Maybe the hardest thing, without calculating, is to see the values for kilometres.	Metacognitive skills: Estimating answers Affect – feeling of difficulty
(5) Even though it can be quickly calculated, it feels like 0.75 is not that much less than 1.02. ...	Mathematical knowledge

Alex executed his plans successfully, using his resources and heuristics. He actively monitored his calculations and solutions, for instance by estimating the correctness and the effectiveness of an answer, and making sure he had considered all options for reaching the answer. If the task did not require exact values as a result, he aimed for understanding the situation instead of doing precise

calculations and markings on paper (e.g. calculating with coefficient 1 instead of 1.02; requires conceptual understanding). When solving a more complicated task, he was also self-questioning and reflecting on the knowledge (see Table 3). His confidence changed only slightly towards more positive from the orienting phase, the average remaining close to 8 (on a scale of 0-10, see an example in Fig. 1).

Table 3

Excerpt on executing from an interview (Braking, task 50; FIER)

Excerpt	Behaviour
(1) I looked first for the overall distance. In my head, I looked how much it... for every 10 kilometres... how steeply the distance grows.	Sense making Mathematical knowledge Metacognitive skills – sub-goaling
(2) First, there are, roughly calculated, about 7 metres. Second, roughly calculated, 9. Third, about 10. Fourth, about 12, 13, 15, 16, 19, no 17.	Mathematical knowledge Organising information Metacognitive skills – recalculating
(3) Then it doesn't grow much at that point. 17.	Sense making
(4) There, does it shorten there already? Hmm. (long pause) There is the biggest leap. No, it isn't. ... Isn't it there... And then next is 23, it is quite big there. Next is also 23, right? No, 22. Have I miscalculated?	Reflecting information Self-questioning Metacognitive skills – recalculating
(5) However, not very big. ...	Mathematical knowledge Affect – confidence to move on

Alex checked the correctness of an answer only after all of the given tasks were solved. Sometimes checking engaged him, for instance, in verifying prior estimations by calculating and rewriting written answers for aesthetical reasons. However, during execution he actively reflected how reasonable the results were, which sometimes resulted in returning to the planning phase. After solving a task, Alex felt a bit more confident than before (average of 9 on a scale 0-10, see an example in Fig. 1).

5.2 University: The development

Alex's problem solving in university is very much similar to the way he solved problems in comprehensive school. The most significant change happened with mathematical knowledge and affect (mathematical intimacy). The following descriptions focus on the changes that affect problem solving.

As a university student, Alex uses more time for orienting and reading the task descriptions than in comprehensive school. For instance, while earlier he just looked at the titles in a table, now he tests if the table actually shows what it is supposed to show (Indonesia). This refers to higher mathematical intimacy with the tasks than in comprehensive school, as well as to high mathematical integrity. He does not always remember the tasks from comprehensive school and expresses similar confidence as before. Planning has not changed from comprehensive school to university.

During the execution phase, the most evident change is related to mathematical knowledge. Alex approaches tasks similarly as before (e.g. calculating percentages with trial-and-error in Growing up), but the growth of mathematical knowledge influences the way he actually solves the task or analyses the data. The most representative example is from Braking (task 50; FIER). While in comprehensive school Alex talked about 'constant growing' or 'curves going up or down' and based his decisions purely on calculations, in university Alex talks about direct proportionality and refers to properties of a line graph. He reads and analyses the given graphs (e.g. 'This could be some kind of x^2 -kind of a

graph... these don't represent the same kind of change...') and is affected by his knowledge from school physics. Alex also shows growth in mathematical knowledge in the area of checking: he checks his work by calculating the task in a different way than how he solved it (Growing up).

Even though problem solving does not show any changes related to heuristics and monitoring, it does not mean that no development has taken place. For comparison reasons, the tasks solved at both levels were PISA-tasks. More challenging tasks might have resulted in different results.

6. Results: View of mathematics

Since Alex's view of mathematics in comprehensive school is already discussed in an earlier publication (Viitala 2013), the analysis below emphasises the developmental aspects of his view of mathematics from comprehensive school to university.

6.1 Student profile

Even though the student profile was created last in the analysis, it is reported first for its explanatory value (background information; cf. Pehkonen 1995). It also covers the category 'view of oneself as a learner and user of mathematics' (ibid.).

In the earlier publication, Alex was described as a student with a positive view of mathematics:

He enjoys learning mathematics and is motivated to study it. ... He is very aware of his own mathematical thinking ... [and] learning process ... [and] seems to have a clear and organized (mathematical) thinking and net of knowledge. (Viitala 2013, p. 79)

While mathematics was one of Alex's favourite subjects in comprehensive school, during upper secondary school his interest moved towards physics and chemistry. This had an impact on Alex's motivation to learn mathematics for the sake of mathematics. Additionally, Alex's confidence decreased and he needed more time and effort to learn mathematics than in comprehensive school. Even so, Alex still considered mathematics as important and interesting, enjoyed learning mathematics, believed in his abilities and success in mathematics, was confident (8 on a scale of 0-10), and got excellent grades (average of 9.4 from all mathematics courses in upper secondary school on a scale of 4-10). Thus, following Rösken et al. (2011) and Alex's own description:

Alex is a successful student who thinks that mathematics is exciting and challenging in an interesting way. He is self-motivated and diligent in learning mathematics but motivated mostly by a good grade. He sees mathematics as a tool and an inseparable part of physics and chemistry.

6.2 Mathematics

Alex's view of mathematics has changed significantly from comprehensive school. The same kind of development can be seen in Alex's view of his mathematical knowledge, how he sees mathematics as a science, and what the role of mathematics in his everyday life is.

First, in comprehensive school, Alex explained that 'rather than changed, mathematics has expanded during the 9 years in school' (Viitala 2013, p. 75). Before university, this view of a growing *plane* has changed into a view of a growing *space*. Mathematics is not numbers and formulas anymore; it is expanded by new dimensions of mathematics (e.g. geometrical thinking). Secondly, after comprehensive school, Alex became more aware of the tool value of mathematics in his life. However, in addition of seeing mathematics as 'explaining different problems or natural phenomena,

or such, with the assistance of calculations' (Viitala 2013, p. 75), he also sees mathematics as a way of thinking (e.g. logical thinking in everyday situations). Thirdly, in university, Alex sees mathematics everywhere in real life giving examples (that do not involve calculations), for instance, from architecture, arts, philosophy, traffic control and medicine. As an example from his future occupation:

In forming a medical diagnosis, I see a mathematical problem-solving model in which [diagnoses] are excluded, finally reaching the correct answer that is not a number in this case but a diagnosis.

Additionally, a significant change has happened concerning Alex's overall interest in mathematics. Whereas in comprehensive school Alex was described to have a 'highly positive affect in mathematics in school without being that interested in it in everyday life' (Viitala 2013, p. 80), during the following four years this seems to have changed to the other way around. While Alex's affect in school mathematics seems to have somewhat decreased, the role of mathematics in real life has increased significantly: Alex refers to mathematics as 'a buttress' that he can trust and lean on in many things. He feels that mathematics brings confidence and order to a hectic world. He also thinks that mathematical thinking is at the core of all learning (e.g. through logical thinking). Mathematics is not viewed through school subject anymore, it is much more:

It is exciting how mathematics can be used to describe the world, with numbers invented by humans, and still it feels like there is some truth behind it. Somehow, nature is based on that even though it is invented by humans. It is exciting!

6.3 Learning and teaching mathematics

Key aspects of Alex's learning in comprehensive school were described earlier as follows (Viitala 2013, pp. 76-77):

Alex is very aware of his learning in mathematics and he can explain it in two levels: the overall process of learning and connecting new knowledge to prior knowledge. ... He has a lot of self-confidence and trusts his own reasoning more than his calculations. ... Alex is motivated to learn mathematics and he aims for understanding. He also recognizes that he is responsible for his own learning.

In many ways, this description of Alex's experience of learning mathematics (Viitala 2013, pp. 76-78) still described his learning in upper secondary school. However, learning mathematics was not always that easy anymore, especially if Alex had problems connecting new knowledge to the old (e.g. with vectors). His self-confidence and motivation in mathematics somewhat decreased. Nonetheless, he still remained 'quite confident', 'knowing what I know and what I don't know'.

The role of teaching also decreased after comprehensive school from central to not important at all (university). Alex explains how he got more initiative in upper secondary school and started to study at home. He emphasises the role of inner motivation to learn mathematics, and when it comes to teaching, he stresses the importance of motivating students to learn on their own. Even though he acknowledges inequality, he thinks that teaching should be different for students from different achievement levels. He explains how it would improve competitiveness in a world that aims for efficiency.

6.4 Problem solving

Problem solving was first discussed as a separate part of the view of mathematics in university. However, the results from Alex's view of mathematics also supported his problem-solving results in comprehensive school. For example, Alex explained how he solves more complicated tasks in pieces, if possible (heuristics, cf. (1) in Table 1), estimates the magnitude of an answer before starting to solve a task (cf. (1) in Table 2) which also brings confidence in problem solving (cf. (2) in Table 2), solves all tasks in a test before checking the answers (as shown with problem-solving behaviour), trusts his own deduction more than calculations (cf. not always calculating with exact values) and thinks mathematics can be beautiful on paper (cf. aesthetics in a written answer).

In university Alex describes himself as a good problem solver (at the level of upper secondary school). He is persistent, and not succeeding in solving a problem annoys him. He tries as long as it takes to reach an answer. This supports the findings from problem solving concerning mathematical intimacy and integrity. He also explains how, in connection to university entrance examinations, he has started to think tasks through before starting to solve them to prevent careless errors, which explains the increased time used for orienting in university. Alex thinks that the ability to solve problems is essential in a world where reasonable results should be reached quickly.

7. Trustworthiness: Comments from Alex

When the first article (Viitala 2013) was sent to Alex, he had just finished three years of upper secondary school studies with high marks and had also had his first university entrance examinations. In a personal email, Alex commented the article as follows:

Your article about 'Alex' describes me and my relationship with mathematics and learning well. During the years in upper secondary school, I have recognised the nature of my mathematical thinking even more, and you have discovered that already from a boy in lower secondary school. ... Nowadays, I am very interested in natural sciences, especially physics and chemistry because they are exact mathematical sciences. Also, my interest in mathematics itself has increased, but it is still most importantly a tool in other sciences and life in general. ...

One year later, Alex comments on the results presented in this article in a personal email as follows:

In the article, you have succeeded in condensing our long discussion well. ... The quotes especially, described my thinking well. I think that the quote at the end of chapter 6.2 best synthesises my abstract perception of mathematics. It synthesises my interest in deeper mathematics (mathematics outside everyday mathematics). The quote also shows why mathematics creates positive or sometimes confusing feelings. ...

The development of my mathematical thinking in four years is very similar as I have understood it. ... [The description of the development of orienting] is completely true and maybe the only aspect of my problem solving abilities I have consciously developed through the years... In the future, in problem solving (probably more in diagnosing patients than with numbers), [planning] would be good to develop.

8. Summary and discussion

In 2005, Carlson and Bloom studied mathematicians' problem-solving behaviour and offered a multidimensional problem-solving framework for analysing mathematical behaviour that also

characterised 'how resources, affect, heuristics, and monitoring influence the solutions path of the solver' (p. 69). Ten years later, Viitala (2015b) studied comprehensive school students' mathematical thinking through problem solving and view of mathematics and showed how the same attributes can affect the development of mathematical thinking differently with different students. The framework for the study discussed in this article was formed based on theories from these two studies. However, the construction of theory on affect has changed since Carlson and Bloom's study and affect is considered as a more dynamic construction than before (see e.g. Hannula 2011; 2012; Pepin and Rösken-Winter 2015). This development also contributed to obtaining more diversity and clarity concerning the use of theory in the new framework.

The case of Alex was analysed using the new framework. In problem solving, Alex's development was best characterised by the growth of mathematical knowledge. The role of well-connected conceptual knowledge in the conjecture cycle and in problem solving in general was also highlighted by Carlson and Bloom (2005). In comprehensive school, the conjecture cycle was not visible even though Alex did estimate answers before starting to solve tasks. However, after upper secondary school Alex started to think tasks through as thoroughly as possible to avoid careless errors. This activity indicates a more proper use of the conjecture cycle as Carlson and Bloom define it, thus moving Alex closer to the activities of experienced problem-solvers.

The most crucial change in the view of mathematics that influences Alex's mathematical thinking concerns his view of mathematics as a school subject and mathematics as a science. In comprehensive school Alex was interested in school mathematics but the interest did not exceed the school boundaries. During upper secondary school Alex's motivation to study school mathematics decreased. He became more aware of the role of mathematics in his life (a tool) and he became more interested in mathematics in real life. Furthermore, a development of more general study skills opened new perspectives in seeing mathematics as ways of thinking and learning and Alex seemed to become excited to use mathematical thinking more in his everyday life (e.g. reading a book called *Logicomix*).

The change in Alex's view of mathematics raises the question of the goals in school mathematics. International assessments show how Finnish students are successful in problem solving and applying mathematics (PISA) but perform modestly in relation to a more conventional curriculum material (TIMSS; see e.g. Andrews, Ryve, Hemmi and Sayers 2014). Finnish mathematicians have criticised the curriculum development that is moving from exact definitions and proofs to a more descriptive mathematics curriculum where, for instance, geometry is neglected (e.g. Martio 2009; cf. geometrical thinking affecting Alex's view of mathematics as a science in upper secondary school). Also, the most recent national and international assessments on mathematics learning outcomes at the end of comprehensive school in Finland support the interpretation that mathematical knowledge and skills are declining (Hirvonen 2012; Rautopuro 2013; Välijärvi 2014). In the upcoming Finnish mathematics curriculum (FNBE 2015), the development of mathematical thinking and real-life problem solving are emphasised even more than in the current curriculum (FNBE 2004). Thus, the question for educational policymakers is: what mathematics do we want our students to learn in school, and for what purpose?

9. Concluding remarks

This study contributes new insight into the development of mathematical thinking by offering a framework with which problem solving and view of mathematics can be studied at different educational levels through the four attributes (resources, heuristics, metacognition and affect) that influence mathematical thinking. The article also reports on the development of one student's mathematical thinking from comprehensive school to university through the use of this framework.

Based on the present study, the framework seems to be an effective way to analyse the development of mathematical thinking at different educational levels. In addition to longitudinal studies, the framework can also be used for cross-sectional studies on mathematical thinking (cf. Carlson and Bloom 2005; Viitala 2015b; Viitala, in press). As shown in the article, however, the results are highly dependent on the tasks used, and thus tasks should be given much consideration, especially when studying the development of mathematical thinking. I hope that the framework will be useful for future studies on the development of mathematical thinking at different educational levels.

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Article 7

A case study on Finnish pupils' mathematical thinking: Problem solving and view of mathematics

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In this article, the mathematical thinking of four Finnish pupils is reported using two temporally different data sets: problem-solving processes and view of mathematics. While the pupils seem similar on the surface level (high achievers, successful problem solvers, enjoy mathematics, motivated to learn mathematics), a closer look at their problem-solving processes and view of mathematics reveal very different strengths and weaknesses in their mathematical thinking. Most of the similarities in this study were found in individual pupils' problem-solving processes and view of mathematics.

Keywords: Problem solving, view of mathematics, affect, metacognition, meta-affect.

Introduction

Developing mathematical thinking is one of the key tasks for mathematics instruction in the Finnish curriculum (FNBE, 2014, 2004). And indeed, Finnish pupils have succeeded well in international studies that assess pupils' mathematical thinking (PISA and TIMSS; see e.g. OECD, 2014; Mullis, Martin, Foy, & Arora, 2012). However, the most recent national and international studies show that the mathematics performance of Finnish pupils is descending (e.g. Välijärvi, 2014; Rautopuro, 2013). Additionally to the alarming trend in mathematics performance, we know very little about Finnish comprehensive school pupils' mathematical thinking that go *beyond* paper tests. Thus, a quantitative research study was conducted with the aim of describing what characterises Finnish 15-year-old pupils' mathematical thinking.

On the way to describe what characterises Finnish pupils' mathematical thinking, the study reported in this article examines four high-achieving Finnish pupils' mathematical thinking through the intertwined relationships of problem-solving processes and view of mathematics. While some of the results of individual pupils' mathematical thinking have been discussed in previous publications (Viitala, 2013; 2015; 2016a), the purpose of this paper is to bring the results together, and answer *what similarities and differences related to mathematical thinking can be found between these pupils*. With this question, we can reveal some of the possible trends in skills and competences that the Finnish high-achieving pupils might have in their mathematical thinking.

Theoretical framework

Developing pupils' mathematical thinking is in the heart of mathematics education, also according to the Finnish curriculum (FNBE, 2014). While research in mathematics education does not seem to have a common understanding of the meaning of mathematical thinking, Schoenfeld (1992) recognised five aspects that are important in a study on mathematical thinking: the knowledge base, problem-solving strategies, monitoring and control, beliefs and affects, and practices. Similar findings have also been found in connection to literature on problem-solving performance (Lester 1994), and are also listed as part of final-assessment criteria in the Finnish curriculum (see FNBE 2014, pp. 433-434).

Similarly as the most recent theories on affect, mathematical thinking can be viewed through two temporally different aspects: state and trait (cf. Hannula, 2011; 2012). On one hand, mathematical thinking is always situational (state). Following Schoenfeld's (1992) categorisation, it is influenced by the pupils' knowledge base and heuristics, and guided by their metacognitive skills, affects and classroom practices. In this study, mathematical thinking is studied through problem-solving processes. In other words, 'pupils' activities, actions and explanations during problem solving are interpreted as visible signs or expressions of their mathematical thinking' (Viitala, 2015, p. 138).

Pupils' problem-solving behaviour is influenced by pupils' metacognition, affect and meta-affect that occur in a problem-solving situation. The successful application of problem-solving activities at the correct moment is a result of metacognitive skilfulness (e.g. van der Stel, Veenman, Deelen, & Haenen, 2010), affect influence the problem-solving situation for instance through the feeling of confidence, and meta-affect transforms individuals' emotional feelings (DeBellis & Goldin, 2006) and directs problem solving behaviour (Carlson & Bloom, 2005).

On the other hand, problem-solving situations can show patterns of thought that can be interpreted as signs of more stable ways of thinking. Some of these patterns can also be revealed through pupils' view of mathematics (see e.g. Viitala, 2016a). View of mathematics draws from psychological theories. It is a mixture of cognitive, motivational and emotional processes that include for instance beliefs, attitudes, values, feelings and motivation (Hannula, 2011; 2012). In this study, view of mathematics is studied through four components: mathematics (as science and as a school subject), oneself as a learner and user of mathematics, learning mathematics, and teaching mathematics (Pehkonen, 1995, cf. Op't Eynde, de Corte, & Verschaffel, 2002).

Methods

Data collection

At the time of data collection, the four pupils (Alex, Daniel, Emma and Nora) were 15 years old and in their 9th and final year of compulsory school in Finland. Additionally, they were all high achievers (mathematics grades between 9 and 10 on a whole number scale of 4 to 10).

The data was collected in three cycles over the course of three months. In each cycle, one mathematical task was solved in an ordinary classroom situation as a 'main task'. The pupils solved the tasks individually but they were allowed to talk about the tasks with a friend or ask for help from the teacher. In each cycle, the pupils were video recorded while they solved the task(s) in class and their solutions on paper were collected. Below, there is an example of a main task (School Excursion, OECD, 2006, p. 87).

A school class wants to rent a coach for an excursion, and three companies are contacted for information about prices.

Company A charges an initial rate of 375 zed plus 0.5 zed per kilometre driven. Company B charges an initial rate of 250 zed plus 0.75 zed per kilometre driven. Company C charges a flat rate of 350 zed up to 200 kilometres, plus 1.02 zed per kilometre beyond 200 km.

Which company should the class choose, if the excursion involves a total travel distance of somewhere between 400 and 600 km?

In each cycle, the pupils were interviewed individually. The interviews took place either on the same day, or on the next day after solving the task in the classroom. The interviews contained two parts. The first part concentrated on affective traits and treated the following themes: pupil's background, mathematical thinking, and pupil's view of mathematics (following the categorization of Pehkonen, 1995; see example questions in Table 1, Viitala, 2016a, p. 1295). This part of the interview was semi-structured and focused (Kvale & Brinkmann, 2009).

Theme	Example questions
Background	Tell me about your family.
Mathematical thinking	What does mathematical thinking mean? / How do you recognise it?
Mathematics	What is mathematics as a science? / Does it exist outside of school? (How? Where?)
Oneself and mathematics	Is mathematics important to you? / Does it help you think logically? (How?)
Learning mathematics	How do you learn mathematics? / Is it most important to get a correct answer?
Teaching mathematics	Does teaching matter to your learning? (How?) / What is good teaching?

Table 1: Interview themes and example questions.

The second part of the interview was about problem solving. The classroom data was used as stimuli when the pupil's problem-solving process was discussed. The pupils were asked to explain their thinking and actions during the problem-solving situation and additional questions were asked (e.g. what are you thinking now? Why are you doing so? What did you feel when you read the task? Did you think about your own thinking when solving the task?).

Finally, in each interview, the pupils were asked to assess their confidence before, during and after solving the problem, as well as their confidence in school mathematics using a 10 cm line segment (scale from 'I couldn't do it at all' to 'I could do it perfectly'). All interviews were video recorded.

Analysis

Following the state and trait aspects of the study, the analysis was divided into two sections: problem solving and view of mathematics. The problem-solving processes were analysed first by going through the problem-solving phases introduced by Carlson and Bloom (2005): orienting, planning, executing and checking (cf. Polya, 1957). Then the results on problem-solving behaviour were complemented with metacognitive activities (orientation, planning, evaluating and elaboration van der Stel et al., 2010), affect (state and trait, as well as cognition, emotion, motivation; Hannula, 2011; 2012) and meta-affect (DeBellis & Goldin, 2006) emerging in problem-solving processes. Finally, the pupils' confidence to solve the problems was analysed.

The first analysis of the pupils' view of mathematics followed the themes of data collection (Pehkonen, 1995). After condensing the results, a pupil profile was created to be used as background information about the pupil. Pupil profile is a short description of the pupil that is based on the pupil's mathematics grade, motivation to learn mathematics, and the core of his view of himself as a learner of mathematics (ability, success, difficulty of mathematics, and enjoyment of mathematics, following Rösken, Hannula, & Pehkonen, 2011).

In the end, the results of problem solving and view of mathematics were compared to see if there are similarities in pupil's problem-solving skills (state) and competences found through pupil's view of mathematics (trait). More details of the methods used in the study are reported for instance in Viitala (in press).

Results

On a surface level, Alex, Daniel, Emma and Nora seem quite similar: they are all high achievers in mathematics, they enjoy mathematics, and they are motivated to learn mathematics (see excerpts in Table 2). They are also successful problem solvers, that is, they could solve all the problems given to them in the study and justify their answers and solutions. However, a deeper look at their problem solving and view of mathematics introduce four pupils with a very different skills and competences. In the following, the key results of each pupil will be introduced individually.

Alex is very fluent and thorough mathematics learner and problem solver. He can move naturally between different phases of problem solving. He is aware of his own thinking and fluent in explaining and justifying his cognitive and metacognitive actions in problem solving. Similarly, when explaining his learning of mathematics, he says he is actively seeking for connections between new knowledge and prior knowledge, and he is able to spontaneously give examples of this behaviour. He says he trusts his own thinking more than his calculations, and shows to be able to direct his behaviour according to his affects in problem solving. He is confident in school mathematics but in the interviews, he constantly compares his abilities to mathematics as a science and recognises that there is much more than school mathematics (more results in Viitala, 2013; 2016b).

Whereas Alex seems to be very fluent in every aspect of mathematical thinking studied in this research project, from a similar starting point, Daniel shows somewhat different strengths in mathematics. Unlike any of the three other pupils, he is extremely confident in mathematics. He says that mathematics is easy for him, and he shows to be very aware of his success in mathematics. His confidence seems to guide also his problem-solving processes. He is able to move fluently back and forth between problem-solving phases and is skilful in performing metacognitive acts. However, even though (or because of) learning mathematics and solving problems are easy for him, he cannot explain the processes he goes through in or for learning, and he has problems in explaining his problem-solving actions after the problem-solving situation. An illustrative example of this situation is Daniel's explanation about how he learns mathematics: pieces just click together or things become familiar (more results in Viitala, in press).

Similarly as in Daniel's case, also Emma's learning of mathematics and problem solving are strongly influenced by her confidence in mathematics, or more precisely, her lack of confidence. Because of the uncertainty in mathematics, for Emma, learning takes time and effort. She says she learns every topic as a separate entity, and she is able explain the steps that are needed for her to learn a new thing. Similarly, she uses a considerable amount of time for orienting and planning in problem solving. After understanding the problem and the given data, she is able to follow her plan through and check her solution. It seems that Emma's uncertainty in mathematics makes her work harder, and through hard work, she succeeds in mathematics. Moreover, she says that succeeding in mathematics and understanding it, makes it worthwhile studying. On the other hand, affect can also be an obstacle in her problem solving, since she does not seem to have efficient tools to overcome the feeling of getting stuck (more results in Viitala, 2015; 2016a).

Also for Nora, learning mathematics takes time and effort but after learning something, applying is easy. She says that she is quite confident in mathematics and likes learning mathematics very much.

She is capable in explaining her thinking and problem solving, and connecting mathematics to her own life. She also has a diverse view of mathematics as a science. In problem solving, she is flexible in directing her actions based on the affective states occurring in problem-solving situations. She is also fluent in moving between orienting, planning and executing in problem solving. However, given the choices she had made while planning, she is happy with the first answer she gets, and does not check her results (more results in Viitala, 2015).

	Ability and success	Difficulty of mathematics	Enjoyment of mathematics	Motivation to learn mathematics
Alex	Confident in math; deserves the high grade: knows school math quite thoroughly	Learning ‘a separate thing’ is easy, connecting it to ‘other things’ might take time	Learning math is fun and interesting; routine learning is boring	Good grade and future studies, also understanding the issue at hand
Daniel	Very confident in math; can do math well; deserves the high grade (active learner, succeeds in tests)	Learning math is easy and it does not take much time or effort	Math is enjoyable, even fun	Math is needed through life; the most important school subject
Emma	Not confident in math; could not get a better grade in math	Learning math takes time and effort	Learning math is irritating and tiring; succeeding and understanding is fun	Wants to succeed in mathematics and be proud of herself; future studies
Nora	Quite confident in math; not perfect in math but deserves the high grade in school math (active learner, succeeds in tests)	Math can be easy or difficult, more on the easy side; learning takes time and effort, applying after that does not	Learning math is interesting, likes math very much	Good grade; wants to learn math

Table 2: Examples of pupils’ own statements about their view of mathematics (cf. pupil profile).

Some reflections of the results

In addition to forming descriptions of pupils’ mathematical thinking, and showing pupils’ strengths, the study also revealed issues that pupils could work with in order to develop their mathematical thinking. For instance, even though Alex was fluent in problem solving and school mathematics, he did not relate the problems to real life and his view of mathematics outside school was quite limited (see Viitala, 2013, 2016b). Recognising mathematics more in his own life could enrich Alex’s view of mathematics, and through that, also his understanding of school mathematics might develop. Daniel, on the other hand, had problems explaining his thinking after the problem-solving situation and had similar problems with explaining his mathematics learning (see Viitala, in press). Problem solving and learning mathematics might be easy for Daniel in compulsory school, but what happens if (when) the situation changes? Becoming aware of his own learning and problem-solving processes could help him cope in new situations and develop his metacognitive skills not only in mathematics but also in other school subjects.

Emma's weak point was her uncertainty which she had turned into success in problem solving and learning of mathematics. She had overcome some of the uncertainty with the support of her family (see Viitala, 2016a). However, because she was not confident in mathematics, she learnt every topic in mathematics as its own entity, and did not connect it to prior knowledge. This might also hinder her learning. Hence, supporting Emma emotionally could open doors to more thorough learning and understanding of mathematics. Finally, Nora's results were not always correct, and both her activities and explanations showed that she does not evaluate her problem-solving process or check her results (see Viitala, 2015). Supporting her to look back, and perhaps exposing her more to, for instance, open problems, might help her to become more reflective user and learner of mathematics.

Summary and discussion

The purpose of the paper was to answer the question what similarities and differences related to mathematical thinking can be found between the four Finnish high-achieving pupils. Mathematical thinking was studied through two temporally different data sets: problem-solving processes (state) and view of mathematics (trait). The results showed that the similarities between the pupils were found to be mainly on a surface level: all the pupils liked mathematics, were motivated to learn it, enjoyed doing mathematics and were successful problem solvers. However, after a deeper look into their problem-solving processes and view of mathematics, the study revealed a great deal of differences between the pupils, and showed different competences: Alex is a very conscious thinker and learner of mathematics, and excellent in justifying his thinking and actions in mathematics. Daniel is extremely confident and metacognitive skills are prominent in his problem solving. Emma is an unsure but very thorough problem solver and learner of mathematics. Nora is fluent in expressing her thoughts and connecting mathematics to real life.

In addition to the strengths found in these four pupils, the framework also revealed some of their weaknesses. The strengths, together with the weaknesses can be used to support individual pupils' development in mathematics. For instance, Alex seemed to see mathematics only as a tool to solve something and his view of mathematics outside school was quite limited (see Viitala, 2013, 2016b). This knowledge can be used to develop pupil's mathematical thinking. Four years after the data collection of this research project, I met Alex again. At this point, Alex was as a university student. He explained that only after realising the tool value that mathematics had for him, and learning that mathematics is not just calculations but also ways of thinking, he began to see mathematics everywhere in his real life, and he began to use his mathematical thinking more creatively (see Viitala, 2016b).

All in all, the results showed that even though the pupils seem similar on the surface level, on a closer look, they have very different skills and competences in mathematics. This is an indication that the framework allows different pupils to show different strengths, and also different weaknesses in problem solving and learning of mathematics. Hence, the framework could assist also teachers to pay attention to the aspects that pupils might need help with in developing their mathematical thinking, which in turn can help the pupils to recognise the knowledge, skills and affects that might need further developing (cf. FNBE, 2014, p. 377; Viitala, in press; see also Viitala, 2015). An example of how teachers can use this framework to support their teaching is presented in Viitala (in press).

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