OSCILLATION OF FOURTH-ORDER QUASILINEAR DIFFERENTIAL EQUATIONS

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(Received September 30, 2013)

Abstract. We study oscillatory behavior of a class of fourth-order quasilinear differential equations without imposing restrictive conditions on the deviated argument. This allows applications to functional differential equations with delayed and advanced arguments, and not only these. New theorems are based on a thorough analysis of possible behavior of nonoscillatory solutions; they complement and improve a number of results reported in the literature. Three illustrative examples are presented.

Keywords: oscillation; quasilinear functional differential equation; delayed argument; advanced argument

MSC 2010: 34K11

1. Introduction

This paper is concerned with the oscillation of a fourth-order quasilinear differential equation

$$(r(t)[x'''(t)]^{\alpha})' + q(t)x^{\beta}(g(t)) = 0, \tag{1.1}$$

where $t \in \mathbb{I} = [t_0, \infty)$, $t_0 \in \mathbb{R}$, $r \in \mathrm{C}^1(\mathbb{I}, \mathbb{R}_+)$, $r'(t) \geqslant 0$, $q \in \mathrm{C}(\mathbb{I}, \mathbb{R}_0)$, $q(t)$ does not vanish eventually, $g \in \mathrm{C}(\mathbb{I}, \mathbb{R})$, and $\lim\limits_{t\to\infty} g(t) = \infty$. Here $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}_0 = [0, \infty)$. We also assume that $\alpha, \beta \in \mathfrak{R}$, where $\mathfrak{R}$ is the set containing all ratios of odd natural numbers.

This research was supported in part by the National Fundamental Research Program of P. R. China (2013CB035604), the NNSF of P. R. China (Grants 61503171, 61034007, 51277116, and 51107069), the Doctor Scientific Research Foundation of Linyi University (LYDX2015BS001), and the AMEP of Linyi University, P. R. China.

By a solution of (1.1) we mean a function $x \in \mathrm{C}^3([T_x, \infty), \mathbb{R})$, $T_x \geqslant t_0$, which has the property $r(t)[x'''(t)]^\alpha \in \mathrm{C}^1([T_x, \infty), \mathbb{R})$ and satisfies (1.1) on $[T_x, \infty)$. We consider only those solutions x of (1.1) which do not vanish eventually; we tacitly assume that (1.1) possesses such solutions. As usual, a solution $x(t)$ of (1.1) is called oscillatory if it does not have the largest zero on $[T_x, \infty)$; otherwise, it is called nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

Fourth-order differential equations are quite often encountered in mathematical models of various physical, biological, and chemical phenomena. Applications include, for instance, problems of elasticity, deformation of structures, or soil settlement; see [6]. In mechanical and engineering problems, questions related to the existence of oscillatory and nonoscillatory solutions play an important role. As a result, many theoretical studies have been undertaken during the last decades. We refer the reader to the monographs [1], [3], [12], [15], papers [2], [4]–[11], [13], [14], [16]–[23], and the references cited therein.

In what follows, we briefly comment on the results that motivated the research in this paper. For a compact presentation of conditions, we use the notation

$$R(t) := \int_t^\infty r^{-1/\alpha}(s)\,\mathrm{d}s.$$

To the best of our knowledge, papers by Onose [17], [18] published in the late seventies were among the first contributions dealing with the oscillation of fourth-order functional differential equations. In these papers, two classes of functional differential equations,

$$(r(t)x''(t))'' + f(t, x(g(t))) = 0$$

and

$$(r(t)x''(t))'' + p(t)f(x(g(t))) = q(t),$$

were studied under the assumption that $\int_T^\infty (s/r(s))\,\mathrm{d}s = \infty$ for some $T > 0$.

Since then, many authors were concerned with the oscillation and nonoscillation of fourth-order and higher-order functional differential equations. Properties of solutions to different classes of equations were explored by using a wide spectrum of approaches. In particular, interesting results for the fourth-order functional differential equations

$$\biggl(\Bigl(\frac{1}{r_3(t)}\Bigl(\Bigl(\frac{1}{r_2(t)}\Bigl(\Bigl(\frac{1}{r_1(t)}x'(t)\Bigr)^{\alpha_1}\Bigr)'\Bigr)^{\alpha_2}\Bigr)'\Bigr)^{\alpha_3}\biggr)' \pm q(t)f(x(g(t))) = 0$$

were obtained by Agarwal et al. [2]. Grace et al. [7] considered the oscillation properties of a fourth-order nonlinear differential equation

$$(r(t)[x'(t)]^\alpha)''' + q(t)f(x(g(t))) = 0.$$

Kamo and Usami [10], [11], Kusano et al. [14], and Wu [20] studied the oscillation of a fourth-order nonlinear differential equation

$$(r(t)[x''(t)]^{\alpha})'' + q(t)x^{\beta}(t) = 0,$$

whereas a more general equation

$$(r(t)[x''(t)]^{\alpha})'' + q(t)f(x(g(t))) = 0$$

was considered by Agarwal et al. [5].

Agarwal et al. [4] and Zhang et al. [23] investigated the oscillatory behavior of a higher-order differential equation

$$(r(t)[x^{(n-1)}(t)]^{\alpha})' + q(t)x^{\beta}(\tau(t)) = 0, \tag{1.2}$$

considering separately the cases where

$$\lim_{t\to\infty} R(t) = \infty$$

and

$$\lim_{t\to\infty} R(t) < \infty. \tag{1.3}$$

In particular, assuming that $\tau(t) < t$, $\alpha \geqslant \beta$, and (1.3) holds, Zhang et al. [23] obtained results which ensure that every solution x of (1.2) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Recently, oscillation of all unbounded solutions to (1.1) was established by Li et al. [16], who assumed that $\alpha = \beta = 1$ and $g(t) \leqslant t$, whereas Zhang et al. [22] derived oscillation criteria for equation (1.1) under the assumptions that $\alpha = \beta = 1$ and $g(t) = t$. Finally, Zhang et al. [21] analysed oscillation of equation (1.2) for $\alpha \geqslant \beta$ and $g(t) < t$.

Our principal goal in this paper is to derive new oscillation criteria for equation (1.1) without imposing restrictive conditions on the deviated argument $g(t)$. Our methods are based on a thorough analysis of possible behavior of nonoscillatory solutions and comparison with oscillation theorems that are available in the literature. We conclude the paper by providing three illustrative examples that explain advantages of the new oscillation results.

2. Main results

We need the following auxiliary result extracted from Agarwal et al. [3], Lemma 2.2.3.

Lemma 2.1. *Let $f \in C^n(\mathbb{I}, \mathbb{R}_+)$. Assume that $f^{(n)}(t)$ is eventually of one sign for all large t, and there exists a $t_1 \geqslant t_0$ such that*

$$f^{(n-1)}(t)f^{(n)}(t) \leqslant 0,$$

for all $t \geqslant t_1$. If

$$\lim_{t\to\infty} f(t) \neq 0,$$

then, for every $\lambda \in (0,1)$, there exists a $t_\lambda \in [t_1, \infty)$ such that

$$f(t) \geqslant \frac{\lambda}{(n-1)!} t^{n-1} f^{(n-1)}(t),$$

for all $t \in [t_\lambda, \infty)$.

Observe that if $x(t)$ is a solution of (1.1), then $-x(t)$ is also a solution. Therefore, without loss of generality, we can assume from now on that nonoscillatory solutions of (1.1) are eventually positive. In what follows, it is tacitly supposed that all functional inequalities are satisfied for all t large enough and we use the following notation: given a solution $x(t)$ of (1.1), we define

$$y(t) \stackrel{\text{def}}{=} -z(t) \stackrel{\text{def}}{=} r(t)[x'''(t)]^\alpha. \tag{2.1}$$

Theorem 2.1. *Assume that* (1.3) *holds and there exist a number $\gamma \in \mathfrak{R}$, $\gamma > \alpha > \beta$ and two functions $\tau, \sigma \in C(\mathbb{I}, \mathbb{R})$ such that*

$$\tau(t) \leqslant g(t) \leqslant \sigma(t), \quad \tau(t) < t < \sigma(t), \quad \lim_{t\to\infty} \tau(t) = \infty. \tag{2.2}$$

Suppose further that

$$\int_{t_0}^{\infty} q(t)\left(\frac{\tau^3(t)}{r^{1/\alpha}(\tau(t))}\right)^\beta \mathrm{d}t = \infty \tag{2.3}$$

and

$$\int_{t_0}^{\infty} q(t) g^{2\beta}(t) R^\gamma(\sigma(t))\, \mathrm{d}t = \infty. \tag{2.4}$$

If, in addition, either

$$\int_{t_0}^{\infty} R(t)\,\mathrm{d}t = \infty, \tag{2.5}$$

or

$$\int_{t_0}^{\infty}\int_{u}^{\infty} R(s)\,\mathrm{d}s\,\mathrm{d}u = \infty, \tag{2.6}$$

or

$$\int_{t_0}^{\infty} q(t)\biggl(\int_{\sigma(t)}^{\infty}\int_{u}^{\infty} R(s)\,\mathrm{d}s\,\mathrm{d}u\biggr)^{\gamma}\mathrm{d}t = \infty \tag{2.7}$$

holds, equation (1.1) *is oscillatory.*

P r o o f. Let $x(t)$ be a nonoscillatory solution of equation (1.1) which is eventually positive. It follows from (1.1) that

$$(r(t)[x'''(t)]^{\alpha})' = -q(t)x^{\beta}(g(t)) \leqslant 0.$$

Hence, we have either $x'''(t) > 0$ or $x'''(t) < 0$. Assume first that $x'''(t) > 0$. Then there exists a $t_1 \geqslant t_0$ such that, for all $t \geqslant t_1$,

$$x(t) > 0, \quad x'(t) > 0, \quad x'''(t) > 0, \quad x^{(4)}(t) \leqslant 0, \quad (r(t)[x'''(t)]^{\alpha})' \leqslant 0. \tag{2.8}$$

On the other hand, if we assume now that $x'''(t) < 0$, then, for all $t \geqslant t_1$, either

$$x(t) > 0, \quad x'(t) < 0, \quad x''(t) > 0, \quad x'''(t) < 0, \quad (r(t)[x'''(t)]^{\alpha})' \leqslant 0, \tag{2.9}$$

or

$$x(t) > 0, \quad x'(t) > 0, \quad x''(t) > 0, \quad x'''(t) < 0, \quad (r(t)[x'''(t)]^{\alpha})' \leqslant 0. \tag{2.10}$$

Assume that (2.8) holds. Taking into account that $\lim\limits_{t\to\infty} x(t) \neq 0$ and also $x'''(t)x^{(4)}(t) \leqslant 0$ and applying Lemma 2.1, we have, for every $\lambda \in (0,1)$ and for all sufficiently large t,

$$x(t) \geqslant \frac{\lambda t^3}{6r^{1/\alpha}(t)} r^{1/\alpha}(t)x'''(t). \tag{2.11}$$

Let y be defined by (2.1). It follows from (1.1) and (2.11) that $y(t)$ is a positive solution of a differential inequality

$$u'(t) + q(t)\left(\frac{\lambda\tau^3(t)}{6r^{1/\alpha}(\tau(t))}\right)^{\beta} u^{\beta/\alpha}(\tau(t)) \leqslant 0. \tag{2.12}$$

However, using (2.3) and [13], Theorem 2, we conclude that delay differential inequality (2.12) has no positive solutions, which is a contradiction.

Assume now that (2.9) holds. Taking into account the fact that the function $r(t)[x'''(t)]^{\alpha}$ is nonincreasing, we deduce that

$$r^{1/\alpha}(s)x'''(s) \leqslant r^{1/\alpha}(t)x'''(t), \quad s \geqslant t \geqslant t_1. \tag{2.13}$$

Dividing both sides of (2.13) by $r^{1/\alpha}(s)$ and integrating the resulting inequality from t to l, we obtain

$$x''(l) \leqslant x''(t) + r^{1/\alpha}(t)x'''(t)\int_t^l r^{-1/\alpha}(s)\,\mathrm{d}s.$$

Passing to the limit as $l \to \infty$, we conclude that

$$x''(t) \geqslant -r^{1/\alpha}(t)x'''(t)R(t). \tag{2.14}$$

Hence, by the monotonicity of $r^{1/\alpha}(t)x'''(t)$, there exists a constant $k > 0$ such that

$$x''(t) \geqslant kR(t). \tag{2.15}$$

Integrating (2.15) from t_0 to t, we obtain

$$x'(t) - x'(t_0) \geqslant k\int_{t_0}^t R(s)\,\mathrm{d}s. \tag{2.16}$$

Inequality (2.16) yields

$$-x'(t_0) \geqslant k\int_{t_0}^t R(s)\,\mathrm{d}s,$$

which contradicts assumption (2.5). Integrating (2.15) from t to ∞, we arrive at the inequality

$$-x'(t) \geqslant k\int_t^{\infty} R(s)\,\mathrm{d}s.$$

Another integration from t_0 to t yields

$$-x(t) + x(t_0) \geqslant k\int_{t_0}^t\int_u^{\infty} R(s)\,\mathrm{d}s\,\mathrm{d}u,$$

which implies that

$$x(t_0) \geqslant k \int_{t_0}^{t} \int_u^\infty R(s)\,\mathrm{d}s\,\mathrm{d}u.$$

But the latter inequality contradicts (2.6).

Integrating (2.14) twice from t to ∞, we obtain

$$-x'(t) \geqslant \int_t^\infty -r^{1/\alpha}(s)x'''(s)R(s)\,\mathrm{d}s \geqslant -r^{1/\alpha}(t)x'''(t)\int_t^\infty R(s)\,\mathrm{d}s \tag{2.17}$$

and

$$\begin{aligned} x(t) &\geqslant \int_t^\infty -r^{1/\alpha}(u)x'''(u)\int_u^\infty R(s)\,\mathrm{d}s\,\mathrm{d}u \\ &\geqslant -r^{1/\alpha}(t)x'''(t)\int_t^\infty\int_u^\infty R(s)\,\mathrm{d}s\,\mathrm{d}u. \end{aligned} \tag{2.18}$$

It follows from the fact that $x'(t) < 0$ that there exists a constant $c_1 > 0$ such that

$$x^\beta(g(t)) \geqslant x^\beta(\sigma(t)) = x^\gamma(\sigma(t))x^{\beta-\gamma}(\sigma(t)) \geqslant c_1 x^\gamma(\sigma(t)).$$

Therefore, it follows from (1.1) and (2.17) that the function $z(t)$ defined by (2.1) is a positive solution of a differential inequality

$$u'(t) - c_1 q(t)\biggl(\int_{\sigma(t)}^\infty\int_u^\infty R(s)\,\mathrm{d}s\,\mathrm{d}u\biggr)^\gamma u^{\gamma/\alpha}(\sigma(t)) \geqslant 0. \tag{2.19}$$

On the other hand, using (2.7) and [13], Theorem 1, we conclude that advanced differential inequality (2.18) has no positive solutions. This is a contradiction.

Assume that (2.10) holds. We have already established that (2.14) holds. Note that $\lim\limits_{t\to\infty} x(t) \neq 0$ and $x''(t)x'''(t) < 0$. Therefore, by virtue of Lemma 2.1, we conclude that

$$x(t) \geqslant \frac{\lambda}{2}t^2x''(t), \tag{2.20}$$

for every $\lambda \in (0,1)$ and for all sufficiently large t. It follows now from (2.14) and (2.19) that

$$x(t) \geqslant -\frac{\lambda}{2}t^2R(t)r^{1/\alpha}(t)x'''(t), \tag{2.21}$$

for every $\lambda \in (0,1)$ and for all sufficiently large t. Using conditions (2.10) and a well-known result in [12], we deduce that, for all t large enough, $x(t)/x'(t) \geqslant t/2$, and thus

$(x/t^2)' \leqslant 0$ eventually. Hence, there exists a constant $c_2 > 0$ such that $x(t)/t^2 \leqslant c_2$ for all sufficiently large t. Consequently,

$$\frac{x^\beta(g(t))}{g^{2\beta}(t)} \geqslant \frac{x^\beta(\sigma(t))}{\sigma^{2\beta}(t)} = \frac{x^\gamma(\sigma(t))}{\sigma^{2\gamma}(t)} \frac{x^{\beta-\gamma}(\sigma(t))}{\sigma^{2(\beta-\gamma)}(\sigma(t))} \geqslant c_3 \frac{x^\gamma(\sigma(t))}{\sigma^{2\gamma}(t)},$$

where $c_3 = c_2^{\beta-\gamma}$. Writing the latter inequality in the form

$$x^\beta(g(t)) \geqslant c_3 \frac{g^{2\beta}(t)}{\sigma^{2\gamma}(t)} x^\gamma(\sigma(t))$$

and using (1.1) and (2.20), we observe that $y(t)$ is a negative solution of a differential inequality

$$u'(t) - c_3 q(t) g^{2\beta}(t) \Big(\frac{\lambda}{2} R(\sigma(t))\Big)^\gamma u^{\gamma/\alpha}(\sigma(t)) \leqslant 0.$$

Hence, the differential inequality

$$v'(t) - c_3 q(t) g^{2\beta}(t) \Big(\frac{\lambda}{2} R(\sigma(t))\Big)^\gamma v^{\gamma/\alpha}(\sigma(t)) \geqslant 0 \tag{2.22}$$

has a positive solution $z(t)$. Note that in view of (2.4) and [13], Theorem 1, advanced differential inequality (2.21) has no positive solutions, which is a contradiction. Therefore, we established that in all possible cases equation (1.1) is oscillatory. $\square$

Theorem 2.2. *Assume that* (1.3) *holds and there exist a number* $\gamma \in \mathfrak{R}$, $\gamma < \alpha < \beta$, *and two functions* $\tau, \sigma \in C(\mathbb{I}, \mathbb{R})$ *such that* (2.2) *holds. Assume further that*

$$\int_{t_0}^{\infty} q(t) \Big(\frac{\tau^3(t)}{r^{1/\alpha}(\tau(t))}\Big)^\gamma \,\mathrm{d}t = \infty \tag{2.23}$$

and

$$\int_{t_0}^{\infty} q(t)(g^2(t) R(\sigma(t)))^\beta \,\mathrm{d}t = \infty. \tag{2.24}$$

If either (2.5), (2.6), *or*

$$\int_{t_0}^{\infty} q(t) \Big(\int_{\sigma(t)}^{\infty} \int_u^{\infty} R(s) \,\mathrm{d}s \,\mathrm{d}u\Big)^\beta \,\mathrm{d}t = \infty \tag{2.25}$$

holds, equation (1.1) *is oscillatory.*

P r o o f. Assume that equation (1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that x is eventually positive. As in the proof of Theorem 2.1, the structure of equation (1.1) implies three possible cases described by (2.8), (2.9), and (2.10).

We start by assuming that (2.8) holds. It has been established in the proof of Theorem 2.1 that (2.11) is satisfied for every $\lambda \in (0,1)$ and for all sufficiently large t. It follows from the condition that $x'(t) > 0$ that there exists a constant c_4 such that

$$x^{\beta}(g(t)) \geqslant x^{\beta}(\tau(t)) = x^{\gamma}(\tau(t))x^{\beta-\gamma}(\tau(t)) \geqslant c_4 x^{\gamma}(\tau(t)).$$

Hence, by (1.1) and (2.11), we observe that the function $y(t)$ is a positive solution of a differential inequality

$$u'(t) + c_4 q(t)\left(\frac{\lambda\tau^3(t)}{6r^{1/\alpha}(\tau(t))}\right)^{\gamma} u^{\gamma/\alpha}(\tau(t)) \leqslant 0. \tag{2.26}$$

However, (2.22) and [13], Theorem 2, imply that delay differential inequality (2.25) has no positive solutions, which is a contradiction.

Assume now that (2.9) holds. Proceeding as in the proof of Theorem 2.1, we obtain a contradiction with our assumptions (2.5) and (2.6). On the other hand, we established in Theorem 2.1 that (2.17) is satisfied. Hence, applying (1.1), (2.17), and using the property $x(g(t)) \geqslant x(\sigma(t))$, we conclude that a differential inequality

$$u'(t) - q(t)\left(\int_{\sigma(t)}^{\infty}\int_{u}^{\infty} R(s)\,\mathrm{d}s\,\mathrm{d}u\right)^{\beta} u^{\beta/\alpha}(\sigma(t)) \geqslant 0 \tag{2.27}$$

has a positive solution $z(t)$. On the other hand, (2.24) and [13], Theorem 1, imply that advanced differential inequality (2.26) has no positive solutions. This is a contradiction.

Finally, assume that (2.10) holds. Following the same lines as in the proof of Theorem 2.1, we deduce that, for every $\lambda \in (0,1)$ and for all sufficiently large t, (2.20) holds. Furthemore,

$$x(g(t))/g^2(t) \geqslant x(\sigma(t))/\sigma^2(t).$$

Consequently, it follows from (1.1) and (2.20) that a differential inequality

$$u'(t) - q(t)\left(\frac{\lambda}{2}g^2(t)R(\sigma(t))\right)^{\beta} u^{\beta/\alpha}(\sigma(t)) \geqslant 0 \tag{2.28}$$

has a positive solution $z(t)$. However, (2.23) and [13], Theorem 1, imply that advanced differential inequality (2.27) has no positive solutions. This contradiction completes the proof. $\square$

Theorem 2.3. *Suppose that* $\alpha=\beta$ *and* (1.3) *holds. Assume that there exist two functions* $\tau,\sigma\in C(\mathbb{I},\mathbb{R})$ *such that* (2.2) *holds. Furthermore, assume that*

$$\frac{1}{6^{\beta}}\liminf_{t\to\infty}\int_{\tau(t)}^{t} q(s)\left(\frac{\tau^3(s)}{r^{1/\alpha}(\tau(s))}\right)^{\beta}\,\mathrm{d}s>\frac{1}{\mathrm{e}} \tag{2.29}$$

and

$$\frac{1}{2^{\beta}}\liminf_{t\to\infty}\int_{t}^{\sigma(t)} q(v)(g^2(v)R(\sigma(v)))^{\beta}\,\mathrm{d}v>\frac{1}{\mathrm{e}}. \tag{2.30}$$

If either (2.5), (2.6), *or*

$$\liminf_{t\to\infty}\int_{t}^{\sigma(t)} q(v)\left(\int_{\sigma(v)}^{\infty}\int_{u}^{\infty}R(s)\,\mathrm{d}s\,\mathrm{d}u\right)^{\beta}\,\mathrm{d}v>\frac{1}{\mathrm{e}} \tag{2.31}$$

holds, equation (1.1) *is oscillatory.*

P r o o f. Suppose that equation (1.1) has a nonoscillatory solution $x(t)$ which, without loss of generality, may be assumed to be eventually positive. As above, differential equation (1.1) "induces" three possible cases described by conditions (2.8), (2.9), and (2.10).

Assume that (2.8) holds. We know from the proof of Theorem 2.1 that, for every $\lambda\in(0,1)$ and for all sufficiently large t, (2.11) is satisfied. Hence, by (1.1) and (2.11), we conclude that $y(t)$ is a positive solution of a differential inequality

$$u'(t)+q(t)\left(\frac{\lambda\tau^3(t)}{6r^{1/\alpha}(\tau(t))}\right)^{\beta}u(\tau(t))\leqslant 0. \tag{2.32}$$

Application of (2.28) and [15], Theorem 2.1.1, yields that delay differential inequality (2.31) has no positive solutions, which is a contradiction.

Assume now that (2.9) holds. As in the proof of Theorem 2.1, we obtain first a contradiction with (2.5) and then with (2.6). On the other hand, by (1.1) and (2.17), a differential inequality

$$u'(t)-q(t)\left(\int_{\sigma(t)}^{\infty}\int_{u}^{\infty}R(s)\,\mathrm{d}s\,\mathrm{d}u\right)^{\beta}u(\sigma(t))\geqslant 0 \tag{2.33}$$

has a positive solution $z(t)$. Using [15], Theorem 2.4.1, and (2.30), we see that advanced differential inequality (2.32) has no positive solutions, which is a contradiction.

Finally, suppose that we have the case (2.10). In the proof of Theorem 2.1, we established that, for every $\lambda \in (0,1)$ and for all sufficiently large t, (2.20) holds and thus, $x(g(t))/g^2(t) \geqslant x(\sigma(t))/\sigma^2(t)$. By virtue of (1.1) and (2.20), a differential inequality

$$u'(t) - q(t)\Big(\frac{\lambda}{2}g^2(t)R(\sigma(t))\Big)^{\beta}u(\sigma(t)) \geqslant 0 \tag{2.34}$$

has a positive solution $z(t)$. Using (2.29) and [15], Theorem 2.4.1, we conclude that advanced differential inequality (2.33) has no positive solutions. This contradiction completes the proof of the fact that equation (1.1) is oscillatory. □

3. Examples and discussion

In this section, we provide three examples that illustrate the main results reported in this paper. Using these examples, we compare the efficiency of our oscillation theorems to that of the criteria reported recently in the papers by Li et al. [16], Zhang et al. [21], [22], and Zhang et al. [23].

E x a m p l e 3.1. Consider an advanced differential equation

$$(t^{2\alpha}(x'''(t))^{\alpha})' + t^{\gamma_0-2\beta-1}x^{\beta}(2t) = 0, \quad t \geqslant 1. \tag{3.1}$$

Here the numbers α, β, $\gamma_0 \in \mathfrak{R}$ are such that $\gamma_0 > \alpha > \beta$, and, in the notation adopted in the paper, $r(t) = t^{2\alpha}$, $q(t) = t^{\gamma_0-2\beta-1}$, $g(t) = 2t$, $\tau(t) = t/2$, and $\sigma(t) = 2t$. Let $\gamma = \gamma_0$, then $R(t) = t^{-1}$. Then

$$\int_{t_0}^{\infty} R(t)\,\mathrm{d}t = \int_1^{\infty} t^{-1}\,\mathrm{d}t = \infty.$$

Since $\gamma_0 > \beta$, a straightforward calculation yields

$$\int_{t_0}^{\infty} q(t)\Big(\frac{\tau^3(t)}{r^{1/\alpha}(\tau(t))}\Big)^{\beta}\mathrm{d}t = \frac{1}{2^{\beta}}\int_1^{\infty} t^{\gamma_0-\beta-1}\,\mathrm{d}t = \infty$$

and

$$\int_{t_0}^{\infty} q(t)g^{2\beta}(t)R^{\gamma}(\sigma(t))\,\mathrm{d}t = 2^{2\beta-\gamma_0}\int_1^{\infty} t^{-1}\,\mathrm{d}t = \infty.$$

Hence, equation (3.1) is oscillatory by Theorem 2.1. Note that the oscillatory nature of this equation cannot be deduced from the results reported in the papers by Li et al. [16], Zhang et al. [21], [22], and Zhang et al. [23] because in our example $g(t) = 2t > t$.

E x a m p l e 3.2. For $t \geqslant 2$, consider a differential equation with an argument $g(t) = t - \sin t$ that alternates between advanced and delayed,

$$(t^{2\alpha}(x'''(t))^{\alpha})' + t^{-1-\gamma_0} x^{\beta}(t - \sin t) = 0. \tag{3.2}$$

We assume that α, β, $\gamma_0 \in \mathfrak{R}$ are such that $\gamma_0 < \alpha < \beta$. Here $r(t) = t^{2\alpha}$, $q(t) = t^{-1-\gamma_0}$, $g(t) = t - \sin t$, $\tau(t) = t/2$, and $\sigma(t) = 2t$. Then $R(t) = 1/t$ and

$$\int_{t_0}^{\infty} R(t)\,\mathrm{d}t = \int_{2}^{\infty} t^{-1}\,\mathrm{d}t = \infty.$$

Let $\gamma = \gamma_0$. Then

$$\int_{t_0}^{\infty} q(t)\Big(\frac{\tau^3(t)}{r^{1/\alpha}(\tau(t))}\Big)^{\gamma}\,\mathrm{d}t = \int_{t_0}^{\infty} q(t)\Big(\frac{\tau^3(t)}{r^{1/\alpha}(\tau(t))}\Big)^{\gamma_0}\,\mathrm{d}t = \frac{1}{2^{\gamma_0}}\int_{2}^{\infty} t^{-1}\,\mathrm{d}t = \infty$$

and, since $\beta > \gamma_0$,

$$\int_{t_0}^{\infty} q(t)(g^2(t)R(\sigma(t)))^{\beta}\,\mathrm{d}t \geqslant 2^{-3\beta}\int_{2}^{\infty} t^{\beta-\gamma_0-1}\,\mathrm{d}t = \infty.$$

Therefore, we conclude that equation (3.2) is oscillatory by Theorem 2.2. Note that results reported by Zhang et al. [23] cannot be applied to (3.2), since all results in the cited paper require that $\beta \leqslant \alpha$ and $g(t) < t$.

E x a m p l e 3.3. For $t \geqslant 1$, consider a delay differential equation

$$(\mathrm{e}^{t}x'''(t))' + 2\sqrt{10}\,\mathrm{e}^{t+\arcsin(\sqrt{10}/10)}x\big(t - \arcsin\big(\sqrt{10}/10\big)\big) = 0. \tag{3.3}$$

Let $\alpha = \beta = 1$, $r(t) = \mathrm{e}^{t}$, $q(t) = 2\sqrt{10}\,\mathrm{e}^{t+\arcsin(\sqrt{10}/10)}$, $g(t) = t - \arcsin\big(\sqrt{10}/10\big)$, $\tau(t) = t/2$, and $\sigma(t) = t + \arcsin\big(\sqrt{10}/10\big)$. Then $R(t) = \mathrm{e}^{-t}$,

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} q(s)\Big(\frac{\tau^3(s)}{r^{1/\alpha}(\tau(s))}\Big)^{\alpha}\,\mathrm{d}s = \frac{\sqrt{10}}{4}\liminf_{t\to\infty}\int_{t/2}^{t} \mathrm{e}^{s/2+\arcsin(\sqrt{10}/10)}s^3\,\mathrm{d}s = \infty,$$

$$\liminf_{t\to\infty}\int_{t}^{\sigma(t)} q(v)(g^2(v)R(\sigma(v)))^{\alpha}\,\mathrm{d}v \geqslant \frac{1}{4}\liminf_{t\to\infty}\int_{t}^{t+\arcsin(\sqrt{10}/10)} v^2\,\mathrm{d}v = \infty,$$

and

$$\begin{aligned}\liminf_{t\to\infty}&\int_{t}^{\sigma(t)} q(v)\Big(\int_{\sigma(v)}^{\infty}\int_{u}^{\infty} R(s)\,\mathrm{d}s\,\mathrm{d}u\Big)^{\alpha}\,\mathrm{d}v\\ &= 2\sqrt{10}\liminf_{t\to\infty}\int_{t}^{t+\arcsin(\sqrt{10}/10)}\mathrm{d}v = 2\sqrt{10}\arcsin\frac{\sqrt{10}}{10} > \frac{1}{\mathrm{e}}.\end{aligned}$$

Hence, equation (3.3) is oscillatory by Theorem 2.3, and $x(t) = \mathrm{e}^t \sin t$ is one such solution. Note that results in the paper by Zhang et al. [23] ensure that every solution of (3.3) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$ and cannot guarantee oscillatory nature of the equation. Our result is stronger since it excludes existence of solutions $x(t)$ satisfying $\lim_{t\to\infty} x(t) = 0$.

R e m a r k 3.1. In this paper, employing a number of known comparison theorems, we derived three new oscillation theorems for a class of fourth-order quasilinear functional differential equations (1.1). These criteria supplement and improve the results obtained by Li et al. [16], Zhang et al. [21], [22], and Zhang et al. [23]. A distinguishing feature of our results is that we do not impose specific restrictions on the deviating argument g, that is, g may be delayed, advanced, or change back and forth from advanced to delayed, as in Example 3.2.

R e m a r k 3.2. As fairly noticed by the referee, equation (1.1) can be viewed as a particular case of a more general class of equations

$$(r(t)\varphi_\alpha(x'''(t)))' + q(t)\varphi_\beta(x(g(t))) = 0, \tag{3.4}$$

where $\varphi_\gamma(u) = |u|^{\gamma-1} u$, $\gamma > 0$. However, techniques used in this paper do not allow a straightforward extension of our results to equation (3.4); this remains an open problem for further research.

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