

The micro-culture of a mathematics classroom
Artefacts and Activity in Meaning making and Problem solving

Sharada Gade

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Preface

I submit this doctoral thesis on the 25th August 2006 at the Agder University College, Kristiansand, Norway towards a Doctorate of Philosophy in Mathematics Education.

My three year doctoral study (2003-2006), was funded by the Norwegian Student Loan Fund under a program supporting citizens from the developing countries. My research however is independent of any research council, educational or developmental project in Norway.

Towards the completion of my study and thesis, I wish to first express deep appreciation and thanks to the teachers and students of the 1MX mathematics class of 2004-2005, with whom I conducted my fieldwork. I hope my thesis speaks for the enriching time spent there.

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Of course I could have had a few more days ...

Sharada Gade
Kristiansand, Norway
25th August 2006

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1. An Introduction

As an introduction to my thesis, I begin with a brief summary of my teaching experiences at my school in India, followed by those specifically in the mathematics classroom that have a bearing on my current study. I then sketch the development and formulation of my research questions. I finally offer an overview of the chapters in my thesis.

My school and teaching experience

I taught at an experimental school in Hyderabad, India run by its founder, Shanta Rameshwar Rao, who was inspired by the Indian philosopher and thinker Jiddu Krishnamurti. A compulsory school, it prepared students for a recognised school leaving examination at Grade 10. In contrast to more routinised schools, our school appeared ‘different’ as we put in practice equal enrolment of boys and girls, no formal examinations till the 8th grade, no ranking of students, no student prefects and no school uniform to wear. Our morning assemblies were not stereotyped; interesting topics were offered and brought to discussion by teachers. By such a school practice we questioned the norm.

Apart from visible differences there were deeper issues of not making the students compete but cooperate with each other. As teachers we had the freedom not to follow ‘the’ textbook and to bring in other material. We attempted to make learning rewarding in and by itself and do away with awards or punishment. We encouraged every student to question and explore, be responsible and speak for her or himself. In addressing some of the many irritants that came in the way of ‘schooling’, we were left with a focus on learning and teaching alone, which was not only creative and enjoyable but also difficult and frustrating.

The underlying philosophy was one intended to recognise that to live was to be related to one another (Krishnamurti, 1974), bring order within oneself and be responsible for one’s own transformation (As in Shotton, 1998). One was to allow experience to awaken intelligence, free ourselves from fear, by understanding ourselves and see the individual as important and not the system. Education was to inquire, not cease to question, see the significance of life as a whole and integrate experience in the understanding of oneself (Krishnamurti, 1953).

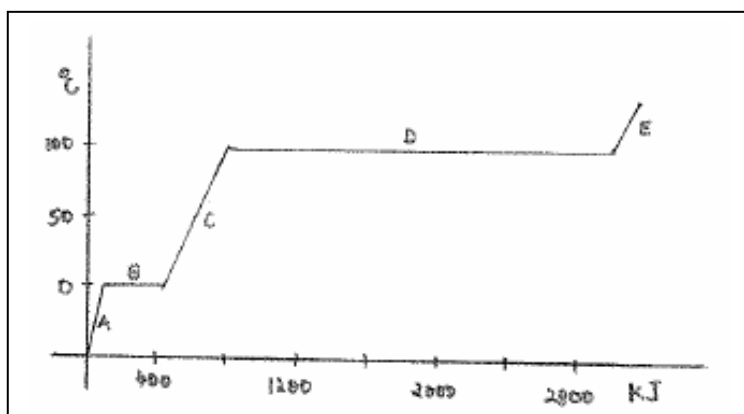
It goes without saying that our school itself was immersed in other sources of values in our culture, which distinguished between knowledge as of things, acts and relations, and wisdom as that of the creative self beyond these. In general, the participation of a person leading to knowledge revolved around the qualities of the mind in search for knowledge and not knowledge per se. The simple metaphor was one of moving from darkness to light. The ambient envelope was one of non-violence.

Our school provided us teachers, the freedom to try our ideas and address the complexity of education by organising teaching-learning based upon our personal development. The philosophical impetus of knowing oneself, led to a focus on meta-cognitive aspects with respect to our selves and our work. A reflective stance towards teaching-learning and the issues related to teaching-learning, led to an interest in *creating* the classroom. In a compilation of my efforts (Sharada, 2004), I argued for the need to provide opportunities that enabled reflection and exploration by the students. Of the rewarding experience in such teaching-learning, I observed that ‘we never learn until we teach’.

In the mathematics classroom

From within teaching-learning in the classroom, I relate two experiences relevant to the areas I focus upon: meaning making and problem solving. I trace my interest of understanding the meaning making processes in the classroom to an incident as a ‘stand-in’ teacher. I was asked on one occasion to ‘teach’ the topic of Latent Heat in Physics to an examination going class (Grade 10) in two teaching periods. Though I recognised the impossibility of the task at hand, I accepted the responsibility, contemplating that the experience I could offer may be better than none for the students. The students were familiar to me, as I had taught them mathematics and science when they were in their middle school.

The teaching-learning of the topic had two deliverables: an ‘understanding’ of the topic, and the ability of the students to apply relevant



formulae in solving numerical problems. To attempt the first I began with the adjacent graph called the heating and cooling curve. It was during my teaching, when the students were in their middle school,

that the students had learnt how to plot points, draw straight line graphs and make use of the co-ordinate system. The curve showed a succession of stages of rise in temperature with the absorption of heat, interspersed by absorption of heat with no rise corresponding to changes in state.

In my teaching of this topic the graph became my starting point, with which I tried to elicit what the students already ‘knew’ about the heating and cooling curve and the topic as was represented by the graph. We then drew a comparison between various points on the graph and related these to everyday experiences we had with ice, water and steam. We ex-

tended such observations to the concept of ‘latent’ or ‘hidden’ heat. Our discussion provided a basic understanding, which the students were able to build upon in preparation for their examination.

I continued to reflect upon how the students and I were able to draw so much ‘knowledge’ and ‘knowing’ (I deal with these concepts in my thesis) from the given graph. We were able to bring our real life experiences to the graph, towards illuminating points both on the graph and in our understanding of the topic of Latent Heat. Though the concept being taught was understood with the help of the graph, the graph itself seemed transparent or immaterial to the understanding of the concept once some understanding had been reached about the transitions of state it represented. That so much meaning and knowing was enabled by the graph representing the transitions, attracted my attention for a closer study.

The second experience that I discuss relates to my focus on problem solving in my teaching-learning practice; of providing and creating opportunities for students to attempt, observe, conjecture and discuss mathematical patterns. My interest in this area was furthered by the reading and implementation of the discussion and arguments offered by Mason (1988) in his inviting book: *Learning and doing mathematics*. Specific instances are too numerous to quote here, but my attempts and enriching experiences in the classroom leading to my amateur writing, have been inspired by the happy union of two factors: the inviting and creative nature of mathematics and the freedom to explore such creativity within teaching-learning at my school.

The development of research questions

I extend my reflective experiences in classroom teaching-learning and outline the formation and formulation of the research questions pursued in my current study in three ways. Firstly, my personal reasons for considering classroom teaching-learning as important, secondly, the reading that steered my thinking before I took up doctoral work and thirdly, my thinking associated with the opportunities that were available in the classroom in which I conducted my field work.

My recognition of the importance of classroom teaching-learning is on three counts. Firstly, I recognise that traditional and ethnic systems of teaching-learning mathematics are fast disappearing with the adoption of the post-industrial model of schooling. In this I admit a concern of ‘schooling’ everybody, by which school takes over the responsibility of education by replacing traditional systems in societal praxis.

Secondly, the problem of poor enrolment beyond compulsory school (in many countries) probably makes school the last location for the learning of mathematics by many. In symbolising the significance of this factor, I subscribe to Ian Stewart’s equation ‘maths = school’ (Mankiewicz,

2000). Thirdly, though people do have opportunities to use and do mathematics outside school, I argue that it is only in school (or in the mathematics classrooms within schools) that students have the opportunity to appreciate various relationships among mathematical concepts, leading to knowing mathematics as a subject, discipline and science. I now discuss my related and amateur reading.

Towards a need to understand the mathematics classroom so as to provide opportunities for learning and creating mathematics, I draw on two arguments from which my thesis is a departure. I begin with description of the nature of mathematics by Hardy (1992, p 84) ‘A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with *ideas*.’ Apart from the notion that mathematics is a body of ideas, I attend on the other hand to the tools used in mathematics. That a body of ideas is inseparable from the tools used in mathematical praxis is highlighted by Davies and Hersh (1998, p 13) ‘The ruler and compass are built into the axioms at the foundation of Euclidean geometry. Euclidian geometry can be defined as the science of ruler-and-compass constructions.’ By the two views I quote above, I point to the two extremities of ideality and materiality of mathematics with which I proceeded to understand the teaching-learning of mathematics in the classroom.

Towards the role of human activity in the creation of knowledge, I mention mathematician and historian Bronowski (1973; 1978) who argues that it is in the evolution of symbolic language which includes mathematics, that human culture has had the most selective influence in making human beings what they are. Bronowski sees the human mind as the instrument for understanding and knowledge as human destiny, for which the experience of the arts and the explanations of science are to be integrated. I owe to my reading of *The ascent of man*, my questioning and re-thinking about the nature and processes of knowledge building, as a collective and cultural achievement of humankind.

Having touched upon an emphasis on ideas, tools and knowledge building, I turn finally to Kuhn (1996; Okasha, 2002) who explains the concept of a paradigm and the praxis of any normal science. Kuhn argues that the praxis of any science is an act of faith on part of the scientist; an agreement on how future research in the field should proceed. Kuhn offered his doctrine of paradigm shifts as a relativistic view of the history of science and argued that all data is theory laden with truth always relative to a particular culture.

The conception of a relative view, in addition to my thinking of the teaching-learning of mathematics in terms of ideas, tools and knowledge building, formed part of my unarticulated agenda for research in mathematics education. As observed by Bruner (1979) these ideas constituted

the few intellectual themes that persisted and remained with me (and probably will) when I took up my doctoral study. I now turn to outline the other factor that influenced the development of my research questions: the opportunity that I had for fieldwork.

It was possible for me to conduct an empirical study in a mathematics classroom at an upper secondary school, taught bilingually (Norwegian and English) by two teachers who laid emphasis on cooperative learning by the students at group-tasks. I allowed the focus of the two teachers to guide my research questions and help address the unarticulated issues of teaching-learning in the mathematics classroom that I brought with me to research. By research, as argued by Kilpatrick (1992), I mean disciplined inquiry into the teaching and learning of mathematics, by which it is my intention to participate in the larger discourse of professionals and practitioners engaged in the study of teaching-learning of mathematics.

In finalising my research questions I brought together two aspects: the distinctive features of the classroom that in my view would give its teaching-learning a unique character and the opportunities of observation available to me. Such thinking brought three factors into consideration. Firstly, the class was taught bilingually and I had access to only the English language. Such a reality was a constraint, yet put me in a position to be hard pressed as an observer to consider and rely on communication beyond language alone. I conjectured in addition that the students being native Norwegian speakers, their participating in English in addition to mathematics might make them choose or put them in a position to communicate in modes in addition to their native Norwegian. This led me to not restrict myself to language alone and adopt in research a broader view to the communication processes in the classroom.

Secondly, and in line with my agreeing to adopt the focus of the teachers, I conjectured that the classroom would provide multiple opportunities for a study of interpersonal communication. There was the possibility of studying student groups in teaching-learning and in multiple student groups working simultaneously at the very same topic. Some questions that logically followed from this were: How did the teachers organise teaching-learning? How did the students participate and address the goals set out for them in this particular classroom?

Thirdly, the questions that I began to ask above were coupled with the opportunity of conducting a longitudinal study and observe teaching-learning for a whole year. This brought in the additional possibility of studying not only the teaching-learning of mathematics, but also the development of teaching-learning of mathematics as the year progressed. Having its own dynamics and routines, I envisaged this classroom to have a distinctive culture of its own. I therefore began to ask how mean-

ing making and problem solving was brought about in the teaching-learning of mathematics in this particular classroom.

My interests in meaning making and problem solving embedded in the unique aspects of the classroom I was to conduct my fieldwork in, was informed in the course of doctoral studies by socio-cultural-historical perspectives. As I elaborate in the following chapter, these perspectives offered the possibility of analytically addressing two diverse aspects on one common basis: on one hand, the cultural and social relationships that would be part of teaching-learning and on the other, the interpersonal communication that would constitute teaching-learning. I adopted the construct of ‘artefacts’, which as I shall discuss offered a common and material basis for the analysis of all communication. I formulated my primary research questions as: How do artefacts mediate meaning making in the mathematics classroom? How do artefacts mediate problem solving in the mathematics classroom?

However, conducting fieldwork coupled with collecting and analysing data extended my reading and in turn led me to consider another construct from socio-cultural-historical perspectives: that of ‘activity’. These deliberations over a period of time in turn allowed me to ask and enabled me to address more specifically the following research questions:

‘Within a collaborative teaching-learning practice in the mathematics classroom, how do artefacts and activity mediate:

- meaning making in participation,
- consolidation of meaning made,
- development of problem solving know-how, and
- cooperation in problem solving.’

The chapters in my thesis elaborate and detail the theoretical, methodological and analytical aspects that enabled me to address and respond to the above research questions. As a result of my synthesis I refer to the definitive statement that I make about the teaching-learning of my classroom as a ‘micro-culture’. By a ‘micro-culture’, I theorise about those unique aspects, which constituted the teaching-learning of mathematics in my classroom. By such a distinction, I also distinguish the micro-culture of my classroom, from the larger praxis and culture of mathematics. I now turn to detail the eight chapters that make up my thesis.

Overview of thesis chapters

To enable the locating of the various aspects of my thesis I offer below the rationale and logical partitioning in my writing.

In Chapter 1: *An introduction*, of which this section is a part, I offer my teaching experiences leading to research, trace the formation, formulation and development of my research questions. I also offer an overview of the chapters in my thesis.

In Chapter 2: *Theoretical perspectives*, I outline contemporary considerations for instruction in the societal praxis of mathematics and discuss socio-cultural-historical perspectives with which I conducted my study. I then deliberate upon my two areas of focus in teaching-learning of mathematics in the classroom: meaning making and problem solving.

In Chapter 3: *Methodology, methods and fieldwork*, I discuss methodological implications of my theoretical perspectives and offer my four units of analysis. I then draw upon the philosophy of educational research and argue for a naturalistic and qualitative study. I follow this by outlining the appropriate methods of data collection, triangulation and case study reporting, while also describing the opportunity available for fieldwork, the extent of data collected and issues of values related to data collection. I finally outline the data and analysis chapters that follow, offering transcription codes and formats of data presentation. I conclude with an overview of my analytical process.

In Chapter 4: *A collaborative classroom practice*, my first data and analysis chapter, I offer the sequentiality and nature of norms and practices established by the teachers for the teaching-learning of mathematics in the classroom. This chapter addresses the first research question and forms the basis and backdrop against which the three analytical chapters that follow are elaborated upon and discussed.

In Chapter 5: *The consolidation of meaning*, my second data and analysis chapter, I address the second research question. I discuss specific efforts and actions in the teaching-learning of mathematics in the classroom towards the consolidation of personal meaning of the students, leading to their meaning or knowing of its more propositional form.

In Chapter 6: *Problem solving know-how*, I discuss my third research question with instances and actions of teaching-learning, which led to the development of problem solving-know-how in mathematics, in a Zone of Proximal Development or ZPD, a concept I detail in Chapter 2.

In Chapter 7: *Cooperation in problem solving*, my last data and analysis chapter, I discuss the nature of cooperation and communication among students towards shared goals, again in a ZPD, while attempting specially designed group-tasks and address my final research question.

In my concluding Chapter 8: *A micro-culture*, I draw upon my data and analysis chapters; offer a synthesis and formulate my definitive statement about the teaching-learning of mathematics in the classroom.

The above chapters are followed by *References*, wherein I offer references to literature cited in my writing. This is followed by *Appendices* where I enclose approvals, permissions, group-tasks and relevant student and other data referred to and called upon in my writing. In my thesis I follow in addition, the practice of **highlighting theoretical constructs** of importance to the arguments that I make throughout.

2. Theoretical perspectives

I discuss in this chapter relevant perspectives from the field of mathematics education and theories of human development upon which I base my study. I first outline contemporary views about the societal praxis of instruction in mathematics. I then elaborate socio-cultural-historical perspectives with which I view teaching-learning in the classroom. I finally make a choice of studies in mathematics education that allow discussion of issues relevant to my two focus areas: meaning making and problem solving. I outline the contents of the above mentioned sections while elaborating upon each. In following the above sequence I recognise up front the vastness of literature, more broadly within mathematics education, classroom studies in particular and socio-cultural-historical perspectives. Any selection I make is thus representative and purposefully chosen towards the issues I wish to discuss, allowing me to draw from and build the arguments of my thesis upon.

I had mentioned in the previous chapter my impetus for the study of communication and my adopting the conception of artefacts towards an understanding of teaching-learning in the classroom. Making this choice however brought along with the choice theoretical complexity, since artefacts are conceptualised in the different though related fields of cultural psychology, sociocultural studies and activity theory. Choosing appropriate constructs from these sub fields therefore necessitated an appropriate rationale for making the said choice. Towards such an effort I allowed myself to be driven by data, so as to retain those constructs that enabled me to synthesise and portray the complexity of teaching-learning in the classroom, as I observed the case to be.

As also mentioned earlier, I view the mathematics classroom as a culture and term its teaching-learning as a **micro-culture** since the classroom is in turn located within the societal praxis of instruction and the cultural inheritance of mathematics. The micro-culture of the classroom is not my unit of analysis, but a construct constituted to examine the teaching-learning of mathematics in the classroom I study. It is within this cultural overview that I employ perspectives from socio-cultural-historical and activity theory. Such an ordering of choices is with an objective of allowing the constitution of the micro-culture of the classroom, to remain without rigid definition and accommodate constructs that lend to recognition of its growth and synthesis.

Within the widest conception of the classroom as a micro-culture, I discuss at the next level relevant social aspects of teaching-learning. For in depth analysis at a finer level, I adopt a more discursive approach. In discussing a cultural, social and discursive ordering I explain relevant terminology as and where appropriate and necessary.

Contemporary considerations for instruction

To situate my study in current mathematics education research, I discuss in this section contemporary considerations about the nature of mathematics followed by related implications for instruction. Such a view is necessitated on two counts: firstly, as a collective of human knowledge with its origins in pre-history, mathematics has undergone and continues to undergo changes in addressing the philosophical ‘what is mathematics’ question. Secondly, such changes trickle by way of informed debate into syllabi and school curriculum. The struggles and relationships realised in the teaching-learning of mathematics I theorise upon in my thesis, are against a backdrop of these considerations.

For a brief discussion about the nature of mathematics, I take the views of Popper (1972) and Pólya (1971) as my point of departure. Popper and Pólya rejected the influence of positivism prevailing in the then praxis of sciences and a formalistic approach to the growth of mathematics. Their writings spurred Lakatos (1976) to reject the disconnect between the history of mathematics and its philosophy and argue that a situational emphasis is essential and imperative for the growth of informal mathematical thought. Following the pattern of inductive arguments of Popper in science and the heuristic approach of Pólya in mathematics, Lakatos argued for the **growth of mathematics in human terms**, not fossilised axioms. Davis and Hersh (1998) observe that the formalist style of mathematics became identified with its philosophy and penetrated instruction as ‘new math’. Though logic is essential, the central problem of philosophy they argue is the analysis of meaning.

A view of mathematics concerned with meaning and experience in its teaching-learning, is also furthered by debating misconceptions about its history. Advocating an epistemological shift in the viewing of history, dealing with growth and reflecting how mathematics is learnt, Crowe (1988) offers an account of mathematical knowledge as fallible, **tentative and evolving**. In concert Kitcher (1986; 1988) highlights the wisdom of following in mathematics, the patterns of development in science, arguing for a focus on practices. He notes that an a priori emphasis diverts attention from rejected theories, internal struggles and noncumulative changes. Kitcher observes that in its growth, mathematics accommodates former theories into later ones by a rational practice, generating mathematical content based on **organisation of understanding**.

A shift to a humanistic view of mathematics, recognising and giving importance to processes of heuristics, growth, learning and practices seem to correspond with the later philosophy of Wittgenstein (Grayling, 2001), who advocated a **relational view of meaning** premised upon usage. Wittgenstein argued that to understand meaning one had to learn how to use words as part of language games they belonged to.

A focus upon usage, for the development of meaning, seems compatible with the arguments of Fields medallist Gowers, who observes that it is quite possible to use mathematical concepts without being able to say what they mean. In classroom practice, Gowers (2002) advises an emphasis on the consequences of rules in place of their justification. In advocating such a stand he argues for the importance of the **enabling of meaning by conferring existence** to mathematical objects, which Thom (1973) emphasised as the real problem of teaching-learning.

More recently an emphasis of how mathematics is known has drawn fruitfully from its conception as a cultural praxis. Such a view in mathematics education has had contributions from ethnic cultures on the one hand, and an analysis of the classroom as a micro-culture on the other. Coining the term **ethnomathematics** and urging for a bridge with anthropology, D'Ambrosio (2004) has drawn attention to the fact that cognitive mechanisms are prevalent in ethnic cultures. Keeping in mind that cognition is not culture-free, he argues for the reframing of curriculum sensitive to such needs. Elaborating mathematics education in schools to be as a cultural phenomena, Bishop (1988; 2004) offers two useful conceptions: **enculturation** which refers to a culture where its meaning, values and symbolic systems are familiar and become inclusive in a way of knowing (I discuss this conception at length in my thesis) and **acculturation** where the culture involved is alien.

In any observation of teaching-learning of mathematics, Skovsmose (1993; 1994) draws attention to issues of power, behaviour and communication to which traditional approaches have been blind. He highlights the blind spots in a teacher-centric or monological approach, and argues that a **theory of knowing** in instructional praxis **needs to be dialogical**, involving both the teacher and the student. He also argues for reflective knowing in instruction, in order to promote voluntary disposition and interest for the learner so as to enable democracy for conjoined living, both in the classroom and in society as a whole.

Summarising the **social turn** away from a formalistic approach, Ernest (1994; 1998) draws attention to the need for the student to evaluate what is being learnt and build **tacit and personal components** in addition to propositional or societally accepted knowledge. As a consequence, intentional activity in the classroom needs to respect prior knowledge and encourage negotiation of inner meanings and include an assessment of one's learning in addition to content and communication.

It is to attend to issues of cognition and theorise about the micro-culture of teaching-learning in the classroom, that I adopt an anthropological stance. Towards including in analysis the cultural, social and discursive aspects discussed above, I now turn to relevant constructs in socio-cultural-historical perspectives.

Socio-cultural-historical perspectives

The analysis of cultural, social and discursive aspects as constitutive to human development and thereby the teaching-learning of mathematics in the classroom, is ably supported by socio-cultural-historical perspectives. Drawn largely though not exclusively from the work of 20th century Russian scholars Vygotsky, Luria, Leont'ev and their contemporary Bakhtin; these perspectives account analytically for the cultural, social and historical aspects and relationships in the development of individual consciousness and communication. Contributions to the growing collective of these perspectives come also from anthropological studies, cross-cultural and cultural psychology, situative models of cognition and semiotic perspectives. In the seven sub-sections that follow, I discuss relevant arguments that contribute to my analysis of the material and communicational aspects in teaching-learning, enabling me to synthesise the micro-culture of the classroom. In my presentation I first discuss various constructs of Vygotsky, Luria, Leont'ev and Bakhtin, which form the basis for a cultural conception of human development that I envisage and in which I incorporate various social and discursive aspects.

Analytical priority to the social

In taking the seminal views of Lev Vygotsky as my point of departure, I briefly outline his basis for giving analytical importance to the social, cultural and historical aspects in human development. I shall draw upon and discuss his other formulations like that of zone of proximal development (ZPD) in later sub-sections. The attempt in my thesis to combine constructs from related disciplines is a Vygotskian enterprise, based on his premise that psychology learn from human praxis.

Vygotsky's analysis, of the role of the environment in individual development, recognised the **learner as a participant in a process**:

In education there is nothing passive or inactive. Even lifeless objects, when they are brought into the educational area, when they are assigned an educational role, acquire a sense of purpose and become effective participants in this process. ...

An active role is the lot of the teacher. ...The teacher fashions, takes apart and puts together, shreds, and carves out elements of the environment, and combines them together in the most diverse ways in order to reach whatever goal he has to reach. Thus is the educational process an active one on three levels: the student is active, the teacher is active, and the environment created between them is an active one. (Vygotsky, 1997a, pp. 47-58)

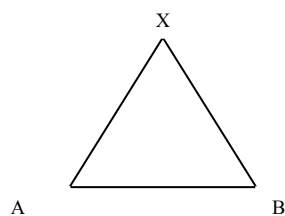
In addition to recognising the active role of the student and teacher, Vygotsky (1971; 1994c; Kozulin, 1986; 1990) elaborated upon the active role of the environment on three counts: the nature of practical material activity, the role of communication and the rationale behind being conscious of oneself. Drawing on a framework that the psyche of man is fundamentally social, Vygotsky brought into his formulation purposeful rational behaviour (material tool action) and cognition (intersubjective

communication, change of meanings). The significance of this approach is that language and symbolic mediators serve not only in communication but also in thinking. On the notion of self, Vygotsky elaborated:

The mechanism of consciousness of the self (self-consciousness) and the cognition of others is the same; we are conscious of ourselves because we are conscious of others, because we are the same vis-à-vis ourselves as others vis-à-vis us. We are conscious of ourselves only to the extent that we are another to ourselves, that is to the extent that we can again perceive our own reflexes as stimuli. (Vygotsky, 1994c, pp. 35-36)

While I discuss shortly Vygotsky's explanation of mediation by auxiliary means, I turn presently to the importance of **speech as both signs and means**. Vygotsky (1978) differentiated between animal existence as one of being, from human existence as one of becoming through the assimilation of social meaning. He specified human behaviour and cognition in terms of semiotic or sign based mediation and argued that as a symbolic mediator, speech not only accompanied practical activity but played an important role in carrying out action. Speech helped attain goals by guiding, planning and dominating the course of action. As auxiliary means, speech was an essential part of cognition and perception, making human behaviour a product of socio-cultural development.

Vygotsky (1981a; 1981b; Kozulin, 1984) extended the role of auxiliary means to all tools both physical (by analogy) and intellectual (like speech), to explain **the instrumental act** wherein the flow of mental functions was altered. In place of a stimulus and response situation between A and B, in a mediated process involving an auxiliary means X, the direct connection A-B was replaced by two connections A-X and X-B. Such a mediated process helped achieve the same result, but by a different path. The process of mediation was thus instrumental in introducing an auxiliary means external to the individual to regulate behaviour.



In explaining the crucial and important shift of control from within the individual to control or regulation from the outside, Vygotsky (1978) stressed however that the intent of using auxiliary means resided in the individual. The cause for mediated action with auxiliary means or artefacts, based on intent, was towards meeting goals that arose from specific social conditions. Knotted handkerchiefs, notches, symbolic gestures or signs all signified meaning in the social circumstances (of their origin) and became part of culture thereafter upon subsequent and continued use. The artefacts of a specific culture and the history of development in that culture thus determined an individual's behaviour.

Based on his explanation of mediation, Vygotsky outlined two kinds of behaviour or **mental functions**: **lower** (natural and not mediated) and **higher** (cultural and mediated). Human development, Vygotsky (1981b;

Luria, 1994; Vygotsky & Luria, 1994a) finally argued, is a dynamic of changes and reversals where natural processes of development are in a dialectical relationship: raised, increased, internalised and widened to higher mental functions. **Education** was therefore the development of lower functions to higher or **cultural forms of behaviour**.

Vygotsky's analytical formulations led to three profound consequences. Firstly, it led him to formulate a **law of human development**:

Any function in the child's cultural development appears twice, or on two planes. First it appears on the social plane, and then on the psychological plane. First it appears between people as an interpsychological category, and then within the child as an intrapsychological category. ... We may consider this position as a law in the full sense of the word, but it goes without saying that *internalisation transforms the process itself* and changes its structure and function. Social relations or *relations among people underlie all higher functions* and their relationships. ... Therefore *sociogenesis of higher forms of behaviour is the basic goal toward which the child's cultural development leads us*. (Vygotsky, 1981a, pp. 163-164, emphasis added)

It is the process of internalisation mentioned above, that forms the basis for cultural development. Since it was **interpsychological functions** that led to **intrapsychological functions**, Vygotsky simultaneously pointed to two related facets: social relationships and social goals which underlie analysis of human cultural development.

The second consequence was in connecting the role of mediated activity (symbolic mediators and tools) to **voluntary behaviour**. Vygotsky (Vygotsky, 1994c; 1999b) argued that all sign activities like reading, writing and counting and voluntary attention, logical memory and higher forms of perception and movement, were one kind of phenomena. These functions realised as a result of social and cultural dialectical processes had two features. Firstly, speech raised action formerly independent of speech to a higher level. Secondly, in so doing speech subordinated human action to individual will, making human action voluntary. Vygotsky (1999a, p 68) argued: 'if the act independent of the word, stands at the beginning of development, then at its end stands the word becoming the act. The word, which makes the action of man free.'

The third consequence of Vygotsky's formulation is related to the use of auxiliary means. Vygotsky (1994d; 1997b) explained that in the use of tools humans overcame their physical limitations and entered a form of development where biological and cultural processes merged. In recognition of such a process, Vygotsky detailed the **role of culture** as follows:

The conclusion is this: culture creates special forms of behaviour, it modifies the activity of mental functions, it constructs new superstructures in the developing system of human behaviour. ... In the process of historical development, social man changes the methods and devices of his behaviour, transforms natural instincts and functions, and develops and creates new forms of behaviour – specifically cultural (Vygotsky, 1997b, p 18)

Culture creates nothing it modifies natural data to conform to the goals of man (Vygotsky, 1997b, p 107)

The importance of Vygotsky's formulation brings into the analysis of teaching-learning, recognition of the active role of the environment, the importance of intersubjective meaning in individual consciousness and the instrumental role of speech on par with physical tools and the external control of behaviour in goal directed activity. Analyses of participation in these relationships that develop historically, enable identifying the cultural development of the individual. It is the analytical importance given to the role of social interaction, goals of development and the emphasis on symbolic mediators in voluntary action that I build my thesis upon. I presently turn to the role of speech in practical activity.

Role of speech in solving problems

Towards the analysis of communication, Luria (1973; 1976; 1979; 1981; 1994) offers two relevant constructs: the role of speech in being conscious of ones own abilities and verbal thinking in concrete processes.

In discussing the role of speech in being conscious of ones own abilities, Luria began by pointing to the **functional importance of a word**. He observed that the active widening of vocabulary by a child, at around the age of five, is a consequence of the child became conscious of its functional importance as a sign. **School** experiences in particular, Luria explained, provided children **the opportunity to realise abilities** and mobilise functions hitherto not known to them. In the use of auxiliary means in practical activity, Luria pointed to the fact that outward techniques became psychological, leading to a functional utilisation by children of their own behaviour. In such utilisation thought itself was formed with the help of words. A consequence of this formulation was that language enabled children to go beyond the limits of their experience, and derive conclusions on the basis of verbal-logical constructions. In such a transition children (humans in general) are not only conscious active agents, but raise their actions **to a higher level of consciousness**.

Arguing against the notion of thinking being a purely mental act, Luria formulated a stage-wise study of thinking as a dynamic act in concrete material activity. Luria's formulation which I present below traces the use of speech in relation to the goals of a task, the importance of the meaning and choice of words and the use of algorithms. Luria's model is akin to Pólya's well known stage-wise process of problem solving in mathematics (which I mention in a later section on problem solving). However the process outlined by Luria incorporates the **active role of speech in solving problems**, an aspect not addressed in Pólya's model. My reference to Luria above is with an intention of exploring instances of teaching-learning, where with the use of speech students raise their actions to a higher level of consciousness in solving problems.

Luria's stage wise study of concrete processes of thinking

- 1 The origin of thought begins in the presence of certain conditions of the task, when the subject investigates paths leading to an adequate solution.
- 2 A direct attempt to respond is not made and impulsive responses are restrained. This leads to investigating conditions, analysing its components, recognising the essential features and correlating them. This preliminary investigation is a vital and essential step without which no intellectual act takes place.
- 3 Selection is made from a number of possible alternatives and a plan (scheme) for the performance is created; alternatives are decided or rejected. This phase of the intellectual act is regarded as its most essential component: word meanings participate in thought, making the intellectual act understandable. This analysis and choice of possible alternatives constitutes the essence of heuristics.
- 4 Choosing appropriate methods and considering operations adequate for affecting solution. Operations include ready-made algorithms (linguistic, logical and numerical) which have evolved in social history. The choosing of operations is called tactics, distinguishing it from the strategy for the solution of the problem.
- 5 Use of operations is the operative rather than creative stage. Successive external actions (trials and errors) progress towards internal speech. Subject obtains assistance from ready made systems: linguistic and logical codes in discursive thinking; numerical codes. Well assimilated internal codes form basis for intellectual operations and provide foundation for the operative stage.
- 6 The use of algorithms leads to the actual solution to the problem or discovery of the answer to the question embodied in the concrete or practical task.
- 7 Comparison of the results obtained with the original conditions of the task. If the results agree the intellectual act is complete or the process must continue until an adequate solution is found.

(Modified from Luria, 1973, pp. 325-329)

Making meaning in 'activity'

In an analytical approach not premised on the role of auxiliary means (artefacts, speech) in behaviour, I utilise the analytical conception of 'activity' (in quotes to distinguish from everyday usage) of Leont'ev. I discuss Leont'ev's other conception of appropriation later. My interest in discussing 'activity' is with an intention of applying this conception as both location and object of analysis to elicit relational transformations that take place in student attempts at goal-directed tasks.

Based on the writings of Leont'ev (1978; 1981b; 1981c; 1994) his son Leont'ev (1981a) and Kozulin (1990), I first discuss his emphasis on human consciousness as **social consciousness** or co-knowledge. Drawing upon Vygotsky, Leont'ev explained that in any practical activity consciousness is not given from the beginning and it is not produced by nature: it is product of society, it is *produced* ... it is a problem for psychology - an object of concrete investigation ... internalization is not the *transferral* of an external activity to a preexisting, internal 'plane of consciousness': it is the process in which this internal plane is *formed*. (Leont'ev, 1981c, pp. 56-57)

Leont'ev premised labour as the mediating point between a subject and the external world in any material, social and object-driven activity. He argued that the development of meaning could not be explained by ver-

bal communication alone, but also by the child's objective reality. In 'activity' man presented himself as an object of adaptation in which he created conditions and means **consciously realisable** in practice.

Within 'activity' Leont'ev distinguished three aspects; **activities** that formed the basis of motives towards which they were oriented; **actions** that are carried out on the basis of goals and **operations** that are conducted on the basis of instrumental conditions. Wertsch (1981) observes the three level analysis of 'activity', to be an original contribution of Leont'ev, and points out that the study of structural characteristics of practical activity is a feature in Russian psychology absent in Western psychology, where behaviour is investigated regardless of goals.

The premise of the theory of 'activity' is that knowledge of the world is mediated by our **interaction with its materiality**. Leont'ev explained that the formation of ideas was from material practice and that the structure of human thinking changed in correspondence with the structure of social interaction and could not exist outside its materiality. 'Activity' mediated the connections between subject and object to corresponding goals and means. In 'activity' the object was transformed into its subjective form and at the same time converted into objective results and products. In summary within 'activity' **the individual is objectivised** and in the individual, **the object is subjectivised**.

Leont'ev argued against understanding human activity as relationships between individuals and society. He observed that in society humans do not find external conditions to which they have to adapt; but social conditions that bear motives and goals of 'activity'. Society produces the activity of individuals and brings humans into practical contact with objects, which change, enrich and transform the 'activity'.

Leont'ev explained that 'activities' differ depending on their objects, where a specific 'activity' answers a specific need of the active agent, moves towards the object of its need and is terminated when satisfied. In any concrete process, actions or chains of actions are internally connected and one and the same action can be instrumental in realising different 'activities'. As part of any well-developed 'activity' every action apart from having an intentional part of what must be done, has an operational aspect of how it can be done. Such intentionality is **defined** not by the goal itself, but **by the objective circumstances** or conditions under which the 'activity' is carried out.

Leont'ev further argued that it is not sufficient for a subject to be absorbed into 'activity' and its material properties, but should be transformed **in a way recognisable to the subject**. Such a transformation he said took place through language, which carries meaning about content liberated from materiality. Leont'ev argued that meanings interpreted the world and were the most important 'formers' of human consciousness.

Finally Leont'ev argued that although language carried meaning, **language was not the creator of meaning**. Behind linguistic meaning hide socially developed methods of action in which people perceive objective reality. Meanings are not a subject in psychology, but become its subject when taken in the system of social relations. In other words meanings represent an ideal form of the existence of the objective world, its properties, connections and relationships, disclosed by cooperative social practice, transformed and hidden in the material of language. (Leont'ev, 1978, p 85)

Leont'ev explained that the process of mastering meanings by the child in external activity involves four stages: concrete meaning from objects, mastery of purely logical operations, internalisation of external meanings and concepts, and finally internal mental activity in the plane of consciousness. In relation to the concept of human consciousness with which I began, Leont'ev argued that at the beginning of the formation of consciousness (social consciousness), meaning merged with personal sense which connected the reality of a subject's own life in his world.

Personal sense was always a sense of something.

My interest in the 'activity' of Leont'ev, is to afford in analysis an emphasis on the concrete operations that constitute specific activity and the transformation of meaning and actions within specific tasks, in pursuit of both immediate and long term goals. Though intended for analysis of goal-directed tasks, the perspectives of 'activity' underpin both problem solving and meaning making. However, any application is with an objective of understanding the teaching-learning of the classroom as a micro-culture, whose linguistic aspects I now turn discussion to.

Consciousness and utterance

From the writings of Bakhtin (1986; 1994; Vološinov, 1973; Holquist, 2002) I refer to the notion of consciousness, the situatedness of utterances and meaning, the materiality of signs and his notion of speech genres. I draw on Bakhtin again while discussing the ZPD of Vygotsky. My interest in discussing these conceptions of Bakhtin is towards analysis of the nature of utterances and meaning, in any communication towards the teaching-learning of mathematics in the classroom.

In a framework since called **dialogism**, Bakhtin's approach to any situation is not an absolute one but a relative one, deeply connected to what is said, what is meant and the notion of self and consciousness:

the utterance is a deed, it is active, productive: it resolves a situation, brings it to an evaluative conclusion (for the moment at least), or extends action into the future. In other words, consciousness is the medium and utterance the specific means by which two otherwise disparate elements – the quickness of experience and the materiality of language – are harnessed into a volatile unity. Discourse does not reflect a situation, it is a situation. (Holquist, 2002, p 63)

In describing **consciousness as medium** and **discourse the situation**, Bakhtin extends the analysis of meaning making as actively produced

and constituted by the meeting or dialogue of two consciousnesses. In acknowledging the notion of self as being in relation to another, Bakhtin is in agreement with Vygotsky and Leont'ev. He however forwards the notion that language and self as the site of meaning exist in order to mean, and that the contexts of any dialogue are without limit.

Bakhtin argued that there is never one meaning alone, but a continuous struggle between collections of contested meanings in any situation. Towards analysis of the nature of meaning made by individuals in any given context or situation, Bakhtin (1986, p 160, emphasis added) observed 'The interpretation of *contextual meanings* cannot be scientific, but it is *profoundly cognitive*. It can directly serve practice, practice that deals with things.' In stressing the **contextual nature of meaning**, Bakhtin drew attention to the cognitive nature of meaning making in material practices, an aspect argued in 'activity' by Leont'ev.

As to the importance of contextual, cognitive or personal meaning which as argued earlier formed the basis for propositional knowing, Bakhtin (1986, p 162) argued: 'Complete maximum reification would inevitably lead to the disappearance of the infinitude and bottomlessness of meaning (any meaning).' Bakhtin also explained that **any meaning** came about only through the **medium and materiality of signs**. Signs existed only if they meant something to those who used them. Based upon this Holquist (2002, p 49) argues: 'Meaning comes about in both the individual psyche and in shared social experience through the medium of the sign, for in both spheres understanding comes about as a response to a sign with signs.'

In signs being medium, Holquist stresses two features. Firstly, that signs are the medium for both individual and shared social experiences. Secondly, that understanding can be analysed by a response to signs with signs. This explanation attests to the active nature of understanding in any setting, where Bakhtin's dictum is that 'every word provokes its counter word'. As a carrier of meaning Bakhtin said **words** were involved in every sphere of activity and were the **most sensitive index of social change**. However unlike the 'higher' synthesis of Vygotsky and Luria, Bakhtin recommended an open-ended approach, leading to his other famous dictum that 'there is neither a first word nor a last word'.

Bakhtin argued for an impossibility of closure in meaning and meaning making. He however recognised the stability of forms of utterances associated with a particular sphere of communication. Such utterances which formed a **speech genre** were inherently interactive, anticipated a response and presupposed another. Occupying a definite position in a sphere of communication, they were determined and related to not only preceding but subsequent links in communication. Being so informed, I now discuss the potentiality of a cultural conception.

A cultural conception: meaning, cognition and artefacts

In discussing a cultural conception and in keeping with my focus on communication, I discuss the significance of meaning making in any culture followed by the role of artefacts in mediating shared cognition. I focus on those features that a cultural conception provides which extend the perspectives I have discussed so far.

Bruner (1986; 1990; 1991; 1993; 1995a; 1996; 1997) argues the process of meaning making as being central to the constitution of any culture. Unlike causal theories which provide a view from nowhere, theories of culture he argues are interpretative and illuminate the meaning-making narrative of participants. These interpretations are relative to the culture participated in and preoccupied with situated action. Within any culture, **action** is the **intentional counterpart of behaviour**, with the quest for meaning its cause and human nature its condition.

Bruner describes culture as the implicit knowledge of the world in which individuals through negotiation act in satisfactory ways. This enables making meaning and provides the template with which meaning is made. Since not only content but a stance towards that content is gathered, culture leads to consequences and transformations. Any active selection of negotiated meaning is deeply connected to the notion of 'self' and to membership in the community. The interweaving of 'selves' and meaning makes the nature of **meaning making intersubjective**. Culture thereby enables individuals to express solidarity and becomes a forum for the recreating of meaning. Understanding one another becomes an important precondition and crucial to individual acts of meaning. **Meaning making and individuals** therefore **converge in a culture**.

Such a conception of culture has three consequences. Firstly, the premise that individuals participating in and realising mental powers through culture, makes a study of the individual alone incomplete. Consequently, any analysis is to be organised around the public and shared nature of meaning making that connects the individual and culture. Finally, consideration of the nature and consequences of intentional states of participants enables reflecting on a culture's way of knowing.

In any culture the **idealisation of personal meaning** is **consolidated into a propositional form** by the imposition of syntactic rules and conceptual systems. Bruner argues that it is this transcending from a personal to a propositional form that is a source of conflict to the individual. However in making such a transition, culture provides **representational ready-mades** and **symbolic means**, so that individuals don't start from scratch. In adopting such a stance, education becomes an embodiment of culture and not a preparation for participating in one. Consequently pedagogy based upon a selection of meaning, negotiation, inquiry and recreation treats individuals 'as if' they had intentional states.

Taking the intentional states and understanding of both teachers and students into account, Olson (2003) extends Bruner's arguments and explains pedagogy as the competence of taking timely and informed decisions towards drawing minds and cultural resources together. He explains that the route to consolidate meaning into a propositional form is via a kind of teaching-learning, which depends on the **formations of joint intentions**, to which both teacher and student contribute to and accept responsibility. Such joint intentions in a shared vocabulary provide a bridge between the personal meaning made by the students and the propositional form of knowledge. Further, joint activity allows students to take the responsibility of learning society's legitimised knowledge.

My interest in discussing Bruner is to bring to fore the centrality of meaning making in the constitution of a culture. The public and shared nature of **intersubjective meaning informs analysis** related to the transformation of personal meaning into prepositional forms of knowledge. Both Bakhtin and Bruner point to this transition. While Bakhtin observes that complete reification leads to the disappearance of personal meaning, Bruner observes that such a transition at an individual level is a source of conflict. However Bruner also observes such transition to be accompanied by representational means which mediate and bring about as Vygotsky argued, the development of higher or cultural forms of behaviour. In the bringing together of cultural resources, Olson guides analysis of the establishment of joint intentionality and the acceptance of responsibility in the transformation of personal meaning to prepositional forms.

I now turn to discuss a **conception of culture** inclusive of intending persons and material embodiments. In describing psyche and culture to be seamlessly and dialectically constituted, Shweder argues:

Culture is the constituted scheme of things for intending persons, ... Culture refers to persons, society, and nature as lit up, and made possible by some already-there intentional world composed of conceptions, evaluations, judgements, goals, and other mental representations already embodied in socially inherited institutions, practices, artifacts, technologies, art forms, texts, and modes of discourse. (Shweder, 1990, p 20)

A **conception of an artefact** within any culture is offered by Cole (1999, p 90): '... a material object that has been modified by human beings as a means of regulating their interactions with the world and each other'. My intention of discussing the conception of a culture and an artefact at this juncture is on two counts. Firstly, to highlight the considerations that need to be accommodated in the understanding of the classroom as a culture. Secondly, given that the conception of an artefact is general, to underscore that my emphasis is on those material objects and interactions that contribute to the teaching-learning of mathematics in the classroom. I shall discuss shortly the nature and role of artefacts as outlined by Cole and presently turn to be informed from related fields.

In describing the role of artefacts in the constitution and sustenance of a culture, philosopher Wartofsky (1979; 1983a; 1983b; 1987) compares the role of a gene in biological evolution to that of an artefact in cultural evolution. He explains that human beings come to know via embodiments of cognitive activity and any study of **cognitive practice** proceeds **by way of representations or artefacts**: linguistic, pictorial, gestural or theoretical. Wartofsky also argues that since human praxis has a history, modes of cognitive practice, thought, ways of seeing and ways of knowing also have a history in any cultural practice.

The conception of artefacts by Wartofsky and Cole are polysemous with the 'ready-mades and means' of Bruner. Vygotsky himself used the term **tool** when referring to external and physical mediation and **symbol** while referring to internal or intellectual mediation. He (1978, pp. 53-54) observed: 'Distinctions between tools as means of labour of mastering nature, and language as a means of social intercourse become dissolved in the general concept of artefacts or artificial adaptations.' Towards a cultural conception of the classroom I adopt the term artefact as inclusive of tools and symbols and offer my rationale and operational classification of artefacts in my next chapter on methodology.

Tomasello (1999) an evolutionary anthropologist says that tools, symbolic mediators and social practices played a key part in the cultural origins of human cognition. Arguing that **cultural nurture is part of human biological nature**, Tomasello argues that cultural artefacts create the most distinctive and important cognitive products and processes. As created objects artefacts become imbued with intentionality; cognitive adaptation to them has changed the process of cognitive human evolution. Along with the human trait of recognising others as intentional agents, the inheritance of cultural artefacts forms the twin basis of sociogenesis. In agreement with Vygotsky, Tomasello says that cognition is a product of three processes: genetic events over evolutionary time, cultural events over historical time and personal events in ontogenetic time.

Donald (1998; 2000) a cognitive neuroscientist also argues that material culture (artefacts, language) play a seminal role in the formation of the human mind. He argues that the use of symbolic culture such as **language reshapes mental life** and constitutes a trait specific to humans. In agreement with Vygotsky and Bruner, Donald says that language has a qualitative impact on cognition and the symbiosis of mind and culture. Symbolic language is drawn into cognitive activity and symbolic thought that originates in external action is internalised. Any adequate science of culture can therefore not leave out the nature of such cognition.

Cole (1983; 1990; 1991; 1993; Cole & Engeström, 1993; 1995; 1996; 1999) provides a rationale for the **convergence of individuals with culture through cognition**. He includes patterns of interaction, transforma-

tions shared within individuals and among individuals, artefacts and social institutions. In addition to the convergence of culture and meaning making, I adopt following Cole those premises that allow analysis of the convergence of culture and cognition: mediation with artefacts, historical development, practical activity, social processes and ZPD.

Cole elaborates his definition of **artefacts** and describes them as being both **ideal and material**. In referring to the ideality of artefacts, Cole refers to that concept which was behind the first instance of creation of any artefact (e.g. a chair for sitting) which is mediated later upon subsequent use (in choosing a chair to sit on). While referring to materiality Cole includes both language and more visible forms of material culture. As a repository of artefacts, culture consequently allows individuals to interact with their past as well as with their future. Cole finally argues that it is by the use of artefacts that human beings participate in a double world, which is both objective or artificial, and subjective or natural.

Cole clarifies that the interaction of individuals with artefacts though patterned culturally, is far from uniform because face-to-face interactions are locally heterogeneous. Such interactions refer to those artefacts present in any culture and also those experienced by individuals in their personal development. Cole therefore argues for analysis of cultural mediation in specific contexts, the incorporation of which regards **cognition as distributed** across artefacts, contexts and accompanying social rules.

An analysis that takes into account distributed cognition is a way of coping with complexity, argues Pea (1993), that enables a shift of focus in educational practice from intelligence-as-substance, to learners as inventors of distributed-intelligence-as-tool. The need for attending to both individual and distributed cognition and their reciprocal interplay simultaneously in the same framework, is also argued by Salomon (1993).

Säljö (1998) explains the importance of a distributed study in situated activity. He begins by arguing that **to learn in situated activity**, means to appropriate artefacts and the conceptual resources constituted within the activity. He then argues that in any study of thinking pursued with and through artefacts, it is the sharing of artefacts as mediational means that becomes the object of analysis. Since it is possible to transform the capacity of intellectual action with artefacts, Säljö finally argues that it is possible to understand how individuals expand their intellectual repertoires and practical skills through collective participation.

Russian philosopher Ilyenkov (Bakhurst, 1991; 1995) clarifies that as created objects artefacts owe their status to material activity, where their significance or ideal is on account of their incorporation. For example words are artefacts invested with meaning, where such meaning is attributed to them by humans. Ilyenkov thus argues that although **human significance** is objectified or **reified in artefacts**, such significance cannot

be reduced to their material form. As a consequence individuals exist not only in nature, but in humanised nature embodied in artefacts. The primary object of **thinking with artefacts** thus becomes thinking not only of their material form, but rather of their humanised form or ideal.

My interest in discussing the nature and role of artefacts is to elaborate the consequence of considering their role both in a culture and in individual cognition. Apart from emphasis in analysis on practical activity and historical development; human mediation and thinking with their ideal is sought in contexts with accompanying social rules. It is to be informed by the nature of interactions and social relationships in situated contexts that I now turn. I elaborate upon the collaborative conception of the ZPD in the sub-section that is to follow.

Situative perspective: appropriation and participation

In adopting a cultural conception, I had extended basic Vygotskian theory in the previous sub-section. In addition to what a culture can be a location for, I now focus on how a **culture** can be a **location for meaning making and cognition**. In discussing the more contextual, social and situative aspects of any culture, I begin again with Vygotskian perspectives yet extend these with contributions by later scholars.

In arguing that intrapsychological functions follow interpsychological functions, Vygotsky pointed to two features of the **social environment**: its function and its role. Describing its **function** he said

everything that is cultural is social. Culture is the product of social life and human social activity. That is why just by raising the question of the cultural development of behaviour we are directly introducing the social plane of development. (Vygotsky, 1981a, p 164)

In elaborating its **role**, Vygotsky (1994e) argued against the social environment being regarded as a condition of development, but for a relative analysis of relationships that exist between the individual and the environment. Since the individual and the individual's environment both change in development, it is necessary to gather insight about the environment in addition to the individual. Vygotsky explained that as a source of development, the **social environment possesses the ideal**.

Stetsenko (1993) explains that **ideal forms in any culture** are objectified in habits, values, norms and artefacts, with culturally fixed ways of handling these. As such, a culture and its social practices contain the arsenal of human psychological functions: a system of behavioural, cognitive and communicative patterns, which every human being has to acquire in order to become a member of society and thereby culture.

Apart from the ideal invested in artefacts, the above arguments bring to attention another ideal: that of the environment. Objectified as in social rules and practices, the continuity of changing relationships between individuals and the environment is thus brought into analytical focus.

Two shortcomings of Vygotskian views are also observed. Firstly, Goodnow (1990) says that while accommodating a social view in analysis, the environment cannot be seen as benign and relatively neutral. Cognitive development in any environment is marked by qualitative aspects like the **acquisition of values** (e.g. what constitutes as intelligent action, why some approaches to problem solving are better than others) and social identity (related to the presence of power relations). As an example Cole (1990) has observed that school could be and has been be a source of social disruption and human misery.

Secondly, Wertsch and Toma (1995) suggest that social phenomena cannot be equated with intermental functioning. Such functioning which is the basis for **intramental functioning** is always **situated in cultural, historical and institutional settings**. In individual cognition with artefacts Wertsch and Tulviste (1996) point out that though tools fundamentally shape mediated action, action itself cannot be mechanistically determined by tools. Such action always involves an inherent tension between tools and their use in **unique and concrete instances**.

Towards analysing explicit and implicit values and the existing constraints of concrete situations in educational praxis, Moll (2000) recommends a **dynamical analysis** which recognises voices of unity and discord (understandings and misunderstandings) that develop in (adaptive or maladaptive) classroom practices. Moll suggests seeking culture in human practices, since it is people's involvement in contexts that constitute the social world. In integrating institutional processes in educational praxis, Forman (Minick, Stone, & Forman, 1993; 1996; Forman & Ansell, 2001; 2003) argues that the form and function of instruction, influences not merely cognitive but **motivational, affective and normative factors**. Towards such analysis she suggests two approaches: firstly, the structural interactions between individuals, artefacts and existing social practices. Secondly, the discursive and semiotic activities that are tied to these concrete social practices. I discuss analysis of social practices below and discursive aspects in the next sub-section.

Resnick (1987; 1991; Resnick & Gall, 2004) argues that only in an understanding of the circumstances and participants' construal of the situation, can any valid interpretation of cognitive activity be made. In emphasising socialisation for higher order skills, Resnick argues for a view of **intelligence as social construction**: incorporation of individuals as a member of a community through, observation, cooperative participation and intersubjectivity. Miller and Goodnow (1995) argue that development in addition to cognition is integrated in social practices. They define **practices as actions**, shared and invested with normative expectations with significances that go beyond immediate goals. Drawing upon language studies, they explain the importance of practices to a culture:

1: Practices provide a way of describing development-in-context, without separating child and context and without separating development into a variety of separate domains. 2: Practices reflect or instantiate a social and moral order. 3: Practices provide the route by which children come to participate in a culture, allowing the culture to be 'reproduced' or 'transformed'. 4: Practices do not exist in isolation. 5: The nature of participation has consequences. (Miller & Goodnow, 1995, pp. 8-13)

My intention in discussing the above arguments is to point to the analytical importance of social practices to cognition, intersubjectivity, development and culture in any environment.

With special reference to teaching-learning situations, Mercer (1992) explains that any learning faced by students is never decontextualised, because the learner necessarily invokes prior experience in making sense of the task. Exemplifying the concept of appropriation (which I discuss shortly) in classroom talk, Mercer argues that **learning is in the talk and the talk is heavily contextualised** by the learners. Mercer argues that any serious interest in how children gain educationally relevant knowledge and understanding needs to attend to and not ignore the meaning of classroom tasks to the students. Mercer argues that educational advancement depends on educational experience in situational contexts.

The **concept of appropriation** of cultural capacities was forwarded by Leont'ev (1981b) in addition to the conception of 'activity'. Leont'ev argued with an example, that the formation of language in individuals needs the existence of language in their environment. In possessing language, the environment objectively posed itself to the individual, who had to acquire the same in an active process. Such ability was formed in joint activity with others, with specific goals to acquire required skills, where in the process human capacities are formed. In contrast to biological adaptation Leont'ev elaborated appropriation as below:

The child does not adapt itself to the world of human objects and phenomena around it, but makes it its own, i.e. *appropriates* it. ... It is a process that has its end result the individual's *reproduction* of historically formed human properties, capacities, and modes of behaviour. (Leont'ev, 1981b, p 422)

I find two historical dimensions enter any discussion of concrete activity in contexts: the historical experience of the student which he brings to the present context and that of human inheritance which a student appropriates in a given context. In extending the scope of analysis that incorporate cultural, social and historical contexts Butterworth (1992) recommends the inclusion of everyday reasoning, logical structures or ways of knowing as contexts. In taking such a view, content and context become inseparable where both are the public frames of reference which recruit an individual's thinking. Butterworth explains that when both content and context are inseparable, they form part of the **everyday or intuitive knowledge** of the individual. Having a heuristic value they form an interface between the novice and expert stages in learning.

My discussion so far has been with an intention of recognising, following Vygotsky, perspectives that are analytically important to the relationships between the environment and the individual. Elaborating upon this importance, Goodnow and Warton (1992) recommend a **pluralistic view to relationships** between context and cognition since a single position is an exception and not a norm. They also recognise ambiguity that could arise from cultural models of anthropologists and social representations of social psychologists, that address these relationships and which I now discuss. I begin with the cultural models of Rogoff, followed by the social representations of Lave and Wenger and conclude with a discussion of the situative framework of Greeno.

Rogoff (1990; 1995; 1999; 2003) describes the context of any problem as its physical, conceptual structure and purpose in any sociocultural milieu; integral to cognitive aspects and not a nuisance variable. I outline two of her constructs corresponding to interpersonal and community processes. Rogoff advances the construct of **guided participation** as the guided construction of meaning leading to cognitive development. Being intersubjective such a model makes the interdependence and independence of an individual culturally given and suited to my study. In **participatory appropriation**, Rogoff draws on Bakhtin and Leont'ev and focuses on the dynamic processes of transformations that take place by the participation of individuals in any culture:

Cognitive development consists of individuals *changing their ways of* understanding, perceiving, noticing, thinking, remembering, classifying, reflecting, problem setting and solving, planning, and so on – in shared endeavours with other people building on the cultural practices and traditions in communities. (Rogoff, 2003, p 237, emphasis added)

Lave (1990; Lave & Wenger, 1991; 1991; 1993) defines context as the way in which individuals organise themselves and in which they are part of and not something they are put into. Her model of **participation** focuses not on transformations but relationships arising in local practice:

Instead of asking, What is the constitutive relationship between persons acting and the contexts with which they act? the question becomes, *What are the relationships between local practices that contextualise the ways people act together, both in and across contexts?* (Lave, 1993, p 22, emphasis added)

Lave's model sees knowledge building as part of the process of becoming a member of a community. Knowing in such practices is located in relations among practitioners, their practices and the artefacts of that practice; the understanding of the world as experienced. Wenger (1998) extends the social model of learning and argues that unlike knowledge which is more a matter of competence in valued enterprises, **knowing is a matter of participating in meaning**. He argues that since learning is a central aspect of being social beings and a central aspect of participation, its reification leads to knowing.

While my discussion of Rogoff emphasises on transformations in cultural practices, that of Lave emphasises on participation in social practices. Yet as Miller and Goodnow have argued, practices allow for development in a context and provide the route by which individuals participate in a culture. In my cultural view of teaching-learning of mathematics in the classroom, I adopt where appropriate the models of transformations in a cultural practice and participation in the social practice as heuristic devices with which tease out various implications.

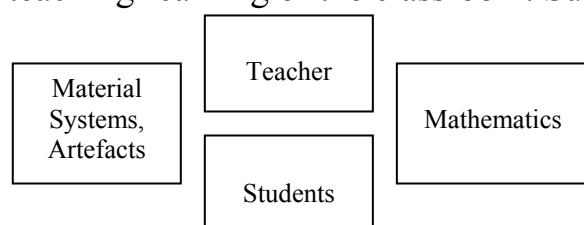
As to the importance of mediated action with artefacts which I find as applicable to either model, Brown, Collins and Dugid (1989) argue that in any situated activity, it is mediation with artefacts that determines how participants are enculturated. Since such mediation is epistemologically prior to any conceptualisation, bypassing actions of mediation they argue has the danger of bypassing ways of knowing. I now extend my discussion of ‘enuclturing participants’ with my focus on knowing.

In analysing learning as a consequence of the use of knowledge is an aspect in participation, Greeno (1998; 2003) elaborates as below:

I prefer the word *situative* as a modifier of *perspective, framework* or *theory*, rather than *situated*, as a modifier for *action, cognition* or *learning*. Phrases such as ‘situated learning’ invite the misconception that there are some kinds of learning that are situated and others that are not situated. Instead the *situative* perspective assumes all learning and cognition is situated; the differences have to do with where and how the processes are situated and not whether they are.

(Greeno, 2003, p 315, emphasis in original)

My interest in discussing the **situative analysis** of Greeno, is in highlighting the ‘where and how’ processes with respect to knowing in the teaching-learning of the classroom. Such an analysis incorporates a relational view of the teachers and students, the culture of mathematics and the artefacts and material practices (shown alongside) that constitute teaching-learning. It is successful participation and the negotiation of



(Adaptation of Figure 21.1, Greeno, 2003, p 321)

meaning in these relationships, that Greeno says leads to knowing, as also to changes in patterns of discourse and understanding (I discuss these shortly). Apart from linking the importance of such an analysis to discursive elements, Greeno observes a situative emphasis to have consequences: not only for what students learn but what kind of learners they become, and how they understand what it means to learn and know.

Starting from my focus on meaning leading to knowing, made possible in social practices in the classroom, which together constitute a micro-culture of teaching-learning, within the larger cultural praxis of mathematics, I now turn to discuss the discursive aspects within.

Discursive framework: collaboration and the ZPD

In conclusion to socio-cultural-historical perspectives, I discuss the narrowest focus of my analytical lens. Yet before discussing the nature of this focus I briefly recount the rationale of arguments made so far. Having conceived the teaching-learning of the classroom as a micro-culture, I first discussed the broadest analytical conception of a culture towards the development of individuals, meaning making and cognition with artefacts following Vygotsky, Bruner and Cole. I also discussed the instrumentality of speech by Luria, the materiality of meaning and ‘activity’ by Leont’ev and the situated nature of meaning and consciousness by Bakhtin. The arguments made by each have helped me highlight the issues that need to be taken into analysis of the nature and kinds of sociability that constitute individual human development.

I then turned to analytical issues of the individual-in-social perspective, by discussing models that address possible relationships between the individual and the environment. Either by analysing transformation, enculturation or participation, my objective has however been to focus on the processes of knowing, relevant to teaching-learning of mathematics in the classroom. Having discussed more cultural and social aspects of analysis I now turn to the interpersonal or collaborative efforts of individuals in teaching-learning. I begin with discursive and semiotic processes of collaboration, followed by the Vygotskian distinction of everyday and scientific concepts and his formulation of zone of proximal development (ZPD) extended also by other scholars.

For any discipline to be scientific its praxis needs to be viewed as a larger ongoing discourse explains Bereiter (1994; 1997). Discourse in classrooms thus needs to be seen, not as preparation for participation in an eventual discourse but as current participation in the societal discourse. Such a framework involves bringing students to clarify observations, examine assumptions and resolve doubts. Consequently teaching-learning is a **collaborative and developmental continuum** of creating, building and adding value towards knowledge construction.

A discursive conception as I have outlined above, enables the study of what constitutes meaning making, thinking and reasoning in socially communicative activity, argues Lerman (1998a; 1998b; 1999; 2000a; 2000b; 2002). Since such collaboration is embedded in classroom practices Lerman argues for a holistic analysis. Firstly, since meaning made in practices is neither static nor independent of them, he recommends analysis of **practice-in-person-in-practice**. Secondly, in analysing the situated nature of knowledge, he recommends a view of both what is focused and what is not in the zoom of the analytical lens. For example mediation he says is a generalising principle that can look for similarities, and ‘activity’ for specific analysis, a focus I adopt.

In any discursive analysis of the classroom, Seeger (1998a; 1998b; 2001a; Seeger, Voigt, & Waschescio, 1998) cautions that discourse alone does not make the world of the classroom, and argues for the study in addition of semiotic or mediated activity with artefacts, which are not only appropriated but constructed, reconstructed and remediated in the process of appropriation. He observes the nature of **teaching-learning** in a classroom to be **both horizontal** (between peers) **and vertical** (mediated thinking). Describing the communication within to be a hybrid of three types of learning: mimetic (observing, performing), discursive (speech genres, narrative) and theoretic (externalisation of knowledge), Seeger argues for the need to grasp crucial moments and unique turning points in the **slow developing process of teaching-learning** in relation to changes in classroom practices, wherein under the influence of the theoretic learning, the meaning of the mimetic and the discursive change. In adopting an analytical view that communication in the classroom is part of a larger societal discourse, where in its situated nature in the classroom both horizontal and vertical elements are to be accommodated in its development, I turn to draw from semiotic perspectives.

Semiotic perspectives focus on communicative activity of representations and signs. Explaining that all sign use is socially located and can never exist in isolation, Ernest (2006) describes any semiotic system to comprise of **a set of signs** (uttered, drawn), **a set of rules** (e.g. cancelling in fractions) and **a set of relationships** (meanings in underlying structure). Ernest describes the classroom practice of historically developed semiotic systems like mathematics to consist of three transitions between four stages. The public structure of mathematical theory is first *recontextualised* as curriculum for school mathematics. This public curriculum is then *realised* as taught topics. Such topics are privately *appropriated* by the students. In this process meaning is made afresh by each individual who in turn mobilises earlier elements of meaning and understanding.

That meaning making is both an intersubjective and a semiotic-cultural construct, playing a central role in the **production of objects of knowledge** is pointed out by Radford (2002; 2003; 2006). He argues that signs, artefacts and linguistic devices are means of objectification, intentionally used in order to achieve awareness and carry out actions. It is through meaning that the individual and culture on one hand and knowing (more of this shortly) and knowledge are realised. Knowledge is a product of sustained reflection on their mediation in cognitive praxis.

Both Ernest and Radford qualify the social and objectifying processes of meaning making and emphasise the (re)creation of meaning towards knowing. Their arguments are in agreement with those of Bruner, Leont'ev and Bakhtin who emphasise the material nature and making public of meaning in shared and concrete activity.

In qualifying the nature of collaborative processes towards knowledge construction in particular, Wells (1999) argues against a focus on knowledge (as abstractions or theory) in favour of a fruitful analytical focus on knowledge construction through collaborative processes, in which **knowing** is involved. Wells summarises the activity of knowing in two ways. Firstly, as modes of actions that involve **representing**, recognising, hypothesising and concluding, in which he is in agreement with semiotic perspectives above. Secondly, he also argues that although knowing is necessarily individual, its purpose and fullest realisation is in its **socially-oriented creation** generated in concrete practice, in which he is in agreement with Bruner and Leont'ev.

Wells also argues that knowledge as the object of activity is a situated process, that involves the creation of physical and intellectual artefacts, that do not in themselves constitute knowledge, but mediate the **activity of knowing**. He argues that theoretical knowledge construction most frequently occurs in the context of a problem of some significance. It takes the form of a dialogue in which solutions are proposed and responded to. Knowing, as substantive knowledge, is thus embedded in practical activity and becomes material for a more detached, context-independent form of knowledge building as **knowledge artefacts**. Created in one cycle these mediate the next cycle of knowledge building activity, and mark the beginning of theoretical knowing. The activity of knowing and creating knowledge artefacts preserve the outcomes of knowing, which illuminates both current and prospective practice of teaching-learning. In agreement with Bishop and Greeno on a focus of participation and representation towards knowing, I shall elaborate upon the use in analysis of knowing and knowledge artefacts in Chapter 3.

Drawing upon the perspectives of language in the analysis of knowing, I draw upon Edwards and Mercer (Edwards & Mercer, 1987; Edwards, 1993; Mercer, 1995) who argue that any construction of knowledge through language, involves the cultural and social circumstances that bring about knowing. They observe that the extent to which knowing is made common amongst students through collaboration and discourse, is a measure of the effectiveness of a classroom. They term the development of knowing constructed through joint activity and discourse as **common knowledge**, the development of which they say is one of the main purposes of education. Classroom practices like group work, which make cooperation and reasoning with each other necessary, provide such opportunities besides the **sharing of power** in teaching-learning.

It is my interest to elicit and analyse those events in my classroom that bring about common knowledge (as above) in the teaching-learning of mathematics and address the imbalance of power in the processes of collaboration and knowing as pointed out by Skovsmose.

Having drawn upon more recent arguments that address the analysis of discourse and collaboration, and since the students I observe are adolescents, I turn to the nature of their learning as argued by Vygotsky, before discussing finally and as promised his formulation of the ZPD.

Vygotsky (1994a; 1994b) described **learning in adolescents** as **conceptual** and qualitatively different from the sensory sources that learning is based upon. Adolescent learning he says is goal directed, progressively abstract and formed during a process of finding a solution. Vygotsky observed that the presence of a raised problem, a reality which stimulates the necessity of solving the problem and the goals put forward by the social environment make adolescents take initiative in the development of their thinking. Concept formation starts with the meaning of words (not fully defined), passes a stage when relationships are formed and ends in analysis and abstraction (unity of form and content). In addition to the personal, propositional and material nature of meaning making Vygotsky informs analysis by drawing attention to its conceptual nature. I now discuss his differentiation of these concepts into two categories and the transformations from one to the other in the ZPD.

In formulating teaching-learning which arose from concrete experiences, Vygotsky (1986; 1987; Kozulin, 1990; 1998) distinguished two groups of concepts: spontaneous (everyday) and scientific (academic). **Spontaneous concepts** emerged from reflecting on everyday experience. They were rich, unsystematic, highly contextual, empirical and practical. **Scientific concepts** originated in highly structured and specialised activity and imposed on the child concepts that were logically defined. They had a formal and decontextualised structure and were conscious and deliberate. Scientific concepts that developed in teaching-learning activity worked downward towards greater concreteness, while spontaneous concepts worked upwards towards greater abstractness.

Vygotsky also argued that while scientific concepts represented education, spontaneous concepts represented development. Such a view had two consequences. Firstly, that classroom instruction depends on natural cognitive development. Secondly, that **education runs ahead of development** and raised the perception of students' ability. Teaching-learning bridged the gap between spontaneous and scientific concepts. As an example, Vygotsky (Vygotsky & Luria, 1994b) explained that the acquisition of a word was not the end but the beginning of knowing related to the word. Vygotsky called the development resulting from such human collaboration a potential of development, which he called the zone of nearest development or the **zone of proximal development: the ZPD**.

Various references are found in literature to the analysis of teaching-learning with reference to the ZPD. It is to discuss those arguments that I draw upon in my thesis that I now turn to.

In analysing the ZPD in my study, I continue with my emphasis on teaching-learning towards knowing of mathematics in the classroom, and begin with Leont'ev (the son) and Luria (1968, pp. 365-366, emphasis added), who explain Vygotsky to argue that a child's inability to solve any problem may be: '... the result of the child having insufficient knowledge and *know-how* that prevents the child from finding the necessary solution independently.' This is circumvented they say by the ZPD, which provides a dynamic principle and scientific basis for the teacher to foresee a child's development, go beyond the child's current knowledge and intervene actively.

In extending an understanding of the nature of intervention that is possible in the ZPD and in a version called **scaffolding**, Bruner (1984; 1985; 1995b; 1999) envisaged a vicarious sharing of not only knowledge but also consciousness in the ZPD, providing more abstract ground and newer uses of language leading to newer consciousness. Bruner observes that via the ZPD, Vygotsky intended for society to take collective responsibility of the growth of the child, wherein language is the collective tool. Language is also the means by which the child could be lured into a ZPD, enabling an opportunity to recognise not only what others did but their intentions as well. The greater consciousness achieved in such dialogue and collaborative effort through the semiotic means of language within a ZPD, is what Holquist (2002) relates to as Bakhtin's notion of meaning of an utterance, by which individuals relate to each other and the future. Acquisition of language in such a manner, is thus not only learning to talk but **under loan of consciousness, learning to think**.

The importance of goal-directed activity, wherein students progress beyond themselves, upon the creation of a ZPD is observed by Cole (Griffin & Cole, 1984; 1985; Newman, Griffin, & Cole, 1989). Cole argues that a sequence of pedagogical steps or sequenced tasks provide the context for making meaning and the opportunity for teachers and students to **appropriate each other's understandings**. The social organisation and materiality available, brings about cognitive change and enables diversity. In explaining that material tools (artefacts) appropriated in the ZPD are representative of cultural ways of doing things, Stetsenko (1999) refers to their meaning as **objects-that-can-be-used-for-a-certain-purpose**. Underscoring the importance of education running ahead of development, her stand is in agreement with Chaiklin (2003) who locates the practices of **imitation**, accompanied by understanding and premised upon development, as being central to the ZPD.

My interest in presenting the above arguments is with an intention of discussing various models that bring into analysis, the use by students of various artefacts (words, tools and signs) under the guidance of the teachers or cooperation with each other in the classroom.

As to the objectives of such guidance I turn finally to Rowlands (2004) who argues that the ZPD is also a **pedagogical methodology** where necessary steps are taken to facilitate the construction of the target concept in the completion of a task. He argues that since the difference between scientific concepts and spontaneous concepts is the difference between the teaching-learning or not of these concepts, it is in this very difference within which teaching-learning must be situated.

Towards my focus on ways of knowing in the teaching-learning of mathematics in the classroom, I find the conception of ZPD to allow for analysis of how the spontaneous or everyday nature of personal meaning; is consolidated into the scientific, academic nature or propositional forms of knowing. The models of sharing know-how, scaffolding, learning to use language, learning to think or how to use objects allows for analysis of events between the teacher and the students or between students, in the teaching-learning of mathematics in the classroom.

Taking ZPD as a culmination to my discussion of developmental perspectives, I now turn to the first of my two areas of focus: meaning making in the teaching-learning of mathematics.

Meaning making in teaching-learning

In discussing literature in meaning making relevant to my study, I first relate one of the earliest studies of meaning in mathematics education. I then outline research which outlines meaningful activity and later the importance of classrooms norms and practices that promote meaningful activity in the teaching-learning of mathematics. I finally discuss the consolidation of meaning leading to mathematical knowing.

One of the earliest studies of meaning aimed at making arithmetic meaningful was conducted by Brownell (1942; 2004; Kilpatrick, 1992). I refer to Brownell's writings since they represent a time when memorisation in mathematics held sway, which Brownell wished to change along with a belief that instruction could be improved. Brownell was critical of Piaget's analysis of problem solving by children. He argued that laboratory studies were untenable since they studied a single variable holding everything else constant, which did not depict learning situations in classrooms at school. I consider Brownell's writings significant on two counts. Firstly, they recognise and argue that **meanings** are not absolute but **relative and cumulative** in nature and only eventually become concise in teaching-learning. Secondly, these writings and the issues they attempt to address seem representative of earlier research, which could have benefited from socio-cultural-historical perspectives.

I now turn to discuss more recent studies which draw upon socio-cultural-historical perspectives and address the development of meaning making in mathematics, in classroom teaching-learning.

While recognising communication as a collaborative endeavour in publicly pooled meanings, van Oers (1996; 2001) like Bruner earlier, makes a distinction between cultural meaning (propositional) and personal meaning. Learning mathematics, as a meaningful activity, is the process of mastering cultural meaning and attaching personal meaning to the actions involved. In such teaching-learning the teacher represents a cultural voice, that helps create a mathematical attitude in the students by personal conduct and creating expectations in shared activity. **Meaningful mathematics** van Oers observes, is not the link with meaningful problem situations but the observance of particular rules, concepts, tools and values in a discourse which defines whether one is doing mathematics or not. In my study I observe the occurrence of instances and actions in teaching-learning, which lead to students appropriating and **participating meaningfully** under the guidance of the teachers.

From a focus on what constitutes meaningful activity, I now turn to the realisation of **meaningful thinking in the classroom**. Discussing classrooms that promote meaning and understanding, Romberg and Kaput (1999), Fennema, Sowder and Carpenter (1999) and Carpenter and Lehrer (1999) collectively argue for the redefinition of mathematics as both object and means. Analysis of such teaching-learning they say needs to recognise tasks, tools and normative practices and the personal involvement and social negotiation that help students author their own learning. Teachers they say play an active role in establishing such a classroom, where their own understanding of the teaching-learning process is a goal as well. Following the above arguments and those of Leont'ev and Cole earlier, I theorise upon the role of tasks, artefacts and practices, which lend themselves to meaningful teaching-learning.

Evidencing that students develop understanding from classroom experience, Schoenfeld (1988; 1991; 1992) argues for the importance of research, in understanding the mechanisms and design of classroom cultures that work. Observing that everyday practices and the cultural milieu define and give meaning to the subject matter taught and have instructional roots, Schoenfeld urges for research to seek a way to develop **classrooms** that are **microcosms of mathematical sense making**.

That any development of meaning in the classroom cannot be decontextualised, but is a result of contextualisation and situatedness of personal and historical experience, is argued by Otte and Seeger (1994). They underline an epistemological principle of relational thinking (relation between things) as essential in modern studies. From the perspective of the student, such relational thinking is both the knowing of **knowledge**, and the knowing about knowledge or **meta-knowledge**. These arguments along with those of Schoenfeld, Vygotsky, Bruner and Bakhtin earlier, speak for the importance in research of the cultivation of class-

rooms that are multi-voiced for meaning making. It is my interest to study the enabling of reflection and relational thinking in the classroom.

Any micro-culture of sense making entails the establishing, maintaining and sustaining of norms of action and interaction, observes Lampert (1990; 2001; Lampert & Cobb, 2003). She argues that the practice of knowing needs to be brought closer to what it means to know. Teaching she argues is not only about content but also about what a lesson is and how to participate in the lesson. **Norms and practices intertwined with goals and means**, she says, consists of deliberately establishing and sustaining physical, social and linguistic routines towards enabling a classroom culture in which a teacher can teach and students can study. Placing emphasis upon communication, Lampert describes **content**:

As it is enacted in classroom relationships while students work on problems, the content is more than a series of topics. When students engage with mathematics in a problem, the content is located in a mathematical territory where ideas are used and understood based on their relationships to one another within a field of study. (Lampert, 2001, p 431, emphasis in original)

As an example of deliberately connecting content across lessons, Lampert discusses anticipation of connections that can be made in a problem context. Lampert's arguments extend to the mathematics classroom, the emphasis on participation in joint intentionality in teaching-learning norms and practices that could lead to greater knowing, as argued by Bruner, Olson, Wenger, Greeno and van Oers.

Evidence that classroom practices influence everyday use of mathematics is had from Boaler (1999; Boaler & Greeno, 2000) who found that in **different classroom cultures**, students had **different affordances and constraints** which affected learner identity and the way students perceived mathematics. In highlighting the distinctive features of classrooms that create constant pressures, Doyle (1986; 1988) points to: multidimensionality, simultaneity, immediacy, unpredictability, publicness and history. He argues for the importance of studying **situational forces that shape curriculum** and hold it in place as a classroom event. Recognising that shared meanings of teachers and students govern their interaction and create the culture of the classroom, Nickson (1992) simultaneously argues for an understanding of the uniqueness and diversity of **teacher-student and student-student interactions** in classrooms. The above arguments are in agreement with the situatedness of meaning, as argued by Bakhtin and the importance of the concrete instances of mediated action, as argued by Tulviste, Moll and Forman.

In my discussion of the importance of classroom practices to the teaching-learning of mathematics, I finally bring in the notion of **sphere of practice** as outlined by Skovsmose (2005). Making a clear distinction between the meaning of a mathematical concept and the meaning of a mathematical task in educational practice, Skovsmose argues that for

students **to ascribe meaning** to concepts they learn, it is **essential to provide meaning** to the educational situation in which the students are involved. Any event or action when analytically isolated, he says, may appear without meaning but makes sense within a sphere of practice or network of tasks. An understanding of activities that provide experience are important Skovsmose argues, because they bring to fore students intentions and recognise **a meaning-producing foreground** to be part of negotiations within classroom teaching-learning. Skovsmose's arguments in mathematics education parallel and define the importance of practices discussed in socio-cultural-historical perspectives. They allow in analysis the highlighting of the person and practice relationship argued by Lerman, the enabling of personal and tacit components in knowing argued by Ernest, Otte and Seeger and the importance of common knowledge as argued by Butterworth, Edwards and Mercer.

In discussing the importance of social practices towards cultural human development and meaning making in socio-cultural-historical perspectives, along with the importance for the same within mathematics education, I prepare ground and basis for attending to classroom norms and practices established towards teaching-learning in the classroom. I provide data and analysis of these norms and practices as they evolved in my classroom in Chapter 4: *A collaborative classroom practice*. These norms and practices form the basis for two aspects. Firstly, they sketch the time-line and the kind of experiences upon which the micro-culture of the classroom is constituted. Secondly, their analysis informs the basis upon which I discuss other thematic data and analysis chapters to follow: Chapter 5: *The consolidation of meaning*, Chapter 6: *Problem solving know-how* and Chapter 7: *Cooperative problem solving*.

Having premised the relationships of a micro-culture of teaching-learning, in classroom norms and practices, I now turn to discuss ways of knowing in teaching-learning. In a shift in emphasis from technique and doing mathematics, to meaning and a way of knowing mathematics, Bishop (1988; 2004) describes mathematics education as a process of **enculturation**, that needs to incorporate the following five aspects:

- be interpersonal and interactional;
- take significant account of its social context;
- be formal, institutionalised, intentional, accountable;
- be concerned with concepts, meanings, processes and values;
- be for all. (Bishop, 1988, p 124)

Bishop argues that although a way of doing is inextricably related to a way of knowing, the goals of the two are considerably different. A **way of knowing** is a socially constructed set of ideas and meanings, where meaning is achieved by connecting particular mathematical ideas under discussion, to the remainder of the individual's personal knowledge.

Bishop's arguments present in mathematics education the emphasis on the social construction of meaning and knowing as argued by Bruner,

Greeno, Brown, Collins and Dugid and Wells and the interconnectedness of ideas in mathematics as argued by Hardy.

Extending his concern for knowing, Bishop argues that the processes of enculturation have equal responsibility towards both the child and the culture of mathematics. **Mediated by teachers** such processes are necessarily an interpersonal affair, where teaching contributes to the awareness of the cultural history of mathematics, as well as the development of mathematics in the micro-culture constituted in the classroom. Bishop finally argues for a fundamental explicitness about values in an education: about mathematics, through mathematics and with mathematics. Acknowledging the complexity of such a task Bishop urges the **development of values and awareness** so as to develop the capacity to **reflect and make choices** in mathematics and not merely train to adopt certain values. In highlighting the importance of the teacher's role, the arguments of Bishop are in agreement with their role in establishing norms as argued by Lampert, representing the cultural voice of mathematics as argued by van Oers, guiding collaborative enquiry as argued by Wells and Cole and finally contributing to the acquisition of values that constitute cognitive development as argued by Goodnow.

In the importance of teaching-learning and collaborative practices in schooling to facilitate knowing, rather than merely acquiring knowledge I finally quote Burton (1999a; 1999b; 1999c; 2002) who advocates a **narrative approach to learning**. A consequence of the approach she takes, is that meaning made in the micro-culture of any classroom is negotiable, uncertain, suffused with feeling, a complex of relationships and non-homogenous. Burton underscores that if the purpose of learning is to make mathematical meaning, then research in classrooms needs to reveal how a student is positioned to be an **agent of one's own learning**. She also argues for the creating of conditions where learners are encouraged **to value and explore** their **intuitive processes** and the means by which to gather these processes towards greater knowing. Burton's arguments are in agreement with the analysis of the meaning making narrative of individuals in a culture as argued by Bruner, the personal and tacit components as argued by Ernest, the component of meta-knowledge in addition to knowledge as argued by Otte and Seeger, and intuitive knowledge of individuals as argued by Butterworth.

I discuss the above arguments towards bringing together the analytical issues that are related to my focus of the process of transformation of intuitive and personal knowledge to cultural or propositional forms of knowledge in the teaching-learning of mathematics, encompassed in a ZPD as argued by Vygotsky. I offer data and analyse specific instances of these in the classroom I study, made possible under the guidance of the teachers, in Chapter 5: *The consolidation of meaning*.

In concluding discussion on my first area of focus: meaning making and before moving over to the second: problem solving I discuss here an aspect crucial to the social construction and consolidation of meaning in any cultural conception, the notion of **common sense**. Describing common sense as the taken for granted concepts in any micro-culture Keitel and Kilpatrick (2005) argue that the making of common sense is not a problem of the individual but **a collective process**, enabling the developing and challenging of assumptions commonly held. They argue that neglecting the development of common sense during teaching-learning, either by implicitly or deliberately referring to it or by strongly rejecting it hinders sense making. Along with Keitel and Kilpatrick above, Kilpatrick, Hoyles, Skovsmose, and Valero (2005) argue that common sense compliments school mathematics and forms the **corpus of social or cultural knowledge and values**. Common sense understanding is based on familiar correspondences, where **to know means to justify conclusions** already formed. I incorporate the study of common sense in my analysis of meaning making, leading to knowing in teaching-learning. That such a reference is also made in the processes of problem solving, is something I premise in the discussion of my second area of interest: problem solving in the teaching-learning of mathematics.

Problem solving in teaching-learning

Consequent to the nature of meaning made in the classroom, with a focus on personal meaning becoming propositional as knowing, I discuss in this section the application of meaning made and knowing, in teaching-learning towards problem solving in mathematics. My particular interest in problem solving within the micro-culture of teaching-learning in the classroom I study is on two counts: the development of problem solving know-how attended to in the teaching-learning of the mathematics and the cooperation and argumentation of students as a group at attempting problems. My focus on know-how is broadly commensurate with the problem of know-how as argued by Vygotsky, the functional use of speech as argued by Luria and the loan of consciousness as argued by Bruner. My focus of cooperation is commensurate with the nature of intersubjectivity between students, as argued Bruner and Bakhtin and the nature of these towards goals as argued by Vygotsky, Leont'ev and Cole. In addressing how problems are attempted in my classroom, I now draw from literature in mathematics education.

I begin with Pólya (1987) who outlined the correlation of teaching-learning in the mathematics classroom to three factors: learning situations, learning teaching and their long-term educational effects. Pólya treated teaching as an art and not a science and foremost about teaching students to think. Pólya also believed a grain of discovery to lie in the

solution to any problem and that problem solving can be taught. Solving problems appropriate to ones knowledge with heuristic or discovery methods, Pólya said, led to independent thinking. His **four stage method** consisted of: understanding the problem, devising a plan, carrying out the plan and looking back. I make reference to Pólya's four stage method in my data and analysis Chapter 6: *Problem solving know-how*. I had made reference to Pólya's model earlier while discussing the role of speech in solving problems by Luria, where I mentioned that unlike Pólya's model that of Luria's incorporated the use of speech. While expressing my intention of addressing this theme in Chapter 6, I presently turn to the importance of the problem solving in teaching-learning.

Kilpatrick (1987) argues that Pólya was concerned with high school mathematics teaching, wherein Pólya felt the goal was more than mathematical knowledge but the willingness to do something with that knowledge. Pólya argued for **know-how on problem solving** to be acquired in teaching-learning practice. This meant primarily the ability to formulate, solve and critically reflect on problem solving process. Lack of mathematical know-how was for Pólya the worst gap in the preparation of high school teachers. Consequently how a teacher teaches is more important than what the teacher teaches. Pólya argued that **to know mathematics meant to do mathematics; to do mathematics in turn meant know-how and know-how in turn meant problem-solving**.

It is towards the purpose of knowing, doing and solving problems that I draw upon socio-cultural-historical perspectives, which address the issues of how this can be achieved in the ZPD as primarily argued by Vygotsky, where the use of words as argued by Luria and Bruner, the loan of consciousness as argued by Bakhtin allow for the use of artefacts as argued by Cole, Wells, Stetsenko and Chaiklin.

My interest in discussing the above is in analysing how problem solving know-how, is built and developed in the high school classroom I observe, as directed by Pólya. However, addressing 'know-how' is as much a problem solving process or ability, as it is an epistemological process and ability. The Oxford English Dictionary Online (© 2006) defines epistemology as 'the theory or science of the method or grounds of knowledge'. In conceptualising such processes in mathematical knowledge and thereby its teaching-learning, Ernest (1998) identifies tacit know-how as part of personal knowledge and argues:

The motivation of including tacit 'know-how' as well as propositional knowledge as part of mathematical knowledge is that it takes human understanding, activity, and experience to make or justify mathematics; in short know-how is needed. ... Most personal knowledge ... consists of tacit knowledge of methods, approaches, and procedures, which can be applied to new situations or problems. ... Thus what an individual knows ... in addition to publicly stated propositional knowledge, includes her mathematical 'know-how'. (Ernest, 1998, pp. 248-249)

In voicing the importance of know-how in solving problems, Ernest argues for the building of methods as argued by Pólya and the adding of a personal and intuitive component of meaning as discussed by many earlier. That such meaning is a result of the materiality in ‘activity’ is argued by Leont’ev and in graded tasks by Cole, is something that I draw upon in analysis in discussing as pointed out by Vygotsky: that a child’s inability to solve any problem may be the result of the having insufficient know-how in finding the necessary solution independently.

Drawing upon socio-cultural theories and arguing for student participation in the **tacit classroom culture of mathematics**, is also argued by Goos, Galbrith and Renshaw (1999) who observe that the use of cultural tools or **artefacts fundamentally change** the nature of **the task** and the requirements to complete the task. They argue for the importance of teacher’s actions in conceptualising of the classroom as a community of practice. Such a stance they observe provides students with sense making goals, support for appropriation and metacognitive strategies (more on this shortly), enabling the learning of mathematics in a meaningful way. These arguments in mathematics education parallel the arguments of situated activity with artefacts made by Wertsch, Tulviste and Säljö.

In also addressing epistemological issues of problem solving in the micro-culture of teaching-learning, Schoenfeld (1985; 1989; 1992; 1994) describes **metacognition** as the knowledge about one’s own thought processes. He discusses the need for instruction to encourage metacognitive abilities and the importance of the teacher’s role in such a process. By getting students to understand the problem and embark on a solution, taking care of appropriate representations (more of this later) and capitalising on opportunities, Schoenfeld believes school mathematics is **simultaneously a cultural and cognitive phenomenon**; contextually bound to practices, where teachers’ actions could lead to internalisation in the Vygotskian sense by the students. If learning is culturally shaped and people develop understandings by participating in the same, then Schoenfeld says that **membership** in that community **constitutes mathematical thinking and knowing**. As such ‘natural’ classroom realities that shape learning and problem solving need to be explored. Schoenfeld also observes that as a science of patterns, relevant mathematical activity is one where connections are seen, structures perceived and valued. If instructional practice is **deprived of an apprenticeship** where the teacher can provide students this access, then the students are also **denied access to doing and knowing** mathematics.

In making the above arguments Schoenfeld is in agreement with the importance of the teacher’s voice as argued by van Oers, the benefits of membership as argued by Luria, Rogoff, Lave, Wenger and Greeno, towards the cultural nature of development as argued by Vygotsky.

As a summary to my focus on problem solving know-how and before attending to my other focus on cooperation of students, I discuss references to literature in mathematics education that refer to the methodology and importance of the same. Discussing methodological considerations Lester (1985) advocates a naturalistic inquiry on behalf of the researcher that relies on qualitative methods, relevance instead of rigour, researcher's tacit knowledge in the formulation of theory grounded in data. Of the three categories of factors that are involved in problem-solving instruction: antecedents, classroom processes and outcomes (Appendix A.1), I attempt to theorise upon **classroom processes** that contribute to the development of problem solving know-how. Such analysis involves the actions of teachers and students in the classroom, which in turn constitute the micro-culture of teaching-learning.

Discussing inadequately or neglected themes in problem-solving research, Grouws (1985) argues that teachers are the most important influence on students' acquisition and advancement of problem solving. He argues for research that throws light on **how time is devoted to problem solving** in the classroom (an issue crucial to teaching-learning and argued by Seeger) and argues for the recognition of **problem solving** that occurs **in small episodes within lessons** which emphasise skills and concepts. Many worthwhile student measures to solve problems, he says, can be observed in classroom settings which may not surface while studying non-routine problem solving.

In concluding my discussion on problem solving in teaching-learning, I turn to issues that I address in Chapter 7: *Cooperative problem solving*. As different from an emphasis on classroom teaching-learning, in discussing student cooperation I shift my observations methodologically (which I discuss in my next chapter) and my focus to the discursive aspects of student cooperation. I adopt such an approach to analyse the cooperation of students at problems or goal-directed tasks designed for the group. In conducting such analysis I study from a socio-cultural-historical perspective the attempts of students' not only **goal-directed tasks** but also at **shared goals**. Commensurate with the aims of my thesis, I attend also to the role of representations, both given and created by the students in their cooperation, as intellectual artefacts, that mediate or are intended to mediate specific outcomes.

My interest in the analysis of student cooperation, follows Vygotsky (1997b) who elaborated that both direct and mediated relations are possible between people. The direct were based on instinctive forms of expressive movement and the higher on mediated relations established by means of artefacts towards converting social relations into mental functions. In the place of how a child behaved in a group Vygotsky asked: **how does the group create higher mental functions** in the child?

In mathematics education literature the terms figures, notations, symbols, inscriptions, tools, constructs, schemes, configurations and artefacts have been used polysemously (at times interchangeably, at times distinctively) with **representations**, depending on and indicative of the theoretical frameworks referred to. My interest in referring to these **as artefacts** is to explore the act of representing, and the role of representations as mediators within the communicational aspects I study.

For a brief discussion on the nature and role of representations in research prior to accommodating a discursive view, I take the writings of Goldin (1998b; 2003; Goldin & Kaput, 1996; Goldin & Janvier, 1998) as my point of departure. Goldin describes various interpretations of the term representation to include: external situations embodying mathematical ideas, linguistic embodiments with emphasis on mathematical syntax and semantics, systems of mathematical constructs such as a system of symbols and individual cognitive configurations. Recognising the shift from what representations are to **how representations evolve** in shared exploration and problem solving contexts, Goldin forwards a three stage development of representations: an inventive-semiotic stage of assigning meaning, a period of structuring and using symbols developed earlier and an autonomous stage of transfer of meaning.

As both a process and product of mathematical activity, Goldin argues that while ambiguity is undesirable in mathematics, its role becomes inescapable when part of teaching-learning. While asserting that the best classroom activities provide opportunities for rich affective representation, Goldin recommends a balance between notational representations (which most school mathematics is still devoted to) and representing (which deals with development of representational modes). He concludes that representing does not oppose but strengthens dealing with representations. My interest in discussing Goldin is to bring to analysis a shift in emphasis **from representations to representing**. That representing, is part of the semiotic process of externalising ones thinking, has been pointed out by Vygotsky, Bakhtin, Ernest and Radford. That artefacts are essential in human cognition is argued by Cole and Säljö.

On the importance of bringing about representing in school, Vergnaud (1998) stresses the role of the teacher as a mediator in helping students to develop their **repository of representations** and providing them with fruitful situations in which to develop these. On the importance of representing, Smith (2003) argues that the act of representing by the students is one of re-presenting their own understanding. As products of problem solving situations, Smith argues that representations can both **enable and constrain** problem-solving processes and solutions, yet can be **gateways to abstraction and generalisation**. He outlines two rela-

tionships of transition: first between personal and conventional representations and second, between contextualised and abstract reasoning.

It has been possible for me to discuss in my data and analysis Chapter 7: *Cooperative problem solving*, three kinds of representations: diagrams, the graph and an algebraic model. In my discussion of their **discursive role**, I turn first to literature in mathematics education that relates to the graph on such a basis, followed by the dependence of mathematical representations on the discourse associated with their use.

The example of a graph as a representation in a cognitive and social process, inextricably linked to the situation represented is given by Monk (2003). As a mediating artefact that is both a product and topic to be taught in mathematics, Monk observes that the **graph** is **both a medium** for communicating information **and a tool** for generating meaning. As a powerful aid to problem solving, Monk lists the six modes of meaning making towards problem solving: exploring aspects of a context not apparent, questioning the context through the process of representing, deepening understanding about a graph, construction of new entities with features of the graph, elaboration of understanding of both context and graph through an interactive process and joint reference to the phenomena in a context. Exemplifying the interactive process, Nemirovsky and Monk (2000) outline the use of a graph in a **trail making process of symbol use**, in which idiosyncratic meaning is foregrounded which brings about interplay in the meaning making process.

I conclude my focus on problem solving with the importance of representations or objects as artefacts in mathematical learning, to bring reality into being as argued by Sfard (2000; Sfard & McClain, 2002). As tools in mediating communication and learning, she argues that objects are inseparable from the social, cultural and historical factors, making them culturally shaped phenomena even if the related actions are performed by a single individual. Sfard makes a distinction between **actual reality**, one that the objects as artefacts mediate perceptually; and **virtual reality**, one that is understood with the help of the objects as symbolic substitutes. Sfard observes that **artefacts as mathematical objects** are **not independent of the discourse** they belong to. She points to an inherent circularity in the process of constructing sense of the object and the discursive practice in which discussion is embedded, bridging the dialogue between actual and virtual reality. In such a process the world is not represented with symbols but **symbolised into being**.

My interest in discussing Sfard, in exploring the bridging processes in student cooperation at goal-directed tasks, is commensurate with the methodology I adopt. Her arguments also inform analysis of artefacts as both medium and tool as argued by Monk, which enable, constrain or are gateways to generalisations as argued by Smith. Such a process is also

opportunity in addition, for the study of appropriation of meaning as argued by Mercer or of each others words as argued by Bakhtin.

It would be inappropriate to conclude my discussion of cooperation in problem solving, without reference to the considerable literature in cooperative learning and locate my position with respect to the same. Beyond the benefits of cooperative learning (CL) as promoting an increase in socialisation and verbalisation, Johnson and Johnson (1990) have argued for the need to implement CL within a conceptual framework in mathematics. Good, Mulryan and McCaslin (1992) also argue for the need for a richer definition of CL including what such instances of learning facilitate. In arguing for a strategy by which CL does not just alter but supplants traditional whole-class teaching, Sharan (1990) has asked for research on students thinking while working in small groups.

Recognising the broader concerns and objectives that the teachers in my classroom may have in their conduct of CL, it is my intention however, to observe not the implementation of CL per se, but the argumentation of students towards shared goals within the same. In such a focus I attempt to trace the ZPD that was possible to create, in their cooperation as has been the focus of Goos, Galbrith and Renshaw (2002) who argue for investigation of the **conditions of interaction**. Following Saxe (2002) I investigate in addition, the **emerging mathematical goals** of students in structured group-tasks and the **complexity of their cooperation** involving artefacts. Based methodologically on the externalisation of student thinking, by way of representations and dialogue, such an analysis is commensurate with a ZPD formed within a group of students, situated within the micro-culture constituted in the classroom.

My interest in concluding my discussion with the above arguments is to highlight once more at this juncture, two aspects that I have been addressing all along my discussion. Firstly, the importance of cultural and social factors that analytically account for the study of an **individual-in-social** perspective. Secondly, the **historical nature** of the educational process, that leads to human development (more of this also in the next chapter). Commensurate with these perspectives and a focus on both problem solving and meaning making in the teaching-learning of mathematics, I now turn to the methodological aspects of my study.

3. Methodology, methods and fieldwork

In this chapter I discuss methodology, methods and issues related to the conduct of fieldwork. I begin with implications of theoretical perspectives outlined in Chapter 2 and discuss my four units of analysis. Drawing on the philosophy of educational research, I then argue for a naturalistic and qualitative approach. I follow this by elaborating my choice of methods for data collection, triangulation and reporting. I then describe the classroom in which I conducted my fieldwork, including the obtaining of consent for data collection. I finally outline data collected, ethical issues in collection and the structure of data and analysis chapters. I conclude with my formats of data presentation and my analytical process.

Implications of theoretical perspectives

In choosing to theorise the micro-culture of the classroom with socio-cultural-historical perspectives, it is imperative that I draw upon related methodological considerations that enable an individual-in-social study as considered across Chapter 2. Towards this objective I discuss below the emphasis Vygotsky gave in his methodology to practical activity as the basis for study, followed by his method of double stimulation. I then discuss the importance and significance of the study of historical processes followed by their application to my study.

Calling his **methodology experimental-developmental**, Vygotsky (1978) laid emphasis on concrete or practical activity as basis for study:

The search for method becomes one of the most important problems of the entire enterprise of understanding the *uniquely human* forms of psychological activity.

In this case, the method is simultaneously prerequisite and product, the tool and the result of the study. (Vygotsky, 1978, p 65, emphasis added)

In addition to the importance of the study of human or cultural forms of behaviour in concrete activity, Vygotsky also specified the position of the **student as the subject** in concrete activity:

Here the subject is put in the position of an observer; he is the observer, the subject, and not the object of the experiment; the experimenter only observes and records what happens. Here instead of facts we get ready-made theories.

(Vygotsky, 1994c, pp. 43-44)

In addition to locating the student as subject in his methodology, Vygotsky argued for the observation and theorisation of student behaviour, by the experimenter or researcher, in a method he called double stimulation, where it was the student who achieved set goals in a given task.

Elaborating **the method of double stimulation**, Vygotsky (1978; 1997b; Vygotsky et al., 1994b) suggested that the student be offered a task more difficult than he or she could be accomplish at a particular point of time. In this task **auxiliary stimuli** or artefacts were offered in addition to the **stimuli of speech**. The advantage of this method (as

against stimulus response methods) was that in the actions of the student towards achieving goals, the task objectified his or her inner psychological processes. In the conduct of the task as experiment the researcher in addition to guiding student behaviour through speech, had the advantage of access to higher mental functions of the student. This was possible since these functions were created in a dialectical manner and mediated by auxiliary means and built upon biological or lower mental functions.

In advocating the method of double-stimulation, Vygotsky parallelly argued for the **study of processes**. Three points of significance of such a study are observed by Valsiner (2000). Firstly, the student as subject encountered the complexity of the whole field of experiment and not just some elements of the field. This enabled recording of the student's choice of elements in the problem. Secondly, the student was active and the experimenter a guide. This allowed the student and not the researcher to be active in driving the proceedings. Thirdly, the construction and reconstruction of meaning was done both by the student and by the experimenter or researcher throughout the goal-directed process.

In conducting a study of processes or a sequence of such processes, Valsiner also argues against a simple empirical description of events and for the construction, testing and modification of models that incorporate time-inclusive processes. In clarifying the objective of a **historical, genetic or developmental study**, Valsiner (2000, p 58, emphasis in original) outlines the following axiom: 'The axiom of historicity: the study of *the time course of formation* of selected phenomena can *explain the present state* of these phenomena.'

With reference to my study of teaching-learning in the classroom, the above methodology and method directs a strategy of conducting prolonged observations of practical activity. Such observations can be made both in the classroom and in specifically designed goal-directed tasks. In classroom teaching-learning, the teacher provides the students tasks and instructions, where as researcher I record the proceedings and reactions of students to the teacher's instructions and tasks (double stimulation). In specific tasks (in which the teacher is absent), I as the experimenter conceptualise, design and conduct goal-directed tasks taking assistance in recording of the attempts of students (e.g. audio-recoding).

The implementation of the above allows me to unify theory, methods and phenomena into one scheme of knowledge construction and enables me to analyse classroom phenomena on the basis of the teacher's and my own reasoning, assumptions and intuitions. It also allows me to retain a focus on the material communication underlying the individual and social environment constituted and use a microgenetic method in conducting problem solving tasks. Drawing upon this strategy, I now turn to specific units of analysis with which to analyse classroom phenomena.

Units of analysis and classification of artefacts

In discussing my rationale and choice of appropriate units of analysis, I first discuss Vygotsky's conception of any unit of analysis. I then outline the classification of artefacts that I use in my study and conclude with my four units of analysis: mediated action and agency, participation in context, knowing and knowledge artefacts and 'activity'.

Vygotsky advocated that in defining and choosing a **unit of analysis**, any unit analyse processes so as to explain development. He argued for analysis of process, not thing, analysis that discloses the real causal-dynamic connection and relation, but does not break up the external traits of the process and is, consequently, an explanatory, not a descriptive analysis, and, finally, genetic analysis, which turns to the initial point and re-establishes all processes of development of any form that is a psychological fossil in the given form. (Vygotsky, 1997b, p 72)

In elaborating and extending Vygotsky's characterisation, Zinchenko (1985) explains that any choice of units could have internal contradictions and be heterogeneous, but needed to be a living part of the phenomena being studied. Each unit needed to reflect characteristics of cognition, sensation, volition or purpose, intelligence, and activation so as to lead to a synthesis that could exhibit the relationship between participants and reality. In making such a choice Veer (2001) emphasises the need to consider the social environment of the child or student as part of its living environment and never external to him or her. On the **importance of social units** rather than individual units within a distributive framework, Resnick, Pontecorvo and Säljö elaborate as below:

The basic unit of analysis must connect thinking to action in the world and contribute to clarifying precisely how cognition enters into and is part of the diverse set of tasks in which people engage. Furthermore, because virtually all activity is socially distributed, social units rather than individuals become the appropriate units of analysis. (Resnick, Pontecorvo, & Säljö, 1997, p 4)

Minick (1987) observes that Vygotsky's own conceptualisation about an appropriate unit of analysis underwent a shift in emphasis. From an initial emphasis on the instrumental act leading to higher mental functions, there is a later emphasis on the nature of speech and the socio-historical nature of psychological processes. Building on Vygotsky's argument that **word meaning** in school is learnt not as a means of communication but **as part of a system** of knowledge, and not through direct experience but through other words, Minick advocates an analytical emphasis to the development of meaning associated with social interaction.

With an objective of using units of analysis as theoretical lenses, I incorporate the above mentioned concerns and reflect upon meaningful units of the teaching-learning. Before elaborating upon these living units that constitute the micro-culture of the classroom, I turn presently to discuss the classification of artefacts that I employ in my units and study.

Classification of artefacts

The basis and rationale for my classification of artefacts, as discussed in Chapter 2, is with intention of being able to conceptualise classroom teaching-learning as a micro-culture. Towards this objective I make a simple classification of artefacts in two kinds: physical and intellectual.

As **physical artefacts**, I include those artefacts in the classroom like the textbook, blackboard and calculator with respect to which actions of individuals are directed outwards and involve physical activity. As **intellectual artefacts**, I include those artefacts and representations with respect to which actions of individuals are directed inwards, involve intellectual activity and are described by Vygotsky as psychological tools:

language; various systems of counting, mnemonic techniques, algebraic symbol systems, works of art, writing, schemes, diagrams, maps and mechanical drawings, all sort of conventional signs etc. (Vygotsky, 1981b, p 137)

The simple classification above allows me to locate and analyse physical and intellectual activity related to the teaching-learning of mathematics. In the social environment of the classroom within which a micro-culture is constituted, these physical and intellectual artefacts are a given, embodied with ideals and passed down in historical time and mathematical praxis. Employed in the enculturation of mathematics their use is to be acquired and appropriated in classroom practice.

I resort to a simple classification for one another reason: the lack of consensus in my reading about a classification of artefacts based on War-tofsky who identifies three kinds of artefacts as below:

Primary artefacts are those that result from a transformation of part of the environment for the purpose of successful production and reproduction of the means of existence. ... *Secondary artefacts* are by contrast, objects created or used for the purpose of preserving the skills and practices involved in the production and use of primary artefacts and of transmitting those skills and practices from one generation to the next. Such artefacts are reflexive representations in that they are produced intentionally as symbolic externalisations of the primary modes of action and serve an informative and pedagogic function. ... *Tertiary artefacts* take this process one stage further, as representing comes to constitute a relatively autonomous 'off-line' world of imaginative activity, as in science or art, in which formal properties of the representations are manipulated 'playfully', without immediate concern for their direct application to the 'actual' world. (As in Wells, 1999, p 69)

For the purpose of classroom teaching-learning of mathematics, I would argue that primary artefacts, are those that are used in the successful and primary production of mathematics e.g. calculator and graphs; secondary artefacts, are those modes of actions and practices in mathematics associated with primary artefacts e.g. ways of operating the calculator, ways of plotting graphs; and tertiary artefacts, as those constructs created with the use of primary and secondary artefacts but have an existence independent of them e.g. properties of proportional quantities.

I would then propose a working classification of artefacts as below.

Category	Description with example
Primary	Having a representation and meant for production
Physical	Textbook, blackboard, calculator, notebook
Intellectual	Language, mnemonic techniques, algebraic symbol systems, writing, diagrams, maps, conventional signs
Secondary	Ways of working with primary artefacts
	Ways of using the calculator, ways of plotting graphs
Tertiary	Derived and abstracted with primary and secondary artefacts
	Models in reasoning, problem solving strategies, knowledge artefacts

However I do not find explicit agreement for the above in literature, where the reason for disagreement is in the interpretation of secondary artefacts. I discuss both views. In disagreement is the classification by Engeström (1987) who equates the physiological tools of Vygotsky to the secondary artefacts of Wartofsky, a classification followed by some in literature e.g. Bartolini Bussi et al. (2005) and McDonald et al.(2005).

My argument is that in the context of the classroom, the psychological tools of Vygotsky are a given and therefore ‘primary’ in nature. My related interpretation of secondary artefacts is thus with an intention of tracing those actions towards enculturation in mathematics, which play a role in ways of knowing with primary artefacts. In what appears to be some agreement, Wells (1999, p 67) as quoted earlier argues for a classification of artefacts on the basis of knowing and representing. I share the emphasis adopted by Wells in the use of artefacts towards knowing and his conception of knowledge artefacts as my third unit of analysis.

Cole (1996, p 121) also sees secondary artefacts as playing a central role in preserving and transmitting modes of action, including recipes, norms, constitutions and the like. Arguing for the historical aspect of activity embedded in traditions of use, Hedegaard (2001, p 18) emphasises that it is the traditions with artefacts and not artefacts in themselves that are important in conceptualising a theory of learning. She observes Wartofsky’s concept of artefact to be a synonym of Vygotsky’s concept of tool but does not discuss their classification. The problem and difficulty of classification of artefacts is also recognised by Seeger (2001b, p 41), who observes that though the classification of Wartofsky may be useful, it does not address how these artefacts are related to each other.

My conjecture is that any classification of artefacts is relative to the context of use and tentative. Actions such as adding numerical quantities, mediated by physical artefacts say in early stages of counting, become intellectual upon internalisation. In my intention of theorising the constitution of a micro-culture in the classroom, I thus retain a simple classification of physical and intellectual artefacts. I now turn to discuss my first unit of analysis related to action mediated by artefacts.

Mediated action and agency

The first of my four units of analysis: **mediated action**, deals with actions related to the mediation of artefacts with which to study classroom communication. Explaining mediated action as the dialectic between action and the mediational means or artefacts which mediate action, Wertsch (1998, p 24) outlines its aim in any socio-cultural-historical analysis: ‘... to explicate the relationships between human *action*, on the one hand, and the cultural, institutional and historical contexts in which this action occurs, on the other.’

In bringing about cultural forms of behaviour or higher mental functions, Vygotsky (1989) explained that by ‘function’ he meant ‘modes of action’. He argued that when humans regulate their behaviour through mediated action, development proceeds from ‘me’ as a member in a society to ‘I’ as an individual; that it was not ‘mentalistic thought’ doing the thinking, but a **person thinking in mediated action**. The role of the individual was central in behaviour becoming intellectual:

We master a function to the degree that it is intellectualised. ... To say that memory is intellectualised in school is exactly the same as to say that voluntary recall emerges; to say that attention becomes voluntary in school age is exactly the same as saying that it depends more and more on thought, that is, on intellect. (Vygotsky as in Wertsch, 1985, p 26)

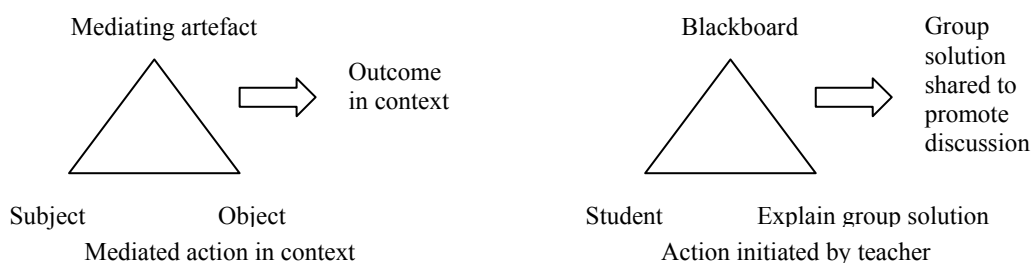
Wertsch (1985; 1998; Wertsch, Rio, & Alvarez, 1995; Wertsch & Tulviste, 1998) observes that mediated action and related conscious realisation as a consequence, is prior to the development and emergence of higher mental functions. In emphasising **higher mental functions as modes of action**, Vygotsky observed that it is only later that action is voluntary and intellectual in nature. This makes the study of mediated action central to both higher mental functions and the individual.

Drawing attention to the nature of artefacts as mediational means, Wertsch also explains that since artefacts are carriers of socio-cultural patterns and knowledge, they **constrain as well as enable** action. They need to be appropriated and are characterised by their mastery by the individual. Since artefacts can transform, create meaning and are situated in developmental paths, they are inherently related to and shape actions. They could serve multiple goals and are associated with power and authority. The shaping of action by artefacts as mediational means in any practical activity does not mean that action can be reduced to or determined by artefacts. Hence the socio-cultural-historical setting is important. Mediated actions always involve an inherent tension between the means and individual using them in concrete activity. Wertsch therefore argues that it is appropriate to think of mediated action not in isolation, but as a **moment of embedded action**. Such moments allow the linking of action (including mental action) to the cultural, institutional and historical contexts and allows for examining them as they interact.

I now address the concept of agency which enhances the concept of mediated action in socio-cultural-historical settings. Wertsch (1991; Wertsch & Rupert, 1993; Wertsch, Tulviste, & Hagstrom, 1993) extends the notion of mediated action or individuals carrying out action, to **mediated agency** or individuals carrying out action with mediational means. Mediated agency enables the taking into account of the situatedness of mediated action and the distribution of intelligence. Any analysis of agency points to a related notion of authority, since it is no longer an individual alone, but **individual(s)-operating-with-mediational-means**.

In addition to the notion of authority as power, Wertsch also discusses the notion of authority as scripting, based on the situated nature of utterances following Bakhtin. For Bakhtin an utterance is a 'voice' that appropriates a genre of speaking. Wertsch develops Bakhtin's notion of voice (both written and spoken) in the conception of mediated agency where again it is not just the individual, but **individual(s)-speaking-with-mediational-means**. In any social milieu that shapes utterances, this refers to the adoption of languages that are prevalent and privileged, where **privileging** refers to specific kinds of social speech that are more appropriate in a context and accessible to conscious reflection.

With an intention of applying the unit of mediated action and agency and taking into account the situatedness of individual action, I utilise the symbolisation shown below. It is a modification of the diagram representing the instrumental act discussed in Chapter 2 by Vygotsky (1981b) and extended by Engeström (1999, p 30) in his version of 'activity'. The modification to the basic mediational triangle incorporates an emphasis on the context in which action is mediated. I extend the 'outcome' of mediated action, to **outcome in context**. Such a modification allows for the accommodation of the contexts in which action is embedded. For example the actions of a student presenting her group's solution at the blackboard, under the teacher's instruction, could be to bring about discussion of that solution, with those of other groups in the classroom.



I do not offer the above symbolisation in my data chapters, but employ it all the same to analyse those instances in teaching-learning where action is mediated and the nature of agency afforded to the individual upon mediation. As against action with artefacts, I now turn to analyse the participation of individuals in various contexts in the classroom.

Participation in context

My second unit of analysis deals with the participation of individuals, both teachers and students, in social practices and the contexts of teaching-learning prevalent in the classroom. I elaborate upon both below.

I focus on individual **participation in social practices** in two ways: by individuals in social practices and simultaneously of social practices set up intentionally by the teachers. I had discussed the significance of social practices in Chapter 2 and mentioned my dedicating a whole chapter towards the elaboration of teaching-learning practices in the classroom. In Chapter 4: *A collaborative classroom practice*, I analyse instances of practice that bring about individual participation. These include the establishment and realisation of individual, group or whole class participation as teaching-learning progresses.

As for contexts of participation I consider the various situational constraints within **teaching-learning** in the classroom **as context for participation**. By this I mean the participation of students for example in a problem with a given representation, a problem in which they had to represent; a question they had to attempt without prior knowledge about the nature of the problem or in their attempting a problem modelled upon the solution of another. Similarly, the context for participation of teachers is either in whole class teaching, when the students are working in groups at their tables or when discussion among groups is being encouraged. Such contexts are accounted for by this unit of analysis.

It is appropriate to mention here a particular and consistent context for participation in my classroom, that of **bilingual teaching-learning** of mathematics wherein students participate in English in addition to mathematics. John-Steiner (1985) observes Vygotsky to argue that, though native and foreign languages are materially different they are united in meaning, making the foreign (English) dependent on the native (Norwegian). Unlike Norwegian learnt unconsciously and brought to conscious use, learning English would be conscious till used freely. Levine (1993) observes that in such a context formal communicative aspects allow for conscious learning. Though the study of bilingualism is neither possible by me nor my objective, by this unit I incorporate the **conscious participation** of individuals in the English language.

My objective of using this unit of analysis is to tease out individual participation, in situational contexts and social practices within the classroom. Since these are more numerous than those having a bearing on teaching-learning of mathematics, I naturally restrict my analysis to those instances which are related to the advancement of knowing in mathematics. As argued in Chapter 2, the growing relationships between individuals and the enabling of knowing in the classroom are focused upon. I now discuss analysis of theoretical knowing in mathematics.

Theoretical knowing and knowledge artefacts

My third unit of analysis focuses on knowing and knowledge artefacts in the classroom. Corresponding to my discussion in Chapter 2, this unit analyses the discursive aspects of communication towards theoretical collaboration, leading to knowing (as against objectified knowledge) and the constitution of a ZPD. Following the units of mediated action and participation, by this unit I analyse those mediated and situated actions in teaching-learning that contribute to knowing. Actions that contribute to the **spiral of knowing** towards greater and greater theoretical knowing as offered by Wells (Appendix A.2) are of interest.

Wells (1999; 2000) who bases his spiral on the genetic approach to development of Vygotsky, agrees with Leont'ev on the significance and value of joint interaction that contributes to co-knowledge. In dwelling on the relationship between knowledge building and experience he discusses Ueno, who says artefacts are embedded in practice and do not always appear the same. How artefacts detach themselves from the ground and under what social organisation and collaboration, are important questions Ueno (1995) says in ideological and classroom practice.

Wells argues for the analysis of classroom activities that organically build on modes of knowing that students deploy. This is dependent on substantial practical information in relation to the activity of knowing and the means of creating artefacts that preserve the outcomes of knowing. Embedded in practical activity and becoming material for a context-independent form of knowledge building, Wells argues that **knowledge artefacts** (discussed in Chapter 2) created in one cycle mediate the next cycle of knowledge building activity. They mark the beginning of theoretical knowing and illuminate both current and prospective practice, making situated activities have the potential for transformation.

Emphasis on ways of knowing as a unit of analysis is also drawn from Bishop (1988), who emphasises the importance of a way of knowing in the enculturation of mathematics as different from a way of doing. As discussed in Chapter 2, Bishop observes a **way of knowing** as a socially constructed set of ideas, achieved by connecting particular mathematical ideas under discussion to personal knowledge.

As also discussed in Chapter 2, my emphasis on the discursive aspects of theoretical collaboration is with an intention of analysing those actions of teachers and peers which contribute to the loan of consciousness in the ZPD, under the assistance of which know-how is shared about the cultural ways of doing and knowing. Analysis of such actions in teaching-learning, allows for insight into the nature of theoretical collaboration in mathematics. This includes the bridge between the personal and propositional forms of knowing or the spontaneous and scientific concepts. It is to focus on such specific transformations that I now turn.

‘Activity’

The three level ‘activity’ of Leont’ev is my fourth and final unit of analysis. Applied across both classroom teaching-learning and specific goal-directed tasks, this unit allows for analysis of **transformations in the pursuit of goals**. Within ‘activity’ I identify as explained in Chapter 2, the following sub-units in pairs: Activity – Motive; Action – Goal; Operation – Conditions. I distinguish ‘activity’ from its sub-unit by enclosing its name and label in quotes along with a descriptive indicating what the analysis pertains to e.g. ‘Activity of similar triangles’.

My objective of employing ‘activity’ is to analyse as observed by Scribner (1997b; 1997c) the structural and behavioural units which include both external and internal processes. This is premised on the understanding that human activity belongs to not two but one sphere of reality, leading to the genesis and forms of thinking. Scribner (1997a, p 386) observes: ‘If thinking is an aspect of concrete activities, and we want to understand its genesis and forms, we need to begin with an analysis of the activities and actions in which it is embedded.’

As discussed in Chapter 2, Rogoff (1995) also observes that ‘activity’ allows analysis of the relation between the individual and the environments in which each is inherently involved in the definition of the other. Russell (2004) argues ‘activity’ as a valuable tool in identifying social and cultural interactions, resulting in the kind of changes called learning, where it is for the researcher to decide the focus of the theoretical lens. In ‘activity’ I analyse how individual means, capacities and skills are created, mediated and transformed by relations entered into in any teaching-learning activity and qualified by Leont’ev (1981b, p 300): ‘In order to make them *his own* means, *his own* capacities, and *his own* skills he must enter into relations with people and with objective human reality.’

I am aware of the application of ‘activity’ to larger systems encompassing the community as a whole and follow Kaptelinin (2005) who observes; that in the model of Engeström communities participate with an objective of producing organisational change, whereas in the model of Leont’ev, the participants are individuals whose motive (sub-unit of ‘activity’) is related to their motivation in their psychological domain. I adopt the later model and analyse the participation of individuals in goal-directed group-tasks or classroom tasks taken together as ‘activity’. I tabulate below an analytical schematic (made up of two tasks) which I employ, before identifying the relationships and transformations within.

‘Activity of two group-tasks’ (See example in Chapter 7, p 200)					
Activity	Motive	Action	Goal	Operations	Conditions
Of both	Of both	Task (1) Task (2)	Task (1) Task (2)	Task (1) Task (2)	Task (1) Task (2)

Empirical methodology

In elaborating empirical methodology, I discuss the philosophy of educational research before arguing for a naturalistic and qualitative approach.

On the philosophy of educational research

Consequent to methodological implications of theoretical perspectives, I discuss the nature of educational enquiry that I conduct.

Pring (2000) argues that though educational research draws on social sciences, its nature needs to be determined by the subject matter of teaching-learning. If **learning** is coming to understand and a struggle to grasp the meaning of ideas and concepts; then research about learning must attend to what it means to have learnt, along with acquisition of virtues such as concern for the truth and openness to criticism. Similarly, if **teaching** is the conscious effort to bridge the gap between student and subject matter; then educational research should centrally (but not exclusively) be about those transactions between teacher and learner, by which the learner comes to see the world in a more valuable way.

Pring observes educational practice as a complex phenomenon and advises a close examination of key ideas that lie at its centre: learning, teaching, personal and social development and culture.

This subtle interconnection between the public and the private, the objective and the subjective, the physical and the mental, the personal and the social, is too often neglected by those who espouse 'research paradigms' which embrace one side of the dichotomy to the exclusion of the other. (Pring, 2000, p 37)

Any educational practice, Pring observes, also embodies a **way of thinking about learning**: its aims, what constitutes as having learnt successfully, what skills, knowledge and values it incorporates. He argues that at the heart of educational research lies **professional judgement**, which in turn is informed by what is relevant. He endorses the central position of the teacher as researcher, who is able to judge both values (public and private) and practices, in light of systematically obtained evidence.

Pring's arguments direct my study of the material view of communication, towards a study of what it means to be a participant in teaching-learning of mathematics, by incorporating an examination of values about ways of thinking. I now discuss my approach to this.

A naturalistic approach

Drawing upon theoretical and methodological socio-cultural-historical perspectives that I have outlined so far, makes any observation of teaching-learning in the classroom I study heavily theory laden. Yet I draw upon the theoretical constructs mentioned therein, to analyse the events of the classroom, with an objective of theorising how teaching-learning plays out in the reality of my classroom. As mentioned before my interest is in theorising how and what sort of micro-culture is constituted in the classroom in terms of the theoretical constructs discussed.

Towards evolving such a theory **grounded in data**, gathered and analysed over time and derived from the teaching-learning of the classroom, I pursue a naturalistic study whose axioms are outlined below.

Axioms about	Naturalistic paradigm
The nature of reality	Realities are multiple, constructed, and holistic.
The possibility of causal linkages	All entities are in a state of mutual simultaneous shaping, so that it is impossible to distinguish causes from effects.
The relationship of knower to the known	Knower and known are interactive, inseparable.
The role of values	Inquiry is value bound.
The possibility of generalisation	Only time-and context-bound working hypothesis (ideographic statements) are possible.

(Lincoln & Guba, 1985, modified from Table 1.1 p 37)

The above axioms have the following **advantages**. Firstly, they recognise the dynamics of the classroom and enable me to address multiple realities, as and when they occur in the context of teaching-learning. Secondly, they recognise the mutual and simultaneous shaping of events, allow for a relational study between individuals and the environment. These enable me to take a holistic view which Vygotsky observed as constantly changing in the historicity of events leading to development. Thirdly, they recognise the relationship between the knower and known as interactive and inseparable. Such a stand is in agreement with the constructs of Vygotsky, Luria and Leont'ev and allows for an interpretation of meaning in a situative perspective. Fourthly, they recognise the nature of inquiry as value bound allowing me to employ my theoretical perspectives and as argued by Pring my professional judgement as a teacher. Finally, recognition of the fact that only time and context bound working hypothesis are possible, is in agreement with Valsiner who demands not just a description but a time-bound explanation of current events.

In adopting the axioms of a naturalistic approach I also face the need to circumvent the possibility of being subjective in my observations, analysis and synthesis. Wellington (2000) offers a way to deal with my theory and value-bound position. He clarifies the position of a researcher as being both **reflective** and **reflexive** as below:

Being reflective involves thinking critically about the research process; how it was done and why, and how it could have been improved. ... But an important part, or subset, of 'reflectivity' is the notion of 'reflexivity'. This involves reflecting on the self, the researcher, the person who did it, the me or the I.
(Wellington, 2000, pp. 42-43)

In distinguishing the processes of reflectivity and reflexivity, Wellington encourages me to make public what I am doing and why, at the level of the observer and at the level of the research process. Such a distinction enables me to adopt socio-cultural-historical perspectives that I conduct analysis with, and build a theory grounded in data and research.

Strauss explains the advantage of building a theory grounded in data as an **effective means of understanding** the phenomenon being studied.

Theory to me, in the broadest sense, is one of the most *effective means for understanding phenomena*. When studying human behaviour, one has to interpret what one sees, hears, overhears, discovers, ferrets out, is told, reads. Analysis of data, of all kinds, can lead to theory if the data are systematically conceptualised. ... when carefully and systematically done, and grounded in equally careful and systematic collection of data, then this conceptualisation can give a deep understanding of our subject matters. Theory, then, consists of systematic relating of concepts, grounded in data. (Strauss, 1995, p 22, emphasis added)

Strauss points to the many characteristics of teaching-learning that make my building of a theory grounded in data a qualitative process. It is to discuss the advantages and criteria of the same that I now turn to.

Qualitative strategy and quality criteria

A qualitative strategy following Bryman (2001) affords the following advantages. Firstly, it allows for an **inductive relationship** between my theoretical perspectives and research. Such a stand enables the generation of a theory grounded in data, emerging from my research processes. Secondly, it allows for an **interpretative** stand in bringing forth meaning. Such a stand allows me to understand and model the classroom I study. Thirdly, it implies an ontological position which states that my interpretations are **outcomes of interactions** between individuals studied. Such a premise is not in conflict but concurrent with socio-cultural-historical perspectives which recognise the analytical importance of relations between individuals and their social environment.

A qualitative strategy necessitates appropriate quality criteria. Bryman (2001) following Lincoln and Guba (1985) lists these as credibility, transferability, dependability. The criterion of **credibility** is an attempt to make my study feasible and acceptable to the others in the research fraternity. I intend to bring about credibility by my choice of methods and respondent validation by students and not depend solely upon interpretation. I also achieve credibility by both method triangulation and data triangulation which I shall discuss shortly.

The criterion of **transferability** is an attempt to make my research reproducible by others. I intend to achieve this in two ways, provide a thick description of situated events (I discuss this also shortly) and offer transparency with the procedures and conduct of my research. It is with these aspects in addition to respondent validation and triangulation, that I make my research satisfy the criteria of **dependability** and reliability.

Finally it is through the reasoning that I employ with which I address the criteria of confirmability or **objectivity**. I call upon theory, methodology, data, analysis and synthesis in formulating an argument for readers well acquainted with teaching-learning. I now turn to discuss the methods with which I achieve the above objectives.

A mix of methods

In outlining appropriate methods for data collection within a qualitative strategy I have been guided by three criteria. Firstly, that I adopt an ethnographic practice as a researcher. Secondly, that I look beyond the qualitative and quantitative divide and let my research questions guide my choice of methods: participant observation, questionnaires and the conduct of problem solving tasks. Thirdly, that data and reporting be a result of triangulation. I discuss each of these issues below.

Ethnographic researcher practice

The need to make prolonged investigations of classroom activity, towards building a theory grounded in data and taking into account my professional judgement, makes my researcher practice ethnographical. Towards discussing this I draw on two research traditions: anthropology and the ethnographic research tradition in mathematics education.

In elaborating its interpretative task, Geertz (1973; 2000) argues ethnography to be a practice beyond techniques and procedures and an intellectual effort that provides a **thick description**. The object of such an effort is to provide a hierarchy of meaningful structures, in which the culture being studied is produced, perceived and interpreted. In the task of observing cultures through people, Geertz says humans are defined neither by their innate capacities nor by actual behaviours but by the link between them i.e. the way capacities are transformed to behaviour. In the making of such interpretations he draws attention to two aspects. Firstly, to be human is not to be every man, but a particular kind of man. Secondly, the interpretative task is second or even third order; only the student as native makes the first one, since it is his or her micro-culture.

Geertz concludes that ethnographers don't study cultures: they study 'in' cultures. Their role is one of a scribe where the nature of interpretation needs to convey the flow of what is said in perusable terms, where small facts are made to speak to large audiences. The aim of such writing is to draw conclusions from densely textured facts, where the objective is not to answer deep questions but **to record what has been said**. I see Geertz point to the importance of the writing process both on and off field on two counts. Firstly, to make my writing draw from the thick description in terms of theoretical constructs. Secondly, to make my writing be part of a larger debate about the phenomena I study.

Defining the scope of ethnographic research within mathematics education, Eisenhart (1988) argues that an anthropologist has two roles: interpreting appropriate behaviour in the classroom and making the classroom understandable to outsiders. This involves **being** in the classroom **as an insider** and **reflecting** upon the classroom **as an outsider**. As a consequence, the classroom needs to be explicated and interpreted in terms of social relations and culture for the research community. Eisen-

hart suggests four methods which I implicitly adopt, with which to find 'what is going on' to then trace and interpret the intersubjective meanings that underlie these 'goings on': participant observation, ethnographic interviewing, search for artefacts and researcher introspection.

Eisenhart also observes that a holistic understanding and long periods of exposure make the ethnographic practice **strong on reliability**. Since there is risk of making sense where there is none, she argues for the need of method triangulation, employed flexibly along with data collection and analysis as an ongoing process throughout fieldwork. Towards building of a theory, Eisenhart advises three stages: organisation and analysis of data collected into domains of meaning for an outsider to make sense, categorisation in the light of socio-cultural theories and refinement of categories and relationships as activity unfolds in time.

In the role of ethnography to build models and theories, Weisner (1996) argues for ethnography to be more than a process of being open-minded and shaped by circumstances alone and as a practice of bringing ground to figure. Not limited to the description of local meanings, he argues that **ethnography** can and should also be **question driven** where it is a process of matching the prior evolving schema in the mind of the ethnographer, against the changing and evolving meaning gained from field experience. It is towards achieving an evolving model or models that I now discuss the taking of field notes as a participant observer.

Participant observation

As a researcher taking **field notes**, I describe my first method of data collection as participant observation following Denzin (1970, p 156): 'Participant observation is a commitment to adopt the perspective of those studied by sharing their day-to-day experiences.' As my primary instrument of data collection, this method enabled me to immerse myself in the social environment of the classroom and record as Silverman (2000; 2001) argues **naturally occurring data**. The focus was on how participants dealt with one another and the skills they used in everyday communication. Okely (1994) describes the nature of such writing as **combined action and contemplation**, freed from the division of fieldwork and analysis, allowing for intuitive reminders and connections while moving back and forth from evidence to ideas being modelled.

The reflective nature of my writing while making field notes, as one of continuously questioning observations made and notes taken is articulated as **dialectical in nature** by Carr and Kemmis below:

Dialectical thinking involves searching out contradictions ... it is an open and questioning form of thinking which demands reflection back and forth between elements like *part* and *whole*, *knowledge* and *action*, *process* and *product*, *subject* and *object*, *being* and *becoming*, *rhetoric* and *reality*, or *structure* and *function*. (Carr & Kemmis, 1986, p 33, emphasis in original)

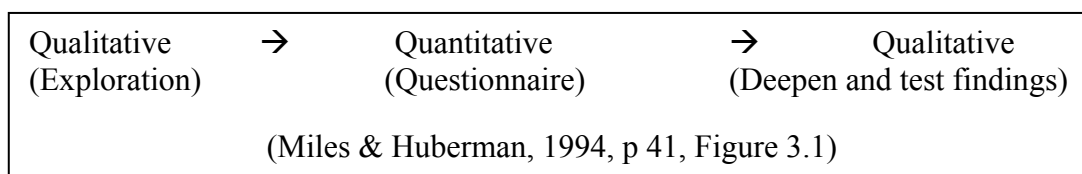
My decision of taking field notes was a natural choice towards understanding the participation of individuals in various practices, routines and norms allowing for a wider focus, within which I observed teaching-learning of mathematics in the classroom. Taking field notes also helped me to overcome by interviewing and triangulation, my lack of speaking the Norwegian language. My **role while taking notes** did not involve my taking an active part in the tasks meant for the students and involving the offering of specific assistance when asked is best described below:

The *participant-as-observer* enters into the social life of those studied, sometimes assuming an insider role, but often playing the part of a snoop, shadow, or historian – roles normally found in the group but familiar enough to participants to allow comfortable interactions. Under these circumstances, the participant-as-observer is known to be a researcher, can address ethical issues more directly, and can request access to the whole group, to negotiate data collecting and recording and to seek feedback on what is seen and how it is interpreted.

(LeCompte, Preissle, & Tesch, 1993, pp. 93-94, emphasis added)

Survey by questionnaires

An anticipated strength of 45 students in my classroom attracted me towards a quantitative element, in an overall qualitative strategy. As the second data collection instrument, the conduct of questionnaires enabled **respondent validation** by the students. Miles and Huberman (1994) argue for the usefulness of a quantitative element in an overall qualitative study and describe the links between the two elements below:



My decision of using questionnaires was to ‘observe’ students whom I could not reach physically as a participant observer. Collected at regular intervals, the intention was to allow for benefits of co-relational research. As shown above I intended the qualitative side of data collection, to help in the design of the questionnaires by which I could test and either ratify or refute my conjectures as a researcher. Along with respondent validation, the data collected by questionnaires could avoid both elite bias (showing preference and privileging certain responses) and holistic fallacy (making generalisations across the entire class). The use of questionnaires as survey, also afforded the freedom of designing them to suit the current issue being studied in the classroom. Following Gorard (2001), I could begin with ‘old favourites’ or more predictable questions and end with more ‘open-ended’ questions. Responses to questionnaires allowed, for fact finding by numbers and provided an overview while allowing for two kinds of triangulation: person data triangulation and space data triangulation, both of which I discuss shortly.

Following a naturalistic approach, the implementation of questionnaires was dictated by the flow of classroom teaching-learning. Initially I designed **feedback worksheets** (See Appendix A.3) upon observing student responses at their school tests. I later let the **responses of students to group-tasks**, designed by the teachers and given out to all the students on worksheets stand in for **survey data**. Since these worksheets were instruments from within the teaching-learning process, their collection as data was valuable besides weighing favourably with the teachers as it avoided administration of an additional instrument to the students.

Apart from stronger claims that I could make by basing my observations on a mix of methods (Gorard & Taylor, 2004) and the benefit of gaining access to data from within the teaching-learning processes, such data collection did not burden students as pointed out by BERA:

Researchers must recognise concerns relating to the ‘bureaucratic burden’ of much research, especially survey research, and must seek to minimise the impact of their research on the normal working and workload of participants. (BERA, 2004, point 19, p 7) (<http://www.bera.ac.uk>)

The nature of survey data did change over my data collection. Firstly, the feedback worksheet I designed after the school tests, lost steam over time since its completion by students was left deliberately as voluntary. The group-tasks designed by teachers and handed out as worksheets also reduced after the first four chapters, since their design and conduct came with the intent of encouraging cooperation amongst students. Over time such cooperation became normative and such a practice ceased.

Problem solving tasks

To deepen findings in a qualitative cycle after a quantitative and survey cycle as described above, I designed and conducted a problem solving task (PST) as my third instrument of data collection. I envisaged the conduct of the PST to have four benefits: deepen insight about particular phenomena, strengthen existing findings, overcome any weakness of my not knowing Norwegian and enable discourse analysis of students’ cooperation. Based upon the principle of double stimulation, I decided to conduct the PST outside the teaching-learning of the classroom, so as to not burden the teaching-learning within the classroom and allow for analysis of cooperation towards shared goals in goal directed tasks.

As to the design and conduct of the PST, I drew on Goldin:

We simply have the choice of proceeding unscientifically, choosing tasks that seem interesting and just “seeing what happens,” or trying to proceed systematically with tasks explicitly described and designed to elicit behaviours that are to some extent anticipated. (Goldin, 1998a, pp. 57-8 emphasis in original)

I envisaged the **design of task** as a **crucial stage in analysis**, depending upon my insight about the students and the teaching-learning of mathematics in the classroom, enabling also an understanding of social, psychological and contextual features in problem solving. In being part of

the task I wished to deal with issues of language and joint construction of meaning following Mishler (1986) who argues that empowerment of respondents and meaning are contextually bound. Silverman (2001) also observes that the analytical objective of any discourse is not merely to describe the situated production of talk, but to show how what is being said relates to the experiences and lives being studied.

The conduct of the PST involved the logistical problems of when to conduct the group-task, who were the students to conduct the PST with and the issue of what topic the PST would be designed around. The PST was conducted in the longer interval of 40 minutes the students had during their day. I conducted the PST with the group of students with whom I observed the teaching-learning of the classroom. I call these groups of students my **group-in-focus** and explain shortly how I collected my data in cycles with each group-in-focus for a different chapter taught. I designed the PST on a sub-topic, which I observed my group-in-focus participate in. I handed out instructions in worksheets and audio-recorded their attempts and made field notes of the proceedings.

Data triangulation

In discussing and elaborating upon the triangulation of data that was possible to bring about in my data collection, I follow Denzin (1970) who lists four kinds of data triangulation: data, investigator, theory and methodological. Being the sole researcher and adopting only one theoretical tradition I discuss the two kinds of triangulation that are applicable to my study: methodological and data triangulation.

Two kinds of **methodological triangulation** are outlined by Denzin, both of which were possible in the conduct of my study: within method and between method. **Within method** triangulation, refers to the same method used on different occasions. I was able to make field notes, collect student responses to questionnaires and conduct problem solving tasks for every chapter of teaching-learning, each of which formed a cycle of data collection. **Between method** triangulation, refers to the use of different methods on the same object of study. I made field notes, collected student responses to questionnaires and conducted problem solving tasks for the same chapter, in each of the seven cycles of observation.

Denzin subdivides **data triangulation** into three categories: time, space and person. In **time** triangulation, he considers the influence of time using cross-sectional and longitudinal design. In **space** triangulation, some form of comparative study is made within the data collected. In **person** triangulation, Denzin suggests analysis at three levels: individual level, interactive level (among groups) and the collective level (across groups). It was possible to achieve different levels of data triangulation mentioned above. I detail these in Appendix A.4.

Case study reporting

I now discuss my choice of case study reporting and quote Bryman (2001, p 49): ‘I would prefer to reserve the term ‘case study’ for those instances where the ‘case’ is the focus of interest in its own right.’ In a study of classroom teaching-learning, I argue for the significance of an event as having a stronger appeal and greater importance than a quantification of occurrences, which may neither have insight nor any in-depth or unique features. Cohen, Morrison and Manion (2000, p 185) observe: ‘Significance rather than frequency is a hallmark of case studies, offering the researcher an insight into the real dynamics of situations and people.’

My choice of case study reporting affords flexibility in tune with the flexibility of the conduct of research that I have argued for earlier. In a **case being of educational value** I follow Stenhouse:

Educational case study [is where] many researchers using case study methods are concerned neither with social theory nor with evaluative judgement, but rather with the understanding of educational action ... They are concerned to enrich the thinking and discourse of educators either by the development of educational theory or by refinement of prudence through the systematic and reflective documentation of evidence. (Stenhouse, 1985 as in; Bassey, 1999, p 28)

In summary, my choice of case study reporting is desirable on three counts: allowing me as researcher to make a choice of what the case is, focus on building educational theory and account for contexts of teaching-learning. Besides an inductive approach to the relationship between theory and praxis, there is also an added advantage of choosing a collective of case studies while reporting as argued by Wellington (2000).

To the sense of paradox that may emerge due to a collective of case studies, Simons (1996) observes paradoxes in case study are possible yet inherent in people and crucial to understanding. As with data collection, I present my analysis in the four thematic chapters in my thesis, from a triangulation of three cases in each. This allows me to present a time bound development of teaching-learning in the classroom. I now turn to discuss the classroom in which the grounded themes evolved.

Opportunity available for fieldwork

In elaborating the nature of opportunity available for fieldwork, I first locate my class in the educational system in Norway and offer an overview of the classroom before discussing the issue of obtaining consent.

The 1MX class

Within the Norwegian system of education, students attend a ten year compulsory school (grunnskole og ungdomsskole) before a three year upper secondary school (videregående skole). The latter qualifies them to enrol for a Bachelor’s degree at a University College (høgskole) or University (universitet). Most instruction at all levels is in Norwegian.

The class in which I conducted my fieldwork was at the first level at the upper secondary school. Called the 1MX, in this class mathematics was compulsory (felles allmenne fag) for all students. After the 1MX, students could either continue or opt out of studying mathematics. At the end of the year students would undertake an examination (eksamen) conducted by the authorised examining body (Utdanningsdirektoratet), syllabus for which is available at <http://www.uddanningsdirektoratet.no>.

My class had two distinct advantages that made my study of its teaching-learning possible. It was located in a school four buildings from my place of work and followed an older practice of the bilingual teaching-learning of mathematics (English and Norwegian) making it possible for a non-Norwegian English speaking person to conduct fieldwork.

Classroom realities

My class belonged to a state run school with around 500 students. The students attended thirty five periods of teaching, between 8:00 and 15:00, in a five day week, where each teaching period was of 45 minute duration. The students attended five periods for mathematics: two double periods on Mondays and Wednesdays and a single period on Thursday.

Two teachers shared teaching-learning responsibilities in the classroom. They laid stress on cooperative or group learning by students, for which they designed specific tasks. By this the teachers acknowledged a reduction in teaching time and an increase in the time spent by the students at mathematics and by them with the students. The teachers combined group and traditional blackboard teaching-learning strategies depending on the topic being taught. The students sat in groups during teaching and over time also developed rules for collaboration. At the commencement of the academic year, the teachers spoke to all the students enrolled at the 1MX level about the option of learning mathematics bilingually with a focus on cooperative learning. Thirty two students opted for this class and came from the different sections into which they were divided. In such a manner the class had a transient existence, not gathering for any other subject in the same formation.

My classroom was located in a two storied building with two large staircases connecting the levels. It had large corridors lined with paintings with classrooms on either side. Each classroom looked to the outside through windows running the length of the classroom. The side of the classroom along the corridor had cupboards and a pin up board. The green board (tavla) ran the breadth of the classroom with a raised platform for the teacher and his table. An address system was fixed on one side of the board. The white screens needed for the overhead projector and geographical maps were found rolled up in the false ceiling. The rooms were centrally heated. Curtains, lined dustbins and paintings helped make the room comfortable, which was also well kept.

A recess (friminutt) of 15 minutes was given after every teaching period, with a longer recess towards the middle of the day. The bell indicating the time for meeting and dispersing was a jingle. Students, both boys and girls dressed casually and appropriately for the weather. During recess some listened to their choice of music from the state of the art MP3 players. When hungry they ate their sandwich (matpakke) and drank milk or juice. Their bags lay around their desks which were joined for group work. They had no student 'prefects' or 'monitors'.

The students used a main textbook (Grunnbok) and a supplementary book with additional exercises (Oppgavesamling). They were allowed to use a formula book (Formelsamling). Apart from referring to the prescribed books, the teachers encouraged students to access questions with solutions meant for practice (Kontrolloppgaver) and available for the students to download from the Internet at <http://sinus.cappelen.no>.

The formula book was *Formelsamling i matematikk* © Gyldendal Norsk Forlag AS, 2001. The mathematics textbook that were being used was *Sinus Matematikk: Imxy Grunnbok*, by Tore Oldervoll, Odd Orskaug and Audhild Vaaje © J.W.Cappelens Forlag A.S, Oslo 2001. The book for additional exercises was *coSinus Matematikk: Imxy oppgavesamling*, by Tore Oldervoll, Odd Orskaug and Audhild Vaaje © J.W.Cappelens Forlag A.S, Oslo 2000. Permission from the publishers obtained to quote from these books, is given in Appendix A.5.

A graphic calculator (TI-83 Plus) was used in the teaching-learning of mathematics and was incorporated both in the textbook and in classroom practices. This was in addition to regular writing materials held in a pouch (penal) with either notebook or a file to hold writing together. Group tasks that were designed by the teachers were handed to them as worksheets. Vocabulary lists giving correspondence between Norwegian and English terms were given from time to time. A common test for all students (whether bilingual or not) was conducted by the school at regular intervals. Solutions (fasit) to these test questions were provided to the students. Classwork was taken at an easy pace with not more than half an hour's workload for homework, between teaching periods.

The obtaining of consent

Under Norwegian law, the Norwegian Social Science Data Services (Norsk samfunnsvitenskapelig datatjeneste AS or NSD) regulates all social science data collection (<http://www.nsd.uib.no>). In order to collect data in school I had to obtain their permission.

My application (meldeskjema) to the NSD needed to spell out the nature of my study, the instruments of data collection, age of my students and my funding agency. The NSD required that active consent be obtained from the students above 18 years of age and the parents to be informed. The NSD also mandated some kind of data as reportable (mel-

depliktig) and needed that these be destroyed. This applied to the audio-recording I intended. I argued against the destruction of data in order to keep data in its original form and not in the form of its transcription. Data collection being an involved process, I wished to retain data with the opportunity to (re)analyse my data at a later date.

The NSD agreed to my request. The students were needed to give active consent with passive consent from their parents. They vetted my letter and format of consent slip which I enclose in Appendix A.6. I enclose the consent and conditions for my collecting data from the NSD in Appendix A.7. On the completion of my data collection and anonymisation process, I was asked about the status of my data by the NSD as in Appendix A.8. I confirmed the anonymisation my data, which has since been acknowledged as in Appendix A.9. The process of anonymisation included the masking out all references to individual names in my field notes and audio recordings, replacing these with pseudonyms.

However, it is one thing to obtain permission from the NSD and quite another to obtain consent from a class of 32 adolescents. I wanted the students to have a fair chance of knowing what I was there for and not accept me because their teacher said so. Towards this objective I set out some topics on the blackboard (Appendix A.10) to break the ice and allowed them to question me about my presence and work. I obtained consent from all except one (Thor), who did give his consent at a later date.

Data collection, values, outline of chapters

I discuss below the extent of data collected and my rationale for collection, values and ethical issues towards collecting this data and the outline of data and analysis chapters that I present in my thesis.

Data collection in cycles

In keeping with a naturalistic approach towards building a theory about the micro-culture grounded in data, I let my data collection be dictated by the functioning and organisation of teaching-learning within the classroom. I collected data in seven cycles. I provide a summary of the data collected through the entire year in Appendix A.11 and discuss below the conditions in the classroom that lent a rationale for collection.

At the commencement of the academic year, of the two teachers of the classroom Olaf and Knut (both pseudonyms), Knut was on leave and Olaf taught the class alone. The students however sat in eight groups around their cluster of tables. During this period I sat to one side of the classroom and observed Olaf conduct teaching-learning alone. This period involved the teaching-learning of the topic in first chapter of the textbook and ended with the conduct of a school test. The data collected in this period constituted my preliminary and first cycle.

Knut joined teaching-learning after the first school test and from the teaching-learning of the second chapter. This event marked my next cycle of data collection. Henceforth, I sectioned my data to correspond with the chapters of the textbook and allowed the topic dealt within each, be the background for both questionnaires and problem solving task.

The presence of eight student groups, made me decide in addition to observe the class with one group for every new chapter. I decided to focus on the teaching-learning of the classroom through the activities of each of these groups, calling each of them **group-in-focus**. I observed teaching-learning of mathematics with a new group-in-focus with every new chapter taught in the school year, from the second cycle onwards. My cycles of data collection thus began with a new group-in-focus, chapter and topic and ended with the test administered by the school. I now discuss values and ethical issues that underlined data collection.

Values and ethical issues

I consider what I value and the issues of ethics that I discuss, deeply intertwined and exhibited, in my practice of data collection and research. I mention these aspects towards the end of this chapter since they are for me the most important and the least I need to explain the 'why' of. As argued by Fog (1993), I consider the issue of ethical standing both superior and prior to the issue of scientific demands on research. I also argue that my ethical stance is best supported by how I deal with such issues. My choice of methods of data collection was based on a constant reflection of two aspects: my research questions on one hand and the kind of data that was both relevant and possible to collect on the other. In the making of these choices, I considered the right of my students to classroom teaching-learning as primary, irrespective of my study.

Within an ethnographical researcher practice, it is in the relationships entered into with the teachers and students that the issue of values and ethics comes to the fore. I had met Olaf and Knut two weeks before classes were to start. As a gesture to be part of their team, I designed and shared with them a set of mathematical tasks that could be attempted in their classroom (Appendix A.12). We also reached an understanding of not entering into discussion before the class was to commence. I entered into a similar working relationship with the students: sharing books, at times chewing gum and exchanging challenges in mathematics.

I tried to resolve any dilemmas in values and ethical issues from an epistemological stand and the conception that knowing is part of a long and complex process of learning. Following Molander (1993) I took a pragmatic conception of knowledge where there is no pure knowledge, knowledge is always in the making. By this I lay emphasis on the dynamics of knowledge, knowledge-in-action or knowing which was being understood in the numerous events of teaching-learning.

Outline of data chapters in thesis

In the present chapter in which I detail the various issues relevant to methodology and field work, I find it relevant to present an outline of the data and analysis chapters (Numbered 4, 5, 6 and 7) that are to follow.

As mentioned before, I discuss in my data and analysis chapters four themes: a collaborative classroom practice, the consolidation of meaning, problem solving know-how and cooperative problem solving. These grounded themes, theoretical perspectives of which I discussed in Chapter 2, evolved and developed in the teaching-learning in the classroom. I present and discuss each of these four themes upon a triangulation of three cases each. I offer the titles of each of the four chapter headings and their respective cases in the table below. I provide my synthesis of theoretical perspectives, methodology and analysis in Chapter 8.

Chapter and case number	Chapter and case title
Chapter 4	A collaborative classroom practice
Case 1	A single teacher
Case 2	A team of teachers
Case 3	Cooperative learning is formalised
Chapter 5	The consolidation of meaning
Case 1	Consolidation at the blackboard
Case 2	Consolidation of meaning making
Case 3	Consolidation of intuitive knowing
Chapter 6	Problem solving-know-how
Case 1	Solutions to questions
Case 2	Applying known solutions
Case 3	Questions to problems
Chapter 7	Cooperative problem solving
Case 1	When together and how heavy
Case 2	Two bodies in motion
Case 3	SA/V ratio and metabolism

Formats of data presentation

In my penultimate section of my chapter on methodology, methods and fieldwork I discuss the formats and codes I use, with which I present data in the four chapters that follow. I discuss below two kinds of codes. The first refers to the codes used in my transcriptions of audio recordings. The second refers to a particular two column format of data presentation, that I had to design in order to portray, the multiplicity of media utilised in communication and teaching-learning in the classroom.

Transcription codes in audio recording

The codes used in transcriptions of audio recording (Chapter 7) are:

Code	Description
((...))	Researcher comments
[...]	Simultaneous utterances
----	Marks the abandoning of an utterance
....	Indication of a pause
@@	Laughter
!!	An exclamation
= =	Lengthening of a word

Two column format in classroom teaching-learning

The presentation of data that represents a study of teaching-learning by taking into account the various physical and intellectual artefacts while accounting for their relationships in the classroom provides a challenge. In designing and implementing the two column format (in Chapters 4, 5 and 6), I had three objectives. Firstly, to achieve transparency both in the presentation and communication of the events that transpired in the classroom. The format thus needed to accommodate both the existence and contribution of various artefacts to teaching-learning. Secondly, following Luria (Chapter 2) to distinguish what is said from what is done, to accommodate a study of how speech is instrumental in raising action. Though both what is said (utterance) and what is done (action) are actions of individuals, in their division I intended an analysis of their mutual relationship. Thirdly, to represent the sequentiality and complexity of events as they unfolded in the classroom. In sharing my two-column format, I first list the abbreviations and conventions that I have utilised and then offer an example which I elaborate upon.

Abbreviation-Convention	Description
CW	Class work
HW	Home work
NOR	Norwegian
Q	Question
RES	Researcher
STD	Student , when name is not identified
((...))	Researcher comments to clarify or amplify event
...	Writing on the board
<i>But I think ...</i>	Student utterances audible and meant for group mates
...	Writing in student notebooks
@@@	Laughter
...	Excerpt from textbook
But I think ...	Student utterances audible to the whole class
...	Excerpt from worksheet

I present below an extract from classroom teaching-learning (Given in Chapter 4) in which most of the abbreviations and conventions mentioned above are used. I follow the extract by an explanation.

Event	Person	Utterance	Action
1	Olaf	It is important to use formulae	
2	RES	((NOR: Formelsamling i matematikk))	Notes Olaf draws the students attention to a Formula book
3	Olaf	((Q1.41 (d): Calculate))	$3^5 \left(\frac{x}{3}\right)^4 = 3^5 \times \frac{x^4}{3^4} = \frac{3^5 \times x^4}{3^4}$
4	Olaf	What is the rule	
5	Olaf		$= 3^{5-4} \times x^4$ $\therefore 3x^4$
6	RES		Notes many students are still not using rules. Works with Per who seems to be looking for assistance
7	Per	((Working in his notebook at Q 1.41(b)))	$\frac{2^5 \left(\frac{5}{2}\right)^3}{5^2} = \frac{2^5 5^3}{5^2 2^3}$
8	RES	((Shows an example in Per's notebook))	$\frac{6^4}{6^3} = 6^{4-3}$
9	Per	((Per continues))	$= 2^{5-3} \times 5^{3-2}$ $= 2^2 \times 5$ $= 20$
10	Per	<i>I am not very good at mathematics. I don't have confidence</i>	
11	RES	<i>Just go ahead and try</i>	

The above extract relates 11 events and describes the actions of three people: Olaf the teacher, Per a student and me the researcher (RES). The utterances of Olaf spoken loudly for the whole class are seen in events 1 and 4, while those of Per spoken to me and not meant for the whole class, are seen in events 10 and 11 (Marked in italics). The actions of Olaf at the backboard are seen in events 3 and 4 (Marked in grey) while those of Per and myself in Per's notebook are seen in events 7, 8 and 9 (Marked with broken lines). The title of the formula book in Norwegian is given in event 2 (Marked NOR). Researcher comments are offered in events 2, 3, 6, 7, 8 and 9 ((Mentioned in double brackets)). While Olaf was discussing the question numbered Q 1.41(d) in event 3, the above extract shows Per's attempt at Q 1.41(b) in his notebook, in event 7.

With the above extract representative of the complexity that I observed in classroom teaching-learning, I now discuss the analytical process that I adopted in the four data and analysis chapters that follow.

Analysis and writing

The thesis that I currently present is a result of analysis spread over time, inclusive of my taking field notes and the making of my present arguments. I can only attempt to outline below the involved and creative nature of the process which I have found similar to finding patterns and evolving strategies in problem solving. I make an attempt all the same.

My first step in analysis was in the classroom, **interpreting** the actions of teachers and students while recording the observations I made. This was followed by highlighting on the same afternoon, the utterances and actions of teachers, students and my own along with any events that I surmised as significant with coloured markers. I followed the above pattern for the three days a week that I observed the class and made a weekly summary. In this summary, I clubbed the utterances and actions I highlighted daily along with any reflections I had at that point of time.

I **sectioned** the above process of weekly summarising into the chapters of the book, referred to as my data cycles and then reviewed the data cycles with an objective of designing the problem solving task (PST). In this I narrowed down the choice of possible areas in each chapter or cycle to two or three. I was also able to peruse some answer scripts of the students at their school test. In addition to my knowledge about the questions that the students were doing from the textbook this gave me an idea of the kind of questions the students were attempting and their responses. As mentioned before I set for the first few cycles feedback worksheets which were a result of my analysis of the teaching-learning till then.

I then **narrowed** down the specific area in which to conceptualise and design the PST with appropriate intellectual artefact or model. These included on different occasions: a speed and time graph, an algebraic model, a probability tree, a Cartesian graph, a logarithmic graph and a numerical pattern. The final design of the PST was based on my judgement of the abilities of the students I would set the PST for, the artefact or model as mentioned above and the context and goals of the problem which I thought would hold the attention and interest of the students.

The selection of data from my field notes in every chapter (cycle) in the two column format, followed by the transcription of the PST, together formed a sequence of teaching-learning events which I treated as a topic to report about in my thesis. Towards the **reduction of data**, I had the necessity of using only that data which was available to me in English. This narrowed my choice of raw data though with an unavoidable feeling that I may have some data relevant but inaccessible. The data I offer in my two column format thus has mention of utterances that were made in Norwegian and which I had no access to, since I did not disturb the teaching-learning in the classroom. However, while transcribing the PST, I took professional help for utterances in Norwegian.

To the sequence of topics extracted from raw data as above, I **applied my units of analysis** of mediated action, participation in context and knowing and knowledge artefacts. As mentioned before it was in this process of such an analysis that I found the need for an additional unit of analysis, with which to analyse the actions of students both in classroom teaching-learning at their group-tasks and at my PST. I found the unit of ‘activity’ appropriate and applicable for the same.

Though I had an intuitive idea of analysis the seven cycles of observation offered towards writing my thesis, I then analysed in depth the selection of data identified as topics for the first five chapters and cycles. With the **importance of historicity and sequentially** of events in my mind, I then decided to report on the teaching-learning as it transpired from the commencement of the school year. As I discuss in Chapter 4, the nature of teaching-learning in the classroom was teacher driven in the first topic I report on. The practice of group work was established in the second topic and consolidated by the end of the third topic I report. From then on group-work became normative in classroom teaching-learning. I thus **chose the first three topics to report on in my thesis** since they also revealed how a teacher driven practice was shifted to group work in the teaching-learning of mathematics in the classroom.

Externalising my preliminary findings at a seminar in my department, from a sequential analysis of the first three topics, provided the opportunity for me to move away from data and reflect about the synthesis provided thus far of analysis in theoretical terms. It was a combination of such a synthesis and personal reflection that led to the recognition of the **grounded themes** that I report as my four data and analysis chapters. In writing my thesis chapters, I have been aware of the filtering process that my analysis and synthesis simultaneously encouraged. By this I refer to my choosing those themes that strongly emerged from analysis grounded in data and my need to cast aside for later reporting those events of teaching-learning, that were none the less informative in their own right. It is to the credit of such a grounded approach, that I am able to discuss the micro-culture of teaching-learning in the classroom on the strength of the four grounded themes. It is also to the same approach that I attribute my **inability to discuss in depth** the three topics of number understanding, equations and proportionality and scale factor in similar figures. Though I do offer a summary of the teaching-learning of these three topics in Chapter 8, I remain satisfied in Vygotskian terms to offer the degree of socialisation that was possible in the teaching-learning of these topics in the classroom. It is **the breadth of such a view**, that I discuss in the coming data and analysis chapters.

4. A collaborative classroom practice

In the first of my data and analysis chapters, I elaborate as mentioned earlier upon specific practices adhered to in teaching-learning, participation in which enabled meaning making of mathematics. In such an attempt I account for two kinds of data: that which establishes a chronological ordering of events and that which details specific kinds of teaching-learning practices within such an ordering. I offer such an account prior to other thematic expositions about the consolidation of meaning, the development of problem-solving know-how and the cooperation of students at problem solving, presented in the chapters that follow.

My objective in this chapter is two fold. First, to trace the flow of classroom activity within which the teaching-learning of mathematics was structured and constituted in the classroom. Second, to elicit the intentionality of the teachers by inferring from their actions as also those of the students in response to the teachers, since it was the teachers who steered the events in the classroom. In short I explore the when and what of the social and material practices of teaching-learning in the classroom and the related nature of individual participation possible.

In elaborating the collaborative practice, I make a deliberate choice of reporting on my first three cycles of observation after which the practices I elaborate upon become normative. In such an exposition I have the opportunity to account for the establishment of the practice itself, since teaching-learning in the first cycle was driven by one of the two teachers Olaf, in a largely teacher-driven manner. It was upon the arrival of Knut, who joined teaching-learning from the second cycle, that a collaborative classroom practice was set in place. The shift from a largely teacher-driven to a largely student-centered classroom practice is thus portrayed. The mathematical topics dealt with in the three cycles I report on are: number understanding, equations and proportionality and scale factor in similar figures. It is in these very topics that my thematic expositions in the coming chapters are built upon, also providing the opportunity for their triangulation within each theme.

I use the term **collaborative practice** to encompass the teaching-learning classroom practice initiated by the two teachers, the participation of eight student groups at their group tables at all times and the instructional importance given by the teachers to specially designed group-tasks within the practice. It is the working of the students with peers in groups, at specially designed group-tasks and at tasks not specially designed for group work, which the teachers and I refer to as **cooperative learning**. The ground rules of cooperating at given classroom assignments (both group and other tasks) were discussed and formalised by the students, during the teaching-learning of the third case I report on.

The present chapter thus enables a wide angle view of the classroom within which I sketch the nature of teaching-learning in mathematics enabled by such a practice. I locate in this chapter the artefacts of use along with those outcomes of their mediation which became the basis for subsequent participation in the collaborative classroom practice. Over the development of such a practice, I locate along with the position of artefacts in the practice established, the shifting position of both students and teachers in their participation. In doing so I detail the extent and nature of shift brought about in the teaching-learning of mathematics, corresponding to the shift from a largely teacher-driven instructional practice to a more student-centered practice. In the exposition of such a shift, I sketch the ground in which the micro-culture of teaching-learning was constituted in the classroom. I build my arguments for this micro-culture beginning with this chapter, continue reflection in other data chapters as well and consolidate my findings in the final chapter.

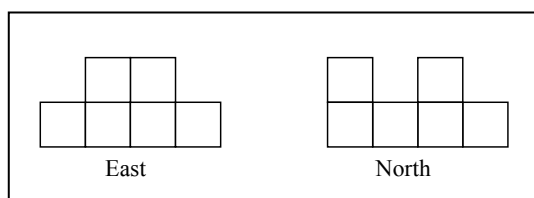
I present the establishment of the collaborative classroom practice as a succession of three cases and sections titled: *A single teacher*, *A team of teachers* and *Cooperative learning is formalised*. These three cases elaborated upon are illustrative of: the establishment of the intentionality of the two teachers Olaf and Knut, the participation of students in the teachers' intentionality and the participation of students with their own intentionality. I triangulate these cases in my concluding discussion.

A single teacher

In a year-ahead interview, both Olaf and Knut expressed their emphasis on cooperative learning by students with group-tasks in their teaching practice. It is in light of the related collaborative classroom to be established, when Knut joined teaching, that I view the present actions of Olaf. In six sub-sections below, I identify specific artefacts and outcomes, in the use of which Olaf expressed his intentionality. Parallel to the flow of work in the teaching-learning of the classroom at group-tasks which the teachers designed, I also elaborate more routine assignments that the students took part in the progression of teaching-learning.

How many cubes: the formation of student groups

On the first day of school the students were asked to form groups. Upon students forming groups of 3 or 4, Olaf gave the groups a group-task to attempt. The group-task asked the students to guess the number of cubes present in a formation represented by the diagram given above. This problem had been implemented by Knut in an earlier class of Olaf's as part of his Master dissertation. It is from the following day that I started taking field notes.



If you have a problem: the textbook

Commencing with this sub-section I offer extracts (Codes in Chapter 3) from classroom teaching-learning, indicative of the nature of teaching-learning in mathematics that emerged in Olaf's instructional practice. On the first day beyond greeting his students Olaf began as below:

Event	Person	Utterance	Action
1	Olaf	Turn to page 14 ... there are some rules in the box	
2	RES		Notes the rules given in the book summarise the four operations of addition, subtraction, multiplication and division with fractions.
3	Olaf	If you have a problem, box first, partners next, then me.	
4	RES	Olaf now goes over the following examples worked in the book: (a) $\frac{7}{12} + \frac{3}{8}$ (b) $3 + \frac{5}{6} + \frac{4}{9}$ (c) $3 \times \frac{17}{18}$ (d) $\frac{14}{15} \times \frac{6}{49}$ (e) $\frac{35}{12} \div \frac{28}{27}$	

The above extract demonstrates the use of **the textbook** in the classroom, the first physical artefact I identify, by the actions mediated and uses it was put to: to draw attention of the students as a class (1), as a source of rules that are to be followed while working with fractions (1, 2) and as a source of examples that have been worked out (4). As part of the norms and practices he was establishing Olaf spelt out two rules: the use of the textbook and the order in which the students were to seek help from each other in their participation in his classroom (3). I now discuss the first use of the blackboard in teaching-learning.

How else can we do this: the blackboard

The second physical artefact that I identify in the teaching-learning of mathematics in the classroom is **the blackboard**. Choosing to discuss the operation of multiplication in fractions, as was being discussed in the previous extract Olaf approached the blackboard as below:

Event	Person	Utterance	Action
1	Olaf	How do we multiply this	
2	Olaf	((Writing on the blackboard))	$\frac{16}{15} \times \frac{5}{8}$
3	STD1	We multiply the numerators and the denominators	
4	Olaf		$\frac{16}{15} \times \frac{5}{8} = \frac{80}{120} = \frac{2}{3}$
5	Olaf	How else can we do this	
6	STD2	Reduce first	
7	Olaf	A good idea	

8	Olaf	((Now obtains by reducing the fractions with the common factors 5 and 8))	$\frac{16}{15} \times \frac{5}{8} = \frac{2}{3}$
9	Olaf	Some students mentioned to me that they were TOLD to multiply. What do you think?	((Addressing the students))
10	Olaf	((Works out another example demonstrating the use of common factors 3 and 7))	$\frac{14}{15} \times \frac{6}{49} = \frac{4}{35}$

The above extract evidences the actions mediated by the blackboard. Olaf used the blackboard not only as a medium of display to an audience of 32 students (2, 4, 8 and 10) but also to bring about discussion (1, 5 and 9). He questioned the students by using examples in an attempt to look more closely at the rules or ways of operating fractions, in order to focus on the use of common factors to reduce numerators and denominators in the multiplication of fractions.

In addition to demonstrating the use of common factors (8 and 10) Olaf elicited how the students who were new to his teaching-learning would attempt his questions. Considering that the topic of fractions and ways of operating fractions would be part of the curriculum for the students at lower secondary school, Olaf guided the participation of students beyond the doing of multiplication, to ways of knowing how to multiply fractions subsequent to reducing them with common factors.

I single out **ways of operating fractions** as the first of the many 'ways' in which the teaching-learning in the classroom allowed for the appropriation of artefacts. The artefacts involved in the above extract were **fractions** and were intellectual since they intellectually mediated the part of the whole they represented. I now turn to discuss how Olaf advanced the participation of students in mathematics in his classroom.

Drawing parallels and alternate strategies

In the extract below, Olaf draws attention to parallel structures within mathematics and the possibility of alternative ways of operating with fractions. This he does to discuss a question in the textbook that he had set for the students to attempt as part of classwork. Olaf is seen calling for and acting upon a student's suggestion.

Event	Person	Utterance	Action
1	Olaf	((Q1.21(a) Combine))	$2\left(\frac{3}{8} + \frac{1}{4}\right)$
2	Olaf	((Olaf explains in NOR))	$3(a + 2b) = 3a + 6b$
3	Olaf	What else could we do?	
4	STD1	((Offers alternative in NOR))	

5	Olaf		$2\left(\frac{3}{8} + \frac{2}{8}\right)$ $= 2\left(\frac{5}{8}\right)$ $= \frac{10}{8}$
6	Olaf	Is it OK to leave it there?	
7	Olaf		$\frac{5}{4}$
8	Olaf	It is nice to see you using different strategies	

The use of **the textbook** in the above extract mediated two different yet parallel outcomes, though both were meant for the students. Firstly, Olaf used the textbook to simultaneously set questions to all the students in his class. Secondly and because of prior instruction, the students used the textbook not only as a source of questions and a source of referring to ways of operating fractions but also as a source of how ways of operating fractions were applied in specific examples.

By his actions Olaf advanced the meaning making of his students in four ways. Firstly, he drew attention of his students to structures in algebra that paralleled those in arithmetic (2). Secondly, he abandoned an approach which he himself acknowledged (2) and asked for and adopted a student's suggestion (3-5). Thirdly, he drew attention to the convention of expressing fractions in their lowest terms (6-7). Finally, by discussing alternative ways of operating fractions, Olaf demonstrated both the existence of alternative ways of operating fractions and the use of easier ways of operating fractions. By guiding student participation towards **simpler strategies** and **conventions**, Olaf guided the values that came along with their use towards ways of knowing. I now turn to the use of students' notebooks, the third physical artefact.

Remove the brackets: the students' notebooks

The extract below transpires towards the end of teaching-learning of the topic of number understanding within which the sub-topic of exponents followed the sub-topic of fractions. I offer my discussion with three students as a stand-in teacher and also reflect on Olaf's actions. The students are working at a question from the textbook in preparation for the school test announced for the next teaching period. The question asked for expressing the product of $(2 \times 10^4)(5 \times 10^3)$ as a single exponent.

Event	Person	Utterance	Action
1	RES	<i>What is the confusion</i>	
2	STD1	<i>Does 2 multiply 4</i>	
3	RES	<i>Remove the brackets</i>	

4	STD1	((Q 1.31(e) Express as a single exponent))	$2 \times 10^4 \times 5 \times 10^3$
5	RES		Notes STD1 hesitates
6	RES	<i>Do you get a 10</i>	
7	STD1	((In her notebook))	$= 10 \times 10^4 \times 10^3$ $= 10^8$
8	RES		Is now with STD2
9	STD2	<i>Does 2 multiply the power 4</i>	
10	RES	<i>Just remove the brackets</i>	
11	STD2		$= 2 \times 10^4 \times 5 \times 10^3$
12	RES	<i>Do you get a 10</i>	
13	STD2	((Got without working any intermediate steps and a 'yes' meaning 'got it'))	10^8
14	RES		Now with STD3 who has written the question but is hesitant to ask
15	RES	<i>Remove the brackets</i>	
16	STD3		$2 \times 10^4 \times 5 \times 10^3$
17	RES	<i>Do you get a 10</i>	
18	STD3		$= 10 \times 10^{4+3}$ $= 10^8$
19	RES	<i>What is the power in ...</i>	Pointing to the first ten
20	STD3	<i>One</i>	
21	RES		Notes that $a = a^1$ is not explicitly mentioned as a rule but called upon or left to be discovered while working with exponents
22	RES		Notes Olaf at the board calling attention to same question
23	Olaf	Do we need the brackets	
24	Olaf		$(2 \times 10^4)(5 \times 10^3)$
25	Olaf	No in multiplication. Yes in addition	
26	Olaf		$= 2 \times 10^4 \times 5 \times 10^3$ $= 2 \times 5 \times 10^4 \times 10^3$ $= 10 \times 10^4 \times 10^3$ $= 10 \times 10^7$ $= 10^8$
27	Olaf	That's a hundred million	
28	Thor	A lot of zeros!	

Unlike the use of fractions earlier, the above extract deals with the use of **exponents** as intellectual artefacts, signifying the repeated multiplication of the same number as in the base of the exponent. The other intellectual artefact that is dealt with (to a greater extent than previously) is **the**

bracket. From its use in bracketing two fractions, the goal of Q 1.31(e) above is to do away with the bracket and express the terms remaining as a single exponent. The removal of the bracket towards this goal is evidenced in three instances above followed by a multiplication process which shows diversity in individual attempts (7, 13 and 18). Though the question being attempted by all the students was the same, each applied an individual **way of operating exponents** and used the laws of exponents commensurate with the personal meaning being made.

The above extract also offers insight about the teaching-learning relationships that have begun to be established in the classroom. It evidences the view a teacher might obtain while gaining access to **students' work** in their **notebooks**. Such access evidences students' efforts in relation to the work that was set by Olaf 'to be' accomplished by the students, and what was 'being' accomplished by the students. By observing the events above as a stand-in teacher, I found my reading as a teacher (21) different from that made by Olaf (23-26). While I inferred that $a = a^1$ was left as an implicit assumption, not explicitly dealt with, Olaf's actions evidence his concern for the lack of explicit mention of an operative sign between the two brackets in the question.

Based on the above extract I make two conjectures. Firstly, just as my reading of the teaching-learning of mathematics was based on my interaction with the students, Olaf's actions reveal his reading based on observations he had made. Secondly and as a consequence of the first, Olaf had begun extending his teaching at the blackboard on the basis of students' working in their notebooks. Even in a largely teacher-driven instructional practice the later is evidence of Olaf pursuing a student-centered practice. Olaf's actions guided the participation of all the students in the class based on his mediation of an attempt made by one of them. In this the blackboard was now mediating the attempts of the students in the classroom, while the students' notebook was mediating students' understanding. I now turn to bring in the fourth physical artefact used in the teaching-learning of mathematics in this classroom.

I don't have confidence: the formula book

My final extract that concludes the exposition of the first case, records the use of **the formula book**. The extract transpires on the same day as the one above and evidences Olaf's concern that the students use and apply the rules being discussed. Olaf's instruction upon his return after a recess and my own interaction with one of the students is offered.

Event	Person	Utterance	Action
1	Olaf	It is important to use formulae	
2	RES	((NOR: Formelsamling i matematikk))	Notes Olaf draws the students attention to a Formula book

3	Olaf	((Q1.41 (d) Calculate))	$3^5 \left(\frac{x}{3}\right)^4 = 3^5 \times \frac{x^4}{3^4} = \frac{3^5 \times x^4}{3^4}$
4	Olaf	What is the rule	
5	Olaf		$= 3^{5-4} \times x^4$ $\therefore 3x^4$
6	RES		Notes many students are still not using rules. Works with Per who seems to be looking for assistance
7	Per	((Working in his notebook at Q 1.41(b)))	$\frac{2^5}{5^2} \left(\frac{5}{2}\right)^3 = \frac{2^5}{5^2} \frac{5^3}{2^3}$
8	RES	((Shows an example in Per's notebook))	$\frac{6^4}{6^3} = 6^{4-3}$
9	Per	((Per continues))	$= 2^{5-3} \times 5^{3-2}$ $= 2^2 \times 5$ $= 20$
10	Per	<i>I am not very good at mathematics. I don't have confidence</i>	
11	RES	<i>Just go ahead and try</i>	

The above extracts show that within the topic of number understanding, teaching-learning which initially dealt with the sub-topic of fractions and moved over to the sub-topic of exponents, was now demanding the application of knowing related to the use of brackets in both . Unlike ways of operating fractions for which the textbook was used to mediate both rules and application, the rules with exponents or ways of operating with exponents were built by Olaf during the teaching-learning of exponents. I relate the building of these rules in Chapter 5, where I detail the consolidation of meaning in the classroom.

Though the above extract evidences that Olaf highlights the usefulness of applying rules by calling attention to their use from the formula book (2), it is Per's notebook that is of importance. Per's mathematical problem (7) and his perceived learning difficulty (10) show that Per benefited from the example shown to him in his notebook (8). Along with his successful attempt at the question (9) Per was able to reflect on two aspects: the mathematics he was learning and also his own learning of mathematics. As with the observations made by Olaf and me in the previous extract, the above extract evidences the centrality of the **students' notebooks** as a mediating artefact in the teaching-learning of mathematics in the classroom. By drawing on students' attempts at questions or problems in them, Olaf and I were able to guide the personal meaning and shift the participation of the students, in their attempts to towards the more propositional form.

Brief summary of the first case

In the first case discussed above, there was evidence of the use of four physical artefacts: the textbook, the blackboard, students' notebooks and the formula book. Apart from their use cited at their first instance, the sequence of extracts saw them mediate different uses over time. **The textbook** was a source of rules, had a definite position in the attempts students were making in the classroom, a source of worked examples and a source of questions, which the teacher could set simultaneously for all the students at the same time. The use of **the blackboard** for display of writing to all the students in the class was later extended by Olaf in displaying the attempts of one of the students as instruction for the others. The **students' notebooks** mediated their personal meaning or current understanding of the topic being taught. The working in these not only showed diversity but also provided them an opportunity to reflect on their own learning. The notebooks helped mediate personal meaning to the teacher when he drew from them for teaching-learning in the classroom. **The formula book** was used as a ready-reckoner for rules.

The specific mathematical and intellectual artefacts that were brought to discussion were **fractions**, **the bracket** and **exponents**. Student learning of their use and application was guided over time in a largely teacher-driven practice which was characterised by discussions at the blackboard. Alternate methods were encouraged and a small extent of work at the blackboard was driven by Olaf's observations of the working of students. Olaf's efforts in guiding the participation of students were centered on **ways of operating fractions** and **ways of operating exponents**. It was discussion about these 'ways' that contributed largely to ways of knowing mathematics in the teaching-learning that transpired.

Akin to a teacher driven classroom, the instructional practice evidenced in this case displayed a **larger share of teacher intention**, with direction on how to seek help while working, use of different strategies in attempting questions and that the formula book should be used. Students' **verbal participation** in the teaching-learning in the classroom was **restricted to brief answers** to questions that Olaf asked, directed by his intentions. Student participation in the teaching-learning of mathematics was personal in nature, restricted largely to the personal meaning they displayed in their notebooks. Though the students reflected on their knowing, these reflections did not form part of the teaching-learning taking place in the classroom. It was Olaf who drew observations from students' notebooks for the teaching-learning of mathematics.

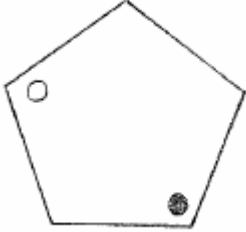
I now turn to the second case that deals with the topic of proportionality. As mentioned before this case evidences the arrival of Knut (for about 40% of time), his sharing of teaching-learning responsibilities and a greater emphasis on cooperative and group learning by the students.

A team of teachers

The joining of Knut brought about the functioning of two kinds of groups in teaching-learning: the team of two teachers and the eight groups of students. My use of the term ‘team’ for the teacher group and not for the groups of students is deliberate and will be discussed later. As I sketch in five sub-sections below, the progress of classroom practices in the teaching-learning of equations and proportionality, I sketch the use of group-tasks by the teachers, but do not elaborate upon all of them. This I do to maintain the sequentiality of events in this chapter, choosing to delve upon their role and issues of deeper understanding in the teaching-learning of mathematics, in the thematic chapters that follow.

When together: cooperating at a representation

The intention of using group-work in the classroom (as mentioned by Olaf and Knut in the year-ahead interview and briefly implemented by the group-task on cubes on the first day) was now pursued more consciously towards establishing a collaborative classroom practice. In pursuit of such an objective and coinciding with the arrival of Knut, the following group-task was set for the student groups.

When together	
<ul style="list-style-type: none">• In the pentagon are two dots, black and white, on the move.• The black moves two corners counter clockwise. The white moves three corners clockwise.• After how many moves are the two dots together?	

I discuss the outcomes of the above group-task in Chapter 7 (dealing with the cooperation of students in problem solving) and mention here that in the conduct of the above group-task by the teachers, the students cooperated at the given representation of the hexagon with the two dots. Such actions of the students were in contrast to their actions in a group-task that was to soon follow: *How heavy*. In attempting *How heavy* the students had to represent in order to cooperate. I now offer an extract that transpired in-between the conduct of the two group-tasks.

Will you show it on the board: speaking for oneself

Under the topic of proportionality, the sub-topic of algebraic expressions was reviewed prior to the commencement of the sub-topic of equations. By the time of the extract below both Olaf and Knut are recorded by me as visiting students’ tables during instruction. I evidence below an extension of the use of the blackboard accompanied by a shift in the position of the students, vis-à-vis classroom teaching-learning. It is also with this extract that I offer observations made by sitting with my group-in-focus, of three girls Anja, Lea, Stine and a boy Egil.

Event	Person	Utterance	Action
1	Knut	((Example (c) on page 47. Calculate))	$(2x - 3)(x + 2)$
2	RES		Notes students work on their own and the teachers visit group tables.
3	Knut	<i>Will you show it in the board</i>	((Talking to one of the students in my group-in-focus))
4	Knut	OK we take it on the board and Anja will present it	
5	Anja	((Anja at the blackboard))	$2x^2 + 4x - 3x - 6$ $2x^2 - x - 6$
6	Knut	Can you tell us how you got this	((Standing at the black board))
7	RES		Notes that <i>as</i> Anja explains she realises she should have + x and not - x. She corrects her solution.
8	Anja	((She finally has))	$2x^2 + x - 6$
9	Olaf	You may have forgotten this	((Standing at the teacher's table))
10	RES		Notes Olaf now walk up to the board
11	Olaf	You multiply each term in this one with each term in the other	((Pointing to the two brackets))
12	Knut	Now we do a little more difficult question	
13	Knut	((Example (d) page 47. Calculate))	$2y^2 - (y + 3)(2y - 1)$
14	RES		Notes students work at the question with teachers visiting group tables
15	Knut	Now Tia will present the solution	
16	RES		Notes murmur in classroom dies down and everyone pays attention
17	Tia	((Tia at the blackboard))	$2y^2 - (y + 3)(2y - 1)$ $= 2y^2 - (y \times 2y - y \times 1 + 3 \times 2y - 3 \times 1)$ $= 2y^2 - 2y^2 + y - 6y + 3$ $= -5y + 3$

The above extract evidences the shift in the position of students in the teaching-learning practices of the classroom. Instead of Olaf discussing the working of students' in the students' notebooks at the blackboard, the students now presented their work at the blackboard (5, 8 and 17). By being asked to explain their working the students spoke for themselves and had the opportunity to offer their personal meaning along with offering reason for their working. Upon Knut's asking (6-8), Anja corrected herself as she explained her working. A process of **justifying ones knowing** was being encouraged and established in teaching-learning.

The above extract also evidences how Knut requested Anja (in my group-in-focus) to show her working at the board (3-4). Knut's actions encouraging Anja, makes me conjecture that he may have done so with Tia a member of another group as well (15). I conjecture that these actions of Knut are critical to and indicative of, the **transparency** with which the new shifts in practice of the participation of students, in the teaching-learning of mathematics were being brought about.

I discuss here the cooperation between Olaf and Knut during teaching, which is a practice I observed the two teachers follow throughout the year. In such cooperation the teacher who was not teaching (Olaf) stood by and viewed the other teacher (Knut) at the blackboard. There were two noteworthy aspects to this. Firstly, it was possible for Olaf to observe the working of students in their notebooks, while Knut was teaching at the blackboard. Secondly, Olaf and Knut demonstrated a cooperative effort that was instructional, in the kind of cooperation they were expecting of their students. I conjecture that in working as a team, Olaf and Knut's cooperation was both visible and instructive in **group cooperation** becoming the **classroom norm**. I now discuss the second group-task, in which the students had to represent to cooperate.

How heavy: representing to cooperate

Subsequent to the revision of the sub-topic of algebraic expressions, I now discuss how the teaching-learning of the sub-topic of equations began with the conduct of the following group-task.

How heavy?

If a brick balances with three-quarters of a brick and three quarters of a pound, then how much does the brick weigh?

By the conduct of the above group-task (whose outcomes I elaborate in Chapter 7) on the third day following *When together*, the teachers were

$$\begin{aligned}
 B &= \frac{3}{4}B + \frac{3}{4}P \\
 B - \frac{3}{4}B &= \frac{3}{4}P \\
 \frac{1}{4}B &= \frac{3}{4}P \\
 \frac{1}{4}B \cdot 4 &= \frac{3}{4}P \cdot 4 \\
 \underline{\underline{B}} &= \underline{\underline{3P}}
 \end{aligned}$$

able to consolidate the practice of having the students cooperate at group-tasks. Not given any representation as part of the group-task, the students offered their personal representations to mediate their thinking in order to cooperate with others in the group and reach a solution.

In my discussion of the shift in classroom practices, I presently discuss the manner in which Olaf concluded the conduct of the above

group-task. Of the many representations made by the students, Olaf asked Tia to present her solution at the blackboard. Tia offered her solu-

tion starting with the equation $B = \frac{3}{4} B + \frac{3}{4} P$ and concluded with $B = 3P$. Following Tia, Olaf extended discussion around Tia's solution. After explaining her solution, Olaf rubbed out the 'P' in each step of the solution and further explained that the equation without 'P' was now an equation in the variable 'B'. Taking examples of various terms in the five steps, Olaf then revised the application of the four operations of addition, subtraction, multiplication and division in simple equations.

The use of the above group-task by Olaf to commence the sub-topic of equations, in addition to having the students cooperate was obvious after he concluded its implementation in the above manner. The above extract also evidences how a student was again encouraged, to offer her explanation at the blackboard. However in addition to speaking for oneself as in the previous extract, in the above extract, Tia as a student was encouraged to explain the solution she and her group had arrived at upon cooperation. I now discuss the consolidation of this practice.

What will x be now: speaking for the group

The shift in student participation in the classroom, where students offered their **personal and group meaning** at the blackboard, was consolidated by the teachers by encouraging a student (Ulrik) to offer at the blackboard his groups solution to a question attempted from the text-book. Though I present the data and analysis of this attempt in Chapter 6, I mention here that unlike speaking for the working of one's group at a specially designed group-task, Ulrik offered his groups solution at a question which he and his group had attempted from the text-book.

Such a practice had four significant implications for teaching-learning in the classroom. Firstly, Ulrik spoke both as an individual and on behalf of his group. Secondly, other members of Ulrik's group saw 'their' solution being discussed at the blackboard. Thirdly, the remaining students in the classroom had an opportunity to compare either their individual or group solution with the one presented by Ulrik. Finally, the offering of a group solution to a question from the textbook is exemplary of how group-work in the classroom was becoming more **common place and routine**. In encouraging such actions there is evidence once more, of how teaching-learning continued to become more **student-centered**. I now discuss the conduct of three group-tasks in succession.

Three group-tasks in succession

The sub-topic of simple equations was followed by the sub-topic of proportionality. This sub-topic was addressed in the teaching-learning of the classroom by three group-tasks in quick succession. As a culmination to my present case it is my intention to present a gist of the working of these three group-tasks, in line with the trajectory of the collaborative classroom practice, that Olaf and Knut are seen establishing. The three group-tasks I make mention of are titled as below:

1. Proportionality
2. Inverse proportionality
3. Follow Up

The conduct of the above titled group-tasks had three associated didactical motives. Firstly, they reinforced the opportunity the students had to cooperate at group-tasks for the teaching-learning of mathematics in the classroom. This was then, followed by the consolidation of the meaning made by the students at the group-tasks by the teachers. Finally, the group-task: *Follow Up*, allowed students the opportunity to apply their meaning making acquired and consolidated in the first two group-tasks, to yet another group-task and again upon cooperation.

To reduce complexity of reporting the data and analysis of the above group-tasks and in keeping with the didactical aims, I report the teaching-learning of the three group-tasks in two parts. In Chapter 5, I highlight the participation of students at the first two group-tasks. This offers insight on student participation at the group-tasks, after which I offer the consolidation of their meaning by the teachers. The latter corresponds thematically with the building of knowing in teaching-learning of mathematics in the classroom.

In Chapter 6, I elaborate upon student attempts at the third group-task. Since the group-task *Follow Up* demanded application of meaning and knowing developed in classroom teaching-learning, its data and analysis lends itself to problem solving know-how being developed in the classroom. I conjecture that the three group-tasks (Appendix A.13) designed by the teachers, evidence specific instructional aims. While on the one hand the sub-tasks within each group-task guided the development of meaning, on the other the sequencing of group-tasks guided and enabled for the application of knowing, that the first two group-tasks enabled. I now proceed to summarise the second case of this chapter.

A brief summary of the second case

The new physical artefact that I highlight, in the extracts presented as part of the second case and reflecting the shift in practices in classroom teaching-learning, is **the group-task**. The implementation of a succession of group-tasks, each with a different design, mediated different outcomes. While the conduct of *When together* initiated the cooperation of students, the conduct of *How heavy* consolidated the value and practice of cooperating. The outcomes of the latter were also utilised by the teachers to introduce equations: the algebraic form of the personal meaning made by the students. The conduct of the third, fourth and fifth group-tasks: *Proportionality*, *Inverse Proportionality* and *Follow Up*, was in quick succession and with well-defined objectives: bringing about personal meaning, consolidation of personal meaning into a propositional form and the subsequent application of the propositional form.

Apart from the group-task the use of the textbook, students' notebooks and formula book did not explicitly figure in this case. The use of **the blackboard** was however extended. From the uses mentioned earlier which mediated the working of students, the use of the blackboard now mediated students to speak for themselves as well as on behalf of their group. As mentioned, such a practice brought with it the opportunity of the **justification** of individual and **personal meaning** as also that of the **group meaning** made with other individuals in their group.

The actions of Olaf and Knut brought with them the value of **transparency**, expressed in three ways. Firstly, the students were gradually led to sharing and expressing meaning made in mathematics both within their groups and with the whole class. Secondly, the implementation of the group tasks first initiated; then consolidated and finally capitalised on cooperation by the students at the group-tasks. Finally, the teachers exhibited cooperation within their teaching which lent to the making of **cooperation** the **norm in the collaborative practice** being established.

Along with the intentionality of Olaf and Knut being expressed through their actions, this case also saw the participation of the students with their personal meaning and therefore their intentionality as well. The various shifts in the position of the students with respect to the teaching-learning of the classroom, made the **practice more student-centered**. Yet though the students were being given more and more opportunities to participate, it was in the didactical intentions of the two teachers Olaf and Knut that the students were participating.

Cooperative learning is formalised

The third and final case which I present in three sub-sections below relates to the topic of scale factor in similar figures. As mentioned earlier it is by the end of this case that the rules of cooperating are discussed and formalised by the students and put up for display on the pin-up board. As with the two earlier cases, I shall trace below the noteworthy shifts in the collaborative classroom practice that was to become the norm subsequent to this case, in the teaching-learning of mathematics in the classroom. My observations below span two groups-in-focus: the first with three boys Kim, Levi and Thor and a girl Nora and the second Levi, Thor and another boy Dan. This was necessitated by the regrouping of student groups by the teachers and my continuity in observation.

Similar triangles: to start a topic with

The topic of scale factor in similar figures began without prior notice, on a day prior to the school test on equations and proportionality. In a 40 minute period the students were given a group-task with which to start the topic. In contrast to the use of prior group-tasks for the didactical aims of first having students cooperate, then consolidating cooperating at

group-tasks and subsequently applying the concepts developed in a group-tasks towards the end of the topic, the topic of scale factor in similar figures began with a group-task. In the conduct of this group-task prior knowing of the students on the topic of similarity was called upon.

As an example of student participation at ‘a’ group-task, I elaborate the actions of students in my group-in-focus and describe the consolidation of their meaning made in this group-task in Chapter 5. In order to better follow the data and arguments that I present, I offer the group-task containing three sub-tasks in its entirety in Appendix A.14. I elaborate only on the first of the three sub-tasks (Task 1) below. Student working at all three is given in Appendix A.15 for reference.

On being given the group-task in separate worksheets, I record Nora and Levi conjecture about that the lengths of the missing sides of the triangles whose diagrams were given as part of Task 1. I draw attention to the fact that Task 1 asked to explain, why the two triangles were similar; implicitly stating that they were for a fact similar triangles.

In the extract below I begin with a brief interlude by Olaf. The extract also shows the beginning of two levels of participation by the students: one within the **group** (utterances shown in italics) and the other offered as participants of the **whole class** (shown without italics). Subsequent to Olaf’s brief interlude, the discussion around Task 1 soon shifted to the four students in my group-in-focus: Kim, Levi and Thor and Nora. I present below their deliberation and follow it by my observations. I follow my discussion with the written attempts of the four students.

Event	Person	Utterance	Action
1	Olaf	The angles of $\triangle ABC$ are similar to ...	
2	Levi	Those of XYZ	
3	Thor	<i>The angles of the two triangles are equal</i>	
4	Thor	<i>They are similar</i>	
5	Kim	<i>The reason is they add up to 180 degrees</i>	
6	Thor	<i>Values?</i>	((Addressing RES))
7	RES	<i>Values</i> ((Nodding))	Confirming usage in English
8	RES		Offers Kim a pen to write with
9	Levi	((Questions Knut, who is on his visit to their group tables, in NOR if his response to the third question $y = x/4$ is correct))	
10	RES		Notes Knut ask Levi in NOR to reconsider his answer
11	Levi	<i>Omvendt!</i>	((NOR word for Inverse or ‘the other way around’))

12	RES		Notes Kim to pull out his book and look at the contents page
13	RES		Notes Thor work at the calculator but soon give it up.
14	RES		Notes Levi and Nora to discuss the next task between them.
15	Thor	<i>Are angles and values the same as vinkles?</i>	((Use vinkles for vinkler which is the plural for angles in NOR))
16	STDs		((Discuss some dialogues from some fantasy or musical movie))
17	RES	((Symbol for multiplication is the dot and not the cross))	Notes student responses below.

In addition to sharing conjectures with Nora in his group, the above extract evidences Levi respond to Olaf (2). It also shows how Levi interacts with Knut who is on his **visits to various group tables** in the class. With the assistance of Knut, Levi modified his conjecture (9-11). The above extract also shows Thor make his observations of the given diagram (3, 4), and continuously search for the appropriate words to use and apply, both in Norwegian and English (3, 4, 6, 7 and 15). Towards his attempts at making meaning, in a manner similar to Levi taking assistance of Knut, Thor takes the help of the researcher at hand. I shall elaborate upon Thor's use of the calculator later in my thesis and present the working of the four students in their worksheets at Task 1 below.

Similar Triangles

TASK 1

Explain why the two triangles are similar.

The triangles are similar because it is always 180°

Calculate the length of YZ.

SpottBcewit4.4.

Kim

Similar Triangles

TASK 1

Explain why the two triangles are similar.

The unknown angle is the same as the angle on the other one.

Calculate the length of YZ.

Levi

Find the length of a side (y) in $\triangle ABC$ in terms of the corresponding side (x) in $\triangle XYZ$.

$32/8 = 4$

$40/4 = 10$

Find the length of a side (y) in $\triangle ABC$ in terms of the corresponding side (x) in $\triangle XYZ$.

$x = 8$

$y = x \cdot 4$

$10 = \frac{40}{4}$

$\frac{GRIS}{KATT} = \frac{GRIS}{KATT}$

Similar Triangles

TASK 1

Explain why the two triangles are similar.
The angles of the two triangles are the same, ergo they are similar.

Calculate the length of YZ.

Find the length of a side (y) in $\triangle ABC$ in terms of the corresponding side (x) in $\triangle XYZ$.

$x = \frac{y}{4}$
 $y = x \cdot 4$

$32/8 = 4$
 $40/4 = 10$

Similar Triangles

TASK 1

Explain why the two triangles are similar.

Because of same angles / sides

Calculate the length of YZ.

$\frac{32}{8} = 4 \mid \frac{40}{4} = 10$

Find the length of a side (y) in $\triangle ABC$ in terms of the corresponding side (x) in $\triangle XYZ$.

$y = 4x$
 $y = 4x$

I mention some noteworthy features with respect to the two extracts above. Firstly, the attempts of students at the above group-task took place in a relaxed atmosphere along with some background music (16). Secondly, the four students used their **group cooperation** to conjecture, test, verify and borrow ideas; in short learn from and with each other. This learning however was neither the same nor uniform, as can be seen by their responses which I discuss in more detail below.

The responses of students above reflect **individual thinking**. Taking the prior discussion along with individual responses into consideration, Kim's response seems to offer as explanation to the first question, a relationship about the angles of a triangle which he recalled as a fact (5). In Levi's response one can notice his correction of the algebraic relationship between the sides of the two triangles, for which as discussed before he involved Knut (9-11). Levi does not offer an explanation (as desired by the question) but evidences how he obtained the unknown angle in either $\triangle ABC$ or $\triangle XYZ$. His statement, that the unknown angle is the same as the corresponding angle on the other triangle (where it is known), evidences his **personal meaning making**. Though Levi's response might not explain why the triangles are similar, it does tell us how he inferred and concluded that they were similar.

Nora's response evidences her designation of the sides of $\triangle ABC$ as 'y' and those of $\triangle XYZ$ as 'x'. The corresponding angles are marked and her explanation shows that her argument for the similarity of triangles rests on the angles being the same. Thor's utterances reveal that he first observed the angles to be equal (3) followed by the use of the word 'similar' (4) used by Olaf and Levi before. Thor is then seen attempting a **conscious use of terms** in English, evidenced by his inquiry about the word 'value' (6-7) and later questioning if the words 'angles' and 'values' were words that could be used to refer to or in place of 'vinkler' (15). Further evidence of Thor's conscious use of English is present in his written response where his explanation for similarity, is that the two triangles have the same 'vinkles/angles'.

The actions of the students in my group-in-focus along with their written responses evidence the nature of meaning they were each making in attempting Task 1, **towards realising goals** set out by the group-task. There was evidence of how both Levi and Thor explicitly utilized the **presence of others** (Olaf, Nora, Knut and me) and both **physical and intellectual artefacts** (calculator, diagrams of similar triangles and words) in their meaning making processes. I call the enabling and realisation of meaning to happen in a construct I term **group space**, the use and extension of which I shall soon elaborate.

Having evidenced the **diversity** in the attempts of students in the above group-task, in the next sub-section I evidence the attempts of my group-in-focus soon upon their reorganisation by the teachers. I follow Levi and Thor who are now joined in their new group by Dan.

Pythagoras' theorem: upon regrouping and the recess

Towards fostering better cooperation, Olaf and Knut reorganised the student groups based on their observations in the teaching-learning of mathematics in the classroom (coinciding with my first two cycles of data collection) and the performances of students at their school tests.

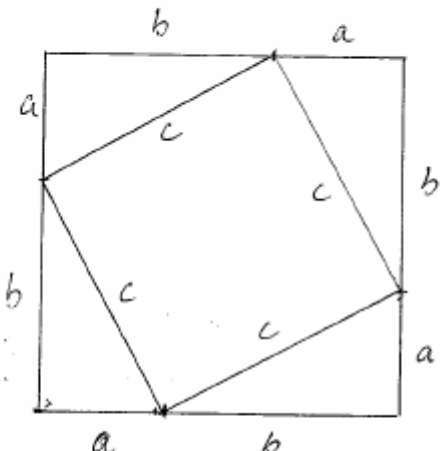
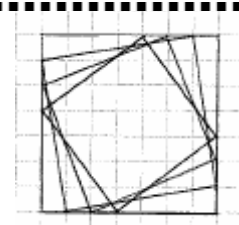
Yet before I present the ensuing data and analysis, I report at this juncture a particular practice of Olaf and Knut which I call **engagement**. Apart from keeping the students interested during classroom teaching-learning with wit and necessary humour, the engagement I speak about had two distinctive features. Firstly, while on their visit to various group tables and in their interaction with the students, both Olaf and Knut subtly elicited various dynamics about group and interpersonal interaction. These informal conversations took a more 'mathematical' turn in the recess, which is the second feature I draw attention to.

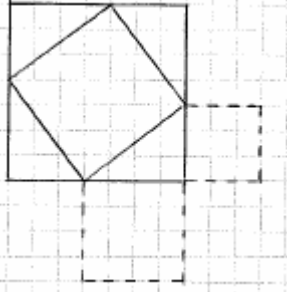
During recess and while erasing the writing at the blackboard, Olaf and Knut moved over at times to writing on the blackboard, some aspect of mathematics that a particular student may have wanted to discuss with them. Sometimes there was just one student and sometimes a whole

group. I mention the incidence of this practice as something which the students seem to extend. On one occasion, I record Rolf (from another group) discuss with Levi the following at the blackboard:

When is $14 - 1 + 1 = 12$? [Solution: $14 - (1 + 1) = 12$]

I mention the use of the engagement I relate above, since I notice such a practice turn into a **larger group space** during the recess I evidence in the extract below. Olaf and Knut presented the following group-task upon their regrouping of the students. I detail the working of my group-in-focus in their cooperation and conjectures in their attempt to 'prove' Pythagoras' theorem. The conduct of the group-task starts in the first teaching period and continues into the second, where the recess in-between is put to use by the students and the teachers.

Event	Person	Utterance	Action
		<p style="text-align: right;">Assessment ©</p> <p style="text-align: center;"><u>PYTHAGORAS' THEOREM - PROOF</u></p>  <p>The figure shows a square inside another square.</p> <p>Use the figure to prove that $a^2 + b^2 = c^2$</p> <p>Hint: Calculate the area of the smallest square in two different ways.</p>	
2	Thor	((Draws as alongside on the worksheet in which the group-task was handed out individually))	
3	Levi	((Offers in NOR his knowing about a square))	

4	Thor	<i>What is hypotenuse</i>	
5	Thor		((Meanwhile indicates $a = 5$ and $b = 10$ on his worksheet))
6	Dan	<i>c is hypotenuse</i>	
7	Thor	<i>I support him</i>	
8	RES		Notes Thor write down $c = 15$ cm
9	Levi		((Calls Knut who is close by their tables and shares what they know of the problem))
10	Knut	((Discusses in NOR))	((Runs his finger along the perimeter of the inner square))
11	Levi	<i>c multiplied by c or c^2</i>	
12	Knut	((Now asks in NOR for the area in another way))	
13	RES		Class breaks for the interval/recess
14	RES	((Observes Rolf having a discussion with the teachers, in which Levi gets interested. They discuss together in NOR Rolf's diagram, which is as alongside))	
15	Rolf	((Discusses the application of Pythagoras' theorem in NOR))	
16	Olaf	((Asks Rolf in NOR to prove Pythagoras' theorem))	((Olaf and Knut then leave for their coffee break))
17	Levi		((Raises his eyebrow in acknowledgement of the distinction))
18	RES		Olaf and Knut return after recess
19	Knut	((Asks them in NOR about their attempts))	((Olaf and Knut move around the group-tables looking for a possible solution))
20	Knut	Now Tia will demonstrate her solution on the board	
21	Tia		$(a + b)(a + b)$ $= a^2 + ba + ab + b^2$ $= a^2 + b^2 + 2ab$
22	Tia	((Explains in NOR how she would subtract $2ab$ from her algebraic expression equal to the four triangles))	$\left[4 \times \frac{1}{2} ab \right]$
23	Tia	((Explains in NOR that upon this subtraction the area of square remaining would be equal to c^2))	

In recounting how Olaf and Knut gradually and systematically developed a collaborative practice, the above extract evidences how for the teaching-learning of mathematics their classroom practices proved beneficial. As also seen in the previous sub-section, the above extract evidences how students had the opportunity of conjecturing in the safety of their group space (2-7). I conjecture that this 'space' was extended to the interaction of students across groups as can be seen by the support offered by the nature and practice of engagement I mentioned above.

In the recess, Olaf and Knut discussed Rolf's argument about Pythagoras' theorem (14-17). Rolf belonged to another group and his table was diagonally across where Levi and his group (my group-in-focus) were sitting. Levi rose to take part in the discussion initiated by Rolf in which he got the teachers interested. Upon joining them, Rolf explained the theorem with his diagram. Olaf and Knut on other hand were asking not for its application but its explanation or proof, the reasoning for which was subsequently provided by Tia (21-23).

I also mention that Rolf and Levi continued to exchange their interests in mathematics throughout the year ahead. This included challenges they shared with me (I share some in Appendix A.16), in preparation for the KappAbel, the Nordic competition in mathematics for school classes (<http://www.kappabel.com>) which Olaf encouraged his students to participate in. The recess was also useful for Per (*I don't have confidence*) and me to work at some mathematics now and then.

Cooperative learning: the rules lived by

I conclude my third case with the formalisation of the rules for cooperation in group-work by the students. The final form of the rules, that I detail shortly, were arrived at by discussion amongst the students in their respective groups. Their discussion was initiated by the following questionnaire handed out to each of the groups in a worksheet.

Cooperative learning in mathematics – why and how?

1. About the way of working in cooperative learning:
 - a. What are the arguments that cooperative learning be a good method to use in mathematics
 - b. What are the counter arguments
2. About the demands on the participants:
 - a. What characterises bad cooperation in the group
 - b. List four points that have to be followed for cooperation in a group to be best possible

The discussion amongst the student groups lasted ten minutes, after which Olaf and Knut held a whole class discussion involving all the student groups in the classroom. Though not privy to the discussion which was held in Norwegian, I offer the rules that were agreed upon and subsequently displayed in large letters on the pin-up board:

Cooperative Learning

- Everyone must be treated with respect.
- Everyone must contribute
- All ideas must be considered by the group
- Everyone must be aware of what transpires before the group moves ahead
- Everyone must be able to present the work of the group
- Everyone must ask the others in the group before seeking help from the teachers

As mentioned earlier the collaborative practice in the classroom, which depended on its deliberate establishment on the cooperative learning of students at group-tasks, was brought to conclusion in the above formalisation. As also mentioned earlier, the collaborative classroom practice whose establishment I have attempted to unravel in the three cases bore fruit in its continuance throughout the year. The practice of **cooperative learning within a collaborative classroom practice became normative** for the teaching-learning of mathematics.

A brief summary of the third case

Though not evidencing newer uses of the textbook, blackboard, students' notebooks and formula book, I state here that the uses being mediated did not discontinue but became a part of everyday practice. The use of **the group-task** as artefact in the third case, soon upon regrouping of the students, seemed to mediate and implicitly convey the reason behind the constitution of the new groups by Olaf and Knut.

The newer intellectual artefacts used in the second and third case were **equations** and **geometrical diagrams**. Though I discuss the use of equations in later chapters, I point to a specific feature that surfaced with the use of geometrical diagrams. When used, geometrical diagrams demanded associated terminology and specific usage. The use of 'same values of angles', 'corresponding sides', 'similar triangles', 'smallest square', 'hypotenuse' and 'proof' came with specific and associated **reasoning situated** in the context of the mathematical task and its goals. In this these terms were not just words but **intellectual artefacts** which mediated specific and propositional meaning.

The third case evidenced in particular, the use and extension of what I termed **group space**. This was evidenced in two occasions and used by students in safety to observe and conjecture with which to share and build personal meaning with. The first instance was the use Levi and Thor made, in their attempts at the task on similar triangles and the second was the use Rolf and Levi made of the recess for discussing Pythagoras' theorem. As argued earlier, both instances were made possible and nurtured directly and indirectly by the processes of **engagement** that I mentioned the teachers to practice. The participation of the students in both these constructs was useful in their meaning making.

The **consolidation of a collaborative practice** stands out as the instructional impetus in the third case outlined above. In addition to speaking for themselves and on behalf of their groups, there was evidence in the third case of the students participating with their own and increased intentionality. By formalising their cooperation the students implicitly acknowledged the **practice of cooperating and making meaning** with each other towards the goals of the mathematical tasks.

Concluding discussion

In the data and analysis presented in three cases above, there was a perceptible shift of the establishment of the intentionality of Olaf and Knut in the first, the participation of students in the teachers' intentionality in the second, and the participation of students with their own intentionality in the third. I now elaborate upon these cases on the basis of socio-cultural-historical and related perspectives discussed in Chapter 2.

In ways pointed out by Lampert (1990), the above cases evidenced the **establishment of norms and practices**, participating with and within which, it was possible for the students to make meaning of both teaching-learning and of mathematics. The means, by which participation was achieved, were also the goals which the teachers had set out to achieve. The collaborative practice that was established in the classroom enabled the students to become aware of the intentionality of the teachers, and participate in the same with their own. The acceptance of responsibility by the students in such a practice seemed to evidence, what Olson (2003) argued to be, a result of the establishment of **joint intentionality** between teachers and students within teaching-learning.

The participation of students in the above norms, routines and practice allowed for two important aspects: firstly, the making of **personal meaning** and **group meaning** and secondly the **making public of such meaning**. This involved various kinds of participation such as justifying one knowing, speaking for oneself and speaking on behalf of one's group, participating in one's group as also in whole class discussion and finally making individual meaning even while working at group-tasks. As argued by Bruner (1990), these aspects of practice provided for the possibility of greater intersubjectivity, which in turn provided the basis for the formation of a micro-culture premised on meaning making.

In line with the arguments of Boaler (1999), the classroom practice **afforded a particular kind of experience** in the teaching-learning of mathematics, where the students, teachers and the classroom had greater and greater access to the personal meaning making of the students. This was evidenced by group work becoming routine and by the gradual shifting of the teaching-learning in the classroom, where it was the students' participation that became more and more central.

In line with the arguments of Skovsmose (2005), a **sphere of practice** was established in which by providing meaning to the various actions being participated in, it was possible for the students to ascribe meaning to their own participation. This was evidenced by their first taking part in cooperative learning and later debating the reasons and rules for cooperation. The nature of such participation brought the meaning making of the students of the sphere of practice, in addition to the personal meaning in mathematics that transpired in the teaching-learning of the classroom. Argued as important to knowing by Skovsmose, such a teaching-learning classroom practice enabled the meaning making of the students to be foregrounded for the teaching-learning of mathematics.

The opportunities provided for participation as above, in turn positioned the students as **agents in their own learning**, the benefits of which have been argued by Burton (1999b). This was evidenced by the utilisation by students, of other students and teachers, in developing their understanding, as well as the utilisation of available physical and intellectual artefacts towards achieving the goals of the task at hand.

The possibility of participation in meaning making at the individual level, as well as at the level of the classroom, was evidence in turn of the establishment of a classroom that was and termed by Schoenfeld (1992) as a **sense making microcosm**. As argued by him such a classroom provided opportunities for knowing by the students, since by participation in the classroom, the possibility of being apprenticed by the teacher in the relationships made possible by the teaching-learning practice were many.

Apart from the sense and meaning making afforded in the general teaching-learning classroom practice, the specific nature of outcomes through the mediation of artefacts also contributed to meaning making. The utilisation of **physical artefacts** included: the location of the textbook within classroom collaboration, the blackboard as a forum for presenting and justifying students personal meaning, the sharing of personal meaning externalised in the students' notebooks, the wisdom of using the formula book for rules of working and the deployment of the group-task for didactical aims. Along with **intellectual artefacts** (fractions, exponents, brackets, geometrical diagrams and terms specifically applicable) whose outcomes of mediation I extend in chapters to come, these artefacts were crucial to meaning making and cognition in the culture being constituted as argued by Cole (1996), Wartofsky (1979), Bakhurst (1991), Bruner (1996) and Vygotsky (1978). Their very utilisation provided for numerous **moments of mediated action** as argued by Wertsch (1998) which Salomon (1993), Pea (1993), Wertsch and Tulviste (1996), Stetsenko (1999), Ueno (1995) and Säljö (1998) have also argued as crucial to distributed cognition afforded in situated activity, incorporating both social and cultural affordances.

The teachers' role, in bringing about the above mentioned outcomes of meaning making in the classroom, was evidenced beyond the very establishment of the practice in many ways. This included the value of **transparency** in the establishment of the practice evidenced by their visiting students' group tables, the value of gradually shifting the teaching-learning to become more **student-centered** and the value and practice of what I have termed as **engagement**. The importance of the presence of values in any environment has been argued by Goodnow (1990) and Bishop (1988) alike. As also argued by Bishop the establishment of the above practice made the teaching-learning within constitute a micro-culture, where the students were made to feel inclusive and enabling **enculturation**, with their alienation or acculturation less possible.

Towards ways of knowing mathematics beyond doing, where knowing is the basis for participation as argued by Greeno (2003) and Bishop, there was evidence of the attention teachers gave to **ways of operating fractions** and **ways of operating exponents**. The highlighting of these actions and routines as the cultural voice of mathematics in the classroom, made the participation of students in teaching-learning, their participation in **meaningful mathematics** as argued by van Oers (1996).

Numerous interpersonal relationships were established between the teachers and students: teacher and student in the whole class, teacher and teacher in the whole class, teachers and students in the whole class and student and student in a group. Such relationships entered into, allowed for a greater access to what Vygotsky (1981a) termed as social relationships upon which internalisation and **individual development** was premised. It was the analysis of these social relationships that Minick (1987) argued was crucial to an understanding of the development of meaning. It was towards these very opportunities for meaning making in a group for which I had pointed to a construct called **group space**.

The possibility of the above relationships for the meaning making in mathematics allowed also for the very **formation of consciousness**, which gave rise to contextual meaning as argued by Bakhtin (1986), the interaction with materiality as argued by Leont'ev (1978) and allowed for the possibility of school being a place where students realised abilities not hitherto known to them as argued by Luria (1994). The conscious search for the meaning and appropriate use of words as terms by students, is only one example of this concept.

In concluding discussion in this chapter and before going over to the next, I mention three aspects whose incidence begins in this chapter yet whose analysis I recall and discuss in later chapters. The first is the use of utterances by the teacher or researcher, to raise the actions of the students to a greater level of consciousness. I summarise my discussion of these instances in Chapter 6: *Problem solving know-how*. The second is a

detailed analysis of the cooperation of students at group-tasks, beyond the mere incidence of these in teaching-learning practice established. I discuss this aspect in Chapter 7: *Cooperative problem solving*. Finally the participation of the students in what I have termed as group space. I discuss these in the concluding Chapter 8: *A micro-culture*.

My final synthesis of the data and analysis of the present chapter is the following. There is evidence that a collaborative practice was established in the teaching-learning of the classroom. Such a practice was also goal-directed towards the cooperation of individuals for the teaching-learning of mathematics. The establishment of a goal-directed practice was pursued by deliberate actions and constituted both by the teachers and students. In such a practice it was the students meaning making which was gradually brought central to the teaching-learning of mathematics in the classroom. Over time **meaning making and cooperating at mathematics became the ideal of the environment** in the classroom.

I now turn to discussion in the coming chapter which allows deliberation of how in the teaching-learning described above, personal forms of meaning by the students were consolidated into propositional forms.

5. The consolidation of meaning

In the second of my data and analysis chapters, I discuss those events that were steered by the teachers which enabled the consolidation of meaning in the teaching-learning of mathematics in the classroom. The events that I elaborate in this chapter play out in the collaborative classroom practice that I detailed in Chapter 4. My exposition remains chronological in the relating of the three cases, yet focuses on the consolidation of personal meaning made, leading to its propositional form. In such an elaboration I draw on both whole class teaching-learning and the conduct of group-tasks, outlining the nature of participation by the students and the nature of consolidation brought about by the teachers.

As in Chapter 4, I present three cases each corresponding with the consolidation of meaning in the topics: number understanding, equations and proportionality and scale factor in similar figures. The background information that is relevant is that during the teaching-learning of the topic number understanding, Olaf taught alone when correspondingly the teaching-learning was largely teacher-driven.

Upon the joining of Knut the teaching-learning during the topic equations and proportionality involved three group-tasks: *Proportionality*, *Inverse proportionality* and *Follow Up*. I deal with consolidation of meaning made in the first two group-tasks in this chapter and the subsequent application of knowing (Follow Up) in the next chapter.

The teaching-learning of scale factor in similar figures was initiated with the group-task: *Similar triangles*, partly discussed in Chapter 4. In elaborating consolidation in the topic of scale factor across lengths, areas and volume, I include this group-task along with other teaching-learning activities, with which I discuss the first ‘activity’ in this chapter.

In taking a more thematic look at the consolidation of personal or spontaneous meaning during teaching-learning in the classroom, I discuss how students gain membership to the propositional or scientific forms of mathematics in each topic. In so doing I continue and extend my exposition of the micro-culture being constituted, which as mentioned before I reflect upon and consolidate in the final chapter.

I present the consolidation of meaning within the collaborative classroom practice, as a succession of three cases and sections titled: *Consolidation at the blackboard*, *Consolidation of students’ meaning* and *Consolidation of intuitive knowing*. The three cases elaborated upon are illustrative of: the building of meaning in a teacher-driven practice, the building upon of the meaning making experiences of students by the teacher and finally the building upon of students’ intuitive knowing by the teachers. As also in chapter 4, I summarise each of the three cases and triangulate them in my concluding discussion.

Consolidation at the blackboard

In elaborating my first case towards the consolidation of meaning, I discuss the teaching-learning of the sub-topic of fractions and exponents in four sub-sections. In the first, I extend the discussion about ways of operating fractions encountered earlier in Chapter 4. In the second, I extend discussion about ways of operating exponents, also mentioned in Chapter 4, along with an instance of bilingual learning. In the third, I discuss the calling upon of common sense in the teaching-learning of exponents and in the fourth, the highlighting of a genre specific to the teaching-learning of exponents. The consolidation of meaning in each is driven largely by Olaf and takes place at the blackboard.

Homework and surprise: a prior finding is useful

In the teaching-learning of fractions I relate below not one but two extracts. In the first I discuss the arguments of Olaf relating to Q1.21 of which part (a) was discussed in Chapter 4. By his discussion Olaf extends teaching-learning of ways of operating fractions, simultaneously calling upon the attempts of students at their homework. An event that transpires in this extract is then made use of in the extract that follows.

Event	Person	Utterance	Action
1	Olaf	But what was the easiest way	
2	Olaf	((Q1.21(d) Combine))	$\left(\frac{7}{6} - \frac{2}{9}\right)\left(\frac{1}{5} + \frac{1}{4}\right)$
3	Rolf	LCM	
4	Olaf	Do we have to find the LCM of all four	
5	Rolf	No	
6	Olaf	It is sufficient if we find of those within the bracket	
7	Olaf	What is the common denominator	
8	STD	Eighteen	
9	Olaf		$\left(\frac{7 \times 3}{6 \times 3} - \frac{2 \times 2}{9 \times 2}\right)\left(\frac{1 \times 4}{5 \times 4} + \frac{1 \times 5}{4 \times 5}\right)$ $= \frac{21-4}{18} \times \frac{4+5}{20}$
10	Olaf	((Olaf pauses and corrects his working))	$= \frac{21-4}{18} \times \frac{4+5}{20}$ $= \frac{17}{18} \times \frac{9}{20}$ $= \frac{17}{40}$
11	Olaf	Is there some mistake?	
12	Olaf	Is it possible to reduce this	
13	Rolf	17 is a prime number	

14	Olaf	Correct!	
15	RES		Wonders what 'correct' refers to: 17 being prime or the fact that the fraction cannot be reduced because 17 is prime or that 17 and 40 have no common factors

In the above extract Olaf began with laying stress and asking what the **easiest way** of attempting the questions set for homework was (1). In response Rolf explained the need to find the lowest common multiple or LCM (3). Olaf's rejoinder then ascertained if it was necessary to find the LCM of all four denominators (4). This was followed by his statement that **it is sufficient** to find the LCM of only those within the bracket (6). In these there is a display of **intersubjectivity** and agreement that the LCM of 'all four' and 'of those' refers to the 'denominators' of the fractions. That there is an implicit understanding of such a reference is also evidenced when Olaf asked for the 'common' denominator (7). The response by a student that followed (8) extended the acceptance and understanding of this reference. In addition to being able to find the LCM, the student offered the LCM of the denominators of the fractions in the first bracket alone, something which Olaf had earlier declared as sufficient.

The intersubjectivity displayed above is continued within the extract. After applying the common denominator to the fractions in the first bracket Olaf applied **similar reasoning** to the second bracket without further explanation (9). Along with correcting his error (9-11) Olaf then returned to giving importance to convention, as he did in Chapter 4 of expressing a fraction in its lowest terms (12). By the many events in this extract Olaf recalled for the students the many newer aspects that were appropriate to the application of **ways of operating fractions**.

In continuation of the limited participation of students by this time of teaching-learning in the year, the above extract also evidences student utterances as being made in response to Olaf's questions (3, 5, 8 and 13). However in Olaf's asking whether the fraction could be reduced any further (12), Rolf offered his knowing about prime numbers (13) though this concept was not particularly referred to in the ongoing discussion. Yet both Olaf's question and Rolf's response left some conceptual issues unresolved (15). It was not clear what was correct; that the number 17 was prime, or because 17 was prime it could not reduce the fraction, or that the numerator 17 and the denominator 40 had no common factors. It is regarding the prime nature of 17, which was extended in the teaching-learning to follow that I offer the second extract below. In so doing I argue for a value of teaching-learning that I term as **continuity**. By continuity, I refer to those didactical and pedagogical actions, by which the teaching-learning of mathematics is sustained in the classroom.

The first incidence of continuity I refer to, involves and extends the use of the textbook, an artefact I singled out in the previous chapter. The above extract evidences the use of the textbook as a source of questions that students could attempt at home. Such a use provides for a sense of continuity between the teaching-learning in the classroom and the time spent at homework. The second incidence of continuity relates to the practice of the teachers discussing the attempts made by the students at homework in the classroom the next day. This allows for the establishment of continuity between the mathematics attempted at home, possibly individually, with the teaching-learning with peers and teachers in the classroom. The above extract evidences how such an opportunity was utilised to revise and extend the sub-topic of fractions by Olaf.

The third incidence of continuity I refer to, is the utilisation of the existence of intersubjectivity towards the teaching-learning of mathematics. I conjecture that the understanding of references: LCM, all four, of those and common denominators is premised upon continuity as are also the more mathematical aspects: the easiest way, it is sufficient and the application of similar reasoning. The consolidation of meaning which I discuss in this chapter evidences the utilisation and builds upon of the value of continuity within teaching-learning. It is for a fourth incidence of continuity, related to the prime number 17, that I now turn to.

The extract below offers discussion that transpires in the same teaching period and follows the one in the above extract. By the time of this extract the students were attempting the next question (Q1.22) from the textbook as part of their classwork. Olaf's emphasis on adopting the **easiest possible way** in attempting any question as discussed in Chapter 4 is evidenced again. He also extends the application of LCM of the denominators of a complex fraction enclosed in a bracket.

Event	Person	Utterance	Action
22	Olaf	((Q 1.22(c) Calculate both with and without a calculator))	$\left(\frac{\frac{3}{2} + \frac{5}{8}}{\frac{1}{4} + \frac{25}{2}} \right)$
23	Olaf	There is a simpler way, what is the common denominator	
24	STD	Eight	
25	Olaf	How do I multiply by a whole number in the numerator and the denominator	
26	Olaf		$= \frac{8}{8} \left(\frac{\frac{3}{2} + \frac{5}{8}}{\frac{1}{4} + \frac{25}{2}} \right)$

27	Olaf		$= \left(\frac{3 \times 8}{2} + \frac{5 \times 8}{8} \right)$ $= \left(\frac{1 \times 8}{4} + \frac{25 \times 8}{2} \right)$
28	Olaf	What can I do now ... I can reduce these	
29	RES		Notes Olaf cancel each 8 with respective denominators
30	Olaf	What is the result of these	
31	Olaf		$= \frac{12 + 5}{2 + 100}$ $= \frac{17}{102}$
32	Olaf	Is it possible to reduce this?	
33	Olaf	What is 17×6	
34	Olaf		$\frac{17}{102} = \frac{1}{6}$
35	RES		Observes Rolf is taken aback

As mentioned in Chapter 4, I recorded Olaf proceed to the blackboard in the above extract after his making observations of the attempts of students in their notebooks. Olaf's utterance that there is **a simpler way** (23) is a reference to an attempt made by the student. In demonstrating another simpler way, Olaf allowed his actions to be mediated by the student's working, towards building upon the meaning being made of his teaching-learning. In his attempts, Olaf's actions address two aspects simultaneously (26-27). Apart from demonstrating a newer application of the bracket, Olaf also demonstrates a newer application of the lowest common multiple or LCM across the denominators of fractions that form a complex fraction. By so doing Olaf extended **ways of using brackets** along with **ways of operating fractions**.

The fourth incidence of the consolidation of meaning, exemplary of the value of continuity, and as evidenced in the above extract, came as a surprise. In the previous extract I had drawn attention to the fact, that what was correct in the context of question being discussed was left unresolved. Such a resolution followed when in attempting to find a solution to Q1.22(c) Olaf showed how 17 being a prime number could reduce a fraction, if it was a common divisor of both the numerator and denominator (31-35). In the surprise expressed by Rolf (35), there is room to conjecture that Rolf may have thought that because 17 is a prime number, it was not possible to reduce the fraction in its earlier instance.

In bringing my observations of the incident related to Rolf, and the prime number 17, to Olaf's notice after the teaching-period, Olaf expressed surprise. The opportunity that presented itself was purely coinci-

dental. However in the events related to the nature of the number 17 and its role in the context I describe, it is not the nature of resolution or surprise that is significant but the benefit of **continuity** in its occurrence. It was the earlier occurrence of events related to the number 17 which when built upon in classroom teaching-learning, though quite coincidentally, that led to an **instance of knowing mathematics**. The occurrence of discussion related to the first incidence, followed by the building upon its nature subsequently was an incidence of the building upon of meaning that enabled greater knowing, though may be only for Rolf. I now turn to the three sub-sections related to the sub-topic of exponents.

How do you read 2^4 : bilingual learning

The following extract relates to the teaching-learning of ways of operating exponents. It evidences how Olaf uses the bilingual teaching-learning in the classroom to further students' knowing in mathematics.

Event	Person	Utterance	Action
1	Olaf	What is 2^4 in Norwegian	
2	Olaf		2^4
3	STD	Potenser	
4	RES		Notes Olaf point to the above and explain in NOR that in English 4 is called the exponent and 2 is called the base
5	Olaf	How do you read 2^4 ?	
6	Olaf	Two to the power four	
7	Olaf	Two to the fourth power	
8	RES		Olaf checks usage with RES who agrees with either usage
9	Olaf	Do you remember from ungdomsskole	((NOR for secondary school))
10	Olaf		$2^3 \times 2^4 = 2^{3+4} = 2^7$
11	Olaf	It only works when you have the same ...	
12	STD1	Number	
13	STD2	Base	
14	Olaf	Base	
15	Olaf		$\frac{2^7}{2^5} = 2^{7-5} = 2^2 = 4$

The above extract allows me to point to yet another instance of the **value of continuity**, though this time it is brought about by specific attention to language. Olaf recalled the prior knowing of his students in their secondary school, of the multiplication and division of exponents (9-15). He then used language to mediate prior knowing in two ways both of which I now turn to discuss. In the first, Olaf began the teaching-learning of exponents by asking his students to identify in English terms the students

would be familiar with in Norwegian (1-8). By these actions Olaf guided the participation of his students in English on the basis of earlier references and meaning made in Norwegian.

In the second instance Olaf stressed on language usage towards continuity in the teaching-learning of **ways of operating exponents** (11). In response to his question Olaf waited for the students to offer the word ‘base’ in addition to ‘number’ (12-15). In repeating the term ‘base’ Olaf used his authority to show his approval and preference for the second term. He **privileged** the usage of the word ‘base’ as part of guiding the participation of his students in mathematics and in English.

Along with the implicit use of appropriate terminology like LCM and common denominator in the first extract of this case (like same angles and similar triangles in the previous chapter) by the events of the above extract Olaf explicitly highlighted and drew attention to **ways of speaking** in the teaching-learning of mathematics. In addition to the many ‘ways’ discussed so far, ways of speaking specified the ways in which specific words as terms were intellectual artefacts, mediating specific meaning and outcomes. I shall address the issue of language use in the fourth section again and presently turn to an extract that transpired in-between and exhibited the calling upon of common sense.

Why is it one ... it is correct: common sense

Towards the consolidation of students’ meaning in his teaching-learning of exponents, Olaf asks for the meaning of 2^0 in the extract below. Olaf identifies a conjecture which is correct and subsequently demonstrates its meaning, by applying rules or ways of operating exponents. In such a demonstration he appeals to common sense.

Event	Person	Utterance	Action
1	Olaf	Remember this ... the rules are given in the book like	
2	Olaf		$a^m \times a^n = a^{m+n}$ $\frac{a^m}{a^n} = a^{m-n}$
3	RES		Observes the students to locate the above rules in their text books
4	Olaf	We have a problem here ... what does this mean	
5	Olaf		2^0
6	STD	Zero	
7	Olaf	That is a suggestion but is it correct ... explain that	
8	Olaf		$\frac{2^4}{2^4}$
9	STD	One	
10	Olaf	Why is it one ... it is correct	
11	Egil	Four divided by four is one	

12	Olaf	The numerator and the denominator are the same	
13	Olaf		$\frac{2^4}{2^4} = \frac{16}{16} = 1$
14	Olaf	Now apply the rule	
15	Olaf		$\frac{2^4}{2^4} = 2^{4-4} = 2^0 = 1$
16	Olaf	What does this show us	
17	Olaf	((Notes Olaf draw two arrows between the left hand and right hand sides of both equations, showing equality of the two left and right sides and concluding that 2^0 is 1))	$\frac{2^4}{2^4} = 1$ $\frac{2^4}{2^4} = 2^0$

In the above extract Olaf first pointed to rules given in the textbook. In drawing attention to rules Olaf's reference helped mediate how rules were a **generalisation** of all the specific examples, which had been discussed so far in the classroom (1-3). Based on the general rule Olaf then brought the **particular case** of 2^0 for discussion of its meaning (4, 5). In response to Olaf's question there is evidence of students **conjecturing**. Olaf responded to the first conjecture by calling it a suggestion and asking for its explanation (6, 7). After pre-empting a more accurate conjecture, by expressing the special case of 2^0 in the form of a fraction (8), Olaf responded to the second conjecture, by stating that the suggestion of 1 was correct, but needed explanation (9, 10). In a third attempt Egil conjectured that the value of 1 could be obtained by dividing the equal powers of 4 (11). Olaf responded to this by drawing attention to the equality of not just the power of the exponents, but the equal values of the exponents as numerator and denominator in the fraction (12, 13).

It is in the ensuing discussion related to the application of the rules being discussed, that Olaf appealed to common sense. Olaf at first demonstrated the application of the general rule to the specific exponent of 2^0 being discussed (14, 15). In asking what the result then obtained demonstrated (16, 17) Olaf equated two expressions: one obtained by applying the rule and the other obtained by prior knowing of fractions. In this I conjecture that Olaf appealed to the **common sense** of equal values of different representations made them equal to each other.

The goal of understanding what 2^0 meant in the above teaching-learning activity, was resolved by a process of simplification and appealing to common sense which was then not questioned. In displaying how the general rule also applied to a special case, the above discussion while concluding with and relying upon common sense had two notable and associated processes. The first was the pedagogical practice in which

Olaf not only allowed for conjecturing by the students but also guided them. The other was the intellectual nature of reasoning employed. The teaching-learning evidenced in the above extract dealt with no physical artefacts. Neither was ‘what’ the intellectual artefacts of fractions and exponents ‘meant’ an issue. In mediating the meaning being made of a new form of intellectual artefact namely 2^0 , the references called upon were not even remotely physical but entirely **intellectual in nature**. I now turn to the fourth section that I conclude this case with.

When we multiply we add: a genre

In the extract below there is an evidence of the development of ways of speaking in the teaching-learning of mathematics to a genre of speaking.

Event	Person	Utterance	Action
1	Olaf	Anything you'd like me to explain or do before we go on	
2	Olaf		$2^3 \times 2^{-4}$
3	Olaf	We apply the rules we know before	
4	Olaf	When we multiply we ...	
5	STD1	Add	
6	Olaf		$= 2^{3+(-4)}$ $= 2^{-1}$
7	Olaf	We can do more ...	
8	STD2	Fraction	
9	Olaf		$\frac{1}{2}$
10	STD3	Zero point five	
11	RES		Notes Olaf wait for explicit equivalence of the exponent, fraction and decimal forms of representation
12	Olaf	I want to do c	((Referring to Q1.33(c)))
13	Olaf	((Q1.33(c) Calculate))	$\frac{3^{-2}}{3^{-3}}$
14	Olaf	What do we do when we divide	
15	STD	Subtract	
16	Olaf		$= 3^{-2-(-3)}$ $= 3^1$
17	Olaf	Subtract because we have division	

The above extract evidences a rarefied use of ways of speaking that I brought attention to earlier. By rarefied I refer to the cryptic way of speaking that contributed to a **genre** made possible by the sharing of understanding between the students and the teachers. The acceptance of the

equality of ‘multiply’ with ‘add’ (4-5) and ‘divide’ with ‘subtract’ (15-16) was possible because of the understating and socialisation associated with their usage. The acceptance of what the specific utterances meant was a result of the presence of social agreement within a **sphere of practice**. Towards an understanding of the existence of the above socialisation I also draw attention to the absence of rules like $a^m \times a^n = a^{m+n}$ in the above extract, based upon which the utterances became cryptic. I conjecture that the possibility of meaningful communication above is evidence of the internalisation of the same rule by many if not all of the participants and upon which an agreement of what could be added or subtracted could be reached. The rules as intellectual artefacts acted as psychological tools. From a semiotic perspective an **algebraic symbolisation** was now signified in **verbal symbolisation** and resulted in a genre. The existence of a genre was in turn evidence of the existence of a meaningful yet situated teaching-learning activity, since the usage of such a genre could be called to question outside similar classroom activity.

Apart from building upon and consolidating meaning by privileging a genre, Olaf also established in the above extract the equivalence of three different intellectual artefacts, as he paused in his teaching until he obtained from the students **equivalent representations** in different notational systems (6-11). In calling for different representations that were equivalent Olaf was able to mediate the commonality in their meaning. As an indication of the **active nature** of meaning made by students by this stage of teaching-learning I conclude with the following extract.

Event	Person	Utterance	Action
1	Olaf	Look here if I have this what does this mean	
2	Olaf		$(2^2)^3 = 2^2 \times 2^2 \times 2^2 = 2^6$
3	Olaf	What do we do with the indices	
4	Olaf	What is the connection between 6 and the two numbers ... multiply them	((Pointing to 2 within the bracket and 3 outside))
5	STDs	!!	((Some in agreement))
6	Olaf	You are not to confuse between the two rules	((Referring to rule for addition and subtraction of exponents))
7	Rolf	What about 2 to the power 3 into 2 to the power 3	
8	Olaf		$2^3 \times 2^3 = (2^3)^2$
9	Olaf	In this case they are the same	

In offering his alternative of arriving at 2^6 , something brought to discussion by Olaf; Rolf’s actions above are a reflection of two features: the nature of participation that Olaf had been encouraging so far and an introduction of Rolf’s own participation in teaching-learning.

Brief summary of the first case

The above case began with evidencing the use of a textbook not discussed in Chapter 4, by which it was possible to bring about a value in teaching-learning that I identified as **continuity**. Describing this value by didactical and pedagogical actions that contributed to the sustaining of teaching-learning, these included: the use of the textbook by the students at home, the extending of the homework done by the teacher in the class, the use of the incidence of prior mathematical concepts (like the prime nature of numbers), the reference to mathematics learnt prior to the present class and the utilisation of intersubjectivity. The occurrence of intersubjectivity or the knowing of each other's intent was itself evidenced by the 'absence' of questioning of the meaning of phrases like: all four, of those and common denominators in a given context, and accompanied by the drawing upon of **values** in mathematical reasoning such as the easiest or simplest way, sufficiency and the exercise of similar reasoning.

The 'ways' in which intellectual artefacts were to be used and which included those of operating fractions and exponents was now accompanied by **ways of using brackets**, examples of which were also seen in the previous chapter. The search for the use of appropriate terms in the previous chapter, led to a more formal acknowledgement by Olaf by his paying attention to **ways of speaking**. When accompanied by the internalisation of rules and a shared understanding; a rarefied and cryptic way of speaking resulted in a verbal and social agreement, leading to the existence of **a genre** by which meaning was shared in particular contexts.

Two kinds of **semiotic transition** were observed. Firstly and as discussed above, the reassigning of understanding present first in the algebraic rules of exponents into a verbal genre that implicitly included socialisation. Secondly, the representation in different notational systems of mathematical quantities that had the same value such as the symbols: 2^{-1} , $\frac{1}{2}$ and 0.5. The activities that brought about such transitions were largely intellectual in nature and associated with related practices that led to greater number understanding. These practices included the utilisation of bilingual teaching-learning, allowing students to conjecture and rely upon of **common sense**. In all such instances, the personal and spontaneous nature of meaning made was called upon towards its consolidation, into its more propositional or scientific form.

The actions of the teacher in bringing about the above, involved the use of his authority in **privileging** certain aspects, in the sphere of practice being established in the classroom and by attempting to make the nature of **meaning making by the students active**. Yet events leading to the consolidation of meaning were driven largely by the teacher and at the blackboard, not at students' tables. I now discuss the bringing about of the latter in classroom teaching-learning.

Consolidation of students' meaning

I elaborate my second case on the consolidation of meaning in four sub-sections, by referring to the attempts of the students in my group-in-focus at two group-tasks: Proportionality and Inverse proportionality. The meaning made by the students in their attempts at each of these group-tasks is followed by the consolidation of their meaning by Olaf.

Proportionality: depending on the concrete

The group-task *Proportionality* as given to the students is given for reference in Appendix A.13. I present an extract below and discuss thereafter the actions of the four students in my group-in-focus: Anja, Egil, Lea and Stine, at each of the sub-tasks sequentially.

Proportionality
Here is a typical set of results for an experiment – stretching a spring.

Task 1: Complete the table

Force (N)	Extension (cm)	$\frac{\text{Extension}}{\text{Force}}$ cm/N
0	0	-----
0.5	2	
1	4	
1.5	6	
2	8	
2.5	10	

Task 2: Draw a graph based on the table Use the force-values along the x-axis and the extension-values along the y-axis.

Task 3: What can you say about the graph?

These are the results for a particular spring. There could have different values for another spring.

Task 4: What would the graph look like for a stiffer spring?

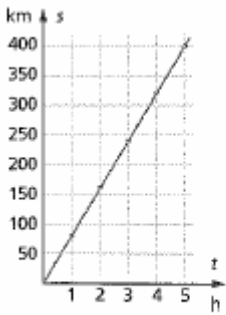
Task 5: What is the ratio y/x between the extension (y) and the force (x)? (Use the table)

Task 6: Use your result from Task 5 to find the formula for y in terms of x ($y = \dots$)

In this case the quantities x and y are said to be proportional.

In working at the first two sub-tasks (Task 1 and Task 2) I record Egil stretch between his fingers a rubber band he pulled out from his pencil box and Lea seek and find a pencil with a spring. Egil then filled his table and asked the others if his working seemed OK. On filling up her table Lea exclaimed when she noticed that the all the ratios she had found in her table had the same value of 4. Subsequent to filling her table Anja wondered aloud in the group as to what the word ‘force’ meant. In parallel to the above working of my students, Olaf drew a **laboratory spring balance** on the blackboard and discussed the meaning and prior knowing of the students. He thereby mediated their personal experience of the extension of a spring on the application of a weight as force, an

aspect referred to in Task 1. After filling their tables the four students also drew appropriate **graphs**. I now elaborate the attempts of my students at Task 3. I follow their discussion by their written attempts.

Event	Person	Utterance	Action
1		Task 3: What can you say about the graph?	
2	RES		Notes Stine place her pencil along the straight line she drew in her graph sheet. This is also done by Anja.
3	Anja	<i>Is it a line?</i>	
4	RES		Notes Egil to pull out his textbook and look up the section titled 'Proportionality' and show the others in his group the graph on page 55
5	RES		Notes the graph and writing in the textbook is as below
6		 <p>Vi ser at grafen er ei rett linje gjennom origo.</p> <p>((We see the graph is a straight line passing through the origin.))</p>	
7	RES		Notes Stine ask which page number it is. She then pulls out her textbook and reads the text herself. The responses of the four students are below:
8	Anja	<p>This graph is a proportional graph. When we double the force, the extension is doubled as well. The graph is a straight line.</p>	
9	Egil	<p>When we double the force, the extension is doubled as well. It increases proportionally. The graph goes straight through the origin, linear.</p>	
10	Lea	<p>It's a straight line going through the origin. It's linear.</p>	
11	Stine	<p>It's a steep straight line, and it goes through origin.</p>	

Considering that the above group-task was given to the students for group work there is evidence as in chapter 4 of the students' use of their **group space**: to reach out for materials that stretch, check with others if their working was OK, ask what the word force meant and if what they had drawn was indeed a line (2-3). On his part Olaf used the diagram of a spring balance to mediate the experience the students may have had with a spring balance in the laboratory. In the context of this group-task the diagram of the spring balance was therefore an intellectual artefact that recalled the experience of the stretching of a spring on the application of a weight or force. Unlike the rubber band which was a physical artefact mediating the physical activity of stretching, the diagram was an intellectual artefact mediating the intellectual experience had with the spring balance as physical artefact by the students and prior to this class.

In the above extract there is also evidence of the use of the textbook by the students in two ways both of which I now discuss. The goal set by Task 3 was to answer a question which asked what could be said about the graph (1). Towards this while Stine and Anja wondered if their graph was a line (2-3), Egil pulled out **his textbook** and turned to the graph on a page titled Proportionality (3-5). The practical sense of using the textbook towards responding to the given task was realised by Stine as well who imitated Egil's actions (6). These events provide evidence of another use the textbook was put to by the students: as a source of information that can be relied upon towards attempting the goals of the task.

In addition to the actions the students took towards addressing Task 3, I now draw attention to the nature of Task 3, which demanded what could be 'said' about the graph. In the context of the bilingual teaching-learning in the classroom, such a task made two kinds of demands upon the students. Firstly, the students were to 'say' something about the graph, secondly the students had to 'say' the same in English. For native Norwegian speaking students this necessitated a reliance on the **textbook as a source of formal language in addition to information**, evidencing the other use the textbook was put to by the students.

Using the textbook towards addressing the task at hand however did not resolve the problem the students had at hand. The requirement of responding in English precluded my students from using the Norwegian writing given in the textbook (6). It is in this context that I discuss the responses of the four students. Though the textbook confirmed that the graph they had each drawn was a straight line that passed through the origin, the responses of the students show diversity. There is evidence of the use of a combination of phrases like 'proportional graph', 'extension will double as well', 'increases proportionally', 'through the origin', 'linear' and 'steep straight line' (8-11). Since none of the responses is the

same I conjecture that the diverse responses reflect the **personal meaning** made by the students in their individual attempts at the task.

I point to two actions above that are didactically significant and related in the attempts of my students at Task 3. The first I term ways of writing and the second re-appropriation, which I now discuss. In addition to the many ways of knowing that I have been highlighting so far, I single out **ways of writing** as the attention demanded of the students of expressing themselves in the written form. However the context of expressing oneself in the written form for the students was different from the context in which ways of speaking was guided (as in the previous subsection). Whereas the **teacher guided the participation** of the students in ways of speaking, it was the textbook that was to guide the actions of my students in the present. In such an eventuality **it was for the students to appropriate** the formal writing available in the textbook.

The actions of the students in my group-in-focus towards appropriating the propositional or scientific nature of writing mathematics, was in turn accompanied by the individual nature of meaning made by each of them. As evidenced in the extract above though all of them appropriated the writing in the textbook to express oneself, I conjecture and use the term **re-appropriation** to identify the individual nature of their actions. Having elaborated in detail upon the actions of my students in Task 3, I now discuss their attempts in the remaining two tasks of the group-task.

Task 4 asked the students to conjecture about how the graph would look like for a stiffer spring. Three of them responded by saying that the line would not increase with the same steepness, while Stine affirmed that the line would not be as steep as the one drawn earlier (See Appendix A.17). I however pause here briefly, to reflect on the use of the term ‘line’ by the students. After making meaning in concrete terms, I conjecture this as evidence of students thinking in terms of an intellectual artefact they had made meaning ‘of’ during concrete activity.

Towards Task 5 the students expressed the ratio between extension (y) and force (x) as ‘4’ and as demanded by Task 6, expressed the variable y in terms of x , as **the equation** $y = 4x$. From a semiotic perspective being able to respond to questions in a more formal and symbolic system is I conjecture also an evidence of mathematical understanding. Having elaborated upon the meaning made by the students at this group-task I now discuss the consolidation of such meaning by Olaf.

Consolidation by Olaf: graph, table and formula

Olaf’s consolidation of the group-task: *Proportionality* began after all the groups in the classroom had had time to make their attempts.

Olaf began his discussion with the students by promising them a vocabulary list correlating English and Norwegian terms. He began consolidation by stating that with the line representing the extension of a

stiffer spring (Task 4) would ‘go up less’ since the extension of the spring would be less. He then showed how if $y/x = 4$, then y would be four times x , or $y = 4x$ and equated the terms ‘gradient’ or ‘slope’ to ‘stignigstall’ in Norwegian. It was subsequent to discussing these questions in Task 5 and Task 6 also, that Olaf widened the scope of his consolidation. Olaf first discussed the **meaning of the word** ‘proportional’ as indicative of an increase in one quantity and related to a corresponding increase in the other, as was the case of extension and force in the spring. He then discussed other examples of quantities that exhibited such a relationship, like price of a commodity corresponding to weight and the number of wheels corresponding to a number of cars.

Finally, Olaf turned to spelling out how one could ‘know’ or recognise that any two quantities were proportional or not. By this time the students observed Knut write the following list on the blackboard:

- 1: Graph 2: Table 3: Formula.

Olaf and Knut then asked the students how they could elicit from a graph when two quantities were proportional. The students responded with the help of their attempts at Task 2 and Task 3 of the group-task. The graph would be linear and pass through the origin. Upon further discussion they agreed that in a table comparing proportional quantities, the ratio between the two quantities being compared would always be constant. For a general formula representing proportional quantities, they collectively agreed that the constant would not necessarily be 4 as in the group-task, but an algebraic term ‘k’ or ‘a’. This led to two equations either of which expressed proportional quantities $y = kx$ and $y = ax$. I offer below an extract from Egil’s writing, following the above discussion.

How can we tell that a graph is proportional?

① Graph
It is linear, and goes through the origin

② Table
 $\frac{y}{x} = a \rightarrow$ ~~the~~ If the number is the same each time, the graph is proportional

③ Formula
* $y = ax$, a : constant (positive)

Inverse proportionality: earlier group-task mediates

After the consolidation of meaning at the above group-task, the group-task of *Inverse Proportionality* was given out as before in worksheets. (See Appendix A.13) As before I offer an extract followed by the attempts of the students in my group-in-focus sequentially below.

Inverse Proportionality

Here is a typical set of results for an experiment – measuring pressure and volume for mass of gas

Task 1: Complete the table

Volume	Pressure	pressure · volume
1	40	
1.5	27	
2	20	
2,5	16	
3	13	

Task 2: Draw a graph based on the table Use the volume-values along the x-axis and the extension-values along the y-axis.

Task 3: What can you say about the graph?

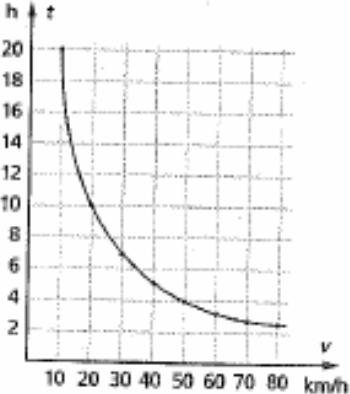
Task 4: What is the product $x \cdot y$ between the volume (x) and pressure (y)? (Use the table)

Task 5: Use your result from Task 4 to find the formula for y in terms of x ($y = \dots$)

In this case the quantities x and y are said to be inversely proportional.

As with the earlier group-task, the attempts of my students at the first two sub-tasks (Task 1 and 2) was accompanied by Olaf drawing the diagram of **a piston and gas cylinder** on the blackboard to make reference to related knowing the students had in Physics. At their attempts at these sub-tasks the students however did not have as many doubts or conjectures as they had in the previous group-task. They filled in the table and drew the graph as required in quick succession. They also made use of the textbook in responding to Task 3, which as before asked the students what they could ‘say’ about the graph.

Since Egil did not bring his work the day I collected their worksheets I offer on the following page the attempts of the other three students (See Appendix A.18). The extract evidences as before the initiative of the students in consulting the textbook, the use they make of the writing in the textbook towards **ways of writing** and the diversity in responses based on personal meaning. The phrases used this time included: ‘inverse proportional’, ‘graph is curved’, ‘it decreases’, ‘not straight’ and ‘goes like a bow’ and evidence again what I termed in the first sub-section as **re-appropriation**. Taken with the earlier group-task the extract allows me to draw attention to another ‘way’ which has been acknowledged though not evidenced in both: **ways of plotting graphs**; which constituted the teaching-learning of mathematics in the classroom.

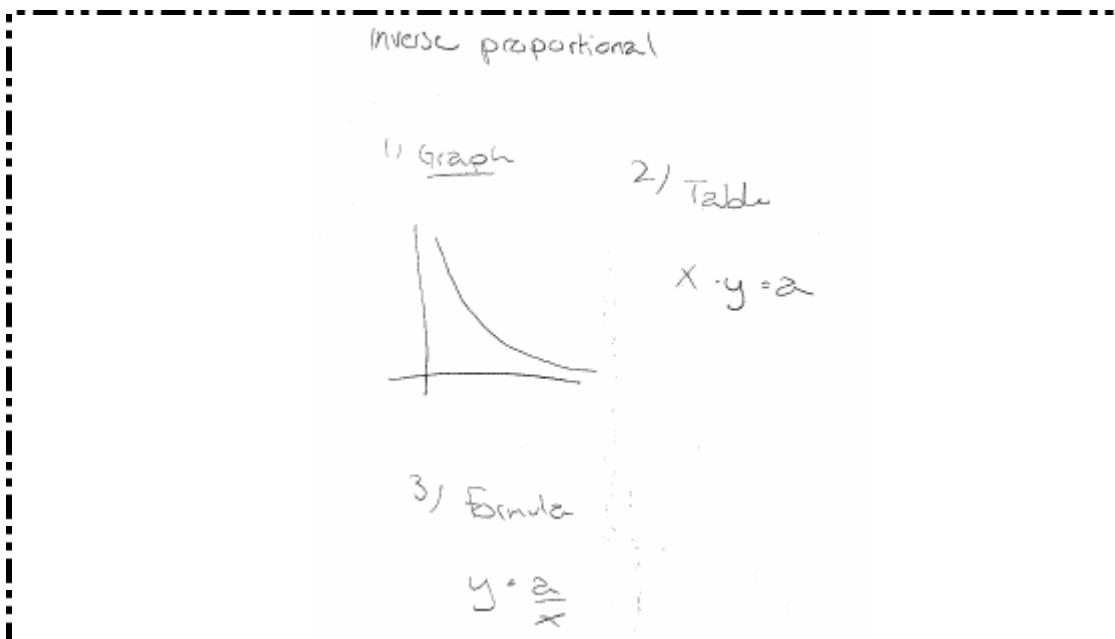
Event	Person	Utterance	Action
1		Task 3: What can you say about the graph?	
2	RES		Lea quickly opens her book and confirms 'Inverse Proportionality'
3	RES	 <p>((Rule: Two unknown quantities x and y are <i>inversely proportional</i> if there exists a number a, such that $y = a/x$ for all corresponding values of x and y. Number a, is called the <i>proportionality constant</i>. The graph of the corresponding values of x and y is a hyperbola.)) ((A graph in this way is called a hyperbola.))</p> <p>((Translations of text in Norwegian))</p>	
4	RES		Notes Anja attempt something with her calculator, which she often does.
5	Anja	This is a inverse proportional graph. The graph is curved.	
6	Lea	It's an inverse proportional graph. It decreases.	
7	Stine	It's not straight, it goes in a bow.	

The group-task on inverse proportionality had no sub-task which asked the students to conjecture. The two final sub-tasks (Task 4 and 5) demanded as before, a **generalisation of the relationship** between inversely proportional quantities: the product of volume (x) and pressure (y) and the formula for y in terms of x . The students found the product to be '40' and the formula to be $y = 40/x$ with one of them responding in addition that $y = a/x$ evidencing the use of the propositional form that was introduced in the previous group-task.

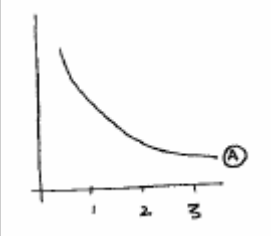
If I have been able to convey that the above group-task was completed in quick succession, it is with intention, since that was how the students attempted the group-task. Such an occurrence is evidence of how the working of the students at the earlier **group-task** on proportionality **mediated** the **attempts and meaning** made by students both of the second group-task and at their individual attempts.

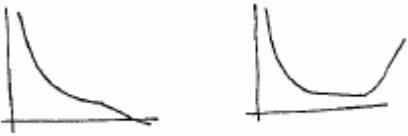
Consolidation by Olaf: graph, table and formula

As in the earlier group-task, Olaf and Knut took up discussion and consolidation after the students had had time to attempt the group-task. The consolidation of the meaning was also patterned similarly and identified in terms of: graph, table and formula. As with the attempts of students at the group-task the consolidation by Olaf and Knut also happened in quick succession. I offer below Anja's summary.



In the consolidation by Olaf and Knut of the meaning made by the students, there was evidence in addition of the use of specific mathematical artefacts by Olaf and three students ‘to speak with’. I offer below an extract that evidences the incidence of this, and take Olaf’s drawing a graph on the blackboard as my point of departure.

Event	Person	Utterance	Action
1	Olaf		
2	RES		Notes Olaf discuss issues of scale and rough shape of graph
3	Olaf	When you were asked to describe it ... you can do it in many ways	((Referring to Task 3 of group-task))
4	Olaf	If volume increases, pressure decreases ... what can we say beyond point (A)	((Pointing to A marked on the graph drawn))
5	Olaf	Will it go like	

6	Olaf		
7	Jan	It will go down but never touch the line	
8	Olaf	Yes	
9	RES		Notes Olaf remind the students of the curves parabola and hyperbola which they would have met in their secondary school
10	Olaf	How can you use the table	
11	Rolf	The product of the two will be almost constant	
12	Olaf		$x \times y = \text{constant}$
13	Anja	<i>We didn't really get a constant number</i>	((Upon her attempts at her calculator))

The above extract evidences as mentioned before, the specific artefacts which various participants in the classroom chose **to speak with** by this time of teaching-learning in the classroom. In bringing about discussion with students about inversely proportional quantities, Olaf mediated the relationship between inversely proportional quantities with a graph (1). As against the use of the 'table' or 'formula' Olaf spoke with the graph to extend what the students knew about the relationship between inversely proportional quantities (1-6) at the point marked A.

In explaining himself he offered two options (6) to which Jan offered his prior knowing about such a graph (7). Jan's response highlights the fact that, though he may not have the English terms to describe the relationship of the quantities at the point marked A on the graph (wanted by Olaf (3) and offered later (9)), he was able to articulate the asymptotic nature of the graph beyond the point marked (A).

In Olaf and Jan both speaking with the graph, there is evidence of an exchange of meaning about inversely proportional quantities beyond the graph. I draw attention to the fact that just as Olaf and Jan used the graph as their intellectual artefact to communicate, the above extract shows the preference of those by Rolf and Anja as well. While Rolf expressed his knowing of inversely proportional quantities by the use of the formula (11), Anja is seen 'making' meaning of the same by her use of the calculator (13). The graph, formula and calculator were the mediational means used by Olaf, Jan, Rolf and Anja to speak with and externalise their thinking, about inversely proportional quantities. Just as Olaf and Jan were: **speaking-with-the-graph**, Rolf: **spoke-with-the-formula** and Anja: **spoke-with-her-calculator**. I now summarise the above case.

A brief summary of the second case

It is in Chapter 6 that I discuss a three task ‘activity’ analysis that consists of the two group-tasks discussed above, along with the attempts at the third (Follow-Up) that I offer in that chapter. Presently and as summary to this case, I point to and discuss the marked shift in the consolidation of meaning being made by the students in the classroom.

Two new uses of **the textbook** which were not discussed earlier were evidenced in this case. The first was as a source of information for the students about the nature and graphs of proportionality and inverse proportionality. The other was as source of formal expression and propositional language written in mathematics. Two new ‘ways’ leading to ways of knowing mathematics were also evidenced: **ways of plotting graphs** and **ways of writing**. It is in addressing ways of writing that the students appropriated the writing presented in the textbook along with their personal and individual meaning making. I termed and explained the use of the writing in the textbook by the students as **re-appropriation** and argued that while the participation of the students was guided by the teacher it was for the students to appropriate the writing in the textbook.

The individual nature of meaning made by the students was also evidenced in the choice of artefacts made by the participants, with which to communicate their ideas with. There was evidence of **speaking-with-the-graph**, **speaking-with-the-formula** and **speaking-with-the-calculator** to express personal meaning with others about mathematical concepts. In such use the students and Olaf were able to express their personal meaning in more propositional forms idealised in them.


As against Olaf consolidating the verbal exchange and recall of students at the blackboard, in this case there was evidence of Olaf and Knut consolidating the meaning made by students at group-tasks at their group tables. This was expressed in the formal terms of **graph, table and formula**, where the goal-directed activity in the first group-task became a template with which the students attempted the second. In this manner the first group-task, became a **knowledge artefact** for the second.

Consolidation of intuitive knowing

In my final case concerning the consolidation of meaning I discuss the teaching-learning of scale factor in similar figures in the classroom. This topic began with a group-task on similar triangles that I discussed in Chapter 4. The extract offered was representative of an activity dealing with the scale factor in one dimension: the lengths of sides in similar triangles. In two sub-sections below I extend the concept of scale factor as applicable to higher dimensions: area and volume. I present an ‘activity’ analysis of the progression of teaching-learning of scale factor in all three dimensions as part of my concluding discussion in this chapter.

Scale factor in 2 dimensions: individual meaning

After the concept of scale factor was introduced with one dimension (lengths of similar triangles) its formulation was extended to two dimensions or area. This was initiated by Knut with a task to be attempted first individually and then together by the students. In relating the attempts of my group-in-focus (Dan, Levi and Thor) I record the emergence of two parallel sets of events. The first relates to meaning being made at the given task and the other to meaning being made independent of the task. Though the incidence of the two 'stories' were intertwined I offer them in sequence. I first offer the task given by Knut (below) and follow the same by the story about an issue of parallel interest to the students.

Similar figures and area *Answer me* 

Task 1

Draw a square where the length of the sides $s = 2$. (Use the length of one square on your paper as unit)

What is the area of the square?

Draw another square where $s = 6$. What is the area?

Find the ratio (forholdet) between the lengths of the sides (the scale factor) in the big and in the small square.

What is the ratio between the area of the big and the small square?

Carry out the same calculations for a square with $s = 2$ and one with $s = 8$.

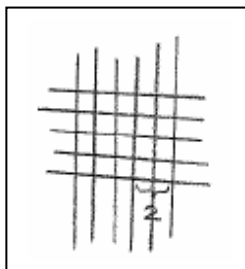
Do you see any connection between the ratio between lengths (the scale factor) and the ratio between areas?

Task 2

Carry out the same calculations as above with two squares, one where $s = 2$ and another one where the side s is f times as long as in the first one.

What connection do you see?

The attempt of students in the classroom at the above task was accompanied by Knut explaining the word **unit** with a diagram (given alongside) at the blackboard. Taking cue from the diagram Levi drew squares as required in Task 1. Thor drew squares though not to scale, while Dan did not make any mathematically relevant attempts. After allowing students time to work individually Knut summarised Task 1 at the blackboard showing that for two squares with sides 2 and 6 the scale factor or ratio between their sides was $f_{\text{side}} = 3$, and the scale factor between their areas would be f_{area}



= 9. Knut then proceeded to find similar scale factors between two squares of side 2 and side 8. The two ratios were now $f_{\text{side}} = 4$ and $f_{\text{area}} = 16$. Knut finally discussed Task 2 with two squares of lengths f and $2f$ and obtained $f_{\text{side}} = f$ and $f_{\text{area}} = f^2$. By the above calculations, Knut demonstrated the **generalisation** of the relationship: for two similar figures whose scale factor was f , the scale factor between the respective areas was the square of f or f^2 .

I explain the above in detail to discuss the actions of Levi. On his own and quite independent of the work that Knut was demonstrating at the blackboard, Levi as demanded by the given task made his own comparison. The ‘connection’ as demanded by the Task that he obtained in his notebook, was the inverse of the one Knut had obtained at the blackboard. By this he compared not the larger side and area to the smaller side and area of the figures but the smaller side and area to the larger. Thus his comparison of the square of side 2 and 6 resulted in $f_{\text{side}} = 1/3$ and $f_{\text{area}} = 1/9$. Levi drew a comparison of many such squares in this manner and obtained as generalisation: $f_{\text{side}} = 1/f$ and $f_{\text{area}} = 1/f^2$.

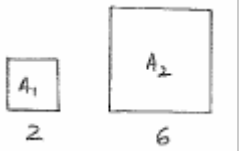
Levi did not offer his **individual formulation** with either Knut or his group mates Thor and Dan. Not only that, when the concept of scale factor was applied in many questions comparing sides and areas of triangles, rectangles, farmland, painted walls etc, Levi kept using his own formulation. At many occasions he checked his formulation by asking ‘But what about the other way’ which the teachers recognised as a good question. I elaborate such an occurrence at length to evidence how Levi as a student in the classroom by this time of teaching-learning had achieved an independent version of the meaning being ‘officially’ mediated by the teachers and tasks in the classroom. The **independent nature** of Levi’s thinking was further exemplified in a parallel set of events which I elaborate now as my second story.

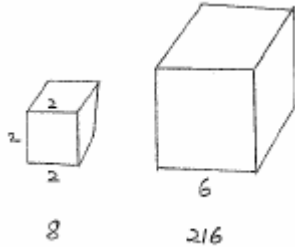
As background to this story I recall it was Dan, Levi and Thor who now belonged to my group-in-focus. My observations of this new group led me to record the frequent use of the calculator by Thor. On the day of the above group-task and before its conduct, Thor arrived at and was discussing the error message ‘**overflow**’ in his calculator with the others. As briefly mentioned Dan made no attempt at the task given by Knut and wrote a string of two numerals 9 and 0 along the edge of his writing paper. He then tried to express the number formed thereof by 1000 raised to the power of 1000 many times over as nested powers. Parallel to his individual working at the task above, Levi considered that the number Dan was attempting to symbolise was a googol. Thor for his part attempted to calculate with his calculator the representation of 1000 raised to many subsequent powers of 1000 expressed by Dan. To his great joy Thor obtained ‘overflow’ yet again in his calculator.

My objective of describing the two stories above is to highlight many new features of the teaching-learning of mathematics in the classroom by the time of teaching-learning in this case. Firstly, though the case relating to this topic showed Levi begin to participate both in his group and in the classroom in the previous chapter, the present case shows a greater **involvement** of the students at their group-tables. Such an involvement also reflected a level of **independence** and was evidenced in two ways. Firstly, Levi was working at his version of meaning in scale factor, one that was not the version being discussed by the teacher in the classroom. Secondly, greater involvement by working at their group tables afforded the students in the group to work at and make meaning of exciting parallel stories not necessarily mandated by the teacher.

Scale factor in 3 dimensions: intuitive meaning

I conclude this case by elaborating upon the consolidation of meaning by Olaf, of scale factor in three dimensions. By the time of occurrence of the extract I offer below, the teachers and students had worked at many applications of the concept, two of which I relate in Chapter 6 which deals with problem solving know-how. I take the diagram drawn by Olaf for such a consolidation as my point of departure.

Event	Person	Utterance	Action
1	Olaf		
2	Olaf	If we remember the square	
3	Olaf	What is the scale factor of the side	
4	STD1	Three	
5	STD2	Three	
6	Levi	Or one by three	
7	Olaf	What is the scale factor of the area	
8	Olaf		$f = 3$ $A_2 = A_1 f^2$ $= A_1 3^2$
9	Olaf	The ratio of the areas of the square is the square of the scale factor	
10	Olaf	What is the scale factor of the volume	
11	Ulrik	Cube	
12	Olaf	Cube ... sure	
13	Ulrik	No	
14	Olaf	Don't you trust yourself	

15	Olaf	((Draws at another location at the blackboard))	
16	Olaf	What is the scale factor between the volumes 8 and 216	
17	RES		Notes Olaf recount the ratio between the two cubes of corresponding sides and areas of face
18	Olaf		$\frac{216}{8}$
19	Olaf	How much is this	
20	Dan	27	
21	Olaf	27!	
22	Olaf		$\frac{216}{8} = 3^3$
23	Olaf	So you were right Ulrik	
24	Olaf	So if you have the volume of one of them we can calculate the value of the other	
25	Levi	What if we have to do it the other way	
26	Olaf	If we know volume of larger we find volume of smaller	
27	Olaf		$\frac{2}{6} = \frac{1}{3}$ $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$
28	Olaf	Good question	

In the above extract Olaf first recalled the prior knowing of students related to the scale factor of lengths (1-6). As an extension to this, he then summarised the scale factor of area in its algebraic form (7-9), as by the time of the above extract the students had worked at many applications based on the concept of scale factor in two dimensional areas. In continuation, Olaf then asked for the scale factor of the three dimensions involved in volume (10). Ulrik offered his **conjecture** of a ‘cube’ though was unsure of his conjecture (11-14). Olaf then followed the strategy of using **geometrical drawings** followed by the numerical value in question to represent and mediate his question (15-19). Dan from my group then conjectured the numerical value of the scale factor as 27 (18-21). In response to Dan’s conjecture, to which he showed surprise, Olaf represented the answer as a cube of the scale factor in one dimension (22-24).

I now highlight an aspect that I dwelt upon in the previous subsection: that of Levi having a scale factor or ratio independent of the version had in the teaching-learning of the classroom. On two occasions in the above extract Levi stopped Olaf, and asked what if the ratio was the inverse or the other way (6 and 25). Not addressing the question in the first instance, Olaf explained in which situation such a ratio can be applied, also explaining this with calculation (26-28). In light of the two teaching-learning events discussed the incidence of the above evidences how Levi not only had an independent ratio but also **used the teaching-learning** of the classroom **to check and ratify** his own formulation. I now turn to summarising the third case.

A brief summary of the third case

In continuation with arguments related to the consolidation of meaning, the third case evidences three significant shifts in the nature of meaning being made by the students in the teaching-learning in the classroom.

Firstly, and as an extension to the meaning being made by the students at their student tables the meaning being made was **individual** in nature. This was evidenced by the nature of meaning made by Levi, Dan and Thor. Secondly, and possibly as a result of the individual meaning making, there was also evidence of meaning being made **independent** of that being discussed in the classroom. There is evidence of such an instance in Levi's formulation both in the first and the second sections of this case. Finally, and probably because of the nature of meaning that was being allowed by this time in teaching-learning, the meaning being made by the students was **intuitive**. It is these individual, intuitive and at times independent meaning that was consolidated by Olaf. By the very nature of such consolidation I make two conjectures: firstly, that individual, intuitive and independent meaning was possible because of greater **involvement** of the students, and secondly that in the nature of consolidation, Olaf was not only consolidating just students' meaning but to some extent students' mathematical **knowing**.

I now turn to summarise the present chapter in which I also discuss the 'activity of scale factor' as promised.

Concluding discussion

Evidence of the consolidation of meaning made by the students, a theme that I dwell upon in this chapter, is built upon three cases which were illustrative of the building of meaning in a teacher-driven practice in the first, the building upon of the meaning making experiences of students by the teachers in the second and finally the building upon of students' intuitive knowing by the teachers in the third. I now offer discussion across the three cases with arguments from literature in mathematics education and socio-cultural-historical perspectives.

Any effort towards the consolidation of meaning in the classroom, whose practices I described in the previous chapter, were extended by the mediation of newer artefacts or the newer mediation of previously cited artefacts. I discuss the newer uses of the textbook in detail shortly, and mention that the other **physical artefact** that was briefly used in mediating the understanding of the concept of extension was the rubber band. The **intellectual artefacts** used were the graph, table and formula, the diagrams of the spring balance and piston, along with geometrical diagrams with which to mediate a unit, the square and the cube. As evidenced the use of the table, graph was made with an intention of both understanding the task to be solved and in the solving of the task. I call upon instances of these later to discuss two features of artefacts, which I term as ‘mobility’ and ‘positioning’ in Chapter 8: *A micro-culture*.

The ‘ways’ that I had identified so far that constituted ways of knowing mathematics as argued by Bishop (1988), were extended in the present chapter with **ways of plotting graphs** (in a limited manner), **ways of using brackets**, **ways of writing** and **ways of speaking**. As found, though not mentioned in the previous chapter, the participation of students in these newer ways was however guided by the teacher the benefits of which have been argued by Rogoff (1995) and towards greater knowing in the teaching-learning of mathematics as argued by Greeno (2003). I now discuss the manner in which the textbook was used by the students towards ways of writing (Case 2). In the incidence of this use there is room for discussion of three theoretical constructs: participatory appropriation as also argued by Rogoff (1995), appropriation of artefacts as argued by Säljö (1998) and the making of meaning in goal directed activity as argued by Leont’ev (1981b).

In discussing the incidence of the use of the textbook, I first point to the very occurrence of the actions of students. It was in the **presence of a goal** of answering a question set to them that the students carried out their actions. Such an eventuality as argued by Leont’ev provided the need for dynamic actions of the students. In the use of their textbook to find both information and the written form of propositional knowledge, the students appropriated the language in the textbook with which they **enhanced their intellectual capacities** as argued by Säljö. Such a shared endeavour between individuals and cultural resources was as a result of students changing their ways. Rogoff termed such changes that in turn lead to cognitive development as **participatory appropriation**, since it was in participation that the need to appropriate was addressed.

I continue with my above discussion about appropriation, with evidence in this chapter of a feature I have termed as **re-appropriation**. I have conjectured that based upon the personal meaning, by the process of re-appropriation each student displayed an appropriation of proposi-

tional knowing suited to his or her own goals and purpose. In such a re-appropriation both diversity and individuality was displayed. I also argued that unlike the guided participation of students by the teacher it was the students who had to be active and appropriate cultural resources such as the writing in the textbook. Having referred to the opportunity provided by the teachers to ways of writing by the students, I now turn to discuss the actions taken by the teachers towards ways of speaking in the propositional form. In the teaching-learning of mathematics these were exemplified by three related processes: privileging, genre and agency.

In extending ways of speaking, there is evidence of the teacher using his authority and **privileging** as argued by Wertsch (1991), the appropriate terms (base) as well as associated usage leading to ways of speaking. Such actions also allowed the teacher to stress the importance of conventions in mathematics. From the incidence of appropriate ways of speaking, I now turn my attention to 'ways of speaking with' certain artefacts of use. There was evidence of both teachers and students utilising artefacts like graph, formula or calculator with which to communicate personal meaning in a more propositional form. Such actions involved the choice made by the individuals and exhibited the incidence of **agency** as argued by Wertsch et al. (1993), since it was not the individual alone who spoke. Ways of speaking towards knowing, in the teaching-learning of mathematics in the classroom, was also furthered by use and privileging of a genre. As argued by Bakhtin (1986), the use of a **genre** enhanced the situated meaning associated with the use of speech and was indicative of the generalisation of understanding before its socialisation.

The use of writing in the textbook and the use of a genre were demonstrative of instances of the **representation, reformulation and objectification** of one kind of semiotic phenomenon into another as argued by Ernest (2006) and Radford (2003). I had conjectured that the existence of a genre could be an indication of the internalisation of the same rule (intellectual artefact) as argued by Vygotsky (1981a) as a psychological tool by many individuals. The simultaneous incidence of both objectification and genre, evidence Seeger's argument (2001) of the incidence at the same time in teaching-learning of **horizontal actions** between individuals and **vertical actions** in mediated activity. These actions evidence his other argument that the discursive activity of teaching-learning is not restricted to discourse alone but mediated activity as well.

The instances that I mention above would not have been possible, without the attempt of the students at the group-tasks, conducted by the teacher. They exemplify the opportunities that schooling could provide to help students mobilise functions not hitherto known to them as argued by Luria (1994). Made possible by participation in the teaching-learning of the classroom they enabled what Stetsenko (1999) argued as students

being able to know of cultural ways of doing things or about **objects-that-can-be-use-for-a-certain-purpose**. These actions of students also exhibited what Bruner (1990) termed as the transformation of personal meaning to its propositional form with the help of cultural ready-mades.

In the use of artefacts to mediate meaning as argued by Bruner (1990) and cognition as argued by Cole (1996), the presence of a culture was evidenced, and extended the possibility of **enculturation** in the classroom as argued by Bishop (1988). The consolidation of personal meaning to propositional forms was the idealisation of meaning making, as argued by Bruner in the above micro-culture. Participation in meaning making and consolidation processes also evidenced three important arguments of Vygotsky (1981a; 1997a). Firstly, that there is **nothing passive in education**. The teacher, the student and the social environment are all active. Secondly, as is the goal of education there was evidence of opportunities for the development of lower functions to higher or **cultural forms of behaviour**. Thirdly, that role of a culture provided for the existence of special forms of behaviour.

I conclude my discussion with the ‘activity of scale factor’. Yet before offering the same, I draw attention to two aspects that enabled the participation in and continuation of, the above mentioned micro-culture of teaching-learning: continuity and common sense. I had explained **continuity** by those actions with which the teaching-learning of mathematics was sustained in the classroom. These actions were of two kinds: those that included a kind of use of the textbook, classwork and homework and those which were more mathematical like elaboration of concepts in different notational systems, the easiest way to attempt questions, know what is sufficient to do, employ the same reasoning to similar situations, reflect between general rules and special cases and allow for the conjecturing of students. The existence of these values ratifies the arguments made by D’Ambrosio (2004) and Goodnow (1990) that cognition is not value free in any culture. They also helped identify those actions in the teaching-learning of mathematics in the classroom, which Bishop argued as ‘of’ value for the enculturation of mathematics.

There was evidence of dependence in teaching-learning also of **common sense**. This was displayed in the acceptance the existence of parallel terminology in two languages, Norwegian and English (e.g. exponents), the calling upon of different symbols that represent the same numerical value (0.5 and $\frac{1}{2}$) and the relying upon of the fact that symbols representing the same and equal value had to be equal to each other. These concepts seemed part of the collective and taken for granted correspondences as argued by Keitel et al. (2005) and Kilpatrick et al. (2005) and narrowed the distance between context and content, to make such aspects everyday as argued by Butterworth (1992).

In presenting the ‘activity of scale factor’ I analyse the consolidation of the meaning of scale factor in similar figures across length, area and volume. For length I take the data related to the group-task on similar triangles (Chapter 4). For area and volume I refer to the consolidation of meaning steered by Olaf as a summary and elaborated in the two extracts of the third case in this Chapter. In presenting the ‘activity’ analysis I first represent the facts I draw upon in a table below. I take the discussion relating to the three dimensions as the basis of three rows that make up the ‘activity’. In describing the three rows I enlist the operations-conditions and actions-goals of each row that together constitute a single activity-motive. I follow the table by discussion.

‘Activity of scale factor’					
<i>Objective</i>				<i>Subjective</i>	
Activity	Motive	Action	Goal	Operations	Conditions
Classroom teaching-learning spread over many days	To formalise the conception of scale factor in all three dimensions	To draw upon similarity in triangles to formalise the concept of scale factor of their sides	To formalise the conception of scale factor in one dimension	To utilise the notion of similarity and calculate lengths of corresponding sides.	Group-task on similar triangles
		To draw upon the lengths and corresponding areas of squares and formalise the concept of scale factor of length and area	To formalise the conception of scale factor in two dimensions	To draw squares with lengths of different sides and compare corresponding lengths and areas	Classroom activity on lengths and corresponding areas of squares
		To draw upon lengths and corresponding volumes of cubes to formalise the concept of scale factor of length and volume	To formalise the conception of scale factor in three dimensions	To recall the nature of scale factor applicable to areas and conjecture and find its nature as applicable to volumes	Blackboard discussion about lengths and corresponding volumes of cubes

In discussing the nature of transformations that took place in the ‘activity of scale factor’ I move from right to left in my arguments. This enables a subjective to objective shift in the analysis of the ‘activity’.

As shown in the table at the level of **operations-conditions** the ‘activity’ allowed for the formalisation of scale factor in each of the three dimensions and differed in the nature of what was called upon to enable the students to arrive at such formalisation. In the first dimension, the students were asked to call upon prior knowing of similar triangles and obtain a ratio between corresponding sides without any reference to the concept of scale factor. Such a concept was introduced by the teachers while consolidating the activity of Similar Triangles by **identifying** the ratio obtained by the students in the activity as scale factor.

In the second dimension (first extract of the third case in this chapter) the ratio between lengths of the squares that were drawn as part of the group-task was **designated** as scale factor. The activity demanded that the students consciously look for the designated scale factor and formalise the patterns that evolved within the given task. The scale factor in the third dimension (second extract of the third case in this chapter) was formalised on the blackboard. The scale factor in the second dimension was first called upon and an opportunity provided for the students to **conjecture** on its nature in the third dimension. Their conjecture was then verified leading to formalisation. The transformation of scale factor was thus from identification in one dimension, followed by designation in two dimensions, leading to conjecturing in three dimensions.

At the level of **actions-goals** the formalisation of scale factor depended on **reasoning with different entities**. It depended on the comparison of lengths of similar triangles in the first dimension, comparison of areas of similar squares in the second and the comparison of volumes of similar cubes in the third dimension. Along with the varying nature of operation-conditions in each dimension, such reasoning was applied by the students in attempting many questions from the textbook as part of teaching-learning spread over many days in the classroom.

The **activity-motive** that constituted ‘activity’ led to the formalisation of the concept of scale factor (f , f^2 , f^3) in a progressive manner. As argued by Leont’ev (1978; 1981c), I conclude that in seeking its application in various dimensions and in different figures in geometry the above ‘activity’ subjectivised the scientific and propositional nature of scale factor and in drawing upon the concept in various dimensions, the everyday and personal nature of scale factor was objectified in propositional terms. The ability of the students to conjecture the scale factor in the third dimension was evidence that the formalisation of the concept was made possible in the communication associated with the materiality of the ‘activity’ and also recognisable to the participants themselves.

Before concluding this discussion, I make a final mention of the nature of transition from personal to propositional forms of meaning. Firstly and as argued before, the nature of transitions that were evidenced in the chapter showed the **increasingly intellectual nature of the transitions** being made. Secondly, there was evidence of not just smaller instances of knowing but of a whole group-task act as a **knowledge artefact** which mediated the working of the students at the next. Thirdly, with personal involvement there was evidence of an **independent, individual and finally intuitive** nature of participation by the students. Taken along with the instances of personal choice in the examples of agency, I conjecture that this evidences what Vygotsky (As in Wertsch, 1985) called as the shift in the intellectual nature of actions from 'me' as a member of the classroom to 'I' as the individual.

My final synthesis of the data and analysis of the present chapter is the following. There is evidence that it is possible to bring forward and draw upon the personal meaning making of the students, to bring about more propositional forms of meaning. Such an eventuality was possible in the pursuit of goals by the students at suitably designed group-tasks, conducted at their group-tables. An important constituent of student participation towards consolidation was the role of the teachers in formalising the teaching-learning, in ways recognisable to the students. Participation included the freedom of making a choice of cultural resources available, with the help of which personal meaning was communicated in more propositional forms. Such possibilities made **participation and consolidation in teaching-learning increasing intellectual in nature**.

I now turn to discuss in my next thematic chapter those instances and events in teaching-learning, by which teachers and students exchange how to solve questions and problems in mathematics in the classroom.

6. Problem solving know-how

In my third data and analysis chapter, I focus on those specific incidents in the teaching-learning of mathematics which address know-how specific to problem solving processes in the classroom. In addressing this theme I follow as earlier, the pattern of offering three cases situated as before in the topics of number understanding, equations and proportionality and scale factor in similar figures.

I recollect my analysis in the two previous data chapters. In Chapter 4 there was evidence of how over the three topics, the teaching-learning of mathematics in the classroom, was transformed by the teachers Olaf and Knut from a largely teacher-driven practice to a more student-centered practice. In the collaborative classroom practice that was established the personal meaning made by the students in teaching-learning became more central vis-à-vis the teaching-learning of mathematics.

In Chapter 5 there was evidence of how Olaf and Knut consolidated the meaning made by the students in their teaching-learning. From the consolidation of meaning at the blackboard, there was a shift in the kind of meaning made by the students that was consolidated, since the meaning was now being made at the students' tables. The personal meaning made by the students at specially designed group-tasks was then consolidated by the teachers into a more propositional form. There was evidence of consolidation of the intuitive knowing of the students by the final case. These themes form the backdrop for the present chapter.

The three cases I discuss in this chapter are titled: *Solutions to questions*, *Applying known solutions* and *From questions to problems*. These cases elaborated as before in sub-sections, allow me to discuss how problem solving know-how evolved and developed in a gradual manner in the teaching-learning of the classroom, by being pursued diligently over numerous teaching-learning events and actions as was the case in earlier chapters. The three cases allow me to discuss how towards the development of problem solving know-how: Olaf first discussed and built rules by turning them into questions, then helped model solutions which were subsequently applied by the students and finally helped transform questions to problems that the students could solve.

As has been my objective, I shall theorise upon the micro-culture of the classroom as my synthesis in the final chapter. In addition to the themes of a collaborative classroom practice and the consolidation of meaning elaborated upon so far, the present chapter dealing with the development of problem solving know-how extends the understanding of the micro-culture being constituted within. As different from the making of meaning, it is know-how related to the application of meaning and knowing that is of interest. I now turn to discuss my first case.

Solutions to questions

In my first case dealing with the topic of number understanding, I present extracts in two sub-sections. In the first, I evidence how Olaf problematises the issue of the meaning of a negative exponent. Offered as a question, he reformulates the question of meaning in known terms and follows it with a demonstration of a solution. In the second, I evidence how Olaf discusses what could be an incorrect way of using or applying a bracket in a question.

But what does this mean: how to solve it

In the teaching-learning of exponents Olaf put up the meaning of an exponent with a negative power, 2^{-3} , for discussion as below:

Event	Person	Utterance	Action
1	Olaf	But what does this mean?	
2	Olaf		2^{-3}
3	Olaf	Does it make any sense ... two multiplied by itself negative three times?	
4	Rolf	We should take one divided by two to the power three	
5	Olaf	Why ... not sure	
6	Olaf	We can do it similar to the example	((Referring to an example from the textbook))
7	Olaf	For getting 2^{-3}	
8	Olaf		$2^{-3} = \frac{2^2}{?}$
9	Olaf	What do I do out here	((Pointing to the place marked by the question mark))
10	Olaf		$\frac{2^2}{2^5} = \frac{2 \times 2}{2 \times 2 \times 2 \times 2 \times 2} = \frac{1}{2^3}$
11	Olaf		$\frac{2^2}{2^5} = 2^{-3}$
12	RES		Observes Olaf draw two arrows to show the correspondence between (10) and (11) above
13	Olaf	They must be the same	
14	Olaf		$a^{-n} = \frac{1}{a^n}$
15	Olaf	So here we have another rule	
16	Olaf	Now we have two more rules in addition to the earlier ones	
17	Olaf		$a^0 = 1$ $a^{-n} = \frac{1}{a^n}$
18	Olaf	That's all you need to know in life for the time being	
19	STDs	@@@	

The above extract evidences how in the teaching-learning of exponents Olaf constantly shifted teaching-learning from specific examples to the general formula and vice versa. Just as he had discussed as in Chapter 5 the meaning of 2^0 in the context of applying rules, the above extract evidences the importance Olaf gives to the meaning of an exponent with a negative power, deriving a general rule from the same.

In discussing the meaning of a negative exponent Olaf asked if there was any sense in 2 being multiplied by itself negative three times (1-3). Rolf who expressed what the term meant equivalently (4) is in turn asked by Olaf for a reason. Olaf then reformulated his question and asks how one would obtain 2^{-3} from a fraction in which he offered the numerator and not the denominator (6-9). He finally offered not one but two solutions to his question (10-11) resorting to common sense as had been resorted to earlier as justification (12-13). Since the two expressions 'had to be the same' Olaf generalised the above case as another rule (13-15) which he added to the many rules he was building over time with his students in the teaching-learning of mathematics in the classroom (16-19).

However, in addition to the meaning being made of 2^{-3} the above extract is also significant in the application of a four step problem solving model. In problematising the meaning of 2^{-3} Olaf implicitly followed the problem solving model of Pólya. In asking for the meaning of 2^{-3} Olaf took the step of **posing a problem**. By reformulating the problem and asking for the denominator of a fraction he **devised a plan** with which he was able to instruct students to arrive at the meaning required. These two steps were followed by his **carrying out the plan** of finding the denominator of the required fraction. In equating the two exponents with the two fractions Olaf arrived at the general rule by **looking back**.

I present two conjectures in relation to the above extract. Firstly, I conjecture that the very action of a teacher demonstrating problem solving, is the primary step in the building of problem solving know-how. That it implicitly demonstrated the use of a problem solving model came as an advantage since such a model could be used as a general strategy. Secondly, I conjecture that the problem solving process discussed above is similar to the process applied in Chapter 5 related to the meaning of 2^0 . My intention of discussing the treatment of meaning of 2^0 and 2^{-3} in different chapters is to parallel and finally conjecture that the processes of **consolidation of meaning and problem solving are reflexive**. Both processes called upon earlier knowing of students in fractions to mediate current knowing in exponents. While in the case of building meaning of 2^0 the general rule $\frac{a^m}{a^n} = a^{m-n}$ was applied, in the case of solving the problem of 2^{-3} the general rule $a^{-n} = \frac{1}{a^n}$ was derived.

The wrong possibility: attention to application

The extract I offer below deals with the application of what I had identified in Chapter 5 as ways of using brackets. Combined with ways of operating exponents as discussed in Chapter 4 the extract also introduces the attention given by Olaf to ways of operating the calculator in utilising its functions towards the teaching-learning of mathematics.

Event	Person	Utterance	Action
1	Olaf	((Q1.12 (a) Calculate both with and without a calculator))	4×2^2
2	Olaf	Are there two possibilities ... explain	
3	Levi	((Offers in NOR two possibilities that yield the same value))	
4	Olaf	What is the other possibility	
5	Levi	The wrong possibility!	
6	Olaf		$4 \times 2^2 = 4 \times 4 = 16$
7	Olaf	The wrong possibility is multiply 4 and 2 and then square	
8	Olaf	For that what do you need	
9	Levi	Parenthesis	
10	Olaf	Then it would be	
11	Olaf		$(4 \times 2)^2$
12	RES		Notes Olaf now explain the use of the 'hat' or ^ or exponent button on the calculator
13	Olaf	3^5 , how do you do that on your calculator?	
14	Olaf		3^5
15	Olaf	Now ...	
16	Olaf	((Q1.12(b) Calculate both with and without a calculator))	$4 \times (-2)^2$
17	Olaf	How much is	
18	Olaf		$(-2)^2$
19	STD	Four	
20	Olaf	Negative or positive	
21	STD	Positive	
22	RES	((Stor minus and lille minus in NOR for big and small minus))	Notes Olaf now explain the difference between the two buttons (-) and (-) on the calculator

The above extract allows me to first evidence the confidence in the response of Levi by the last day of the teaching-learning of number understanding. In his response to Olaf's question and challenge (1-4), Levi suggested that any possibility other than the one offered by Olaf was the wrong possibility (5). Olaf then explained what an incorrect application of using brackets could be and asked if the students knew what would be

needed in the ‘wrong’ possibility being discussed (7-8). Levi’s response evidenced that he knew how the bracket could be applied in the option offered by Olaf (9). In such a response I conjecture that Levi evidenced two things simultaneously: his understanding of Olaf’s argument as well as the application and misapplication of ways of using brackets.

The above extract also evidences how Olaf introduced yet another physical artefact, **the calculator**, into the teaching-learning of the classroom. Though I evidence Olaf referring to the calculator earlier in the above abstract he is seen guiding the participation of his students in **ways of operating the calculator** (12-22). I add ways of operating the calculator to the list of ‘ways’ (operating fractions, operating exponents, speaking, writing, using brackets and plotting graphs) that constitute ways of knowing in the teaching-learning of this classroom.

The attention Olaf was giving to the use of the calculator above is also evidence of two other things. Firstly, by the care with which Olaf showed how the calculator was to be used, Olaf implicitly acknowledged the importance of the calculator in the teaching-learning of mathematics and **privileged** its use. Secondly, Olaf stressed the importance of the proper application of the calculator in attempting the question at hand, just as he had done with the bracket. I conjecture that Olaf treated both the bracket and calculator, towards their application in solving the questions at hand. Such an approach was not towards building meaning or consolidating the same but towards their use and application.

Brief summary of the first case

The calculator was the new physical artefact introduced in the above case. Attention was given to the use of various functions related to the buttons as also to the use of the appropriate function or button. Such attempts contributed to **ways of operating the calculator** and added another ‘way’ to the long list that I have been identifying in my thesis.

In referring to the (mis)application of a bracket, issues related to ways of using brackets were also extended in this case. Though not explicitly mentioned to the students a four step model of problem solving was introduced to discuss the meaning of an exponent with a negative power 2^{-3} . Akin to the consolidation of meaning made towards the understanding of 2^0 the two extracts evidenced the importance being given in teaching-learning to the **generalisation** of specific cases by drawing upon general rules and the **specialisation** of general cases.

In connection with the above I had conjectured that the meaning making processes and problem solving processes were reflexive. In meaning making the application of a rule was demonstrated and in problem solving the meaning of rules was consolidated. It was the meaning making processes in the first, which led to the problem solving; and problem solving processes in the second, which led to meaning making.

I draw attention to another significant factor in the kind of focus provided in the above extracts. In the extracts above the utilisation of the ways of using the calculator, brackets or even the four step model was different from how the many ‘ways’ identified so far were used. In the above extracts in addition to a focus of knowing as an extension of meaning, the ways were also **geared towards their application to resolve** or address specific questions. Such an approach therefore led them to contribute to the problem solving know-how being developed in the teaching-learning of mathematics in this classroom.

I now turn to the application of not just physical and intellectual artefacts to problems but solutions to questions as well.

Applying known solutions

As against applying specific tools to solve questions, in the three sub-sections below I discuss the application of solutions to earlier questions as models to attempt and solve later questions. In the first, two solutions are built and are put to use in attempting later questions. In the second, the use of a mnemonic as a cultural artefact is encouraged in attempting questions. In the third, the consolidation of meaning made in the two group-tasks on proportionality (discussed in Chapter 5) is applied in the group-task: *Follow Up*. I present analysis of these three group-tasks as ‘activity’ in my concluding discussion of this chapter.

What will x be now: using a solution as a tool

I present three extracts in this sub-section and discuss the sub-topic of simple equations. I do not unscramble the three extracts which deal with two sets of solutions and offer them sequentially, as an example of the complexity of events in teaching-learning as they transpire in the reality of the classroom. The first and third extracts are both related to the resolution of Q2.224 from the supplementary textbook. The second extract in-between relates to Q2.35 (a) and Q 2.35 (b) from the main textbook. These three questions are set by Olaf and Knut to be attempted by the students for classwork.

In the extract below I offer attempts at Q2.224 by Anja, Lea, Stine and Egil of my group-in-focus at their tables, followed by a brief discussion by Olaf and Knut. It is in the third extract that I relate the discussion at the blackboard by Ulrik, of his group’s solution to the same question.

Event	Person	Utterance	Action
1	RES	Q2.224: Per weighs 5 kg more than Hans and 20 kg more than Grete. Together they weigh 200 kg. How much does each weigh?	
2	RES		Notes that the students use the variable x from the beginning.

3	Stine	((Attempts in her note book))	$P = x + 20 + 5$ $H = x + 20$ $G = x$												
4	Lea	((Attempts in her note book))	$200 = x + x - 5 + x + 20$												
5	Egil	((Egil makes an attempt in his notebook and shares it with the others))	<table style="border-collapse: collapse; margin-left: auto; margin-right: auto;"> <tr> <td style="padding-right: 10px;">x</td> <td style="border-left: 1px solid black; padding-left: 10px;"></td> <td style="padding-left: 10px; text-align: right;">75</td> </tr> <tr> <td>$x - 5$</td> <td style="border-left: 1px solid black;"></td> <td style="text-align: right;">70</td> </tr> <tr> <td>$x - 20$</td> <td style="border-left: 1px solid black;"></td> <td style="text-align: right;"><u>55</u></td> </tr> <tr> <td></td> <td></td> <td style="text-align: right;">200</td> </tr> </table>	x		75	$x - 5$		70	$x - 20$		<u>55</u>			200
x		75													
$x - 5$		70													
$x - 20$		<u>55</u>													
		200													
6	Lea and Stine	((Lea and Stine correct their working in their notebooks))	$200 = x + x - 5 + x - 20$ $200 = 3x - 25$ $225 = 3x$ $\frac{225}{3} = x$ $75 = x$												
7	RES		Olaf and Knut bring group work across the class to conclusion												
8	Olaf	There are many ways in doing the question													
9	Olaf	The question doesn't say use an equation.	((Addressing Knut and speaking))												
10	Knut		((Nods in agreement))												
11	Olaf	We can use any one way as long as we can guess and check	((Olaf draws parallel in NOR to the phrase 'gjette og sjekke'))												

The above extract evidences the use of the unknown x in the formulation of Q 2.224 by Stine, Egil and Lea as **an equation**. Each of them made independent attempts (3-5) in which only Egil arrived at a solution. Upon observing Egil's solution, Lea and Stine rework their equations based on Lea's attempt (4, 6) and arrived at the appropriate value of x . There is no evidence to show if Stine and Lea 'realised' the error in their earlier conjectures, though there is evidence that the two corrected their attempts. However in their correction they do not abandon their earlier approach but modify the same by drawing upon Egil's solution.

The above extract also evidences how Olaf and Knut brought the attempts of the various groups at Q2.224 to a close, indicating that the teaching-learning would move over to the next event. As a conclusion Olaf continued with his encouragement of alternate strategies (8) and pointed in particular to the nature of the given question. Saying that the question did not ask for the use of an equation (9, 10) Olaf referred to the wisdom of relying on the common sense of guess and check.

In his mention of guess and check (11) Olaf alluded to the phrase of 'gjette og sjekke' used in the Norwegian language. In such a reference Olaf and Knut encouraged their students to call upon a method the students would have used by then. The use of **guess and check** is a method

resorted to in everyday life and not necessarily used in classroom teaching-learning. However in the teachers **privileging** ‘gjette of sjekke’, the use of guess and check becomes a valid method of solving problems and contributed to problem solving know-how in the classroom.

The sub-topic of equations was continued with the attempt of two questions on consecutive numbers from the textbook as below:

Event	Person	Utterance	Action
1	RES	Q2.35 (a) Find three whole numbers which follow one another so that the sum of the numbers is 123. Q2.35 (b) Find five even numbers which follow one another so that the sum of the numbers is 240.	
2	RES		Olaf introduces and discusses the word ‘consecutive’ since in NOR, the concept is implied in the text of the question.
3	Olaf	Someone said the question could be done as $x+y+z=123$	
4	Olaf		$x + y + z = 123$
5	Olaf	Is it a good idea?	
6	STD1	No	
7	Olaf	Why is it not?	
8	STD2	Because you won’t find the answer	
9	Olaf	((Explains at a separate location))	17 18 19 17 + 1 17 + 2
10	Olaf	Thus ...	
11	Olaf	((Explains while writing))	$x + (x + 1) + (x + 2) = 123$ $x + x + 1 + x + 2 = 123$ $3x + 3 = 123$ $3x = 120$ $x = 40$
12	Olaf	Someone suggested the middle be x	
13	Olaf	((Beneath writing in event 4))	$x + y + z = 123$ $(x - 1) + x + (x + 1)$
14	Olaf	What will x be now?	
15	STD	Forty one	
16	RES		Observes my group-in-focus who did not have any written solutions so far to copy the above solution in their notebooks
17	RES		Notes Olaf asks students to work at 2:35(b) and concludes as below
18	Olaf	((Explains while writing))	$x + (x + 2) + (x + 4) + (x + 6) + (x + 8) = 240$ $5x + 20 = 240$ $5x = 220$ $x = 44$

By calling attention to the board in the above extract (2) Olaf began his discussion about the solution to Q 2.35(a). Yet by the way in which Olaf began his utterance ‘Someone said...’ (3), I conjecture two possibilities: either some student did attempt the question with the equation $x + y + z = 123$ in his or her notebook, or Olaf used the equation as a **rhetorical device**. Though I have no way of establishing either, Olaf’s utterance is useful in drawing attention of the students to the relationship that exists between **consecutive numbers**. Having called attention (3-8), Olaf then offered a numerical example (9) to discuss how consecutive numbers are related. After showing how one number followed another numerically, Olaf formulated and solved the required equation (10, 11).

The manner in which Olaf used the **numerical example** to formulate the algebraic equation is an example of mediating knowing. By this I mean Olaf used the numerical example (9) to mediate the knowing about the numerical relationships between consecutive numbers, to formulate a corresponding relationship between consecutive terms algebraically. This enabled him to formulate an equation for Q 2.35(a). The numerical example that mediated knowing is an example of a **knowledge artefact**. Its use by Olaf mediated prior numerical knowing to further the meaning and algebraic knowing necessary to solve the problem at hand.

The extract evidences in addition that Olaf proceeded to check whether his students had understood the equation he has formulated. By mentioning ‘Someone suggested ...’ (12) Olaf approached the previous **‘rhetorical’ equation** (4) and designated the middle term as x . In such a designation Olaf formulated the previous term as $x - 1$ and the following term as $x + 1$ (13). On Olaf’s asking what x would be in such a formulation (14) Olaf’s actions addressed two factors of importance in teaching-learning. Firstly, they elicited if the students could identify what x would be in the new designation. Secondly, they implicitly demonstrated that a variable like x could designate a number of one’s choice. On finding confirmation from one student (15), Olaf then set Q 2.35(b) for all the students to attempt. By these actions Olaf used his entire discussion leading up to the solution of Q2.35 (a) as another **knowledge artefact**. The entire solution of Q2.35 (a) now mediated the knowing that enabled the students to arrive at an appropriate solution to Q2.35 (b).

I now offer the second extract dealing with the solution to Q 2.224. As mentioned before, this extract evidences Ulrik discuss his group-solution at the blackboard, followed by a brief consolidation by Olaf.

Event	Person	Utterances	Action
1	Olaf	Perhaps we can take your problem now	((Addressing a particular group))
2	RES		Ulrik comes forward and explains his group’s solution while writing

3	Ulrik	They weigh 200 together. So ...	
4	Ulrik		$200 = P + (P - 5) + (P - 20)$
5	Ulrik	This is because --	
6	Ulrik	((Offers at a separate location on the blackboard an explanation as alongside))	$P = \text{Per}$ $\text{Hans} = P - 5$ $\text{Grete} = P - 20$
7	Ulrik	((Under the equation shown in event 4))	$200 = 3P - 25$
8	Ulrik	Move this over to the other side	((Pointing to $- 25$ above))
9	Ulrik	((Notes Ulrik explain in NOR as he goes along))	$\frac{225}{3} = P$ $75 = P$
10	RES		Notes Olaf acknowledge Ulrik's solution and go up to the board
11	Olaf	You could have written this the other way	
12	Olaf	((At a separate location))	$P + (P - 5) + (P - 20) = 200$

As mentioned in Chapter 4, the above extract shows how student participation in the teaching-learning of the classroom is enhanced in two ways. Firstly, Ulrik spoke for himself and on behalf of his group of students. As a consequence students in his group saw 'their' work being presented, leading to the possibility of their agreeing or disagreeing with the work presented if they so wished. Secondly, students not in the group presenting its solution (Ulrik's in this case) had the opportunity to analyse, compare and discuss their own or group-solution with the one being presented. This allowed comparison at an individual and group level.

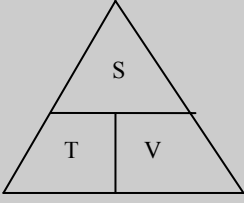
The above sequence of three extracts allows me to focus on two other aspects of importance to the teaching-learning of mathematics. Firstly, Olaf uses Ulrik's solution to draw the attention of students to an oft-used **convention** in algebra: of writing the variables on the left hand side of the equation (10-12). This action in turn leads me to question the selection by Olaf, of Ulrik's group-solution among many possible students' solution for presentation at the blackboard. I conjecture that in the use of Ulrik's group-solution Olaf availed of the opportunity to draw attention to convention. Olaf utilised Ulrik's group-solution to further the knowing of conventions to all students in the classroom. Ulrik's group-solution became a **knowledge artefact**, which by drawing attention to conventions, mediated further knowing.

Secondly, the selection of questions, and the selection of solutions to the questions in the above extracts, were both used as intellectual artefacts in addressing subsequent questions. By applying the solutions of

questions as knowledge artefacts to solve other questions, the problem solving know-how in the teaching-learning of mathematics was extended further in the classroom. I now turn to the privileging of another cultural artefact like ‘gjette og sjekke’ in problem solving: a mnemonic.

Siv has a TV in the basement: using a mnemonic

Apart from the use of common sense and guess and check the short extract below elaborates the use of a mnemonic in teaching-learning. The mnemonic I mention came up for discussion in the sub-topic of conversion of formulae wherein one had to express a given formula in terms of other variables. I recall its use in my summary of this case.

Event	Person	Utterances	Action
1	RES		Notes Olaf and Knut call attention of the students to how Rolf and his group were attempting a given question.
2	Olaf	((Olaf draws the diagram Rolf and his group have in their working))	
3	RES		Notes Olaf explain that this mnemonic pertains to Q2.40 and asks Rolf to offer it with the class
4	Rolf	Siv has a TV in the basement	
5	RES	Q2.40: If we travel t hours with a speed v km/hr, then the distance s in kilometres is given by $s = vt$. Find the speed if we travel 259 km in 3.5 hours. Find the formula for speed v . Find the speed if we travel 364 km in 4 hr 40 min.	

The discussion pertaining to the application of the above **mnemonic** to Q 2.40 by Knut was largely in Norwegian. Yet the context of the question and subsequent actions, lead me to conclude that the mnemonic pertained to the formula equating distance (s) as the product of speed (v) and time (t). While discussing the mnemonic that Rolf and his group were using, Olaf and Knut shared two representations: **visual and verbal** (2-4). By offering the mnemonic being used by one group of students with all the students in the classroom, Olaf and Knut **privileged** its use as a way of remembering the relationship between the variables s , v and t in the teaching-learning of mathematics in the classroom.

I label **ways of remembering** as another of the many ‘ways’ that I have been highlighting as constituent to ways of knowing in the classroom. Within the context of teaching-learning, in the above extract the discussion around the speed-distance formula $s = vt$, was not about deriving the relationship between the quantities being discussed, but with

an objective of finding the formula for speed v , as asked for in the question. Towards such a goal, remembering the formula had both a necessary and practical aim. In privileging the use of the mnemonic in teaching-learning Olaf and Knut evidence that they recognise the possibility of **easily remembering** the required relationship. In so doing they continue with their value of finding easier ways of attempting questions.

Follow Up: applying what is known

In the final sub-section of the second case I discuss as mentioned the attempts by the students at the group-task *Follow Up*. Having three sub-tasks (See Appendix A.13) this group-task asked the students to apply the known relationships between directly proportional and inversely proportional quantities by using: graph, table and formula. I offer sequentially below the extracts of the three questions (Task 1, 2 and 3) from the group-task, followed by the responses of my students in my group-in-focus. It is by drawing upon student responses at this group-task and the two group-tasks reported in Chapter 5 that I offer the ‘Activity of applying graph, table and formula’ in my concluding discussion.

Task 1	Are x and y proportional, inversely proportional or none of the two?				
	a) $y = 12x$	b) $y = 8x + 4$	c) $y = \frac{10}{x}$	d) $y = \frac{\pi}{180}x$	e) $x \cdot y = 0.9$
	a) $y = 12x$ <i>P</i>	b) $y = 8x + 4$	c) $y = \frac{10}{x}$ <i>IP</i>	d) $y = \frac{\pi}{180}x$ <i>0</i>	e) $x \cdot y = 0.9$ <i>IP</i>
Anja	a) proportional b) not proportional c) inversely proportional d) prop e) inversely prop.				
	a) $y = 12x$	b) $y = 8x + 4$	c) $y = \frac{10}{x}$	d) $y = \frac{\pi}{180}x$	e) $x \cdot y = 0.9$
Lea	proportional	neither	prop inv-prop	neither prop.	inv-prop
	a) $y = 12x$ <i>P</i>	b) $y = 8x + 4$	c) $y = \frac{10}{x}$ <i>I.P</i>	d) $y = \frac{\pi}{180}x$ <i>P</i>	e) $x \cdot y = 0.9$ <i>I.P</i>
Stine	a) proportional b) none c) inversely prop d) proportional e) inversely prop.				

In discussing the application of meaning consolidated in the group-tasks discussed in Chapter 5, the above extract evidences the problems ‘of’ know-how. By this I mean the issues relating to the ‘how’ in know-how. Of the three ways by which proportional and inversely proportional quantities were being distinguished: **graph, table and formula**, Task 1 in the group-task above asked for the application of ‘formula’. Of the four student responses above which reveal a fair application, I draw attention to the fact that three of the students, identify the equation $y = (\pi/180) x$ as being inversely proportional ‘before’ scoring out ‘inverse’. Anja’s response is the only one that shows no such scoring out.

I draw on my being a participant observer and analyse Anja’s response. I record Anja working at the value of $\pi/180$ on her calculator, something which the others did not. I conjecture that on performing the said calculation, the value 0.0174532925 on the calculator made Anja conjecture, that the value she obtained was the ‘a’ in the formula ‘ $y = ax$ ’. This in turn made her mark the equation $y = (\pi/180) x$ as representing proportional quantities. My conjecture is based on the attention I have been evidencing earlier (Chapter 5) that Anja was constantly found using and relying on her calculator. It is possible that the other three students (less proficient with the calculator at that time) may have identified the equation that I discuss, as having the form $y = a/x$ because of the fractional nature of the coefficient. This may have resulted in their initially marking the said equation as inversely proportional.

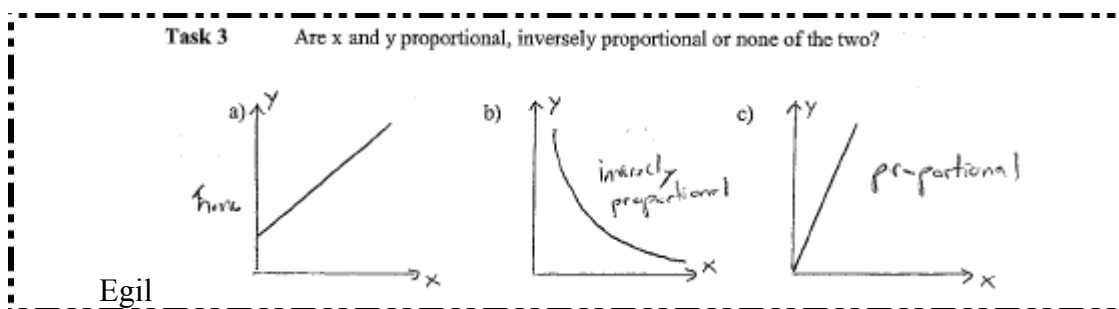
However the purpose of my argument above is neither to prove what Anja did nor what the others did not do. I can never know this with certainty anyway. My intention of the above discussion is on two other counts. Firstly, it was the collaborative nature of application of the ‘formula’ or the propositional form of meaning or knowing with which I wish to view the above ‘application’. It is likely that when Anja did use the calculator (for $\pi/180$), either her actions or the result obtained on the authority of a calculator, made the others accept the outcome and correct their response. Such a possibility may not have been available if students were to attempt the group-task alone with no **opportunity to cross-check**. Secondly, in the context of the application of knowing leading to problem solving know-how, Anja put her calculator to her use. The calculator mediated her participation in the teaching-learning of mathematics and its application. However, the benefits of her application depended on her familiarity with the calculator, which in turn was enabled by her use of the calculator to mediate personal **meaning and knowing towards goals** offered by the group-task.

I now discuss the attempts of my students at Task 2 of the group-task. I offer only Stine’s response below with the responses of others in Appendix A.19 for reference.

Event	Person	Utterance	Action																					
Task 2 Are x and y proportional, inversely proportional or none of the two?																								
	a)	<table border="1"> <tr><td>x</td><td>2</td><td>5</td><td>8</td><td>12</td></tr> <tr><td>y</td><td>15</td><td>45</td><td>65</td><td>80</td></tr> </table>	x	2	5	8	12	y	15	45	65	80	b)	<table border="1"> <tr><td>x</td><td>3</td><td>10</td><td>15</td><td>20</td></tr> <tr><td>y</td><td>6</td><td>1,8</td><td>1,2</td><td>0,9</td></tr> </table>	x	3	10	15	20	y	6	1,8	1,2	0,9
x	2	5	8	12																				
y	15	45	65	80																				
x	3	10	15	20																				
y	6	1,8	1,2	0,9																				
	c)	<table border="1"> <tr><td>x</td><td>5</td><td>12</td><td>18</td><td>30</td></tr> <tr><td>y</td><td>60</td><td>144</td><td>216</td><td>360</td></tr> </table>	x	5	12	18	30	y	60	144	216	360	d)	<table border="1"> <tr><td>x</td><td>40</td><td>60</td><td>90</td><td>130</td></tr> <tr><td>y</td><td>10</td><td>15</td><td>22,5</td><td>32,5</td></tr> </table>	x	40	60	90	130	y	10	15	22,5	32,5
x	5	12	18	30																				
y	60	144	216	360																				
x	40	60	90	130																				
y	10	15	22,5	32,5																				
2	Anja	<i>Is the product a constant ?</i>	((Thinking aloud))																					
3	Lea		((Works in correspondence with the 'rules' she has written down))																					
4	Stine		((Uses calculator to find product))																					
5	Olaf	A comment on Task 2 ... the best idea is to add an extra row																						
6	Olaf		<table border="1"> <tr> <td>$\frac{y}{x}$</td> <td></td> <td>$x \cdot y$</td> <td></td> </tr> </table>	$\frac{y}{x}$		$x \cdot y$																		
$\frac{y}{x}$		$x \cdot y$																						
7	Stine		((Stine was doing so already))																					
Stine																								

Anja's concern of the nature of the 'constant' value in the above extract gives further credence that her actions are based on her use of **the calculator** yet again. In the present discussion where I relate the application of knowing of various artefacts towards solving problems, I contrast Anja's focus with that of Lea, who is working with **a set of rules** (3). As a participant observer I record Lea to write down the rules that were derived in teaching-learning, on sheets in a separate file. Similarly I draw attention to Stine's approach. Prior to Olaf's instructing the use of the extra row (5, 6) I record Stine make use of such a row. In such use she evidences the use of **the table** as her intellectual artefact of use in attempting the question. Stine's participation with the table paralleled Lea's participation with rules and that of Anja's with the calculator.

In discussing Task 3 of the group-task I offer only Egil's response below with those of the others referenced in Appendix A.19. Unlike the application of 'formula' and 'table' the application of the 'graph' as propositional form and know-how seemed to be without much debate.



Brief summary of the second case

In the above case it was possible to look beyond specific physical and intellectual artefacts in the building of know-how. The use of solutions to questions as models to solve latter questions with was part of the problem solving know-how being deployed, in the teaching-learning of mathematics in the classroom. As a consequence the selection of questions, and the selection of solutions to questions, became important to the value of **continuity** being cultivated in the classroom.

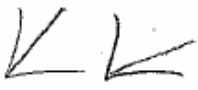
I had conjectured that in the use of solutions to prior questions, to solve latter questions, the earlier solutions were **knowledge artefacts** in that they mediated prior knowing, towards knowing necessary for the current problem at hand. The kind of knowledge artefact used in the development of problem solving know-how, included the use of a numerical sequence of consecutive numbers to mediate the corresponding algebraic sequence, the entire solution of a question and only the end result of the solution of a question. I conjectured in Chapter 5 that the propositional forms of meaning as graph, table and formula were a result of attempts at group-tasks themselves acting as knowledge artefacts. It was the **objectification** of such a process as graph, table and formula that was applied above and as part of **problem solving know-how**.

The application of artefacts or models to solve problems was accompanied by the **privileging** of more 'cultural' methods of solving problems. In this I include **guess and check**, the use of a **rhetorical device** and a **mnemonic**. The application of the mnemonic contributed also to another of the many 'ways' leading to knowing: **ways of remembering**.

Individual preferences in the choice of artefacts to solve problems were evidenced in the application of graph, table and formula in the group-task: *Follow Up*. Such preferences like Anja using her calculator seem to have a longer history of usage than in application to problem solving alone. I conjecture that just as it was with the calculator that Anja was making meaning (Chapter 5), it was with the same choice of artefact that afforded her another instance of **agency**, with which she was now attempting to solve problems. The logical follow through of the above is another conjecture that **speaking-with-the-calculator** lead Anja to **solving-problems-with-the-calculator**.

As I conjecture about the individual choices and preferences of artefacts with which participants participated either in making meaning or solving problems, I find it appropriate to discuss survey data connected with the attempts of students at the three group-tasks on the sub-topic of proportionality. I draw from two sets collected from all the students in the classroom of their attempts at each of the three group tasks. The first data I offer, relates to the question of what would the graph look like for

Task 4: What would the graph look like for a stiffer spring?



a stiffer spring (Task 4 in group-task *P proportionality*). Of the 29 responses I col-


lected 15 students responded to this question graphically (accurate: 14, inaccurate: 1) as above. In contrast to the rest of the students who gave a verbal explanation there is evidence of greater **agency** with graph since almost half the students in the classroom **spoke-with-their-graphs**.

The second survey data I refer to is the response of Tia to Task 5 in the group-task: *Inverse Proportionality*. There is evidence in Tia's response given below that Tia used the **mnemonic privileged** by Olaf and Knut in their classroom teaching-learning of mathematics. Tia's response

Task 5: Use your result from task 4 to find a formula for y in terms of x (y = ...)

$y = \frac{a}{x}$ $y = \frac{40}{x}$ $x \cdot y = a$
 $y = \frac{a}{x}$

In this case the quantities x and y are said to be inversely proportional.



indicates her use of the mnemonic to reflect on the relationship

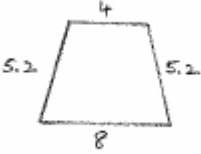
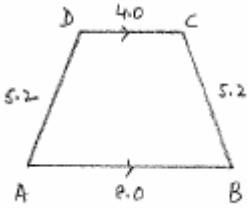
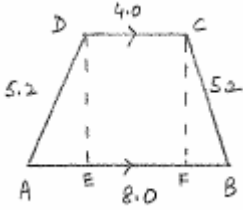
between the variables x , y and a in inversely proportional quantities. Tia's response also strengthens my claim that in Tia appropriating the mnemonic privileged by Olaf and Knut, she **re-appropriated** the cultural artefact to suit her individual goals. I now turn to the nature of independence achieved by students in problem solving and the know-how that enabled the same.

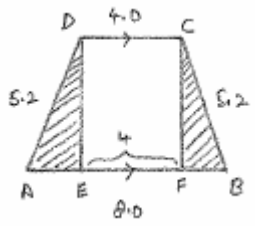
Questions to problems

As my third and final case, I offer two extracts and discuss the development of problem solving know-how in the teaching-learning of scale factor in similar figures. In the first, I relate how a question from the textbook became a problem that a student could solve in the course of discussion. In the second, I relate to the intuitive nature of participation in problem solving that I have discussed as present by this time in the trajectory of teaching-learning of mathematics in the classroom.

What do we do now: becoming a problem

In the extract below, I discuss know-how related to and built upon the written text of a question. I elaborate Olaf's actions at the blackboard in relation to Q 4.213 from the textbook. Though the question has four parts (offered below) I discuss only the first in which the area of the trapezium is to be calculated. As in earlier cases relating to scale factor in similar figures my group-in-focus has Dan, Levi and Thor.

Event	Person	Utterance	Action
1	RES	Q 4.213: Sides AB and CD of trapezium ABCD are parallel. AB=8.0cm and CD=4.0cm. Further AD=BC=5.2cm. 1) Draw the figure and find the area of the trapezium 2) Construct the trapezium The trapezium is a sketch of a larger building in which AB is 60m 3) What is the scale factor used in the working drawing 4) Find the area of the base of the building	
2	Levi	((Asks RES how to find the area of the figure which he draws as alongside))	
3	RES	<i>We can use a general formula</i>	
4	RES	((Offer assistance towards solution in Levi's notebook))	$\frac{h}{2}(a + b)$
5	RES	<i>I think</i>	
6	Levi	<i>Sure or I think</i>	
7	RES	<i>I think</i>	
8	Olaf	((Initial diagram which Olaf builds upon subsequently. Olaf's drawing of the figure calls attention of the students))	
9	Olaf	What do we need	
10	STD	h	
11	Olaf		$A = \frac{h}{2}(a + b)$
12	Levi		((Nods in agreement with RES when he finds the formula on the board))
13	Olaf	What do we do now	
14	Olaf	((Addition to earlier diagram in event 8))	

15	Olaf	What else do we know	
16	Olaf	((Inferred from the text of Q 4.213))	$DC = \frac{AB}{2}$
17	Levi	EF= 4 cm, AE + FB = 4 cm	
18	Olaf	((Addition to earlier diagram in event 14))	
19	Levi	((Explains in NOR that since AED and BFC are congruent and the excess of EF on either side of could be shared))	
20	Olaf		$AE = FB = 2.0$ $h^2 + 2.0^2 = 5.2^2$ $h^2 = 5.2^2 - 2.0^2$
21	Olaf	Has anyone done this	
22	STD	$h = 4.8$	
23	Olaf		$A = \frac{(8.0 + 4.0)4.8}{2}$ $= 28.8\text{cm}^2 = 29\text{cm}^2$

The above extract evidences how the question relating to the area of a trapezium becomes a problem that Levi could solve. As discussed in Chapter 4 and 5, the above extract also evidences Levi call upon where possible experienced help to attempt his question at hand. My offering Levi the formula that needed to be applied to find the required area (2-7) offered Levi some direction with what may be required to find the area but left the challenge of how to apply the formula to him. In was upon Olaf's directions that the question became a problem that he could solve.

Olaf's directions which made the question a problem that Levi could solve, took root in six steps as I now recount.

First, there was a difference between the diagrams drawn by Levi and Olaf. The one made by Olaf (8) unlike that made by Levi (2) acknowledged that the sides of the trapezium AB and CD are parallel. This is the first step. Upon Olaf asking 'what do we need' (9) a student expressed the need for the altitude 'h' of the trapezium (10). Upon response from a student Olaf wrote down **the formula** necessary for finding the area of the trapezium to calculate the required area (11) one which was also found in the formula book. Incidentally, I found that although the formula did indicate 'h' as the altitude, in the formula book the accompanying diagram did not indicate the right angle at its base nor mark the two sides as parallel with corresponding arrows. Olaf's actions of relating the formula with an appropriate diagram is the second step.

Having related the formula with the appropriate diagram, Olaf then asked what was needed to be done ‘now’ (13). He followed this question by drawing and then labelling the two altitudes DE and CF in his first diagram (14). This was the third step. I need to clarify here that Olaf modified in successive steps the first diagram that he drew (8). The pace at which he conducted his discussion was easy to follow. As a participant observer I recorded that the pace at which Olaf was making changes to the original diagram, allowed the students enough time to become aware of reasons for which he was making the said changes.

In order to obtain the lengths DE and CF Olaf then asked what else is ‘known’ (15). Though the given lengths of DC and AB were four and eight centimetres respectively, Olaf chose to express DC as half of AB (16). This was the fourth step. Evidence that by this step the question was now ‘becoming’ a problem that Levi could solve is found in Levi’s response (17). Levi’s response of first offering the length of EF shows that he was able to take into account the equal lengths of EF and DC. Simultaneously he inferred that the resultant sum of lengths of EF and FB would be equal and each would measure four centimetres. This is the fifth step. In response to Levi’s inference and explanation, Olaf shaded the two triangles on either side of the altitudes and designated EF as four centimetres (18). This was the sixth and final step.


Subsequent to the sixth step the question became a problem that Levi could solve. Levi drew upon the **congruence** of the two triangles and argued that the lengths of AE and FB were each two centimetres in length (19). I emphasise that the question became a problem with the sixth step since the calculation of the height of the similar triangles or the altitude of the trapezium (using Pythagoras’ theorem) had become routine for the students by then (20). That such a calculation had **become routine** in teaching-learning is evidenced also by Olaf’s offering the approach to the calculations at the blackboard, and calling upon the students to use their calculators and give him the numerical answer (21).

I make two additional observations of significance while discussing the above reformulation of the question into a problem. Firstly, the above question was attempted within the context of the topic of similarity and congruence. The questions that Olaf successively asked (which led to such a resolution) drew attention to the **situatedness** of the textbook **question** within the sub-topics. It is reasonable to conjecture that as a stand alone problem the above question may have been approached in other ways as well. Secondly, and in correspondence to the shift in teaching-learning practice to a more student-centered practice in the classroom, Olaf left the calculation of the altitude and the area of the trapezium to the students. He used the value that the students had obtained in their calculators, at the solution he was discussing at the blackboard.

The history of teaching-learning in the classroom would justify these actions of Olaf since he had paid attention to ways of operating the calculator and the **value of students refuting a conjecture** in his teaching-learning. I now turn to discuss my second extract.

Somehow had that feeling: becoming intuitive

In Chapter 5 where I discussed the consolidation of the scale factor in three dimensions, the students evidenced an intuitive understanding that the scale factor in three dimensions was a cube of the scale factor in one dimension. The extract I offer below relates to a time before such consolidation by Olaf. This extract allows me to discuss yet again the use of the calculator as intrinsic to the collaborative classroom practice. I discuss the attempt of Thor at Q4.52 from the textbook given below:

Event	Person	Utterance	Action
1	RES	Q 4.52: A tank shaped like a cylinder has a volume of 250 l. The radius of the base is 31 cm. Calculate the altitude and surface area of the tank.	
2	Olaf	Lets look at 4.52	
3	Olaf		
4	Olaf	If we cut it out and spread it	
5	Olaf	((As also given in the formula book))	
6	Olaf	If we look at the formula	
7	Olaf	((As in formula book))	$O = 2\pi r^2 + 2\pi rh$
8	Olaf	What is this	((Circles $2\pi r^2$ in the formula))
9	Levi	The circle	
10	Olaf	Very often if there is an open box what do you do then	
11	Levi	We remove πr^2	
12	Olaf	This formula is only for when there is both a top and a bottom	
13	Thor	<i>The teacher said</i>	((Addressing RES))
14	Thor		$1\ell = 1\text{dm}^3 = 1000\text{cm}^3$
15	Thor	<i>Is that correct</i>	
16	Levi		((Turns the pages of the textbook looking for the appropriate page))
17	RES	<i>Page 29</i>	
18	RES		1liter = 1dm^3 , 1cl = 0.01litre 1dl = 0.1liter

19	Thor	<i>Can I ask you for help</i>	
20	RES	<i>Yes</i>	
21	Thor	((Directs attention of RES to Q 4.52 being discussed))	
22	RES	((Refer to the formula book and mention that we had the quantities V and r given in the question))	$V = \pi r^2 h$
23	RES	((Show in Thor's notebook))	$h = \frac{V}{\pi r^2}$
24	Thor	<i>Is that all</i>	
25	Thor		((Obtains πr^2 on calculator as 3019.07054))
26	Thor	((Now uses calculator to divide and obtain the value as shown alongside))	$\frac{V}{\pi r^2} = \frac{250}{3019.07054}$ $= 0.0828069423$
27	RES	<i>Look at the fasit</i>	((Fasit: answers or solutions found at the end of the textbook))
28	Thor		Answer to Q4.52 is 83
29	Thor	<i>OK, OK!</i>	
30	Thor	<i>Should multiply by three zeros</i>	
31	Thor	<i>Somehow had that feeling</i>	

It was in the teaching-learning of an earlier sub-topic that the conversion in metric measures was introduced. The above extract evidences Thor searching for the meaning and possible **way of applying the conversion** of one litre being equal to a thousand cubic centimetres (13-15). In response Levi looked for the appropriate page where the book said that one litre was a thousand cubic centimetres (16-18). The extract then allows me to conjecture that unlike Levi who may have been taking the book for granted, the intentions of Thor were to know the meaning of that equality in relation to his question at hand (21-23). Towards a resolution of 'his' goal or question, I then directed Thor to the appropriate formula which had to be applied and mentioned to him that of the three variables or unknown in the formula, the question offered two known values. In an attempt at making the question a problem, I then showed Thor the formula for 'h' in terms of the known variables 'V' and 'r' (23).

Since over my observations as a participant observer I record Thor to make good use of the calculator, the reformulation I offered to Thor did not pose a problem to him (24-26). I conjecture that my interventions enabled to **make the question at hand, a problem** that Thor could solve with the calculator. Thor used his calculator and found his calculation to be a long decimal (26). Assessing that Thor may be wanting to make meaning of the conversion of the volume measure of litre in terms of cu-

bic centimetres, I then directed Thor (27) to check the answers given at the end of the textbook (fasit in Norwegian). The answer of 83 to the question (28) seemed to help Thor make an intuitive leap of meaning and understand the requirement for three zeros (29-31). By this I mean he was able to account for or factor in three places of decimal in the value on his calculator, with the three zeros in the thousand cubic centimetres of the volume measure of a litre. The above extract helps me identify **ways of applying conversions** to the many ways of knowing in the classroom. I conjecture that it was Thor's familiarity with the calculator (as in Chapter 5) that made Thor, like Anja, use the same artefact in attempting problems. I now turn to summarising this case.

Brief summary of the third case

The last case in the discussion relating to problem solving know-how, evidenced how by this time of teaching-learning, it was **possible to turn questions to problems that the students could solve**. This of course did not happen without experienced assistance or guidance of the teachers or adult peers. The two extracts presented a similar pattern of actions given as assistance by experienced peers which mediated greater awareness in the students. By their utterances, the teacher and I were able to raise the previous actions of the students to a greater level of consciousness. In their newer actions the students were informed by two kinds of artefacts. In the first, **successive diagrams** drawn by Olaf representing the formulation that was being currently discussed mediated more and more meaning and knowing about the goals of the question. In the second, the use of **formulae** combined with the **calculator** and **textbook** mediated more information, with the help of which Thor was able to make meaning. This led to my identifying **ways of applying conversions** as one more of the ways of knowing mathematics.

The sets of actions of both Olaf and myself, and Levi and Thor were however different yet equally significant to the problem solving know-how being continually developed in the classroom. In the first, **more and more meaning** of the same question emerged when various aspects of the given question were successively incorporated in what was initially 'known'. In the second, it was Thor who **wanted to know how to put his calculator to use** in attempting the problem at hand.

As conjectured earlier, the problem solving process in both the extracts above showed a reflexive connection with the meaning making processes involved of the problem at hand. While in the first extract it was the **meaning of the problem** that needed attention, in the second it was the **problem of applying** and extending the use of an artefact or tool towards solving the problem. Thor strengthens my conjecture that the artefact with which he was making meaning, was also the one with which he was solving problems. I now turn to conclude the chapter.

Concluding discussion

The three cases that I present in this chapter, allow me to discuss how the development of problem solving know-how took place in stages, of discussing and building rules by turning them into questions in the first; of building solutions which were subsequently applied in the second, and of transforming questions to problems that the students could solve in the third. As in the previous two data and analysis chapters, I now offer discussion across the three cases with arguments from literature in mathematics education and socio-cultural-historical perspectives.

In discussing the development of problem solving know-how, I have evidenced how in addition to the consolidation of meaning making into knowing the emphasis in the building of know-how in teaching-learning was towards **applying what is known in addition to knowing**. It is such an added emphasis that made the artefacts of use in this chapter to have greater instrumentality than that encountered in the previous chapter. I begin discussion with the new **physical artefact** that I have evidenced in this chapter: **the calculator**. Both in its use by the teacher and by the student (Anja, Thor) it was the application of the calculator towards the goals of the problem at hand that was of emphasis.

The emphasis of use of **intellectual artefacts** was similar: the appropriate use of **the bracket** was towards preventing its misapplication, **the numerical example** mediated the knowing necessary for its corresponding algebraic formulation in order to solve the question posed; **the algebraic equation** was of use in reaching the solution of the question of consecutive numbers, **the graph, table and formula** had become objectified with respect to directly and inversely proportional quantities and were applied to identify these relationships in given graphs, tables and formulae, the **formulae of areas and volumes** were similarly used towards finding respective areas and volumes and **successive geometrical diagrams** mediated greater and greater knowing towards knowing how to attempt the question at hand.

In addition to the above artefacts, there was evidence of the use of artefacts that were not necessarily mathematical, but because of their use in the teaching-learning of mathematics in the classroom became part of the problem solving know-how. As intellectual artefacts: **guess and check, rhetorical questions** and **the mnemonic** (visual and verbal) mediated the processes available in any social praxis or culture that were associated with them. The use of these 'cultural' artefacts or models was as mentioned, to act as tools to solve questions in mathematics. It was such an emphasis, that made these artefacts part of the problem solving know-how being developed in the classroom. That this was the case was evidenced by their being **privileged**, as argued by Wertsch (1991), towards the solving of questions or problems at hand.

Ways of knowing mathematics, as argued by Bishop (1988) evidenced in this chapter included: **ways of operating the calculator**, **ways of remembering** and **ways of applying conversions** were as argued earlier these constituted **ways of knowing how-to** in addition to ways of knowing as applicable to meaning in the propositional form.

I make three points of significance, with respect to the above mentioned emphasis of applying what is known, in addition to knowing. Firstly, it was the **presence of goals** in which each of these artefacts or ways was put to use that is important to recognise. As argued by Vygotsky (1981a) and Leont'ev (1981b) these provided either the need for use or the materiality in which they were found useful. Secondly, the use of the above mentioned artefacts by the students was shown by the teacher as useful towards particular goals. The invitation of deploying the artefacts for specific use was associated with the lure of language as argued by Bruner (1984) or through appropriating each others understanding as argued by Cole (1985; Newman et al., 1989). Their use as objects that could be used for a specific purpose as argued by Stetsenko (1999) was part of the dynamic principle of sharing know-how as argued by Vygotsky and quoted by Leont'ev (son) and Luria (1968). It is such an eventuality that allowed for the above events to be a constituent of a **zone of proximal development or ZPD** as argued by Vygotsky (1978). Finally, in the above mentioned emphasis on application, it was the know-how of problem solving that was the context. The actions of teachers and students alike contributed to special forms of cultural behaviour in the micro-culture of the classroom which as Vygotsky (1994d) and Luria (1994) argued constituted **education**. The above mentioned actions that were specific to problem solving know-how also helped students realise abilities hitherto not known to them as argued by Luria.

However as argued by Goodnow (1990) and by Bishop such an emphasis of problem solving know-how in the social environment of the classroom did not come without **specific values**. These included attention to the relationships between special cases and general rules, attention to convention, promoting the ease with which to attempt questions, the allowing of the refutation of conjectures and the making routine of certain procedures in the process of teaching-learning of know-how.

The value of using knowledge artefacts in addition to physical and intellectual artefacts and of knowing how-to was also visibly evidenced. In mediating prior knowing to latter use as argued by Wells (1999), these **knowledge artefacts** included numerical and algebraic patterns, entire solutions to questions; the end result of solutions to questions and objectified results of group-tasks (graph, table, formula). The use of these as knowledge artefacts for problem solving made them specific to the teaching-learning of problem solving know-how in the classroom. They

evidenced in addition the arguments of Lester (1985) and Grows (1985) that problem solving occurs in small episodes **in routine teaching-learning** and not necessarily in isolated episodes of problem solving.

The collaborative nature of sharing know-how of problem solving, also provided for the **apprenticeship** that Schoenfeld (1991) argued as crucial and necessary for the students to know mathematics. The manner evidenced of the teaching-learning of problem solving know-how also contributed to problem solving and **know-how becoming a tacit** part of classroom culture as argued by Goos et al. (1999) since the know-how and problem solving were both part of teaching-learning. Evidence of the latter is had from how towards the end of three topics of teaching-learning, the instruction in the classroom allowed for the questions at hand to become problems that the students could solve.

The special nature of know-how transacted in evolving the micro-culture of the classroom was however not independent of the features of classroom practice outlined in Chapter 4 and the consolidation of meaning discussed in Chapter 5. This has led to evidence and conjecture the **reflexive nature** of meaning making and problem solving process. There was evidence of the meaning making of a rule or a question (2^{-3} , Levi and trapezium) lead to problem solving and the problem solving of a rule and question (2^0 , Thor and conversion) lead to meaning making.

There are three issues I turn to in concluding my discussion about the teaching-learning of problem solving know-how in the classroom: the emphasis of Pólya, the analysis of ‘activity’ and the arguments of Luria.

In addressing the concern of Pólya (1987) that problem solving **know-how is important**, even crucial in high school mathematics and more importantly the possibility that **problem solving can be taught**, I think this chapter provides evidence in the affirmative. However my conjecture is that such a concern can be analytically addressed, only by widening the lens and including socio-cultural-historical perspectives which account for the value of increasing of one’s intellectual repertoires by the use of artefacts as argued by Säljö (1998), moments of mediated action as argued by Wertsch (1985), the value of making public one’s meaning as argued by Bruner (1990; 1996) and the establishment of joint intentionality as argued by Olson (2003).

The actions I refer to were in turn situated in the context of teaching-learning in the classroom, and were accompanied by values as discussed above. It was in the establishing of the ZPD that the teacher was able to bring about a loan of consciousness of his own knowing, of that of the other students and of the discipline of mathematics as argued by Holquist (2002) who drew on Bakhtin. Such analysis may begin to address the concern shared by Brownell (2004) in his writings that meaning making is a cumulative process and that problem solving can be improved.

It is to continue discussion about the reflexive nature of meaning making and problem solving, that I now turn to the ‘activity’ based on the three group-tasks on proportionality. As before I present the facts in a tabular form before discussing the transformations brought about.

‘Activity of applying graph, table and formula’					
<i>Objective</i>				<i>Subjective</i>	
Activity	Motive	Action	Goal	Operations	Conditions
To work at the three group-tasks	To compare, relate and identify quantities that were either proportional, inversely proportional or not in three ways: table, graph and equation	To complete a group-task which in turn was made up of six tasks	To relate the relationship between proportional quantities	To complete a table, plot a graph, comment on the graph, conjecture, find the ratio between y and x and express y in terms of x	To use a calculator, graph sheet, writing in textbook, graph drawn earlier, filled in table and compare quantities
		To complete a group-task which in turn was made up of five tasks	To relate the relationship between inversely proportional quantities	To complete a table, plot a graph, comment on the graph, find the ratio between y and x and express y in terms of x	To use a calculator, graph sheet, writing in textbook, graph drawn earlier, filled in table and compare quantities
		To complete a group-task which in turn was made up of three distinct tasks	To distinguish the incidence or not of proportional quantities from inversely proportional quantities in equations, tabulated values and graphs	To identify quantities that are proportional, inversely proportional or not with the help of a formula, table or graph	To gauge formula by using the calculator, numerical relationship with table and calculator and graph based upon characteristics

Based on the arguments of Leont'ev (1978; 1981c) I have modelled the 'activity' in three rows in the above tabulation: the first relating to the group-task Proportionality, the second to the one on Inverse Proportionality and the third: Follow up. I discuss the table as before from right to left and top down wards towards analysing the transformations involved. I choose to discuss the first two rows together before the third.

At the level of **operations-conditions** there are three differences in the first two rows or group-tasks. In the first row comprehension began with concrete examples (rubber band, diagram of a spring balance) and also asked for knowing generated within the group-task in concrete terms (stiffer spring). Such an interpretation was not sought in the second row. In a similar manner, in the first row the textbook was found to be of use whereas in the second its usefulness was taken as a given and acted upon. Finally, in expressing the relationships between y and x in the first row the students deliberated upon 'what' was being related in the ratio where by the second row 'that' they were related was taken for granted and expressed. The operations-conditions in the second row drew on the knowing generated in the first.

At the level of **actions-goals**, the first row established the relationships between proportional quantities in three distinct ways: table, graph and equation. The level of formalisation was from the concrete to the mathematical. In the second row the same relationships were not only established and formalised but held in contrast to the relationships realised in the previous row. The 'inverse' nature of proportionality was brought to conscious reflection.

The knowing that resulted at the level of operations-conditions and actions-goals were both utilised in the third row. The group-task in this row first asked for identification in the three established ways of graph, table and formula. It then required that their identification be made on application of appropriate routines in operations-conditions. In such an order there is a reversal of direction in the *Follow Up* group task. Instead of operations-conditions necessary for realising the actions-goals, in the third row the actions-goals were applied and realised in operations-conditions, recognisable to the students themselves.

At the **activity-motive** level, the motive of the activity was realised in two directions. In the first direction, realisation moved from concrete examples to the more formalised relationships, while in the second a formalised relationship was realised (or not) in the given examples. While the first was derived from the task, in the second their existence and recognition was demanded in the given examples. The concrete materials with which the 'activity' began was replaced by relationships. In 'activity' the personal observations of the students made were objectified and the formal relationships in mathematics were subjectivised.

As a culmination to discussion in Chapter 6 and before moving to Chapter 7 where the data I offer is different, I take up an aspect of teaching-learning argued by Luria (1973; Vygotsky et al., 1994b): that of speech raising actions to a higher level of consciousness and being instrumental in solving problems. Drawing upon data that evidences the nature of interaction between teachers and students and among the students, the two column presentation that I have used all along allows for such an analysis and discussion. In discussing examples of utterances and the subsequent actions that were raised, where greater consciousness was instrumental towards solving problems, I compile a selection across Chapters 4, 5 and 6 below. In each instance I evidence the many small yet discernable steps by which utterances were **instrumental in bringing in more cognition** to the problem being solved or attempted. It was this difference in approach to solving problems, which I had pointed to earlier. In Pólya's model the role of speech is absent.

Page	Event	Utterance	Action raised
88	1	Turn to page 14	<i>Turn</i> to the appropriate page in the textbook and <i>locate</i> rules with which to attempt questions
89	9	Some students mentioned to me that ... What do you think?	<i>Weigh</i> two options, <i>consider</i> the value of either, <i>decide</i> which is more fruitful and <i>argue</i> for the same.
90	6	Is it OK to leave it here	<i>Gauge</i> whether the given fraction satisfies convention for which one is to <i>recollect</i>
91	23	Do we need brackets	<i>Identify</i> that there is no sign between the two brackets given and <i>recall</i> that this signifies multiplication
91	27	That a hundred million	<i>Accept</i> that a hundred million is 10^8
93	4	What is the rule	<i>Recollect</i> the rule applicable to the expression being discussed
96	6	Can you tell us how you got this	<i>Reason</i> for the working just offered at the blackboard
101	1	The angles of $\triangle ABC$ are similar to ...	<i>Locate</i> angles of the same measure in another triangle which make the two similar in comparison
114	4	Do we have to find the LCM of all four	<i>Recognise</i> the presence of different denominators in the fractions of the two brackets being multiplied and <i>decide</i> for which of those the LCM is necessary
117	33	What is 17×6	<i>Calculate</i> in order to find out if the fraction can be reduced by a common factor
118	11	It only works when you have the same ...	<i>Acknowledge</i> when the given expression is valid
119	4	We have a problem here ... what does this mean	<i>Express</i> the knowing related to the representation being discussed

121	7	We can do more	<i>Offer</i> greater understanding than is currently being shared
122	3	What do we do with the indices	<i>Parallel</i> that the sum of the indices of the three exponents being multiplied add up to the single power in the resultant exponent expressed
131	4	If volume increases, pressure decreases ... what can we say beyond point (A)	<i>Relate</i> the increase in volume and decrease in pressure of a gas to the inverse relationship represented by the graph and <i>conjecture</i> its nature at a marked point
136	1	If we remember the square	<i>Recall</i> the scale factor derived in prior activity by comparing the lengths and areas of similar squares
146	6	We can do it similar to the example	<i>Follow</i> in the question being currently discussed, a similar procedure as applied to the one shown in the example
148	4	What is the other possibility	<i>Acknowledge</i> if there is any other possibility than the one being discussed
152	14	What will x be now?	<i>Justify</i> the current designation of x
154	11	You could have written this another way	<i>Reconsider</i> what has been presently written in another manner of writing
158	5	A comment on Task 2 ... the best idea is to add an extra row	<i>Add</i> an extra row, the use of which makes it easy to <i>compare</i>
161	9	What do we need	<i>Set</i> a goal that has to be achieved
161	13	What do we do now	<i>Exemplify</i> diagram
162	15	What else do we know	<i>Relate</i> information of relevance
162	21	Has anyone done this	<i>Give</i> the calculation arrived at
165	23	$h = \frac{V}{\pi r^2}$	<i>Admit</i> simplicity
165	27	<i>Look at the fasit</i>	<i>Compare</i> personal and textbook answer
165	28	Answer to Q4.52 is 83	<i>Associate</i> number of zeros in the denominator of a fraction with number of shifts in the decimal equivalent

My final synthesis of the data and analysis of the present chapter is the following. That know-how of problem solving can be taught in the classroom. Embedded in small episodes in teaching-learning, the process involves a combined understanding and application of meaning, knowing and knowing how-to in the presence and recognition of goals. This is possible with presence and use of artefacts including speech, which mediate both meaning making and cognition and whose contribution to ways of knowing is accompanied with associated values. It is **within the micro-culture** so constituted, **that a ZPD enables know-how**.

Having observed the ZPD at the level of the classroom, I now turn to my final data and analysis chapter where I discuss the nature of cooperation and ZPD between student peers at group-tasks.

7. Cooperative problem solving

In my final data and analysis chapter, I elaborate upon the cooperation of students as a group and in a possible ZPD at group-tasks. Each of these group-tasks was conducted towards the end of the teaching-learning of the three topics I have been reporting all along: number understanding, equations and proportionality and scale factor in similar figures.

I discuss the group-tasks in a succession of three cases: *When together and How heavy*, *Two bodies in motion* and *SA/V ratio and metabolism*. As mentioned in Chapter 4 the group-tasks *When together* and *How heavy* were conducted by Olaf and Knut as part of the teaching-learning of the classroom and involved all the student groups in the class. They were conducted after the teaching-learning of the topic of number understanding and towards commencing of the sub-topic of equations. As also mentioned it was in their conduct that the cooperation of students as a group was initiated and consolidated for the first time in teaching-learning. The later two group-tasks *Two bodies in motion* and *SA/V ratio and metabolism* were conducted only for the groups-in-focus with whom I observed the teaching-learning of the second and third topics: equations and proportionality and scale factor in similar figures.

As discussed in Chapter 3 the rationale behind my design of the group-tasks was to focus on the argumentation related to a particular artefact and observe the cooperation among the students. I attend in particular to the meaning made by the students of the problem presented, the role of artefacts in mediating the problem at hand and the arguments of the students in relation to the shared goals of the group-task. The artefacts involved in each of the tasks are intellectual: diagrams, graphs, tables and formulae. While the goals in the first two group-tasks (the first case) are specific, the goals of the tasks in the second and third group-tasks are open-ended towards the end of the group-task. An ‘activity’ analysis is presented for the two group-tasks of the first case.

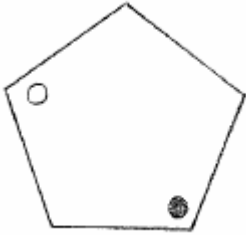
The backdrop for the three cases and the theme that I discuss in this chapter are: the participation of the students in the collaborative classroom practice (outlined in Chapter 4), where students’ meaning making became more central to classroom teaching-learning; the consolidation of personal meaning of students (discussed in Chapter 5), where the consolidation by the teachers shifted from that done at the blackboard to students knowing; and the development of problem solving know-how (elaborated in Chapter 6), where its conduct lent greater and greater freedom to the students. The cases in this chapter together with those in the earlier chapters constitute the micro-culture of the mathematics classroom in which I conducted my study. As mentioned before I discuss its final constitution in my coming and last chapter.

When together and How heavy

The group-tasks I discuss in the two sub-sections below were conducted as mentioned to first initiate and then consolidate for the first time, the cooperation of students as groups. The topic of number understanding had concluded before the conduct of *When together* and the sub-topic of equations commenced with the conduct of *How heavy*.

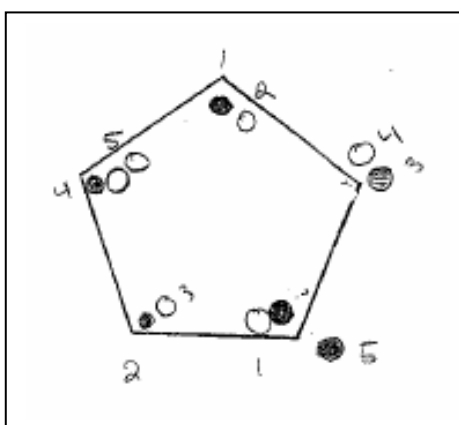
When together: cooperating at a representation

The group-task presently being discussed and given below was handed out as individual worksheets to all the students seated in their groups. I offer and discuss subsequently the working of six student groups.

When together	
<ul style="list-style-type: none"> • In the pentagon are two dots, black and white, on the move. • The black moves two corners counter clockwise. The white moves three corners clockwise. • After how many moves are the two dots together? 	

Of the six group solutions collected, the first (Group A) had no diagram or written solution. On my inquiring the group who showed some surprise and disbelief and told me that the two dots **would never meet**. I surmised that the students did cooperate however since they did have a solution when asked. The written solutions by Groups B to F offer more evidence to draw conclusions from which I now turn to elaborate.

There is evidence (given below) that Group B arrived at their solution with the help of black and white dots with corresponding positions

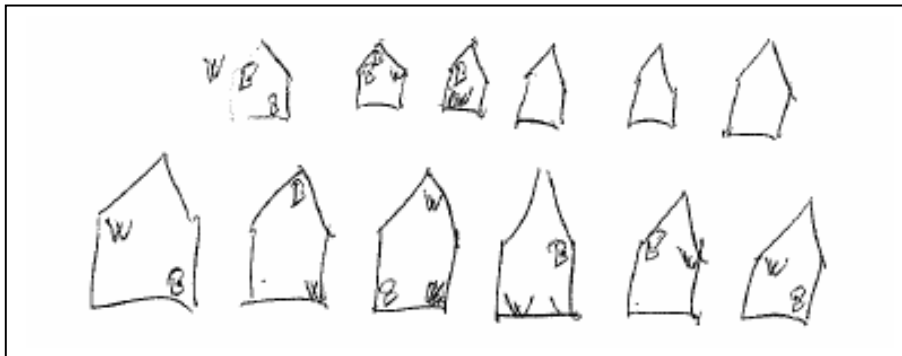


After the fifth move the dots are back in their starting positions!
Answer: They can never meet.

numbered. Their explanation makes me conjecture their realisation that the black and white dots return to their starting positions and ‘thus’ can never meet. In their conclusion (alongside) the students seem to recognise implicitly that the relative positions numbered 1, 2 etc of the two dots are never at the same corner of the pentagon. Such an eventuality seems to have made them argue that the dots would never meet. The students admit the **circularity of movement** as a hopeless one in terms of being together, making them arrive at the conclusion

that after the fifth move the dots are back to the starting position.

As a participant observer I record the students of Group C redraw the given pentagon on a larger scale on a separate sheet. They carried out the

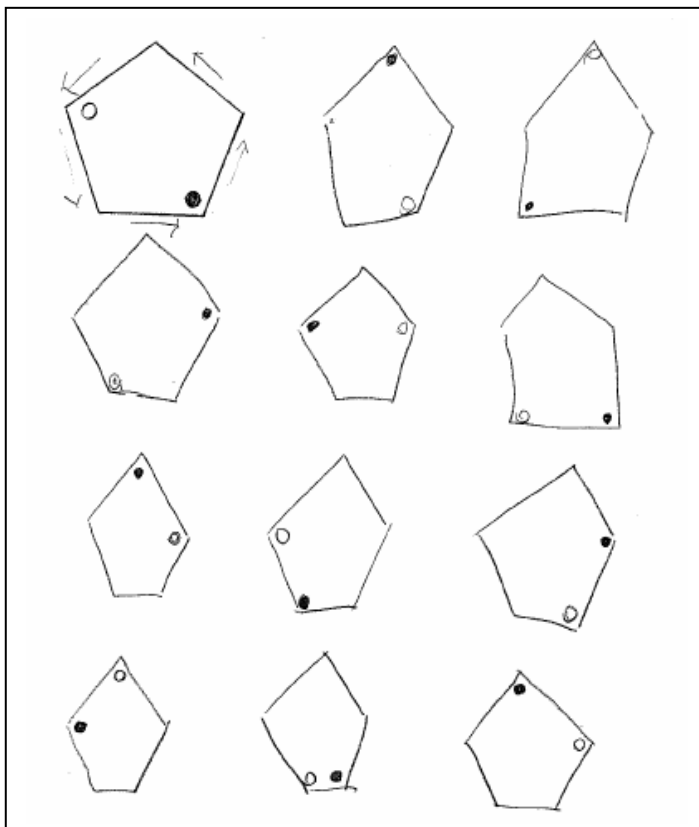


They hit each other at the first move, but they never end up in the same corner. They land at the point they start after the sixth move.

task in physical and concrete terms, crouching over their sheet and use a pen which moved as 'black'

and a pencil which moved as 'white'. I observed the hands of the students' criss-cross each other during discussion. The data offered by them (shown above) evidences two rows. In the first the students attempt to represent the movement of the black dot (B) while indicating the white's position (W) outside the pentagon. The non-completion of the first row evidences confusion and the need for a fresh attempt which is made in the second row. The presence of confusion is confirmed by the clarity in the second row, in which the 'start' condition is followed by pentagons showing accurate and simultaneous moves of both black and white.

The written conclusion of Group C has three parts. The phrase mentioning that the dots hit each other evidences the physical movement of

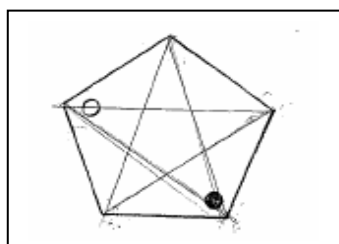


pen and pencil. The second row of pentagons represents why the dots represented by B and W are never in the same corner. The circularity of movement evidenced therein is also mentioned by them. They however seem to mention this as their sixth 'move' where in fact it is the fifth move but sixth 'position'.

Group D made successive pentagons (as alongside) and moved black and white dots simultaneously and correctly in the first four

moves. As is evident from the diagram in the fifth move the black moves correctly but not the white. This however is not apparent since the diagram evidences the correct moves of black and white in each of the steps that follow. The crucial fifth move where the ‘original’ position is observed (realised by Groups B and C) is precisely the one in which the **inadvertent mistake** is made. However the dots are never together.

The solution offered by Group E (given below) denotes a pattern that traces the path to the destinations of the dots. It is not clear which dot the students started drawing the path for; a closer look at the original shows two sets of star patterns one in blue ink and the other in pencil. I con-



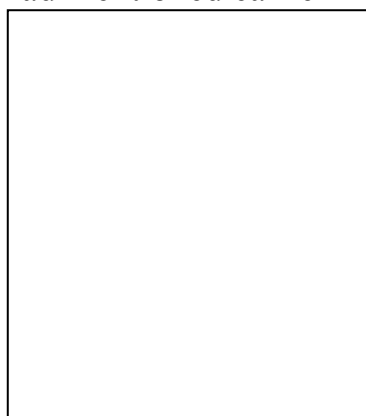
The dots will never meet as they move in the same pattern. When one of the dots catches up with the other, the other one will just move away from it again and this results in an unbreakable pattern. We therefore believe that this is a trick-question.

ture that the blue ink and

pencil paths being the same made the students articulate ‘same pattern’ in their conclusion. The students in this group seem to however recognise that the **patterns in themselves do not establish** that the dots do not meet and explain how such an eventuality results from the dots moving away from each other resulting in the pattern being unbreakable.

They do not hide their (dis)belief in their response in which they consider the question as a trick-question. The hyphen in the word also evidences the possibility of an implicit categorisation of question by them.

Group F was one of the two groups (other than Group C) that on requested by Olaf and Knut to explain their solution at the blackboard. I had mentioned earlier in Chapter 4 how it was during this time in teaching-learning that the students were being asked to first speak for themselves at the blackboard and then also on behalf of their group as well. The diagram offered by the group does not indicate any specific method beyond dots and dashes drawn while discussion but not representative of anything significant in particular. Their other written conjecture evidence that students designated corners with numbers:



black = 0. When Ulrik offered the solution on behalf on Group F at the blackboard he questioned the conclusion reached by other groups that the dots never meet. His group had concluded that the black and white dots do get together **if the white moves first**.

The conduct of the above group task and the next was separated by one teaching period. I now turn to discuss the second.

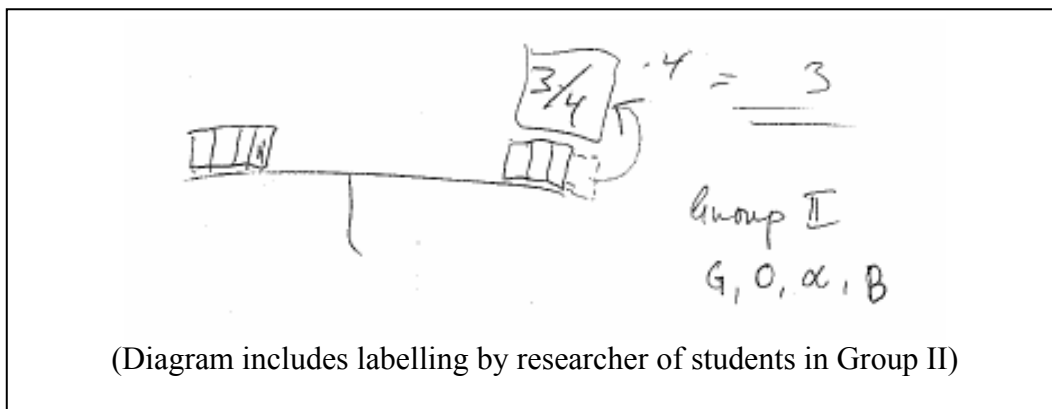
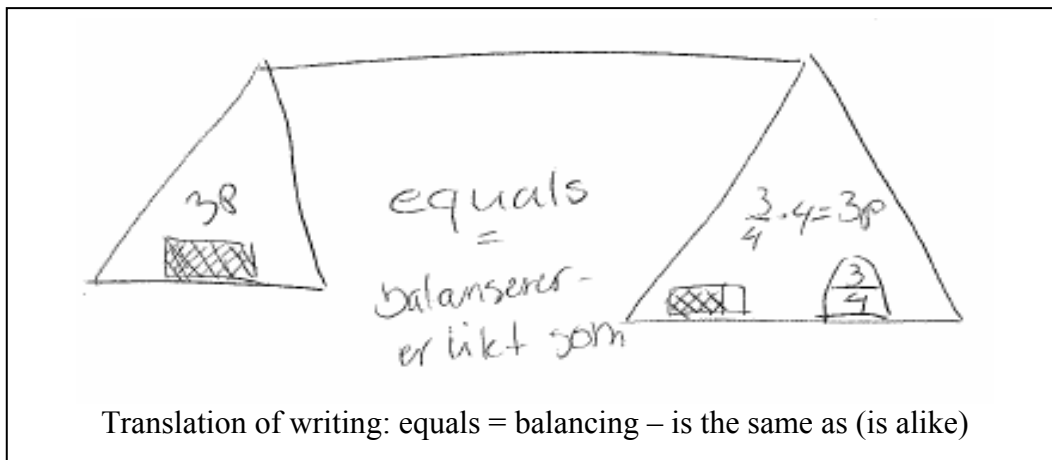
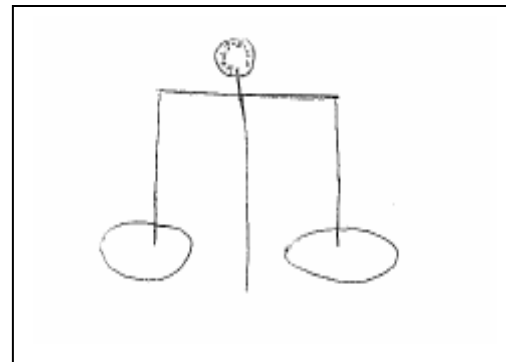
How heavy: representing to cooperate

As before the group-task I discuss and give below was handed over individually to the students seated in their respective groups.

How heavy?

If a brick balances with three-quarters of a brick and three quarters of a pound, then how much does the brick weigh?

Unlike the unique solutions to the previous group-task the working of students at this group-task lent themselves to a categorisation. The three categories were: not algebraic (X) becoming algebraic (Y) and algebraic (Z). I present below the three categories in succession and begin with data contributing to the **not algebraic** category X.



If the brick balances with $\frac{3}{4}$ of itself + $\frac{3}{4}$ of a pound, then $\frac{1}{4}$ of the brick balances with $\frac{3}{4}$ lbs.

$$\frac{3}{4} \text{ lbs.} \cdot 4 = \underline{\underline{3 \text{ lbs.}}}$$

Answer: The brick weighs 3 lbs.

The first three diagrams forming category X, evidence a beam balance with two weighing pans. There is evidence of the equality of $\frac{3}{4}$ (0.75) of four (pounds) and three pounds as in the first diagram. In the third diagram the word 'equals' in English is equated with the Norwegian word 'balanserer'. In the fourth there is evidence of equating the brick in one pan with a brick with a missing quarter and a $\frac{3}{4}$ pound weight in the other pan. Gard was one of the two students who was asked to present his groups solution at the blackboard and explained:

$\frac{1}{4}$ th missing must be equal to $\frac{3}{4}$ of a pound.

Therefore $\frac{3}{4}$ times 4 was equal to 3 pounds.

The above argument seems to be based on the following reasoning:

IF: The brick balances with $\frac{3}{4}$ of itself + $\frac{3}{4}$ of a pound

THEN: $\frac{1}{4}$ of the brick balances with $\frac{3}{4}$ pounds.

THEREFORE: $\frac{3}{4}$ pounds times 4 is equal to 3 pounds

ANSWER: The brick weighs 3 pounds

In the fifth diagram and solution the beam balance is absent as a diagram, yet seems represented in the solution since there is nonetheless evidence of the use of the idea of a balance. I conjecture that the balance as an intellectual artefact is acting as a **psychological tool**.

I now present the only 'becoming algebraic' solution (Y) discussed by Anja, Lea, Stine and Egil in my group-in-focus. The three diagrams I offer are culled from Egil's and Stine's worksheet. Towards understanding the working of Egil and Stine and the other group members I draw from my field notes and record Lea asking: How much is a pound?

$$1 \text{ brick} = \frac{3}{4} \text{ brick} + \frac{3}{4} \text{ pound}$$

$$\text{⊕} = \text{⊗} = \underline{\underline{3 \text{ pound}}}$$

I record no takers for her question but her subsequent assertion that the brick was x accompanied by her tapping her finger on her table. Egil expressed himself by equating a brick to $\frac{3}{4}$ a brick and $\frac{3}{4}$ a pound. In

$$\text{brick} = \text{brick} + \text{pound}$$

$$x = \text{brick}$$

$$x = x \cdot \frac{3}{4} + x \cdot \frac{1}{4}$$

$$\frac{3}{4} \cdot 4 = \frac{12}{4} = \underline{3 \text{ pounds}}$$

their representations expressing equality Egil and Stine evidence their attempts at ‘becoming algebraic’ though neither is accurate or successful. I may have missed some discussion in the Norwegian, but there is evidence of the group equating the missing quarter of the brick as three quarters of the pound, and the weight of the brick as three pounds. It is the evident

struggle with their personal and algebraic representations that makes me categorise their attempt as becoming algebraic.

I finally turn to offer the two solutions that I have **categorised as algebraic (Z)** before following them with my analysis.

$$\frac{4}{4}b = \frac{3}{4}b + \frac{3}{4}$$

$$\frac{4}{4}b - \frac{3}{4}b = \frac{3}{4}$$

$$\frac{1}{4}b = \frac{3}{4}$$

$$\boxed{1b = 3}$$

$$B = \frac{3}{4}B + \frac{3}{4}P$$

$$B - \frac{3}{4}B = \frac{3}{4}P$$

$$\frac{1}{4}B = \frac{3}{4}P$$

$$\frac{1}{4}B \cdot 4 = \frac{3}{4}P \cdot 4$$

$$\underline{\underline{B = 3P}}$$

In the two solutions offered above there is a **visible use of algebra**. In the first (left) the equation is clearly stated followed by an appropriate use of the LCM to arrive at the weight of the brick. I read the answer that is highlighted in a box $1b = 3$ as expressing one brick weighs 3 pounds. In the second attempt (right) the brick is designated as B and the standard pound weight as P . The algebraic solution finally arrived at is

$B = 3P$ or the brick weighs three pounds. As mentioned before in Chapter 4, it was the second solution which was offered by Tia at the blackboard upon the request of Olaf. I had also explained there how Olaf used the above solution to start the sub-topic of simple equations. I now turn to summarising the first case of students' cooperation in groups.

Brief summary of the first case

The resolution of: *When together*, by various groups was governed by the pentagon or representation and the rules that accompanied the group-task. The attempts of students evidenced diversity. There was no use of any particular method and evidence of a **reliance on pattern and common sense**. In two instances the students redrew the given representation to a larger size or a sequence of the same. Their written arguments evidence a dependence on the **given representation** and the concrete experiences that accompanied the attempts of the students at the same.

In the conduct of the task the motives of Olaf and Knut in **having their students to collaborate** were met. The seemingly fun or trick task allowed students to elicit and demonstrate the **existence of patterns** and draw them into discussion and debate, leading to **verbalisation and argumentation** of the problem and its resolution.

The goals in the group-task: *How heavy* were different. No diagram or rules were offered as part of the group-task. The students **drew diagrams to represent and communicate** their understanding. Their representations taken together allowed for a categorisation, evidencing their everyday experiences and algebra on the basis of which they proceeded to arrive at a solution. In one of the written solutions there was evidence of the 'balance' as represented by many others as a diagram acting as a **psychological tool**, evidenced in the written argument offered.

In the conduct of the above group-task, Olaf was able to have students **consolidate** for the first time **the practice of cooperating**. It was in the succession of the two group-tasks with which Olaf and Knut achieved their objectives of having students cooperate in groups. I offer analysis of the above two tasks as 'activity' in the concluding discussion and presently turn to my second case of cooperation.

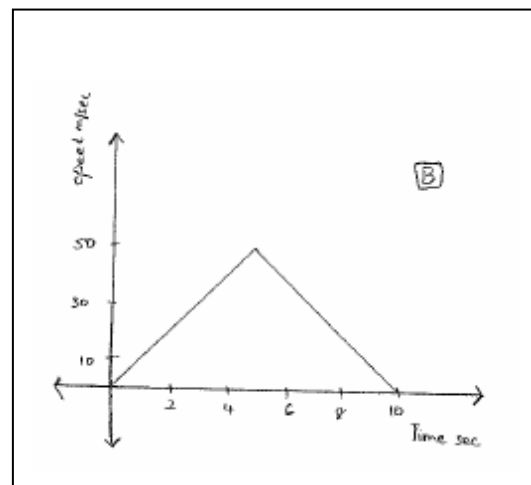
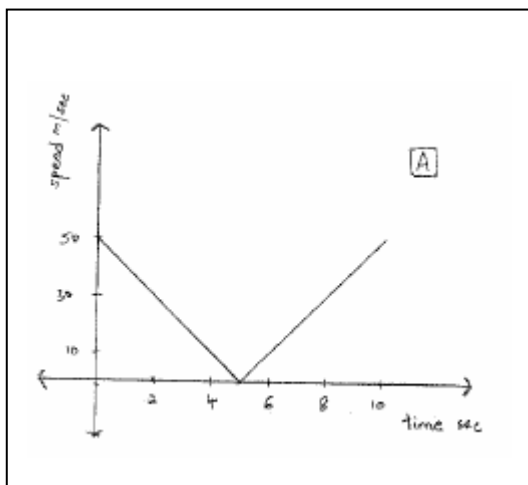
Two bodies in motion

The group-task related to my second case as presented and conducted with Anja, Egil, Lea and Stine is given in Appendix A.20. Towards my discussion relating to the cooperation of students I present below an extract of the task and their arguments in three sub-sections: identification of the two graphs given, movement of elevator between five levels and ball thrown on the moon. As different from the data I have offered so far I present and draw from transcripts of audio-recordings. I present the transcript in four columns: time, person, turn and utterance.

What can we say from the given graphs?

- You will be given two graphs A and B.
- The graphs relate to the motion of two different bodies.
 - One of the graphs describes the motion of an elevator travelling between two floors.
 - The other graph describes the motion of a ball thrown up in the air and caught on its return.
- Discuss the following:
 - Which of the two graphs given to you shows the movements of the elevator and ball mentioned above and why?
 - Refer to graph A and explain what may be happening at the points marked X, Y and Z.
 - Refer to graph B and explain what may be happening at the points marked P, Q and R.
 - In graph A what is the significance of the point marked Y?
 - In graph B what is the significance of the point marked Q?
 - In what ways are the two graphs A and B different?
 - In what ways are the two graphs A and B similar?
 - The graph of the elevator shows its motion between two floors; across one level. How will the graph differ if it were to show the elevator moving between six floors or across five levels?
 - The graph of the ball shows its motion when thrown from your play ground. How would the graph differ if the ball were thrown from the surface of the Moon instead of the Earth?
- At the end of the task session lasting 30 minutes, you will be given 5 minutes to share and summarise your experience with the task.
- (The values of speed and time in the graphs are not actual but for the sake of discussion.)

I offer below a schematic of the two graphs referred to as A and B in the above task. The two graphs were given to the students on A4 size sheets with the possibility for them to draw on those very graphs. Copies of the graphs presented to the students together with their working are offered along with my discussion of their attempts.



Which graph is which: reasoning with real life experiences

As can be seen from the design of the group-task, the first goal set for the students was to identify and distinguish the two graphs marked A and B with the movement of two bodies in motion: the ball and the elevator. The cooperation of the students and their arguments towards this goal were driven largely by Egil as can be seen below:

Time	Person	Turn	Utterance
03:59	RES	37	That is A and that is B
04:07	Stine	38	Elevator
04:15	Lea	39	Hmm, hmm ... and that's the ball
04:17	Anja	40	And that's the ball
04:18	Egil	41	No, that's the ball that's the ball because it starts fast and mm mm – ((referring to graph A))
	Egil	42	--arh if you throw up a ball it will go up like this ... this you know
	Egil	43	--and it will kind of stop like this
04:30	Anja	44	On the ground?
	Egil	45	On the top
04:31	Egil	46	On the top and then it will fall down again
04:34	Anja	47	((Inaudible))
04:37	Egil	48	It says here ... ball thrown up in the air
	Egil	49	You throw it and it sort of stops and falls down again
04:44	Egil	50	So I think this is the ball definitely
04:46	Egil	51	And because that ... arh the elevator will increase in speed
04:50	Anja	52	-- and stop
04:51	Egil	53	It'll stop and then it'll go up again
04:58	RES	54	What do you think? Is that OK?
05:00	Many	55	Yes, ya

In the above extract there is evidence that Egil was in **disagreement with Anja** (40-41). He asserted that the motion in Graph A represented the ball because 'it' (the ball) started fast (41). His assertion in turn evidenced his recognition that the y-axis in the graph represents the velocity, a claim strengthened by the fact that he mentions that the ball would go up and stop 'like this' (42-43). The two points being referred to by Egil were the points marked X (where the ball starts fast) and point Y (where the ball is at the top). At the point marked Y when Egil said the ball would stop (43) Anja asked if that position would be located on the ground (44). Egil explained that after reaching the top, the ball would fall down by gathering speed (45-46) concluding that Graph A was definitely the ball (48-50). The above extract evidences that Egil was able to **relate the actual motion of the ball to its representation** on the graph.

After **convincing the others** that Graph A depicted the motion of the ball, Egil explained the motion of Graph B as that of the elevator. He brought his experience of the increase in the speed of an elevator to the graph, and concluded that the elevator will stop halfway and go 'up'

again (51-53). While Egil argued that it was the point Y which was the topmost point the ball would reach when thrown in the air, Anja seemed to argue that this point (located on the x-axis in graph A) was on the ground. Similarly Egil argued that the elevator will increase in speed till it stopped, which could refer either to the halting of the elevator or a stop in the increase in speed so as to come to a halt. In such an eventuality the next level reached by the elevator would be ‘up’.

By the time and turn of the arguments above I draw attention to the **use of words** like top, stop and up by the students in correlating their personal experience to the two motions represented. However it is not evident that there is a shared understanding as yet on whether the words used by them refer to the same locations on the graph or the movements that are being referred to. Though Stine, Lea and Anja agree with Egil’s arguments in turn 55 above it is **only later** that two of them begin to offer their **agreement**. It is in turn 73 that Stine offered the following:

06:10 Stine 73 Now I understand what you meant earlier ‘cause you throw it up

The agreement by Lea was only subsequent to Stine as below:

06:44 Egil 91 And here it falls down
RES 92 OK
06:45 Lea 93 And ... and then you catch it
06:46 Egil 94 And then at point Z you catch it

On her part Anja still had doubts about the whole argument being agreed upon by the others. She asserted as below that the ball started at zero speed. To Anja’s expressed doubts I offer Egil and Stine’s responses:

Time	Person	Turn	Utterance
07:12	Anja	106	[That] is at least the ball ... when you throw it up it starts at zero and increases
07:15	Egil	107	[No it doesn’t ‘cause when you throw it starts with a speed]
	Stine	108	[No, no, no because you throw it you know]
07:18	Egil	109	You don’t start from ... like a car if you start a car
07:21	Stine	110	[Then it goes ((makes a sound ranging from low pitch to high pitch))]
	Egil	111	[Then it goes ((makes a sound ranging from low pitch to high pitch))]
07:23	Egil	112	But a ball goes like ...
07:25	Stine	113	[@@]
	Lea	114	[@@]

The above extract brings in an element that I had not anticipated as an ‘argument’. There is evidence of at first Egil and then Stine calling upon their personal and common knowledge of the **sound of a car accelerat-**

ing in argument. While wanting to argue that the ball does not start with ‘no’ speed, Egil contrasted the movement of the ball to a car (109). Egil and Stine then made a sound from a low pitch to a high pitch showing how a car would sound when it accelerated from start (110-111). Egil then argued ‘but a ball’ does not do so (112) intending to convey that a ball does not accelerate in the same manner as a car whose sound he and Stine were making. It is with these arguments that Egil argues that unlike a car which starts from zero and increases in speed, a ball starts at a certain speed and stops on reaching zero speed.

The creative use of sound in **expressing personal experiences to convey and share meaning so as to convince others** was called upon again (as below) to explain the movement of the elevator between floors.

Time	Person	Turn	Utterance
07:32	Egil	119	So Graph B then
	RES	120	Ya
07:38	Anja	121	OK the P is when it’s on the first floor
07:43	RES	122	So what about P, Q and R
07:49	Egil	123	Well, here it increases arh an elevator well is similar to the car it will start and it goes faster and faster...
	Egil	124	And then it will stop and then it will arh ...
	Lea	125	((Inaudible)) it will slow down
	Egil	126	Ya
08:13	RES	127	So P is when you say it is starting?
	Egil	128	Ya ... and Q is ...
	Stine	129	[When it is at the top]
	Anja	130	[When it is at the top]
08:19	Anja	131	Stops and then
08:28	Egil	132	Ya, ‘cause the elevator has to slow down in order to stop at the next floor
08:33	Lea	133	((Makes sound of an elevator between two floors, from low pitch to high and then from high pitch to low))
	Egil	134	So this must be first floor
	Anja	135	second floor
	Egil	136	And this the second floor
	Anja	137	Arh ya the second floor
08:41	Egil	138	Cause, this is when it holds top speed and then it has to
08:45			slow down in order to ...
	Anja	139	OK
	Egil	140	Stop at the second floor
	Lea	141	<i>Doying!</i> ((Sound of elevator coming to a halt))

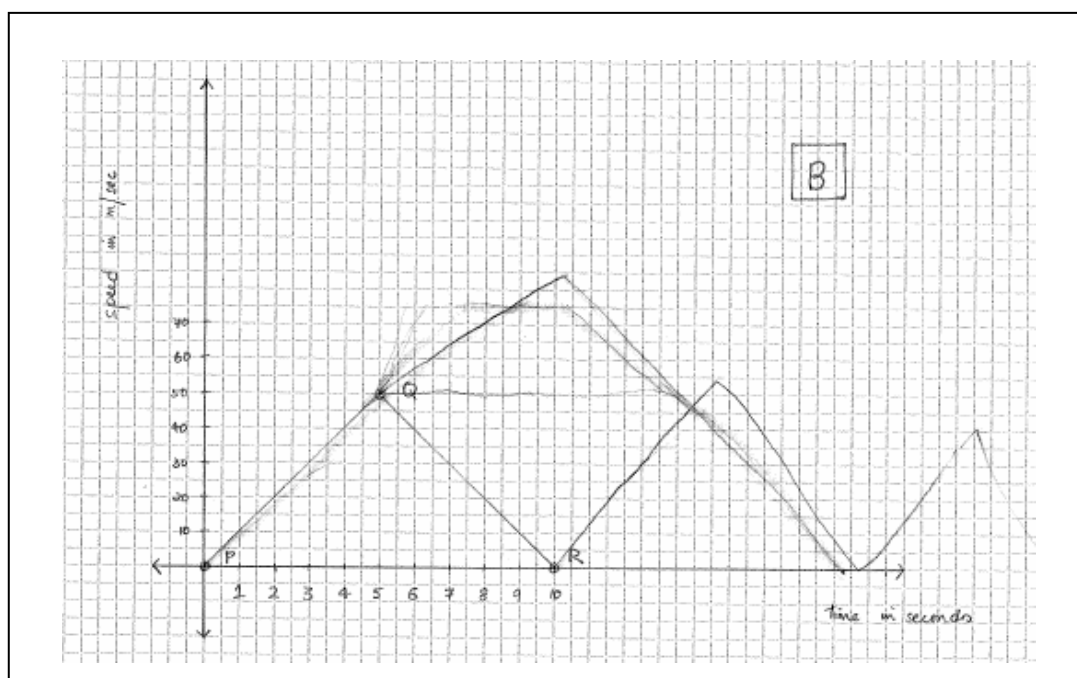
Apart from the use of sound which I shortly discuss the above extract also evidences a shift in Anja’s approach. As different from arguing with the others and observing the others to agree, Anja **explained the graph to herself** in concrete terms (121). Towards assisting Anja, Egil recalled his example of a car and explained that an elevator is similar to a car in that it increases in speed after its starts from a stationary position (123).

On my asking if P was the location where Egil thought the elevator was starting (127) Egil agreed and continued with what he thought would happen at Q (128). Stine and Anja's **agreement** with Egil is again evidenced by their completing Egil's utterance (129-130). Anja continued with her attempt to understand the graphs in her asking if the elevator stopped at Q what would happen to the motion of the elevator then (131). It at this juncture that Lea **relied on another sound** to explain how the elevator first increased in speed (like the car) and then decreased in speed to come to a stop (133). Lea **appropriated a technique** that Egil and Stine put to effective use earlier to make a convincing and 'sound' argument. Just as Egil referred to the elevator halting (140), Lea used '**Doying!**' to express the same experience in sound.

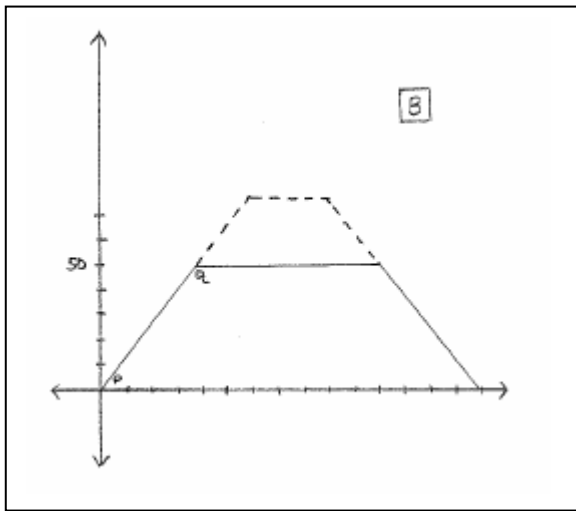
The two sounds used by the students provide for interesting analysis. While the use of the **first sound** was to **share personal meaning** so as to convey and convince others, by the time of the **second sound** its use became an **accepted strategy** in their cooperation. I evidence below how Lea's sound of the elevator coming to a halt (*Doying!*) was further used to explain the movement of the elevator between floors.

Movement of elevator: in terms of the graph

Having identified the two graphs by explaining the motions they represented with sounds in addition to words, in this sub-section I elaborate the discussion had by the students while addressing how the graph of an elevator moving between six floors or across five levels would look like. In addition to the arguments that the students would present orally, I encouraged the students to represent their conjectures graphically as well. I present their graph below and follow the same with discussion.



In analysing the attempts of my students in the above graph I extract a relevant redrawing of mine (shown below) to keep my discussion focused on the arguments brought made by the students. Egil was the first to



conjecture that for the elevator to cross five levels it would have to hold top speed longer. Incidentally the use of the words ‘top speed’ by Egil referring to the peak speed that the elevator could reach can be traced to a literal translation of ‘toppfart’ in Norwegian. I record an agreement between the students and prior to the above drawing that the top speed reached by the elevator represented in the given

graph was 50 meters per second. Upon Egil’s conjecture that the elevator would hold top speed, Lea drew a graph representing an elevator holding top speed of about 75 meters per second (dotted line in redrawing). In response Egil argued that the movement of the elevator between five floors would not change its top speed, making him draw his version with a top speed of 50 meters per second (solid line in redrawing). Upon his drawing Egil **sought the approval** of others.

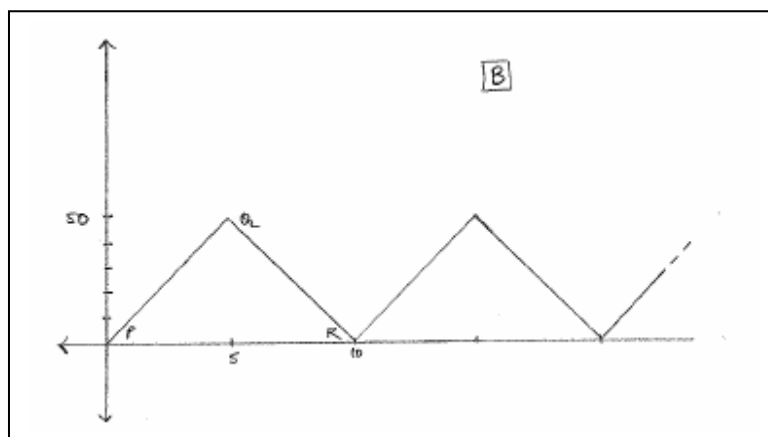
At this point of time when asked by me if Egil’s solution was the only situation, Anja **considered what might happen** if the elevator did stop between two floors. I offer the discussion that followed:

Time	Person	Turn	Utterance
	Anja	253	But that doesn’t stop between each floor couldn’t it be that it stops on all the floors ((referring to Egil’s graph))
15:13	Stine	254	No
15:18	Egil	255	It doesn’t stop in all floors on its way
	RES	256	Suppose it did
15:20	Lea	257	And then it will go <i>Doying!</i> ... <i>Doying!</i> ... <i>Doying!</i>
15:22	Egil	258	And it will be very ...
15:23	RES	259	And then it would be what <i>Doying!</i> .. <i>Doying!</i> ... <i>Doying!</i>
15:24	All	260	@@@@
15:26	Lea	261	Up and down and up and down ((in a sing song manner))
15:29	RES	262	Yes ... how would that be
15:33	Egil	263	You could really just copy the P ... P, Q and R

The above extract evidences how Egil, drawing from his personal experience, mentions that an elevator did not stop at the all floors while crossing many levels at a time (253). Anticipating a graph different from the ones drawn until then, when asked what if the elevator did stop at all the floors in-between (256) Lea responded with her newly acquired and

preferred mode of communication: **a sequence of ‘Doying!’ sounds** (257) with an explanation of what she meant later (261). On being pursued again about how such an eventuality would like (262) Egil’s responded that one had **to just copy P, Q and R** (263).

In the extract offered above there is evidence of both Lea and Egil using constructs (sound of the elevator halting, *Doying!* and P, Q and R) to **mediate their experiences from real life**. However and in addition to



the sound conveying meaning to the four students in the group, I conjecture that the **usage of sound** in their arguments was **also successful**. Having succeeded in using the first sound to

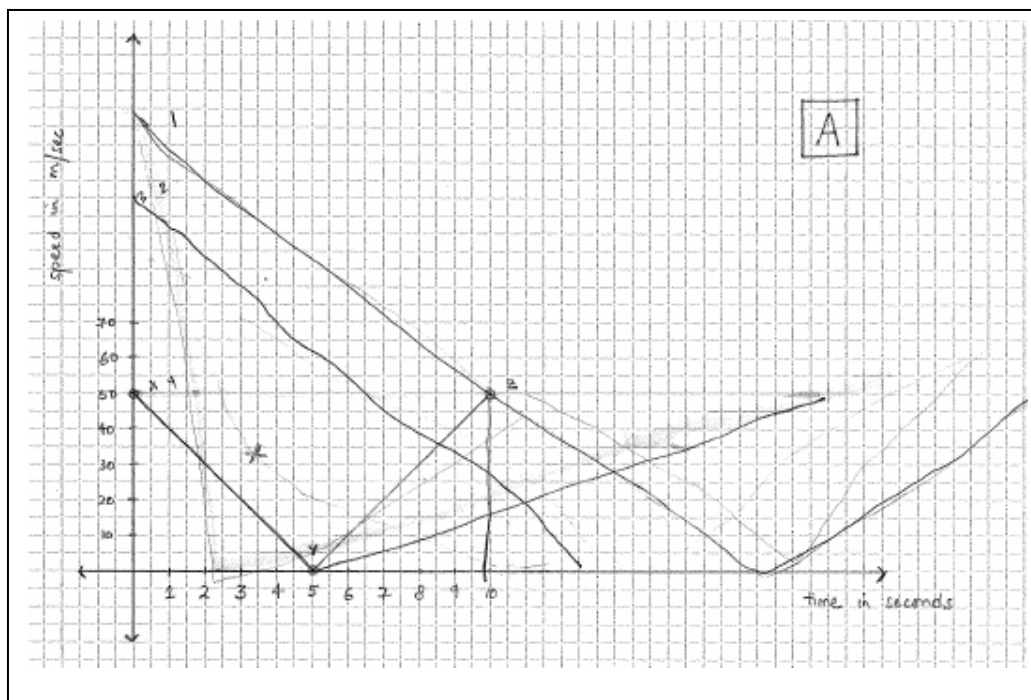
convince the others in their group, the students experimented and used a second sound towards addressing the next question at hand.

The arguments that I have presented so far bring to discussion many points of interest. In the first they evidence the existence of very little intersubjectivity among the students at the graphs given in the group-task at the commencement of the group-task. Shared understanding was arrived at gradually and by bringing personal experiences of the students to the graphs given. It is only upon incorporation of personal experiences and subsequent arguments that the graphs began to represent the two situations that were in the group-task. It seems fair to conclude that the given **graphs as intellectual artefacts** began to mediate meaning and act as a psychological tool for the students **only upon the actions** of the students **towards the goals provided by the task**.

The **presence of shared meaning** is the pursuit of the goals that were set is the second point of interest. This is evidenced by the implementation of sound in convincing others towards one’s point of view, followed by a repetition of a similar strategy within a short duration of time. The successful use of sound leads me to the third and fourth arguments. I conjecture that the use of the ‘*Doying!*’ sound by Lea is yet another instance of **re-appropriation**. Upon observing the success of the first sound in the sharing of understanding, Lea used the presence of the new artefact to meet her own goals by generating another. Finally and as pointed out before, there was evidence of **individual preference in the use of mediational means**. Just as Egil spoke-with-the-graph, Stine, Egil and Lea spoke at different times with the sounds they made.

Ball thrown on the moon: experience found wanting

I present in the last sub-section, the arguments of students at the question of how the graph of the ball would look like if the ball were thrown up on the Moon instead of the Earth. The attempts of students to this question did not reach any conclusion agreeable to everyone. Though the two graphs presented with the group-task had begun to be used quite effectively as an intellectual artefact to mediate their personal meaning, in the attempts of students at this question **many new and conceptual issues became part of the problem** such as gravity on moon and earth, force, strength and top speed. I present below the graph of the students followed by an assortment of utterances indicative of the kind of arguments that were made by the students in their attempts.



In sharing the assortment of utterances made by the students (below) I attempt to make no cohesive argument since none was made, as also evidenced by the graph above. Though the use of the graph itself did not limit the students, I conjecture that any **lack of common and shared understanding** between the students was due to a **lack of** two practical aspects: firstly, a **personal experience** related to the new situation (gravity on the moon) and secondly a **common sharing of these aspects** as was possible 'with' the sound of the car and elevator.

16:07	Anja	268	Well the moon has less [gravity] than the earth
	Lea	269	[gravity]
16:24	Anja	276	Then it would have to start higher
16:35	Egil	280	Well there is this gravity on the Moon and on the Earth, it will fall well it will go much higher and it will fall slower
16:43	Egil	282	It would go up ... slowly and then I think it will fall slower

17:16	Egil	292	It would take a longer time for the ball to go up and will take longer time for it to fall
17:26	Egil	298	You know the astronauts they are jumping you know should they just be flying away
18:31	Stine	322	It will be much more like long
19:43	Egil	345	[I think it will finally]
		346	[Why does it take longer] ... then to go up it will go faster cause then it goes <i>Doying!</i>
21:38	Anja	389	It will go choooooh!
21:50	Anja	393	You throw it just like you did on Earth
22:04	Lea	396	What is it that we have done
	All	397	@@@@
22:42	Anja	411	But when you throw it up then its at fifty
23:52	Egil	438	The gravity is very strong on earth and ...if you throw the ball up it will fall down quite fast
23:56	Egil	439	While on the moon the gravity is not so strong ... so it would take more, long time
24:11	Anja	442	But actually I think that is the ball ((referring to Graph B))
25:02	Egil	454	But it won't increase in speed, it will, it will decrease in speed because it holds top speed at the moment you throw it
26:36	Egil	487	Almost the same speed except that it changes all the time
27:00	Egil	496	But we can agree that a ball when you throw it up then it will have greater speed than on the Earth
27:03	Egil	497	And when it comes down it will have less speed than on the Earth
27:14	Egil	502	You'll have less speed on the way down on the moon than on the Earth
27:29	Egil	508	I think this is really quite confusing @

Brief summary of the second case

In the second case of cooperation of students towards the goals of the group-task many aspects surfaced simultaneously which I discuss.

All along the discourse that I have offered, it was possible to identify the kind of argumentation that took place between the students. There was evidence of **agreement and disagreement, and of consciousness** on behalf of the students that that there was agreement and disagreement. This was evidenced by the stance taken by Egil to convince Anja and the stance taken later by Anja to explain the situation to herself when she found she that all others except she had reached an understanding. A crucial part played in the reaching upon of understanding was what the words were referring to. It was only when the use of words like top, stop, down and up referred to the same locations or aspects and mediated respective meaning that there was the **existence of intersubjectivity** which led to **sharing of understanding** and **reaching of agreement**. The aspects of discourse that were associated with the above were convincing others, convincing one self and considering a stated position.

The use of personal experience to both understand and explain the given task was the other significant feature. It was only when it was possible to share personal experience that it was possible **to bring meaning to the given representation** (graph) of the situation explained. The **lack of such an experience** (on the moon) hampered both the understanding of the situation expressed and the corresponding resolution of the question in the group-task. As a consequence the **lack of a shared experience** in relation to the same was also a deterrent to understanding.

The **effective and creative use of sound** came along with the actions of sharing one's understanding. The sound of the accelerating car so embodied in the students was used by them to articulate their opinions. This instance was so successful that the use of sound even became a strategy in the short duration of the group-task to make further arguments. In such use the **re-appropriation** of an earlier strategy was evidenced. It was with the goal of convincing others to one's own point of view that a choice of mediational means was also evidenced. It was the use of sound by one and the use of the graph by the other that offered **agency** or the enhanced opportunity to **speak-with-graph** and **speak-with-sound**.

It was only on considerable and **conscious actions of behalf of the students** as described above that the graph became a mediational means. This involved a two part process of first bringing personal experiences to the graph and then **speaking of personal experiences through the graph**. The incidence of the later was evidenced by arguments being made more in terms of the graph. That the graphs began to mediate meaning or the realities, they did or did not represent, came with the actions of students in pursuit of the goals of the group-task.

In their cooperation towards goals there was evidence of understanding emerge because of the actions and utterances that helped raise the awareness and arguments of the others. However in achieving this nature of cooperation, the coming upon and reaching of a shared understanding was crucial. This was possible in a gradual manner and depended upon an **increase in intersubjectivity**, a perception of the **shared understanding** and **creation of artefacts** where found necessary.

SA/V ratio and metabolism

The third and final case with which I discuss cooperation between students was at another group-task which as presented to the students is given in Appendix A.21. I offer an extract of the task below and the attempts of Dan, Levi and Thor at the group-task subsequently in three subsections. I first elaborate the obtaining and inferring from the ratios of surface area (SA) and volume (V). I then relate students questioning reality based upon the ratio arrived. I finally discuss arguments based on the ratio about the nature of cells which could have better metabolism.

What can we find out about surface area and volume?

For any given sphere of radius r

- the surface area (SA) is given by the formula $4\pi r^2$
- the volume (V) is given by the formula $\frac{4\pi r^3}{3}$

Fill in the table below for spheres of different radii:

SN	Radius (r_x)	Surface Area (SA_x) in 2 decimals	Volume (V_x) in 2 decimals	$\frac{SA_x}{V_x}$
1	1 unit			
2	2 units			
3	3 units			
4	4 units			
5	5 units			
6	6 units			

Compare and calculate the following ratios for a sphere of radius 5 units (r_5) and a unit sphere of radius 1 unit (r_1):

$$r_5 / r_1 = \underline{\hspace{2cm}}$$

$$SA_5 / SA_1 = \underline{\hspace{2cm}}$$

$$V_5 / V_1 = \underline{\hspace{2cm}}$$

In what way is the ratio SA_5 / SA_1 and V_5 / V_1 related to r_5 / r_1 .

How does the ratio of the surface-area-to-volume (SA / V) change as the measure of the radius increases?

Work with the formulae for Surface Area and Volume and express (SA / V) as a ratio.

Does the ratio just calculated agree with your results in the table above?

If we assume that living cells in our bodies are spherical what happens to the surface-area-to-volume (SA/ V) ratio as the cells get larger and larger? Why?

For any living cell, metabolism is the rate of chemical activity in the cell. Metabolism maintains life. For metabolism to take place materials like oxygen and water need to be absorbed. Of the two values Surface Area and Volume:

- Which value will you think determines the cell's metabolism:
- Which value controls how much material gets in and out of the cell:

For a spherical cell how would a larger surface-area-to-volume (SA / V) affect its metabolism?

Do you think it is advisable for organisms to have large cells or small ones?

Why?

Surface Area and Volume ratio: intuitive

In this sub-section I relate the arguments of my students towards the goal of finding the surface area (SA) to volume (V) ratio. Of the two stories that emerge in cooperation, I deal with the sharing of responsibilities at a later stage and follow in the current extracts the goal of finding the ratio. At the commencement of the group-task I record Dan asking:

01:58	Dan	13	Are we gonna work together or --
	RES	14	Together, together
	Dan	15	OK

After the above clarification the table given was filled in as below:

- Fill in the table below for spheres of different radii:

SN	Radius (r_x)	Surface Area (SA_x) in 2 decimals	Volume (V_x) in 2 decimals	$\frac{SA_x}{V_x}$
1	1 unit	12,57	4,19	3
2	2 units	50,27	33,51	1,5
3	3 units	113,09	113,1	1
4	4 units	201,06	269,09	0,75
5	5 units	314,16	523,6	0,6
6	6 units	452,39	904,78	0,5

While filling the table with required values of surface area and volume for increasing radii, Levi had an **initial conjecture**:

05:19 Levi 67 That's pretty wrong ... it's hmm in the power of three

On filling in some more values he **finalised his pattern** and observed:

05:56 Levi 79 One is to the power of two and the other to the power three

On my suggestion that the students share the responsibility of carrying out the required calculations, Levi decided to take 'Volume' and:

06:24 Thor 91 Ok, I'll take up surface area

06:25 Dan 92 I can watch

All 93 @@@

Levi 94 Exactly!

Thor 95 You'll observe!

The above 'sharing' notwithstanding, Levi soon asked:

06:51 Levi 99 Is there any connection here

It was while filling in the last column (SA/V) that Dan suggested:

11:12 Dan 159 You have to ... like ... round off

By the time of filling the SA/V ratio for a radius of 6 units:

11:30 Levi 164 That should be five

The above utterances evidence that along with Levi filling in the table with values (67) he was simultaneously conjecturing about their nature (79). Levi then drew upon the values he had filled in (99) and by the time he filled in the SA/V for a radius of 6 units, mentioned that its value 'should' be five (164). I would like to reflect on the nature of the current responses of Levi in light of the **intuitive nature** of student responses by this time of teaching-learning as evidenced in Chapter 5. I conjecture that the manner in which the ratio was arrived at by the students was intuitive, as was their responses to the next two questions:

- Compare and calculate the following ratios for a sphere of radius 5 units (r_5) and a unit sphere of radius 1 unit (r_1):
 - $r_5 / r_1 = \frac{5}{1} = 5$
 - $SA_5 / SA_1 = \frac{100}{4} = 25$
 - $V_5 / V_1 = \frac{125}{1} = 125$
- In what way is the ratio SA_5 / SA_1 and V_5 / V_1 related to r_5 / r_1 .

Upon finding the ratio of radii 5/1 as 5, Levi and Thor rounded off the fractional values they obtained in the quotient of $314.16/12.57 (= 24.992)$ as 25, and $523.6/4.19 (= 124.964)$ as 125. They then addressed in what 'way' the ratio of SA_5 / SA_1 and V_5/V_1 and r_5 / r_1 were related in algebraic terms. Levi **summarised his observations with ease** as:

14:28 Levi 240 x ... x in two and x in three

Levi and Thor then responded to the next three questions as below:

- How does the ratio of the surface-area-to-volume (SA / V) change as the measure of the radius increases? $\frac{3}{V}$
- Work with the formulae for Surface Area and Volume and express (SA / V) as a ratio: $\frac{3}{r}$
- Does the ratio just calculated agree with your results in the table above? $\frac{4\pi r^2}{\frac{4}{3}\pi r^3}$

Towards completing worksheet A, the students then discussed the question of how the ratio of SA/V changed as a measure of the radius:

Time	Person	Turn	Utterance
14:49	Thor	248	it lowers , it lowers [down]
	Levi	249	[half]
14:50	Thor	250	[Half]
	Levi	251	[There] it is half ((Ratio of 1.5 compared to 3))
	RES	252	hm hm
14:55	Levi	253	Its hm..one third ((Ratio of 1 compared to 3))
	RES	254	hm hm
14:58	Levi	255	One fourth ((Ratio of 0.75 to 3))

As with the calculation of the numerical ratios and algebraic ratios the observation of the change in the ratios was done with ease by the students. Subsequent to the above the students also agreed that the **algebraic ratio** they had obtained **agreed with the numerical values** they had found. I now turn to arguments related to the term metabolism.

Metabolism: questioning reality

In proceeding with the goals of the group-task the students wished to know what metabolism was. I explained briefly that **metabolism** was an energy giving chemical reaction inside living cells. Following my expla-

nation, I let Thor who had some idea of what the term meant, explain the same in Norwegian to the others. The question of the application of this concept to living cells was the next goal. Of the two parameters surface area and volume being discussed and compared so far, the above explanation of the term metabolism led the students to agree that it was the **volume of the cell** that determined a **cell's metabolism**. As a consequence (they said the second choice was obvious) they agreed that it was the **surface area of the cells** that controlled the **exchange of materials** like oxygen and water into the cells. I offer below their discussion on how the SA/V ratio would affect the metabolism of a spherical cell.

Time	Person	Turn	Utterance
	RES	472	volume, how would that affect metabolism
23:32	Levi	473	it will be lesser ... energy
23:36	RES	474	lesser energy if, if the ratio is less [or if] the ratio is more
23:41	Thor	475	[ya]
	Levi	476	[ya]
	Thor	477	Oh what you mean is if the bigger the cells are
	RES	478	Hm mm
23:50	Thor	479	the worse you have it
	RES	480	Right why do say so
23:55	Thor	481	Because you just said that the bigger the radius is..
	RES	482	Hm hm
	Thor	483	the lower the metabolism effect is

The above extract evidences the conclusion drawn by Thor (477, 479). He goes on to explain what happens when the radius of a spherical cell decreases and makes the connection that the **nature of metabolism is related to the dimension of the radius** (481, 483). On my wanting to ascertain if the others agreed with Thor's conclusion, Dan responded:

24:12	Dan	488	I agree
	All	489	@@@

On his part Levi argued as below:

Time	Person	Turn	Utterance
24:28	Levi	497	I agree
	RES	498	you agree.. why?
24:30	Levi	499	because metabolism is good and the metabolism is big with while when the radius is low
24:32	RES	500	Ok and why is that
	Levi	501	I don't know @@ I have no clue
24:40	RES	502	@ No no you have it
24:41	Thor	503	@ it is from you said you said SA / V
	RES	504	yes the ratio
	RES	505	So can you explain in one line what's the relationship between the ratio and the rate of metabolism
24:51	Levi	506	It increases when the radius ... goes ... down
	RES	507	It goes down and you have a reason for that

	Levi	508	This ((Levi points to the worksheets))
25:08	Thor	509	[@@]
	RES	510	[What's the reason]
25:10	Thor	511	This ... mathematics
25:13	RES	512	Yes, the table
	Thor	513	the table it gives us a solution and answer for it

Towards an understanding of the relationship between metabolism and the SA/V ratio being discussed there is evidence in the above discussion that Levi first acknowledged that the **metabolism of the cell was better when the cells had a smaller radius** (499, 506) though he did not give any explicit reason (501). On my pursuance both Levi and Thor used the discussion thus far to form the basis for conclusions (508, 511 and 513). Such a 'conclusion' was however subject to further doubt by Levi, who continued to question the arguments being made as follows:

Time	Person	Turn	Utterance
26:36	Levi	543	but, ya, ya, are they
	RES	544	Well what is the word here we say
	RES	545	we assume
	Thor	546	[we assume]
26:39	RES	547	[we ass=ume]
	Thor	548	[We don't know] but
26:41	Levi	549	[We don't know]if we [assume]
	RES	550	[it seems correct]
	Thor	551	[we don't know]
26:43	Levi	552	Ya, if we make the assumption then this table says what we have ... [concluded]
	RES	553	[Ya]
	Thor	554	Its like a theory, ya
26:49	Levi	555	But is it like that

The above discussion evidences that Levi questions not so much the connection between metabolism and SA/V ratio but **whether cells are really spherical** (543, 555). When as an interviewer I add that in the group-task there is an assumption being made (544), my argument finds agreement with Thor who says it's like a theory (546, 554). On his part Levi himself acknowledges the mathematical relationship (552) yet questions the spherical nature of living cells. He even asserts himself:

Levi 562 I don't think so

The sequence and flow of arguments I present above allow me to evidence how the students progressed from calculating the SA/V ratio systematically. This ratio then led them to model the condition that the metabolism in living cells was better when the radius of the cell was small. They then came to realise that the model was **based upon an assumption**. That such an assumption was based on the **'mathematics in the**

tables' made them further suggest the existence of a **'theory'**. However ratio, mathematics, table and theory notwithstanding, Levi **firmly questioned the reality** that living cells were indeed spherical.

Living cells: reason based on assumption

In the last sub-section of my final case I offer discussion of the students after the above arguments and model of reasoning. I discuss students' responses to a question I posed during the conduct of the task: whether the cells in a large animal like an elephant would be larger or smaller.

Time	Person	Turn	Utterance
27:10	Thor	569	they have smaller cells
	RES	570	Just because the animal is larger do you think the cells would be larger
	Levi	571	No, it should be
	RES	572	what does this tell us
27:16	Levi	573	Since the metabolism is got to higher
	RES	574	yes, yes
	Thor	575	If that has to be higher then
	Levi	576	then the cells got to be low ... smaller... ya
	Thor	577	Hmm hmm
	Thor	578	The bigger the animal ... the smaller the cells are

Subsequent to Thor's **conclusion** in the above extract (578), I discuss now the issue of the sharing of the group-task by Dan, Thor and Levi. On hindsight there is reason to believe that Dan's asking if the group-task had to be done together (13) was to take a **free ride**. When asked to share responsibilities, Dan simply admitted that he would watch (92) a role his group-mates seemed to be familiar with (94-95). On hindsight it also seems that when Dan suggested that the value of the ratios should be rounded off (159) he could very well be suggesting that, Levi or Thor round off the values being obtained. The **passive role** of Dan towards the whole task is ironically evidenced when he in a way **had to agree** (488) with the arguments of the others. The above group-task ended with my thanking the students and Thor mentioning that though he thought he would be bored he actually found the task interesting. When Dan was asked he responded by saying that he **'just had to observe'** admitting what working at the group-task **'together'** may have meant to him.

Brief summary of the third case

The flow of arguments in the third group-task evidenced three phases. The first phase was the **intuitive nature of calculating** displayed by Levi and Thor. By this I refer to the conjectures of the students: that the values they were getting at first were incorrect, then that the values in question 'should' be something they anticipated, and finally that their observations were related in a particular mathematical way.

The second phase was the **modelling phase** where many mathematical relationships and relative models were both built and considered. The first of these was the **calculating** of numerical and ratios of spheres with hypothetical radii. The next was **comparing** the numerical ratios and expressing the same algebraically. This was followed by **agreeing** that comparing the two ratios represented the same relationship. This was followed by **identifying** the parameters of the cell, with those of a living cell, wherein the volume of the cell was found crucial to its metabolism and its surface area to the exchange of material in and out of the cell. The model that followed was in **recognising** that the metabolism of the cell was related to the radius of a spherical cell. The **reasoning** that followed from this was that cells of smaller radii were more beneficial to metabolism. That the above was possible on **assuming** that the cells were spherical finally came to pass. The above mathematical arguments and ‘theory’ however led to the third and final phase.

In the final phase the students **conjectured** about the size of cells in large animals. In this phase they **questioned** the reality of the spherical shape of living cells. There was evidence that they were able to apply the mathematical model to the case of large animals. The model was not in question, yet even while accepting the model they questioned reality.

Simultaneous with the three phases however there was evidence that Dan took a **free ride** by ascertaining that the group-task had to be done together. For him the group-task meant that he ‘just had to observe’.

Concluding discussion

The nature of cooperation among students at group-tasks, a theme that I highlighted and discussed in this chapter was brought about differently in the three cases. In the first case, the first of the two group-tasks demanded the use of a representation to cooperate while in the second the two group-tasks demanded the students represent to cooperate. The representations involved did not belong to the propositional form of mathematics. It was in the attempts at the latter that there was a mix of both personal and propositional forms (equation) of solving problems. I present an ‘activity’ analysis of these towards the end of my discussion.

It was in the second and third case that more verbal arguments were evidenced and discussed. The second case dealt with bringing real life experience to a graph and later conjecturing based on the understanding generated around and with the graph. The third and final case drew on the relationship between surface area and volume of spheres. The ratio obtained was then utilised to both understand metabolism of living cells and later conjecture the size of living cells based on the ratio previously found. Where the second case brought experience to the group-task, the third questioned reality with the model built in the group-task.

In discussing my three cases and referring to literature in mathematics education and socio-cultural-historical perspectives, I begin with the nature and role of intellectual artefacts encountered in the cooperation of students in above group-tasks. The use of **intellectual artefacts** like the geometrical diagrams, algebraic equations, graphs, arithmetical and algebraic ratios in the above group-tasks went beyond their role in mediating the making of meaning, as in Chapter 5, or even the application of knowing, as in Chapter 6, and **contributed specifically to modelling** the real or hypothetical situation they were called upon to represent.

In such a role they were able to mediate between the **actual reality** they represented, as argued by Sfard (2000); and **virtual reality** they could possibly represent. This was evidenced by and not independent of the discourse they belonged to as also argued by her. The questioning of the two dots meeting if the white moves first, or the calling upon of personal meaning and symbolising the same in sounds or in questioning reality based upon a model that was built, evidence also the back and forth processes between meaning making and symbolising she observes.

As also argued by Goldin (2003), it was in the processes of representing the 'equation' in the problems of weight that it was possible to **formalise the representation of the algebraic equation**. In agreement with the arguments of Smith (2003) there was evidence also of using the equations offered by a student to **enable instruction of greater abstraction and generalisation**. The **graph** as argued by Monk (2003) was the centre of considerable debate, in which it was both **medium and tool**.

The above process of symbolising, representing or modelling was not independent either of the ability of forming patterns and using embodied sounds which were part of the **collective common sense** and justification of knowing as argued by Kilpatrick et al. (2005). Such evidence lends credence to the arguments of Resnick et al. (2004) that **intelligence is socially constructed**, intrinsically related to the tension that is embedded in concrete circumstances as argued by Wertsch et al. (1998) and the motivational factors in the tasks as well as the **normative nature of cooperative learning** in this classroom as argued by Forman (1996).

As discussed before, two significant aspects emerged in the cooperation of the students. Firstly, it was the **presence of goals** offered by the group-tasks, as argued all along that the meaning of words and the task at hand led to greater and greater objectification. As also argued by Wells (1999), it was the collaborative nature of inquiry and the socialisation that was part of the same that led to **shared experience and understanding** in order to compare, identify, recognise, reason, assume, agree, disagree, conjecture and question. These more mathematical processes involved imitation as argued by Chaiklin (2003) or an appropriation of each others understanding as argued by Newman et al. (1989)

The ability to make one's **personal meaning** (everyday) and experience into more **objectified** though not necessarily propositional forms (academic) was evidence of the formation of ZPD as argued by Rowlands (2004), in which the cooperation in the group did bring forward **newer cultural forms of behaviour**. Meaning and cooperation in such a situated ZPD happened only upon the reaching of **intersubjectivity** as argued by Bruner (1996) and making the words and sounds as medium and **discourse the situation** as argued by Bakhtin (1986).

Many arguments of Bakhtin were strongly evidenced when it was **outsiderness** that made a student (Anja) explain the situation to oneself. It was also the presence of opposing arguments that provided opportunity to strengthen one's own arguments (Egil). There was evidence of the **borrowing of others words to convince them**. I give three examples. Firstly, in response to 'The elevator will increase in speed' 'and stop' the counter response was 'It'll stop and then it'll go up again'. Secondly, in response to 'stops and then' the counter response was 'the elevator has to slow down in order to stop at the next floor'. Finally, in response to 'And ... then you catch it' the counter response was 'And then at point Z you catch it'. These examples evidenced Bakhtin's dictum that there is never a final word and that **every word provokes its counter word**.

Towards strengthening my discussion about the nature of goal directness and towards concluding discussion in this chapter, I turn finally to the analysis of the two group-tasks *When together* and *How heavy* as 'activity'. I follow the practice of presenting the aspects of the two tasks in a table below before following the same by discussion. I present data representative of the two tasks as the two rows in the table.

Activity of using a representation and representing					
<i>Objective</i>				<i>Subjective</i>	
Activity	Motive	Action	Goal	Operations	Conditions
Have students participate at the two tasks 'When together' and 'How heavy'	To establish and consolidate cooperative learning in group-tasks	Students attempt the task 'When together'	To have students to cooperate	Students draw on patterns they find in the diagram given	Students use diagram and governing rules given in the task
		Students attempt the task 'How heavy'	To consolidate group cooperation and introduce algebraic equations	Students represent their understanding to cooperate, some of which shows the use of algebra	Students use diagrams and equations which they represent

I argue as before from right to left across the first and the second rows or group-tasks. At the level of **operations-conditions** while in the first row the students used a given representation to cooperate, in the second the students were representing to cooperate. At the level of **actions-goals** while in the first row the students were asked to cooperate in the second the students were collaborating. At the level of **activity-motive** in the first row or group-task the teachers' motive was met by having the students to cooperate while in the second their motive was met by an extension of students' outcomes towards introducing algebra in classroom teaching-learning. In the 'activity' it was the goals of the teachers which were subjectivised and the personal meaning and cooperation of students which was objectivised, with the concept of cooperation in a group recognisable to the students.

The above 'activity' analysis allows me to highlight two significant aspects not mentioned before in my thesis. Firstly, that the **group tasks** conducted by the teachers, either towards the teaching-learning of direct and inverse proportionality or in establishing and consolidating the practice of group cooperation were **intrinsically a teachers artefact**. It was with the group-task that the teachers mediated their objectives of teaching-learning. In the first, the aims were mathematical and thus didactical. In the second, they were to do with conduct or working or pedagogical.

Secondly, the presence of the ZPD in the **cooperation of students** was possible and embedded within a larger ZPD of **collaboration in the classroom** that was built and pursued in the establishment of practices, the consolidation of meaning and the application of knowing. There was a consistent pursuit of making or allowing the personal, spontaneous and everyday to be transformed into propositional, scientific or academic. I conjecture that it was in this 'zone' that the teaching-learning of the classroom was positioned and located.

My final synthesis of the data and analysis of the present chapter is the following. That cooperative problem solving is premised on the establishment of shared goals, where a state of virtually no intersubjectivity, is led through cooperation, to greater and greater intersubjectivity upon mutual inquiry and dialogue. Positioned so as to allow the transformations of personal to propositional, spontaneous to scientific or everyday to academic, in such a process, a ZPD is created with consideration of each others words, artefacts, models, reasoning and actions. Thus objectification of meaning towards resolution, in **cooperative problem solving is premised upon drawing each other into one's own ZPD**.

I now turn to the final synthesis of my thesis.

8. A micro-culture

I theorise upon the micro-culture of teaching-learning in my final chapter. Situated in the classroom I described across my four data and analysis chapters, its preliminary synthesis was in the form of the corresponding grounded themes. I now portray the micro-culture across the four grounded themes. Towards such an attempt I first focus on the three topics of mathematics in which the grounded themes developed. I then synthesise findings from my four units of analysis. With the grounded themes as background, I finally discuss the micro-culture of teaching-learning constituted towards meaning making and problem solving. I conclude by reflecting on some implications of my study.

The three topics in mathematics

The three topics in mathematics discussed in my thesis are: number understanding, equations and proportionality and scale factor in similar figures. In discussing the teaching-learning of these topics as it transpired in the classroom, I attempt to consolidate their contribution to the four grounded themes. As mentioned earlier these topics were nested in larger chapters of the curriculum whose details I offer in Appendix A.11. In my discussion below I offer their ‘quadrangulation’ as it were across the themes of a collaborative practice, the consolidation of meaning, problem solving know-how and cooperation at problem solving.

Number understanding

The sub-topics under the heading **number understanding** included the teaching-learning of fractions and exponents.

The teaching-learning of the sub-topic of **fractions** began with the summarisation of the four operations, the introduction of which was part of the curriculum for the students in their secondary school. The appropriate application of the four rules of operation was thus the focus in teaching-learning. Attention was given to the role of common factors while performing operations, leading to the value of dealing with a smaller magnitude in numerators and denominators and the convention of expressing fractions in their lowest terms. The nature of a prime number came up for discussion in the context of a prime number being a common factor to both the numerator and denominator of a fraction.

The appropriate utilisation of **the bracket** was addressed along with the operations of fractions. This involved the application of the LCM to the necessary denominators of fractions within a bracket and the appropriate ways of distributing the multiplication of a number, across a complex fraction enclosed in brackets. The use of brackets with fractions was also paralleled in similar usage with terms in algebra.

The sub-topic of **exponents** began with attention in teaching-learning to the existence of their parallel **terminology** in English and Norwegian. The presence of parallel terminology in both languages was built upon by the teachers, with its existence not questioned by the students. The topic of exponents, whose notation was part of the curriculum in secondary school, was developed conceptually with deliberation on rules and newer forms of notation. Numerous examples of operations with exponents were **generalised as rules** along with examples of application of general rules to **specific cases**. Special attention was given to the meaning of exponents whose power was zero or negative. In arriving at, explaining or understanding their significance, the equivalents of exponents in fractional terms were discussed in which common sense was called upon for their justification. A four step problem solving model was implicitly introduced in the elaboration of one such case.

The importance of the use of rules as applications of generalisations was reached upon collaborative inquiry by the students and teachers, over many instances of teaching-learning. The **common and shared understanding** that this resulted in, was evidenced by cryptic references and socialisation in speech, resulting in a genre. The use of such a genre was both highlighted and **privileged** by the teachers. Besides capitalising on linguistic aspects, attention in teaching-learning was also given to the fact that the same mathematical concept lay behind older fractional representations and the more newly used forms of exponential notation.

The appropriate **application** of the bracket was addressed in the teaching-learning of exponents as well. Apart from its application to nested powers, attention was given to its possible misapplication. Proper use and application of **the calculator** in operations related to exponents was also attended to with reference to the use of appropriate buttons or functions. While care and attention was given by the teachers to the many issues detailed above, the attempts of students at resolving expressions like $(2 \times 10^4)(5 \times 10^3)$ evidenced the numerous ways in which students 'got a ten'. This was indicative of the **existence of personal meaning** and the **individual nature of its becoming propositional**.

The common sense that was called upon in the teaching-learning of the above sub-topics was also used in the two group-tasks with which the topic ended. It was the **common sense** of the students that was called upon in the recognition of a pattern, and the resolution of the question based on a polygon. Common sense was also called upon in the resolution of the unknown weight of a brick. The attempts of the students, which evidenced the use of both common sense and algebra, were called upon by the teachers, to be presented by the students at the blackboard. The teaching-learning of equations in the classroom, began with the teacher's discussion of their algebraic version.

Equations and proportionality

The sub-topics discussed under the heading **equations and proportionality** included the teaching-learning of algebraic expressions, simple equations and the relationships of direct and inverse proportionality.

In the sub-topic of **algebraic expressions** attention was given to the collection of like terms and their segregation from unlike terms. Special attention was also given to the proper application of a negative sign when the sign was the **coefficient of a bracket** within which terms were both negative and positive. In substituting known values for variables in a formula, so as to arrive at the value of **an unknown**, the use of a **mnemonic for remembering** the formula was privileged by the teachers.

The sub-topic of equations started with the designation of an unknown quantity with a variable, to form **simple equations**. The application of this model was extended to the use of variables in representing arithmetical patterns such as in the case of consecutive numbers. In such an application, attention was given to the **relative designation** of the first variable and the **consequent designation** of others in relation to the first. The application of equations to different questions was again accompanied by an appeal to the **common sense of guess and check**.

The two relationships of direct and inverse proportionality were dealt with comprehensively. The concept of direct proportionality proceeded from **reference to concrete experiences** with artefacts like the spring balance. The constant value of ratio, in a table comparing the two variables, was represented graphically as a straight line passing through the origin. The graph of a stiffer spring was **conjectured** and the relationship between proportional quantities **formalised** in an equation. The identification of directly proportional quantities was **abstracted** in three ways: graph, table and formula. The concept of inverse proportionality was introduced with the constant nature of the product of two inversely proportional quantities. Their graphical representation as a hyperbola, which was identified as asymptotic, was followed by the formalisation of another equation. The identification of inversely proportional quantities was **abstracted in similar manner** as proportional quantities: graph, table and formula. A mix of graphs, tables and formulae were subsequently **identified** as directly, inversely or not proportional.

The **cooperation** of students at two graphs representing the motions of the elevator and ball thrown up was outside the curriculum. The identification of the graphs as representing the motions corresponding to them, **drew heavily on the personal meaning and experiences** of the students prior to their cooperating at this task. It was the **lack of personal and shared meaning that hindered resolution** of a graph that depicted a ball thrown up on the moon and not the earth. I now turn to the teaching-learning of relationships in geometrical figures.

Scale factor in similar figures

The sub-topics discussed under the heading of **scale factor in similar figures** included the teaching-learning of scale factor of lengths, areas and volumes, Pythagoras' theorem and problems of application of algebraic relationships to geometrical questions.

The concept of **scale factor or ratio** between similar and corresponding parameters of similar figures developed over numerous independent examples in one, two and three dimensions (length, area and volume). The development of the concept of scale factor of length was based on the property of similar triangles, part of the curriculum of students at secondary school. The ratio of lengths of corresponding sides of similar triangles was then **designated** as scale factor in one dimension. In such recognition the angle sum property of triangles was referred to. The search for appropriate terms in geometry and its relevant usage in application was part of the teaching-learning of this group-task.

The concept of scale factor or ratio of areas developed over many practical examples, and was **consolidated in its application** to squares of various sizes. In comparison to the scale factor of lengths, the scale factor of areas of similar squares (in two dimensions) was found to be a 'square' of the scale factor of the ratio of the corresponding sides. The concept of the scale factor or ratio of volumes, was again developed over many practical examples, and consolidated in its application to cubes. In comparison to the scale factor of their sides, the scale factor of the corresponding volumes of cubes (three dimensions) was found to be a 'cube' of the scale factor of their respective sides. There was evidence of the enabling of conjecturing and the associated **intuitive** nature of students' thinking, with regards to the development of the above ratios.

The application of algebraic relationships to geometrical figures was exemplified with a group-task based on Pythagoras' theorem. The conduct of this very group-task upon the regrouping of students implicitly emphasised the cooperation demanded of the students, in their regrouping. The application of the concept of congruence was exemplified in finding out the area of a trapezium, along with the application of appropriate formula. The application of formulae, metric conversions and the calculator was exemplified in finding the volume of a cylinder.

The particular case of the decreasing ratio of surface area and volume, of spherical bodies with increasing radii, lay outside school curriculum. This ratio was **calculated numerically** and **expressed algebraically** and used to **model** the size of living cells for better metabolism. The model in turn helped question the size of living cells in reality.

I now synthesise my four units of analysis, with the help of which I analysed teaching-learning in the classroom. In my elaboration I offer their triangulation in the topics of mathematics just discussed.

Mediated action and agency

With the first of my units of analysis I discuss those physical and intellectual artefacts that were employed in teaching-learning along with the outcomes they mediated. I then discuss two features of artefacts I term mobility and positioning. I conclude with instances of agency that were evidenced in the teaching-learning of mathematics in the classroom.

Physical artefacts

Towards summarising and synthesising the outcomes of mediation of physical artefacts, I maintain sequentiality of their first occurrence and begin with **the textbook**. The outcomes of mediation by the textbook commenced with its use as a source of rules and examples, utilised to recount the sub-topic of fractions which was part of earlier curriculum. As a source of **reference** for the students, even before their peers and teacher, the textbook took place of preference in the participation of students in classroom teaching-learning. The textbook was also the source of the very same set of questions to all the students. The setting of questions simultaneously to all the students by the teacher, allowed for the creation of a **group space**, a construct I shall elaborate upon later.

Beyond its role in the classroom the textbook contributed to the value of **continuity** which also I discuss later. As a source of questions, rules to work these with and demonstrative examples, the textbook provided continuity between classroom teaching-learning and the responsibility of work done individually by the student at home. In the teacher discussing homework on the following day and extending the concepts dealt with earlier, the actions that the textbook mediated, brought action back to the classroom where students and teachers collaborated again.

The second physical artefact I evidenced was **the blackboard**. As a large medium of display for a classroom of students, its role was to mediate the content of the teachers' talk and discussion. However, and as evidenced, its role in the collaborative practice that was established in the classroom was pivotal. The blackboard was also used as a medium through which individual students could display their working at mathematics, leading to the **externalisation of thinking** and the possibility of **justification** that were crucial to the collaborative practice.

In the students' **speaking** at the blackboard **on behalf of** the attempts made by their group at the group tables, the blackboard mediated the **discussion** of individuals of the group, and between the groups in the classroom. Its use was also extended by its enabling the display of **cooperation between the teachers** while teaching, and the group debates that ensued, allowing for the possibility of collaboration at mathematics in teaching-learning within the classroom. Finally, the blackboard was also the location for a construct that I have termed as **engagement** which I shall bring to discussion shortly.

The **students' notebooks** were the third physical artefact singled out, along with their central role in mediating the current understanding of the students in teaching-learning. Students' attempts in them, at the mathematics being discussed in the classroom, displayed their **personal knowledge**. Such an externalisation of their thinking, allowed for personal **reflection** and meta-cognition about one's self confidence. Personal knowledge in the written form also provided the students and the teachers, the basis upon which **a sense of judgement** was arrived at, on the difference between the goals that were set to be achieved and those that were being achieved. Drawing upon students' notebooks was thus the first step in making classroom teaching-learning more student-centric. When drawn upon, students' notebooks were a resource for the teacher with which to gauge current understanding of the students, in order to take suitable actions towards the goals of the curriculum.

As a physical artefact **the formula book** was referred to and its use incorporated in classroom practice as a **ready reckoner** of formulae or rules. In teaching-learning the practice of drawing upon specific cases and discussing the subsequent emergence of rules, the formula book mediated generalizations that could in turn be applied to specific instances. The other artefact instrumental in carrying out various calculations **was the calculator**, with its utilisation dictated by the mathematics in question. However any arguments I make on the actions mediated by the calculator remain restricted on two counts. Firstly, in the current study I observed the role of the calculator as one of the many artefacts in the classroom. Secondly, I conjecture that any comprehensive study of actions mediated by the calculator need methods other than those I have used.

The group-task was the last of the physical artefacts discussed. Used **to mediate goal-directed activity** its use towards specific mathematical ends, allowed for the mediation **of intended outcomes**. These objectives were however based on prior and appropriate design of the group-task. In a generic form the group-task was **the teacher's artefact** with which to capitalise upon concrete experiences of the students, towards the intended goals of teaching-learning. In its implementation the group-task had the advantage of being used at a time of the teacher's choosing, in the continuum of teaching-learning in the classroom. The making of such a choice had strategic advantages of utilising the group-task in bringing forward personal meaning of the students, so as to mediate their prior knowing towards intended goals of their propositional form.

In my objective of identifying all those artefacts that mediated physical outcomes, towards the teaching-learning of mathematics in the classroom, I have not identified many physical artefacts like the pen, pencil, eraser, ruler, compass and graph sheet which were implicit and taken for granted in the micro-culture of the classroom on which I report.

Intellectual artefacts

In identifying artefacts which mediated intellectual activity in the teaching-learning of mathematics, I recognise up front the difficulty faced in their identification, since words are intellectual artefacts whose use is common place and their mediation specific to mathematics difficult to isolate. Faced with such a predicament I have attempted to narrow my focus to those which had a specific role in teaching-learning.

The first group of intellectual artefacts evidenced in the data were **fractions**. Mediating the concept of a part in reference to their respective wholes these representations were intellectual in that they were removed from their concrete references. The equivalence and reference of fractions to decimal notation made the **decimals**, the other group of intellectual artefacts in this classroom. **Exponents** as cryptic representations of the continued product of integers were the third group of intellectual artefacts in teaching-learning. **Brackets** or parenthesis were the other intellectual artefact, used to mediate operations applicable to any intended selection of fractions, decimals, exponents or algebraic terms.

Discussed also in the teaching-learning of mathematics were **simple equations**, which as intellectual artefacts mediated the equality of relationships, in an algebraic form. Their use in relation to concrete examples, mediated the relationships found in the reality they represented. The group of **ratios**, as intellectual artefacts, was referred to as **scale factor** in reference to geometrical figures. The teaching-learning of scale factor in a succession of group-tasks, brought with the respective activities, the use of **graphs** as intellectual artefacts which visually represented corresponding relationships between a set of quantities.

Conventional geometrical diagrams which represented lengths, angles, perimeters or volumes in the teaching-learning of mathematics were the next group of intellectual artefacts. On occasions they were less conventional and modified to real-life applications of painting houses, mowing lawns or filling swimming pools. In such representations they were meant to mediate the mathematics abstracted from their reality.

Specific **mathematical symbols** such as those used to indicate perpendicularity, equal lengths, parallel sides in geometrical figures, in addition to **relational signs** for equality, approximately equal to, decimal point, the ten Hindu-Arabic **numerals** were part of the teaching-learning of mathematics. The specificity of their utilisation made the elaboration of their teaching-learning in terms of prior knowing.

Parallel to specific usage of symbols and signs, teaching-learning in the classroom also included specific usage of **terminology**. By this I refer to the specific relationships that were meant to be mediated with the usage of words like denominators, LCM, similar triangles, straight line, hyperbola, asymptote, parallel, perpendicular and altitude.

My discussion of intellectual artefacts brings to light two features of significance which I discuss presently: their conceptual and intellectual nature. By the **conceptual nature** of intellectual artefacts I refer to the nature of ideas in mathematics that are mediated by their utilisation in teaching-learning. Since the meaning mediated by the appropriate use of these, is in turn embedded in a network of interrelated ideas recognised as mathematics, it is not just the immediate meaning mediated by the term or intellectual artefact but also their relations with other concepts that is also implicitly mediated and brought into consideration. For example the word ‘similar’ as applicable to say two triangles is a specific case of the generic concept of ‘similarity’ in mathematics.

I had evidenced the increasingly **intellectual nature** of classroom activity in Chapter 5. Even though a group-task began in the personal meaning and concrete experience of students, the nature of activity that made the personal form propositional, was increasingly intellectual in that more and more intellectual artefacts were used in the transformation. The presence of such immediate and increasing intellectual nature is of **significance** to the nature of activity that is teaching-learning.

Intellectual artefacts that were conceptual in nature and demanded **associated reasoning** were found in use as well. In making such a distinction I include the parallel mediated in mathematics say between specific arithmetical relationships and their generic algebraic equivalents. For example consecutive numbers exemplified by a 16, 17 and 18 or as x , $x+1$ and $x+2$ or even as $x-1$, x and $x+1$. I extend discussion of **models of reasoning** as artefacts later again when I discuss knowledge artefacts.

In concluding my identification of intellectual artefacts I refer to the use of constructs like the process of **guess and check**, the asking of **rhetorical questions** and the use of a **mnemonic** in remembering. Being privileged by teachers, such artefacts became part of the intellectual artefacts that constituted teaching-learning. The use of these artefacts however stands in contrast with those created by the students (*Doying!* and P, Q and R), which did mediate specific meaning in the group in which their construction was necessitated. I conjecture that the use of *Doying!* and P, Q and R more widely across the classroom, would require a higher degree of generalisation and a wider degree of success, to communicate across the group and become intellectual artefacts of the micro-culture of the classroom. A contrast to the utilisation of these two constructs is also found in the utilisation in teaching-learning of **a genre**. Premised upon a wider internalisation of the same intellectual artefact (rules of exponents), such utilisation was a result of a higher degree of generalisation and resulted in a higher degree of socialisation. Though the presence of a genre evidenced common usage, I recognise that such usage may be limited to the micro-culture that I am reporting on.

Mobility and positioning

Before discussing those instances of the above mediated actions that evidenced agency in individuals, I presently discuss two features related to artefacts that I have termed earlier as ‘mobility’ and ‘positioning’.

By **mobility** I refer to that feature of any artefact, which acknowledges in analysis the transformation of the artefact from its original form (in most cases physical to intellectual). In some sense this feature may be seen as accounting for the decreasing concreteness of an artefact and its increasing ideality. As example I offer the shift from a physical and more material recognition in teaching-learning of the reference to the ratio of five units (be they stones, pencils, or fruit) in comparison to ten. Such a concrete experience develops as a mathematical notion and could be expressed over the longer duration of teaching-learning as perhaps $\frac{1}{2}$ or 1:2, or even 0.5 or later 50% and still later 2^{-1} . I conjecture that accounting for such ‘mobility’ is essential to the analysis of teaching-learning in three **significant** ways. Firstly, accounting for mobility in current teaching-learning, brings into consideration those prior concepts that are necessary for the present form to mediate meaning, and depending upon which the present can be understood (2^{-1} on the basis of $\frac{1}{2}$). Secondly such analysis allows for viewing the sequential and increasing intellectual nature of prior transitions related to the present. Finally, accounting for mobility allows for locating transitions being made by the students between the personal and propositional forms of meaning.

By **positioning** I refer to another analytical distinction: that of the current nature, physical or intellectual, of the artefact in relation to the teaching-learning activity. I clarify this distinction with another example. In Chapter 5 the personal experience with the extension of a spring was brought about with the diagram of a spring balance. By the diagram on the blackboard, the teacher mediated the intellectual activity associated with the personal experience the students would have had earlier with the spring balance in the laboratory. At the earlier occasion when students had concrete experience in the laboratory, the classification of the spring balance in teaching-learning would be as a physical artefact. As a diagram the representation of the spring balance in its present context, was appropriately as an intellectual artefact. The ‘positioning’ of the spring balance in either teaching-learning activity is different. I therefore conjecture that analytically accounting for the positioning of any artefact has **significance** for the teaching-learning of mathematics. Firstly, such an analysis allows for accounting in teaching-learning of the current experience, physical or intellectual, that is being called upon in the student. Secondly, such an analysis again allows as above, the locating of the transitions being demanded of the students, between the personal and propositional forms of meaning in teaching-learning.

My discussion of ‘mobility’ and ‘positioning’ of artefacts in analysis in turn leads me to a discussion of the **limitations** of labelling. For example the artefacts that I have identified in my study as physical, like the textbook, blackboard, notebook and calculator all embed intellectual artefacts and in turn also mediate intellectual activity. Highlighting those uses that are only physical does not adequately distinguish intellectual from physical activity being mediated by their use. In most instances both physical and intellectual activity is mediated. Yet acknowledging that such a classification is dependent on usage in reality, I conjecture that the very effort of applying a distinction, allows for pinpointing and isolating mediated actions in the analysis of teaching-learning.

Any **analytical isolation** of artefacts is in turn related to the contexts of use and mediation within teaching-learning. In the reality of artefacts mediating parallel outcomes their analysis involves contextual **judgement**. For example the use of a textbook in classroom teaching-learning is different for the teachers than it is for the students. While the use of the textbook enables the teacher to set the very same question to all the students to work with, its use affords the students, in addition to the questions to be attempted, the presence of associated examples and generalised rules for working. Accounting for mediated action in analysis thus requires taking **contextual goals** into consideration.

In analytically pinpointing actions and associated outcomes mediated by artefacts, the issue of **inability in access** to certain uses and outcomes is also a related problem. For example the use of words or language as signs associated with any mediated action is to some extent unavailable for analysis. Even though there is a part of thinking that is externalised verbally by the individual, which may be subject to analysis upon extended observation, there remains a part of thinking that is not externalised yet mediated by signs. Though such thinking, internal to the individual may remain unavailable for external analysis, its existence is none the less mediated by signs, words or intellectual artefacts.

A final **practical distinction** seems necessary in isolating mediated action with either physical or intellectual artefacts. In the contexts of teaching-learning two kinds of artefacts are used. Firstly, those which mediate the understating of the goals of the mathematical task at hand. Secondly, those that mediate required actions of the students towards achieving the goals of any task. I give two examples. Firstly, the diagram of the spring balance mediated the task of proportionality and the calculator mediated the calculations necessary for arriving at the required constant. Secondly, the two graphs (elevator and ball) represented the task related to their motions, for whose resolution sounds were used.

From a general overview of mediated action, I now discuss analysis of individual communication involving instances of individual choice.

Agency

While the study of mediated action allows for analysis of events in the trajectory of teaching-learning, the study of **agency** allows accounting for the nature and opportunities of individual participation.

I had cited in my data and analysis chapters various occasions which evidenced the nature of participation of individuals as and when they were found: *speaking-with-the graph*, *speaking-with-the-formula*, *speaking-with-a-calculator* and *speaking-with-the-table*. Such instances of agency afforded in the actions described enabled students to **externalise their thinking and communicate** personal meaning and knowing **in terms of the mediational capabilities afforded** by the artefact. Incidence of agency was also afforded when the teachers used the group-task to mediate various mathematical and pedagogical outcomes, which I called teacher-speaking-with or *teaching-with-a-group-task*.

The cooperation of students at group-tasks added another dimension to agency, that of creation or invention of artefacts with which to speak and communicate. Irrespective of whether the artefact used was a sound or verbal construct, it is fair to conclude that individuals-spoke-with-mediational means so as to **enhance communication** of their personal meaning. In such instances of agency, the outcomes of mediation of the individuals and artefacts were also unified. The individuals did not speak or act alone but in conjunction with their mediational means.

There were three significant aspects to teaching-learning associated with instances of agency: that of intention, that of choice and that of goals. I find these three aspects to reside in the individual and connected to each other as I explain. Firstly, it is only with personal **intention** that any individual would act and use mediational means towards imminent needs. Secondly, towards satisfying his or her intention any individual simultaneously makes a **choice** or exhibits personal preference towards the use of one artefact amongst the many present. Finally, personal intent in the actions of choice is exhibited towards a **goal**. I conjecture that it is a combination of intention, choice and goals that is associated with instances of agency, which allows for **explaining the existence of diversity** in student responses and also **accounting for the diversity of responses** across students in the classroom. The significance of the above is that the occurrence of these instances is evidence that students in turn have the freedom to do so in the first place. In my study the freedom that I mention was available at two levels: in classroom collaboration and in group cooperation. I conjecture that it is the **existence of freedom** that allows for agency and affords benefits to personal communication.

Beyond communication of personal meaning with the mediation of available artefacts, I now discuss specific instances of knowing and the mediation of such knowing in the teaching-learning of the classroom.

Theoretical knowing and knowledge artefacts

In the importance given in my perspectives to the historical development of students in the trajectory of teaching-learning of mathematics in the classroom, I had adopted the epistemological principle that knowledge is not once and for all, but always in the making. It is towards the analysis of the progression of both personal and propositional knowing, that I had chosen to analyse and evidence various aspects in teaching-learning that contributed to and constituted the development of mathematical knowing in the classroom. As part of this unit of analysis, I therefore also singled out instances of knowing that were at times objectified as ‘knowledge artefacts’ which mediated knowing of the past into the present. I elaborate upon both these kinds of knowing below.

Ways of knowing mathematics

Across my data and analysis chapters I have pointed, on various occasions, to what I termed as ways of knowing mathematics. It is a synthesis of these ‘ways’ that I discuss here and which I have been able to categorise into four: didactical ways of knowing, pedagogical ways of knowing, linguistic ways of knowing, and values. I elaborate upon each below with recognition that these constructs have at times resulted from the composite of units of analysis that I have employed. The ‘ways’ I highlight find mention here since these are related to knowing.

By **didactical ways of knowing** I refer to those ways of knowing which pertain to the teaching-learning of the discipline of mathematics. In my data and analysis chapters I have highlighted these to include *ways of operating fractions*, *ways of operating exponents*, *ways of using brackets*, *ways of plotting graphs*, *ways of operating the calculator* and *ways of following convention*. In listing the above I make mention of two facts. Firstly, the list I offer above is limited to the topics I have had the opportunity to discuss in my thesis. As a consequence, didactical ways of knowing could include in other instances of teaching-learning other ‘ways’ such as ways of constructing geometrical diagrams or ways of transposing matrices. Secondly, though the above ways are essential to the teaching-learning of mathematics, it is obvious that such constructs are also part of the teaching-learning of other sciences as well.

By **pedagogical ways of knowing** I refer to those constructs that relate to implementation in teaching-learning. In contrast to didactical ways of knowing there is nothing in their implementation that restricts them to the teaching-learning of the discipline of mathematics. Such ‘ways’ could provide opportunities in the teaching-learning of other subjects as well. Yet in providing opportunities specially suited to the perception of instruction that I have argued, I had first pointed to the enabling of a *group space* for students to observe, imitate and conjecture in safety. This was followed by the process of *engagement*, initiated and

nourished by the teachers, towards keeping an ear on the communication of mathematics between either interested or doubting students.

Specific to the classroom in addition to group space and engagement the teaching-learning in my classroom was accompanied by a visible *use of students' attempts* to mediate the teaching-learning of mathematics to all other students in the classroom. Such a practice began with the setting and utilisation of *homework* and *classwork* and was strengthened by what I have termed as privileging. By *privileging* importance was given by the teachers with their authority to certain ways of knowing. Instances of privileging in my data and analysis chapters included linguistic ways of knowing which I discuss shortly, and the common sense of *guess and check*, use of *rhetorical questions* and remembering with a *mnemonic*.

The third way of knowing that I singled out is **linguistic ways of knowing** and refers to those ways of knowing related specifically to the use of language to further the teaching-learning of mathematics in the classroom. These ways included the attention given to the *use of terminology* specific to the topic being addressed, the benefits of which relate to following convention and as discussed earlier the **network of concepts** that is implicitly **co-opted by such usage**. I conjecture that it was the concept of scale factor explored in the first dimension, which when applied to higher dimensions than length, allowed the students to intuitively conjecture: if the scale factor was a 'square' in two dimensions then the expression of its ratio would be a 'cube' in three dimensions.

Apart from the specific use of terminology there was evidence of attention to *ways of speaking*. Such ways included parallels drawn in usage within a language (two to the power four or two to the fourth power) and of parallel structures across languages (Norwegian and English). I highlighted the importance given to the manner of externalisation of thinking in the written form, which accompanied the oral or verbal form in teaching-learning of mathematics as *ways of writing*. I had also argued that it was the re-appropriation of formal written expression and attention to ways of writing that enables the transformation of an individual's personal meaning into its more propositional form. Finally, a *genre* applicable to specific topics was acknowledged and even privileged by the teachers. The privileging of a genre of speaking was as argued, evidence of consolidating the shared understanding of the topic being discussed.

The **significance of values** which to some extent cuts across and accompanies all didactical, pedagogical and linguistic ways of knowing is the final and crucial component I record, as contributing to ways of knowing mathematics in its teaching-learning in the classroom. In highlighting these values I have identified the value of what I termed as *continuity* which refers to those actions which sustain the trajectory of teaching-learning in the classroom. There was evidence of continuity in the

pursuit of a concept within classroom teaching-learning (prime nature of 17), across classwork and homework (but what is the easiest way), across curriculum (You would have done this in ungdomskole) and also across topics (arithmetic and algebra). The *transparency* associated with teachers' actions is the other value that I evidenced, by which the students and teachers were able to establish intersubjectivity and build expectations of each other. I conjecture that it is the establishment of such joint intentionality in teaching-learning, which is valuable in building intersubjectivity in goal-directed discourse and a sense of predictability to the events of the teaching-learning in the classroom.

The establishment of joint intentionality was possible in addition by the value given to *common sense*. In privileging the many constructs that I have referred to above, I conjecture the presence of an implicit value of alluding to and an explicit value of calling upon common sense. Classrooms where common sense is superseded by unnecessary rigour or undue importance of obtaining marks, allow one to contrast and help appreciate the simple wisdom of common sense.

There were values evidenced that were more mathematical as well. These included the *making and correction of errors* by the teacher and visible to the students, the discussion, use and *encouragement of different strategies* towards addressing the problem at hand. These were accompanied by conscious efforts that led towards *easier and simpler ways of attempting* questions, the bringing in practice of the idea of *what is sufficient to do* in any question and the visible *application of similar reasoning* across similar situations and structures.

In summary to the above evidenced ways of knowing mathematics, it is appropriate for me to mention two aspects, one methodological and the other theoretical, that were reflected in the above constructs. Methodologically, a study of the individual-in-social construct allowed for the incorporation of a **wider unit of analysis** with which to observe the teaching-learning in the classroom. Theoretically, the study of the individual-in-social demanded the importance of **two dimensions of relationships** in which the individual was involved, the first among people and the second across time. I conjecture that it is this combined view of both interpersonal communication combined with time-bound analysis, that has enabled an analysis beyond a division of conceptual and procedural knowledge and allowed for identifying ways of knowing that constitute the enculturation of individuals into the culture of mathematics.

I now turn to discuss a particular aspect of knowing that, as mentioned above, is placed in the development of knowing across time and attends to the conscious building of prior knowing. I discuss instances of occurrence and use of knowledge artefacts or instances of the objectification of knowing so as to mediate past or future knowing.

Knowledge artefacts

I recall four different instances in teaching-learning that illustrate the use and application of the construct called knowledge artefact.

The first instance is the use of **examples in textbook** by the teacher and the students in the teaching-learning of fractions. As part of the curriculum of secondary and not the upper secondary school, the teaching-learning of fractions needed the remembering of the rules of operations and their appropriate application in the current context. I conjecture that the rules in the textbook along with associated examples worked in the textbook were knowledge artefacts. Their use by the teachers and the students in the current context mediated the prior knowing of students regarding operations of fraction into current teaching-learning.

The second instance of the use of a knowledge artefact is the use of the **mnemonic** by a student. Privileged by the teachers in two forms (verbal and symbolic) so as to easily remember the relationships between quantities that were inversely related, a student evidenced (Chapter 6) the use of its symbolic form to represent an inverse relationship of the two quantities that she came across later in teaching-learning. In her use the symbolic form, the mnemonic acted as her knowledge artefact to mediate prior knowing to the quantities related inversely, and apply the symbolic form to two other quantities inversely related.

Unlike the first and second example, the third and fourth examples I offer, were not existent or readily available for use in the classroom. As knowledge artefacts they were built with intention and their instances of occurrence were purposefully chosen towards goals. As the third example, I refer to the solution of the question about consecutive numbers which needed the formulation of an equation. The process of arriving at the required equation and building of **the solution**, served as a knowledge artefact for attempting a subsequent question. Similarly, it was the students' attempts at **the group-task** on Proportionality and its objectification as graph, table and formula, which acted as a knowledge artefact while the students worked at the group-task on inverse proportionality. These examples were, as mentioned before, models of reasoning.

In bringing my discussion on knowing and knowledge artefacts to a close, I turn to another conjecture for which I had laid the grounds in my methodology chapter. I conjecture that if it is possible to take both physical and intellectual artefacts as primary artefacts, and ways of knowing as secondary artefacts, then it is possible to consider the knowledge artefact as the tertiary artefact of Wartofsky (Chapter 3, page 65). Such artefacts are independent of, but built with the appropriate use and application of both the primary and secondary artefacts.

I now discuss the nature of participation of students and teachers in the classroom, brought about in the contexts of teaching-learning.

Participation in context

By participation in context as my third unit of analysis, I had indented to analyse the nature of participation made possible by the classroom practice established, as well as that made possible by the contexts of teaching-learning of mathematics within the practice. I discuss both below.

Classroom practice

It is with an entire data and analysis chapter dedicated to the collaborative practice of the classroom (Chapter 4), that I have evidenced the manner in which the participation of students and teachers was enabled. Having observed the functioning of the practice for a whole year, I turn to highlight some significant aspects that stand out and enable discussion about the nature of individual participation in the classroom.

As was evidenced, the **collaborative practice** that came to prevail in the classroom was constituted deliberately, and over a period of time. The actions of the teachers were intentional and pursued consistently over time. As also evidenced, such actions involved the constant guidance of the two teachers during and in-between teaching periods, accompanied by the design and implementation of goal-directed activity.

The teachers' actions, directed towards enabling the students to cooperate at the social level, formed the basis upon which the **group-tasks** set out to achieve more selective goals, in the teaching-learning of mathematics at the individual and group level. Group-tasks were used to both initiate and consolidate the personal meaning and cooperation of the students. The seriousness with which cooperation was pursued was evidenced in the regrouping of students, based upon observation of student cooperation followed by the conduct of a group-task soon upon regrouping. The transparency, with which the above practices were deployed, was accompanied by the transparency with which the teachers made the personal meaning of students' central, to classroom teaching-learning.

Individual participation at group-tasks along with the outcomes of group cooperation at group-tasks formed the basis upon which students externalised their **personal meaning**. Argumentation and justification, at the individual and group level, firstly allowed for participation and secondly allowed for such participation **to have a history** in classroom teaching-learning. It was also upon greater involvement over time, that the participation of students became intuitive and independent.

It was the materiality of the above collaborative practice that allowed students to have access to artefacts as mediational means. In the possibility of choosing and using artefacts as mediational means, the collaborative classroom practice enabled **ways of knowing**. Such a structure in turn enabled **democratic participation** of the students. The participation of the students in the above collaborative practice meant that students cooperate, making the constitution of **a micro-culture possible**.

Contexts of teaching-learning

The context of teaching-learning in the above featured classroom practice was an example of an intentional effort, guided towards desired and most often well defined goals and outcomes in mathematics.

The collective nature of participation allowed for the possibility of perceiving mathematics as a subject that was spoken, shared, conjectured and refuted. This allowed participation of personal meanings with opportunities for justification and ratification because of such participation. In such an opportunity the **teaching-learning of mathematics** was not centered in the teacher or the textbook or other student ‘authorities’ in the classroom but **shared, common and accessible** to everybody. The instance of rarefied ways of speaking that I have identified as a genre was evidence of shared meaning which only subsequently become a genre. Its existence implied the existence of socially shared meaning.

In order to gradually shift the participation of the personal meaning of students in order to be deliberate upon these, the **teachers visited students’ tables**. Visits to students’ tables by teachers were two aspects rolled in one. On the one hand they enabled, their offering human assistance, while simultaneously guiding student behaviour both in group cooperation and within mathematics. On the other hand, they allowed the teacher the possibility to assess the teaching-learning activity they initiated by observing the outcomes of instruction at the group-tables.

Two kinds of meanings and experiences constituted the above participation. The first was the **prior meaning and experience** that enabled participation and other, the meaning and experience made in such participation. Orthogonal to the everyday nature of the first, the second was a result of participation in **an intentional environment**. Participation of students in goal directed group-tasks, guided the participation and communication of students ‘for’ mathematics. In the communication so demanded, the students simultaneously exhibited their understanding of other’s intentions as well as their own: intentions, meaning or knowing.

Yet another kind of participation in the classroom was of the nature of **following conventions** by which participation in the classroom was simultaneously participation in the **larger mathematical praxis**. I now discuss instances evidenced by my final units of analysis.

‘Activity’ and appropriation

Unlike the focus on behaviour as a result of mediated action in the first, ways of knowing in enculturation in the second, and the possibilities of participation in a collaborative practice in the third, I analysed with the fourth unit of analysis the transformations that were a result of students’ encounter with materiality. I summarise below instances of ‘activity’ in my thematic chapters and conclude with instances of re-appropriation.

‘Activity’

In the last three of my data and analysis chapters, I provided detailed analysis of three kinds of ‘activity’. Such analysis was indicative of the nature of material conditions that were encountered in the achievement of goals, making practical activity both the object of study and analysis. I discuss below the three different applications I offered of ‘activity’.

In the first ‘activity’ the concept of scale factor in all three dimensions was analysed across many instances of teaching-learning. Towards the first dimension, a group-task was conducted to commence the topic. Calling upon prior knowing the scale factor of lengths was *designated* in this instance. The scale factor of two dimensions was *consolidated* over many problems on area and formalised in its application to a square. The scale factor in three dimensions was the culmination of the ‘activity’. In discussing the volume of a cube, the students were able to conjecture its scale factor. The ‘activity’ allowed the analysis of transformation in which the intuitive and **personal meaning** of students was **objectified** and the **propositional meaning** of scale factor was **subjectivised**.

In the second ‘activity’ the concept of direct and inverse proportionality was both objectified and applied. The ‘activity’ analysis was applied to three group-tasks conducted in quick succession. In the first, the personal meaning of the students of proportional quantities was *expressed* in the propositional terms of graph, table and formula. In the second, the same three propositional parameters were *sought* in inversely proportional quantities. Finally, these propositional parameters were *applied* to relationships offered in the three propositional forms, in which personal meaning was objectified. The ‘activity’ not only allowed the analysis of transformations that took place when personal meaning was objectified and the propositional meaning was subjectivised, but also when **propositional meaning** was **applied** to given examples.

The final ‘activity’ was constituted by the conduct in quick succession, of two group-tasks (When together and How heavy) with seemingly independent goals. This brought about the *initiation* of student cooperation in groups and the *consolidation* of such a practice. The success of the initiation, followed by its consolidation in the second, led its outcomes to initiate the transformation of common sense and personal meaning into a propositional form of teaching-learning. In the conduct of ‘activity’ it was the **students’ participation** and meanings which were **objectified** and the **teachers’ intentions** which were **subjectivised**.

The observations highlighted in the above examples of ‘activity’, evidence how it was the material structure of the tasks, in addition to verbal communication, that brought forward meaning in a form recognisable to the participants. I now discuss the nature of meaning made in individual instances of re-appropriation.

Re-appropriation

I had used the term re-appropriation to extend and identify the individual nature of actions associated with appropriation, by which available cultural capacities were made one's own. I highlighted four examples.

In the first example, four students evidenced appropriation of the **writing in the textbook** in order to respond to 'what could be said' about the directly proportional graph they had drawn, in a written form and in English. In their responses, the students appropriated the propositional form in the Norwegian and evidenced diversity in their responses, displaying personal meaning. It was appropriating the writing, taking into account personal meaning, which I had termed as re-appropriation.

The second and third case of re-appropriation, was evidenced by the attempts of students at conjecturing the movement of an elevator between six floors, with its halting at each of the floors in-between. In the conduct of the task, the motion of the elevator between two floors was represented by a graph labelled P, Q and R. In the discussion of the students a sound was also made to convey a car accelerating and decelerating, in comparison to the sound made by the elevator. In their responses the construct of '**P, Q and R**' was re-appropriated as a sequence, to represent a succession of accelerations and decelerations indicative of the halting at each of the floors. Similarly, the sound of '*Doying!*' of the halting or deceleration of the elevator was used in succession to communicate the halting at each of the floors in-between. In either case the students re-appropriated earlier constructs they communicated with.

The fourth case of re-appropriation, was the use of symbolic form of a **mnemonic**, privileged by the teachers to represent the relationship between speed, distance and time. This symbolic form was re-appropriated by a student in later teaching-learning and applied while formalising the nature of relationship between inversely proportional quantities.

In discussing the above cases of re-appropriation, I had pointed to the significance of the fact that it was the shared endeavour between individuals and cultural resources, which resulted in students changing their ways and bringing about instances of **generic appropriation** which in turn led to instances of **individual re-appropriation**. The incidence of the instances of re-appropriation, were as mentioned before, also a result of participation in circumstances in which the **need to appropriate** existed. Such a need was addressed as also pointed out earlier, to the freedom to act out one's intentions, choices and goals.

Having deliberated upon the nature of analysis evidenced by the four units of analysis, against a backdrop of the three topics of mathematics, and the four grounded themes that evolved in my study and thesis, I now turn to discuss the micro-culture that was constituted in the teaching-learning of the classroom I conducted my study in.

The micro-culture constituted

As a synthesis of my theoretical perspectives, methodology, data and analysis, I theorise that the micro-culture constituted was one of possibilities, nurtured towards the teaching-learning of mathematics. As an ‘as is’ study, where I did not steer the day to day teaching-learning of the classroom, I took upon the role of analysing the actions of others. Revealed on pursuit and reflection, I observed the providing of opportunities for meaning making and problem solving, and the constitution of a comprehensive Zone of Proximal Development. I elaborate below.

The centrality of meaning making

My two data and analysis chapters, on the establishment of a collaborative practice (Chapter 4) and the consolidation of meaning (Chapter 5), evidenced how the making of meaning, both **of the practice** and that made **in the practice** was central to the micro-culture of teaching-learning in the classroom. The emphasis on meaning was commensurate with the shift in emphasis of perceiving mathematics in humanistic terms as argued by Lakatos (1976), offering opportunities for growth of meaning as argued by Thom (1973) with emphasis on conferring existence in teaching-learning as argued by Gowers (2002).

In such an implementation, meaning was provided in order that students ascribe their personal meaning as argued by Skovsmose (2005). In the sphere of practice so established, personal meaning of the students was brought forward to becoming central to the teaching-learning of mathematics in the classroom. Such an implementation made the democratic participation of students possible, where it was the meaning being made by the students, which the teachers were given attention to.

The establishment of the above in teaching-learning made possible numerous ways of knowing mathematics. Not limited to the instruction of mathematics alone, such teaching-learning was inclusive of didactical, pedagogical and linguistic aspects combined with values. These aspects made possible the existence of a micro-culture and the enculturation of students in knowing mathematics as argued by Bishop (1988; 2004).

The enabling of personal, individual, independent and intuitive components in the meaning making and knowing by the students, was evidence of the nature of classroom interactions that shaped curriculum in this classroom as argued by both Nickson (1992) and Doyle (1988). It was the possibility of such interactions, which were the affordances of the classroom as argued by Boaler (1999), which made the students agents of their own learning as argued by Burton (1999a; 1999c) and which enabled tacit components in learning as argued by Ernest (1994).

Participation in the enabling of meaning and knowing in teaching-learning in the classroom, was importantly accompanied by the making public and sharing of one’s meaning which allowed the micro-culture

and meaning made by the individual to converge as argued by Bruner (1990; 1996). The making public of meaning, and the transparency with which meaning was made public by both students and teachers, in turn made possible the very establishment of joint intentionality as argued by Olson (2003) by which it was possible for students to take responsibility for their learning of the propositional forms of meaning in society.

The making propositional of personal meaning and knowing, was made possible in addition through the mediation of artefacts in practical and concrete activity, which enabled cognition as argued by Cole (1996) where the micro-culture of the classroom and the cognition of the individuals converged. Involvement in practical activity allowed students to recognise their abilities in the structure of materiality, as argued by Leont'ev (1978; 1981c) and realise in schooling, abilities hitherto not known to them as argued by Luria (1994). In possessing the ideal of meaning making, the dynamic environment of teaching-learning in the classroom, allowed for numerous instances where socialisation was possible, which in turn allowed for possibilities by which the natural behaviour of individuals could be modified to cultural forms of behaviour as argued by Vygotsky (1981a; 1997b).

The goal-directness of problem solving

The pursuit of goals gave direction to the processes of meaning making in the micro-culture constituted. In the form of problems in mathematics to be solved, goals regulated the making propositional of personal meaning and the applying of what is known to specific problems. As different from the meaning of participation and meaning made in participation, it was goal-directedness which gave **meaning to teaching-learning**.

It was in the processes of collaboratively understanding and resolving the goals of problems as argued by Vygotsky (1978), Cole (1996) and Wells (1999) that the mediation of artefacts brought about cognition, meaning, knowing and knowing how-to. Exemplified by instances of mediated action and agency as argued by Wertsch (1998), it was the goals of problems towards which the intention and choice of artefacts was created and its utilisation realised. Such utilisation in turn allowed for the expansion of the intellectual repertoires and practical skills of the concerned individuals in situated activity as argued by Säljö (1998).

The functional importance of speech, as one form of mediated action with words as artefacts, in increasing or raising the level of individual consciousness as argued by Vygotsky and Luria (Vygotsky et al., 1994b) was evidenced towards the goals of the problems to be solved. In contrast to regulating functions towards cultural forms of behaviour, it was also the goals set out by the structure of materiality that led to the objectification of personal meaning and subjectification of objects and also the appropriation of cultural capacities as argued by Leont'ev (1981b).

It was the goals of teaching-learning, and those of the teaching-learning of mathematics in particular, that provided the opportunity for appropriation by participation in teaching-learning by the students as argued by Rogoff (1999; 2003). Such opportunities provided, as also argued by Rogoff (1995), the need to guide the participation of the students by the teacher to specific kinds and forms of knowing, so as to enable participation of the students in the larger praxis of mathematics. As the reflexive component of meaning making, the processes of problem solving in the classroom, dictated by application towards goals, made the classroom a location for apprenticing students to greater knowing, in a microcosm of sense making as argued by Schoenfeld (1991; 1992).

A comprehensive Zone of Proximal Development

The premise of meaning making, accompanied by the goal-directness offered by problem solving, was realised and made conscious in a comprehensive ZPD. By this I mean that the micro-culture of teaching-learning, evidenced ways in which the cooperative efforts of the students, and the collaborative practice established in the classroom, created a potential of 'cultural' possibilities and thereby development.

The comprehensive ZPD was situated in making the personal meaning propositional, the spontaneous scientific or the everyday scientific as argued by Vygotsky (1978). Such a transition was possible by the lure of language as argued by Bruner (1984) in which not only knowing but also consciousness was vicariously shared. It was the direction offered by problem solving, by which teachers and students were able to appropriate each other's understanding and meaning making processes as argued by Newman et al. (1989). The utterances between individuals that took place where words were borrowed and countered as argued by Bakhtin (1986; Holquist, 2002), provided the possibility of learning to think, as he said, under a loan of greater and collaborative consciousness.

The presence of discourse in the comprehensive ZPD afforded different possibilities at the level of the classroom teaching-learning and at the level of student cooperation. In classroom teaching-learning, it was possible to bring about the sharing of problem solving know-how, which Pólya (1987) argued as the main problem of high school teaching of mathematics, where in turn teaching problem solving know-how was teaching mathematics. Such a possibility was however realised by including the greater consciousness brought about by the functional use of words as argued by Luria (1973) and therefore possible only upon the establishment of a ZPD, where know-how, consciousness and thinking could be loaned as argued by Bakhtin earlier. I conjecture that the processes of privileging, engagement and continuity, were possible and recognisable to the participants again only in the presence of a ZPD, in which the everyday was being transformed to the academic.

At the level of student cooperation, it was the discourse associated with the use of artefacts as argued by Sfard (2000) that enabled the artefacts to represent both actual and virtual reality. Allowing the modelling of problems and solutions, such an eventuality was again situated in a ZPD. I conjecture once more that the group space, a construct I identified at the stage of identifying classroom practices, made its material existence allow for sharing and imitating, yet become a ZPD only upon the realisation of personal meaning into its more propositional form.

The consistent pursuit of the transformation of the personal to the propositional, in the micro-culture of teaching-learning made the ZPD possible, and in the establishment of the ZPD made the micro-culture of the teaching-learning in the classroom possible. In being pursued over numerous actions and instances of knowing, its establishment allowed finally, importantly and significantly for the genetic development of the individual as argued by Vygotsky (1994e; 1997b). I conjecture that for the behaviour of students like Dan (who took a free ride, Chapter 7), it was the lack of taking on the responsibility of bringing forth personal meaning and making meaning propositional and be part of the ZPD that was missing. It was the success of the ZPD, which constituted and sustained the micro-culture of the teaching-learning in the classroom.

Some implications of my study

It was my personal goal of understanding the teaching-learning process in theoretical terms, which drew me to research and led me to the classroom as a researcher. My presence in the classroom was therefore neither as a student, nor as a teacher, but as someone reflecting on the nature and purposes of teaching-learning of mathematics.

I hope the writing in my thesis has revealed, that in the opportunities I had, it was the processes of the classroom which I spent my time and energy upon: meaning making, problem solving, teaching-learning and the ZPD. The conception of the classroom as a culture which was elusive to me when I began my study; was however found embodied in specific instances and themes, evidenced in terms of the perspectives which I adopted. In concluding my thesis I discuss the implications that my study may have towards the role of the teacher and the student, the nature of teaching-learning, classroom pedagogy, the didactics of mathematics, the existence of a micro-culture, classroom research in mathematics education and the understanding of teaching-learning for educators.

My findings seem to imply that if the personal meaning of the students is to be transformed into its propositional form, then it is important to set up an intentional practice for such a purpose, wherein the personal meaning of the students becomes central to the teaching-learning of the classroom. Such a practice seems to require the empowering of cogni-

tion, with the freedom associated with the choice and use of artefacts, besides allowing democratic participation. This in turn implies that very conscious and goal driven decisions about classroom norms are the role of a teacher, whose 'control' of the classroom is in giving attention to the conceptual development of the meaning making of the students.

My findings also imply that beyond mere socialisation with peers in the classroom, it is for the students to take the responsibility of bringing forth personal meaning and making meaning propositional in the classroom. This in turn implies the necessity for the students to understand that, that is what their participation in the classroom is basically about. It is towards bringing about such conscious realisation, that the goal directed practices established by the teacher are important in addition.

My findings also evidence the difficulty in delineating instances of learning, with no corresponding events of teaching; and events of teaching, with no corresponding events about learning. In emphasising teaching-learning and highlighting not merely the existence, but also the nature of relationships between participants, it is both the teachers and the students who make meaning: both of each other and of mathematics. This in turn implies that the time taken and the opportunities provided for the establishment of goal-directed intersubjectivity is valuable. It is only upon the establishment of such intersubjectivity, that the meanings being made can lead towards sustained conceptual learning, in turn providing the opportunity and possibility for genetic development.

It follows that classroom pedagogy needs to be goal driven and goal-directed. The implications of taking such a stand are in making any group space a goal-directed zone of proximal development. While a group space can exist physically, its utilisation needs to become a ZPD, wherein the associated activity is driven towards the making of personal meaning more propositional, the spontaneous more scientific and the everyday more academic. Classroom pedagogy, whether realised in cooperative group or whole class teaching-learning, thus needs to enable a ZPD and make the constituted ZPD, the basis for teaching-learning.

My findings finally imply that the teaching-learning of mathematics is not to do with instruction in mathematics alone, but with the embedding of its instruction in a micro-culture, which enables ways of knowing that are also didactical, pedagogical and linguistic in nature. With the constitution of a ZPD as the ideal, such a micro-culture is significantly associated with the incorporation of accompanying values. As to the constitution, existence and sustenance of a micro-culture, the making transparent and common of both context and content allows for the continued development of teaching-learning in the classroom.

Being a document of research in mathematics education, I conclude with my take on the nature and benefits afforded by my theoretical per-

spectives and methodology. While the individual-in-social perspective brought in a complex of issues, across numerous disciplines such as mathematics, anthropology, psychology, didactics and pedagogy, the nature of findings that I have argued for are also a result of the adoption of such perspectives. Though the nature of analysis of classroom teaching-learning by adopting these perspectives was complex, and needed long periods of observation and analysis, the creative process involved in its synthesis was fruitful in contributing to the understanding of teaching-learning of mathematics in the classroom.

Getting down to the brass tacks of the teaching-learning of mathematics in the classroom, I finally summarise for teachers and educators as follows. Opportunities need to be provided for students to bring forth their personal meaning for propositional meaning or knowing to be made. The nature of classroom practices, enabling such opportunities therefore needs to allow students to take on the responsibility of bringing forth personal meaning and making knowing propositional. This is possible by establishing and demonstrating in practice, that it is the student's meaning making that is central to teaching-learning in the classroom.

Beyond classroom practices, it is practical activity at the students' tables that allows for meaning making. These activities need to be designed and conducted with intent, and address the transformation of the personal meaning into its propositional form. These activities would in all probability begin with the more concrete and physical experiences for the students and become increasingly intellectual for them. Such activities could also make the processes of meaning making and problem solving reflexive, where increasing meaning made by the students is conducive to the capabilities and processes called for in problem solving.

Problems in mathematics offer numerous opportunities to pursue goals. It is in the pursuit of their goals that students have the possibility of building upon their existing and older capabilities. The nature of such pursuit enables students to not only acquire and build, but importantly become conscious of the capabilities that they are building.

The grouping of students for goal-directed activity has the advantage of placing the students in position to conjecture and appropriate skills and ways of knowing from each other. Implementation of practical group activities towards a ZPD, also allows students to externalise their thinking and use language with which to achieve two very significant aspects in their development: to think with and to solve problems with. In summary, with a balance of meaning making and problem solving it is possible to achieve meaningful teaching-learning of mathematics, along with the crucial contribution of time required to achieve this, in the classroom.

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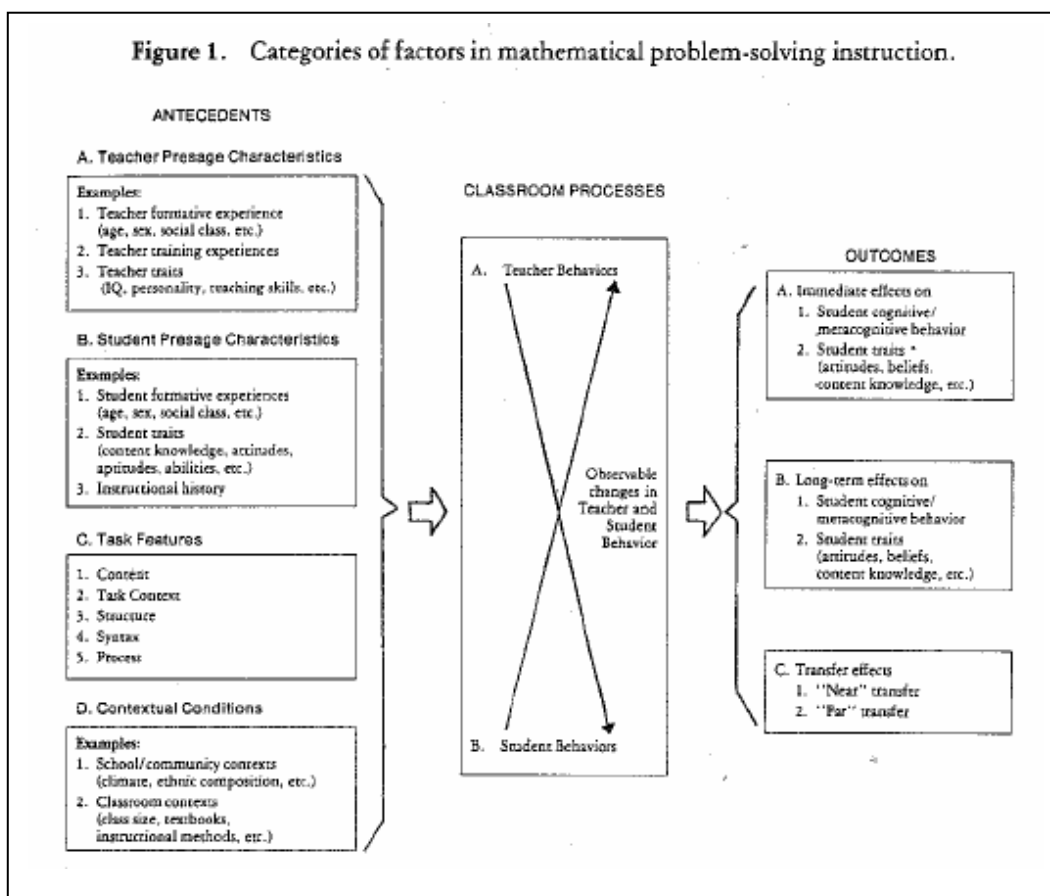
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Appendices

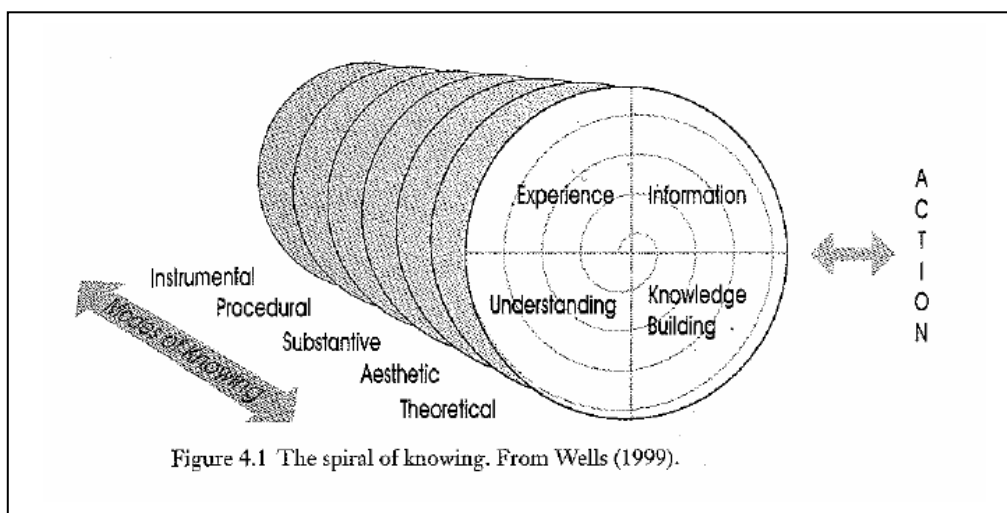
A.1: Categories of factors in problem solving instruction

(From Lester 1985, p.57)



A.2: The spiral of knowing

(From Wells, 2000, p.75)



A.3: Feedback worksheets

(Three worksheets on 8 sides; the first worksheet has student responses)

[A] Fill in the blanks: C

1. $\frac{1}{3} + \frac{4}{5} + \frac{8}{15} = \frac{1}{3 \cdot 5} + \frac{4}{5 \cdot 3} + \frac{8}{15 \cdot 1}$
2. $\frac{\frac{2}{3} + \frac{5}{9}}{1 + \frac{7}{6}} = \frac{\frac{2}{3} \cdot \frac{1}{1} + \frac{5}{9} \cdot \frac{1}{1}}{1 \cdot \frac{1}{1} + \frac{7}{6} \cdot \frac{1}{1}}$
3. $\frac{4^6}{4^2} = 4^3 \checkmark$ *I don't understand 2 thing*
4. $5^4 = \frac{5^4 \cdot 5^0}{5^0} \checkmark$ *but look, no drawings today!*
5. $(2^3)^4 = 2^{12} \checkmark$
6. $(3^2)^2 = 3^4 \checkmark$
7. $\frac{2^3 x^5}{2^5 x^3} = 2^{-2} x^2$
8. $(2,3 \cdot 10^8) \cdot (1,6 \cdot 10^{-6}) = 3,68 \cdot 10^2 \checkmark$
9. $\frac{0,00045 \cdot 0,0012}{27\ 000\ 000} = \frac{4,5 \cdot 1,2}{2,7} \cdot 10^{-7} \checkmark$
10. $9\ 460\ 000\ 000\ 000\ 000 = 9,46 \cdot 10^{14}$

1/16

[B] For the error that is circled, tick the correct answer:
I don't understand the question

1) $\left(\frac{2}{3}\right)^{-1} = \frac{-2}{-3}$ A) $\frac{2}{3}$ B) $\frac{3}{2}$ C) $\frac{-3}{2}$

2) $(-3)^2 = -9$ A) -6 B) $+9$ C) $+6$

3) $-(-5)^3 = -125$ A) -15 B) $+15$ C) $+125$

4) $3^{-1} = -3$ A) $\frac{1}{3}$ B) $+3$ C) $\frac{3}{1}$

5) $x^4 \cdot x^{-4} = x$ A) x^8 B) 1 C) x^{-16}

6) $\left(\frac{1}{2}\right)^{-1} = -0.5$ A) $\frac{1}{2}$ B) $-\frac{1}{2}$ C) 2

7) $x^1 = 1$ A) x B) $\frac{1}{x}$ C) 1^x

8) $11^0 = 11$ A) $\frac{1}{11}$ B) $\frac{11}{1}$ C) 1

9) $8\,700\,000 = 8,7 \cdot 10^5$ A) $8,7 \cdot 10^6$
B) $8,7 \cdot 10^7$
C) $8,7 \cdot 10$

10) $\frac{1}{1,67 \cdot 10^{-27}} = 1,67 \cdot 10^{27}$ A) $\frac{1}{1,67} \cdot 10^{27}$
B) $1,67 \cdot 10^{-27}$
C) $\frac{1}{1,67} \cdot 10^{-27}$

[C] Circle the error in the following:

1. $\frac{\frac{2}{3}}{\frac{3}{4}} = \frac{2 \cdot 3}{3 \cdot 4}$

2. $\left(\frac{3}{2} - \frac{3}{5}\right) : \left(1 + \frac{1}{5}\right) = \left(\frac{3 \cdot 5}{2 \cdot 5} - \frac{3 \cdot 2}{5 \cdot 2}\right) : \left(\frac{10}{10} + \frac{1}{5}\right)$

3. $\frac{a^7 \cdot a^5}{a^3} = \frac{a^{7 \cdot 5}}{a^3}$

4. $\frac{(2a)^2}{2^5} = 2^{2+5} \cdot a^2$

5. $\frac{x^2 \cdot (x^3)^2}{x^5} = x^{2+5-5}$

6. $(3x^2)^5 = 15x^7$

7. $(2x)^2 \cdot (2x^2)^3 = 2^{2+3} \cdot x^{2+3}$

8. $0,0045 = 4,5 \cdot 10^{-2}$

9. $7 \cdot 10^{-3} = \frac{1}{7} \cdot 10^3$

10. $(3 \cdot 10^{-3})^2 \cdot (3 \cdot 10^{-2})^{-1} = 3^{2+1}$

Hz Hz, do you think I was going to stop drawing?

No I am totally dependent

It's better than smoking through



[D] Fill in the blanks:

1) $9 \frac{3}{16} - 10 \frac{7}{8} + 4 \frac{1}{4} = 9 \frac{3}{2} - 10 \frac{7}{2} + 4 \frac{1}{2}$

2) $(-1)^{-} = 1$

3) $x^{\frac{4}{5}} \cdot x^{-\frac{4}{5}} = x^{-}$

4) $\frac{k^{10}}{k^{-}} = k^{3 \cdot 8}$

5) If $2^m = m$ then $m^2 = 2^{-}$

6) If $a = x^2$ then $a^2 = x^{-}$

7) If $a = 6,3$ then $a^2 - 3a = \underline{\hspace{2cm}}$

8) $(\underline{\hspace{1cm}} \cdot 10^4) \cdot (7 \cdot 10^5) = 14\ 000\ 000\ 000$

9) $(5 \cdot 10^7) \cdot (\underline{\hspace{1cm}} \cdot 10^3) = 5\ 000\ 000\ 000$

10) $(5 \cdot 10) \cdot (2 \cdot 10^3) \cdot (\underline{\hspace{1cm}} \cdot 10^4) = 10^{10}$

Let us xyz!

A. Fill in the blanks:

$$\frac{2x}{5} = \frac{\quad}{10} \quad \frac{4a}{5b} = \frac{12a^2}{\quad} \quad \frac{\quad}{7x} = \frac{9xy}{21x^2}$$

B. What 'fellesnevneren' will you use to simplify the following:

1) $\frac{5x}{6} - \frac{x}{4}$

2) $\frac{3}{4a} + \frac{5}{7b}$

3) $\frac{3a-2b}{10ab} + \frac{2b-3c}{15bc}$

C. What will you do to retain x on the left hand side:

1) $3x = 15$ _____

2) $\frac{1}{2}x = 5$ _____

3) $13 + x = 25$ _____

4) $x - 10,5 = 33,8$ _____

D. Simplify if possible

1) $5 + 9y - 3y$ _____

2) $3a + 4b - 6c$ _____

3) $3x - 2 - 7y - 4 + 5z + 6$ _____

4) $a - 2b - 4a + 3c + 4a + 5b$ _____

E. Rewrite the following without the bracket:

1) $4x - (5y - 4)$ _____

2) $5a + (2 - 3b + 6c)$ _____

3) $(6x - 5y) - (7z - 15)$ _____

4) $-(3a + 5b) + (4c - 5d) - (-14)$ _____

F. Solve the following:

1) $8b - (3b + 4) = 11$

2) $14x = (18 - x) - (15 - 6x)$

3) $\frac{x-5}{2} = \frac{x-4}{3}$

G. Fill in the bracket:

- 1) $3a - 2b - 2c = 3a - (\underline{\hspace{2cm}})$
2) $5w - 4x - y + 2z = 5w - (\underline{\hspace{2cm}})$
3) $7x + 12y - 8z = 7x + 4(\underline{\hspace{2cm}})$
4) $12a - 4b - 16c - 8d + 20e = -4(\underline{\hspace{2cm}})$

H. Solve the following:

- 1) $5x + 6 - 3x = 4x + 7 - 4x$
2) $\frac{x}{2} + \frac{7}{4} = \frac{x}{4} + \frac{3}{2}$
3) $\frac{6x-3}{7} - \frac{2x+1}{3} = 0$

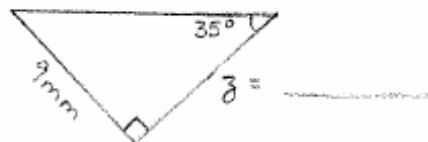
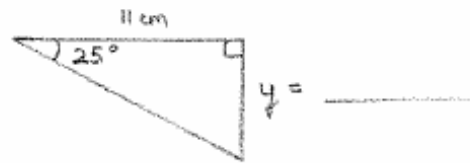
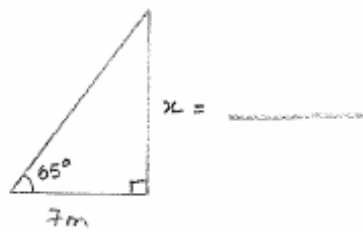
I. Remove the bracket:

- 1) $5(a + 2b - 3c)$ _____
2) $-3(x + y + z)$ _____
3) $7(a - 4b) + 3(x + 5y)$ _____
4) $2(x + 8) - 5(6x - 9)$ _____

J. Solve the following:

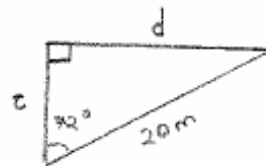
- 1) $\frac{3a-1}{4} - \frac{5a+2}{3} + \frac{7a-3}{6} = 0$
2) $\frac{2(5x-3)}{3} - \frac{3(5x-2)}{5} = \frac{8}{15}$
3) $\frac{5(3x-1)}{6} - \frac{3(5x-3)}{8} = \frac{9x-5}{24}$

I : FIND THE UNKNOWN SIDES



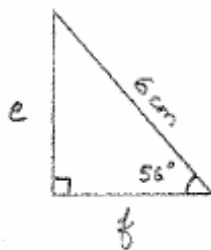
$a = \underline{\hspace{2cm}}$

$b = \underline{\hspace{2cm}}$



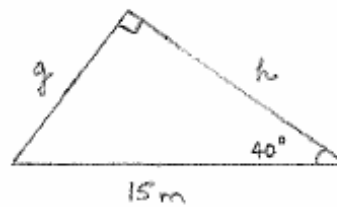
$c = \underline{\hspace{2cm}}$

$d = \underline{\hspace{2cm}}$



$e = \underline{\hspace{2cm}}$

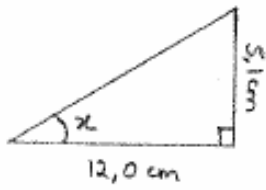
$f = \underline{\hspace{2cm}}$



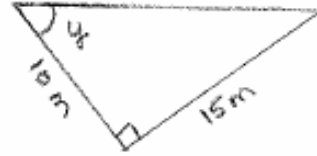
$g = \underline{\hspace{2cm}}$

$h = \underline{\hspace{2cm}}$

II: FIND THE UNKNOWN ANGLES



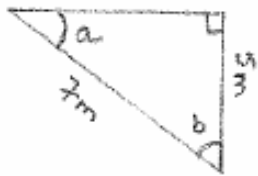
$$x = \underline{\hspace{2cm}}$$



$$y = \underline{\hspace{2cm}}$$

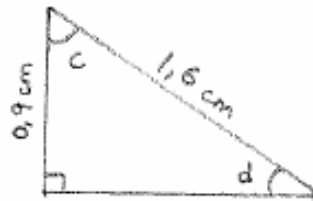


$$z = \underline{\hspace{2cm}}$$



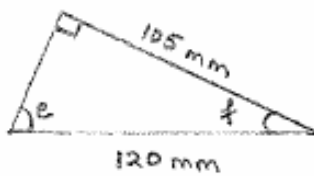
$$a = \underline{\hspace{2cm}}$$

$$b = \underline{\hspace{2cm}}$$



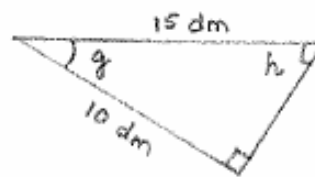
$$c = \underline{\hspace{2cm}}$$

$$d = \underline{\hspace{2cm}}$$



$$e = \underline{\hspace{2cm}}$$

$$f = \underline{\hspace{2cm}}$$

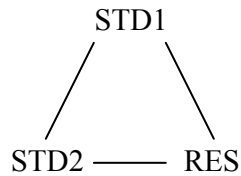


$$g = \underline{\hspace{2cm}}$$

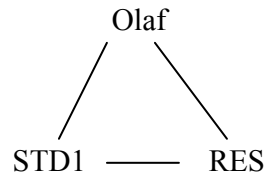
$$h = \underline{\hspace{2cm}}$$

A.4: Kinds of data triangulation

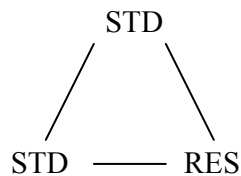
Following Denzin (1970) and as mentioned in Chapter 3, the following kinds of data triangulation were applicable to my study: time, space and person. The first six diagrams below represent person triangulation; the seventh represents space triangulation and the eighth time triangulation. In representing triangulation diagrammatically, the signs used for representing the students are STD, STD1 and STD2. The teachers are represented by their names Olaf and Knut and the researcher by RES.



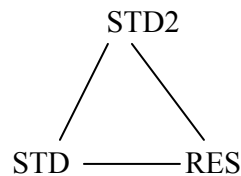
1: Students within a group-in-focus



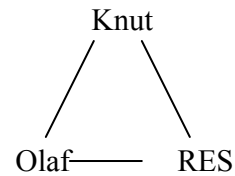
2: Student in group-in-focus and Olaf



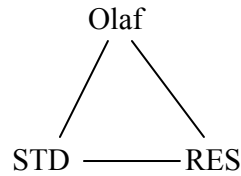
3: Other students in the class



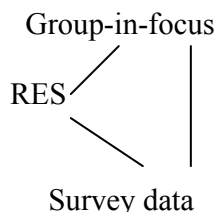
4: Student in class and in group-in-focus



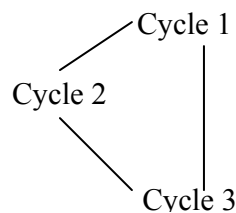
5: The two teachers during teaching-learning



6: Other student in class and Teacher



7: Group task by class and group



8: Cases reported across cycles

A.5: Permission from publishers

(Extract from e-mail)

Subject: [Fwd: [Fwd: Re: [Fwd: Referencing diagrams etc in thesis from SINUS]]]

From: Hans Erik Borgersen <hans.e.borgersen@hia.no>

Date: Tue, 28 Jun 2005 10:24:40 +0200

To: Sharada Gade <sharada.gade@hia.no>, Barbara Jaworski <barbara.jaworski@hia.no>

Dear Sharada,
Audhild has asked Cappelen forlag v/Otto Svorstøl, and he says ok to copy diagrams from Sinus in your thesis if you make references as suggested.
Best wishes,
Hans Erik

----- Opprinnelig melding -----

Emne: [Fwd: Re: [Fwd: Referencing diagrams etc in thesis from SINUS]]

Dato: Tue, 28 Jun 2005 10:00:09 +0200

Fra: Audhild Vaaje <audhild.vaaje@hia.no>

Til: Hans Erik Borgersen <hans.e.borgersen@hia.no>

Kjære Hans Erik,
nå har jeg spurt Cappelen v/Otto Svorstøl og han er enig med deg.
God sommer til Mia og deg!
Hilsen Audhild

----- Opprinnelig melding -----

Emne: Re: [Fwd: Referencing diagrams etc in thesis from SINUS]

Dato: Tue, 28 Jun 2005 09:53:37 +0200

Fra: Otto Svorstøl <otto.svorstol@cappelen.no>

Til: Audhild Vaaje <audhild.vaaje@hia.no>

Referanser: <42C0FEAA.5020104@hia.no>

Synes det er ok ved å referere til læreverket som hun antyder. Trenger ikke ytterligere tillatelse i dette tilfellet.

Otto

A.6: Letter asking for consent and consent slip

(This appendix is on 2 sides)

Forespørsel angående deltagelse i forskningsprosjektet “A study of conjecturing with artefacts in the mathematics classroom”

Kjære studenter

Mitt navn er Sharada Gade og jeg er doktorgradsstudent ved Høgskolen i Agder (HiA) med matematikdidaktikk som fagområde (didaktikk handler om sammenhengen mellom undervisningens begrunnelse, innhold og gjennomføring). Jeg ønsker å samle inn data i forbindelse med matematikkleksjonene i din klasse i løpet av skoleåret 2004-2005.

Formålet med mitt prosjekt er å studere elevenes matematikkforståelse- og læring. I den forbindelse vil jeg ta feltnotater, samle inn data ved hjelp av matematikkprøver, gjøre digitale lydopptak (mini-disc) under intervjuer og fotografere elevenes ferdige eller uferdige matematikkarbeid (jeg vil ikke fotografere elevene). I løpet av prosjektperioden vil alle elever bli tildelt et eget nummer eller et pseudonym som brukes i forhold til de ulike delene av datamaterialet. Nummeret eller pseudonymet viser til en navneliste som oppbevares adskilt fra datamaterialet. Navnelisten og samtykkeerklæringene vil bli slettet senest ved prosjektslutt 30.06.2006.

Deltakelse i prosjektet er frivillig og du kan på et hvilket som helst tidspunkt før prosjektslutt trekke ditt samtykke til deltagelse i prosjektet tilbake uten at dette medfører begrunnelsesplikt eller får noen andre konsekvenser, og du kan i løpet av samme tid kreve opplysninger om deg selv slettet.

Etter fullføring av mitt prosjekt og innlevering av Ph.D.-avhandlingen i 2006, vil datamaterialet bli forsvarlig oppbevart ved HiA, på ubestemt tid. Datamaterialet (råmaterialet, inkludert lydopptak) kan i fremtiden bli utlånt til andre forskere innenfor fagområdet matematikdidaktikk etter avtale med HiA. Det vil ikke lages kopier av lydopptak eller andre deler av materialet. I publikasjoner og rapporter vil ingen opplysninger som fremkommer kunne tilbakeføres til enkeltelever.

Jeg er som forsker underlagt taushetsplikt og alle data vil bli behandlet konfidensielt. Prosjektet er meldt til Personvernombudet for forskning, Norsk samfunnsvitenskapelig datatjeneste AS, Norge.

Jeg vil også be deg om å vise dette brevet til dine foreldre dersom du er under 18 år. Dersom du ønsker å delta i prosjektet ber jeg om ditt skriftlige samtykke (se samtykkeerklæring neste side). Du kan gi samtykkeerklæringen til din lærer som vil videreformidle den til meg. Dersom du eller dine foreldre har noen spørsmål er du/ dere velkommen til å ta kontakt med meg eller med mine veiledere.

Sharada Gade, Stipendiat
Fakultet for Realfag, Høgskolen i Agder,
Gimlemoen 25J, Service Boks 422,
4604 Kristiansand
e-mail: Sharada.Gade@hia.no

Veilederne mine (samme post/ besøksadresse, Høgskolen i Agder):
Barbara Jaworski, Professor (e-mail: Barbara.Jaworski@hia.no) og
Hans Erik Borgersen, Førsteamanuensis (e-mail: Hans.E.Borgersen@hia.no).


Samtykkeerklæring

Jeg _____ (Navn) tillater at Sharada Gade samler inn personopplysninger om meg i forbindelse med prosjektet "A study of conjecturing with artefacts in the mathematics classroom" som beskrevet over, og at datamaterialet blir oppbevart bare ved Høgskolen i Agder og at det i fremtiden kan bli utlånt til andre forskere innenfor fagområdet matematikkdirakikk som beskrevet over.

Underskrift:

A.7: Copy of letter from the NSD

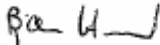
(This appendix is on 4 sides)

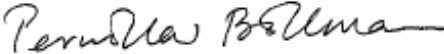
Norsk samfunnsvitenskapelig datatjeneste AS NORWEGIAN SOCIAL SCIENCE DATA SERVICES		
Sharada Gade Kaserneveien 15, Rom 105, Sersjanten 4630 KRISTIANSAND S	Hans Holmboas gate 22 N-5007 Bergen Norway Tel: +47/55 58 21 17 Fax: +47/55 58 96 50 nsd@nsd.uib.no www.nsd.uib.no Org.nr. 985 321 864	
Vår dato: 31.08.2004 Vår ref: 200400723 PS /MH	Deres dato:	Deres ref:
KVITTERING FRA PERSONVERNOMBUDET		
Vi viser til melding om behandling av personopplysninger, mottatt 10.06.2004. All nødvendig informasjon om prosjektet forelå i sin helhet 30.08.2004. Meldingen gjelder prosjektet:		
<i>11168 A study of conjecturing with artifacts in the mathematics classroom</i>		
Norsk samfunnsvitenskapelig datatjeneste AS er utpekt som personvernombud av Høgskolen i Agder, jf. personopplysningsforskriften § 7-12. Ordningen innebærer at meldeplikten til Datatilsynet er erstattet av meldeplikt til personvernombudet.		
Personvernombudets vurdering		
Personvernombudet finner at behandlingen av personopplysningene er meldepliktig i henhold til personopplysningsloven § 31 og at behandlingen tilfredsstiller kravene i personopplysningsloven.		
Personvernombudets vurdering forutsetter at prosjektet gjennomføres slik det er beskrevet i vedlegget.		
Behandlingen av personopplysninger kan settes i gang.		
Ny melding		
Det gjøres oppmerksom på at det skal gis ny melding dersom behandlingen endres i forhold til de punktene som ligger til grunn for personvernombudets vurdering.		
Selv om det ikke skjer endringer i behandlingsopplegget, skal det gis ny melding tre år etter at forrige melding ble gitt dersom prosjektet fortsatt pågår.		
Ny melding skal skje skriftlig til personvernombudet.		
Offentlig register		
Personvernombudet har lagt ut meldingen i et offentlig register, www.nsd.uib.no/personvern/register/		
<small>Avdelingskontorer / District Offices: OSLO: NSD, Universitetet i Oslo, Postboks 1055 Blindern, 0316 Oslo. Tel: +47/22 85 52 11, nsd@uis.no TRONDHEIM: NSD, Norges teknisk-naturvitenskapelige universitet, 7401 Trondheim. Tel: +47/73 59 10 07, lysn.sva@iuhv.ntnu.no TROMSØ: NSD, SVP, Universitetet i Tromsø, 9207 Tromsø. Tel: +47/77 04 43 35, nsdms@sv.uib.no</small>		

Ny kontakt

Personvernombudet vil ved prosjektets avslutning, 30.06.2006, rette en henvendelse angående status for prosjektet.

Vennlig hilsen


Bjørn Henrichsen


Pernilla Bollman

Kontaktperson: Pernilla Bollman tlf: 55583348

Vedlegg: Prosjektbeskrivelse

Kopi: Behandlingsansvarlig Hans Erik Borgersen

Prosjektbeskrivelse

Behandlingsansvarlig:

Hans Erik Borgersen

Fakultet for realfag

Høgskolen i Agder

Serviceboks 422

4604 KRISTIANSAND S

Daglig ansvar/prosjektleder:

Sharada Gade

Fakultet for realfag

Høgskolen i Agder

Serviceboks 422

4604 KRISTIANSAND S

11168 A study of conjecturing with artefacts in the mathematics classroom

The project aims at studying mathematical understanding in students. The use of artefacts as introduced by the teacher in various parts of the syllabus will be observed, and the mathematical understanding obtained in this way will be examined.

The participants/respondents in the project will be 45 students that are taught mathematics bilingually in the first year of Videregående skole. The head of project has already approached the school in which she would like to conduct the study and have had a meeting with the teachers of the class. One of the teachers will present the written information regarding all aspects of the project, as formulated by head of project, to the potential participants/respondents. Parents/custodians will receive the same letter of information regarding all aspects of the research project. The students that are interested in participating will give their written consent.

Data will be collected through the means of observations resulting in field notes, digital audio recordings of specific classroom activities, student questionnaires (mathematics tests) and interviews with students (also digitally recorded). The interviews are made with one, two or three students at a time and involve the solving of mathematical tasks. The information that will be collected includes names of the students, mathematical understanding as observed in the classroom situation, students' responses to specific mathematical tasks and actions and responses in the interviews. The individual work (mathematics tasks) of the students will be photographed (the students themselves will not be photographed).

The data material will be processed on digital audio recordings and on computer connected to intranet and Internet. Directly identifiable personal information (names) will be substituted with pseudonyms or numbers with reference to the actual list of names that will be kept separate from the data material. At the end of the project (30.06.2006) the list of actual names and written consent slips will be destroyed.


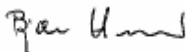
The audio recordings and other data material will be stored at Høgskolen i Agder. We would like to point out that usually the voices on audio recordings are in themselves being considered as indirect personal information. Though we would like to argue that this could be open to discussion in a project like this.

First of all the data cannot be considered as sensitive. The purpose of the Personal Data Act (personopplysningsloven) is to protect the privacy of the individual (see PDL §1). In this case we cannot see that the hypothetical disclosure of personal information would at all violate the protection of privacy of the individuals participating. Also, the consent of each participant regarding storage of the material, and other researchers access to it, will be obtained. Furthermore, the degree to which the voices in the audio recordings will be personally identifiable will decrease over time. For the most part more than one person will be recorded at the same time. Especially on audio recordings of classroom activities the individual voices should be difficult to identify. All in all it seems as though it would require a great deal of work for anyone to be able to identify the students from the recorded voices. It would follow that the audio recordings could be considered de facto unidentifiable, which means that the storage and later use of the audio recordings in themselves will not be subject to notification (meldepikt) according to the Personal Data Act (personopplysningsloven).

When it comes to sharing the data with other researchers in the area of mathematics didactics, we would like to emphasize that the responsibility for the data material rests on Høgskolen i Agder. The audio recordings are not to be duplicated. The recordings can only be lent by other researchers and used for the kind of research described in the information given to the students and their parents. It is being presupposed that Høgskolen i Agder has got adequate routines regarding responsible data storage and usage.

A.8: Status of data

(This appendix is on 2 sides)

Norsk samfunnsvitenskapelig datatjeneste AS NORWEGIAN SOCIAL SCIENCE DATA SERVICES			
Sharada Gade Kaserneveien 15, Rom 105, Sersjanten 4630 KRISTIANSAND S		Herald Hårfagres gate 29 N-5007 Bergen Norway Tel: +47 55 58 21 17 Fax: +47 55 58 96 50 nsd@nsd.uib.no www.nsd.uib.no Org nr. 985 321 884	
Vår dato: 06.06.2006	Vår ref: 11168	Deres dato:	Deres ref:
STATUS FOR BEHANDLING AV PERSONOPPLYSNINGER			
11168 <i>A study of conjecturing with artefacts in the mathematics classroom</i>			
Vi viser til tidligere innsendt meldeskjema for forskningsprosjekt som medfører meldeplikt eller konsesjonsplikt. Videre vises det til vårt svarbrev hvor det gikk frem at vi ville ta kontakt ved prosjektslutt angående prosjektets status.			
Ifølge våre opplysninger skal projektet nå være avsluttet og datamaterialet anonymisert. Personvernombudet for forskning ber om en bekrefteelse på at data er anonymisert.			
Dersom data ikke er anonymisert og det fortsatt er behov for oppbevaring av personopplysninger, må prosjektleder gi en redegjørelse til personvernombudet for hvorfor data ikke kan anonymiseres på nåværende tidspunkt. Denne tilbakemeldingen er nødvendig for at prosjektleder skal ha lovlig grunnlag for behandling av personopplysninger.			
NSD arkiverer forskningsdata for fremtidig bruk. Dersom lagring av data ved NSD er ønskelig ber personvernombudet om at data oversendes sammen med nødvendig dokumentasjon og utfylt Arkiveringskjema. Vi viser til våre nettsider for veiledning www.nsd.uib.no/personvern . Forskere som gjennomfører forskningsprosjekt med støtte fra Norges forskningsråd (NFR) minnes om at arkivering ved NSD er et kontraktsvilkår for den gitte støtte (dersom data er egnet for arkivering ved NSD).			
Tilbakemelding skjer fortrinnsvis på vedlagt statusskjema, men kan også gies pr. brev, e-post eller telefon. Vi ber om tilbakemelding innen 3 uker. Ta gjerne kontakt dersom noe er uklart.			
Vennlig hilsen			
			
Bjørn Henriksen		Siv Midthassel	
Vedlegg: Statusskjema			
Kopi: Hans Erik Borgersen			
<small>Arkiveringkontoret / Datacenter OSLO: NSD, UNIVERSITET I OSLO, Postboks 1055 Blindern, 0316 Oslo. Tel: +47-22 85 52 11. nsd@uio.no TRONDHEIM: NSD, Norges teknisk-naturvitenskapelige universitet, 7491 Trondheim. Tel: +47-73 59 19 07. byrn.warna@ntnu.no TROMSØ: NSD, SVF, Universitetet i Tromsø, 9107 Tromsø. Tel: +47-77 64 43 36. sadtse@svf.uib.no</small>			

A.9: Confirmation of declaration from the NSD

Subject: 11168 A study of conjecturing with artefacts in the mathematics classroom
From: Pernilla Bollman <Pernilla.Bollman@nsd.uib.no>
Date: Tue, 27 Jun 2006 16:41:55 +0200
To: sharada.gade@hia.no

Dear Sharada Gade,

This is to let you know, as requested, that we have registered that your data material is anonymous as of June 26th 2006.

Vennlig hilsen/best regards

Pernilla Bollman
Fagkonsulent
(Specialist Consultant)

Norsk samfunnsvitenskapelig datatjeneste AS
(Norwegian Social Science Data Services)
Personvernombud for forskning
Harald Hårfagres gate 29, 5007 BERGEN

TLF. Direkte: (+47) 55 58 24 10
TLF. sentral: (+47) 55 58 21 17
Faks: (+47) 55 58 96 50
Email: pernilla.bollman@nsd.uib.no
Internettadresse www.nsd.uib.no/personvern

A.10: Layout of topics on blackboard

Roald Dahl	Galdhøpiggen (Mountain peak)	Yes	No
Knut Hamsun	Jostedalbreen (A glacier)	May	be
Pink Floyd	Nordstjernen (North star)	India	
Black Debbath	Lille Bjørn (Little Bear)	Kenya	
	Store Bjørn (Great Bear)	Nepal	
Abel, Galois		Norway	
Ramanujan		USA	

A.11: Data collected during fieldwork

(This appendix is on 3 sides)

The entire extent of data collected in seven cycles, on observation with different groups-in-focus through the year, is summarised below. The table gives details of duration, time, topic content, students observed (pseudonyms), date of school test or examination and data collected. The table also mentions the titles of group-tasks (designed and conducted by the teachers) and the problem solving tasks (designed and conducted with the groups-in-focus by me). Of the data collected, cycles from which topics are reported, or not, have been indicated.

Summary of data collected in the year 2004-2005	
Cycle 1	
Period observed	26 th August 2004 – 16 th September 2004
Teaching time	12 teaching periods
Date of school test	15 th September 2004 (2 teaching periods)
Topic content	Number and number understanding (reckoning with fractions, exponents, rules of calculation with exponents, numbers in standard form, units and measurement, large and small numbers)
Students observed	Students throughout the class (No group-in-focus)
Data collected	Field notes, some responses to class tests, responses to feedback worksheet, responses to group tasks
Group tasks	<i>When together, How heavy</i>
Details of problem solving task	Date: 16 th September 2004 and 22 nd September 2004
	Title: <i>When together</i> and <i>How heavy</i> (Group tasks doubled up as problem solving tasks for this cycle)
	Participants: Random groups in entire class
Cases reported	Yes
Cycle 2	
Period observed	20 th September – 13 th October
Teaching time	14 teaching periods
Date of school test	20 th October 2004 (2 teaching periods)
Topic content	Formulae and equations (algebraic terms, substitution of formula with numbers, linear equations, conversion of formula, proportional quantities, inversely proportional quantities, price index, currency, real income)
Students observed	Group-in-focus: Anja, Egil, Lea, Stine
Data collected	Field notes, some responses to class tests, responses to feedback worksheet, responses to group tasks, audio-recording of and students' workings at problem solving task
Group Tasks	<i>Proportionality, Inverse proportionality, Follow Up</i>
Details of problem solving task	Date: 4 th November 2004
	Title: <i>What can we say from the given graphs?</i>
	Participants: Anja, Egil, Lea, Stine
Cases reported	Yes

Cycle 3	
Period observed	18 th October 2004 – 11 th November 2004
Teaching time	16 teaching periods
Date of school test	17 th November 2004 (2 teaching periods)
Topic content	Geometry (similar figures, similar triangles, Pythagoras' theorem, area, volume), Trigonometry (Sine of an angle, use of sine, area formula, Cosine of an angle, Tangent of an angle)
Students observed	Group-in-focus: Kim, Levi, Nora and Thor (18 th October 2004 – 27 th October 2004) Group-in-focus: Dan, Levi and Thor (27 th October 2004 – 11 th November 2004)
Data collected	Field notes, responses to group task, tasks related to <i>Kappabel</i> contest, vocabulary lists, some responses to class tests, audio-recording of and students' workings at problem solving task
Group tasks	<i>Similar triangles, Pythagoras' Theorem-Proof, Similar figures and area (scale factor)</i>
Details of problem solving task	2 nd December 2004
	<i>What can we find out about surface area and volume?</i>
	Participants: Dan, Levi, Thor
Cases reported	Yes
Cycle 4	
Period observed	15 th November 2004 – 17 th December 2004
Teaching time	17 teaching periods
Date of school test	9 th December 2004 (End of semester examination: 4 hours)
Topic content	Probability (principle of multiplication, probability, events, addition principle, independent events, probability models, conditional probability), Geometry (circle, ellipse, parabola and hyperbola, conic sections through the ages)
Students observed	Group-in-focus: Ria, Tia, Vidar
Data collected	Field notes, some responses to feedback worksheet, some responses to year end examination, additional material on conic sections, vocabulary lists, audio-recording of and students' workings at problem solving task
Group tasks	<i>Probability of an event, Independent events</i> (Both on internet)
Details of problem solving task	13 th January 2005
	<i>What can we find out, probably?</i>
	Participants: Ria, Tia, Vidar
Cases reported	None
Cycle 5	
Period observed	3 rd January 2005 – 17 th February 2005
Teaching time	27 teaching periods
Date of school test	9 th February 2005 (2 teaching periods)
Topic content	Numbers and number understanding (rational and irrational numbers), Expressions and equations (straight lines, to find slope by calculating, equation of a straight line, linear mathematical models, linear regression with calculator, graphical solution of systems of linear equations, method of substitution), Functions and quadratic equations (the concept of a function, graph of a function, zeros, maxima and minima,

	graphical solution, quadratic equation with two terms, quadratic determinant, applications of quadratic equations)
Students observed	Group-in-focus: Jan, Nora, Rolf, Sean
Data collected	Field notes, responses to group task, audio-recording of and students' workings at problem solving task
Group tasks	<i>Straight lines and linear functions, Linear regression, Polynomial functions, Solution of a quadratic-graphic and with calculation, Quadratic equations in practical situations</i>
Details of problem solving task	17 th February 2005
	<i>What can we know from the straight line graph?</i>
	Participants: Nora, Rolf, Sean, Levi (Jan absent)
Cases reported	None
Cycle 6	
Period observed	28 th February 2005 – 17 th March 2005
Teaching time	13 teaching periods
Date of school test	16 th March 2005 (2 teaching periods)
Topic content	Functions and quadratic equations (rational functions), Power functions and exponential functions (power functions, percent and growth factors, percentile change over multiple factors, exponential function, logarithms)
Students observed	Group-in-focus: Dan, Idar, Max, Ulrik
Data collected	Field notes, responses to group tasks, solutions to group task audio-recording of and students' workings at problem solving task
Group tasks	<i>Exponential functions, Logarithms, Graphs in exponential functions</i>
Details of problem solving task	17 th March 2005
	<i>How about another graph?</i>
	Participants: Dan, Idar, Ulrik, Jan (Max declines)
Cases reported	None
Cycle 7	
Period observed	30 th March 2005 – 12 th May 2005
Teaching time	18 teaching periods
Date of school test	Year-end examination (5 hours)
Topic content	Power functions and exponential functions (square roots and roots of higher order), Algebra (more on exponents, quadratic expansions, factorising, perfect squares, zeros and factorisation, zeros and coefficients, rational expressions, rational equations, non-linear equations, proof of quadratic determinant)
Students observed	Group-in-focus: Gard, Mia, Per, Tove
Data collected	Field notes, Rolf's notes on language use, audio-recording of and students' workings at problem solving task
Group tasks	<i>Square laws $[(a + b)^2, (a - b)^2, (a + b)(a - b)]$, Factorisation</i>
Details of problem solving task	12 th May 2005
	<i>Can we find a formula?</i>
	Participants: Aron, Ben, Gard, Mia, Per, Helle
Cases reported	None

A.12: Some of the tasks shared with the teachers

(This appendix contains 12 tasks spread over 8 sides)

Multiplication with Napier's help

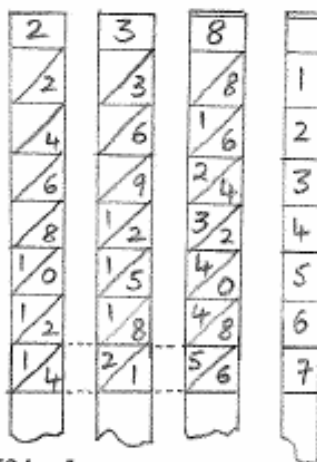
(Lattice method from *The Crest of the Peacock*, ISBN 0-14-012529)

Among many algorithms

Scotsman John Napier (1550 – 1617)

showed multiplication in the following way:

E.g. $238 \times 7 = 1666$



- Find out the product of 238 and

2, 1, 3, 5 and 8

- Make strips of paper and find:

1. 47×8 and 74×6

2. 254×7 , 425×5 , 542×9 , 245×8 , 452×6 and 524×5

3. 7593×6 , 9507×8 , 23546×7 , 51039×5

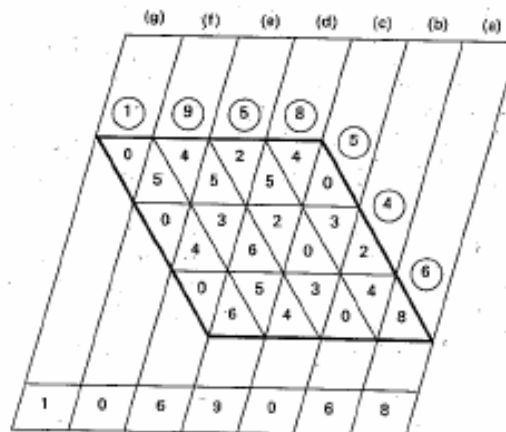
- Find out which product the following adds up to:

$$238 \times 8 = 1904$$

$$238 \times 20 = 4760$$

$$\underline{238 \times 600 = 142800}$$

- Compare the above method to the 'lattice method' shown which was popularised by Arabic mathematicians and probably of Indian origin.



- Now multiply:

1. 45×23 or any two-digit number by another two-digit number

2. Three digit number by another three digit number

- Can you improve on the 'lattice method' shown above?

Decipher this

(Problem form *Reading, Writing and Proving*, ISBN 0-387-00834-9)

- In cryptography, a cipher replaces a piece of information with another and the substitution is controlled by an algorithm or procedure.
- A substitution cipher is a cipher that replaces each 'plain' text symbol for another 'cipher' text symbol.
- A Caesar cipher is a substitution cipher in which the cipher alphabet is merely the plain alphabet rotated left or right by some number of positions.
- For instance, here is a Caesar cipher using a right rotation of three places:
Plain: ABCDEFGHIJKLMNOPQRSTUVWXYZ
Cipher: XYZABCDEFGHIJKLMNQRSTUUVW
- To encipher a message, simply look up each letter of the message in the 'plain' line and write down the corresponding letter in the 'cipher' line.
- To decipher, do the reverse. Decode the message using inverse substitution.
- What does the following say: LK JV TXV QL PZELLI F XQB XK XMMIB
- Now try to decipher: PDEO AJKZEJC WHCKNEPDI EO YWHHAZ W YWAOWN YELDAN. EP EO RANU AWOU PK XNAWG, NECDP?
- Do you have a cryptic message?

Decimal or not?

(Copy of *La Disme* from *Learning activities from the History of Mathematics*, ISBN 0-8251-2264-3)

Although the Hindu-Arabic decimal place value system was introduced into Europe in the twelfth century its acceptance was slow. The use of sexagesimal fractions (fractions with denominators 60 or powers of 60) continued. In 1585, the Belgian born Dutch engineer Simon Stevin (1548 – 1620) strongly urged the acceptance of decimal numbers with his book called *De Thiende, The Art of Tenths* or *La Disme*. Decimal numbers were however accepted only along with the introduction of the metric system by the French revolutionary government. Known as the SI, after *Système International d'Unités* its key agreement the *Convention du Mètre* was signed in Paris on May 20, 1875.

SECONDE PARTIE DE LA DISME DE L'OPERATION.

R A T I O N .

PROPOSITION I, DE L'ADDITION.

Etant donnez nombres de Disme à ajoûter : Trouver leur somme :

Explication du donné. Il y a trois ordres de nombres de Disme, desquels le premier 27 ⑥ 8 ① 4 ② 7 ③, le deuxiesme 37 ⑥ 8 ① 7 ② 5 ③, le troisieme 875 ⑥ 7 ① 8 ② 4 ③.

Explication du requis. Il nous faut trouver leur somme. *Construction.* On mettra les nombres donnez en ordre comme ci joignant, les ajoûtant selon la vulgaire maniere d'ajoûter nombres entiers, en ceste sorte :

$$\begin{array}{r}
 \textcircled{6} \textcircled{1} \textcircled{2} \textcircled{3} \\
 27847 \\
 37675 \\
 875782 \\
 \hline
 941304
 \end{array}$$

Donne somme (par le 1^{er} probleme de l'Arithmetique) 941304, qui sont (ce que demonstrent les signes dessus les nombres) 941 ⑥ 3 ① 0 ② 4 ③. Le di, que les mesmes sont la somme requise. *Demonstration.* Les 27 ⑥ 8 ① 4 ② 7 ③ donnez, font (par la 3^e definition) $27 \frac{8}{10} + \frac{4}{100} + \frac{7}{1000}$, ensemble $27 \frac{847}{1000}$, & par mesme railon les 37 ⑥ 8 ① 7 ② 5 ③ vallent $37 \frac{675}{1000}$, & les 875 ⑥ 7 ① 8 ② 4 ③ feront $875 \frac{782}{1000}$, lesquels trois nombres, comme $27 \frac{847}{1000} + 37 \frac{675}{1000} + 875 \frac{782}{1000}$, font ensemble (par le 10^e probleme de l'Arith.) $941 \frac{304}{1000}$, mais autant vaut aussi la somme 941 ⑥ 3 ① 0 ② 4 ③, c'est

- Give three examples of sexagesimal numbers.
- From Stevin's text pick the two equivalent forms of the decimal numbers 27,847. Also is there an error somewhere?
- Express the following decimal numbers in Stevin's notation:
 1. 25,6
 2. 343
 3. 6,786
 4. 932,6
 5. 0,4567
- To urge the use of decimal numbers Stevin said 'It teaches the easy performance of all reckonings ... without broken numbers...' Do you agree with Stevin? Explain.
- Give your reasons for why Stevin's notation may have been modified to the notation we use today.
- Stevin's idea was to perform multiplication on whole numbers and provide for the decimal point by suggesting that 'the sign of the right-most digit is determined by adding the signs of the right-most digits of the multiplicands'. Give an example of what this may mean and also if you agree with the statement.
- What do you think Stevin may have suggested in his notation and method when division was to be performed.
- In what ways is the decimal notation we use today an improvement of Stevin's notation.
- What do the words 'unit' and 'metre' signify in *Système International d'Unités* and *Convention du Mètre*.

Explain what these numbers mean

Given below are statements we may come across in our everyday lives.

Explain as you would do to a child of say 10 yrs, what these numbers mean.

Use diagrams if you wish in your explanation.

- A. Galdhøpiggen, the highest peak in Norway stands at 2469 m.
- B. The sale at the store is offering a 25% discount.
- C. There are 4 more vowels in the Norwegian language than in English.
- D. *Sunblock 30* provides 30 times natural protection against UV rays.
- E. The birth rate in Norway at present is 1,78 children per woman.
- F. Sounds with frequency between 20 Hertz to 20 000 Hertz is audible to the human ear.
- G. Mainland Norway extends between 58 and 62 degrees north of the equator.
- H. $\frac{3}{4}$ of the capacity of the football stadium was filled with children from our school. Of this $\frac{1}{3}$ were boys.

Find the cryptic number

- Find a ten digit number in which the digit in any place tells you the number of such digits in the number.
- The ten places in the digit from left to right correspond to the ten digits from 0 to 9 that can be used in the number.
- For example a number with the digit 5 in the fourth place (_ _ _ 5 _ _ _ _) would mean that the digit 3 occurs five times in the number.

Find the error

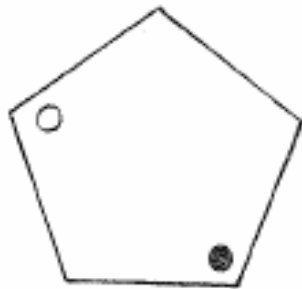
- The following addition of decimal numbers has four errors in that only one decimal point is correct.
- Change the position of four decimal points to make the 'sum' or 'total' correct.
- There are two possible solutions. Find both of them.

$$\begin{array}{r} 37,6 \\ 1921,5 \\ 109,4 \\ \underline{14,7} \\ 3876,9 \end{array}$$

- Make similar questions where three, four or five decimal numbers are being added and where there are two, three and four errors in the 'sum' or 'total'.
- Is it possible to have more than two solutions?

When together?

- In the pentagon below are two dots, black and white, on the move
- The black moves two corners counter clockwise.
- The white moves three corners clockwise.
- After how many moves are the two dots together?



How many times?

(Problems from *Reading, Writing and Proving*, ISBN 0-387-00834-9)

- There are two problems given below. Make your choice and solve one.
 - Later explain how you solved the problem. Also tell us why and how you chose the problem.
 - You could work alone or with your friends.
 - You could solve both the problems if you wish. Go read the problems first.
-
- ❖ Suppose n teams play in a single game elimination tournament, how many games are played?
 - ❖ You are given twelve coins that appear to be identical. However one of the coins is counterfeit (imitation, not genuine) and its weight is slightly different than that of the other eleven.
Using a two-pan balance, what is the smallest number of weighings you would need to find the counterfeit coin?

What is the distance?

(From *Crossing the river with dogs*, ISBN 1-931914-14-1)

Draw a diagram to solve each of the problems below.

- ❖ A ball rebounds one-half the height from which it is dropped. The ball is dropped from a height of 160 feet and keeps on bouncing.

What is the total vertical distance the ball will travel from the moment it is dropped to the moment it hits the floor the fifth time?

- ❖ Becky, Ruby, Isabel, Lani, Alma and Sabrina ran an 800-meter race. Alma beat Isabel by 7 meters. Sabrina beat Becky by 12 meters. Alma finished 5 meters ahead of Lani but 3 meters behind Sabrina. Ruby finished halfway between the first and the last women.

In what order did the women finish? What were the distances between them?

How many ‘fives’ ?

Solve the problem below.

After solving the problem set another problem.

Solve your problem to make sure your problem is solvable.

- ❖ After a game, every member of a winning basket-ball team gives a ‘five’ to each member of the losing team and then to each member of its own team.

In all how many ‘fives’ were there?

How far and how heavy?

(From *Crossing the river with dogs*, ISBN 1-931914-14-1)

- ❖ In your apartment, a round table is shoved into the corner of the room. The table touches the two walls at points that are 17 inches apart.

How far is the centre of the table from the corner?

- ❖ If a brick balances with three-quarters of a brick and three quarters of a pound, then how much does the brick weigh?

Hailstone Numbers

(From *Creativity in Middle School Mathematics*, In Press, G Sharada)

- Consider the problem of Hailstone Numbers given below.
- Complete the exercise and then suggest your own pattern for generating another sequence.

Hailstone Numbers

The origin of Hailstone Numbers, also referred to as the ' $3n + 1$ problem' is shrouded in mystery. However, the method of generating these numbers is not a mystery. Follow these steps to generate Hailstone numbers:

- Think of any number.
- Now apply either of the following:
 - i. If the number is odd, triple it and add one.
 - ii. If the number is even halve it.
- Continue the second step on the resulting number in each step. The Hailstone pattern is generated.

For example 5, an odd number, yields the following pattern: 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, ... Similarly 48, an even number yields 48, 24, 12, 6, 3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, ... pattern. Following the Hailstone method, all numbers will result in the pattern 4, 2, 1, 4, 2, 1, ...

- Think of any number and generate the Hailstone pattern. Do not forget to try the numbers 7 and 27.

A.13: Three group-tasks in succession: proportionality

(This appendix is on 3 sides)

B

Proportionality

Here is a typical set of results for an experiment – stretching a spring

Task 1: Complete the table

Force (N)	Extension (cm)	$\frac{\text{Extension}}{\text{Force}}$ cm/N
0	0	-----
0.5	2	
1	4	
1,5	6	
2	8	
2.5	10	

Task 2: Draw a graph based on the table. Use the force-values along the x-axis and the extension-values along the y-axis.

Task 3: What can you say about the graph?

These are the results for a particular spring. There could have different values for another spring.

Task 4: What would the graph look like for a stiffer spring?

Task 5: What is the ratio $\frac{y}{x}$ between the extension (y) and the force (x)? (use the table)

Task 6: Use your result from task 5 to find a formula for y in terms of x ($y = \dots$)

In this case the quantities x and y are said to be proportional.

E

Inverse Proportionality

Here is a typical set of results for an experiment – measuring pressure and volume for mass of gas

Task 1: Complete the table

Volume	Pressure	pressure · volume
1	40	
1.5	27	
2	20	
2,5	16	
3	13	

Task 2: Draw a graph based on the table. Use the volume-values along the x-axis and the pressure-values along the y-axis. .

Task 3: What can you say about the graph?

Task 4: What is the product $x \cdot y$ between the volume (x) and pressure (y)? (use the table)

Task 5: Use your result from task 4 to find a formula for y in terms of x ($y = \dots$)

In this case the quantities x and y are said to be inversely proportional.

£

FOLLOW-UP

Task 1 Are x and y proportional, inversely proportional or none of the two?

a) $y = 12x$ b) $y = 8x + 4$ c) $y = \frac{10}{x}$ d) $y = \frac{\pi}{180}x$ e) $x \cdot y = 0.9$

Task 2 Are x and y proportional, inversely proportional or none of the two?

a)

x	2	5	8	12
y	15	45	65	80

b)

x	3	10	15	20
y	6	1,8	1,2	0,9

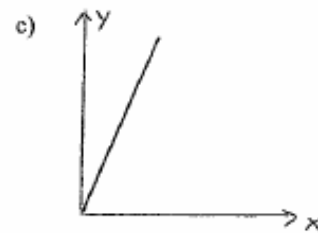
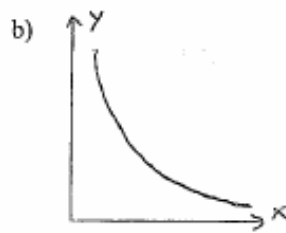
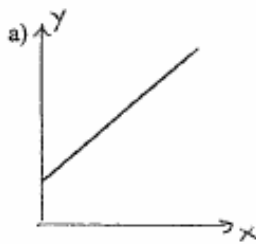
c)

x	5	12	18	30
y	60	144	216	360

d)

x	40	60	90	130
y	10	15	22,5	32,5

Task 3 Are x and y proportional, inversely proportional or none of the two?



A.14: Group-task on similar triangles

(This appendix is on 1 side only)

Annexure (A)

Similar Triangles

TASK 1

Explain why the two triangles are similar.

Calculate the length of YZ.

Find the length of a side (y) in $\triangle ABC$ in terms of the corresponding side (x) in $\triangle XYZ$.

TASK 2

Explain why $\triangle ABC$ and $\triangle ADE$ are similar.

Calculate BC.

TASK 3

Explain why $\triangle PQR$ and $\triangle TSR$ are similar.

Calculate QR and TS.

A.15: Students' responses: Similar triangles
 (This appendix is on 4 sides)

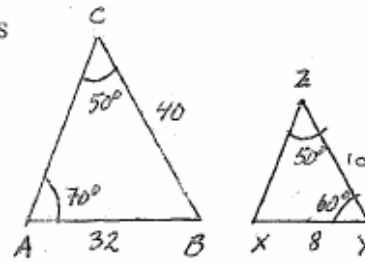
(K)

Similar Triangles

TASK 1

Explain why the two triangles are similar.

The triangles are similar because it is always 180°



Calculate the length of YZ.

Split BC with 4.

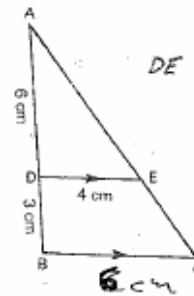
Find the length of a side (y) in $\triangle ABC$ in terms of the corresponding side (x) in $\triangle XYZ$.

TASK 2

Explain why $\triangle ABC$ and $\triangle ADE$ are similar.

Calculate BC.

angles = angles



DE is parallel to BC.

TASK 3

Explain why $\triangle PQR$ and $\triangle TSR$ are similar.

Calculate QR and TS.



19/20

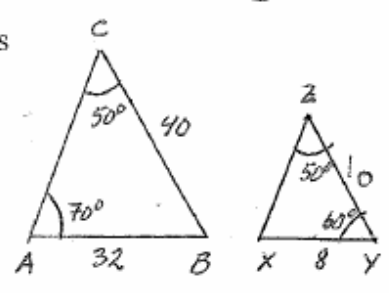
(L)

Similar Triangles

TASK 1

Explain why the two triangles are similar.

The unknown angle is the same as the other angle on the other one.



$32/8 = 4$
 $40/4 = 10$

Find the length of a side (y) in ΔABC in terms of the corresponding side (x) in ΔXYZ .

~~$x = \frac{32}{4}$~~
 $y = x \cdot 4$
 $10 = \frac{40}{4}$

$\frac{GRIS}{KATI} = \frac{GRIS}{KATI}$

~~$\frac{LGF}{AGN} = \frac{F}{G}$~~

~~$\frac{LEIF}{EGIL} = \frac{F}{G}$~~

TASK 2

Explain why ΔABC and ΔADE are similar.

Its the same angles.

Calculate BC.



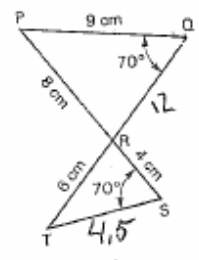
~~$LEIFEGALROVE = L^2E^2$~~

TASK 3

Explain why ΔPQR and ΔTSR are similar.

They are not!

Calculate QR and TS.



~~$(L+E+I+F) \cdot (E+G+I+L) \cdot (R+E+VE) = (R^2EV)$~~

~~$(LEIFEGALROVE = L^2E^2)$~~
 $2L^4E^5FG^4RV$
 $2L^2REGILGILFEGIL \cdot (R+2E+V)$
 $20/20$

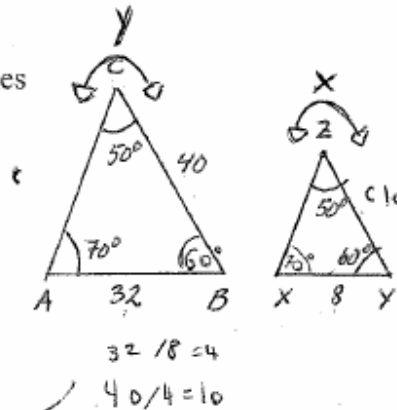
(C)

Similar Triangles

TASK 1

Explain why the two triangles are similar.
 The ~~angles~~ ^{angles} of the two triangles are the same, ergo they are similar.

Calculate the length of YZ.



Find the length of a side (y) in ΔABC in terms of the corresponding side (x) in ΔXYZ.

$$x = \frac{y}{4}$$

$$y = x \cdot 4$$

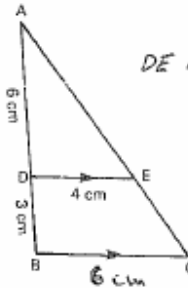
TASK 2

Explain why ΔABC and ΔADE are similar.

The angles are the same because these angles have similar grades.

Calculate BC.

$$BC = AB / 3 \cdot 2$$



DE is parallel to BC.

TASK 3

Explain why ΔPQR and ΔTSR are similar.

$$\angle PRQ = \angle TRS$$

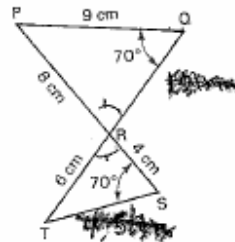
$$\angle Q = \angle S$$

Konklusjon: $\Delta PQR \sim \Delta TRS$

Calculate QR and TS.

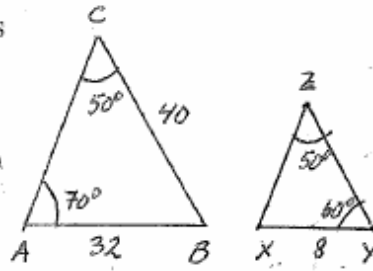
$$\frac{QR}{4} = \frac{8 \cdot 4}{6}$$

$$QR = 5,3$$



(T)

Similar Triangles



TASK 1

Explain why the two triangles are similar.

Because of same angles/angles

Calculate the length of YZ.

$$\frac{32}{8} = 4 \quad \frac{40}{4} = 10$$

Find the length of a side (y) in $\triangle ABC$ in terms of the corresponding side (x) in $\triangle XYZ$.

$$\frac{y}{x} = 4 \quad \underline{y = 4x}$$

TASK 2

Explain why $\triangle ABC$ and $\triangle ADE$ are similar.

Similar angles, Corresponding
 $\angle A =$ common angle

Calculate BC.

$$\frac{AD}{AB} = \frac{DE}{BC} = \frac{3}{6} = \frac{x}{y}$$

TASK 3

$$\frac{36}{6} = 6$$

Explain why $\triangle PQR$ and $\triangle TSR$ are similar.

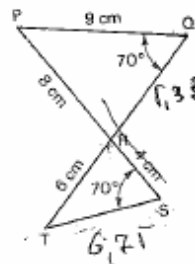
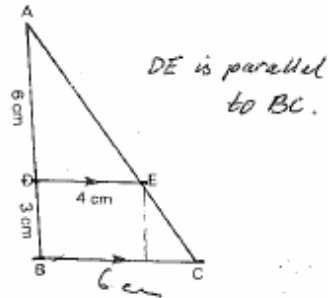
Same angle.

Calculate QR and TS.

$$\frac{8}{4} = \frac{8}{6} \quad 1.33$$

$$4 \cdot \frac{8}{6} = \frac{32}{6} = 5.33$$

$$\frac{9}{1} = \frac{8}{6} \quad \frac{84}{8} = 10.5$$

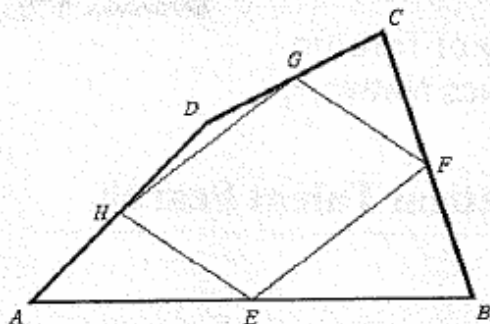


5/20

A.16: Some challenges brought in by students

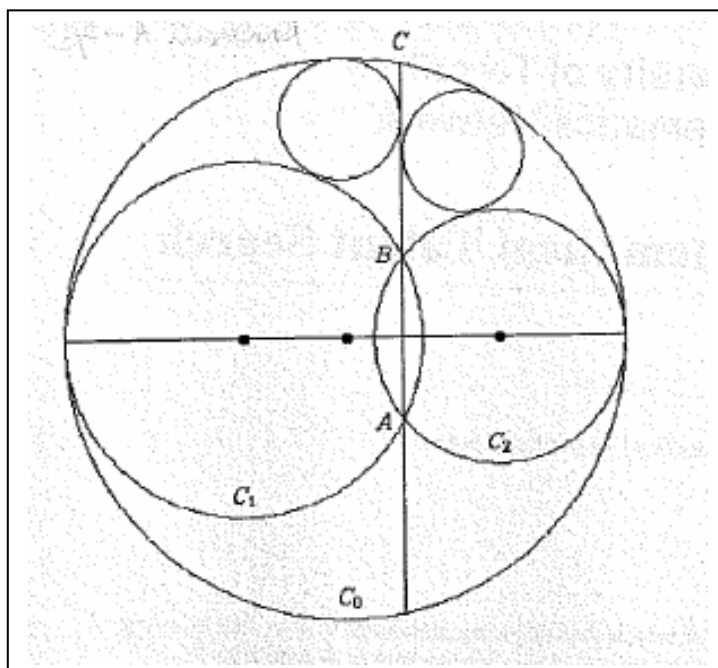
(From Round 16, University of Toronto Mathematics Network)

Problem 4/16. Let $ABCD$ be an arbitrary convex quadrilateral, with E, F, G, H the midpoints of its sides, as shown in the figure below. Prove that one can piece together triangles AEH, BEF, CFG, DGH to form a parallelogram congruent to parallelogram $EFGH$.



Problem 5/16. An equiangular polygon $ABCDEFGH$ has sides of length $2, 2\sqrt{2}, 4, 4\sqrt{2}, 6, 7, 7, 8$. Given that $AB = 8$, find the length of EF .

Problem 5/20. In the figure shown below, the centres of the circles $C_0, C_1,$ and C_2 are collinear, A and B are the points of intersection of C_1 and C_2 , and C is a point of intersection of C_0 and the extension of AB . Prove that the two small circles shown, tangent to C_0, C_1 and BC , and to C_0, C_2 and BC , respectively, are congruent to one another.



A.17: Students' responses to group-task: Proportionality

(This appendix is on 4 sides: Anja, Lea, Stine and Egil)

Proportionality

Here is a typical set of results for an experiment – stretching a spring

Task 1: Complete the table

Force (N)	Extension (cm)	$\frac{\text{Extension}}{\text{Force}}$ cm/N
0	0	-----
0.5	2	4cm/N
1	4	4cm/N
1,5	6	4cm/N
2	8	4cm/N
2.5	10	4cm/N

Task 2: Draw a graph based on the table. Use the force-values along the x-axis and the extension-values along the y-axis.

Task 3: What can you say about the graph?

This graph is a proportional graph.
When we double the force, the extension is doubled as well. The graph is a straight line.

These are the results for a particular spring. There could have different values for another spring.

Task 4: What would the graph look like for a stiffer spring?

It wouldn't have increased with a similar steepness.

Task 5: What is the ratio $\frac{y}{x}$ between the extension (y) and the force (x)? (use the table)

The ratio is 4

Task 6: Use your result from task 5 to find a formula for y in terms of x ($y = \dots$)

$$y = 4x$$

In this case the quantities x and y are said to be proportional.

↓
størrelser

Proportionality

I

Here is a typical set of results for an experiment – stretching a spring

Task 1: Complete the table

Force (N)	Extension (cm)	$\frac{\text{Extension}}{\text{Force}}$ cm/N
0	0	-----
0.5	2	$\frac{2}{0.5} = 4$ cm/N
1	4	$\frac{4}{1} = 4$ cm/N
1.5	6	$\frac{6}{1.5} = 4$ cm/N
2	8	$\frac{8}{2} = 4$ cm/N
2.5	10	$\frac{10}{2.5} = 4$ cm/N

Task 2: Draw a graph based on the table. Use the force-values along the x-axis and the extension-values along the y-axis.

Task 3: What can you say about the graph?

It's a straight line going through the origin.
It's linear.

These are the results for a particular spring. There could have different values for another spring.

Task 4: What would the graph look like for a stiffer spring?

It wouldn't have increase with a similar steepness.

Task 5: What is the ratio $\frac{y}{x}$ between the extension (y) and the force (x)? (use the table)

$$\underline{\underline{4}}$$

Task 6: Use your result from task 5 to find a formula for y in terms of x (y = ...)

$$y = 4x$$

In this case the quantities x and y are said to be proportional.

Proportionality

I

Here is a typical set of results for an experiment – stretching a spring

Task 1: Complete the table

Force (N)	Extension (cm)	$\frac{\text{Extension}}{\text{Force}}$ cm/N
0	0	$\frac{0}{0}$ cm/N = 0
0.5	2	$\frac{2}{0.5}$ cm/N = 4
1	4	$\frac{4}{1}$ cm/N = 4
1.5	6	$\frac{6}{1.5}$ cm/N = 4
2	8	$\frac{8}{2}$ cm/N = 4
2.5	10	$\frac{10}{2.5}$ cm/N = 4

Task 2: Draw a graph based on the table. Use the force-values along the x-axis and the extension-values along the y-axis.

Task 3: What can you say about the graph?

It's a steep straight line, and it goes through origin.

These are the results for a particular spring. There could have different values for another spring.

Task 4: What would the graph look like for a stiffer spring?

The line wouldn't have been so steep as the other line.

Task 5: What is the ratio $\frac{y}{x}$ between the extension (y) and the force (x)? (use the table)

The ratio is 4.

Task 6: Use your result from task 5 to find a formula for y in terms of x (y = ...)

$$y = ax + b$$

$$y = 4x$$

In this case the quantities x and y are said to be proportional.

Proportionality

Here is a typical set of results for an experiment – stretching a spring

Task 1: Complete the table

Force (N)	Extension (cm)	$\frac{\text{Extension}}{\text{Force}}$ cm/N
0	0	0
0.5	2	2/0.5
1	4	4/1
1.5	6	6/1.5
2	8	8/2
2.5	10	10/2.5

Task 2: Draw a graph based on the table. Use the force-values along the x-axis and the extension-values along the y-axis.

Task 3: What can you say about the graph?

When we double the force, the extension is doubled as well
 It increases proportionally
 The graph goes straight through the origin, linear
 These are the results for a particular spring. There could have different values for another spring.

Task 4: What would the graph look like for a stiffer spring?

It wouldn't have increased with ~~the~~ ~~same~~ similar steepness

Task 5: What is the ratio $\frac{y}{x}$ between the extension (y) and the force (x)? (use the table)

~~The~~ The ratio is 4

Task 6: Use your result from task 5 to find a formula for y in terms of x ($y = \dots$)

$$y = 4x$$

In this case the quantities x and y are said to be proportional.

A.18: Students' responses to group-task: Inverse proportionality

(This appendix is on 3 sides: Anja, Lea and Stine)

Inverse Proportionality

Here is a typical set of results for an experiment – measuring pressure and volume for mass of gas

Task 1: Complete the table

Volume x	Pressure y	pressure \cdot volume
1	40	$40 \cdot 1 = 40$
1.5	27	40.5
2	20	40
2.5	16	40
3	13	39

Task 2: Draw a graph based on the table. Use the volume-values along the x-axis and the pressure-values along the y-axis.

Task 3: What can you say about the graph?

This is a inverse proportional graph.
The graph is curved.

Task 4: What is the product $x \cdot y$ between the volume (x) and pressure (y)? (use the table)

40

Task 5: Use your result from task 4 to find a formula for y in terms of x ($y = \dots$)

$$y = \frac{40}{x} / y = \frac{4}{x}$$

In this case the quantities x and y are said to be inversely proportional.

Inverse Proportionality

I

Here is a typical set of results for an experiment – measuring pressure and volume for mass of gas

Task 1: Complete the table

Volume x	Pressure y	pressure · volume
1	40	40
1.5	27	40.5
2	20	40
2.5	16	40
3	13	39

Task 2: Draw a graph based on the table. Use the volume-values along the x-axis and the pressure-values along the y-axis.

Task 3: What can you say about the graph?

It's an inverse proportional graph.
It decreases.

Task 4: What is the product $x \cdot y$ between the volume (x) and pressure (y)? (use the table)

$$y = \frac{40}{x}$$

Task 5: Use your result from task 4 to find a formula for y in terms of x ($y = \dots$)

$$y = \frac{40}{x} \quad (y = \frac{a}{x})$$

In this case the quantities x and y are said to be inversely proportional.

Inverse Proportionality

I

Here is a typical set of results for an experiment – measuring pressure and volume for mass of gas

Task 1: Complete the table

Volume	Pressure	pressure · volume
1	40	$40 \cdot 1 = 40$
1.5	27	$27 \cdot 1.5 = 40.5$
2	20	$20 \cdot 2 = 40$
2.5	16	$16 \cdot 2.5 = 40$
3	13	$13 \cdot 3 = 39$

Task 2: Draw a graph based on the table. Use the volume-values along the x-axis and the pressure-values along the y-axis.

Task 3: What can you say about the graph?

It's not straight, it goes in a bow.

Task 4: What is the product $x \cdot y$ between the volume (x) and pressure (y)? (use the table)

40

Task 5: Use your result from task 4 to find a formula for y in terms of x ($y = \dots$)

$$y = \frac{a}{x} \quad y = \frac{40}{x}$$

In this case the quantities x and y are said to be inversely proportional.

A.19: Students' responses to group-tasks: Follow Up

(This appendix is on 4 sides: Anja, Lea, Stine and Egil)

FOLLOW-UP

Task 1 Are x and y proportional, inversely proportional or none of the two?

a) $y = 12x$ b) $y = 8x + 4$ c) $y = \frac{10}{x}$ d) $y = \frac{\pi}{180}x$ e) $x \cdot y = 0.9$
P *IP* *D* *IP*

Task 2 Are x and y proportional, inversely proportional or none of the two?

a)

x	2	5	8	12
y	15	45	65	80

b)

x	3	10	15	20
y	6	1,8	1,2	0,9

y/x = 2, 1,8, 1,2, 0,9

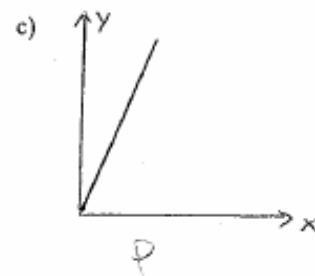
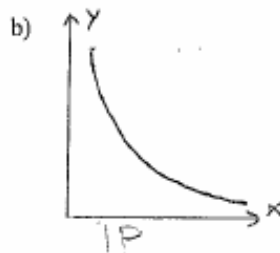
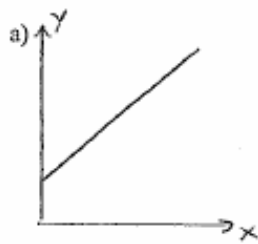
c)

x	5	12	18	30
y	60	144	216	360

d)

x	40	60	90	130
y	10	15	22,5	32,5

Task 3 Are x and y proportional, inversely proportional or none of the two?



FOLLOW-UP

Task 1 Are x and y proportional, inversely proportional or none of the two?

a) $y = 12x$ | b) $y = 8x + 4$ | c) $y = \frac{10}{x}$ | d) $y = \frac{\pi}{180}x$ | e) $x \cdot y = 0.9$
 proportional | neither | ~~direct~~ inv-prop | neither prop. | inv-prop

Task 2 Are x and y proportional, inversely proportional or none of the two?

a) *none*

x	2	5	8	12
y	15	45	65	80

b) **inverse proportional*

x	3	10	15	20
y	6	1,8	1,2	0,9

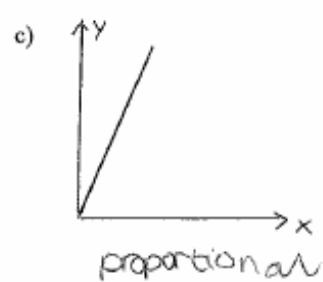
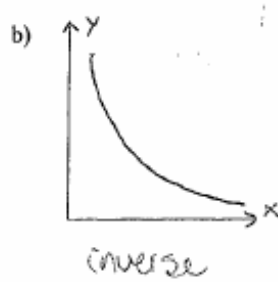
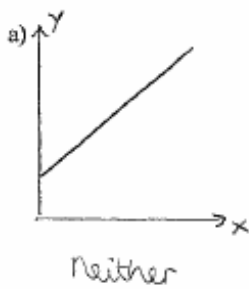
c) **Proportional*

x	5	12	18	30
y	60	144	216	360

d) **Inverse proportional*

x	40	60	90	130
y	10	15	22,5	32,5

Task 3 Are x and y proportional, inversely proportional or none of the two?



FOLLOW-UP

Task 1 Are x and y proportional, inversely proportional or none of the two?

a) $y = 12x$ b) $y = 8x + 4$ c) $y = \frac{10}{x}$ d) $y = \frac{\pi}{180}x$ e) $x \cdot y = 0.9$

P I.P P I.P

- a) proportional
- b) none
- c) inversely prop
- d) ~~proportional~~
- e) inversely prop.

Task 2 Are x and y proportional, inversely proportional or none of the two?

a) *None*

x	2	5	8	12
y	15	45	65	80

I.P.
P

b) *I.P*

x	3	10	15	20
y	.6	1.8	1.2	0.9

a | 18 | 18 | 18 | 18

c) *Proportional*

x	5	12	18	30
y	60	144	216	360

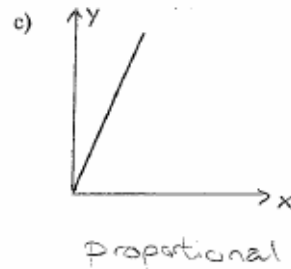
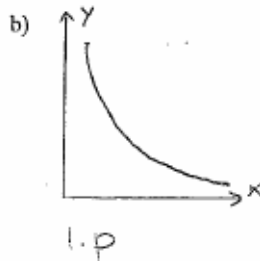
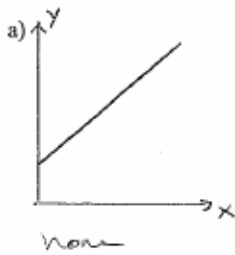
a | 12 | 12 | 12 | 12

d) *Proportional*

x	40	60	90	130
y	10	15	22.5	32.5

a | 0,25 | 0,25 | 0,25 | 0,25

Task 3 Are x and y proportional, inversely proportional or none of the two?



$$I.P = x \cdot y = a$$

$$P = \frac{y}{x} = a$$

FOLLOW-UP

I

Task 1 Are x and y proportional, inversely proportional or none of the two?

a) $y = 12x$ b) $y = 8x + 4$ c) $y = \frac{10}{x}$ d) $y = \frac{\pi}{180}x$ e) $x \cdot y = 0.9$

a) proportional
 b) not proportional
 c) inversely proportional
 d) ~~prop~~ prop
 e) inversely prop.

Task 2 Are x and y proportional, inversely proportional or none of the two?

a) *none*

x	2	5	8	12
y	15	45	65	80

b) *inversely proportional*

x	3	10	15	20
y	6	1,8	1,2	0,9

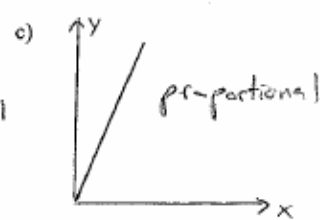
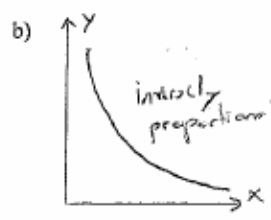
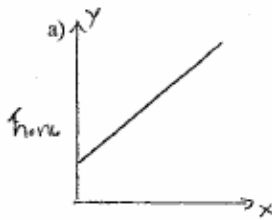
c) *proportional*

x	5	12	18	30
y	60	144	216	360

d) *proportional*

x	40	60	90	130
y	10	15	22,5	32,5

Task 3 Are x and y proportional, inversely proportional or none of the two?



A.20: Two bodies in motion

(This appendix is on 1 side only)

What can we say from the given graphs?

- You will be given two graphs A and B.
- The graphs relate to the motion of two different bodies.
 - One of the graphs describes the motion of an elevator travelling between two floors.
 - The other graph describes the motion of a ball thrown up in the air and caught on its return.
- Discuss the following:
 - Which of the two graphs given to you shows the movements of the elevator and ball mentioned above and why?
 - Refer to graph A and explain what may be happening at the points marked X, Y and Z.
 - Refer to graph B and explain what may be happening at the points marked P, Q and R.
 - In graph A what is the significance of the point marked Y?
 - In graph B what is the significance of the point marked Q?
 - In what ways are the two graphs A and B different?
 - In what ways are the two graphs A and B similar?
 - The graph of the elevator shows its motion between two floors; across one level. How will the graph differ if it were to show the elevator moving between six floors or across five levels?
 - The graph of the ball shows its motion when thrown from your play ground. How would the graph differ if the ball were thrown from the surface of the Moon instead of the Earth?
- At the end of the task session lasting 30 minutes, you will be given 5 minutes to share and summarise your experience with the task.
- (The values of speed and time in the graphs are not actual but for the sake of discussion.)

A.21: SA/V ratio and metabolism

(This appendix is on 3 sides)

What can we find out about surface area and volume?

- You will be given two worksheets A and B.
- Worksheet A relates to the surface area and volume of a sphere.
- In worksheet B you will need to work with the results you obtain in worksheet A.
- Discuss the questions given in the worksheets as you go along.
- You may want to use the calculator and the graph sheet for calculation and explanation.
- At the end of the task session lasting 30 minutes, you will be given 5 minutes to share and summarise your experience with the task.

Worksheet A

- For any given sphere of radius r
 - the surface area (SA) is given by the formula $4\pi r^2$
 - the volume (V) is given by the formula $4\pi r^3/3$

- Fill in the table below for spheres of different radii:

SN	Radius (r_x)	Surface Area (SA_x) in 2 decimals	Volume (V_x) in 2 decimals	$\frac{SA_x}{V_x}$
1	1 unit			
2	2 units			
3	3 units			
4	4 units			
5	5 units			
6	6 units			

- Compare and calculate the following ratios for a sphere of radius 5 units (r_5) and a unit sphere of radius 1 unit (r_1):
 - $r_5 / r_1 =$ _____
 - $SA_5 / SA_1 =$ _____
 - $V_5 / V_1 =$ _____

- In what way is the ratio SA_5 / SA_1 and V_5 / V_1 related to r_5 / r_1 .

- How does the ratio of the surface-area-to-volume (SA / V) change as the measure of the radius increases?

- Work with the formulae for Surface Area and Volume and express (SA / V) as a ratio: _____

- Does the ratio just calculated agree with your results in the table above?

Worksheet B

Exercise 1

- The surface area of any body is an important measure in science.
- What do you think the cat gains by stretching out as shown below?



30°



60°



90°

- Can you offer an explanation in terms of its surface area and volume?
- Do we humans behave similarly to the cat? How and why?
- Cats and humans are warm blooded. Would the above observation change if we were discussing cold blooded animals?

Exercise 2

- If we assume that living cells in our bodies are spherical what happens to the surface-area-to-volume (SA/ V) ratio as the cells get larger and larger? Why?
- For any living cell, metabolism is the rate of chemical activity in the cell. Metabolism maintains life. For metabolism to take place materials like oxygen and water need to be absorbed.

Of the two values Surface Area and Volume:

1. Which value will you think determines the cell's metabolism:

2. Which value controls how much material gets in and out of the cell:

- For a spherical cell how would a larger surface-area-to-volume (SA / V) affect its metabolism?
- Do you think it is advisable for organisms to have large cells or small ones? Why?