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Schauder bases and locally complemented subspaces of Banach spaces

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Preface

When I think that when in the beginning I didn't even knew what a Schauder basis was, it makes me realize how far I have gone, and still, how little I know. I need to thanks my supervisors Olav Nygaard and Trond Abrahamsen for helping and leading me through all this time. It was a long and tiring period, but it had its own fun.

Thanks goes to my family for their support. Also, I don't know why, I need to thank my friends for restricting my study time and made me had sleepless nights. The last thanks goes to Kristina for our short breaks that weren't never short. She was the only friend that understood what I was talking about.

I hope you enjoy reading my theses.

Contents

Abstract

The thesis is about Schauder basis in infinite-dimensional Banach spaces and locally complemented subspaces. It starts with the notion of bases and it proves that it is equivalent with that of Schauder basis. It follows with some general theory about bases, and gives the notion of basic sequences and equivalence of bases. It proves that every Banach space has a basic sequence. Next it gives some general theory about unconditional basis. To give an other version of the definition of complemented subspaces, we present adjoint operators and projections. We prove that c_0 is not complemented in l_∞. The Principle of Local Reflexivity (PLR) is proved and it states that a Bnach space is locally 1-complemented in its didual space. We present Hahn-Banach extension operators and prove that its existence is equivalent with being locally 1-complemented. In the end, the definition for a basic sequence to be (locally) complemented is given and it proves that if a basic sequence is locally complemented, then its biorthogonal functionals can be extended to a basic sequence in the dual space.

Notation

The linear *span* of a subset A of a vector space X, denoted by $span\{A\}$, is the set of all finite linear combinations of elements of A. We will denote the closed linear span of a set A by $[A]$. The dimension of a set $A \subset X$ is denoted by $\dim A$, and its closure by $\overline{A}$. We denote by B_X the closed unit ball on a normed space X.

The dual space of X, denoted by X^*, is the space of all continuous linear functionals from X to a field $\mathbb{F}$. $Q_X : X \to X^{**}$ will denote the canonical embedding of X in its bidual X^{**}, and we will say that X is a subspace of X^{**}.

$B(X, Y)$ will denote the space of all bounded (continuous) linear operators from X to Y, where X, Y are Banach spaces. For an operator $T : X \to Y$, we denote $\ker T = \{x \in X : Tx = 0\}$, and the image of T, by ImT. For a subset A of X, we denote $A^{\perp} = \{x^* \in X^* : x^*(x) = 0,\ \forall x \in A\}$.

Cornerstone theorems of Functional Analysis

In this section we will state some fundamental theorems in Functional Analysis that we will use later in the proofs of others theorems and propositions.

The Open Mapping Theorem Every bounded linear operator from a Banach space onto a Banach space is an open mapping.

In other words, if A is the bounded linear operator, $A : X \to Y$, where X, Y are Banach spaces, then if U is an open set in X, then $A(U)$ is open in Y.

The proof of this theorem you can find it on [8, p. 43].

The Closed Graph Theorem Let X, Y be Banach spaces and $T : M \to Y$ a closed linear operator with domain $M \subset X$. If M is closed in X, then T is bounded.

The Closed Graph Theorem is an application of the Open Mapping Theorem. An other version of this theorem is:

If $T : M \to Y$ is a linear mapping from X into Y, where X, Y are Banach spaces, with the property: whenever (x_n) in X is such that both $x = \lim x_n$ and $y = \lim Tx_n$, exist, it follows that $y = Tx$. Then T is continuous.

The Uniform Boundedness Principle Let X be a Banach space and let Y be a normed vector space. Let $\{T_\alpha : \alpha \in A\}$ be a family of bounded linear operators from X to Y. Suppose that for every $x \in X$, the set $\{T_\alpha x : \alpha \in A\}$ is a bounded subset of Y . Then the set $\{\|T_\alpha\| : \alpha \in A\}$ is bounded.

The proof of this theorem you can find it on [**?**, p. 189].

The first isomorphism theorem for Banach spaces Let X, Y be Banach spaces and T a bounded linear operator, $T \in B(X,Y)$ such that the range of T is closed in Y. Then $X/ker(T)$ is isomorphic to $T(X)$.

The Hahn-Banach Separation Theorem Let X be a normed linear space, and $A, B \subset X$ non-empty disjoint convex subsets. If A is open, then there exist a non-zero continuous linear functional f and a real number α such that

$$f(x) \leq \alpha \leq f(y), \text{ for all } x \in A \; and \; y \in B.$$

The Hahn-Banach Extension Theorem Let Y be a subspace of a real linear space X, and p a positive functional on X such that

$$p(tx) = tp(x) \;\; and \;\; p(x+y) \leq p(x) + p(y) \;\; for \; every \; x, y \in X, \; t \geq 0.$$

If f is a linear functional on Y such that $f(x) \leq p(x)$, for every $x \in Y$, then there is a linear functional F on X such that $F = f$ on Y and $F(x) \leq p(x)$, for every $x \in X$.

An other version of the Hahn-Banach Extension Theorem for normed spaces is:

If Y is a subspace of a normed space X, then for every $y^* \in Y^*$ there exist $x^* \in X^*$ such that $x^*\big|_Y = y^*$ and $\|x^*\| = \|y^*\|$.

1 Bases in Banach spaces

1.1 Introduction

One of the central objects of study in functional analysis are Banach spaces. A *Banach space* is defined as complete normed vector space.

First let us present what a norm is.

A *norm* is a function $\|\cdot\| : X \to \mathbb{F}$, where X is a vector space and $\mathbb{F}$ denotes the real or complex numbers, that has the following properties:

For every $x, y \in X$, $k \in \mathbb{F}$,

(i) $\|x\| \geq 0$, and $\|x\| = 0 \Leftrightarrow x = 0$ (separating points)

(ii) $\|kx\| = |k|\|x\|$ (absolute homogeneity)

(iii) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

For the definition of a complete normed space, we will first give the definition of a Cauchy sequence.

A sequence $(x_n)_{n=1}^{\infty}$ in a normed space X is called *Cauchy* if for every $\epsilon > 0$ there exist $N \in \mathbb{N}$ such that for every $m, n \in \mathbb{N}$ such that $m, n > N$ we have that $\|x_m - x_n\| < \epsilon$.

A normed space is called *complete* if every Cauchy sequence converges in that space.

So a Banach space is a vector space X over the real or complex numbers with a norm $\|\cdot\|$ such that every Cauchy sequence in X converges in X.

Independent from Stefan Banach, such spaces were introduced by Norbert Wiener, however Wiener thought that the spaces would not be of importance and gave up. A long time later Wiener wrote in his memoirs that the spaces quite justly should be named after Banach alone, as sometimes they were called "Banach-Wiener spaces" [2].

We will work with Banach spaces which are infinite-dimensional. First lets present the notion of basis in a Banach space. In linear algebra we are used with the concept of basis, however the spaces under consideration are finite-dimensional. Many generalisation of the basis concept in infinite-dimensional Banach spaces are possible. The one presented in the following definition is the most useful.

Definition. A sequence of elements $(e_n)_{n=1}^{\infty}$ in a infinite-dimensional Banach space X is said to be a *basis* of X if for each $x \in X$ there is a *unique* sequence of scalars $(a_n)_{n=1}^{\infty}$ such that

$$x = \sum_{n=1}^{\infty} a_n e_n.$$

This means that we require that the sequence $(\sum_{n=1}^{N} a_n e_n)_{n=1}^{\infty}$ converges to x in the norm topology of X.

From the uniqueness part of the definition it is clear that a basis consist of linearly independent, and in particular non-zero, vectors.

The following proposition states a necessary condition for a Banach space to have a basis. We will define first what a separable space is.

A space is called *separable* if it contains a countable dense set. In other words, a space X is separable if there is a countable subset $A \subset X$ such that the closure of A, $\bar{A} = X$.

Proposition 1.1. *Every Banach space X with a basis $(e_n)_{n=1}^{\infty}$ is separable.*

Proof. If $\mathbb{Q}$ is the set of rational numbers, we will show that the countable set

$$A := \{\sum_{n=1}^{m} a_n e_n : a_n \in \mathbb{Q},\ 1 \leq n \leq m,\ m \in \mathbb{N}\},$$

is dense in X. Since $(e_n)_{n=1}^{\infty}$ is a basis for X, then for every $x \in X$, $x = \sum_{n=1}^{\infty} a_n e_n$, so we can write that

$$\sum_{n=1}^{\infty} a_n e_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n e_n.$$

From this it follows that $X = [e_n]$. Therefore, it suffices to show that A is a dense subset of *span*$(\{e_n : n \in \mathbb{N}\})$.

Fix $k \in \mathbb{N}$, $e_1, \dots, e_k \in \{e_n : n \in \mathbb{N}\}$ and $a_1, \dots, a_k \in \mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is a sequence $(a_{j,i})_{i\in\mathbb{N}} \in \mathbb{Q}^n$ that converges to a_j for each $j = 1, \dots, k$. From the continuity of the vector space operations it follows that

$$\lim_{i\to\infty} \sum_{j=1}^{k} a_{j,i} e_j = \sum_{j=1}^{k} a_j e_j.$$

This means that an arbitrary element of *span*$(\{e_n : n \in \mathbb{N}\})$ can be written as the limit of a sequence in A, so A is dense in *span*$(\{e_n : n \in \mathbb{N}\})$ □

Another thing to point out is that the order of the basis is important; if we permute the elements of the basis then the new sequence can easily fail to be a basis. We will discuss this phenomenon later on.

Proposition 1.2. *If $(e_n)_{n=1}^{\infty}$ is a basis for a Banach space X and $(k_n)_{n=1}^{\infty}$ is a sequence of nonzero scalars, then $(k_n e_n)_{n=1}^{\infty}$ is also a basis for X.*

Proof. If $(e_n)_{n=1}^{\infty}$ is a basis for X, then for every $x \in X$ there is an unique sequence of scalars $(a_n)_{n=1}^{\infty}$ such that

$$x = \sum_{n=1}^{\infty} a_n e_n = \sum_{n=1}^{\infty} \frac{a_n}{k_n} k_n e_n = \sum_{n=1}^{\infty} b_n k_n e_n$$

where $b_n = \frac{a_n}{k_n}$ for every $n \in \mathbb{N}$. The sequence $(b_n)_{n=1}^{\infty}$ is unique because of the uniqueness of $(a_n)_{n=1}^{\infty}$, so $(k_n e_n)_{n=1}^{\infty}$ is also a basis for X. □

A basis $(e_n)_{n=1}^{\infty}$ is called *normalized* if for every $n \in \mathbb{N}$, $\|e_n\| = 1$. If $(e_n)_{n=1}^{\infty}$ is a basis of X, take $k_n = \|e_n\|^{-1}$, then the sequence $(k_n e_n)_{n=1}^{\infty} = (e_n/\|e_n\|)_{n=1}^{\infty}$ is a normalized basis in X.

Note that if $(e_n)_{n=1}^{\infty}$ is a basis of a Banach space X, the maps $x \mapsto a_n$ are linear functionals on X. Let us write $e_n^{\#}(x) = a_n$. However, it is by no means immediate that the linear functionals $(e_n^{\#})_{n=1}^{\infty}$ are actually continuous [1, p. 2].

Let introduce the *partial sum projections* $(S_n)_{n=0}^{\infty}$ associated to $(e_n)_{n=1}^{\infty}$ defined by $S_0 = 0$ and for $n \geq 1$,

$$S_n(x) = \sum_{k=1}^{n} e_k^{\#}(x) e_k.$$

To be a *projection*, a linear operator $P : X \to X$ should be such that $P(Px) = Px$ for every $x \in X$. For every $n \in \mathbb{N}$, S_n are indeed projections because for every $x \in X$,

$$\begin{aligned} S_n(S_n(x)) &= S_n(\sum_{k=1}^{n} e_k^{\#}(x)e_k) = \sum_{k=1}^{n} e_k^{\#}(\sum_{k=1}^{n} e_k^{\#}(x)e_k)e_k \\ &= \sum_{k=1}^{n}\sum_{i=1}^{n} e_k^{\#}(x)e_k^{\#}(e_i)e_k = \sum_{k=1}^{n} e_k^{\#}(x)e_k = S_n(x), \end{aligned}$$

since $e_k^{\#}(e_i) = 1$ for $k = i$, and $e_k^{\#}(e_i) = 0$ for $k \neq i$, where $1 \leq k, i \leq n$.

In the next proposition we will introduce a new norm for the Banach space X, equivalent with the old one, but it is more convenient to work with. The proof is from [1, p. 3].

Proposition 1.3. *Let X be a Banach space with norm $\|\cdot\|$ and for every $x \in X$ let $|||x||| = sup_{n\geq 1}\|S_n x\|$. Then $|||\cdot|||$ is a norm in X equivalent with $\|\cdot\|$ such that $|||\cdot||| \geq \|\cdot\|$.*

Proof. First let show that $|||\cdot|||$ is indeed a norm. Notice that for every $x \in X$, $|||x||| < +\infty$ because we have that $x \in X$ and $n \in \mathbb{N}$, $\|S_n x\| < +\infty$. The first two requirements of the definition of a norm follow immediately from the fact that $\|\cdot\|$ is a norm. To show the triangle inequality, lets pick $x, y \in X$ having the expansions $x = \sum_{n=1}^{\infty} e_n^{\#}(x)e_n$ and $y = \sum_{n=1}^{\infty} e_n^{\#}(y)e_n$, then

$$
\begin{aligned}
|||x+y||| &= |||\sum_{n=1}^{\infty} e_n^{\#}(x)e_n + \sum_{n=1}^{\infty} e_n^{\#}(y)e_n||| \\
&= \sup_{m\in\mathbb{N}} \|\sum_{n=1}^{m} e_n^{\#}(x)e_n + \sum_{n=1}^{m} e_n^{\#}(y)e_n\| \\
&\leq \sup_{m\in\mathbb{N}} (\|\sum_{n=1}^{m} e_n^{\#}(x)e_n\| + \|\sum_{n=1}^{m} e_n^{\#}(y)e_n\|) \\
&\leq \sup_{m\in\mathbb{N}} \|\sum_{n=1}^{m} e_n^{\#}(x)e_n\| + \sup_{m\in\mathbb{N}} \|\sum_{n=1}^{m} e_n^{\#}(y)e_n\| \\
&= |||\sum_{n=1}^{\infty} e_n^{\#}(x)e_n||| + |||\sum_{n=1}^{\infty} e_n^{\#}(y)e_n||| \\
&= |||x||| + |||y|||
\end{aligned}
$$

It follows from the continuity of the norm $\|\cdot\|$ that for each $x \in X$, we have

$$
\begin{aligned}
|||x||| = |||\sum_{n=1}^{\infty} e_n^{\#}(x)e_n||| &= \sup_{m\in\mathbb{N}} \|\sum_{n=1}^{m} e_n^{\#}(x)e_n\| \\
&\geq \lim_{m\to\infty} \|\sum_{n=1}^{m} e_n^{\#}(x)e_n\| = \|\sum_{n=1}^{\infty} e_n^{\#}(x)e_n\|.
\end{aligned}
$$

So $|||\cdot||| \geq \|\cdot\|$ for all $x \in X$.

We will now show that $(X, |||\cdot|||)$ is complete.

Suppose that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(X, |||\cdot|||)$. $(x_n)_{n=1}^{\infty}$ is indeed convergent to some $x \in X$ for the original norm since $\|\cdot\| \leq |||\cdot|||$. Our goal is to prove that $lim_{n\to\infty}|||x_n - x||| = 0$.

Notice that for each fixed k the sequence $(S_k x_n)_{n=1}^{\infty}$ is convergent in the original norm to some $y_k \in X$,. This is because

$$
\|S_k x_n - S_k x_m\| \leq \sup_{k} \|S_k(x_n - x_m)\| = |||x_n - x_m||| \xrightarrow{n,m\to\infty} 0,
$$

which means that the sequence $(S_k x_n)_{n=1}^{\infty}$ is Cauchy in $(X, \|\cdot\|)$ and therefore convergent.

Note also that $(S_k x_n)_{n=1}^{\infty}$ is contained in the finite-dimensional subspace $[e_1, \ldots, e_k]$. Certainly, the functionals $e_j^{\#}$ are continuous on any finite-dimensional subspace; hence if $1 \le j \le k$ we have

$$\lim_{n\to\infty} e_j^{\#}(x_n) = \lim_{n\to\infty} e_j^{\#}(S_k x_n) = e_j^{\#}(\lim_{n\to\infty} S_k x_n) = e_j^{\#}(y_k) := a_j.$$

Next we argue that $\sum_{j=1}^{\infty} a_j e_j = x$ for the original norm.

Since $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence, for every $\epsilon > 0$, pick an integer n so that if $m \ge n$ then $|||x_m - x_n||| \le \frac{1}{3}\epsilon$, and since $\lim_{k\to\infty} S_k x_n = x_n$, we can take k_0 so that $k \ge k_0$ implies $\|x_n - S_k x_n\| \le \frac{1}{3}\epsilon$. Then for $k \ge k_0$ we have

$$\begin{aligned} \|y_k - x\| &\le \|y_k - S_k x_n + S_k x_n - x_n + x_n - x\| \\ &\le \|y_k - S_k x_n\| + \|S_k x_n - x_n\| + \|x_n - x\| \\ &= \lim_{m\to\infty} \|S_k x_m - S_k x_n\| + \|S_k x_n - x_n\| + \lim_{m\to\infty} \|x_m - x_n\| \le \epsilon \end{aligned}$$

Thus $\lim_{k\to\infty} \|y_k - x\| = 0$ and, by uniqueness of the expansion of x with respect to the basis, $S_k x = y_k$.

Now,

$$|||x_n - x||| = \sup_{k\ge1} \|S_k x_n - S_k x\| \le \limsup_{m\to\infty} \sup_{k\ge1} \|S_k x_n - S_k x_m\|,$$

so $\lim_{n\to\infty} |||x_n - x||| = 0$ and $(X, |||\cdot|||)$ is complete.

The identity map $i : (X, |||\cdot|||) \to (X, \|\cdot\|)$ is a bijective, linear and due to the inequality $|||\cdot||| \ge \|\cdot\|$, a continuous operator. The Open Mapping Theorem ensures that $i^{-1} : (X, \|\cdot\|) \to (X, |||\cdot|||)$ is continuous too, which means that i is an isomorphism, so the two norms $\|\cdot\|$ and $|||\cdot|||$ are equivalent. □

We will follow with presenting the notion of Schauder basis in a Banach space, and prove that it is equivalent with the notion of basis.

Let $(e_n)_{n=1}^{\infty}$ be a sequence in a Banach space X and X^* the space of continuous linear functionals on X. Functionals $(e_n^*)_{n=1}^{\infty} \subset X^*$ are called the *biorthogonal functionals* associated with $(e_n)_{n=1}^{\infty}$ provided that for every $k, j \in \mathbb{N}$,

$$e_k^*(e_j) = \begin{cases} 1 & \textit{if } j = k, \\ 0 & \textit{otherwise.} \end{cases}$$

Definition. A sequence $(e_n)_{n=1}^{\infty}$ in a infinite-dimensional Banach space X is called a *Schauder basis* of X if for every $x \in X$, $x = \sum_{n=1}^{\infty} e_n^*(x) e_n$, where $(e_n^*)_{n=1}^{\infty}$ are the biorthogonal functionals associated with $(e_n)_{n=1}^{\infty}$.

In 1927, Julius Schauder introduced the concept of Schauder basis for Banach spaces and constructed a Schauder basis for $C[0,1]$, [6].

Theorem 1.4. *Let X be a (separable) Banach space. A sequence $(e_n)_{n=1}^{\infty}$ in X is a Schauder basis for X if and only if $(e_n)_{n=1}^{\infty}$ is a basis for X.*

Proof. If $(e_n)_{n=1}^{\infty}$ is a Schauder basis for X, then it follows from the definitions that $(e_n)_{n=1}^{\infty}$ is a basis for X.

Let now assume that $(e_n)_{n=1}^{\infty}$ is a basis for X. From Proposition 1.3 we know that the norm $|||\cdot|||$ in X defined for every $x \in X$ as

$$|||x||| = sup_{n\geq 1}\|S_n x\|,$$

is equivalent to the original norm $\|\cdot\|$ of X. Thus there exists K so that $|||x||| \leq K\|x\|$ for $x \in X$. This implies that

$$\|S_n x\| \leq K\|x\|, \qquad \forall x \in X, n \in \mathbb{N}.$$

In particular,

$$|e_n^{\#}(x)|\|e_n\| = \|S_n x - S_{n-1}x\| \leq 2K\|x\|,$$

hence $\|e_n^{\#}(x)\| \leq 2K\|e_n\|^{-1}$, which means that for every $n \in \mathbb{N}$, $e_n^{\#}$ is a bounded (continuous) linear operator on X, i.e. $e_n^{\#} \in X^*$ for every $n \in \mathbb{N}$.

Next, we have to show that $(e_n^{\#})_{n=1}^{\infty}$ are the biorthogonal functionals associated to the basis $(e_n)_{n=1}^{\infty}$. Since for every $m \in \mathbb{N}$, $e_m = \sum_{n=1}^{\infty} a_n e_n$ implies that $a_m = 1$ and $a_n = 0$ for $n \neq m$, this means that

$$e_n^{\#}(e_m) = \begin{cases} 1 & if\ m = n, \\ 0 & otherwise. \end{cases}$$

□

So, every time that we will talk about a basis $(e_n)_{n=1}^{\infty}$ of a Banach space, it means that $(e_n)_{n=1}^{\infty}$ is actually a Schauder basis.

We will present now some examples of Schauder basis.

Example. For $n \in \mathbb{N}$ let

$$e_n = (\underbrace{0, ..., 0, 1}_{n\ times}, 0, ...) \in \mathbb{R}^{\mathbb{N}}$$

Then e_n is a basis of l_p, $1 \leq p < \infty$ and c_0. Indeed, if we take $x \in c_0$ (same for $x \in l_p$, $1 \leq p < \infty$), then

$$\begin{aligned} x &= (x_1, x_2, \ldots, x_n, \ldots) \\ &= x_1(1, 0, \ldots, 0, \ldots) + x_2(0, 1, \ldots, 0, \ldots) + \ldots + x_n(0, 0, \ldots, 1, \ldots) + \ldots \\ &= x_1 e_1 + x_2 e_2 + \ldots + x_n e_n + \ldots = \sum_{n=1}^{\infty} x_n e_n. \end{aligned}$$

To prove the uniqueness, let $(a_n)_{n=1}^{\infty}$ be scalars such that we have also that $x = \sum_{n=1}^{\infty} a_n e_n$. It follows immediately that $a_n = x_n$ for every $n \in \mathbb{N}$.

We call e_n the canonical basis of l_p and c_0, respectively. Note that e_n is also normalized.

The space l_∞ is not separable, therefore it does not have a Schauder basis.

Example. [5, p. 3] In the space of convergent sequence of scalars c, an example of basis can be given by

$$x_1 = (1, 1, 1, \ldots) \qquad and,\ x_n = e_{n-1},\ for\ n > 1,$$

where $(e_n)_{n=1}^{\infty}$ are as in the example above. With respect to this basis, an element $x = (a_1, a_2, \ldots) \in c$ can be written as

$$x = (\lim_n a_n)x_1 + (a_1 - \lim_n a_n)x_2 + (a_2 - \lim_n a_n)x_3 + \cdots .$$

In the next example, taken from [8, p. 352], we will show that $C([0,1])$, the space of continuous function in $[0,1]$, has a basis. This has some interest, because it can be shown that every separable Banach space is isometrically isomorphic to a subspace of $C([0,1])$.

Example. A Schauder basis $(f_n)_{n=0}^{\infty}$ of $C([0,1])$ may be constructed from "tent" functions. For $n = 0, 1$, we define

$$f_0(x) = 1, \;\; f_1(x) = x.$$

For $2^{k-1} < n \leq 2^k$, where $k \geq 1$, we define

$$f_n(x) = \begin{cases} 2^k(x - (\frac{2n-2}{2^k} - 1)) & \text{if } \frac{2n-2}{2^k} - 1 \leq x < \frac{2n-1}{2^k} - 1, \\ 1 - 2^k(x - (\frac{2n-1}{2^k} - 1)) & \text{if } \frac{2n-1}{2^k} - 1 \leq x < \frac{2n}{2^k} - 1, \\ 0 & \text{otherwise.} \end{cases}$$

The graphs of these functions are form a sequence of "tents" of height one and width 2^{-k+1} that sweeps across the interval $[0,1]$. Take $f \in C([0,1])$. We define a sequence $(p_n)_{n=0}^{\infty}$ in $C([0,1])$ such that

$$\begin{aligned} p_0 &= f(0)f_0, \\ p_1 &= p_0 + (f(1) - p_0(1))f_1, \\ p_2 &= p_1 + (f(\frac{1}{2}) - p_1(\frac{1}{2}))f_2, \\ p_3 &= p_2 + (f(\frac{1}{4}) - p_2(\frac{1}{4}))f_3, \\ p_4 &= p_3 + (f(\frac{3}{4}) - p_3(\frac{3}{4}))f_4, \\ p_5 &= p_4 + (f(\frac{1}{8}) - p_4(\frac{1}{8}))f_5, \\ p_6 &= p_5 + (f(\frac{3}{8}) - p_5(\frac{3}{8}))f_6, \\ p_7 &= p_6 + (f(\frac{5}{8}) - p_6(\frac{5}{8}))f_7, \\ p_8 &= p_7 + (f(\frac{7}{8}) - p_7(\frac{7}{8}))f_8, \end{aligned}$$

and so forth. The p_0 is the constant function that agrees with f at 0, while p_1 agrees with f at 0 and 1 and interpolates linearly in between, and p_2 agrees with f at 0, 1, and $\frac{1}{2}$ and interpolates linearly in between, and so forth.

For each $n \in \mathbb{N}_0$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, let a_n be the coefficient of f_n in the formula for p_n. Then $p_k = \sum_{n=0}^{k} a_n f_n$ for each k. The uniform continuity of f implies that $\lim_k \|p_k - f\| = 0$ and therefore that $f = \sum_{n=0}^{\infty} a_n f_n$.

To check the uniqueness, let $(b_n)_{n=0}^{\infty}$ be an other sequence of scalars such that $f = \sum_{n=0}^{\infty} b_n f_n$. Then $\sum_{n=0}^{\infty}(a_n - b_n) f_n = 0$, which implies that $\sum_{n=0}^{\infty}(a_n - b_n) f_n(x) = 0$ when $x = 0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \ldots$, which follows that $a_n = b_n$ for every n. So $(f_n)_{n=0}^{\infty}$ is a basis for $C([0,1])$.

This basis is normalized because for each $n \in \{0, 1, 2, \ldots\}$ and $x \in [0,1]$, $0 \leq f_n(x) \leq 1$, so $\|f_n\| = 1$.

Example. Lets check if the sequence $1, x, x^2, ...$ is a Schauder basis for $C[a,b]$. If the sequence is a Schauder basis for $C[a,b]$, then for every function $f \in C[a,b]$ there exist an unique sequence of scalars (a_n) such that $f = \sum_{n=0}^{\infty} a_n x^n$. Since functions of this form are analytic and because not all continuous functions are analytic, the sequence $1, x, x^2, ...$ is not a Schauder basis for $C[a,b]$.

In common spaces there exists a Schauder basis. This fact led Banach to pose the question in 1932: "Does every separable Banach space have a basis?" This question it is known as the *basis problem*. The basis problem is closely related to another important problem of functional analysis, the approximation problem.

A Banach space X is said to have *the approximation property* if corresponding to each compact set $K \subset X$ and $\epsilon > 0$ there exists a bounded linear operator $F : X \to X$ with finite dimensional range such that $\|x - F(x)\| < \epsilon$ for all $x \in K$.

We say that a subset $K \subset X$ is *compact* if every net $(x_\alpha)_{\alpha \in I} \subset K$ has a convergent subnet on K.

A Banach space with a basis has the approximation property. In June 1972, Per Enflo, at an analysis conference at Hebrew University, Jerusalem, Israel, announced the result: There exist a Banach space without the approximation property and thus there exist a separable Banach space without a basis [6].

1.2 Schauder bases and basic sequences

This section introduces some general theory about Schauder basis. It states the definition of basic sequences and it proves that every Banach space has a basic sequence.

In the proof of theorem 1.4 it is shown that $(e_n^{\#})$ are the biorthogonal functionals associated with $(e_n)_{n=1}^{\infty}$. So, for every $n \in \mathbb{N}$, $e_n^{\#} = e_n^*$.

Definition. Let X be a Banach space with basis $(e_n)_{n=1}^{\infty}$. For each $n \in \mathbb{N}$ the map $S_n : X \to X : \sum_{k=1}^{\infty} e_k^*(x)e_k \mapsto \sum_{n=1}^{n} e_k^*(x)e_k$ is called the n^{th} *natural projection* associated with $(e_n)_{n\in N}$.

S_n is a continuous linear operator since each e_k^* is continuous.

Proposition 1.5. *Let $(e_n)_{n=1}^{\infty}$ be a Schauder basis for a Banach space X and $(S_n)_{n=1}^{\infty}$ the natural projections associated with it. Then*

$$\sup_n \|S_n\| < \infty.$$

Proof. For a Schauder basis the operators $(S_n)_{n=1}^{\infty}$ are bounded *a priori*. Since $S_n(x) \to x$ for every $x \in X$ we have $\sup_n \|S_n(x)\| < \infty$ for each $x \in X$. The Uniform Boundedness Principle yields that $\sup_n \|S_n\| < \infty$. □

Definition. If $(e_n)_{n=1}^{\infty}$ is a basis for a Banach space X then the number $K = \sup_n \|S_n\|$ is called the *basis constant*. In the optimal case that $K = 1$ the basis $(e_n)_{n=1}^{\infty}$ is said to be *monotone*.

Each natural projection for a basis has norm at least 1 because for each $m \in \mathbb{N}$, there exist $x \in X$ such that $x = \sum_{n=1}^{\infty} a_n e_n = \sum_{n=1}^{m} a_n e_n = S_m(x)$ (for example e_n, $n \leq m$), so the norm can't be smaller than 1. This means that the basis constant is always greater or equal to 1, $K \geq 1$.

Proposition 1.6. *If $(e_n)_{n=1}^{\infty}$ is a basis for a Banach space X and K is the basis constant for $(e_n)_{n=1}^{\infty}$, then K is the smallest real number C such that*

$$\|\sum_{n=1}^{m} a_n e_n\| \leq C \|\sum_{n=1}^{\infty} a_n e_n\|$$

whenever $\sum_{n=1}^{\infty} a_n e_n \in X$ *and* $m \in \mathbb{N}$, *which is in turn the smallest real number* C *such that*

$$\|\sum_{n=1}^{m_1} a_n e_n\| \leq C\|\sum_{n=1}^{m_2} a_n e_n\|$$

whenever $m_1, m_2 \in \mathbb{N}$, $m_1 \leq m_2$, *and* $a_1, \ldots, a_{m_2} \in \mathbb{F}$.

Proof. From the definition of the basic constant we have that

$$K = \sup\{\frac{\|\sum_{n=1}^{m} a_n e_n\|}{\|\sum_{n=1}^{\infty} a_n e_n\|} : \sum_{n=1}^{\infty} a_n e_n \in X\backslash\{0\}, m \in \mathbb{N}\},$$

from which the first part of the proposition follows.

For every $m_1, m_2 \in \mathbb{N}$, $m_1 \leq m_2$, and $a_1, \ldots, a_{m_2} \in \mathbb{F}$ we can write

$$\|\sum_{n=1}^{m_1} a_n e_n\| = \|S_{m_1}(\sum_{n=1}^{m_2} a_n e_n)\| \leq \sup_{m_1} \|S_{m_1}\| \|\sum_{n=1}^{m_2} a_n e_n\|,$$

and the proposition is proved since $K = \sup_{m_1} \|S_{m_1}\|$. □

Remark. We can always renorm a Banach space X with a basis in such a way that the given basis is monotone. Just put

$$|||x||| = \sup_{n \geq 1} \|S_n x\|.$$

From Proposition 1.3 we have that the new norm is equivalent to the old one and $\|x\| \leq |||x|||$ for every $x \in X$. Since

$$|||x||| = \sup_{n \geq 1} \|S_n x\| \leq \sup_{n \geq 1} \|S_n\| \|x\| = K\|x\|,$$

we have that $\|x\| \leq |||x||| \leq K\|x\|$, where K is the basis constant.

To prove that $K = 1$ we need to show that $|||S_n||| = 1$ for $n \in \mathbb{N}$. This follows from

$$\begin{aligned} |||S_n||| &= \sup_{|||x||| \leq 1} |||S_n x||| = \sup_{|||x||| \leq 1} \sup_{m \geq 1} \|S_m S_n x\| \\ &\geq \sup_{|||x||| \leq 1} \sup_{m \geq n \geq 1} \|S_m S_n x\| = \sup_{|||x||| \leq 1} \sup_{m \geq n \geq 1} \|S_n x\| \\ &= \sup_{|||x||| \leq 1} |||x||| = 1. \end{aligned}$$

and since $\|S_m S_n x\| \le |||x|||$ for every m, we can write

$$\sup_{m\ge 1} \|S_m S_n x\| \le |||x||| \Leftrightarrow \|S_n x\| \le |||x||| \text{ for every } x$$
$$\Leftrightarrow |||S_n||| \le 1.$$

If we have a family of projections enjoying the properties of the partial sum operators, one can construct a basis for a Banach space X.

Proposition 1.7. *Suppose $S_n : X \to X, n \in \mathbb{N}$, is a sequence of bounded linear projections on a Banach space X such that*

(i) $\dim S_n(X) = n$ *for each n;*

(ii) $S_n S_m = S_m S_n = S_{min(m,n)}$, *for any integers m and n;*

(iii) $S_n(x) \to x$ *for every $x \in X$.*

Then any nonzero sequence of vectors $(e_n)_{n=1}^{\infty}$ in X chosen inductively so that $e_1 \in S_1(X)$, and $e_k \in S_k(X) \cap S_{k-1}^{-1}(0)$ if $k \ge 2$ is a basis for X with partial sum projections $(S_n)_{n=1}^{\infty}$.

Proof. Let $0 \ne e_1 \in S_1(X)$, $0 \ne e_2 \in S_2(X) \cap S_1^{-1}(0)$ and so on, by induction, $0 \ne e_n \in S_n(X) \cap S_{n-1}^{-1}(0)$. Then for $x \in X$, by (ii) we have

$$S_{n-1}(S_n(x) - S_{n-1}(x)) = S_{n-1}(x) - S_{n-1}(x) = 0, \text{ and}$$
$$S_n(x) - S_{n-1}(x) = S_n(S_n(x) - S_{n-1}(x)) \in S_n(X),$$

and therefore $S_n(x) - S_{n-1}(x) \in S_n(X) \cap S_{n-1}^{-1}(0)$. Thus we can write $S_n(x) - S_{n-1}(x) = a_n e_n$, for $n \in \mathbb{N}$.

If we let $S_0(x) = 0$ for all x, it follows from (iii) that

$$x = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \sum_{k=1}^{n} (S_k(x) - S_{k-1}(x)) = \lim_{n\to\infty} \sum_{k=1}^{n} a_k e_k = \sum_{k=1}^{\infty} a_k e_k.$$

To show the uniqueness of the representation of x, lets assume that $x = \sum_{k=1}^{\infty} b_k e_k$. From the continuity of $S_m - S_{m-1}$, for $m \in \mathbb{N}$ it follows that

$$a_m e_m = (S_m - S_{m-1})(x) = \lim_{n\to\infty} (S_m - S_{m-1})(\sum_{k=1}^{n} b_k e_k) = b_m e_m,$$

and thus $a_m = b_m$ for every $m \in \mathbb{N}$.

Therefore, the sequence $(e_n)_{n=1}^{\infty}$ is a basis and $(S_n)_{n=1}^{\infty}$ its partial sum projections. □

We will now present what a basic sequence is and later on we will prove that every Banach space contains a basic sequence.

Definition. A sequence $(e_k)_{k=1}^{\infty}$ in a Banach space X is called a *basic sequence* if it is a basis for $[e_k]$, the closed linear span of $(e_n)_{n=1}^{\infty}$.

The following proposition is known as *Grunblum's criterion* and it is used as a test for recognising a sequence of elements in a Banach space as a basic sequence [1, p. 6]. For the proof of this proposition we will need the following theorem:

Theorem 1.8. *Let (S_n) be a sequence of operators from a Banach space X into a normed linear space Y such that $\sup_n \|S_n\| < \infty$. Then, if $T : X \to Y$ is another operator, the subspace $\{x \in X : \|S_n x - Tx\| \to 0\}$ is norm-closed in X.*

Proposition 1.9. *A sequence $(e_k)_{k=1}^{\infty}$ of nonzero elements of a Banach space X is basic if and only if there is a positive constant K such that*

$$\|\sum_{k=1}^{m} a_k e_k\| \leq K \|\sum_{k=1}^{n} a_k e_k\|$$

for any sequence of scalars (a_k) and any integers m, n such that $m \leq n$.

Proof. Assume $(e_k)_{k=1}^{\infty}$ is basic, and let $S_N : [e_k] \to [e_k], N = 1, 2, ...,$ be its partial sum projections. Then, if $m \leq n$ we have

$$\|\sum_{k=1}^{m} a_k e_k\| = \|S_m(\sum_{k=1}^{n} a_k e_k)\| \leq \sup_m \|S_m\| \|\sum_{k=1}^{n} a_k e_k\|,$$

so the inequality holds with $K = \sup_m \|S_m\|$.

For the converse, let E be the linear span of $(e_k)_{k=1}^{\infty}$ and $s_m : E \to [e_k]_{k=1}^{m}$ be the finite-rank operator (the bounded linear operator between Banach spaces whose range is finite-dimensional) defined by

$$s_m(\sum_{k=1}^{n} a_k e_k) = \sum_{k=1}^{min(m,n)} a_k e_k, \qquad m, n \in \mathbb{N}.$$

Since $\overline{E} = [e_k]$, E is dense in $[e_k]$, each s_m extends to $S_m : [e_k] \to [e_k]_{k=1}^{m}$ with $\|S_m\| = \|s_m\| \leq K$.

Notice that for each $x \in E$ we have

$$S_n S_m(x) = S_m S_n(x) = S_{min(m,n)}(x), \qquad m, n \in \mathbb{N},$$

so, by density, this equality holds for all $x \in [e_n]$.

$S_n x \to x$ for all $x \in [e_n]$ since the set $x \in [e_n] : S_m(x) \to x$ is closed (it follows from Theorem 1.8, when T is the identity operator) and contains E. Proposition 1.7 yields that (e_k) is a basis for $[e_n]$ with partial sum projections (S_m). □

Does every Banach space contain a basic sequence? A first answer to this question was given by Banach who stated without proof that every infinite dimensional Banach space X contains an infinite dimensional closed subspace with a basis. Different proofs of this were given in 1958 by Gelbaum and by Bessaga and Pelczynski, and in 1962 by Day [6].

Below is a simple proof of the statement taken from [3].

Lemma 1.10. *Let X be an infinite-dimensional Banach space, let E be a finite-dimensional subspace of X, and let $\epsilon > 0$. Then there exists $x \in X$ such that $\|x\| = 1$ and*

$$\|y\| \leq (1+\epsilon)\|y + ax\|,$$

for all $y \in E$ and all scalars a.

Proof. We may suppose that $\epsilon < 1$. As the unit ball of E is compact, there is a finite set $\{y_1, \ldots, y_n\}$ in E such that $\|y_k\| = 1$ where $1 \leq k \leq n$, and

$$\min_{1 \leq k \leq n} \|y - y_k\| < \frac{\epsilon}{2} \quad forevery\ y \in E,\ \|y\| = 1.$$

Pick $y_1^*, \ldots, y_n^* \in X^*$, the dual of X, $\|y_k\| = 1$ for $1 \leq k \leq n$, such that $y_k^*(y_k) = 1$ for each k. Then there exist $x \in X$ with $\|x\| = 1$ and $y_k^*(x) = 0$ for each k. For any $y \in E$, $\|y\| = 1$, pick y_k such that $\|y_k - y\| \leq \frac{\epsilon}{2}$. For a scalar a, we have that

$$\begin{aligned}\|y + ax\| &= \|y - y_k + y_k + ax\| \\ &\geq \|y_k + ax\| - \|y - y_k\| \\ &> \|y_k + ax\| - \frac{\epsilon}{2} \\ &\geq |y_k^*(y_k + ax)| - \frac{\epsilon}{2} \\ &= 1 - \frac{\epsilon}{2} \geq (1 + \epsilon)^{-1},\end{aligned}$$

as required. □

Theorem 1.11. *Every Banach space X contains a basic sequence.*

Proof. We use induction to pick a sequence of norm-one elements $(x_n) \subset X$ such that the inequality of Proposition 1.9 always holds, with $K = 2$ say (the proof works for any $K > 1$).

For $n = 1$, $\|a_1 x_1\| \leq (2 - \epsilon)\|a_1 x_1\|$. The inequality is true.

Suppose we have chosen $x_1, \ldots, x_n$ and $\epsilon > 0$ such that

$$\|\sum_{k=1}^{m} a_k x_k\| \leq (2 - \epsilon)\|\sum_{k=1}^{n} a_k x_k\|,$$

for any $m \leq n$ and any scalars $(a_k)_{k=1}^{n}$.

We now try to find x_{n+1}. We need to ensure that $\|x_{n+1}\| = 1$ and that for some $\epsilon_0 > 0$, we have that

$$\|\sum_{k=1}^{m} a_k x_k\| \leq (2 - \epsilon_0)\|\sum_{k=1}^{n+1} a_k x_k\|,$$

for any $m \leq n$ and any scalars $(a_k)_{k=1}^{n+1}$.

Let E_n be the linear span of $x_1, \ldots, x_n$, a finite-dimensional subspace of X. From Lemma 1.10 we can find $x_{n+1} \in X$, $\|x_{n+1}\| = 1$ such that

$$\|y\| \leq (1 + \delta)\|y + a_{n+1} x_{n+1}\|$$

for every $y \in E_n$ and every scalar a_{n+1}, where $\delta > 0$ is chosen so that $(2-\epsilon)(1+\delta) = 2 - \frac{\epsilon}{2}$,that is $\delta = \frac{\epsilon}{2(2-\epsilon)}$. Then, for a sequence of scalars $(a_k)_{k=1}^{n+1}$, let $y = \sum_{k=1}^{n} a_k x_k \in E_n$, so that for $m \leq n$, we have that

$$\begin{aligned} \|\sum_{k=1}^{m} a_k x_k\| &\leq (2-\epsilon)\|\sum_{k=1}^{n} a_k x_k\| = (2-\epsilon)\|y\| \\ &\leq (2-\epsilon)(1+\delta)\|y + a_{n+1}x_{n+1}\| \\ &= (2-\frac{\epsilon}{2})\|\sum_{k=1}^{n+1} a_k x_k\|, \end{aligned}$$

as required. □

If a Banach space has a basis then it is natural the question about its uniqueness. First lets introduce the notion of equivalence for basis.

Definition. Two basis (basic sequences), $(x_n)_{n=1}^{\infty} \subset X$, $(y_n)_{n=1}^{\infty} \subset Y$, where X, Y are Banach spaces, are called *equivalent* provided that for every sequence of scalars $(a_n)_{n=1}^{\infty}$ the series $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if $\sum_{n=1}^{\infty} a_n y_n$ converges.

Proposition 1.12. *Two basis (or basic sequences) $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in the Banach spaces X and Y respectively, are equivalent if and only if there exists an isomorphism $T : [x_n] \to [y_n]$ such that $Tx_n = y_n$ for every $n \in \mathbb{N}$.*

Proof. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be two equivalent basis. Define $T : X \to Y$ by $T(\sum_{n=1}^{\infty} a_n x_n) = \sum_{n=1}^{\infty} a_n y_n$. It is easily shown that T is a bijection.

To prove that T is continuous we use the Closed Graph Theorem. Let $(u_j)_{j=1}^{\infty}$ be a sequence such that $u_j \to u \in X$ and $Tu_j \to v \in Y$. Furthermore, let $u_j = \sum_{n=1}^{\infty} x_n^*(u_j)x_n$ and $u = \sum_{n=1}^{\infty} x_n^*(u)x_n$. We need to show that $Tu = v$.

From the continuity of biorthofonal functionals associated with $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ respectively, we have that for every $n \in \mathbb{N}$, $x_n^*(u_j) \to x_n^*(u)$ and $y_n^*(Tu_j) = x_n^*(u_j) \to y_n^*(v)$. From the uniqueness of the limit, for every n, $x_n^*(u) = y_n^*(v)$. Therefore $Tu = v$, which means that T is continuous.

Conversely, we have that $T : [x_n] \to [y_n]$ is an isomorphism such that $Tx_n = y_n$ for every $n \in \mathbb{N}$. Let $X = [x_n]$ and $Y = [y_n]$. Then for every series $\sum_{n=1}^{\infty} a_n y_n$ in Y, we can write

$$\sum_{n=1}^{\infty} a_n y_n = \sum_{n=1}^{\infty} a_n T x_n = T(\sum_{n=1}^{\infty} a_n x_n)$$

which means that $\sum_{n=1}^{\infty} a_n y_n$ converges if and only if $\sum_{n=1}^{\infty} a_n x_n$ converges. $\square$

Theorem 1.13. *[7] If X is an infinite-dimensional Banach space with a basis, then there exists two non-equivalent normalized basis.*

1.3 Unconditional basic sequences

As mentioned in the beginning, the order of elements of the basis is important. However, there are bases, called *unconditional bases*, which are bases no matter how you reorder them. The canonical basis for c_0 and l_p, $1 \leq p < \infty$, is an example of such basis.

First, let us present the notion of unconditional convergence.

Definition. Let $(x_n)_{n=1}^{\infty}$ be a sequence in Banach space X. A series $\sum_{n=1}^{\infty} x_n$ in X is said to be *unconditionally convergent* if $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges for every permutation π of $\mathbb{N}$.

Next are some propositions and theorems about unconditional convergence that will be useful later on when we will present the notion of unconditional basis.

Proposition 1.14. *If the series $\sum_{n=1}^{\infty} x_n$ is unconditional convergent in a normed space, then $\sum_{n=1}^{\infty} x_{\pi(n)} = \sum_{n=1}^{\infty} x_n$ for each permutation π of $\mathbb{N}$.*

Proof. Suppose the contrary, that there is a permutation π of $\mathbb{N}$ such that the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} x_{\pi(n)}$ both converge but to different limits. We will show that there is another permutation π' of $\mathbb{N}$ such that $\sum_{n=1}^{\infty} x_{\pi'(n)}$ doesn't converge, which means that $\sum_{n=1}^{\infty} x_n$ is only conditional convergent.

Let $\epsilon = \| \sum_{n=1}^{\infty} x_{\pi(n)} - \sum_{n=1}^{\infty} x_n \|$, and let $p_1 \in \mathbb{N}$ be such that

$$\| \sum_{n=1}^{\infty} x_{\pi(n)} - \sum_{n=1}^{p_1} x_n \| < \frac{\epsilon}{3}.$$

There is a positive integer q_1 such that

$$\{\pi(n) : n \in \mathbb{N}, 1 \leq n \leq p_1\} \subseteq \{n : n \in \mathbb{N}, 1 \leq n \leq q_1\}$$

and

$$\| \sum_{n=1}^{\infty} x_n - \sum_{n=1}^{q_1} x_n \| < \frac{\epsilon}{3}.$$

Then there is a positive integer p_2 such that

$$\{n : n \in \mathbb{N}, 1 \le n \le q_1\} \subseteq \{\pi(n) : n \in \mathbb{N}, 1 \le n \le p_2\}$$

and

$$\|\sum_{n=1}^{\infty} x_{\pi(n)} - \sum_{n=1}^{p_2} x_{\pi(n)}\| < \frac{\epsilon}{3},$$

and a further positive integer q_2 such that

$$\{\pi(n) : n \in \mathbb{N}, 1 \le n \le p_2\} \subseteq \{n : n \in \mathbb{N}, 1 \le n \le q_2\}$$

and

$$\|\sum_{n=1}^{\infty} x_n - \sum_{n=1}^{q_2} x_n\| < \frac{\epsilon}{3}.$$

If we continue in this way, we will get two sequences (p_n) and (q_n).

Now let π' be the permutation of $\mathbb{N}$ by listing $\mathbb{N}$ in the following order. First list $\pi(1), \ldots, \pi(p_1)$, than follow this by the members of $1, \ldots, q_1$ not already listed. Follow this by members of $\pi(1), \ldots, \pi(p_2)$ not already listed, and in turn follow that by the members of $1, \ldots, q_2$ not already listed, and so forth. Since the partial sums of $\sum_{n=1}^{\infty} x_{\pi'(n)}$ swing back and forth between being within $\frac{\epsilon}{3}$ of the series $\sum_{n=1}^{\infty} x_{\pi(n)}$ and being within $\frac{\epsilon}{3}$ of $\sum_{n=1}^{\infty} x_n$, the series $\sum_{n=1}^{\infty} x_{\pi'(n)}$ doesn't converge. □

Theorem 1.15. *For a sequence (x_n) in Banach space X the following statements are equivalent:*

(i) $\sum_{n=1}^{\infty} x_n$ *is unconditionally convergent.*

(ii) *For any $\epsilon > 0$ there exist an $n \in \mathbb{N}$ so that if M is any finite subset of $\{n+1, n+2, \ldots\}$, then $\|\sum_{j \in M} x_j\| < \epsilon$.*

(iii) *For any subsequence (n_j) the series $\sum_{j \in \mathbb{N}} x_{n_j}$ converges.*

(iv) *For sequence $(\epsilon_j) \subset \{\pm 1\}$ the series $\sum_{j=1}^{\infty} \epsilon_j x_j$ converges.*

Proof. "$(i) \Rightarrow (ii)$" Suppose that (ii) is false. Then there exist $\epsilon > 0$ so that for every $n \in \mathbb{N}$ we can find a finite subset M_n of $\{n+1, n+2, \ldots\}$ with

$$\|\sum_{j \in M_n} x_j\| \geq \epsilon.$$

We will build a permutation π of $\mathbb{N}$ so that $\sum_{n=1}^{\infty} x_{\pi(n)}$ diverges.

Take $n_1 = 1$ and let $A_1 = M_{n_1}$. Next pick $n_2 = \max A_1$ and let $B_1 = \{n_1 + 1, \ldots, n_2\} \setminus A_1$. Now repeat the process taking $A_2 = M_{n_2}$, $n_3 = \max A_2$ and $B_2 = \{n_2 + 1, \ldots, n_3\} \setminus A_2$. Iterating we generate a sequence $(n_k)_{k=1}^{\infty}$ and a partition $\{n_k + 1, \ldots, n_{k+1}\} = A_k \cup B_k$. Define π so that π permutes the elements of $\{n_k + 1, \ldots, n_{k+1}\}$ in such a way that A_k precedes B_k. Then the series $\sum_{n=1}^{\infty} x_{\pi(n)}$ is divergent because the Cauchy condition fails.

"$(ii) \Rightarrow (iii)$" Let (n_j) be a subsequence of $\mathbb{N}$. For every $\epsilon > 0$, use condition (ii) and choose $n \in \mathbb{N}$, so that $\|\sum_{j \in M} x_j\| < \epsilon$, whenever M is any finite subset of $\{n+1, n+2, \ldots\}$. This implies that for all $i_0 \leq i < j$, with $i_0 = \min\{s : n_s > n\}$, it follows that $\|\sum_{s=i}^{j} x_{n_s}\| < \epsilon$. Since $\epsilon > 0$ was arbitrary this means that the sequence $(\sum_{s=1}^{j} x_{n_s})_{j \in \mathbb{N}}$ is Cauchy and thus convergent.

"$(iii) \Rightarrow (iv)$" If (ϵ_n) is a sequence of ± 1's, let $N^+ = \{n \in \mathbb{N} : \epsilon_n = 1\}$ and $N^- = \{n \in \mathbb{N} : \epsilon_n = -1\}$. Since

$$\sum_{j=1}^{n} \epsilon_j x_j = \sum_{j \in N^+, j \leq n} x_j - \sum_{j \in N^-, j \leq n} x_j, \ for\ n \in \mathbb{N},$$

and since $\sum_{j \in N^+, j \leq n} x_j$ and $\sum_{j \in N^-, j \leq n} x_j$ converge as $n \to \infty$ by (iii), it follows that $\sum_{j=1}^{n} \epsilon_j x_j$ converges as $n \to \infty$.

"$(iv) \Rightarrow (ii)$" Assume that (ii) is false. Then there is an $\epsilon > 0$ and for every $n \in \mathbb{N}$ there is a finite set M of $\{n+1, n+2, \ldots\}$, so that $\|\sum_{j \in M} x_j\| \geq \epsilon$. As above choose finite subsets M_1, M_2, M_3 etc. so that $\min M_{n+1} > \max M_n$ and $\|\sum_{j \in M_n} x_j\| \geq \epsilon$, for $n \in N$. Assign $\epsilon_n = 1$ if $n \in \cup_{k \in \mathbb{N}} M_k$ and $\epsilon_n = -1$, otherwise.

Note that the series $\sum_{n=1}^{\infty}(1+\epsilon_n)x_n$ cannot converge because

$$\sum_{j=1}^{k}\sum_{i\in M_j} x_i = \frac{1}{2}\sum_{j=1}^{k}\sum_{i\in M_j}(x_j+\epsilon_j x_j) = \sum_{j=1}^{k}\sum_{i\in M_j}(1+\epsilon_j)x_j$$
$$= \frac{1}{2}\sum_{n=1}^{\max M_k}(1+\epsilon_n)x_n, \ \ for\ k\in\mathbb{N}.$$

Thus at least one of the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty}\epsilon_n x_n$ cannot converge.

"$(ii)\Rightarrow(i)$" Assume that $\pi:\mathbb{N}\to\mathbb{N}$ is a permutation for which $\sum x_{\pi(j)}$ is not convergent. Then we can find an $\epsilon>0$ and $0=k_0<k_1<k_2<\dots$ so that

$$\|\sum_{j=k_{n-1}+1}^{k_n} x_{\pi(j)}\|\geq\epsilon.$$

Then choose $M_1=\{\pi(1),\dots,\pi(k_1)\}$ and if $M_1<M_2<\dots<M_n$ have been chosen with $\min M_{j+1}>\max M_j$ and $\|\sum_{i\in M_j} x_i\|\geq\epsilon$, if $i=1,2,\dots,n$, choose $m\in\mathbb{N}$ so that $\pi(j)>\max M_n$ for all $j>k_m$ (we are using the fact that for any permutation π, $\lim_{j\to\infty}\pi(j)=\infty$) and let

$$M_{n+1}=\{\pi(k_m+1),\pi(k_m+2),\dots,\pi(k_{m+1})\},$$

then $\min M_{n+1}>\max M_n$ and $\|\sum_{i\in M_{n+1}} x_i\|\geq\epsilon$.

We constructed the finite sets M_n by induction in such a way that (ii) is not satisfied. □

Proposition 1.16. *A series $\sum_{n=1}^{\infty} x_n$ in a Banach space X is unconditionally convergent if and only if $\sum_{n=1}^{\infty} t_n x_n$ converges (unconditionally) for all $(t_n)\in l_\infty$.*

Proof. Suppose that $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent. Take $(t_n)_{n=1}^{\infty}$ a bounded sequence, $(t_n)\in l_\infty$. For $1\leq r<s$, pick $x^*\in X^*$ such that $\|x^*\|=1$ and

$$\sum_{n=r}^{s} t_n x^*(x_n)=\|\sum_{n=1}^{N} t_n x_n\|, \qquad for\ N\geq s.$$

For every $n \in \mathbb{N}$, let

$$\epsilon_n = \begin{cases} 1 & if\ x^*(x_n) \geq 0 \\ -1 & otherwise. \end{cases}$$

We can write

$$\begin{aligned} \|\sum_{n=r}^{s} t_n x_n\| \leq \|\sum_{n=1}^{N} t_n x_n\| &= \sum_{n=r}^{s} |t_n||x^*(x_n)| \\ &= \sum_{n=r}^{s} |t_n|\epsilon_n x^*(x_n) \leq \sup_{n\in\mathbb{N}} |t_n| \sum_{n=r}^{s} \epsilon_n x^*(x_n) \\ &= \|t_n\|_\infty \|\sum_{n=r}^{s} \epsilon_n x_n\|. \end{aligned}$$

Since $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent, then from Theorem 1.15 (iv) we have that $\sum_{n=1}^{\infty} \epsilon_n x_n$ converges, so we can make $\|\sum_{n=r}^{s} \epsilon_n x_n\|$ as small as we want, $\|\sum_{n=r}^{s} \epsilon_n x_n\| < \frac{\epsilon}{\|t_n\|_\infty}$. So we have that $\|\sum_{n=r}^{s} t_n x_n\| < \epsilon$. From Theorem 1.15 (ii), $\sum_{n=1}^{\infty} t_n x_n$ converges unconditionally.

Conversely, if $\sum_{n=1}^{\infty} t_n x_n$ converges unconditionally, then by taking $t_n \subset \{\pm 1\}$ we have that $\sum_{n=1}^{\infty} x_n$ converges unconditionally. □

Definition. A basis $(e_n)_{n=1}^{\infty}$ of a Banach space X is called *unconditional* if for each $x \in X$ the series $\sum_{n=1}^{\infty} e_n^*(x) e_n$ converges unconditionally.

Obviously, $(e_n)_{n=1}^{\infty}$ is unconditional basis of X if and only if $(e_{\pi(n)})_{n=1}^{\infty}$ is a basis of X for all permutations $\pi : \mathbb{N} \to \mathbb{N}$.

From Proposition 1.14 we have that if $(e_n)_{n=1}^{\infty}$ is an unconditional basis of X, then for every $x = \sum_{n=1}^{\infty} e_n^*(x) e_n \in X$, $x = \sum_{n=1}^{\infty} e_{\pi(n)}^*(x) e_{\pi(n)}$ for every permutations π of $\mathbb{N}$.

Also, from Proposition 1.12 it follows that a basis equivalent to an unconditional basis is itself unconditional.

Bases that are not unconditional are called *conditional.* Below is an example of a conditional basis.

Example. The *summing basis* of c_0, $(f_n)_{n=1}^{\infty}$, defined as

$$f_n = e_1 + \cdots + e_n, \qquad n \in \mathbb{N},$$

is a conditional basis, where $(e_n)_{n=1}^{\infty}$ is the canonical basis.

To see that (f_n) is a basis for c_0 we prove that for each $\xi = (\xi(n))_{n=1}^{\infty} \in c_0$ we have $\xi = \sum_{n=1}^{\infty} f_n^*(\xi) f_n$, where $f_n^* = e_n^* - e_{n+1}^*$ are the biorthogonal functionals of (f_n). Given $N \in \mathbb{N}$,

$$\begin{aligned}
\sum_{n=1}^{N} f_n^*(\xi) f_n &= \sum_{n=1}^{N} (e_n^*(\xi) - e_{n+1}^*(\xi)) f_n \\
&= \sum_{n=1}^{N} (\xi(n) - \xi(n+1)) f_n \\
&= \sum_{n=1}^{N} \xi(n) f_n - \sum_{n=2}^{N+1} \xi(n) f_{n-1} \\
&= \sum_{n=1}^{N} \xi(n)(f_n - f_{n-1}) - \xi(N+1) f_N \\
&= (\sum_{n=1}^{N} \xi(n) e_n) - \xi(N+1) f_N.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\xi - \sum_{n=1}^{N} f_n^*(\xi) f_n\|_\infty &= \| \sum_{N+1}^{\infty} \xi(n) e_n + \xi(N+1) f_N \|_\infty \\
&\leq \| \sum_{N+1}^{\infty} \xi_n e_n \|_\infty + |\xi(N+1)| \|f_N\|_\infty \xrightarrow{N\to\infty} 0,
\end{aligned}$$

and $(f_n)_{n=1}^{\infty}$ is a basis.

Now we will identify the set, S, of coefficient $(\alpha_n)_{n=1}^{\infty}$ such that the series $\sum_{n=1}^{\infty} \alpha_n f_n$ converges. In fact we have that $(\alpha_n) \in S$ if and only if there exist $\xi = (\xi(n)) \in c_0$ so that $\alpha_n = \xi(n) - \xi(n+1)$ for all n. Then, clearly, unless the series $\sum_{n=1}^{\infty} \alpha_n$ converges absolutely, the convergence of $\sum_{n=1}^{\infty} \alpha_n f_n$ in c_0 is not equivalent to the convergence of $\sum_{n=1}^{\infty} \epsilon_n \alpha_n f_n$ for any choice of signs $(\epsilon_n)_{n=1}^{\infty}$. Hence (f_n) cannot be unconditional.

The following result about unconditional basis can be proved easily from the corresponding one about basis.

Proposition 1.17. *If $(e_n)_{n=1}^\infty$ is an unconditional basis for a Banach space X and $(k_n)_{n=1}^\infty$ is a sequence of nonzero scalars, then $(k_n e_n)_{n=1}^\infty$ is also an unconditional basis for X.*

It follows immediately that $(\|e_n\|^{-1}e_n)$ is a normalized unconditional basis for a Banach space X, if $(e_n)_{n=1}^\infty$ is an unconditional basis for X.

Proposition 1.18. *A basis $(e_n)_{n=1}^\infty$ of a Banach space X is unconditional if and only if there is a constant $K \geq 1$ such that for all $N \in \mathbb{N}$, whenever $a_1, \ldots, a_N, b_1, \ldots, b_N$ are scalars satisfying $|a_n| \leq |b_n|$ for $n = 1, \ldots, N$, then the following inequality holds:*

$$\|\sum_{n=1}^N a_n e_n\| \leq K\|\sum_{n=1}^N b_n e_n\|. \tag{1.1}$$

Proof. Assume $(e_n)_{n=1}^\infty$ is unconditional. If $\sum_{n=1}^\infty a_n e_n$ is convergent then $\sum_{n=1}^\infty t_n a_n e_n$ converges for all $(t_n) \in l_\infty$ by Proposition 1.16. By the Uniform Boundedness principle, the linear map

$$T_{(t_n)} : X \to X, \quad \sum_{n=1}^\infty a_n e_n \to \sum_{n=1}^\infty t_n a_n e_n$$

is continuous because $\sup \|T_{(t_n)}\|_X < \infty$ since $\sum_{n=1}^\infty t_n a_n e_n$ converges. Let note $K = \sup \|T_{(t_n)}\|_X$. Now, if we take (t_n) such that $b_n = \frac{1}{t_n} a_n$ for $n = 1, \ldots, N$ and 0 for $n > N$, then

$$\begin{aligned}\|\sum_{n=1}^N a_n e_n\| \leq \|\sum_{n=1}^\infty a_n e_n\| &= \|T_{(t_n)}(\sum_{n=1}^\infty \frac{1}{t_n} a_n e_n)\| \\ &\leq \|T_{(t_n)}\|\|\sum_{n=1}^\infty \frac{1}{t_n} a_n e_n\| \leq K\|\sum_{n=1}^N b_n e_n\|.\end{aligned}$$

Conversely, let us take $x = \sum_{n=1}^\infty a_n e_n$ in X. We are going to prove that for any subsequence $(n_k)_{k=1}^\infty$, the series $\sum_{n=1}^\infty a_{n_k} e_{n_k}$ is convergent. Since the series converges, for every $\epsilon > 0$ there is $N = N(\epsilon) \in \mathbb{N}$ such that if $m_2 > m_1 \geq N$ then

$$\|\sum_{n=m_1+1}^{m_2} a_n e_n\| < \frac{\epsilon}{K}.$$

If $N \leq n_k < \cdots < n_{k+l}$ we have

$$\| \sum_{j=k+1}^{k+l} a_{n_j} e_{n_j} \| \leq K \| \sum_{j=n_k+1}^{n_{k+l}} a_j e_j \| < \epsilon,$$

and so $\sum_{k=1}^{\infty} a_{n_k} e_{n_k}$ is Cauchy, means for every subsequence $(n_k)_{k=1}^{\infty}$ the series $\sum_{k=1}^{\infty} a_{n_k} e_{n_k}$ converges. By Theorem 1.15 we have that the basis $(e_n)_{n=1}^{\infty}$ is unconditional. □

Definition. Let (e_n) be an unconditional basis of a Banach space X. The *unconditional basis constant*, K_u, of (e_n) is the least constant K so that equation (1.1) holds.

We than say that (e_n) is K-*unconditional* whenever $K \geq K_u$.

Remark. *Suppose $(e_n)_{n=1}^{\infty}$ is an unconditional basis for a Banach space X. For each sequence of scalars (α_n) with $|\alpha_n| = 1$, let $T_{(\alpha_n)} : X \to X$ be the defined by $T_{(\alpha_n)}(\sum_{n=1}^{\infty} a_n e_n) = \sum_{n=1}^{\infty} \alpha_n a_n e_n$. Then*

$$K_u = \sup\{\|T_{(\alpha_n),}\| : (\alpha_n) \ scalars, \ |\alpha_n| = 1 \ for \ all \ n\}.$$

If $(e_n)_{n=1}^{\infty}$ is an unconditional basis of X and A is any subset of integers then there is a linear projection P_A from X onto $[e_k : k \in A]$ defined for each $x = \sum_{k=1}^{\infty} e_k^*(x) e_k$ by

$$P_A(x) = \sum_{k \in A} e_k^*(x) e_k.$$

By the Uniform Boundedness principle we have that P_A is bounded.

$\{P_A : A \subset \mathbb{N}\}$ are the natural projections associated to the unconditional basis (e_n) and the number

$$K_s = \sup_A \|P_A\|$$

(which is finite by the Uniform Boundedness principle) is called the *suppression constant* of the basis.

Proposition 1.19. *A basis $(e_n)_{n=1}^{\infty}$ of a Banach space X is unconditional if and only if there is a constant K' such that for all $N \in \mathbb{N}$, all $A \subset \{1, 2, \ldots, N\}$ and all scalars $(a_n)_{n=1}^{N}$, the following inequality holds:*

$$\|\sum_{n \in A} a_n e_n\| \leq K' \|\sum_{n=1}^{N} a_n e_n\|. \tag{1.2}$$

The suppression constant, K_s is the smallest constant K' which satisfies (1.2).

Moreover, we have that $K_s \leq K_u \leq 2K_s$. To prove this relation we use Proposition 1.18 and 1.19.

For $N \in \mathbb{N}$, scalars $(a_n)_{n=1}^{N}$, $A \subset \{1, 2, \ldots, N\}$ and $(\alpha_n)_{n=1}^{N} \subset \{\pm 1\}$ we have

$$\begin{aligned} \|\sum_{n=1}^{N} \alpha_n a_n e_n\| &\leq \|\sum_{n=1, \alpha_n = 1}^{N} a_n e_n\| + \|\sum_{n=1, \alpha_n = -1}^{N} a_n e_n\| \\ &\leq K' \|\sum_{n=1}^{N} a_n e_n\| + K' \|\sum_{n=1}^{N} a_n e_n\| \\ &= 2K' \|\sum_{n=1}^{N} a_n e_n, \| \end{aligned}$$

so $K_u \leq 2K_s$, and from

$$\begin{aligned} \|\sum_{n \in A} a_n e_n\| &\leq \frac{1}{2}(\|\sum_{n \in A} a_n e_n + \sum_{n \in \{1,2,\ldots\} \setminus A} a_n e_n\| \\ &+ \|\sum_{n \in A} a_n e_n - \sum_{n \in \{1,2,\ldots\} \setminus A} a_n e_n\|) \\ &\leq \frac{1}{2}(\|K \sum_{n=1}^{N} a_n e_n\| + K\|\sum_{n=1}^{N} a_n e_n\|) \\ &= K\|\sum_{n=1}^{N} a_n e_n\| \end{aligned}$$

we have that $K_s \leq K_u$.

These inequalities also prove that Proposition 1.18 and 1.19 are equivalent.

2 Complemented subspaces and linear operators

2.1 Adjoint operators

We will introduce the notion of adjoint operators and some of its properties. We will use adjoints to prove some theorems about bases and later on, on defining the notion of complemented subspaces.

Theorem 2.1. *Suppose that X and Y are two normed spaces. To each $T \in B(X, Y)$ corresponds an unique $T^* \in B(Y^*, X^*)$ such that*

$$y^*(Tx) = T^*y^*(x),$$

for all $x \in X$ and all $y^* \in Y^*$. Moreover, $\|T^*\| = \|T\|$.

Proof. If $y^* \in Y^*$ and $T \in B(X, Y)$, define $T^*y^* = y^* \circ T$. Since T^*y^* is the composition of two continuous linear mappings we have that $T^*y^* \in X^*$. Also for every $x \in X$, $T^*y^*(x) = y^*(Tx)$. Since the equality holds for every $x \in X$, it means that T^*y^* uniquely determined.

To prove that $T^* : Y^* \to X^*$ is linear, take $y_1^*, y_2^* \in Y^*$ and for every $x \in X$ we have

$$\begin{aligned}(T^*(y_1^* + y_2^*))(x) &= (y_1^* + y_2^*)(Tx) = y_1^*(Tx) + y_2^*(Tx) \\ &= T^*y_1^*(x) + T^*y_2^*(x) = (T^*y_1^* + T^*y_2^*)(x).\end{aligned}$$

So we have $T^*(y_1^* + y_2^*) = T^*y_1^* + T^*y_2^*$.

For every scalar $\alpha \in \mathbb{F}$, $y^* \in Y^*$ and $x \in X$, we have

$$T^*(\alpha y^*)(x) = (\alpha y^*)(Tx) = \alpha T^* y^*(x),$$

so $T^*(\alpha y^*) = \alpha T^* y^*$.

Finally, if B_X and B_{Y^*} are the closed unit balls of X and Y^* respectively, then

$$\begin{aligned}\|T\| &= \sup_{x \in B_X} \|Tx\| = \sup_{x \in B_X} \sup_{y^* \in B_{Y^*}} |y^*(Tx)| \\ &= \sup_{y^* \in B_{Y^*}} \sup_{x \in B_X} |T^* y^*(x)| = \sup_{y^* \in B_{Y^*}} \|T^* y^*\| = \|T^*\|.\end{aligned}$$

□

Definition. If X and Y are two normed spaces and $T \in B(X, Y)$, then the *adjoint* of T is the bounded linear operator $T^* : Y^* \to X^*$ such that $T^*(y^*) = y^* T$.

The concept of adjoint is in a way a generalization of the notion of the transpose of a matrix of scalars. So, some of the properties of transposes of matrices generalize to adjoints of operators.

Proposition 2.2. *If S and T are bounded linear operators from X into Y, where X and Y are normed spaces, and $\alpha \in \mathbb{F}$, then $(S+T)^* = S^* + T^*$, and $(\alpha S)^* = \alpha S^*$.*

Proof. For every $\alpha \in \mathbb{F}$, $x \in X$ and $y^* \in Y^*$, we can write

$$\begin{aligned}(S+T)^* y^*(x) &= y^*(S+T)(x) = y^*(S(x) + T(x)) = y^*(Sx) + y^*(Tx) \\ &= S^* y^*(x) + T^* y^*(x) = (S^* + T^*) y^*(x),\end{aligned}$$

and

$$(\alpha S)^* y^*(x) = y^*(\alpha S)(x) = y^*(\alpha Sx) = \alpha y^*(Sx) = \alpha S^* y^*(x).$$

□

From Proposition 2.2 and Theorem 2.1 it follows the corollary:

Corollary 2.3. *If X and Y are normed spaces, than the map $T \mapsto T^*$ is an isometric isomorphism from $B(X,Y)$ into $B(Y^*,X^*)$.*

Proposition 2.4. *If X, Y, and Z are normed spaces, and $S \in B(X,Y)$, $T \in B(Y,Z)$, then $(TS)^* = S^*T^*$.*

Proof. For every $x \in X$ and $z^* \in Z^*$ we have

$$(TS)^*z^*(x) = z^*(TSx) = z^*(T(Sx)) = T^*z^*(Sx) = S^*T^*z^*(x),$$

from which it follows that $(TS)^* = S^*T^*$. □

Theorem 2.5. *If X and Y are normed spaces such that there is an isomorphism T from X onto Y, then the adjoint of T, T^* is an isomorphism from Y^* onto X^*. If T is an isometric isomorphism, then so is T^*.*

Proof. We know that $T^* \in B(Y^*,X^*)$ and $\|T^*\| = \|T\|$ from Theorem 2.1. If $y^* \in Y^*$ and $T^*y^* = 0$, then for every $y \in Y$

$$y^*y = y^*(T(T_{-1}y)) = T^*y^*(T^{-1}y) = 0,$$

so $y^* = 0$. It follows that T^* is one-to-one.

If $x^* \in X^*$, then for every $x \in X$

$$x^*x = x^*T_{-1}(Tx) = T^*x^*T^{-1}(x),$$

so $T^*(x^*T^{-1}) = x^*$, thus the operator T^* maps Y^* onto X^*.

Since the dual of a normed space is a Banach space, we have that T^* is a one-to-one bounded linear operator from one Banach space onto another. Is a conclusion of the Open Mapping Theorem that every one-to-one bounded linear operator from a Banach space onto a Banach space is an isomorphism, which means that T^* is an isomorphism.

Finally, suppose that T is an isometric isomorphism. Then for every $y^* \in Y^*$ we have

$$\|T^*y^*\| = \sup_{x \in B_X} |T^*y^*x| = \sup_{x \in B_X} |y^*(Tx)| = \sup_{y \in B_Y} |y^*y| = \|y^*\|.$$

So T^* is also an isometric isomorphism. □

Proposition 2.6. *Suppose that T is an isomorphism from a normed space X onto a normed space Y. Then $(T^{-1})^* = (T^*)^{-1}$.*

Proof. From Theorem 2.5 we have that T^* is an isomorphism from Y^* onto X^*, so $(T^*)^{-1}$ does exist. For every $y \in Y$ and $x^* \in X^*$,

$$\begin{aligned}(T^{-1})^* x^* y = x^*(T^{-1}y) &= T^*(T^*)^{-1}x^*(T^{-1}y) \\ &= (T^*)^{-1}x^*(TT^{-1}y) = (T^*)^{-1}x^*y,\end{aligned}$$

so we have that $(T^{-1})^* = (T^*)^{-1}$. □

Let us present some examples of adjoint operators.

Example. Let I be the identity operator on a normed space X. For every $x \in X$ and $x^* \in X^*$,

$$I^*x^*(x) = x^*(Ix) = x^*(x).$$

So, for each $x^* \in X^*$, $I^*x^* = x^*$, that is, I^* is the identity operator on X^*.

Example. In this example, members of l_1 will also be treated as members of c_0 and l_∞, so a subscript of 0, 1 or ∞ will show whether a sequence is treated as a member of c_0, l_1 or l_∞ respectively. This is just to avoid confusion.

Let T be the map from l_1 into c_0 given by the formula $T((\alpha_n)_1) = (\alpha_n)_0$. T is linear because for every scalar $t \in \mathbb{F}$ and $(\alpha_n)_1, (\beta_n)_1 \in l_1$

$$T((\alpha_n)_1 + (\beta_n)_1) = ((\alpha_n)_1 + (\beta_n)_1)_0 = (\alpha_n)_0 + (\beta_n)_0 = T((\alpha_n)_1) + T((\beta_n)_1)$$

and

$$T(t(\alpha_n)_1) = T((t\alpha_n)_1) = (t\alpha_n)_0 = t(\alpha_n)_0 = tT((\alpha_n)_1).$$

T is bounded because for every $(\alpha_n)_1 \in l_1$, $T((\alpha_n)_1) = (\alpha_n)_0 \in c_0$.

$\|T\| = 1$ because

$$\|T\| = \sup_{\|(\alpha_n)\|_1 \leq 1} |T(\alpha_n)_1| = \sup_{\|(\alpha_n)\|_0 \leq 1} |(\alpha_n)_0| = 1.$$

We know that the dual of l_1 and c_0, l_1^* and c_0^*, are l_∞ and l_1 respectively. Then, for each pair of elements $(\alpha_n)_1$ and $(\beta_n)_1$ of l_1 ($(\alpha_n)_1 \in l_1$, $(\beta_n)_1 \in c_0^* = l_1$), we have

$$T^*(\beta_n)_1(\alpha_n)_1 = (\beta_n)_1(T^*(\alpha_n)_1) = (\beta_n)_1(\alpha_n)_0 = \sum_n \beta_n \alpha_n,$$

so the element $T^*(\beta_n)_1$ of l_1^* can be identified with the element $(\beta_n)_\infty$ of l_∞.

In short, the adjoint of the "identity" map from l_1 into c_0 is the "identity" map from l_1 into l_∞.

$$\begin{aligned}\|T^*\| = \sup_{\|(\alpha_n)\|_1 \le 1} |T^*(\alpha_n)_1| &= \sup_{\|(\alpha_n)\|_1 \le 1} \sup_{\|(\beta_n)\|_1 \le 1} |T^*(\alpha_n)_1(\beta_n)_1| \\ &= \sup_{\|(\alpha_n)\|_1 \le 1} \sup_{\|(\beta_n)_0 \le 1} |(\alpha_n)_1(\beta_n)_0| = 1,\end{aligned}$$

as in Theorem 2.1.

From the two first example, one might get the idea that the adjoints of one-to-one bounded linear operators between normed spaces must itself be one-to-one. The next example shows that this is not the case.

First, lets state what a reflexive space is.

Definition. A normed space X is *reflexive* if the linear isometric embedding $Q_X : X \to X^{**}$ defined by

$$(Q_X(x))(x^*) = x^*(x) \qquad \forall x^* \in X^*,\ \forall x \in X,$$

is surjective (onto).

Also we should have in mind the following result:

A Banach space is reflexive if and only if its dual space is reflexive. You can find the proof at [8, p. 104].

Example. Let X be any nonreflexive Banach space. Let Q_X be the natural map from X into X^{**}, an isometric isomorphism from X onto a closed subspace of X^{**}. Then Q_X^* maps X^{***} into X^*.

Let Q_{X^*} be the natural map from X^* into X^{***}. Then for every $x \in X$ and $x^* \in X^*$,

$$Q_X^* Q_{X^*} x^*(x) = Q_{X^*} x^*(Q_X x) = x^*(Q_X x) = x^*(x),$$

which implies that $Q_X^* Q_{X^*}$ is the identity map on X^* and therefore that Q_X^* maps X^{***} onto X^*.

If Q_X^* were also one-to-one, then Q_{X^*} would have to map X^* onto X^{***}, contradicting the fact X^* is not reflexive. The isomorphic isomorphism Q_X therefore does not have a one-to-one adjoint.

The following Theorem is about the connection between two concepts, adjoint and weak*-to-weak* continuity. We will need the theorem:

Theorem 2.7. *A linear operator from a normed space X into a normed space Y is norm-to-norm continuous if and only if is weak-to-weak continuous.*

Theorem 2.8. *Suppose that X and Y are normed spaces. If $T \in B(X, Y)$, then T^* is weak*-to-weak* continuous. Conversely, If S is a weak*-to-weak* continuous linear operator from Y^* into X^*, then there is a $T \in B(X, Y)$ such that $T^* = S$.*

Proof. Suppose that $T \in B(X, Y)$. Let (y^*_α) be a net in Y^* that is weakly* convergent to some y^*. For every $x \in X$,

$$T^* y^*_\alpha x = y^*_\alpha(Tx) \to y^*(Tx) = T^* y^* x,$$

so $T^* y^*_\alpha \xrightarrow{w*} T^* y^*$. We just proved that T^* is weak*-to-weak* continuous.

Conversely, suppose that S is a weak*-to-weak* continuous linear operator. Let Q_X and Q_Y be the natural maps form X and Y respectively into their second duals. For every $x \in X$, $Q_X(x) : X^* \to \mathbb{F}$ is weakly* continuous on X^*. So the product operator $Q_X(x)S$ is a weakly* continuous linear functional on Y^*, thus is a member of $Q_Y(Y)$, which in turn implies that $Q_Y^{-1}(Q_X(x)S) \in Y$.

We define the operator $T : X \to Y$ by the formula $Tx = Q_Y^{-1}(Q_X(x)S)$. That T is linear follows from the fact that S, $Q_X(x)$, and Q_Y^{-1} are linear. To see that T is bounded, take a net (x_α) in X that converges weakly to some x_0. Then

$$Q_X(x_\alpha) \xrightarrow{w*} Q_X(x_0),$$

so

$$(Q_X(x_\alpha)S)(y^*) \to (Q_X(x_0)S)(y^*), \qquad for\ y^* \in Y^*,$$

which implies that

$$Q_X(x_\alpha)S \xrightarrow{w*} Q_X(x_0)S,$$

and therefore

$$Tx_\alpha = Q_Y^{-1}(Q_X(x_\alpha)S) \xrightarrow{w} Q_Y^{-1}(Q_X(x_0)S) = Tx_0.$$

The operator T is therefore weak-to-weak continuous and from Theorem 2.7 is norm-to-norm continuous, so $T \in B(X, Y)$.

Finally, for every $x \in X$ and $y^* \in Y^*$ we have

$$\begin{aligned} T^*y^*(x) &= y^*(Tx) = y^*(Q_Y^{-1}(Q_X(x)S)) \\ &= (Q_X(x)S)(y^*) = Q_X(x)(Sy^*) \\ &= Sy^*(x), \end{aligned}$$

so $T^* = S$. □

We use the concept of adjoint to prove the following proposition about bases:

Proposition 2.9. *If $(e_n)_{n=1}^\infty$ is a basis for a Banach space X, then the biorthogonal functionals associated to $(e_n)_{n=1}^\infty$, $(e_n^*)_{n=1}^\infty$, form a basic sequence in X^*.*

Proof. Let S_n be the natural projection associated to $(e_n)_{n=1}^\infty$. We know that for every $x \in X$ and $n \in \mathbb{N}$, $S_n(\sum_{i=1}^\infty e_i^*(x)e_i) = \sum_{i=1}^n e_i^*(x)e_i$. For every $n \in \mathbb{N}$ take the adjoint of S_n, S_n^*. Then, for every $x \in X$, scalars $(a_k)_{k=1}^{\infty}$ and every integers m, n such that $n \leq m$ we have that

$$S_n^*(\sum_{k=1}^{m} a_k e_k^*)x = (\sum_{i=1}^{m} a_k e_k^*)S_n x = \sum_{i=1}^{m} a_k e_k^*(\sum_{i=1}^{n} e_i^*(x)e_i)$$
$$= \sum_{i=1}^{m}\sum_{i=1}^{n} a_k e_i^*(x)e_k^*(e_i) = \sum_{i=1}^{n} a_k e_k^*(x).$$

So

$$S_n^*(\sum_{k=1}^{m} a_k e_k^*) = \sum_{i=1}^{n} a_k e_k^*.$$

For every integers m, n such that $n \leq m$ we can write

$$\|\sum_{k=1}^{n} a_k e_k^*\| = \|S_n^*(\sum_{k=1}^{m} a_k e_k^*)\| \leq \sup_n \|S_n^*\| \|\sum_{k=1}^{m} a_k e_k^*\|.$$

From Proposition 1.9 we have that $(e_k^*)_{k=1}^{\infty}$ is a basic sequence in X^*. □

Note that the basis constant is identical to that of $(e_n)_{n=1}^{\infty}$ since for every $n \in \mathbb{N}$, $\|S_n\| = \|S_n^*\|$.

A similar result about unconditional basis:

Proposition 2.10. *If X is a Banach space with an unconditional basis $(e_n)_{n=1}^{\infty}$, then the biorthogonal functionals $(e_n^*)_{n=1}^{\infty}$ form an unconditional basic sequence in X^*, with the same unconditional constant and the same suppression constant.*

Proof. From Proposition 2.9 it follows that if $(e_n)_{n=1}^{\infty}$ is an unconditional basis in X, then the biorthogonal functionals $(e_n^*)_{n=1}^{\infty}$ form an unconditional basic sequence in X^*.

Now, let K_u and K_s be the unconditional basis constant and suppression constant of X, and let K_u^* and K_s^* be the unconditional basis constant and suppression constant of X^*.

For every $x^* \in X^*$, $x^* = \sum_{n=1}^{\infty} b_n e_n^*$, and scalars (α_n), $|\alpha_n| = 1$ we can write

$$\begin{aligned}
\|\sum_{n=1}^{\infty} \alpha_n b_n e_n^*\| &= \sup_{\|x\|\leq 1} |\sum_{n=1}^{\infty} \alpha_n b_n e_n^*(x)| \\
&= \sup_{\|x\|\leq 1} |\sum_{n=1}^{\infty} \alpha_n b_n e_n^*(\sum_{m=1}^{\infty} a_m e_m)| \\
&= \sup_{\|x\|\leq 1} |\sum_{n=1}^{\infty} \alpha_n a_n b_n \sum_{m=1}^{\infty} e_n^*(e_m)| \\
&= \sup_{\|x\|\leq 1} |\sum_{n=1}^{\infty} b_n e_n^*(\sum_{n=1}^{\infty} \alpha_n a_n e_n)| \\
&= \sup_{\|x\|\leq 1} \|\sum_{n=1}^{\infty} b_n e_n^*\| \|\sum_{n=1}^{\infty} \alpha_n a_n e_n\| \\
&\leq K_u \|\sum_{n=1}^{\infty} b_n e_n^*\|.
\end{aligned}$$

So we have that $K_u^* \leq K_u$.

Same way, for every $x \in X$, $x = \sum_{n=1}^{\infty} a_n e_n$,

$$\begin{aligned}
\|\sum_{n=1}^{\infty} \alpha_n a_n e_n\| &= \sup_{\|x^*\|\leq 1} |x^*(\sum_{n=1}^{\infty} \alpha_n a_n e_n)| \\
&= \sup_{\|x^*\|\leq 1} |\sum_{m=1}^{\infty} b_n e_n^*(\sum_{n=1}^{\infty} \alpha_n a_n e_n)| \\
&= \sup_{\|x^*\|\leq 1} |\sum_{n=1}^{\infty} \alpha_n b_n e_n^*(\sum_{n=1}^{\infty} a_n e_n)| \\
&= \sup_{\|x^*\|\leq 1} \|\sum_{n=1}^{\infty} \alpha_n b_n e_n^*\| \|\sum_{n=1}^{\infty} a_n e_n\| \\
&\leq K_u^* \|\sum_{n=1}^{\infty} a_n e_n\|.
\end{aligned}$$

This means that $K_u \leq K_u^*$. So we have that $K_u = K_u^*$.

In a similar way it is shown that $K_s = K_s^*$. □

2.2 Complemented subspaces

To give the definition of complemented subspaces, let us first introduce the notion of algebraic internal direct sum and internal direct sum.

Definition. A vector space X is said to be the *algebraic internal direct sum* of its subspaces $M_1, \dots, M_n$ if $\sum_{k=1}^{n} M_k = X$ and $M_j \cap \sum_{k \neq j} M_k = \{0\}$ when $j = 1, \dots, n$.

If the subspaces $M_1, \dots, M_n$ are closed, then X is said to be the *internal direct sum* of $M_1, \dots, M_n$.

Proposition 2.11. *If $M_1, \dots, M_n$ are subspaces of a vector space X, then the following are equivalent.*

(i) *The space X is the algebraic internal direct sum of $M_1, \dots, M_n$.*

(ii) *For every $x \in X$, there exist unique elements $m_1(x), \dots, m_n(x)$ of $M_1, \dots, M_n$ respectively such that $x = \sum_{k=1}^{n} m_k(x)$.*

Definition. Let X be a Banach space and let M and N be two closed subspaces of X. We say that X is *the complemented sum of M and N*, and we write $X = M \oplus N$, if for every $x \in X$ there are $m \in M$ and $n \in M$, so that $x = m + n$ and so that this representation of x as sum of an element of M and an element of N is unique.

We say that a closed subspace M of X is *complemented* in X if there is a closed subspace N of X so that $X = M \oplus N$.

Remark. If the Banach space X is the complemented sum of two closed subspaces M and N, than this implies that $M \cap N = \{0\}$.

Indeed, if $M \cap N \neq \{0\}$, then there exist x, $0 \neq x \in M \cap N$ such that $x = x + 0$ and $x = 0 + x$ which is in contradiction with the uniqueness of the representation of x.

In other words, from Proposition 2.11 we can say that a closed subspace M of X is complemented in X if there is a closed subspace N of X such that X is the internal direct sum of M and N.

Another way to present the notion of complemented subspaces is through the notion of projections. For this the following theorems and propositions are needed. As mentioned in the first section:

Definition. Let X be a vector space. A linear operator $P : X \to X$ is a *projection* in X if $P(Px) = Px$ for every $x \in X$, that is, $P^2 = P$.

Note that if $P : X \to X$ is a projection then $\|P\| \geq 1$. Indeed, for every $x \in X$,

$$\|Px\| = \|PPx\| \leq \|P\|\|Px\| \Rightarrow 1 \leq \|P\|.$$

Proposition 2.12. *Let X be a vector space and P a linear operator from X into X. Then P is a projection if and only if $I - P$ is a projection.*

Proof. If P is a projection, then for every $x \in X$,

$$\begin{aligned}(I-P)^2(x) &= I^2x - 2IPx + P^2x \\ &= x - 2Px + Px \\ &= x - Px = (I-P)(x),\end{aligned}$$

so $I - P$ is a projection.

Conversely, if $I - P$ is a projection, then since we have that

$$P = I - (I - P),$$

P is a projection. □

Proposition 2.13. *If P is a projection in a vector space X, then $ker(P) = (I-P)(X)$ and $P(X) = ker(I-P)$.*

Proof. If $x \in ker(P)$, then $(I-P)(x) = x$, so $ker(P) \subseteq (I-P)(X)$. Now,

$$P((I-P)(x)) = P(Ix) - P(Px) = P(x) - P(x) = 0$$

for every $x \in X$, so $P((I-P)(X)) = \{0\}$, which means that for every $x \in (I-P)(X)$, $x \in ker(P)$, so $(I-P)(X) \subseteq ker(P)$. From this and the first inclusion we have that $ker(P) = (I-P)(X)$.

From Proposition 2.12, since P is a projection, so is $I - P$. By replacing P with $I - P$ in the equality just proved, we will get

$$ker(I - P) = (I - (I - P))(X) = P(X).$$

□

Corollary 2.14. *If P is a projection in a vector space X, then $P(X) = \{x \in X : Px = x\}$.*

Theorem 2.15. *Let X be a vector space. If P is a projection in X, then X is the algebraic internal direct sum of the range and kernel of P. Conversely, if X is the algebraic internal direct sum of its subspaces M and N, then there is an unique projection in X having range M and kernel N.*

Proof. If P is a projection in X, then from Proposition 2.13 it follows that

$$X = P(X) + (I - P)(X) = P(X) + ker(P)$$

and

$$P(X) \cap ker(P) = ker(I - P) \cap ker(P) = \{0\},$$

so X is the algebraic internal direct sum of $P(X)$ and $ker(P)$.

Conversely, suppose that X is the algebraic internal direct sum of it subspaces M and N. By Proposition 2.11, every $x \in X$ can be represented in an unique way as a sum $m(x)+n(x)$ such that $m(x) \in M$ and $n(x) \in N$. The map $x \mapsto m$, $P : X \to X$, is a projection in X with range M and kernel N, because for every $x \in X$, $P(Px) = P(m(x)) = m(x) \in M$ and if $x \in N$, $P(x) = 0$.

If P_0 is any projection in X with range M and kernel N, then $P_0(x) = P_0(m(x) + n(x)) = P_0(m(x)) = m(x) = P(x)$ for every $x \in X$, which proves the uniqueness. □

Theorem 2.16. *If M and N are complementary subspaces of a Banach space X, then the projection in X with range M and kernel N is bounded.*

Proof. Let P be the projection with range M and kernel N. Let (x_n) be a sequence in X that converges to some x and that (Px_n) converges to some y. Then $(I - P)(x_n) \to x - y$. It follows that $y \in M$ and $x - y \in N$, so $P(x - y) = 0 \Rightarrow Px = Py = y$. So P is a closed linear operator and by the Closed Graph Theorem, since M is closed in X, the operator P is bounded. □

Corollary 2.17. *A subspace M of a Banach space X is complemented if and only if it is the range of a bounded projection in the space.*

Proof. If M is complemented in X, then from Theorem 2.15 there exist an unique projection with range M. Theorem 2.16 assures that the projection is bounded.

Conversely, if M is the range of a bounded projection P, then from Theorem 2.15 we have that X is the algebraic internal direct sum of $M = P(X)$ and $\ker P$. Since P is bounded we have that $P(X)$ and $\ker P$ are closed. □

If the projection has norm λ, then the subspace is called λ-*complemented*.

Corollary 2.18. *If M and N are complementary subspaces of a Banach space X, then M is isomorphic with X/N, $M \cong X/N$.*

Proof. From Theorem 2.16 we have that $P \in B(X, X)$ and since M is closed in X, then from the First Isomorphism Theorem for Banach spaces we have that $X/ker(P) = X/N$ is isomorphic to $P(X) = M$. □

The next theorem will show that c_0 is not complemented in l_∞. The proof is taken form [8].First we will present a notion that will help to prove the theorem.

We will say that a Banach space have property P if X^* has a countable subset A that is a separating family for X. This means that for every $x \in X$ we can find $y^* \in A$, where $A \subset X^*$ is a countable subset, such that $y^*(x) \neq 0$.

If a Banach space X has property P, then every closed subspace of X has property P.

Indeed, if $M \subset X$ is a closed subspace, then $M^* \cap A$ is a countable subset of M^* such that for every $x \in M$, there is a $y^* \in M^* \cap A$ such that $y^*(x) \neq 0$.

If a Banach space X has property P, then every Banach space isomorphic to X has property P.

Indeed, if Y is a Banach space isomorphic to X, then there exist an isomorphism $T : X \to Y$. For every $y \in Y$ there exist $x \in X$ such that $Tx = y$. For that x, there exists $x^* \in A$ such that $x^*(x) \neq 0$.

From Theorem 2.5, the adjoint of T, $T^* : Y^* \to X^*$ is an isomorphism, so for that x^* there exist $y^* \in Y^*$ such that $T^*y^* = x^*$. This means that $T^*y^*(x) \neq 0$, which implies that $y^*(Tx) = y^*(y) \neq 0$. So, the countable subset of Y^* which is a separating family for Y is $(T^*)^{-1}(A)$.

l_∞ has property P because one countable family for l_∞ in l_∞^* is the collection $\{e_n^* : n \in \mathbb{N}\}$, such that for every $x \in l_\infty$, $e_n^*(x) = x_n$. e_n^* is the n-th coordinate functional on l_∞.

Theorem 2.19. *c_0 is not complemented in l_∞.*

Proof. Suppose the contrary, that c_0 is complemented in l_∞. Let $N \subset l_\infty$ be the closed subspace that is complementary to the closed subspace c_0. N has property P because l_∞ have it. From Corollary 2.18, $l_\infty/c_0 \cong N$, so l_∞/c_0 has property P. The theorem will be proved when we will show that l_∞/c_0 cannot have property P, which it will be a contradiction with what we first stated.

First, let show that there is an uncountable family $\{S_\alpha : \alpha \in I\}$ where for every $\alpha \in I$, S_α are infinite subsets subsets of $\mathbb{N}$ such that for every $\alpha, \beta \in I$, $\alpha \neq \beta$, we have that $S_\alpha \cap S_\beta$ is finite.

Indeed, if we write the rational numbers $\mathbb{Q}$ as a sequence $(q_i : i \in \mathbb{N})$, and for each $r \in \mathbb{R}$ (uncountable set) we chose a sequence $(n_k(r) : k \in \mathbb{N})$ such that $(q_{n_k(r)} : k \in \mathbb{N})$ converges to r, then for every $r \in \mathbb{R}$ let $S_r = \{n_k(r) : k \in \mathbb{N}\}$.

Next, for every $\alpha \in I$, let $x_\alpha \in l_\infty$ be such that

$$x_\alpha = (\delta_k^{(\alpha)} : k \in \mathbb{N}) \;\; \textit{where} \;\; \delta_k^{(\alpha)} = \begin{cases} 1 & \text{if } k \in S_\alpha \\ 0 & \text{if } k \notin S_\alpha. \end{cases}$$

Notice that $x_\alpha + c_0 \neq x_\beta + c_0$ when $\alpha \neq \beta$.

Suppose that $y^* \in (l_\infty/c_0)^*$, that $p \in \mathbb{N}$, and $\alpha_1, \ldots, \alpha_q$ are distinct members of I such that $|y^*(x_{\alpha_j} + c_0)| \geq \frac{1}{p}$ when $j = 1, \ldots, q$. Let $\gamma_1, \ldots, \gamma_q$ be scalars of absolute value 1 such that $\gamma_j y^*(x_{\alpha_j} + c_0) = |y^*(x_{\alpha_j} + c_0)|$ for every j.

Because S_α are infinite sets and $S_\alpha \cap S_\beta$ is finite whenever $\alpha \neq \beta$, it assures that infinitely terms of the form $\sum_{j=1}^q \gamma_j x_{\alpha_j}$ of l_∞ have absolute value 1 and that only finitely many have absolute value more than 1. From this it follows that

$$\|(\sum_{j=1}^q \gamma_j x_{\alpha_j}) + c_0\| = d(\sum_{j=1}^q \gamma_j x_{\alpha_j}, c_0) = 1.$$

Therefore

$$\|y^*\| \geq |y^*((\sum_{j=1}^q \gamma_j x_{\alpha_j}) + c_0)| = \sum_{j=1}^q |y^*(x_{\alpha_j}) + c_0)| \geq \frac{q}{p}$$

and so $q \leq p\|y^*\|$. This means that there are only finitely many index elements α such that $|y^*(x_\alpha + c_0)| \geq \frac{1}{p}$. Since $p \in \mathbb{N}$ was arbitrary, it follows that there are only countably many index elements α such that $y^*(x_\alpha + c_0) \neq 0$.

Now suppose that C is a countable subset of $(l_\infty/c_0)^*$. It follows that for $z^* \in C$, there are only countably many elements α of I such that $z^*(x_\alpha + c_0) \neq 0$. Since I is uncountable, there must exist $\alpha_1, \alpha_2 \in I$, $\alpha_1 \neq \alpha_2$ such that $z^*(x_{\alpha_1} + c_0) = z^*(x_{\alpha_2} + c_0) = 0$ for every $z^* \in C$, which shows that C is not a separating family for l_∞/c_0. This means that the space l_∞/c_0 does not have property P. □

Corollary 2.20. *There is no bounded linear operator from l_∞ to c_0 which maps each element of c_0 to itself.*

The next theorem shows that if a linear bounded operator in a normed space is a projection then so is the adjoint of that operator, and conversely. Our purpose is to define λ-complemented subspaces through adjoint operators.

Theorem 2.21. *If X is a normed space and T is a bounded linear operator, then T is a projection if and only if T^* is a projection.*

Proof. First, suppose that T^* is a projection. For every $x \in X$ and $x^* \in X^*$,

$$x^*(T(Tx)) = T^*x^*(Tx) = T^*(T^*x^*)(x) = T^*x^*x = x^*(Tx)$$

which implies that $T(Tx) = Tx$ since the collection of all bounded linear functionals on a normed space is always a separating family for that normed space.

Conversely, if T is a projection, then for every $x \in X$ and $x^* \in X^*$,

$$T^*(T^*x^*)(x) = T^*x^*(Tx) = x^*(T(Tx)) = x^*(Tx) = T^*x^*x$$

which implies that $T^*(T^*x^*) = T^*x^*$, and so T^* is a projection. □

From Theorem 2.21, if $P : X \to X$ is such a projection with $\|P\| = \lambda$, then the adjoint of P, P^* is also a projection, and we know that $\|P^*\| = \|P\| = \lambda$.

Note $P(X) = M \subset X$. Let us see what $\ker P^*$ is. Take

$$\begin{aligned} x^* \in \ker P^* &\Leftrightarrow P^*x^* = 0 \in X^* \Leftrightarrow P^*x^*(x) = 0, \ \forall x \in X \\ &\Leftrightarrow x^*(Px) = 0, \ \forall x \in X. \end{aligned}$$

This means that $\ker P^* = \{x^* \in X^* : x^* = 0 \text{ on } TX\}$. Thus

$$\ker P^* = M^\perp = \{x^* \in X^* : x^*\big|_M = 0\}.$$

To add all, if M is λ complemented in X, then there exist $Q : X^* \to X^*$ such that

(i) Q is a projection,

(ii) $\|Q\| = \lambda$,

(iii) $\ker Q = M^{\perp}$,

(iv) Q is the adjoint of a projection.

Often such an operator exists, but Q is not weak*-to-weak* continuous, which means that (iv) is not fulfilled. c_0 as a subspace of l_∞ is an example of a non-complemented subspace such that its annihilator is the kernel of a norm-one projection in the dual space.

2.3 The Principle of Local Reflexivity

Our purpose in this section is to present and to prove the Principle of Local Reflexivity (PLR). This principle asserts that in a local sense every Banach space is reflexive.

In this section X and Y will denote Banach spaces, and by an operator we will mean a continuous linear operator.

The proof of the PLR is taken from [9]. For that we need the following lemmas.

Lemma 2.22. *Let $T : X \to Y$ be an operator with closed range. If $x^{**} \in X^{**}$ and $y \in Y$ are such that $T^{**}x^{**} = Q_Y y$ and $\|x^{**}\| < 1$ then, there exist an $x \in X$ with $\|x\| < 1$ such that $Tx = y$.*

Proof. Denote by U_X the unit ball of X. To prove the lemma we have to show that $y \in T(U_X)$.

Suppose first that $y \notin T(X)$. Then, there exist $y^* \in Y^*$ such that $T^*y^* = 0$ but $y^*(y) = 1$. In this case we have that

$$T^{**}x^{**}(y^*) = x^{**}(T^*y^*) = 0,$$

and

$$T^{**}x^{**}(y^*) = Q_y(y^*) = y^*(y) = 1$$

which is a contradiction.

Next we suppose that $y \in T(X) \backslash T(U_X)$. By the Open Mapping Theorem, since U_X is open in X, we have that $T(U_X)$ is open in $T(X)$. Since the conditions of the Hahn-Banach Separation Theorem are fulfilled for the subsets $T(U_X)$ and $T(X) \backslash T(U_X)$ of $T(X)$, we can find $y^* \in Y^*$ such that $y^*(Tx) < 1$ for all $x \in U_X$, and $y^*(y) \geq 1$.

Since $y^*(Tx) = T^*y^*(x) < 1$, we have that $\|T^*y^*\| \leq 1$ and so $|x^{**}(T^*y^*)| < 1$ (because $\|x^{**}\| < 1$), which means that $|y^*(y)| < 1$, which is in contradiction with $y^*(y) \geq 1$.

So, we must have $y \in T(U_X)$. □

Note that in Lemma 2.22 we can pick $x \in X$ such that

$$\|x\| < (1+\delta)\|x^{**}\| \quad \text{for every } \delta > 0.$$

If $T : X \to Y$ is a bounded operator with closed range, $T(X)$ is closed, [1, p. 272], this is equivalent to the requirement that T factors through an isomorphism embedding on $X/ker(T)$, which in turn is equivalent to the statement that for some constant C, we have

$$d(x, ker(T)) \leq C\|Tx\|, \quad x \in X.$$

Lemma 2.23. *Let $T : X \to Y$ be an operator with closed range, and $K : X \to Y$ be a finite-rank operator. Than $T + K$ has closed range.*

Proof. Suppose the contrary, that $T + K$ does not have closed range. This means that there is a bounded sequence $(x_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty}(T+K)(x_n) = 0$ but $d(x_n, ker(T + K)) \geq 1$, for every $n \in \mathbb{N}$.

We can pass to a subsequence and assume that $(Kx_n)_{n=1}^{\infty}$ converges to some $y \in Y$ and hence $\lim_{n\to\infty} Tx_n = -y$. Since T has closed range, this implies that there exist $x \in X$ such that $Tx = -y$ and so $\lim_{n\to\infty} \|Tx_n - Tx\| = 0$. From this we can write that $\lim_{n\to\infty} d(x_n - x, ker(T)) = 0$. It follows that $\lim_{n\to\infty} K(x_n - x) \in K(ker(T))$, and so $y - Kx \in K(ker(T))$.

Let $y - Kx = Ku$, where $u \in ker(T)$. Then we have that

$$\lim_{n\to\infty} d(x_n - x - u, ker(T)) = 0,$$

and

$$\lim_{n\to\infty} \|Kx_n - Kx - u\| = 0.$$

Since $K\big|_{ker(T)}$ has closed range, it means that

$$\lim_{n\to\infty} d(x_n - x - u, \ker(T) \cap \ker(K)) = 0.$$

But $T(x + u) = Tx + Tu = -y = -Kx - Ku = -K(x + u)$, so

$$(T + K)(x + u) = T(x + u) + K(x + u) = -y + y = 0,$$

which means that $x + u \in \ker(T + K)$ and therefore

$$\lim_{n\to\infty} d(x_n, ker(T + K)) = 0,$$

which contradicts the assumption that $d(x_n, ker(T + K)) \geq 1$. $\square$

Lemma 2.24. *Let $T : E \to X$ be an operator where E is a finite dimensional normed space such that*

$$(1 + \delta)^{-1} \leq \|Tx_i\| \leq (1 + \delta),$$

$1 \leq i \leq N$, where $(x_i)_{i=1}^N$ is an δ-net for the unit sphere of E. Then T is invertible and

$$\|T\|\|T^{-1}\| \leq (\frac{1+\delta}{1-\delta})(\frac{1}{1+\delta} - \frac{\delta(1+\delta)}{1-\delta})^{-1} = \vartheta(\delta).$$

Proof. Lets take $e \in E$, $\|e\| = 1$. We pick i so that $\|e - x_i\| < \delta$. Then

$$\begin{aligned} \|Te\| &= \|Te - Tx_i + Tx_i\| \leq \|Te - Tx_i\| + \|Tx_i\| \\ &\leq \|Te - Tx_i\| + (1 + \delta), \end{aligned}$$

so

$$\|T\| \leq \|T\|\|e - x_i\| + (1 + \delta) = \|T\|\delta + (1 + \delta)$$

which is equivalent to

$$(1 - \delta)\|T\| \leq 1 + \delta \Leftrightarrow \|T\| \leq \frac{1+\delta}{1-\delta}.$$

On the other hand,

$$\begin{aligned} \|T\| \geq \|Te\| &= \|Te - Tx_i + Tx_i\| \geq \|Tx_i\| - \|Te - Tx_i\| \\ &\geq |Tx_i\| - \|e - x_i\|\|T\| \geq (1 + \delta)^{-1} - \delta\|T\| \\ &\geq \frac{1}{1+\delta} - \frac{\delta(1+\delta)}{1-\delta}. \end{aligned}$$

We know that $T^{-1}T = I_E$ where I_E is the identity operator on E, and $\|I_E\| = 1$. So we have that

$$1 = \|I\| \leq \|T\|\|T^{-1}\| \Leftrightarrow \|T^{-1}\| \geq \frac{1}{\|T\|},$$

from which it follows that

$$\|T^{-1}\| \leq (\frac{1}{1+\delta} - \frac{\delta(1+\delta)}{1-\delta})^{-1}.$$

□

Theorem 2.25. *Let E and F be finite dimensional subspaces of X^{**} and X^*, respectively, and let $\epsilon > 0$. Then there exists an operator $T : E \to X$ such that*

(i) $\|T\|\|T^{-1}\| < 1 + \epsilon$,

(ii) $x^*(Tx^{**}) = x^{**}(x^*)$ *for every* $x^{**} \in E$ *and every* $x^* \in F$,

(iii) $Tx^{**} = x$ *if* $x^{**} \in Q_X X \cap E$.

Proof. Choose $\delta > 0$ so that $\vartheta(\delta) < 1 + \epsilon$ where ϑ is as in Lemma 2.24. Choose $a_1^*, a_2^*, \ldots, a_m^* \in X^*$, $\|a_j^*\| = 1$ for $1 \leq j \leq m$, containing a basis of F and such that

$$\|x^{**}\| < (1+\delta) \sup_{1\leq j\leq m} |x^{**}(a_j^*)|$$

for every $x^{**} \in E$.

Choose $b_1^{**}, b_2^{**}, \ldots, b_n^{**}$ a δ-net for the unit sphere of E such that $b_1^{**}, \ldots, b_k^{**}$ is a basis for $Q_X X \cap E$ and $b_1^{**}, \ldots, b_r^{**}$, $r \geq k$, is a basis for E. Then, for $1 \leq p \leq q = n - r$, we have unique scalars $(t_{p,i})_{i=1}^r$, such that

$$b_{r+p}^{**} = \sum_{1\leq i\leq r} t_{p,i} b_i^{**}.$$

Define for $1 \leq p \leq q$

$$s_{p,i} = \begin{cases} t_{p,i} & i \leq r, \\ -1 & i = r+p, \\ 0 & r < i \leq n \text{ and } i \neq r+p. \end{cases}$$

Define $A_0 : X^n \to X^{k+q}$ by

$$A_0(x_1, \ldots, x_n) = (x_1, \ldots, x_k; (\sum_{1\leq i\leq n} s_{p,i} x_i))$$

for $1 \leq p \leq q$ where X^n and X^{k+q} are the usual product spaces with the sup norm. Since the matrix $(s_{p,i})$ has rank q, the operator A_0 is onto.

Define $A : X^n \to Z = X^{k+q} \times C^{nm}$ by

$$A(x_1, \ldots, x_n) = (A_0(x_1, \ldots, x_n); (a_j^*(x_i)))$$

for $1 \leq j \leq m$ and $1 \leq i \leq n$. By Lemma 2.23, A has closed range.

We observe that $A^{**}(b_1^{**}, \ldots, b_n^{**}) = (A_0^{**}(b_1^{**}, \ldots, b_n^{**}); (a_j^*(b_i^{**}))) = (b_1^{**}, \ldots, b_k^{**}, 0, \ldots, 0, (b_i^{**}(a_j^*)))$, which means that $A^{**}(b_1^{**}, \ldots, b_n^{**})$ is in $Q_Z Z$. Therefore, by Lemma 2.22, there exists $(b_1, \ldots, b_n) \in X^n$,

$$\sup_{1 \leq i \leq n} \|b_i\| < (1+\delta) \sup_{1 \leq i \leq n} \|b_i^{**}\| = 1 + \delta,$$

such that $Q_Z A(b_1, \ldots, b_n) = A^{**}(b_1^{**}, \ldots, b_n^{**})$.

$$\begin{aligned} Q_Z A(b_1, \ldots, b_n) &= Q_Z(A_0(b_1, \ldots, b_n); (a_j^*(b_i))) \\ &= Q_Z(b_1, \ldots, b_k; (\sum_{1 \leq i \leq n} s_{p,i} b_i), (a_j^*(b_i))). \end{aligned}$$

$$A^{**}(b_1^{**}, \ldots, b_n^{**}) = (b_1^{**}, \ldots, b_k^{**}, (\sum_{1 \leq i \leq n} s_{p,i} b_i^{**}), (b_i^{**}(a_j^*))).$$

This means that for $1 \leq i \leq k$, $Q_X b_i = Q_Z b_i = b_i^{**}$ and $a_j^*(Q_Z b_i) = b_i^{**}(a_j^*)$ for $1 \leq i \leq n$, $1 \leq j \leq m$.

Define the operator $T : E \to X$ such that $T b_i^{**} = b_i$ for $1 \leq i \leq r$. For $1 \leq p \leq q$, we have that $\sum_{1 \leq i \leq n} s_{p,i} b_i^{**} = \sum_{1 \leq i \leq r} t_{p,i} b_i^{**} - b_{r+p}^{**} = 0$ and $\sum_{1 \leq i \leq n} s_{p,i} b_i = 0$ which gives that $T b_i^{**} = b_i$ also for $r \leq i \leq n$.

Condition (iii) is fulfilled because for every $x^{**} \in Q_X X \cap E$, $x^{**} = \sum_{1 \leq i \leq k} a_i b_i^{**}$ where $(a_i)_{i=1}^k$ are scalars,

$$T x^{**} = T(\sum_{1 \leq i \leq k} a_i b_i^{**}) = \sum_{1 \leq i \leq k} a_i T b_i^{**} = \sum_{1 \leq i \leq k} a_i b_i = x.$$

Also, from $a_j^*(Q_Z b_i) = b_i^{**}(a_j^*)$ for $1 \leq i \leq n$, $1 \leq j \leq m$ and the fact that $a_1^*, a_2^*, \ldots, a_m^*$ is a basis in F we have that for every $x^{**} \in E$ and every $x^* \in F$, $x^*(T x^{**}) = x^{**}(x^*)$. So, condition (ii) is fulfilled too.

At last we observe also that for $1 \leq i \leq n$,

$$\|Tb_i^{**}\| \geq \sup_{1\leq j\leq m} |a_j(Tb_i^{**})| = \sup_{1\leq j\leq m} |b_i^{**}(a_j^*)| \geq \|b_i^{**}\|(1+\delta)^{-1} = (1+\delta)^{-1}.$$

Now, we apply Lemma 2.24 and we have that $\|T\|\|T^{-1}\| < 1+\epsilon$. $\square$

2.4 Locally Complemented Subspaces

Definition. Let M be a closed subspace of a Banach space X. Then M is called *locally λ-complemented* in X, if for every finite-dimensional subspace F of X, $\lambda \in [1, \infty)$, and $\epsilon > 0$, there exist an operator $T : F \to X$ such that

(i) $x \in F \cap M \Rightarrow Tx = x$,

(ii) $\|T\| \leq \lambda + \epsilon$.

If $\lambda = 1$, then M is called called *locally 1-complemented.*

Note that in terms of locally complemented subspaces, the PLR states that a Banach space X is locally 1-complemented in X^{**}.

The following theorem is from [4]. For the proof we will need the Tychonoff's Theorem:

Theorem 2.26. *If $(X_\alpha)_{\alpha \in I}$ is an arbitrary family of compact spaces, then their product $X := \prod_{\alpha \in I} X_\alpha$ is compact.*

Theorem 2.27. *If M is a closed subspace of a Banach space X, then the following statement are equivalent:*

(i) *$M^\perp$ is the kernel of a projection with norm λ on X^*.*

(ii) *$M^{\perp\perp}$ is the image of a projection with norm λ on X^{**}.*

(iii) *M is locally λ-complemented in X.*

Proof. *(i)* $\Rightarrow$ *(ii)* We have that $M^\perp = \ker P$, where $P : X^* \to X^*$ is a norm-one projection. We know from Theorem 2.21 that the adjoint of P, $P^* : X^{**} \to X^{**}$ is a projection and $\|P^*\| = \|P\| = \lambda$. Moreover $ImP^* = (\ker P)^\perp = (M^\perp)^\perp = M^{\perp\perp}$.

(ii) $\Rightarrow$ *(iii)* Let Q be a projection with norm λ on X^{**} such that $Q(X^{**}) = M^{\perp\perp}$. Let Q_F be the restriction of Q to the finite-dimensional subspace F. Using the PLR we get an operator T_0 such that the operator

$T = Q_F \circ T_0$ satisfies the requirements in the definition of locally λ-complemented.

$(iii) \Rightarrow (i)$ For each finite-dimensional subspace F of X, choose an operator $T_F : F \to X$ which satisfies (i) and (ii) in the definition of locally λ-complemented with $\epsilon = \frac{1}{dimF}$. Let

$$S = \prod_{x \in X} B_{x^{**}}(0, 2\|x\|)$$

where $B_{x^{**}}(0, 2\|x\|)$ is the closed ball in X^{**} with center 0 and radius $2\|x\|$. We equip S with the product weak* topology. Then from Tychonoff's Theorem we can say that S is compact Hausdorff.

For every subspace F as above, and every $x \in X$ we define

$$x_F = \begin{cases} T_F(x) & if\ x \in F, \\ 0 & if x\ \notin F. \end{cases}$$

$(x_F)_{x \in X}$ is a net in S ordered by $(x_F) > (x_G)$ if $G \subseteq F$.

Since S is compact, we can find a subnet $(x_G)_{x \in X}$ that converges to a point $(y_x)_{x \in X}$ in S. For every $x \in X$ and every $x^* \in X^*$ we have that $x^*(x_G) \to y_x(x^*)$. The map $x \mapsto y_x$ from X to X^{**} is linear.

For $x \in X$ and $x^* \in X^*$ we define $(Px^*)(x) = y_x(x^*)$. Then the operator $P : x^* \to Px^*$ is a projection with norm λ in X^* and $\ker P = M^{\perp}$.

P is a projection because for every $x \in X$ and $x^* \in X^*$ we have that $(P(Px^*))(x) = y_x(Px^*)$. This means that there exist a subnet (x_G) such that $Px^*(x_G) \to y_x(Px^*)$. Since for every $x \in X$ and every $x^* \in X^*$ we have that $x^*(x_G) \to y_x(x^*)$, this means that $Px^*(x_G) \to y_x(x^*)$. So, $(P(Px^*))(x) = y_x(x^*) = (Px^*)(x)$.

That $\|P\| = \lambda$ it follows from:

$$\begin{aligned} \|P\| &= \sup_{\|x\| \leq 1} \sup_{\|x^*\| \leq 1} |(Px^*)(x)| = \sup_{\|x\| \leq 1} \sup_{\|x^*\| \leq 1} |y_x(x^*)| \\ &= \sup_{\|x\| \leq 1} \|y_x\| = \sup_{\|x\| \leq 1} \|x_G\| \\ &= \sup_{\|x\| \leq 1} |T_G(x)| \leq \lambda + \frac{1}{\dim G} \end{aligned}$$

To show that $\ker P = M^{\perp}$, take $x^* \in \ker P$, so $Px^* = 0$. This implies that for every $x \in X$, $(Px^*)(x) = y_x(x^*) = 0$, which in turn means that for every $x \in X$, $x^*(x_G) \to y_x(x^*) = 0$. This implies that $x^*(T_G(x)) \to 0$. So, for every $x \in G \cap M \subset M$, $x^*(x) = 0$. This means that $x^* \in M^{\perp}$. $\square$

Remark. From the proof Theorem 2.27 we have that M is locally λ-complemented if there exists $Q : X^* \to X^*$ such that

(i) Q is a projection,

(ii) $\|Q\| = \lambda$,

(iii) $\ker Q = M^{\perp}$.

Note that if M is λ-complemented, then M is locally λ-complemented.

We will now present the concept of a Hahn-Banach extension operator. The purpose is to show that the concept of locally 1-complemented subspaces is equivalent to the existence of a bounded linear Hahn-Banach extension operator.

Let M be a closed subspace of a Banach space X. The Hahn-Banach Extension Theorem assures that for every $y^* \in M^*$ there exist $x^* \in X^*$ such that $x^*\big|_M = y^*$ and $\|x^*\| = \|y^*\|$. We denote by

$$HB(y^*) = \{x^* \in X^* : x^*\big|_M = y^*,\ \|y^*\| = \|x^*\|\}$$

the set of *Hahn-Banach extensions* of y^* to X.

An operator $T : M^* \to X^*$, which for every $y^* \in M^*$ satisfies $Ty^* \in HB(y^*)$, is said to a *Hahn-Banach extension operator from M^* to X^**.

Proposition 2.28. *Let $T : M^* \to X^*$ be a Hahn-Banach extension operator and $y^* \in M^*$, and $R_M : X^* \to M^*$ the natural restriction operator $x^* \mapsto x^*\big|_M$. Then $P = TR_M : X^* \to X^*$ is a norm one projection on X^* with range $T(M^*)$ and $\ker P = M^{\perp}$.*

Conversely, if $P : X^* \to X^*$ is a norm one projection with $\ker P = M^{\perp}$, then there exist a Hahn-Banach extension operator.

Proof. P is a projection because for every $x^* \in X^*$,

$$P(Px^*) = P(T(R_M x^*)) = P(Tx^*\big|_M) = Px^*.$$

Since $\|T\| = 1$ and $\|R_M\| = 1$ because $\|y^*\| = \|x^*\|$, for every $x^* \in X^*$ where $y^* = x^*\big|_M$, we have that $1 \leq \|P\| \leq \|T\|\|R_M\| = 1$, so $\|P\| = 1$.

From the construction of P it is clear that it has range $T(M^*)$.

To see that $\ker P = M^\perp$, take $x^* \in \ker P$.

$$\begin{aligned} x^* \in \ker P \Leftrightarrow Px^* = 0 &\Leftrightarrow T(R_M x^*) = 0 \\ &\Leftrightarrow Tx^*(x) = 0, \ \forall x \in M \\ &\Leftrightarrow x^*(x) = 0, \ \forall x \in M. \end{aligned}$$

Conversely, let $P : X^* \to X^*$ be a norm one projection such that $\ker P = M^\perp$. For $y^* \in M^*$, let $x^* \in HB(y^*)$. Then for every $x \in X$,

$$Px^*(x) = x^*(P^* x) = x^*(x) = y^*(x),$$

because $P^* x = x \in M$.

$$\|Px^*\| \leq \|P\|\|x^*\| = \|x^*\| = \|y^*\|.$$

This means that $Px^* \in HB(y^*)$. $\square$

From Proposition 2.28 it follows that the existence of a Hahn-Banach extension operator $T : M^* \to X^*$ is equivalent with M being locally 1-complemented.

Definition. A basic sequence $(x_n)_{n=1}^\infty$ in a Banach space X is called *(locally) complemented* if the closed linear span $[x_n]$ of $(x_n)_{n=1}^\infty$, is a (locally) complemented subspace of X.

Proposition 2.29. *If $(x_n)_{n=1}^\infty$ is a locally complemented basic sequence equivalent with $(y_n)_{n=1}^\infty$, then $(y_n)_{n=1}^\infty$ is locally complemented.*

Proof. Since $(x_n)_{n=1}^{\infty}$ is a locally complemented, then $[x_n]$ is locally complemented and, from the definition we have that for every finite-dimensional subspaces F of X, there exist an operator $T_1 : F \to X$ such that $T_1 x = x$ for every $x \in F \cap [x_n]$.

Now, since $(x_n)_{n=1}^{\infty}$ is equivalent with $(y_n)_{n=1}^{\infty}$, from Proposition 1.12 there exists an isomorphism $T : [x_n] \to [y_n]$ such that $Tx_n = y_n$, for every n. Then for every finite-dimensional subspaces E of Y, $T_2 = TT_1T^{-1} : E \to Y$ is an operator such that $T_2 y = y$ for every $y \in E \cap [y_n]$. Indeed, for every $y \in E \cap [y_n]$, we have

$$T_2 y = (TT_1T^{-1})(y) = TT_1(T^{-1}y) = TT_1 x = Tx = y.$$

So, $(y_n)_{n=1}^{\infty}$ is locally complemented. □

Theorem 2.30. *If a basic sequence $(x_n)_{n=1}^{\infty}$ in a Banach space X is locally complemented, then the sequence of biorthogonal functionals can be extended to a basic sequence in X^*.*

Proof. We have that $(x_n)_{n=1}^{\infty}$ is a basic sequence in X. By definition, this means that $(x_n)_{n=1}^{\infty}$ is a basis for the closed linear span $[x_n] = M$. From Proposition 2.9 we have that $(x_n^*)_{n=1}^{\infty}$ is a basic sequence in M^*.

Since M is locally complemented, then there exist a Hahn-Banach extension operator $T : M^* \to X^*$. For every x_n^*, pick $y_n^* = Tx^* \in HB(x_n^*)$. Then $(y_n^*)_{n=1}^{\infty}$ is a basic sequence in X^*.

Indeed, for every sequence of scalars (a_n) and any integers m, n such that $m \leq n$, we have

$$\begin{aligned}
\|\sum_{n=1}^{m} a_n y_n^*\| = \|\sum_{n=1}^{m} a_n Tx^*\| &= \|T(\sum_{n=1}^{m} a_n x^*)\| \\
&\leq \|T\|\|(\sum_{n=1}^{m} a_n x^*)\| \leq K\|T\|\|(\sum_{n=1}^{n} a_n x^*)\| \\
&= K\|\sum_{n=1}^{n} a_n y_n^*\|,
\end{aligned}$$

where K is the basis constant for the basic sequence $(x_n^*)_{n=1}^{\infty}$. □

Theorem 2.31. *Let $(x_n)_{n=1}^{\infty}$ be a basic sequence in X. The following are equivalent:*

(i) $(x_n)_{n=1}^{\infty}$ *is complemented.*

(ii) *There exists a sequence $(u_n^*)_{n=1}^{\infty}$ in X^* such that $u_m^*(x_n) = \delta_{mn}$ and $\sum_{n=1}^{\infty} u_n^*(x)x_n < \infty$, for every $x \in X$.*

Proof. $(i) \Rightarrow (ii)$ We have that $(x_n)_{n=1}^{\infty}$ is a complemented basic sequence in a Banach space X. This means that $M = [x_n]$ is a complemented subspace of X. So, there exist a projection $P : X \to X$ such that $PX = M$. Let $(x_n^*)_{n=1}^{\infty} \subset M^*$ be the biorthogonal functionals associated with $(x_n)_{n=1}^{\infty}$.

Denote $u_n^* = x_n^* \circ P$, for every $n \in \mathbb{N}$. Then u_n^* extends each x_n^* to whole X, $(u_n^*)_{n=1}^{\infty} \subset X^*$.

$$u_m^*(x_n) = x_m^*(Px_n) = x_m^*(x_n) = \begin{cases} 1 & m = n, \\ 0 & m \neq n. \end{cases}$$

So, $u_m^*(x_n) = \delta_{mn}$. For every $x \in X$ we have

$$\sum_{n=1}^{\infty} u_n^*(x)x_n = \sum_{n=1}^{\infty} x_n^*(Px)x_n = P(x) < \infty.$$

$(ii) \Rightarrow (i)$ We have that there exists a sequence $(u_n^*)_{n=1}^{\infty} \subset X^*$ such that $u_m^*(x_n) = \delta_{mn}$ and for every $x \in X$, $\sum_{n=1}^{\infty} u_n^*(x)x_n < \infty$. We need to prove that $M = [x_n]$ is complemented in X.

Take $P(x) = \sum_{n=1}^{\infty} u_n^*(x)x_n$. For every $x \in X$ we have

$$\begin{aligned} P(Px) &= P(\sum_{n=1}^{\infty} u_n^*(x)x_n) = \sum_{n=1}^{\infty} u_n^*(x)P(x_n) \\ &= \sum_{n=1}^{\infty} u_n^*(x) \sum_{k=1}^{\infty} u_k^*(x_n)x_n = \sum_{n=1}^{\infty} u_n^*(x)x_n = Px. \end{aligned}$$

So P is a projection.

It is clear that for every $x \in X$, $Px \in M$ since $(x_n)_{n=1}^{\infty}$ is a basic sequence in X, which means that it is a basis for $M = [x_n]$. $\square$

Summary

The concept of basis is fundamental in linear algebra, however the spaces under consideration are finite-dimensional. In the case of infinite-dimensional Banach spaces, we introduced the notion of basis and we proved that it is equivalent with that introduced by Schauder in 1927. We showed that a necessary condition for a Banach space to have a basis is separability and we gave some examples about such basis. We also gave a way to construct a basis if we have a family of projections enjoin the properties of partial sum projections. However, not every Banach space have a basis.

We introduced the notion of basic sequences, and we proved Grunblum's criterion, which is a test for recognising a sequence in a Banach space as a basic sequence. A very important result is that every Banach space have a basic sequence.

We defined the equivalence between two bases (basic sequences), and showed that if a Banach space has a basis, then there exists normalized bases non-equivalent.

Since whenever we permute the element of a basis, it doesn't mean that the new sequence is a basis, we defined unconditional bases. For this we had to define unconditionally convergent series and prove some theorems about unconditional convergence. We also proved a necessary and sufficient condition for a basis to be unconditional. We gave the definitions of unconditional basis constant and suppression constant, and proved the relations between them.

We introduced adjoint operators and we gave some of its properties. An important result, is that for a bounded linear operator to be an adjoins it is equivalent with that operator being weak*-to-weak* continuous. Also, by using the notion of adjoints, we proved some theorems about basis.

We gave the definition of complemented subspaces, and by using the notion of projections and some of its properties, we presented an equivalent definition. Next, it is shown that c_0 is not equivalent with l_∞. Later on, this is used as an example to show that there exists subspaces, which are not complemented, but have some of complemented subspaces properties, when we redefined it by using adjoint operators. In the last section, we defined those subspaces as locally complemented.

Before that, we presented and proved the Principle of Local Reflexivity (PLR). This is an important result that helps to prove equivalent definitions about locally complemented subspaces.

In the last section we defined locally complemented subspaces and showed that if a subspace is λ-complemented then it is locally λ-complemented. Next, we presented Hahn-Banach extension operators and proved that its existence is equivalent with being locally 1-complemented.

At last, we gave the definition for a basic sequence to be (locally) complemented. We proved that if a basic sequence is locally complemented, then its biorthogonal functionals can be extended to a basic sequence in the dual space.

Also, we proved the equivalence of a basis sequence being complemented, with the existence of a sequence in the dual space which extends the biorthogonal functionals to the whole space. Since every complemented space is locally complemented, we have that this extension implies for the basic sequence to be locally complemented.

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