OSCILLATION CRITERIA FOR EVEN-ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. We study oscillatory behavior of solutions to a class of even-order neutral differential equations relating oscillation of higher-order equations to that of a pair of associated first-order delay differential equations. As illustrated with two examples in the final part of the paper, our criteria improve a number of related results reported in the literature.
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1. Introduction

In this paper, we are concerned with the oscillation of solutions to a class of even-order neutral differential equations

$$z^{(n)}(t) + q(t)x(\sigma(t)) = 0, \tag{1.1}$$

where $z(t) = x(t) + p(t)x(\tau(t))$, $t \geq t_0 > 0$ and $n \geq 4$ is an even natural number. We suppose that

(H_1) $p,\ q \in \mathrm{C}([t_0, \infty), \mathbb{R})$, $p(t) > 0$, $q(t) \geq 0$, and $q(t)$ is not identically zero for large t;

(H_2) $\tau,\ \sigma \in \mathrm{C}([t_0, \infty), \mathbb{R})$, $\tau(t) \leq t$, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$.

Let $t_* = \min_{t\in[t_0,\infty)}\{\tau(t), \sigma(t)\}$. By a solution of (1.1), we mean a function $x \in \mathrm{C}([t_*, \infty), \mathbb{R})$ such that $z \in \mathrm{C}^n([t_0, \infty), \mathbb{R})$ and $x(t)$ satisfies (1.1) on $[t_0, \infty)$. In what follows, we suppose that solutions of (1.1) exist and can be continued indefinitely to the right. Furthermore, we consider only solutions $x(t)$ of (1.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq t_0$ and we tacitly assume that (1.1) possesses such solutions. As customary, a solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[t_x, \infty)$ for some $t_x \geq t_0$; otherwise, we call it non-oscillatory. Equation (1.1) is termed oscillatory if all its solutions are oscillatory.

Oscillatory and non-oscillatory behavior of solutions to different classes of differential and functional differential equations always attracted interest of researchers; see, for instance, [1-5, 7-15] and the references cited therein. One of the main reasons for this lies in the fact that differential and functional differential equations arise in many applied problems in natural sciences and engineering, cf. Hale [6]. In what follows, we briefly comment on a number of closely related results which motivated our study. Several interesting oscillation criteria for equation (1.1) have been reported in the recent papers [1, 4, 5, 11, 14] under the assumptions that

$$0 \leq p(t) \leq p_0 < \infty, \qquad \tau \circ \sigma = \sigma \circ \tau, \tag{1.2}$$
$$\tau \in \mathrm{C}^1([t_0, \infty), \mathbb{R}), \qquad \text{and} \qquad \tau'(t) \geq \tau_0 > 0.$$

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For the convenience of the reader, we present below some related results. We use notation τ^{-1} and σ^{-1} for the inverse functions of τ and σ along with

$$f_+(t) = \max\{0, f(t)\},$$

$$Q(t) = \min\{q(t), q(\tau(t))\}, \qquad \tilde{Q}(t) = \min\{q(\sigma^{-1}(t)), q(\sigma^{-1}(\tau(t)))\},$$

$$I_1(t) = t - t_1, \qquad t_1 \gg t_0, \qquad I_i(t) = \int_{t_1}^{t} I_{i-1}(s)\mathrm{d}s, \qquad 2 \le i \le n-1,$$

$$J_2^*(t) = \int_t^\infty \tau'(u) \int_u^\infty Q(s)\mathrm{d}s\mathrm{d}u, \qquad J_i^*(t) = \int_t^\infty \tau'(s) J_{i-1}^*(s)\mathrm{d}s, \qquad 3 \le i \le n,$$

$$Q_{n-1}^*(t) = Q(t) I_{n-1}(\sigma(t)),$$

$$Q_i^*(t) = \frac{1}{i!}\left(\tau^{-1}(\sigma(t)) - t_1\right)^i J_{n-1-i}^*(\tau^{-1}(t)), \qquad 1 \le i \le n-3,$$

$$p_*(t) = \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{(\tau^{-1}(\tau^{-1}(t)))^{n-1}}{(\tau^{-1}(t))^{n-1} p(\tau^{-1}(\tau^{-1}(t)))}\right),$$

and

$$p^*(t) = \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{\tau^{-1}(\tau^{-1}(t))}{\tau^{-1}(t) p(\tau^{-1}(\tau^{-1}(t)))}\right).$$

Theorem 1.1 ([1, Theorem 2.2]). *Let conditions (H_1) and (H_2) be satisfied. Suppose further that $\sigma(t) \le \tau(t)$,*

(1.3)
$$\tau \in \mathrm{C}^1([t_0, \infty), \mathbb{R}), \qquad \tau'(t) > 0, \qquad \text{and} \qquad 1 - \frac{(\tau^{-1}(\tau^{-1}(t)))^{n-1}}{(\tau^{-1}(t))^{n-1} p(\tau^{-1}(\tau^{-1}(t)))} \ge 0.$$

If there exist two functions $\rho, \delta \in \mathrm{C}^1([t_0, \infty), (0, \infty))$ such that, for some $\lambda_0 \in (0, 1)$,

$$\int^\infty \left[\rho(t) q(t) p_*(\sigma(t)) \frac{(\tau^{-1}(\sigma(t)))^{n-1}}{t^{n-1}} - \frac{(n-2)!}{4\lambda_0} \frac{(\rho'_+(t))^2}{t^{n-2}\rho(t)}\right] \mathrm{d}t = \infty$$

and

$$\int^\infty \left[\frac{\delta(t)}{(n-3)!} \int_t^\infty (\eta - t)^{n-3} q(\eta) p^*(\sigma(\eta)) \frac{\tau^{-1}(\sigma(\eta))}{\eta} \mathrm{d}\eta - \frac{(\delta'_+(t))^2}{4\delta(t)}\right] \mathrm{d}t = \infty,$$

then equation (1.1) is oscillatory.

Theorem 1.2 ([5, Corollary 2.8] and [14, Corollary 2.14]). *Assume that $0 \le p(t) \le p_0 < \infty$, $\tau \in \mathrm{C}^1([t_0, \infty), \mathbb{R})$, $\tau'(t) \ge \tau_0 > 0$, and assumptions (H_1) and (H_2) hold. If σ is invertible, $\sigma^{-1} \in \mathrm{C}^1([t_0, \infty), \mathbb{R})$, $(\sigma^{-1}(t))' \ge \sigma_0 > 0$, $\sigma(t) < \tau(t)$, and*

$$\frac{\tau_0 \sigma_0}{(\tau_0 + p_0)(n-1)!} \liminf_{t \to \infty} \int_{\tau^{-1}(\sigma(t))}^{t} \tilde{Q}(s) s^{n-1} \mathrm{d}s > \frac{1}{\mathrm{e}},$$

then equation (1.1) is oscillatory.

Theorem 1.3 ([4, Corollary 2]). *Suppose that $J_n^*(t_0) = \infty$ and assumptions (H_1), (H_2), and (1.2) hold. If $\sigma(t) < \tau(t)$ and*

$$\frac{\tau_0}{\tau_0 + p_0} \liminf_{t \to \infty} \int_{\tau^{-1}(\sigma(t))}^{t} Q_i^*(s) \mathrm{d}s > \frac{1}{\mathrm{e}}$$

for $i = 1, 3, \ldots, n-1$, then equation (1.1) is oscillatory.

The objective of this paper is to establish for equation (1.1) new oscillation criteria that improve Theorems 1.1-1.3. In the sequel, all functional inequalities are supposed to hold for all t large enough. Without loss of generality, we deal only with positive solutions of (1.1) since, under our assumptions, if $x(t)$ is a solution, so is $-x(t)$.

2. Main results

To prove our oscillation criteria, we need the following auxiliary lemmas.

Lemma 2.1 (Philos [12]). *Let $f \in \mathrm{C}^n([t_0,\infty),(0,\infty))$. If the derivative $f^{(n)}(t)$ is eventually of one sign for large t, then there exist a $t_x \geq t_0$ and an integer l, $0 \leq l \leq n$ with $n+l$ even for $f^{(n)}(t) \geq 0$, or $n+l$ odd for $f^{(n)}(t) \leq 0$ such that*

$$l > 0 \qquad \textit{yields} \qquad f^{(k)}(t) > 0, \qquad t \geq t_x, \qquad k = 0,1,\ldots,l-1,$$

and

$$l \leq n-1 \qquad \textit{yields} \qquad (-1)^{l+k} f^{(k)}(t) > 0, \qquad t \geq t_x, \qquad k = l, l+1, \ldots, n-1.$$

Lemma 2.2 ([2, Lemma 2.2.3]). *Let $f \in \mathrm{C}^n([t_0,\infty),(0,\infty))$, $f^{(n)}(t)f^{(n-1)}(t) \leq 0$ for $t \geq t_*$, and assume that $\lim_{t\to\infty} f(t) \neq 0$. Then for every $\lambda \in (0,1)$ there exists a $t_\lambda \in [t_*,\infty)$ such that, for all $t \in [t_\lambda,\infty)$,*

$$f(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} |f^{(n-1)}(t)|.$$

Lemma 2.3 (Kiguradze and Chanturia [9]). *Suppose that the function h satisfies $h^{(i)}(t) > 0$, $i = 0,1,2,\ldots,k$ and $h^{(k+1)}(t) \leq 0$ eventually. Then, for t large enough,*

$$\frac{h(t)}{h'(t)} \geq \frac{t}{k}.$$

Theorem 2.1. *Let conditions (H_1), (H_2), and (1.3) be satisfied. Suppose that there exist functions $\eta \in \mathrm{C}([t_0,\infty),\mathbb{R})$ and $\xi \in \mathrm{C}^1([t_0,\infty),\mathbb{R})$ satisfying*

$$\eta(t) \leq \sigma(t), \qquad \eta(t) < \tau(t), \qquad \xi(t) \leq \sigma(t), \qquad \xi(t) < \tau(t),$$
$$\xi'(t) \geq 0, \qquad \textit{and} \qquad \lim_{t\to\infty} \eta(t) = \lim_{t\to\infty} \xi(t) = \infty.$$

If there exists a $\lambda_0 \in (0,1)$ such that the two first-order delay differential equations

$$y'(t) + \frac{\lambda_0}{(n-1)!} q(t) p_*(\sigma(t)) (\tau^{-1}(\eta(t)))^{n-1} y(\tau^{-1}(\eta(t))) = 0 \tag{2.1}$$

and

$$w'(t) + \frac{1}{(n-3)!} \left(\int_t^\infty (s-t)^{n-3} q(s) p^*(\sigma(s)) \mathrm{d}s \right) \tau^{-1}(\xi(t)) w(\tau^{-1}(\xi(t))) = 0 \tag{2.2}$$

are oscillatory, then equation (1.1) is oscillatory.

Proof. Assume that equation (1.1) has a non-oscillatory solution $x(t)$ which is eventually positive. It follows from (1.1) that

$$z^{(n)}(t) = -q(t)x(\sigma(t)) \leq 0. \tag{2.3}$$

Then, using Lemma 2.1, we conclude that there are two possible cases for the behavior of z and its derivatives for large t, either

$$\text{Case (i)} \qquad z(t) > 0, \qquad z'(t) > 0, \qquad z''(t) > 0,$$
$$z^{(n-1)}(t) > 0, \qquad z^{(n)}(t) \leq 0;$$

or

$$\text{Case (ii)} \qquad z(t) > 0, \qquad z^{(j)}(t) > 0,$$
$$z^{(j+1)}(t) < 0, \qquad \text{for all odd } j \in \{1, 2, \dots, n-3\},$$
$$z^{(n-1)}(t) > 0, \qquad \text{and} \qquad z^{(n)}(t) \le 0.$$

We consider each of the two cases separately starting with case (i). Then

$$\lim_{t\to\infty} z(t) \neq 0,$$

and, by virtue of Lemma 2.2, for every $\lambda \in (0,1)$ and for all large t,

$$z(t) \ge \frac{\lambda}{(n-1)!} t^{n-1} z^{(n-1)}(t). \tag{2.4}$$

It follows from the definition of $z(t)$ that

$$\begin{aligned} x(t) &= \frac{1}{p(\tau^{-1}(t))}\left(z(\tau^{-1}(t)) - x(\tau^{-1}(t))\right) \\ &= \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t))}\left(\frac{z(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(\tau^{-1}(t)))} - \frac{x(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(\tau^{-1}(t)))}\right) \\ &\ge \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t))}\frac{z(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(\tau^{-1}(t)))}. \end{aligned} \tag{2.5}$$

Then, by Lemma 2.3,

$$\frac{z(t)}{z'(t)} \ge \frac{t}{n-1},$$

and we deduce that for all large t,

$$\left(\frac{z(t)}{t^{n-1}}\right)' \le 0. \tag{2.6}$$

Using the condition $\tau^{-1}(t) \le \tau^{-1}(\tau^{-1}(t))$ and (2.6), we conclude that

$$z(\tau^{-1}(\tau^{-1}(t))) \le \frac{(\tau^{-1}(\tau^{-1}(t)))^{n-1}}{(\tau^{-1}(t))^{n-1}} z(\tau^{-1}(t)). \tag{2.7}$$

Substitute first (2.7) into (2.5) to obtain

$$x(t) \ge \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))}\left(1 - \frac{(\tau^{-1}(\tau^{-1}(t)))^{n-1}}{(\tau^{-1}(t))^{n-1} p(\tau^{-1}(\tau^{-1}(t)))}\right) = p_*(t) z(\tau^{-1}(t)). \tag{2.8}$$

Subsequent substitution of (2.8) into (2.3) yields

$$z^{(n)}(t) + q(t) p_*(\sigma(t)) z(\tau^{-1}(\sigma(t))) \le 0.$$

Using conditions $\eta(t) \le \sigma(t)$ and $z'(t) > 0$, we conclude that

$$z^{(n)}(t) + q(t) p_*(\sigma(t)) z(\tau^{-1}(\eta(t))) \le 0. \tag{2.9}$$

It follows from (2.4) and (2.9) that for all $\lambda \in (0,1)$,

$$z^{(n)}(t) + \frac{\lambda}{(n-1)!} q(t) p_*(\sigma(t)) (\tau^{-1}(\eta(t)))^{n-1} z^{(n-1)}(\tau^{-1}(\eta(t))) \le 0.$$

Introduce now a new function $y(t) = z^{(n-1)}(t)$. Clearly, $y(t)$ is a positive solution of the first-order delay differential inequality

$$y'(t) + \frac{\lambda}{(n-1)!} q(t) p_*(\sigma(t)) (\tau^{-1}(\eta(t)))^{n-1} y(\tau^{-1}(\eta(t))) \le 0. \tag{2.10}$$

It follows from [13, Theorem 1] that the associated with (2.10) delay differential equation (2.1) also has a positive solution for all $\lambda_0 \in (0,1)$, but this contradicts our assumption on equation (2.1).

Consider now case (ii). By the definition of $z(t)$, (2.5) holds. It follows from conditions $z(t)>0$, $z'(t)>0$, $z''(t)<0$, and Lemma 2.3 that

$$z(t) \geq t z'(t), \tag{2.11}$$

and hence

$$\left(\frac{z(t)}{t}\right)' \leq 0 \tag{2.12}$$

eventually. By virtue of condition $\tau^{-1}(t) \leq \tau^{-1}(\tau^{-1}(t))$ and (2.12), we deduce that

$$z(\tau^{-1}(\tau^{-1}(t))) \leq \frac{\tau^{-1}(\tau^{-1}(t))}{\tau^{-1}(t)} z(\tau^{-1}(t)). \tag{2.13}$$

Substitution of (2.13) into (2.5) yields

$$x(t) \geq \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))}\left(1 - \frac{\tau^{-1}(\tau^{-1}(t))}{\tau^{-1}(t)p(\tau^{-1}(\tau^{-1}(t)))}\right) = p^*(t) z(\tau^{-1}(t)). \tag{2.14}$$

Substituting now (2.14) into (2.3), we obtain

$$z^{(n)}(t) + q(t)p^*(\sigma(t))z(\tau^{-1}(\sigma(t))) \leq 0.$$

Since $\xi(t) \leq \sigma(t)$ and $z'(t)>0$, we also have

$$z^{(n)}(t) + q(t)p^*(\sigma(t))z(\tau^{-1}(\xi(t))) \leq 0. \tag{2.15}$$

Integrating (2.15) from t to ∞ consecutively $n-2$ times, we deduce that

$$z''(t) + \frac{1}{(n-3)!}\left(\int_t^\infty (s-t)^{n-3} q(s) p^*(\sigma(s))\mathrm{d}s\right) z(\tau^{-1}(\xi(t))) \leq 0. \tag{2.16}$$

Letting $w(t) = z'(t)$ and using (2.11) in (2.16), we conclude that $w(t)$ is a positive solution of a first-order delay differential inequality

$$w'(t) + \frac{1}{(n-3)!}\left(\int_t^\infty (s-t)^{n-3} q(s) p^*(\sigma(s))\mathrm{d}s\right) \tau^{-1}(\xi(t)) w(\tau^{-1}(\xi(t))) \leq 0. \tag{2.17}$$

It follows from [13, Theorem 1] that the associated with (2.17) delay differential equation (2.2) also has a positive solution, which again contradicts our assumption on equation (2.2). Therefore, equation (1.1) is oscillatory. □

Combining Theorem 2.1 with the oscillation criterion reported by Baculíková and Džurina [3, Lemma 4], we obtain the following result.

Corollary 2.1. *Let conditions* (H_1), (H_2), *and* (1.3) *be satisfied. Suppose that there exist functions* η, ξ *as in Theorem 2.1. If*

$$\frac{1}{(n-1)!}\liminf_{t\to\infty}\int_{\tau^{-1}(\eta(t))}^t q(s) p_*(\sigma(s))(\tau^{-1}(\eta(s)))^{n-1}\mathrm{d}s > \frac{1}{\mathrm{e}} \tag{2.18}$$

and

$$\frac{1}{(n-3)!}\liminf_{t\to\infty}\int_{\tau^{-1}(\xi(t))}^t \left(\int_s^\infty (u-s)^{n-3} q(u) p^*(\sigma(u))\mathrm{d}u\right)\tau^{-1}(\xi(s))\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{2.19}$$

then equation (1.1) *is oscillatory.*

Proof. Applying (2.18), (2.19), and [3, Lemma 4], we conclude that (2.1) and (2.2) are oscillatory. Hence, by Theorem 2.1, equation (1.1) is oscillatory. □

3. Examples and discussion

The following examples illustrate theoretical results obtained in the previous section. We assume that $t \geq 1$ and $q_0 > 0$.

Example 3.1. *Consider a fourth-order neutral delay differential equation*

$$\left[x(t) + 16x\left(\frac{t}{2}\right)\right]^{(4)} + \frac{q_0}{t^4} x\left(\frac{t}{6}\right) = 0. \tag{3.1}$$

Let $\eta(t) = \xi(t) = t/6$. An application of Corollary 2.1 yields that equation (3.1) *is oscillatory provided that $q_0 > 5184/(\mathrm{e}\ln 3) \approx 1,736$.*

Remark 3.1. *Our oscillation result for equation* (3.1) *improves known theorems. Indeed, [11, Theorem 2.4] cannot be applied to equation* (3.1) *because $t/6 = \sigma(t) < \tau(t) = t/2$ for $t \geq 1$ and, in addition, one of the assumptions has to be satisfied for arbitrary M. Let $p_0 = 16$, $\tau_0 = 1/2$, and $\sigma_0 = 6$. Theorems 1.2 and 1.3 ensure oscillation of equation* (3.1) *for $q_0 > 42768/(\mathrm{e}\ln 3) \approx 14,321.22$. However, choosing $\rho(t) = t^3$, $\delta(t) = t$, and $\lambda_0 = 144/145$ and using Theorem 1.1, we observe that* (3.1) *is oscillatory for $q_0 > 3,915$. Therefore, our criterion provides a sharper estimate.*

Example 3.2. *Consider a fourth-order neutral differential equation*

$$\left[x(t) + 16x\left(\frac{t}{2}\right)\right]^{(4)} + \frac{q_0}{t^4} x(\sigma(t)) = 0 \tag{3.2}$$

where σ is any continuous function satisfying $\sigma(t) > t/2$ on $[1, \infty)$. Observe that in this case Theorems 1.1-1.3 cannot be applied to equation (3.2)*. Let $\eta(t) = \xi(t) = t/3$. By Corollary 2.1, equation* (3.2) *is oscillatory for $q_0 > 648/(\mathrm{e}\ln 1.5) \approx 588$.*

Remark 3.2. *For a class of even-order neutral functional differential equations* (1.1)*, we derived two new oscillation results which complement and improve those obtained by Agarwal et al. [1], Baculíková and Džurina [4], Baculíková et al. [5], Li et al. [11], and Xing et al. [14]. A distinguishing feature of our criteria is that we do not impose specific restrictions on the deviating argument σ; that is, σ can be delayed, advanced, and even change back and forth from advanced to delayed.*

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References

[1] R.P. Agarwal, M. Bohner, T. Li, and C. Zhang, A new approach in the study of oscillatory behavior of even-order neutral delay differential equations, *Appl. Math. Comput.* **225** (2013), 787-794.

[2] R.P. Agarwal, S.R. Grace, and D. O'Regan, *Oscillation Theory for Difference and Functional Differential Equations*, Kluwer Academic Publishers, Dordrecht, 2000.

[3] B. Baculíková and J. Džurina, Oscillation theorems for second-order nonlinear neutral differential equations, *Comput. Math. Appl.* **62** (2011), 4472-4478.
[4] B. Baculíková and J. Džurina, Oscillation theorems for higher order neutral differential equations, *Appl. Math. Comput.* **219** (2012), 3769-3778.
[5] B. Baculíková, J. Džurina, and T. Li, Oscillation results for even-order quasilinear neutral functional differential equations, *Electron. J. Differential Equations* **2011** (2011), 1-9.
[6] J.K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.
[7] M. Hasanbulli and Yu.V. Rogovchenko, Asymptotic behavior of nonoscillatory solutions to n-th order nonlinear neutral differential equations, *Nonlinear Anal.* **69** (2008), 1208-1218.
[8] M. Hasanbulli and Yu.V. Rogovchenko, Oscillation criteria for second order nonlinear neutral differential equations, *Appl. Math. Comput.* **215** (2010), 4392-4399.
[9] I.T. Kiguradze and T.A. Chanturia, *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*, Kluwer Academic Publishers, Dordrecht, 1993. Translated from the 1985 Russian original.
[10] G.S. Ladde, V. Lakshmikantham, and B.G. Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments*, Marcel Dekker Inc., New York, 1987.
[11] T. Li, Z. Han, P. Zhao, and S. Sun, Oscillation of even-order neutral delay differential equations, *Adv. Difference Equ.* **2010** (2010), 1-9.
[12] Ch.G. Philos, A new criterion for the oscillatory and asymptotic behavior of delay differential equations, *Bull. Acad. Pol. Sci., Sér. Sci. Math.* **39** (1981), 61-64.
[13] Ch.G. Philos, On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays, *Arch. Math.* **36** (1981), 168-178.
[14] G. Xing, T. Li, and C. Zhang, Oscillation of higher-order quasi-linear neutral differential equations, *Adv. Difference Equ.* **2011** (2011), 1-10.
[15] A. Zafer, Oscillation criteria for even order neutral differential equations, *Appl. Math. Lett.* **11** (1998), 21-25.

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