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Eberlein-Šmulian theorem and some of its applications

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Contents

Abstract

The thesis is about Eberlein-Šmulian and some its applications. The goal is to investigate and explain different proofs of the Eberlein-Šmulian theorem. First we introduce the general theory of weak and weak* topology defined on a normed space X. Next we present the definition of a basis and a Schauder basis of a given Banach space. We give some examples and prove the main theorems which are needed to enjoy the proof of the Eberlein-Šmulian theorem given by Pelchynski in 1964. Also we present the proof given by Whitley in 1967. Next there is described the connection between the weak topology and the topology and the topology of pointwise convergence in $C(K)$ for K compact Hausdorff. Then we give a generalization of the Eberlein-Šmulian in the context of $C(K)$ spaces. We end the thesis by providing some examples of applications of the Eberlein-Šmulian theorem to Tauberian operators theory.

1 Introduction

Compactness is a powerful property of spaces, and is used in many ways in different areas of mathematics. One could think of compactness as something that takes local properties to global properties and which let us use finite argument on infinite covers. One of most popular use of compactness is to locate maxima or minima of a function. Another use of compactness is to partially recover the notion of a limit when dealing with non-convergent sequences, by passing to the limit of a subsequence of the original sequence (Even though different subsequences may converge to different limits; compactness guarantees the existence of a limit point, but not uniqueness). Compactness of one object also tends to initiate compactness of other objects; for instance, the image of a compact set under a continuous map is still compact, and the product of finally many or even infinitely many compact sets continues to be compact.
The term *compact* was introduced in 1904 by Maurice Fréchet to describe those spaces in which every sequence has a convergent subsequence. Such spaces are now called *sequentially compact*. The property of compactness has a long and complicated history. Basically, the purpose of defining compactness was to generalize the properties of closed and bounded intervals to general topological spaces. The early attempts to achieve this goal included sequential compactness, the Bolzano-Weierstrass property, and, finally the modern property of compactness, which was introduced by Pavel Alexandroff and Paul Urysohn in 1923 [2].
Let X be a topological space and A an arbitrary indexed set.

Definition 1. *A set $K \subset X$ is compact (C) if each of its open covers* *has a finite subcover i.e.* $K \subset \bigcup_{\alpha \in A} U_\alpha \Rightarrow \exists$ *a fnite subset* $I \subset A$ *such that* $K \subset \bigcup_{i \in I} U_i$.

There is another equivalent definition in terms of nets.

Definition 2. *A set* $K \subset X$ *is compact if every net in* K *has a subnet which converges in* K *i.e.*
$\forall (x_\alpha)_{\alpha \in A} \subset K$ $, \exists$ *a subnet* $(y_\beta)_{\beta \in B} \subset (x_\alpha)_{\alpha \in A}$ *such that* $y_\beta \to y \in K$.

Definition 3. *A set* $K \subset X$ *is sequentially compact (SC) if every sequence in* K *has a convergent subsequence which converges in* K *i.e.*
$\forall (x_n)_{n \in N} \subset K, \exists (x_n)_k \subset (x_n)_{n \in N}$ *such that* $(x_n)_k \to x \in K$

Definition 4. *A set* $K \subset X$ *is countably compact (CC) if every coutable cover has a finite subcover i.e.*
$K \subset \bigcup_{i \in N} U_i$ *there* $\exists$ *a finite number a finite number of sets* U_i, $i = 1, \ldots, n_0$ *such that* $K \subset \bigcup_{i=1}^{n_0} U_i$.

or equivalently in terms of nets and sequences

Definition 5. *A set* $K \subset X$ *is countably compact if every sequence in* K *has a subnet which converges in* K *i.e.*
$\forall (x_n)_{n \in N} \subset K, \exists (x_\alpha)_{\alpha \in A} \subset (x_n)_{n \in N}$ *such that* $x_\alpha \to x \in K$.

In general topological spaces $(C) \Rightarrow (CC) \Leftarrow (SC)$. In metric spaces three types of compactness are equivalent since every sequence can be considered as a special case of a net. For Banach spaces with finite dimensions the weak topology is metrizable so it is quite easy to deal with compactness in that case. Even though the weak topology of Banach spaces with infinite dimensions is never metrizable it behaves like a metrizable topology because in the weak topology the three types of compactness are still equivalent. This surprising result is called the Eberlein-Šmulian theorem. In 1940 Šmulian proved that every weakly compact subset of a Banach space is weakly sequentially compact. In 1947 Eberlein showed the converse.

1.1 Notation and terminology

The notation we use is standard in the theory of Banach spaces.

In a Banach (or normed) space X, we denote the unit sphere by S_X and the closed unit ball by B_X. We denote the dual of a normed space X by X^* and the bidual by X^{**}. For a set A its closure is denoted by $\overline{A}$ and the linear span $[A]$. For closures with respect to other topologies, we mark the topology separately, such as $\overline{A}^{weak^*}$ and $\overline{A}^{weak}$. A C^* algebra is a complex algebra B of continuous linear operators on a complex Hilbert space with two additional properties:

- B is a topologically closed set in the norm topology of operators.
- B is closed under the operation of taking adjoints of operators.

For Banach spaces X and Y, we denote the Banach space of all bounded linear operators from X to Y by $B(X,Y)$. For an operator $T\colon X \to Y$, we denote $\ker T = \{x \in X \colon Tx = 0\}$ and $\mathrm{im}T = \{y \in Y : \quad \exists x \in X \quad \text{with} \quad y = Tx\}$.

1.2 Cornerstones in Functional Analysis

In this section we will present the theorems in Functional Analysis which will be used during the proofs that will be presented during the thesis.

The Hahn-Banach extension Theorem Let Y be a subspace of a real linear space X, and p a positive functional on X such that

$$p(tx) = tp(x) \quad \text{and} \quad p(x+y) \leq p(x) + p(y) \quad \text{for avery} \quad x, y \in X, t \geq 0.$$

If f is a linear functional on Y such that $f(x) \leq p(x)$, for every $x \in Y$, then there is a linear functional F on X such that $F = f$ on Y and $F(x) \leq p(x)$, for every $x \in X$.

Normed-space version of the Hahn-Banach theorem Let y^* be a bounded linear functional on a subspace Y of a normed space X. Then there is $x^* \in X^*$ such that $\|x^*\| = \|y^*\|$ and $x^* \restriction_Y = y^*$.

The Closed Graph Theorem Let X and Y be Banach spaces. Suppose that $T : X \to Y$ is a linear mapping of X into Y with the following property: whenever $(x_n) \subset X$ is such that both $x = \lim x_n$ and $y = \lim Tx_n$ exist, it follows that $y = Tx$. Then T is continuous.

The Uniform Boundedness Principle Suppose $(f_\alpha)_{\alpha \in A}$ is a family of bounded linear operators from a Banach space X into a normed linear space Y. If $sup\{\|f_\alpha(x)\| : \alpha \in A\}$ is finite for each x in X then $sup\{\|f_\alpha\| : \alpha \in A\}$ is finite.

Hahn-Banach separation Theorem Let A and B be disjoint, non-empty, convex subsets of a real topological vector space X.

(i) If A is open, then there exists $f \in X^*$ and a real number α such that $\sup_{x \in A} f(x) \leq \alpha \leq \inf_{y \in B} f(y)$.

(ii) If A is compact, B is closed and X is a locally topological vector space, then there exists $f \in X^*$ such that $\sup_{x \in A} f(x) < \inf_{y \in B} f(y)$.

Stone-Weierstrass Theorem Let X be a compact Hausdorff space. A subalgebra $A \subset C(X)$ is dense if and only if it separates points.

Gelfand-Naimark Theorem Each C^* algebra is isometrically-isomorphic to a closed subalgebra of the algebra $B(H)$ consisting of all bounded operators acting on a Hilbert space H.

Partial Converse of the Banach-Steinhaus Theorem Let (S_n) be a sequence of operators from a Banach space X into a normed linear space Y such that $\sup_n \|S_n\| < \infty$. Then, if $T : X \to Y$ is another operator, the subspace $\{x \in X : \|S_n(x) - T(x)\| \to 0\}$ is normed closed in X.

2 Basics of weak and weak* topologies

The toplogy induced by a norm on a vector space is a very strong topology in the sense that it has many open sets. This brings advantages to the functions whose domain is such a space because for them is easy to be continuous but it brings disadvantages to compactness because the richness of open sets makes it difficult for a set to be compact. For example, an infinite dimensional normed space has so many open sets that its closed unit ball cannot be compact. Because of this, many facts about finite dimensional normed spaces that are based on *Heine-Borel property* cannot be immediately generalized to the infinite-dimensional case.

Compactness is depended on open sets i.e. is depended on the nature of topology. The weaker the topology is the more compact sets we have, this fact motivates us to search for a topology defined on a normed space X which is the weakest topology among all topologies which we can define on X, so we can have the biggest class of compact sets. The topology which gives us the desired result is the *weak topology* defined on X. Another useful topology is the weak* topology of X^*. These two topologies help us to characterize properties of topological spaces with infinite dimensions and provide simple means to check their nature. To present the basics from these topologies we will use [7] and [3].

2.1 The weak topology

Let X be a normed space and F a scalar field. Usually we will take $F = \mathbb{R}$ unless otherwise stated. We define X^{**} as the dual of the dual space X^*. Let x_0 be an element of a normed space X and let $Jx_0 : X^* \to F$ given by formula

$$(J(x_0))(x^*) = x^*(x_0). \tag{2.1}$$

Then it is easy to check that J_{x_0} is a linear functional on X^* and

$$\|J_{x_0}\| = \{\sup |J_{x_0}(x^*)| : x^* \in B_{X^*}\} = \{\sup |x^*(x_0)| : x^* \in B_{X^*}\} = \|x_0\|.$$

Since J_{x_0} is linear and bounded then it is continuous so $J_{x_0} \in X^{**}$. If $J(x)$ is defined similarly for each $x \in X$, then the resulting $J : X \to X^{**}$ is linear. Since it is also norm preserving and J^{-1} is continuous, then it is an isometric isomorhpism. We call the above described function J the *canonical embedding* of X into X^{**}.

Definition 6. *The weak topology for X is the weakest topology on X such that every member of the X^* is continuous. This topology is denoted $\sigma(X, X^*)$.*

The next result tells us the form of the neighborhoods of a point $x \in$ in the weak topology.

Proposition 2.1. *The neighborhood basis at the point $x \in X$ is the collection of sets of the form*

$$W(\epsilon; x_1^*, \dots, x_n^*, x) = \{y \in X,\ |x_i^*(y - x)| < \epsilon,\ i = 1, \dots, n\},$$

where $n \in \mathbb{N}$, $\epsilon > 0$ and $x_1^*, \dots, x_n^* \in X^*$.

Proof. Since $x_1^*, \dots, x_n^*$ are continuous then $W(\epsilon; x_1^*, \dots, x_n^*, x)$ is open as it is a finite intersection of open sets. Furthermore it contains x since

$$|x_i^*(x - x)| = 0 < \epsilon.$$

Now let O be any open set containing x. By definition of the topology $\sigma(X, X^*)$, it is a union of finite intersections of preimages of open sets O_i in $\mathbb{R}$ of bounded linear functionals X on i.e.

$$O = \bigcup_{n\in\mathbb{N}} \bigcap_{i=1}^{n} x_i^{*-1}(O_i)$$

Since $x \in O$ there exist finitely many bounded linear functionals $x_1^*, \ldots, x_n^*$ and open subsets $O_1, \ldots, O_n$ of $\mathbb{R}$ such that

$$x \in \bigcap_{i=1}^{n} x_i^{*-1}(O_i) \Rightarrow x \in x_i^{*-1}(O_i), \;\; \forall i \in 1, \ldots, n$$

so $x_i^*(x) \in O_i, \;\; i = 1, \ldots, n$. Since O_i is an open set in $\mathbb{R}$ containing $x_i^*(x)$ there exists $\epsilon_i > 0$ with

$$\big(x_i^*(x) - \epsilon_i, x_i^*(x) + \epsilon_i\big) \subset O_i$$

Let $\epsilon = \min_{1\leq i\leq n} \epsilon_i$ and we will have

$$\big(x_i^*(x) - \epsilon, x_i^*(x) + \epsilon\big) \subset O_i$$

Now let $y \in W(\epsilon; x_1^*, \ldots, x_n^*, x)$ by definition this means

$$x_i^*(y) \in \big(x_i^*(x) - \epsilon, x_i^*(x) + \epsilon\big) \subset O_i, \;\; i = 1, \ldots, n \Rightarrow$$

$$y \in x_i^{*-1}(O_i), \;\; i = 1, \ldots, n \Rightarrow y \in \bigcap_{i=1}^{n} x_i^{*-1} \subset O \Rightarrow y \in O$$

We proved the implication: $y \in W(\epsilon; x_1^*, \ldots, x_n^*, x) \Rightarrow y \in O$ which is equivalent with $W(\epsilon; x_1^*, \ldots, x_n^*, x) \subset O$ as required.

$\square$

From Proposition 2.1 we already know the weak neighborhoods of a point $x \in$ so we can define the convergence of a net $(x_\alpha)_{\alpha\in A}$ in the weak topology of X.

Definition 7. *We say that the net* $(x_\alpha)_{\alpha\in A} \subset X$ *converges weakly to* $x_0 \in X$ *if for each* $x^* \in X^*$ *we have*

$$\lim_\alpha x^*(x_\alpha) = x^*(x_0)$$

Since the weak topology is linear instead of considering the neighborhoods of arbitrary points we can limit the case to the neighborhoods of 0; translation will carry these neighborhoods throughout X. A typical basic neighborhood of 0 is defined for any given arbitrary $\epsilon > 0$ and $x_1^*, \ldots, x_n^*$ and it is of the form

$$W(0; x_1^*, \ldots, x_n^*; \epsilon) = \{x \in X : |x_1^*(x)|, \ldots, |x_n^*(x)| \leq \epsilon\}.$$

The Hahn-Banach theorem tells us that the dual space X^* of a normed space X is a separating family of functions for X and each x^* maps X into the field F which is completely regular then the weak topology defined on X is completely regular.

Theorem 2.2. *The weak topology of a normed space is a completely regular locally convex subtopology of the norm topology.*

We will see that even though the weak topology is weaker than the norm topology they give rise to the same class of linear functionals.

Proposition 2.3. *A linear functional on a normed space is weak continuous if and only if it is norm continuous.*

Proof. ($\Rightarrow$) Without loss of generality we can assume that the linear functional on the normed space X is such that $f : X \to \mathbb{R}$. We have to prove that f is norm continuous knowing that it is weak continuous.Take a sequence $(x_n) \in X$. Assume that $x_n \to x$ and $f(x_n) \to y$ in the norm topology. According to Closed Graph Theorem if we prove that $y = f(x)$ we are done. Since norm convergence implies weak convergence we have

$$x_n \xrightarrow{weakly} x \quad \text{and} \quad f(x_n) \xrightarrow{weakly} y.$$

On the other hand f is weakly continuous and $x_n \to x$ thus

$$f(x_n) \xrightarrow{weakly} f(x).$$

Now, we are at the point where the real sequence $f(x_n)$ has two limits $f(x)$ and y. Since $\mathbb{R}$ is Hausdorff, these two limits should be the same, so $y = f(x)$.

($\Leftarrow$) Assume f is norm continuous i.e., $\|x_\alpha - x\| \to 0$ and $\|f\| < \infty$. We can write

$$|f(x_\alpha) - f(x)| = |f(x_\alpha - x)| \leq \|f\|\|x_\alpha - x\| \to 0.$$

Passing to the limit in both sides we get that $f(x_\alpha) \to f(x)$ whenever $x_\alpha \to x$ which means that f is weakly continuous. $\square$

Next we will show that norm topology and weak topology have the same class of bounded sets.

Definition 8. *A set $A \subset X$ is said to be weakly bounded if for each $x^* \in X^*$ we have that $x^*(A)$ is a bounded set of scalars.*

Theorem 2.4. *A subset of a normed space is bounded if and only if it is weakly bounded.*

Proof. Since every weakly open subset of a normed space is open then every bounded subset of a normed space is weakly bounded. Conversely, suppose that A is weakly bounded subset of a normed space X. It may be assumed that A is nonempty. Let J be the natural map from X to X^{**}. Then $J(A)$ is a nonempty collection of bounded linear functionals on the Banach space X^*. For each x^* in X^* we have:

$$\sup\{|(J(x))(x^*) : x \in A\} = \sup\{|x^*x| : x \in A\} < \infty.$$

From the Uniform Boundedness Principle it follows that

$$\sup\{\|x\| : x \in A\} = \sup\{\|Jx\| : x \in A\} < \infty,$$

so A is bounded. $\square$

From the above theorem very useful observations follow:

Corollary 2.5. *Weakly compact subsets of a normed space are bounded.*

Corollary 2.6. *In a normed space, weakly Caughy sequences, and so weakly convergent sequences, are bounded.*

Corollary 2.7. *Every nonempty weakly open subset of an infinite-dimensional normed space is unbounded.*

Proof. Let X be an infinite-dimensional normed space. Then X^* is also infinite-dimensional. Since every nonemtpty weakly open subset of X is weakly unbounded (see [7, proposition 2.4.15]), it must be unbounded. □

The following result is known as Mazur's theorem and shows us that there is a class of sets, namely convex sets where the norm and the weak closure agree.

Theorem 2.8. *If K is a convex subset of the normed linear space X, then the closure of K in the norm topology coincides with the weak closure of K: $\overline{K}^{\|\ \|} = \overline{K}^{weak}$*

Proof. Since the weak topology is weaker than the norm topology and the closure of a set is the intersection of all closed sets which contain it then $\overline{K}^{weak} \subset \overline{K}^{\|\ \|}$.
Now it remains to prove that $\overline{K}^{\|\ \|} \subset \overline{K}^{weak}$. Assume the opposite, that exists $x_0 \in \overline{K}^{weak} \backslash \overline{K}^{\|\ \|}$. Then applying the Hahn-Banach separation theorem for disjoint convex sets $\overline{K}^{\|\ \|}$ and $\{x_0\}$ we are guaranteed the existence of $f \in X^*$ satisfying

$$f(y)_{y \in \overline{K}^{\|\ \|}} < f(x_0). \tag{2.2}$$

Since $x_0 \in \overline{K}^{weak}$ there exists a net $(x_\alpha)_{\alpha \in A} \subset K$ such that

$$(x_\alpha) \xrightarrow{\text{weakly}} x_0,$$

therefore by definition of the weak convergence

$$f(x_\alpha) \xrightarrow{\text{weakly}} f(x_0)$$

Since $x_0 \in \overline{K}^{weak}$ then $f(x_0) \in f(\overline{K}^{weak}) \subset f(\overline{K}^{\| \|})$ i.e. $f(x_0) \geq \inf(\overline{K}^{\| \|})$ which is a contradiction to 2.2. This completes the proof. □

Theorems 2.3, 2.4 and Mazur's Theorem tell us that norm and the weak topology have (define) the same:

(i) Class of continuous linear functions,

(ii) Class of bounded sets,

(iii) Closure of convex sets.

Even though the norm and the weak topology agree in the sense described above they are different :

Proposition 2.9. *The norm and weak topologies are the same if and only if the space is finite-dimensional.*

Proof. ($\Leftarrow$) X Assume X is finite dimensional. We have to show that the weak and the norm topology coincide. Norm makes all linear functionals continuous. Since the weak topology is the weakest topology with this property, it is weaker than the norm topology.
Now we have to show the converse. Since X is finite dimensional, it has a basis $(e_1, \dots, e_n)$. Any $x \in X$ has a unique decomposition along this basis, which means

$$\forall x \in X,\ \exists (x_1, \dots, x_n) \in \mathbb{R}^n : x = \sum_{i=1}^{n} x_i e_i,$$

define

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

It is easy to check that it is a norm on X. Recall that, as a consequence of finite dimensionality of X, all norms are equivalent and the norm topology defined on X is the topology defined by any norm. So, if O is any norm open set, it is in particular open for $\| \|_\infty$. This means that $\forall x \in O,\ \exists \epsilon_x > 0$ such that

$$B_\infty(x, \epsilon_x) \subset O,$$

therefore

$$O = \bigcup_{x \in O} B_\infty(x, \epsilon_x).$$

If we show that any norm open ball is weakly open, we get that the norm open set O is weakly open as the union of weakly open sets. Let x be any point in X and $\epsilon > 0$ real number. Then

$$B_\infty(x, \epsilon) = \{y \in X : \|y - x\|_\infty < \epsilon\} = \{y \in X : |y_i - x_i| < \epsilon,\ i = 1, \ldots, n\}.$$

Define functionals $x_1^*, \ldots, x_n^*$ by

$$x_i^*(x) = x_i,\ \forall x = \sum_{i=1}^{n} x_i e_i,\ i = 1, \ldots, n.$$

These functionals are in X^*, and we can write

$$B_\infty(x, \epsilon) = \{y \in X : |x_i^*(y - x)| < \epsilon, i = 1, \ldots, n\},$$

which proves that $B_\infty(x, \epsilon)$ is weakly open by Theorem 2.1. So any norm open set is weakly open and therefore weak topology and norm topology coincide.

($\Rightarrow$) Now we have to prove that if weak and norm topology coincide then X is finite dimensional. Assume the opposite, that X is infinite dimensional, if we prove that the weak and norm topology differ then we are done. Let S be the unit sphere in X, i.e.

$$S = \{x \in X : \|x\|_\infty = 1\}.$$

Then S is norm closed and $0 \notin \overline{S}^{\| \|}$ since $\|0\|_\infty = 0 \neq 1$. Let O be any neighbourhood of 0. By theorem 2.1 there exist $\epsilon > 0$ and $x_1^*, \ldots, x_n^*$ in X^* such that

$$W = \{x \in X : |x_i^*(x) < e, i = 1, \ldots, n\} \subset O.$$

D the map $\Phi : X \to \mathbb{R}^n$ with $\Phi(x) = \big(x_1^*(x), \dots, x_n^*(x)\big)$ this map is linear and

$$\ker \Phi = \{x \in X : x_i^*(x) = 0,\ i = 1, \dots, n\} = \bigcap_{i=1}^{n} \ker x_i^*.$$

By the Rank Duality Theorem

$$\dim(\ker \Phi) + \dim \big(\operatorname{im}(\Phi)\big) = \dim X = \infty.$$

Since $\operatorname{im}(\Phi) \subset \mathbb{R}^n$ then $\dim(\ \operatorname{im}(\Phi)) \leq n$, it follows that $\ker \Phi$ is infinite dimensional, and so it is not equal to $\{0\}$. Therefore there exists $x \neq 0$ such that

$$x_i^*(x) = 0,\ i = 1, \dots, n.$$

Then $\forall \lambda \in \mathbb{R}$ we have

$$|x_i^*(\lambda x)| = |\lambda x_i^*(x)| = |\lambda||x_i^*(x)| = 0\ i = 1, \dots, n,$$

which proves that $\forall \lambda \in \mathbb{R}$, $\lambda x \in W \subset O$. Now taking $\lambda = \frac{1}{\|x\|}$ we will have

$$\|\lambda x\| = |\lambda|\|x\| = \frac{1}{\|x\|}\|x\| = 1,$$

so for $\lambda = \frac{1}{\|x\|}$, $\|\lambda x\| = 1 \Rightarrow \lambda x \in S$. But on the other hand $\lambda x \in O$, therefore $\lambda x \in O \cap S$. We found out that every weak neighbourhood O of 0 has nonempty intersection with S and this means that 0 is in the weak closure of S i.e. $0 \in \overline{S}^{weak}$, but $0 \notin \overline{S}^{\|\ \|}$ and as final conclusion

$$\overline{S}^{weak} \neq \overline{S}^{\|\ \|}$$

This completes the proof. $\square$

When given a normed space X we may wonder whether the weak topology is metrizable.

Proposition 2.10. *The weak topology of a normed space is metrizable if and only if the space is finite-dimensional.*

Proof. The weak toplogogy of a finite-dimensional normed space is the same as the norm topology, and so it is induced by a metric. For the converse, suppose the opposite, that the weak topology of some infinite-dimensional normed space X is induced by a metric d. For each positive integer n, let U_n be the d-open ball in X centered at 0 and with radius n^{-1}. By Corollary 2.7 each U_n contains some x_n such that $\|x_n\| \geq n$ which shows that (x_n) is an unbounded sequence. Since x_n is in the ball with radius n^{-1} then $x_n \xrightarrow{weakly} 0$. So, we have an unbounded sequence converging weakly to 0 which contradicts Corollary 2.6. □

2.2 Weak* topology

If X is a Banach space then X^* is a Banach space too, so we can define the weak* topology on X^*

Definition 9. *Let X be a normed space and let $J : X \to X^{**}$. Then the weakset topology defined on X^* such that every member of $J(X)$ is continuous is called the weak* topology of X^* and is denoted $\sigma(X^*, X)$.*

As with the weak topology of a normed space, the results obtained for a Hausdorff locally convex topology induced by a separating vector space $J(X)$ of linear functionals hold for the weak* topology of the dual space X^* of a normed space X. All properties which are stated for weak topology of normed space X can be adapted and hold for the weak* topology of X^* and the proofs are the same in principle.

Proposition 2.11. *The neighborhood basis at the point $x^* \in X^*$ is the collection of sets of the form*

$$\{\ y^* \in X^*, |(y^* - x^*)x_i| < 1,\ i = 1, \dots, n\},$$

where $n \in N$ and $x_1, \dots, x_n \in X$.

The proof is similar to the one given for the weak neighbourhoods.
The weak* topology is linear by definition so it is enough to describe the neighbourhoods of 0, neighbourhoods of other points in X^* can be obtained by translation. Notice that weak* neighbourhoods of 0 are weak neighbourhoods of 0; in fact they are just the basic neighbourhoods generated by those members on X^{**} that are actually in X. A typical neighbourhood of 0 is defined for any given arbitrary $\epsilon > 0$ and $x_1, \dots, x_n \in X$ and it is of the form

$$W(0; x_1, \dots, x_n; \epsilon) = \{x^* \in X^* : |x^* x_1|, \dots, |x^* x_n| \leq \epsilon\}.$$

Now we can introduce the concept of the weak* convergence:

Definition 10. *We say that the net* $(x^*_\alpha) \subset X^*$ *converges weak* to* $x_0 \in X^*$ *if for each* $x \in X$,

$$x^*_0(x) = \lim_\alpha x^*_\alpha(x).$$

Let J be the map defined in 2.1. We introduce:

Definition 11. *A normed space* X *is reflexive if the natural map from* X *into* X^{**} *is into* X^{**}.

Proposition 2.12. *A Banach space is reflexive if and only if the weak and the weak* topologies are the same.*

Proof. The weak topology of X^* is the weakest topology on X^* such that all members of X^{**} are continuous and the weak* topology of X^* is the weakest topology on X^* is the weakest topology such that all the members of $J(X)$ are continuous. Since $J(X) \subset X^{**}$ then the weak* topology of X^* is weaker than the weak topology of X^*. These topologies are the same if and only if $J(X) = X^{**}$, that is if and only if X is reflexive □

Even though weak and weak* topologies in general are different, it is quite interesting and important to know that actually every point in X^{**} can be approximated by a net (set) of points in X i.e. X is weak* dense in X^{**}. This is expressed in the following results:

Theorem 2.13 (**Goldstine**)**.** *Let* X *be a normed space.* $J(B_X)$ *is weak* dense in* $B_{X^{**}}$ *i.e.* $(\overline{J(B_X)}^{weak*} = B_{X^{**}}$.

Proof. Since by definition it is clear that $\overline{J(B_X)}^{weak*} \subset B_{X^{**}}$ we concentrate to show the converse, that $B_{X^{**}} \subset \overline{J(B_X)}^{weak*}$. Let $x^{**} \in X^{**}$ be any point not in $\overline{J(B_X)}^{weak*}$, if we prove that $\|x^{**}\| > 1$ i.e. $x^{**} \notin \overline{B}_{X^{**}}$ then we are done.
Since $\overline{J(B_X)}^{weak*}$ is closed and convex in weak* topology and $\{x^{**}\}$ is convex and compact in weak* topology then applying Hahn-Banach Separation Theorem we know that there exists a nonzero element x^* in X^* such that

$$\sup\{y^{**}x^* : y^{**} \in \overline{J(B_X)}^{weak*}\} < x^{**}x^*.$$

Since

$$\sup\{|y^{**}x^*| : y^{**} \in \overline{J(B_X)}^{weak*}\} \geq \sup\{|x^*y| : y \in B_X\} = \|x^*\|$$

we have that $\|x^*\| < x^{**}x^*$ which shows that $x^{**}x^*$ has to be a positive number and therefore

$$\|x^*\| < x^{**}x^* = |x^{**}x^*| \leq \|x^{**}\|\|x^*\|.$$

Dividing both sides with $\|x^*\|$ we get $\|x^{**}\| > 1$, i.e. $x^{**} \notin B_{X^{**}}$ □

Corollary 2.14. *Let X be a normed space. Then X is weak* dense in X^{**} i.e., $\overline{J(X)}^{weak*} = X^{**}$*

Proof. Let x^{**} be a nonzero element from X^{**}. Then $\frac{x^{**}}{\|x^{**}\|} \in B_{X^{**}}$. From Goldstine's Theorem there exists a net $(x_\alpha) \subset B_X$ such that

$$(x_\alpha) \xrightarrow{\text{weak*}} \frac{x^{**}}{\|x^{**}\|}.$$

Therefore

$$\|x^{**}\|(x_\alpha) \xrightarrow{\text{weak*}} x^{**}.$$

So we proved that every element $x^{**} \in X^{**}$ is a weak* limit of a net $(x_\alpha) \subset X$. This completes the proof. □

We will need the following result from Topology during the next proof.

Theorem 2.15 (Tychonoff's Theorem). *If $(X_\alpha)_{\alpha\in A}$ is an arbitrary family of compact spaces, then their product $X = \prod_{\alpha\in A} X_\alpha$ is compact.*

As important and useful as Goldstine's theorem is, the most important feature of the weak* topology is contained in the following compactness result.

Theorem 2.16 (Banach-Alaoglu). *In any normed space X B_{X^*} is weak* compact.*

Proof. If $x^* \in B_{X^*}$, then for each $x \in B_X$, $|x^*(x)| \leq 1$. Consequently, each $x^* \in B_{X^*}$ maps B_X into the set $D = \{\alpha \in \mathbb{F},\ |\alpha| \leq 1\}$. We can therefore identify each member of B_{X^*} with a point in the product space D^{B_X} via the map $F : B_{X^*} \to D^{B_X}$ defined by

$$F(x^*) = \{x^*(x) : x \in B_X\}.$$

Since $D = \{\alpha \in \mathbb{F},\ |\alpha| \leq 1\}$ is compact then by Tychonoff Theorem D^{B_X} is compact. On the other hand, if J is the natural map from X to X^{**} then $J(B_X)$ is a separating family of functions on B_{X^*} then the map F defined above is a homeomorphism from B_{X^*} onto a topological subspace of D^{B_X} ([7, see proposition 2.4.4]). Since the closed subsets of a compact space are compact, homeomorphism F assures us that we only need to prove that B_{X^*} is weak* closed in D^{B_X} i.e. we need to show that the linearity (the property specifying B_{X^*} within D^{B_X}) preserves the limit. Let (x_β) be a net in B_{X^*} converging pointwise on B_X to $f \in D^{B_X}$. We need f to be linear and $\|f\| \leq 1$. It is easy to check that f is linear. Let's take $x_1, x_2 \in B_X$ and the scalars α_1, α_2 such that $\alpha_1 x_1 + \alpha_2 x_2 \in B_X$, then

$$\begin{aligned} f(\alpha_1 x_1 + \alpha_2 x_2) &= \lim_\beta x^*_\beta(\alpha_1 x_1 + \alpha_2 x_2) \\ &= \lim_\beta x^*_\beta(\alpha_1 x_1) + \lim_\beta x^*_\beta(\alpha_2 x_2) \\ &= \alpha_1 \lim_\beta x^*_\beta(x_1) + \alpha_2 \lim_\beta x^*_\beta(x_2) \\ &= \alpha_1 f(x_1) + \alpha_2 f(x_2). \end{aligned}$$

Furthermore since $|f(x)| \leq 1$ for $x \in B_X$, then $f \in B_{X^*}$. This completes the proof. □

Corollary 2.17. *Every weak* closed bounded subset of X^* is weak* compact*

.

Now with the power that Banach-Alaouglu and Goldestine provide us, we can develop another test for reflexivity of Banach spaces.

Proposition 2.18. *A Banach space X is reflexive if and only if the closed unit ball B_X is weakly compact.*

Proof. ($\Rightarrow$) If X is reflexive then $J : X \to X^{**}$ is continuous, injective and syrjective. Hence J^{-1} is linear and continuous i.e. J is homeomorphism so we can identify X with X^{**} and $\sigma(X^{**}, X^*)$ and $\sigma(X, X^*)$ have the same topological structure. Since

$$J(B_X) = B_{X^{**}}$$

and by Banach-Alaoglu $B_{X^{**}}$ is weak* compact then B_X is weak compact as well.
($\Leftarrow$) Conversely if B_X is compact then $J(B_X)$ is weak* compact, since weak* topology is Hausdorff then $J(B_X)$ is weak* closed in X^{**} from Goldstine's theorem we will have

$$J(B_X) = \overline{J(B_X)}^{weak*} = B_{X^{**}},$$

which means that X and X^{**} have the same topological structure since their closed unit balls coincide, i.e. X is reflexive. $\square$

3 Schauder Basis Theory

In this chapter we are going to introduce the fundamental notion of a Schauder basis of a Banach space and the corresponding notion of a basic sequence. We will use the properties of basis and basic sequences as a tool to understand the differences and similarities between spaces. Throughout this chapter we will refer to [7] and [1]. The use of basic sequence arguments turns out to simplify some classical theorems and among them Eberlein-Šmulian theorem on weakly compact subsets of a Banach space.

3.1 First Properties

In order to combine the techniques of linear algebra and topological considerations in the context of functional analysis it is necessary to look for a concept to extend the notion of a basis of a finite dimensional vector space.

Definition 12. *A sequence $(e_n)_{n=1}^{\infty}$ in an infinite-dimensional Banach space X is said to be a basis of X if for each $x \in X$ there is a unique sequence of scalars $(a_n)_{n=1}^{\infty}$ such that*

$$x = \sum_{n=1}^{\infty} a_n e_n.$$

This means we require the sequence $\left(\sum_{i=0}^{n} a_i e_i\right)_{n=1}^{\infty}$ converges to x in the norm topology of X i.e.

$$\lim_{n\to\infty} \|x - \sum_{i=0}^{n} a_i e_i\| = 0.$$

From the definition we conclude that a basis consists of linearly independent, and in particular nonzero, vectors. If X has a basis $(e_n)_{n=1}^{\infty}$, then its closed linear span, $[e_n]$, coincides with X and therefore X is separable. The order of the basis is important; if we permute the elements of the basis, then the new sequence can very easily fail to be a basis.
We also note that if $(e_n)_{n=1}^{\infty}$ is a basis of a Banach space X, the maps $x \to a_n$ are linear functionals on X. Let us write, for now, $e_n^{\#}(x) = a_n$. It is by no means clear that the linear functionals $(e_n^{\#})_{n=1}^{\infty}$ are continuous. Let us make the following definition

Definition 13. *Let $(e_n)_{n=1}^{\infty}$ be a sequence in a Banach space X. Suppose there is a sequence $(e_n^*)_{n=1}^{\infty}$ in X^* such that*

(i) $e_k^*(e_j) = \begin{cases} 1 & , j = k \\ 0 & , j \neq k \end{cases}$

(ii) $x = \sum\limits_{n=1}^{\infty} e_n^*(x) e_n$ *for each* $x \in X$

then $(e_n)_{n=1}^{\infty}$ is called Schauder basis for X and the functionals $(e_n^*)_{n=1}^{\infty}$ are called the biorthogonal functionals associated with $(e_n)_{n=1}^{\infty}$.

Now we will give two examples of Schauder basis.

Example 1. *The sequence of unit vectors $(e_n)_{n=1}^{\infty}$ is a Scauder basis for c_0 and l_p such that $1 \leq p < \infty$. The vector e_n has the element 1 at the $n-th$ coordinate:*

$$\begin{aligned} e_1 &= (1, 0, 0 \ldots, 0, \ldots) \\ e_2 &= (0, 1, 0, \ldots, 0, \ldots) \\ &\ldots \\ e_n &= (0, 0, \ldots, 0, 1, 0, \ldots) \end{aligned}$$

First of all it is trivial that the sequence ${e_n}_{n=1}^{\infty}$ is element of both spaces c_0 and l_p. Note that if we pick a sequence $(a_n) = (a_1, a_2, \ldots, a_n, \ldots)$ then we have

$$\begin{aligned}(a_n) &= (a_1, a_2, \dots, a_n, \dots) \\ &= a_1(0,1,0,\dots,..) + a_2(0,1,0\dots,..) + \dots \\ &= \sum_{n=1}^{\infty} a_n e_n\end{aligned}$$

.

Let's show that the decomposition of any $(a_n) \in c_0$ or l_p according to $(e_n)_{n=1}^{\infty}$ is unique. Assume that there exists another sequence of scalars (b_n) with

$$(a_n) = \sum_{n=1}^{\infty} b_n e_n,$$

we have

$$\begin{aligned}(a_n) &= a_1(0,1,0,\dots,..) + a_2(0,1,0\dots,..) + \dots \\ &= b_1(0,1,0,\dots,..) + b_2(0,1,0\dots,..) + \dots \\ &= (b_n).\end{aligned}$$

Which shows that the sequence of vectors $(e_n)_{n=1}^{\infty}$ is a Schauder basis for both c_0 and l_p when $1 \leq p < \infty$.

Example 2. ***The classical Schauder basis for C[0,1].***
Define the sequence $(s_n)_{n=1}^{\infty}$ of members of $C[0,1]$ as follows. Let $s_0(t) = 1$ and $s_1(t) = t$. When $n \geq 2$, define s_n by letting m the positive integer such that $2^{m-1} < n \leq 2^m$, then let

$$s_n(t) = \begin{cases} 2^m\Big(t - (\frac{2n-2}{2^m} - 1)\Big) & \textit{if} \quad \frac{2n-2}{2^m} - 1 \leq t < \frac{2n-1}{2^m} - 1 \\ 1 - 2^m\Big(t - (\frac{2n-2}{2^m} - 1)\Big) & \textit{if} \quad \frac{2n-2}{2^m} - 1 \leq t < \frac{2n-1}{2^m} - 1 \\ 0 & \textit{otherwise} \end{cases}$$

Take a function $f \in C[0,1]$. Define a sequence $(p_n)_{n=1}^{\infty}$ in $C[0,1]$ in the following way. Let

$$
\begin{aligned}
p_0 &= f(0)s_0,\\
p_1 &= p_0 + \big(f(1) - p_0(1)\big)s_1,\\
p_2 &= p_1 + \big(f(\frac{1}{2}) - p_1(\frac{1}{2})\big)s_2,\\
p_3 &= p_2 + \big(f(\frac{1}{4}) - p_2(\frac{1}{4})\big)s_3,\\
p_4 &= p_3 + \big(f(\frac{3}{4}) - p_3(\frac{3}{4})\big)s_4,\\
p_5 &= p_4 + \big(f(\frac{1}{8}) - p_4(\frac{1}{8})\big)s_5,\\
p_6 &= p_5 + \big(f(\frac{3}{8}) - p_5(\frac{3}{8})\big)s_6,\\
p_7 &= p_6 + \big(f(\frac{5}{8}) - p_6(\frac{5}{8})\big)s_7,\\
p_8 &= p_7 + \big(f(\frac{7}{8}) - p_7(\frac{7}{8})\big)s_8,
\end{aligned}
$$

and so forth. Then p_0 is the constant function that agrees with f at 0, while p_1 agrees with f at 0 and 1 and interpolates linearly in between, and p_2 agrees with f at $0, 1$ and $\frac{1}{2}$ and interpolates linearly in between, and so forth. For each nonnegative integer n, let α_n be the coefficient of s_n in the formula for p_n. Then

$$p_m = \sum_{n=0}^{m} \alpha_n s_n$$

for each m. We have

$$\lim_{m \to \infty} \|p_m - f\|_\infty = 0,$$

and therefore

$$f = \sum_{n=0}^{\infty} \alpha_n s_n.$$

Let's show the uniqueness of scalars now. Let $(\beta_n)_{n=0}^{\infty}$ *be any sequence of scalars such that*

$$f = \sum_{n=0}^{\infty} \beta_n s_n.$$

Then $\sum_{n=0}^{\infty}(\alpha_n - \beta_n)s_n = 0$*, which implies that* $\sum_{n=0}^{\infty}(\alpha_n - \beta_n)s_n(t) = 0$ *when* $t = 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \ldots,$ *from which it quickly follows that* $\alpha_n = \beta_n$ *for each* n*. Therefore there is a unique sequence* $(\gamma_n)_{n=0}^{\infty}$ *of scalars such that* $f = \sum_{n=0}^{\infty} \gamma_n s_n$*, so the sequence* $(\gamma_n)_{n=0}^{\infty}$ *is a basis for* $C[0,1]$*.*

If $(e_n)_{n=1}^{\infty}$ is a Schauder basis for X and $x = \sum_{n=1}^{\infty} e_n^*(x)e_n$, the support of x is the subset of integers n such that $e_n^{\#}(x) \neq 0$. We denote it by *supp*(x). If *supp*$(x) < \infty$ we say that x is finitely supported. The name Schauder in the previous definition is in honour of Julius Schauder, who first introduced the concept of a basis in 1927. It turns out that the two definitions are equivalent in separable Banach spaces.

Theorem 3.1. *Let* X *be a separable Banach space. A sequence* $(e_n)_{n=1}^{\infty}$ *in* X *is a Schauder basis for* X *if and only if* $(e_n)_{n=1}^{\infty}$ *is a basis for* X*.*

Proof. Since it is clear that every Schauder basis is a basis then we concentrate in showing the converse. Let us assume that $(e_n)_{n=1}^{\infty}$ is a basis for X and introduce the partial sum projections $(S_n)_{n=1}^{\infty}$ associated to $(e_n)_{n=1}^{\infty}$ defined by $S_0 = 0$ and for $n \geq 2$,

$$S_n(x) = \sum_{k=1}^{n} e_k^{\#}(x)e_k.$$

Let us define a new norm on X given by formula

$$|||x||| = \sup_{n\geq 1} \|s_n(x)\|. \tag{3.1}$$

We have

$$|||x||| = \sup_{n\geq 1} \|s_n(x)\| \geq \lim_{n\to\infty} \|s_n(x)\| = \|x\|.$$

Let's prove that the function : $X \to \mathbb{R}$ given by (3.1) is a norm. First of all for each $x \in X$ we have $|||x||| < \infty$ since for each $n \in \mathbb{N}$, $\|S_n\| < \infty$. Since the two other requirements of the definition are trivial we concentrate in showing the triangle inequality. Pick two elements x and y in X having the expansions $x = \sum_{n=1}^{\infty} a_n e_n$ and $y = \sum_{n=1}^{\infty} b_n e_n$, then

$$\begin{aligned}
|||x+y||| &= \sup_{n\geq 1} \|S_n(x) + S_n(y)\| \\
&\leq \sup_{n\geq 1}\{\|S_n(x)\| + \|S_n(y)\|\} \\
&\leq \sup_{n\geq 1} \|S_n(x)\| + \sup_{n\geq 1} \|S_n(y)\| \\
&= |||x||| + |||y|||.
\end{aligned}$$

We will show that $(X, |||.|||)$ is complete. Suppose that $(x_n)_{n=1}^{\infty}$ is a Caughy sequence in $(X, ||| \,.\, |||)$. Since $||| \,.\, ||| \geq \| \,.\, \|$, then $(x_n)_{n=1}^{\infty}$ is Caughy in $(X, \| \,.\, \|)$, and therefore convergent to some $x \in X$ for the original norm. Our goal is to prove that $\lim_{n\to\infty} |||x_n - x||| = 0$. For each fixed $k \in \mathbb{N}$, look at the sequence $S_k(x_n)$. We can write

$$\begin{aligned}
\|S_k(x_m) - S_k(x_m)\| &= \|S_k(x_m - x_n)\| \\
&\leq \sup_k \|S_k(x_m - x_n)\| \\
&= |||x_m - x_n|||.
\end{aligned}$$

Since (x_n) is Caughy in $(X, ||| \,.\, |||)$ then $S_k(x_n)$ is Caughy in $(X, || \,.\, ||)$ and therefore it converges to some $y_k \in X$. Note also that $(S_k x_n)_{n=1}^{\infty}$ is contained in the finite-dimensional subspace $[e_1, \ldots, e_n]$. The functionals $e_j^{\#}$ are continuous on any finite dimensional subspace; hence if $1 \leq j \leq k$ we have

$$
\begin{aligned}
lim_{n\to\infty} e_j^{\#}(x_n) &= e_j^{\#}(y_k) \\
&= e_j{}^{\#}(\lim_{n\infty} S_k x_n) \\
&= e_j{}^{\#} y_k \\
&= a_j
\end{aligned}
$$

Next we will argue that $\sum_{j=1}^{\infty} a_j e_j = x$ for the original norm. Since $(x_n)_{n=1}^{\infty}$ is Caughy in $(X, |||\ .\ |||)$, then for given $\epsilon > 0$ we can pick an integer n so that if $m \geq n$ then $|||x_m - x_n||| < \frac{1}{3}\epsilon$, which implies

$$\|x_m - x_n\| < \frac{1}{3}\epsilon.$$

On the other hand $(S_k x_n)_{k=1}^{\infty}$ is convergent to x_n in the original norm, now we take k_0 so that $k \geq k_0$ implies

$$\|x_n - S_k x_n\| \leq \frac{1}{3}\epsilon.$$

The sequence $(S_k x_n)_{n=1}^{\infty}$ is convergent too, so for $m \geq n$ we can write

$$\|S_k x_m - S_k x_n\| \leq \frac{1}{3}\epsilon.$$

Therefore

$$
\begin{aligned}
\|y_k - x\| &= \lim_{m\to\infty} \|S_k x_m - x_m\| \\
&= \lim_{m\to\infty} \|S_k x_m - S_k x_n + S_k x_n - x_n + x_n - x_m\| \\
&\leq \lim_{m\to\infty} \|S_k x_m - S_k x_n\| + \|s_k x_n - x_n\| + \lim_{m\to\infty} \|x_m - x_n\| \\
&\leq \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \frac{1}{3}\epsilon \\
&= \epsilon.
\end{aligned}
$$

Thus $\lim_{k\to\infty} \|y_k - x\| = 0$. Since $\lim_{k\to\infty} \|S_k(x) - x\| = 0$, by the uniqueness of the expansion of x with respect to the basis, $S_k(x) = y_k$. Now

$$|||x_n - x||| = \sup_{n\geq 1} \|S_k x_n - S_k x\| \leq \lim_{m\to\infty} \sup \sup_{k\geq 1} \|S_k x_n - S_k x_m\|.$$

So $\lim_{n\to\infty} |||x_n - x||| = 0$ and $(X, |||\,.\,|||)$ is complete. By the closed Graph Theorem, the identity map $i : (X, \|\,.\,\|) \to (X, |||\,.\,|||)$ is bounded, i.e., there exists K so that $|||x||| \leq K\|x\|$ for $x \in X$. This implies that

$$\|S_n(x)\| \leq K\|x\|, \quad x \in X, n \in \mathbb{N}.$$

In particular $|e_n^{\#}(x)|\|e_n\| = \|S_n(x) - S_{n-1}(x)\| \leq 2K\|x\|$, hence $e_n^{\#} \in X^*$ which means that for every $n \in \mathbb{N}$ $e_n^{\#}$ is a bounded continuous linear function on X. □

Let $(e_n)_{n=1}^{\infty}$ be a basis for a Banach space X. The preceding theorem tells us that $(e_n)_{n=1}^{\infty}$ is actually a Schauder basis, hence we use $(e_n^*)_{n=1}^{\infty}$ for the biorthogonal functionals. As above, we consider the partial sum operators $S_n : X \to X$, given by $S_0 = 0$, and for $n \geq 1$

$$S_n\big(\sum_{k=1}^{\infty} e_k^*(x)e_k\big) = \sum_{k=1}^{n} e_k^*(x)e_k.$$

S_n is a continuous linear operator since each e_k^* is continuous. In Theorem 3.1 we proved that $(s_n)_{n=1}^{\infty}$ are uniformly bounded, but we note it for further reference.

Proposition 3.2. *Let $(e_n)_{n=1}^{\infty}$ be a Schauder basis for a Banach space X and $(S_n)_{n=1}^{\infty}$ the natural projections associated with it. Then $\sup_n \|S_n\| < \infty$.*

Proof. For a Schauder basis the operators $(S_n)_{n=1}^{\infty}$ are bounded since e_n^* are bounded. Since $S_n(x) \to x$ for $x \in X$ we have $\sup_n \|S_n(x)\| < \infty$ for each $x \in X$. By the Uniform Boundedness Principle $\sup_n \|S_n\| < \infty$. □

Definition 14. *If $(e_n)_{n=1}^{\infty}$ is a basis for a Banach space X, then the number $K = \sup_n \|S_n\|$ is called the basis constant. In the optimal case that $K = 1$ the basis $(e_n)_{n=1}^{\infty}$ is said to be monotone.*

Remark 1. *We can always renorm a Banach space X with a basis such that the given basis is monotone. Let K be the basis constant. Just put*

$$|||x||| = \sup_{n\geq 1} \|S_n(x)\|.$$

If K is the basis constant then we can write

$$K \geq \|S_n\| = \sup\{\frac{\|S_n(x)\|}{\|x\|} \ : x \neq 0\}$$

$$\Rightarrow \|S_n(x)\| \leq K\|x\| \ \ , n \in \mathbb{N}$$

$$\Rightarrow |||x||| = \sup_{n \geq 1} \|S_n(x)\| \leq K\|x\|.$$

Therefore

$$\|x\| \leq |||x||| \leq K\|x\|,$$

which means that the new norm is equivalent to the old one. On the other hand

$$\begin{aligned}
|||S_n||| &= \sup_{|||x||| \leq 1} |||S_n(x)||| \\
&= \sup_{|||x||| \leq 1} \sup_{m \geq 1} ||S_m S_n(x)|| \\
&\geq \sup_{|||x||| \leq 1} \sup_{m \geq n \geq n} ||S_m S_n(x)|| \\
&\geq \sup_{|||x||| \leq 1} \sup_{m \geq n \geq n} ||S_n(x)|| \\
&= \sup_{|||x||| \leq 1} \sup_{n \geq 1} \|S_n(x)\| \\
&= \sup_{|||x||| \leq 1} |||x||| \\
&= 1.
\end{aligned}$$

So we have

$$|||S_n||| \geq 1. \tag{3.2}$$

On the other hand by the definition of the norm $|||\ .|||$ *we have*

$$||S_m S_n(x)|| \leq |||x||| \quad \textit{for all } m,n \ \in \mathbb{N}.$$

Therefore

$$\begin{aligned}|||S_n(x)||| &= \sup_{m\geq 1} \|S_m S_n(x)\| \\ &\leq |||x|||,\end{aligned}$$

which means that

$$|||S_n||| \leq 1. \tag{3.3}$$

Comparing 3.2 and 3.3 we conclude that $|||S_n||| = 1$ and the basis is monotone related to the new norm.

In the next result we will see that if we have a family of projections satisfying the properties of the partial sum operators then we can construct a basis for a given Banach space X.

Theorem 3.3. *Suppose $S_n : X \to X$,$n \in \mathbb{N}$, is a sequence of bounded linear projections on a Banach space x such that*

(i) $\dim S_n(X) = n$ *for each* $n \in \mathbb{N}$

(ii) $S_n S_m = S_m S_n = S_{\min\{m,n\}}$, *for any integer* m *and* n *and*

(iii) $S_n(x) \to x$ *for every* $x \in X$

Then any nonzero sequence of vectors $(e_k)_{k=1}^{\infty}$ in X chosen inductively so that $e_1 \in S_1(X)$, and $e_k \in S_k X \cap \ker S_{k-1}$ if $k \geq 2$ is a basis for X with partial sum projections $(S_n)_{n=1}^{\infty}$.

Proof. Let $0 \neq e_1 \in S_1(X)$ and define $e_1{}^* : X \to \mathbb{R}$ by $e_1{}^*(x)e_1 = S_1(x)$. Next we pick $0 \neq e_2 \in S_2(X) \cap S_1^{-1}(0)$ and define the functional $e_2^* : X \to \mathbb{R}$ by $e_2{}^*(x)e_2 = S_2(x) - S_1(x)$. This gives us by induction the procedure to extract the basis and its biorthogonal functionals: for each $n \in \mathbb{N}$, we pick $0 \neq e_n \in S_n(X) \cap S_1^{n-1}(0)$ and define $e_n^* : X \to \mathbb{R}$ by $e_n{}^*(x)e_n =$

$S_n(x) - S_{n-1}(x)$. Then we have

$$\begin{aligned}|e_n^*(x)| &= \|S_n(x) - S_{n-1}(x)\| \|e_n\|^{-1} \\ &\leq (\|S_(x)\| + \|S_{n-1}(x)\|) \|e_n\|^{-1} \\ &\leq 2 \sup_n \|S_n\| \|e_n\|^{-1} \|x\| \\ &< \infty\end{aligned}$$

hence $e_n^* \in X^*$. Also $e_k^*(e_j) = \delta_{kj}$ for any two integers k, j. If we let $S_0(x) = 0$ for all x, we can write

$$S_n(x) = \sum_{k=1}^{n} \big(S_k(x) - S_{k-1}(x)\big) = \sum_{k=1}^{n} e_k^*(x) e_k,$$

which, by *(iii)* in the hypothesis, converges to x for every $x \in X$. So the sequence $(e_n)_{n=1}^{\infty}$ is a basis and $(S_n)_{n=1}^{\infty}$ its natural projections.

□

In the next definition we drop the assumption that a basis must span the entire space X.

Definition 15. *A sequence $(e_k)_{k=1}^{\infty}$ in a Banach space X is called a basic sequence if it is a basis for $[e_k]$, the closed linear span of $(e_k)_{k=1}^{\infty}$.*

Now we develop a test which tells us whether a sequence $(e_k)_{k=1}^{\infty}$ of nonzero elements is basic.

Proposition 3.4 (Grunblum's criterion). *A sequence $(e_k)_{k=1}^{\infty}$ of nonzero elements of a Banach space X is basic if and only if there is a positive constant K such that*

$$\| \sum_{k=1}^{m} a_k e_k \| \leq K \| \sum_{k=1}^{n} a_k e_k \| \tag{3.4}$$

for any sequence of scalars $(a_k)_{k=1}^{\infty}$ and any integers m, n such that $m \leq n$.

Proof. ($\Rightarrow$) Let's assume that the sequence $(e_k)_{k=1}^{\infty}$ is basic, and let $S_N : [e_k] \to [e_k]_{k=1}^{\infty}\ , N = 1, 2 \ldots$ be its partial sum projections. Then, for $m \leq n$,

$$\begin{aligned}\|\sum_{k=1}^{m} a_k e_k\| &= \|S_m(\sum_{k=1}^{n} a_k e_k)\| \leq \sum_{k=1}^{n} \|S_m(a_k e_k)\| \\ &\leq \sum_{k=1}^{n} \|S_m\| \|a_k e_k\| \leq \|S_m\| \sum_{k=1}^{n} \|a_k e_k\| \\ &\leq \sup_m \|S_m\| \sum_{k=1}^{n} \|a_k e.k\|\end{aligned}$$

So if we let $K = \sup_m \|S_m\|$, then (3.4) holds.
($\Leftarrow$) Assume that $(e_k)_{k=1}^{\infty}$ is a sequence of nonzero elements in X such that (3.4) holds. Let E be the linear span of $(e_k)_{k=1}^{\infty}$ and $s_m : E \to [e_k]_{k=1}^{m}$ be the finite-rank operator defined by

$$s_m(\sum_{k=1}^{n} a_k e_k) = \sum_{k=1}^{\min\{m,n\}} a_k e_k \quad m, n \in \mathbb{N}.$$

Since E is dense in $[e_k]$ and s_m is bounded, by Hahn-Banach extension Theorem we can extend s_m to $S_m : [e_k] \to [e_k]_{k=1}^{m}$ with $\|S_m\| = \|s_m\| \leq K$. Notice that for each $x \in E$ we have

$$S_n S_m(x) = S_m S_n(x) = S_{\min\{m,n\}}, \quad m, n \in \mathbb{N} \tag{3.5}$$

so, by density (3.5) holds for all $x \in [e_n]$. By Partial converse of the Banach-Steinhaus Theorem the set $\{x \in [e_n]\ : S_m(x) \to x\}$ is closed. It also contains E, which is dense in $[e_n]$, and

$$[e_k] = \overline{E} \subset \{x \in [e_n]\ : S_m(x) \to x\}$$

which implies that $S_n(x) \to x$ for all $x \in [e_n]$. On the other hand, by the construction of S_n, we have $\dim S_n[e_k] = n$. By Proposition 3.3 $(e_k)_{k=1}^{\infty}$ is a basis for $[e_k]$ with partial sum projections $(S_m)_{m=1}^{\infty}$. □

Every basis in a finite-dimensional vector space defines a system of coordinates. Bases in infinite-dimensional Banach spaces work in the same way. Thus, if we have a basis $(e_n)_{n=1}^{\infty}$ of X then we can identify $x \in X$ by its coordinates $\big(e_n^*(x)\big)_{n=1}^{\infty}$. It is not true that every scalar sequence $(a_n)_{n=1}^{\infty}$ defines an element of X. Thus X is coordinatized by a certain sequence space, i.e., a linear subspace of the vector space of all sequences. This leads us to the following definition.

Definition 16. *Two bases (or basic sequences)$(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in the respective Banach spaces X and Y are equivalent if whenever we take a sequence of scalars $(a_n)_{n=1}^{\infty}$, then $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if $\sum_{n=1}^{\infty} a_n y_n$ converges.*

It turns out that if $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are bases of the Banach spaces X and Y respectively and they are equivalent then X and Y must be isomorphic.

Theorem 3.5. *Two bases (or basic sequences), $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are equivalent if and only if there is an isomorphism $T : [x_n] \to [y_n]$ such that $Tx_n = y_n$ for each n.*

Proof. ($\Rightarrow$)Let $X = [x_n]$ and $Y = [y_n]$. Let us assume that there exists an isomorphism T from X to Y such that $Tx_n = y_n$. We want to prove that $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are equivalent. Take a sequence of scalars $(a_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} a_n x_n$ converges to some element $x \in X$ i.e.,

$$\sum_{i=1}^{n} a_i x_i \xrightarrow[n\to\infty]{} x$$

in the norm of X. Since T is isomorphism T is linear and continuous, so

$$\sum_{i=1}^{n} a_i x_i \xrightarrow[n\to\infty]{} Tx$$

in the norm of Y. By the linearity of T we can write

$$T(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} T(a_i x_i) = \sum_{i=1}^{n} a_i T x_i = \sum_{i=1}^{n} a_i y_i \to Tx.$$

So we have that $\sum_{n=1}^{\infty} a_n y_n$ converges in the norm of Y. Taking a sequence of scalars $(a_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} a_n y_n$ converges and using the fact that T is isomorphism and thus T^{-1} is linear and continuous, we conclude that $\sum_{n=1}^{\infty} a_n x_n$ converges. By assumption we have

$$\sum_{i=1}^{n} a_i y_i \xrightarrow[n\to\infty]{} y,$$

which implies

$$T^{-1}(\sum_{i=1}^{n} a_i y_i) \xrightarrow[n\to\infty]{} T^{-1}y.$$

Since T is an isomorphism, there exists an element $x \in X$ such that $T^{-1}y = x$. Using the linearity of T^{-1} we can write

$$T^{-1}\sum_{i=1}^{n} a_i y_i = \sum_{i=1}^{n} T^{-1}(a_i y_i) = \sum_{i=1}^{n} a_i T^{-1} y_i = \sum_{i=1}^{n} a_i x_i \xrightarrow[n\to\infty]{} x,$$

which means that $\sum_{n=1}^{\infty} a_n x_n$ converges in the norm of X.
($\Leftarrow$) Assume that $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are equivalent. Let us define $T : X \to Y$ by

$$T(\sum_{n=1}^{\infty} a_n x_n) = \sum_{n=1}^{\infty} a_n y_n.$$

T is one-to-one: Take a sequence of scalars $(b_n)_{n=1}^{\infty}$ such that $T(\sum_{n=1}^{\infty} a_n x_n) = T(\sum_{n=1}^{\infty} b_n x_n)$. Then, the way that we have defined T implies

$$\sum_{n=1}^{\infty} a_n y_n = \sum_{n=1}^{\infty} b_n y_n.$$

by the unicity of expansion according to the basis $(y_n)_{n=1}^{\infty}$ we have $(a_n)_{n=1}^{\infty} = (b_n)_{n=1}^{\infty}$, and therefore

$$\sum_{n=1}^{\infty} a_n x_n = \sum_{n=1}^{\infty} b_n x_n.$$

T is onto: Take an element $y \in Y$, then, since (y_n) is a basis, there exists a unique sequence of scalars $(a_n)_{n=1}^{\infty}$ such that

$$y = \sum_{n=1}^{\infty} a_n y_n = T(\sum_{n=1}^{\infty} a_n x_n).$$

Since $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are equivalent and $\sum_{n=1}^{\infty} a_n y_n$ converges to y, then $\sum_{n=1}^{\infty} a_n x_n$ converges to some element $x \in X$ and we can write

$$y = \sum_{n=1}^{\infty} a_n y_n = T(\sum_{n=1}^{\infty} a_n x_n) = Tx.$$

So we proved that $\forall y \in Y$, $\exists x \in X$ with $Tx = y$. To prove that T is continuous we use the Closed Graph Theorem. Suppose $(u_n)_{n=1}^{\infty}$ is a sequence such that $u_j \to u$ in X and $Tu_j \to v$ in Y. If we prove that $Tu = v$ then we are done . Let us write

$$u_j = \sum_{n=1}^{\infty} x_n^*(u_j) x_n \quad \text{and} \quad u_j = \sum_{n=1}^{\infty} x_n^*(u) x_n \tag{3.6}$$

Since x_n^* and y_n^* are continuous, then

$$u_j \to u \quad \text{implies} \quad x_n^*(u_j) \to x_n^*(u)$$

and

$$Tu_j \to v \quad \text{implies} \quad y_n^*(Tu_j) \to y_n^*(v).$$

Where

$$\begin{aligned} y_n^*(Tu_j) &= y_n^*\Big(T\big(\sum_{i=1}^{\infty} x_i^*(u_j) x_i\big)\Big) = y_n^*\Big(\sum_{i=1}^{\infty} T\big(x_i^*(u_j) x_i\big)\Big) \\ &= y_n^*\big(\sum_{i=1}^{\infty} x_i^*(u_j) T(x_i)\big) = y_n^*\big(\sum_{i=1}^{\infty} x_i^*(u_j) y_i\big) \\ &= \sum_{i=1}^{\infty} x_i^*(u_j) y_n^*(y_i). \end{aligned}$$

Since $y_n^*(y_i) = \begin{cases} 1 & , n = i \\ 0 & , n \neq i \end{cases}$ we can write

$$\begin{aligned} y_n^*(Tu_j) &= \sum_{i=1}^{\infty} x_i^*(u_j) y_n^*(y_i) = \sum_{i \neq n} x_i^*(u_j) y_n^*(y_i) + x_n^*(u_j) y_n^*(y_n) \\ &= 0 + x_n^*(u_j) \cdot 1 \\ &= x_n^*(u_j). \end{aligned}$$

Now we have

$$y_n^*(Tu_j) = x_(^*u_j) \to y^*(v) \quad \text{and} \quad x_n^*(u_j) \to x_(^*u).$$

Since $\mathbb{R}$ is Hausdorff i.e., the limits are unique, then $y_n^*(v) = x_n^*(u)$ for each n. From (3.6) we can write

$$u = \sum_{i=1}^{\infty} x_i^*(u) x_i \Rightarrow$$

$$Tu = T(\sum_{i=1}^{\infty} x_i^*(u) x_i) = \sum_{i=1}^{\infty} x_i^*(u) y_i = \sum_{i=1}^{\infty} y_i^*(v) y_i = v.$$

So we have proved that $Tu = v$ and therefore T is continuous. Since T is one-to-one and onto then we are sure that $\exists T^{-1}$ and $T^{-1} : Y \to X$ with

$$T^{-1}(\sum_{n=1}^{\infty} a_n y_n) = \sum_{n=1}^{\infty} a_n x_n.$$

It is proved in the same way that T^{-1} is continuous. So T is a linear bicontinuous function and thus it is an isomorphism. To complete the proof let's show that $Tx_n = y_n$,

$$x_n = \sum_{i=1}^{\infty} x_i^*(x_n) x_i \Rightarrow Tx_n = \sum_{i=1}^{\infty} x_i^*(x_n) y_i = y_n.$$

$\square$

3.2 Constructing basic sequences

We will now present a method for constructing basic sequences. We work in the dual X^* of a Banach space for purely technical reasons; ultimately we will apply Lemma 3.6 and Theorem 3.7 to X^{**}.

Lemma 3.6. *Suppose that S is a subset of X^* such that $0 \in \overline{S}^{weak^*}$ but $0 \notin \overline{S}^{\|\,\|}$. Let E be a finite-dimensional subspace of X^*. Then for any given $\epsilon > 0$ there exists $x^* \in S$ such that*

$$|e^* + \lambda x^*\| \geq (1-\epsilon)\|e^*\| \tag{3.7}$$

for all $e^* \in E$ and $\lambda \in \mathbb{R}$.

Proof. First of all, recall that during the proof of Proposition 2.9 we proved that when X is infinite dimensional the weak and the norm topology do not coincide. Moreover, we found out that if $S = \{x \in X : \ \|x\| = 1\}$, then $0 \in \overline{S}^{weak}$ but $0 \notin \overline{S}^{\|\,\|}$. Since X is infinite dimensional X^* is infinite dimensional and we adapt this conclusion for the weak* and norm topology of X^*. So we can say that there exists a set $S \subset X^*$ with $0 \in \overline{S}^{weak^*}$ but $0 \notin \overline{S}^{\|\,\|}$. Recalling that a norm open neighbourhood $V(0,\alpha)$ of 0 is of the type

$$V(0,\alpha) = \{x^* \in X^* : \ \|x^*\| < \alpha\}$$

and since

$$0 \notin \overline{S}^{\|\,\|} = \{y^* \in Y^* : \ O_{y^*} \cap S \neq \emptyset\}$$

where O_{y^*} is any norm open neighbourhood of $y^* \in Y^*$, then there exists $\alpha \in \mathbb{R}^+$ such that

$$V(0,\alpha) \cap S = \{x^* \in X^* : \ \|x^*\| < \alpha\} \cap S = \emptyset$$

which means $\|x^*\| \geq \alpha, \ \forall x^* \in S$. For given $\epsilon > 0$ put

$$\overline{\epsilon} = \frac{\alpha\epsilon}{2(1+\alpha)}.$$

Let $U_E = \{e^* \in E : \|e^*\| = 1\}$ which is a norm closed and bounded set. Since E is a finite-dimensional subspace and in such spaces closed and bounded sets are compact then U_E is norm compact. This means that there exists $N \in \mathbb{N}$ such that U_E is covered by the union of the nighborhoods of $y_1, y_2 \dots, y_N$. So, for each $e^* \in U_e$ we have $\|y_k^* - e^*\| < \overline{\epsilon}, \quad$ for some $\quad k = 1, \dots, N$. Look at the set

$$W(0; y_1^*, \dots, y_N^*; \overline{\epsilon}) = \{x \in X : y_k^*(x) \leq 1 - \overline{\epsilon},\ k = 1, \dots, N\},$$

it is a weak neighbourhood of 0. Since $0 \in \overline{B_X}^{\|\ \|}$,

$$W \cap B_X = \{x \in X :\ y_k^*(x) \leq 1 - \overline{\epsilon},\ k = 1, \dots, N\} \cap B_X \neq \emptyset.$$

Since the sets do not coincide then $W^C \cap B_X \neq \emptyset$ which means

$$W \cap B_X = \{x \in X :\ y_k^*(x) > 1 - \overline{\epsilon},\ k = 1, \dots, N\} \cap B_X \neq \emptyset.$$

So we can pick $x_k \in B_X$ such that $y_k^*(x_k) > 1 - \overline{\epsilon}$. Since $0 \in \overline{S}^{weak*}$ each neighbourhood of 0 in the weak* topology of X^* and in particular $U(0; x_1, \dots, x_N; \overline{\epsilon}) = \{x^* \in X^* : |x^*(x_k)| < \overline{\epsilon},\ k = 1, \dots, N\}$ contains one point of S distinct from 0 i.e.

$$\{x^* \in X^* :\ |x^*(x_k)| < \overline{\epsilon},\ k = 1, \dots, N\} \cap S \neq \emptyset$$

So there exists $x^* \in S$ such that $|x^*(x_k)| < \overline{\epsilon}$ for each $k = 1, \dots, N$. If $e^* \in U_E$ and $|\lambda| \geq \frac{2}{\alpha}$, from the reverse triangle inequality for norms, we have

$$\begin{aligned}
\|e^* + \lambda x^*\| &= \|\lambda x^* - (-e^*\|) \\
&\geq \|\lambda x^*\| - \| - e*\| \\
&= |\lambda|\|x\| - \|e^*\| \\
&\geq \frac{2}{\alpha} \cdot \alpha - 1 \\
&= 1.
\end{aligned}$$

If $|\lambda| < \frac{2}{\alpha}$ we pick y_k such that $\|e^* - y_k\| < \bar{\epsilon}$. The fact that $x_k \in B_X$ allows us to write

$$\begin{aligned}
\|y_k^* + \lambda x^*\| &\geq y_k^*(x_k) + \lambda x^*(x_k) \\
&\geq (1 - \bar{\epsilon}) + \lambda x^*(x_k) \\
&\geq (1 - \bar{\epsilon}) - |\lambda|\bar{\epsilon} \\
&\geq (1 - \bar{\epsilon}) - \frac{2}{\alpha}\bar{\epsilon} \\
&= \big(1 - (1 + \frac{2}{\alpha})\bar{\epsilon}\big),
\end{aligned}$$

and therefore

$$\begin{aligned}
\|e^* + \lambda x^*\| &\geq \Big| \|e^* - y_k^*\| - \|y_k^* + \lambda x^*\| \Big| \\
&\geq \Big| \|y_k^* + \lambda x^*\| - \|e^* - y_k^*\| \Big| \\
&\geq \big(1 - (2 + \frac{2}{\alpha})\bar{\epsilon}\big) - \bar{\epsilon} \\
&= 1 - \big((1 + \frac{2}{\alpha})\bar{\epsilon} + \bar{\epsilon}\big) \\
&= 1 - \frac{2(\alpha + 1)\bar{\epsilon}}{\alpha} \\
&= 1 - \epsilon.
\end{aligned}$$

Now we have proved (3.7) for $e^* \in U_e$. Using the homogenity of the norm it is easy to conclude that (3.7) holds for all $e^* \in E$. $\square$

Now we will show a method for constructing basic sequences in some certain sets of X^*.

Theorem 3.7. *Suppose S is a subset of X^* such that $0 \in \overline{S}^{weak*}$ but $0 \notin \overline{S}^{\|\ \|}$. Then for any $\epsilon > 0$, S contains a basic sequence with basis constant less than $1 + \epsilon$.*

Proof. Fix a decreasing sequence of positive numbers $(\epsilon_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} < \infty$ and so that

$$\prod_{n=1}^{\infty}(1-\epsilon_n) > \frac{1}{(1+\epsilon)}.$$

Pick x_1^* and consider the 1-dimensional space $E = [x_1^*]$. By Lemma 3.6 there exists $x_2^* \in S$ such that

$$\|e^* + \lambda x_2^*\| \geq (1-\epsilon_1)\|e^*\|$$

for all $e^* \in E$ and $\lambda \in \mathbb{R}$. Now let $E_2 = [x_1^*, x_2^*]$. Lemma 3.6 guarantees the existence of $x_3^* \in S$ such that

$$\|e^* + \lambda x_3^*\| \geq (1-\epsilon_2)\|e^*\|.$$

Repeating the process we produce a sequence $(x_n^*)_{n=1}^{\infty}$ in S such that for each $n \in \mathbb{N}$ and any scalars (a_k)

$$\sum_{k=1}^{n+1} a_k x_k^*\| = \|\underbrace{\sum_{k=1}^{n} a_k x_k^*}_{\in[x_1^*,\ldots,x_n^*]} + a_{n+1}x_{n+1}^*\| \geq (1-\epsilon_n)\|\sum_{k=1}^{n} a_k x_k^*\|.$$

Therefore for any two given integers m, n with $m \leq n$, we have

$$\begin{aligned}
\|\sum_{k=1}^{n} a_k x_k^* &= \|\sum_{k=1}^{n-1} a_k x_k^* + a_n x_n^*\| \geq (1-\epsilon_n)\|\sum_{k=1}^{n-1} a_k x_k^*\| \\
&= (1-\epsilon_n)\|\sum_{k=1}^{n-2} a_k x_k^* + a_{n-1} x_{n-1}^*\| \\
&\geq (1-\epsilon_n)(1-\epsilon_{n-1})\|\sum_{k=1}^{n-2} a_k x_k^*\| \\
&\geq (1-\epsilon_n)(1-\epsilon_{n-1}), \ldots, (1-\epsilon_{m+1})\|\sum_{k=1}^{m} a_k x_k^*\| \\
&= \prod_{j=1}^{m+1}(1-\epsilon_j)\|\sum_{k=1}^{n} a_k x_k^*\|,
\end{aligned}$$

which implies

$$\begin{aligned}
\|\sum_{k=1}^{m} a_k x_k^*\| &\leq \frac{1}{\prod\limits_{j=1}^{m+1}(1-\epsilon_j)}\|\sum_{k=1}^{n} a_k x_k^*\| \\
&\leq \frac{1}{\prod\limits_{j=1}^{\infty}(1-\epsilon_j)}\|\sum_{k=1}^{n} a_k x_k^*\| \\
&< (1+\epsilon)\|\sum_{k=1}^{n} a_k x_k^*\|.
\end{aligned}$$

Proposition 3.5 tells us that $(x_n^*)_{n=1}^{\infty}$ is a basic sequence with basis constant at most $1+\epsilon$. □

The following corollary guarantees the existence of basic sequences in infinite-dimensional Banach spaces.

Corollary 3.8. *Every infinite-dimensional Banach space contains, for $\epsilon > 0$, a basic sequence with basis constant less than $1+\epsilon$.*

Proof. During the proof of the proposition 2.9 we proved that if we look at $S = S_X = \{x \in X : \|x\| = 1\}$ then $0 \in \overline{S_X}^{weak}$ and $0 \notin \overline{S_X}^{\|\,\|}$. Thus

$0 \in \overline{S}^{weak*} \subset X^{**}$ and $0 \notin \overline{S}^{\| \|}$. To complete the proof we apply the Theorem 3.7. □

Proposition 3.9. *If $(x_n)_{n=1}^{\infty}$ is a weakly null sequence in an infinite-dimensional Banach space X such that $\inf_n \|x_n\| > 0$ then, for $\epsilon > 0$, $(x_n)_{n=1}^{\infty}$ contains a basic subsequence with basis constant less than $1 + \epsilon$.*

Proof. Look at $S = \{x_n : n \in \mathbb{N}\}$. Let's prove that $0 \notin \overline{S}^{\| \|}$. If we assume the opposite, that $0 \in \overline{S}^{\| \|}$ then $\forall \epsilon > 0$ the norm open neighbourhood of 0. $V(\epsilon) = \{x \in X : \|x\|\}$ would contain at least one element from S i.e., $\forall \epsilon > 0$, $\exists n \in \mathbb{N}$ such that $\|x_n\| < \epsilon$. Thus we can pick a subsequence $(x_{nk})_{k=1}^{\infty} \subset (x_n)_{n=1}^{\infty}$ such that $\|x_{nk}\| < \epsilon, \forall \epsilon > 0$ so

$$\lim_{k \to \infty} \|x_{nk}\| = 0.$$

On the other hand

$$\lim_{k \to \infty} \|x_{nk}\| \geq \inf_n \|x_{nk}\| > 0$$

thus $\lim_{k \to \infty} \|x_{nk}\| > 0$, which is a contradiction to **??** and therefore $0 \notin \overline{S}^{\| \|}$. We know that the weak neighbourhoods of 0 are of the form

$$W(0; x_1^*, \dots, x_k^*) = \{x \in X : |x_i^*(x)| < \epsilon,\ i = 1, \dots, k \quad \text{and} \quad \epsilon > 0\}.$$

Since (x_n) converges weakly to 0 we can write

$$x^*(x_n) \to 0 \quad \text{for all} \quad x^* \in X^*.$$

Therefore $\forall \epsilon > 0$, $\exists k_0 \in \mathbb{N}$ such that $\forall n \geq k_0$ and $x^* \in X^*$, we have

$$|x^*(x_n)| < \epsilon,$$

which means that $x_n \in W(0; x_1^*, \dots, x_k^*; \epsilon)$ whenever $n \geq k_0$. Thus every norm open neighbourhood of 0 contains points from S thus $0 \in \overline{S}^{weak}$. We are now in the conditions of the Theorem 3.7 which assures the existence of a basic sequence contained in S with basis constant at most $1+\epsilon$. To finish the proof, since the basic sequence is a subset of (x_n), we extract the terms of basic sequence in the increasing order and we obtain a basic subsequence of $(x_n)_{n=1}^{\infty}$. □

Lemma 3.10. *Let $(x_n)_{n=1}^{\infty}$ a basic sequence in X. Suppose there exists a linear functional $x^* \in X^*$ such that $x^*(x_n) = 1$ for all $n \in \mathbb{N}$. If $u \notin [x_n]$ then the sequence $(x_n + u)_{n=1}^{\infty}$ is basic.*

Proof. Since $u \notin [x_n]$, without loss of generality we can assume $x^*(u) = 0$. Let $T : X \to X$ be the operator given by $T(x) = x^*(x)u$. T is linear: Take the scalars α, β and $x, y \in X$, then we can write

$$\begin{aligned} T(\alpha x + \beta y) &= x^*(\alpha x + \beta y)u \\ &= \big(x^*(\alpha x) + x^*(\beta y)\big)u \\ &= \big(\alpha x^*(x) + \beta x^*(y)\big)u \\ &= \alpha x^*(x)u + \beta x^*(x)u \\ &= \alpha T(x) + \beta T(y). \end{aligned}$$

Since $x^* \in X^*$ is linear and bounded we can write

$$\begin{aligned} \|Tx\| &= \|x^*(x)u\| \\ &= |x^*(x)|\|u\| \\ &\leq \|x^*\|\|u\|\|x\| \\ &\leq K\|x\|, \end{aligned}$$

which means that T is norm continuous. Let i_x be the identic function on X. Since i_x and T are linear and continuous then $i_x + T$ is linear and continuous. Observe that

$$
\begin{aligned}
(i_x + T)(x_n) &= i_x(x_n) + T(x_n) \\
&= x_n + x^*(x_n)u \\
&= x_n + u.
\end{aligned}
$$

Let's prove that $(x_n + u)_{n=1}^{\infty}$ is a basic sequence. Pick $x \in [x_n + u]$, then there exists $n \in \mathbb{N}$, scalars $(a_i)_{i=1}^{n}$ and $(x_i)_{i=1}^{n}$ such that

$$
\begin{aligned}
x &= \sum_{i=1}^{n} a_i(x_i + u) \\
&= \sum_{i=1}^{n} a_i(i_x + T)(x_i) \\
&= (i_x + T)\sum_{i=1}^{n} a_i x_i.
\end{aligned}
$$

Since $\sum_{i=1}^{n} a_i x_i \in [x_n]$ and $(x_n)_{n=1}^{\infty}$ is a basic sequence, there exists a unique sequence of nonzero scalars $(b_n)_{n=1}^{\infty}$ such that $\sum_{i=1}^{n} a_i x_i = \sum_{n=1}^{\infty} b_n x_n$ and we can write

$$
\begin{aligned}
x &= (i_x + T)\sum_{i=1}^{n} a_i x_i \\
&= (i_x + T)\sum_{n=1}^{\infty} b_n x_n \\
&= \sum_{n=1}^{\infty} b_n (i_x + T)x_n \\
&= \sum_{n=1}^{\infty} b_n (x_n + u).
\end{aligned}
$$

So we proved that if $x \in [x_n + u]$ then there exists a unique sequence of nonzero scalars $(b_n)_{n=1}^{\infty}$ such that $x = \sum_{n=1}^{\infty} b_n(x_n + u)$ therefore $(x_n + u)_{n=1}^{\infty}$ is a basic sequence. $\square$

Now we are going to present a simple and useful strategy which we will apply in the next Theorem as well as in the proof of the Eberlein-Šmulian Theorem. The result is quite easy but we note it for further reference.

Remark 2. *If A is a bounded set in X such that $\overline{A}^{weak*} \subset X$ then A is relatively weakly compact.*

Proof. Take $x^{**} \in \overline{A}^{weak*}$, then there exists a net $(x_\alpha) \subset A$ with

$$(x_\alpha) \xrightarrow{weak^*} x^{**},$$

which implies $\|x^{**}\| \leq \liminf_\alpha \|x_\alpha\| < \infty$. Thus $\overline{A}^{weak*}$ is bounded. By Corollary 2.17 since $\overline{A}^{weak*}$ is weak* closed and bounded then it is weak* compact.
If we consider $\overline{A}^{weak*}$ then every element of $\overline{A}^{weak*}$ is in X. Thus $\overline{A}^{weak*}$ is just $\overline{A}^{weak}$ and therefore $\overline{A}^{weak}$ is weakly compact ,i.e. ,A is relatively weakly compact.

□

Theorem 3.11. *Let S be a bounded subset of a Banach space X such that $0 \notin S^{\|\,\|}$. Then the following are equivalent*

(i) *S fails to contain a basic sequence*

(ii) *$\overline{S}^{weak}$ is weakly compact and fails to contain 0.*

Proof. <u>*(ii) ⇒ (i)*</u>. Suppose $(x_n)_{n=1}^{\infty} \subset S$ is a basic sequence. Since $\overline{S}^{weak}$ is weakly compact the sequence $(x_n)_{n=1}^{\infty}$ has a has a subnet which converges to $x \in \overline{S}^{weak}$ i.e., x is a weak cluster point of $(x_n)_{n=1}^{\infty}$. We can write

$$(x_n)_{n=1}^{\infty} \subset [x_n],$$

so we will have

$$x \in \overline{(x_n)}^{weak} \Rightarrow x \in [x_n].$$

Since $[x_n]$ is a convex set, by Mazur's Theorem $\overline{[x_n]}^{weak} = \overline{[x_n]}^{\|\ \|}$ therefore we can write

$$x = \sum_{n=1}^{\infty} x_n^*(x) x_n.$$

Let B be an index set, since x is a weak cluster point of $(x_m)_{m=1}^{\infty}$ so there exists a net $(x_\beta)_{\beta \in B} \subset (x_m)_{m \in \mathbb{N}}$ such that $x_\beta \to x$, x_n^* are continuous so they preserve limits and we can write

$$x_n^*(x_\beta) \to x_n^*(x).$$

Since $x_n^*(x_\beta) \subset \big(x_n^*(x_m)\big)_{m=1}^{\infty}$ then we have that $x_n^*(x)$ is a weak cluster point of the sequence of scalars $\big(x_n^*(x_m)\big)_{m=1}^{\infty}$ which converges to 0. Thus $x_n^*(x) = 0$, $\forall n \in \mathbb{N}$, as a consequence

$$x = \sum_{n=1}^{\infty} x_n^*(x) x_n = \sum_{n=1}^{\infty} 0 \cdot x_n = 0.$$

This contradicts hypothesis that $\overline{S}^{weak}$ does not contain 0, so S contains no basic sequences.

<u>*(i)* $\Rightarrow$ *(ii)*</u>. Assume S contains no basic sequence. Applying Theorem 3.7 to S considered as a subset of X^{**} equipped with the weak* topology we conclude that 0 cannot be a weak closure point of S. It remains to show that $\overline{S}^{weak}$ is weakly compact i.e. that S is relatively weakly compact. To achieve this we it is sufficient to show that all weak* cluster points of S in X^{**} are already contained in X. Let us suppose x^{**} is a weak* cluster point of S and that $x^{**} \in X^{**} \backslash X$. Consider the set $A = S - x^{**} = \{s - x^{**} : s \in S\}$ in X^{**}. By Theorem 3.7 there exists a sequence $(x_n)_{n=1}^{\infty}$ in S such that the sequence $(x_n - x)_{n=1}^{\infty}$ is basic. We can suppose that $x^{**} \notin [x_n - x^{**} \geq 1]$ because it is certainly true that $x^{**} \notin [x_n - x^{**} \geq N]$ for some choice of N. By the Hahn-Banach Theorem there exists $x^{***} \in X^{***}$ so that $x^{***} \in X^{\perp}$ and $x^{***}(x^{**}) = -1$. This implies that $x^{***}(x_n - x^{**}) = 1$ for all $n \in \mathbb{N}$. Now Lemma 3.10 applies and we deduce that $(x_n)_{n=1}^{\infty}$ is also basic which contradicts our assumption on S. $\square$

Lemma 3.12. *If $(x_n)_{n=1}^{\infty}$ is a basic sequence in a Banach space X and $x \in X$ is a weak cluster point of $(x_n)_{n=1}^{\infty}$ then $x = 0$.*

Proof. Since x is a weak cluster point of $(x_n)_{n=1}^{\infty}$ we can write

$$x \in \overline{\{x_n : n \in \mathbb{N}}^{weak} \Rightarrow x \in \overline{\langle x_n : n \in \mathbb{N}\rangle}^{weak},$$

where $\langle x_n : n \in \mathbb{N}\rangle$ is the linear span of the sequence $(x_n)_{n=1}^{\infty}$. The set $\langle x_n : n \in \mathbb{N}\rangle$ is convex so applying Mazur's Theorem we conclude that x belongs to the norm closed linear span $[x_n]$, of (x_n). Hence

$$x = \sum_{n=1}^{\infty} x_n^*(x) x_n,$$

where (x_n^*) are the biorthogonal functionals of (x_n). Let B be an index set. Since x is a weak cluster point of $(x_m)_{m=1}^{\infty}$ so there exists a net $(x_\beta)_{\beta \in B} \subset (x_m)_{m \in \mathbb{N}}$ such that $x_\beta \to x$, x_n^* are continuous so they preserve limits and we can write

$$x_n^*(x_\beta) \to x_n^*(x).$$

Since $x_n^*(x_\beta) \subset \left(x_n^*(x_m)\right)_{m=1}^{\infty}$ then we have that $x_n^*(x)$ is a weak cluster point of the sequence of scalars $\left(x_n^*(x_m)\right)_{m=1}^{\infty}$ which converges to 0. Thus $x_n^*(x) = 0$, $\forall n \in \mathbb{N}$, as a consequence

$$x = \sum_{n=1}^{\infty} x_n^*(x) x_n = \sum_{n=1}^{\infty} 0 \cdot x_n = 0.$$

□

Lemma 3.13. *Let A be a relatively weakly countably compact subset of a Banach space X. Suppose that $x \in X$ is the only weak cluster point of the sequence $(x_n)_{n=1}^{\infty} \subset A$. Then $(x_n)_{n=1}^{\infty}$ converges weakly to x.*

Proof. Assume the opposite, that (x_n) does not converge weakly to x. Then $\exists x^* \in X^*$ such that $\left(x^*(x_n)\right)_{n=1}^{\infty}$ fails to converge to $x^*(x)$, hence we may pick a subsequence $(x_{nk})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that

$$\inf_k |x^*(x) - x^*(x_{nk})| > 0,$$

which means that x cannot be a weak cluster point of (x_{nk}), contradicting the hypothesis. □

Now we are ready to give the proof of the Eberlein-Šmulian theorem given by Pelczynski in 1964.

Theorem 3.14 (Eberlein-Šmulian). *Let A be a subset of a Banach space X. The following are equivalent:*

(i) *A is [relatively] weakly compact,*

(ii) *A is [relatively] weakly sequentially compact,*

(iii) *A is [relatively] weakly countably compact.*

Proof. Since every sequence can be considered as a special case of a net (i) and (ii) both imply (iii). We have to show that (iii) implies both (ii) and (i). First we will prove the relativized versions and then show that the result for nonrelativized versions can follow easily. Note that each of the statements implies thast A is bounded.
$\underline{(iii) \Rightarrow (ii)}$. Let $(x_n)_{n=1}^{\infty}$ be any sequence in A. Then, by hypothesis, there is a weak cluster point x of $(x_n)_{n=1}^{\infty}$. If x is in the norm closure of the set $\{x_n\}_{n=1}^{\infty}$, then there is a subsequence which converges and we are done. If not, applying Theorem 3.11, we construct a subsequence $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ so that $(y_n - x)_{n=1}^{\infty}$ is a basic sequence. Since $(y_n)_{n=1}^{\infty}$ is in A which is relatively weakly compact then it has a weak cluster point y. So, if B is an arbitray index set, there exists a net $(x_\beta)_{\beta \in B} \subset (y_n)_{n=1}^{\infty}$ with $x_\beta \to y$. We have

$$(x_\beta - x)_{\beta \in B} \subset (y_n - x)_{n=1}^{\infty} \quad \text{and} \quad (x_\beta - x) \to y - x$$

therefore $y - x$ is a weak cluster point of the basic sequence $(y_n - x)_{n=1}^{\infty}$. By Lemma 3.12 $y - x = 0$, therefore $y = x$. Thus x is the only weak cluster point of $(y_n)_{n=1}^{\infty}$. By Lemma 3.13 $(y_n)_{n=1}^{\infty}$ converges to x. So we proved that every sequence $(x_n)_{n=1}^{\infty} \subset A$ has a convergent subsequence $(y_n)_{n=1}^{\infty}$ and therefore A is realtively weakly sequentially compact.
$\underline{(iii) \Rightarrow (i)}$. Suppose the opposite, that A is not relatively weakly compact by Remark 2 then the weak* closure W of A is not contained in X. Thus

there exists $x^{**} \in W \backslash X$. Pick $x^* \in X^*$ so that $x^{**}(x^*) > 1$. Then consider the set $A_0 = \{x \in A : \ x^*(x) > 1\}$. The set is not relatively weakly compact since x^{**} is in its weak* closure. Theorem 3.11 gives us a basic sequence $(x_n)_{n=1}^\infty$ contained in A_0. Since $(x_n)_{n=1}^\infty \subset A_0 \subset A$ and A relatively countably compact then $(x_n)_{n=1}^\infty$ must have a weak cluster point x which by by Lemma 3.12 should be $x = 0$. This is a contradiction since, by construction, $x^*(x) \geq 1$. $\square$

4 Proof of the Eberlein Šmulian theorem due to Whitley

In this section we are going to present the proof of the Eberlein Šmulian theorem due to Whitley. This proof can be found in [9] and uses only basic theorems from the weak and weak* topologies, namely Alaoglu's theorem, Hahn-Banach theorem and Mazurs's theorem. Throughout this section we will also refer to [?]. We will start by showing some basic results and observations, then we will present the proof of the theorem.

Proposition 4.1. *If A is a countable subset of a normed space, then $[A]$ is separable.*

Proof. Since a dense subset of $\langle A\rangle$ is also a dense subset of $[A]$, it is enough to prove that $\langle A\rangle$ is separable. Let Q be the rationals. Let S be the subset of $\langle A\rangle$ consisting of all linear combinations of elements of A formed by using only scalar coefficients from Q. Let's prove that S is countable.
Take $x_1,\dots,x_n \in A$ and $\alpha_1,\dots,\alpha_n \in \mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for each j such that $j=1,\dots,n$ there is a sequence $(\alpha_{j,m})_{m=1}^{\infty} \subset \mathbb{Q}$ converging to α_j, and from the continuity of the vector space operations we have that

$$(\alpha_{1,m}x_1+\alpha_{2,m}x_2+\cdots+\alpha_{n,m}x_n) \xrightarrow[m\to\infty]{} (\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_nx_n).$$

Thus, the countable set S is dense in $\langle A\rangle$. $\square$

Lemma 4.2. *A continuous one-to-one map from a compact space to a Hausdorff space is a homeomorphism.*

Proof. Let X be a compact space, Y a Hausforff space, and f a one-to-one continuous function on X. Take a closed set A in X, since X is compact then A is compact as closed subset of a compact space and $f(A)$ is compact as a continuous image of a compact set. Since Y is Hausdorff and in such spaces compact sets are closed then $f(A)$ is closed. Thus f^{-1} is continuous and f is homeomorphism. □

Definition 17. *A set of continuous linear functionals defined on a Banach space X is total if the only vector in X mapped into 0 by each functional is the zero vector.*

Lemma 4.3. *If X is a separable Banach space then X^* contains a countable total set.*

Proof. Let $(x_n)_{n=1}^{\infty}$ be a sequence of nonzero vectors which is dense in X. Using the Hahn-Banach theorem, for each x_n pick $x_n^* \in X$ so that

$$x_n^*(x_n) = \|x_n\| \quad \text{and} \quad \|x_n^*\| = 1.$$

Suppose $x_n^*(x_n) = 0$. Since $\overline{(x_n)} = X$ then

$$\forall \epsilon > 0 \; \exists m \quad \text{such that} \quad \|x - x_m\| < \epsilon.$$

On the other hand

$$\begin{aligned} |x_n^*(x - x_m)| &= |x_n^*(x) - x_n^*(x_m)| \\ &= |x_n^*(x_m)| = \|x_m\| \\ &\leq \|x_n^*\| \cdot \|x - x_m\| \\ &< \epsilon. \end{aligned}$$

So we have $\|x_m\| < \epsilon$ and we can write

$$\|x\| = \|x - x_m + x_m\| \leq \|x - x_m\| + \|x_m\| < 2\epsilon.$$

So $\forall \epsilon > 0$ we have that $\|x\| < \epsilon \Rightarrow \|x\| = 0 \Leftrightarrow x = 0$. Thus $(x_n^*)_{n=1}^{\infty}$ is total. $\square$

Lemma 4.4. *Let X be a Banach space such that X^* contains a countable total set. Then the weak topology on a weakly compact subset of X is metrizable.*

Proof. Let $(x_n^*)_{n=1}^{\infty}$ be a countable total set of functionals each of norm one. Define a function $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |x_n^*(x-y)|.$$

First of all, the function d is well defined

$$\begin{aligned} d(x,y) &= \sum_{n=1}^{\infty} \frac{1}{2^n} |x_n^*(x-y)| \\ &\leq \sum_{n=1}^{\infty} \frac{1}{2^n} \|x_n^*\| \cdot \|x-y\| \\ &= \|x-y\| \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= \|x-y\| \\ &< \infty. \end{aligned}$$

Let's prove now that d is a metric.

(i) $d(x,y) = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} |x_n^*(x-y)| \geq 0$.

(ii) $d(x,y) = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} |x_n^*(x-y)| = 0 \Leftrightarrow |x_n^*(x-y)| = 0 \quad \forall n$. Since $(x_n^*)_{n=1}^{\infty}$ is total, then $x - y = 0 \Leftrightarrow x = y$.

(iii) $d(x,y) = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} |x_n^*(x-y)| = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} |x_n^*(y-x)| = d(y,x)$.

(iv) From triangle inequality we have $|x_n^*(x-y)| \leq |x_n^*(x-z)|+|x_n^*(z-y)|$, and

$$\begin{aligned} d(x,y) &= \sum_{n=1}^{\infty} \frac{1}{2^n}|x_n^*(x-y)| \\ &\leq \sum_{n=1}^{\infty} \frac{1}{2^n}|x_n^*(x-z)| + \sum_{n=1}^{\infty} \frac{1}{2^n}|x_n^*(z-y)| \\ &= d(x,z) + d(z,y). \end{aligned}$$

Let A be a weakly compact subset of X. Look at the identity $i_d : (A, weak) \to (A, d)$ which maps A with the weak topology one-to-one and onto A with the weak topology induced by d. Take $a \in A$ and an open ball $B(a,\epsilon) = \{x \in A : \; d(x,a) < \epsilon\}$ around a with radius $\epsilon > 0$ and we can write

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n}|x_n^*(x-y)| < \epsilon \Rightarrow |x_n^*(x-a)| < \epsilon \; \forall n,$$

and $i_d\{x \in X : \; |x_n^*(x-a)| < \epsilon\} \subset B(a,\epsilon)$. Thus i_d is continuous. The metric space (A,d) is Hausdorff. Applying Lemma 4 we conclude that i_d is a homeomorphism and this means that the weak and the metric topology are equivalent on A and therefore A is metrizable. □

Lemma 4.5. *Let X be a Banach space and let E be a finite dimensional subspace of X^*. Then there exists a finite set $E' \subset S_X$ such that $\forall x^* \in E$*

$$\frac{\|x^*\|}{2} \leq \max_{x \in E'} |x^*(x)|.$$

Proof. Look at $S_E = \{e^* \in E : \|e^*\| = 1\}$. This is a closed and bounded subset of the finite dimensional subspace E, so it is norm compact. Therefore we can find a $\frac{1}{4}$-net $N = \{x_1^*, \ldots, x_n^*\}$. From proposition 2.9 we know that $0 \in \overline{S_X}^{weak}$ which means that for the weak neighbourhoods $W(0, x_i^*, \frac{3}{4}), i = 1, \ldots, N$ of 0 we have $W(0, x_i^*, \frac{3}{4}) \cap S_X \neq \emptyset$ i.e., for each $i = 1, \ldots, n \; \exists x_i \in S_X$ such that $x_i^*(x_i) > \frac{3}{4}$. Then, whenever $x^* \in S_E$ we have

$$\begin{aligned} x^*(x_i) &= (x_i^* + x^* - x_i^*)(x_i) \\ &= x_i^*(x_i) + (x^* - x_i^*)(x_i) \\ &> \frac{3}{4} - \frac{1}{4} \\ &= \frac{1}{2}. \end{aligned}$$

Therefore we can write

$$\frac{\|x^*\|}{2} \leq \max\{|x^*(x_i)|, i = 1\ldots, n\}.$$

So if we take $E' = \{x_1, \ldots, x_n\}$ the proof is completed. □

Now we have everything we need to prove Eberlein-Šmulian Theorem.

Theorem 4.6 (**Eberlein-Šmulian**). *Let A be a subset of a Banach space X. The following are equivalent.*

(i) *A is [relatively] weakly compact*

(ii) *A is [relatively] weakly sequentially compact*

(iii) *A is [relatively] weakly countably compact.*

.

Proof. <u>*(i) ⇒ (ii)*</u>.
Let $(a_n)_{n=1}^{\infty}$ be a sequence of elements in A which is relatively weakly compact. Look at $[a_n]$ norm closed linear span of $(a_n)_{n=1}^{\infty}$. From Proposition 4.1 $[a_n]$ is separable. Since A is relatively weakly compact then $A \cup [a_n]$ is a relatively weakly compact subset of the separable space $[a_n]$. By Lemma 4.3, $[a_n]$ contains a countable total. Lemma 4.4 $A \cup [a_n]$ tells us that $A \cup [a_n]$ is metrizable. Since in metric spaces compactness and sequential compactness are equivalent and $A \cup [a_n]$ is relatively weakly compact we conclude that $A \cup [a_n]$ is relatively weakly sequentialy compact in the weak topology of $[a_n]$. Since $(a_n)_{n=1}^{\infty} \subset A \cup [a_n]$, $(a_n)_{n=1}^{\infty}$ contains a

subsequence converging to an element x in the weak topology of $[a_n]$ and thus converging to x in the weak topology of X.

$\underline{(ii) \Rightarrow (iii)}$
It is clear that every relatively weakly sequentially compact subset is relatively weakly countably compact since each sequence can be considered as a special case of a net.

$\underline{(iii) \Rightarrow (i)}$
First let's show that A is bounded. Since A is relatively weakly countably compact then for each continuous linear functional x^* the set $x^*(A)$ is a relatively weakly countably compact set of scalars and thus a bounded set of scalars, by the Uniform Boundedness Principle, A is bounded. According to Remark 2 to prove that the bounded set A is realtively weakly compact it is enough to prove that $\overline{A}^{weak*} \subset X$. Take $x^{**} \in \overline{A}^{weak*}$ and let $x_1^* \in S_{X^*}$. Since $x^{**} \in \overline{A}^{weak*}$ each neighbourhood of x^{**} contains an element of A. In particular, the weak* neighbourhood generated by $\epsilon = 1$ and x_1^* $W(x^{**}, x_1^*, \epsilon) = \{y^{**} \in X^{**} : \ |(y^{**} - x^{**})(x_1^*)| < 1\}$ contains a member a_1 of A. So we can write

$$|(x^{**} - a_1)(x_1^*)| < 1.$$

Consider the norm closed linear span $E_1 = [x^{**}, x^{**} - a_1]$ of x^{**} and $x^* - a_1$, this is finite dimensional subspace of X^{**}. Lemma 4.5 assures us the existence of the finite set $E_1{}' = \{x_2^*, x_3^*, \ldots, x_{n(2)}^*\}$ in S_{X^*}, such that for any $y^{**} \in E_1$,

$$\frac{\|y^{**}\|}{2} \leq \max\{|y^{**}(x_i^*)| : 1 \leq i \leq n(2).$$

Look at

$$W_2 = \{y^{**} \in X^{**} : |(y^{**} - x^{**})(x_i^*)| < \frac{1}{2}, \ 1 \leq i \leq n(2)\},$$

this is a weak* neighbourhood of x^{**} so it contains one element $a_2 \in A$. And we can write

$$|(x^{**} - a_2)(x_i^*)| < \frac{1}{2},\ 1 \le i \le n(2)\}.$$

Now consider $E_2 = [x^{**}, x^{**} - a_1, x^{**} - a_2]$ which is a finite dimensional subspace of X^{**}. Applying Lemma 4.5 again we are provided $E_2^{'} = \{x^*_{n(2)+1}, \ldots, x^*_{n(3)}\}$ such that for any $y^{**} \in E_2$

$$\frac{\|y^{**}\|}{2} \le \max\{|y^{**}(x_i^*)| : 1 \le i \le n(3).$$

Look at

$$W_3 = \{y^{**} \in X^{**} : |(y^{**} - x^{**})(x_i^*)| < \frac{1}{3},\ 1 \le i \le n(3)\},$$

this a weak* neighbourhood of x^{**} and therefore it contains one element $a_4 \in A$. We can write

$$|(x^{**} - a_2)(x_i^*)| < \frac{1}{3},\ 1 \le i \le n(3)\}.$$

Applying Lemma 4.5 on the finite dimensional subspace $E_3 = [x^{**}, x^{**} - a_1, x^{**} - a_2, x^{**} - a_3]$ we are provided $E_3^{'} = \{x^*_{n(3)+1}, \ldots, x^*_{n(4)}\} \subset S_{X^*}$ such that for any $y^{**} \in E_3$

$$\frac{\|y^{**}\|}{2} \le \max\{|y^{**}(x_i^*)| : 1 \le i \le n(4).$$

Considering the weak* neighbourhood of x^{**}

$$W_4 = \{y^{**} \in X^{**} : |(y^{**} - x^{**})(x_i^*)| < \frac{1}{4},\ 1 \le i \le n(4)\},$$

we find an element $a_4 \in A$ such that

$$|(x^{**} - a_4)(x_i^*)| < \frac{1}{4},\ 1 \le i \le n(4)\}.$$

Considering $E_4 = [x^{**}, x^{**} - a_1, x^{**} - a_2, x^{**} - a_3, x^{**} - a_4]$ we produce W_5 and thus a_5 and so on.

Now look at the produced sequence $(a_i)_{i=1}^{\infty} \subset A$. Since A is relatively weakly countably compact then is has a weak cluster point $x \in X$, i.e.,

$x \in \overline{(a_n)}^{weak}$, by Mazur's Theorem $x \in [a_n]$ so $x^{**} - x = [x^{**}, x^{**} - a_1, x^{**} - a_2, \ldots, x^{**} - a_n, \ldots,] = C$. By construction

$$\frac{\|y^{**}\|}{2} \leq \sup_m |y^{**}(x_m^*)|,$$

for any $y^{**} \in C$. Since $x^{**} - x \in C$ then we can write

$$\frac{\|x^{**} - x\|}{2} \leq \sup_m |(x^{**} - x)(x_m^*)|.$$

Since x is a weak cluster point of $(a_i)_{i=1}^{\infty}$, there exists $(a_{ik})_{k=1}^{\infty} \subset (a_i)_{i=1}^{\infty}$ so that $a_{ik} \xrightarrow{weakly} x$ i.e.,

$$x_m^*(a_i)_k \to x_m^*(x) \quad m \in \mathbb{N}.$$

Thus, $\forall \epsilon > 0$ we can find a natural number $p(m)$ so that $n_k > p$ implies

$$|x_m^*(a_{ik} - x)| < \frac{\epsilon}{4}.$$

Now, if necessary, during construction process we increase p such that $n_k > p$ implies also

$$|(x^{**} - a_{n_k})(x_m^*)| < \frac{\epsilon}{4}.$$

By triangle inequality

$$\begin{aligned} |(x^{**} - x)(x_m^*)| &= |(x^{**} - a_{n_k})(x_m^*)| + |x_m^*(a_{n_k} - x)| \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} \\ &= \frac{\epsilon}{2}. \end{aligned}$$

Thus $\forall \epsilon > 0$ we can write $\|x^{**} - x\| < \epsilon$ which implies $\|x^{**} - x\| = 0$ and therefore $x^{**} = x$. Proof is completed.

□

5 The weak topology and the topology of pointwise convergence on $C(K)$

In this section we will show the connection between the weak topology and the topology of pointwise convergence in $C(K)$ for K compact Hausdorff. Our main source will be [4]. We denote the weak topology by τ_w and the topology of pointwise convergence by τ_p.
The weak topology and weak convergence on $C(K)$ are defined according to definitions 6 and 7 respectively.
Let us define first the concept of pointwise convergence for a net $(f_\alpha)_{\alpha \in A} \subset C(K)$.

Definition 18. *We say that the net $(f_\alpha)_{\alpha \in A} \subset \mathbb{C}(K)$ converges pointwise to $f \in C(K)$ if*

$$\lim_{\alpha} f_\alpha(x) = f(x) \quad , \forall x \in K.$$

Now we will define a topology τ_p on $C(K)$ such that convergence of nets in τ_p is pointwise and we will call it the topology of pointwise convergence. For each $f \in C(K)$ and $\epsilon > 0$ put

$$W(f, A, \epsilon) = \{g \in C(K) : |f(x) - g(x)| < \epsilon, \quad \forall x \in A\} \tag{5.1}$$

where $A \subset K$ is finite.

Proposition 5.1. *The topology τ_p defined on $C(K)$ such that each $f \in C(K)$ has the neighbourhood basis of the form (5.1) is the topology of pointwise convergence.*

Proof. Assume that the net (f_α) converges to $f \in C(K)$ in the τ_p topology. Then for every $W(f, A, \epsilon)$ there exists an α_0 such that $W(f, A, \epsilon)$ contains all f_α for $\alpha > \alpha_0$. In particular, if $x \in K$, and $\epsilon > 0$, there exists $\alpha_0 = \alpha_0(x, \epsilon)$ such that $\alpha > \alpha_0 \Rightarrow f_\alpha \in W(f, x, \epsilon)$. So, we can write

$$\alpha \geq \alpha_0 \quad \text{implies} \quad |f_\alpha(x) - f(x)| < \epsilon,$$

which means that $f_\alpha \to f$ pointwise.

Now assume that $f_\alpha \to f$ pointwise, let A be finite and $\epsilon > 0$. We should prove that $W(f, A, \epsilon)$ contains all f_α starting from some α_0. For each $x \in A$ there exists $\alpha_0 = \alpha_0(x)$ such that $\alpha > \alpha_0 \Rightarrow |f_\alpha(x) - f(x)| < \epsilon$. Let α_0 be the biggest of $\alpha_0(x)$'s. Then W contains f_α for all $\alpha \geq \alpha_0$. This means that the net f_α converges to f in the τ_p topology. □

Now that we know how τ_w and τ_p are constructed we may wonder which of them is stronger.

Proposition 5.2. *Topology of pointwise convergence is weaker than the weak topology i.e., $\tau_p \subset \tau_w$.*

Proof. Take a net $(f_\alpha)_{\alpha \in A} \subset C(K)$ which converges weakly to $f \in C(K)$ i.e.,

$$w(f_\alpha) \to w(f) \quad , \forall w \in C(K)^*.$$

Whenever $x \in K$ define $w_x : C(K) \to \mathbb{R}$ with

$$w_x(f) = f(x).$$

Then $w_x \in C(K)^*$ and we have

$$f_\alpha(x) \to f(x) \quad , \forall x \in K,$$

therefore f_α converges pointwise to f. □

The converse of the inclusion $\tau_p \subset \tau_w$ does not hold in general and we show it by the following example.

Example 3. *Take $K = [0,1]$ and define Λ to be the set of finite subsets of $[0,1]$ ordered by inclusion. Λ is an index set. For each $\lambda \in \Lambda$ define a function $f_\lambda : [0,1] \to [0,1]$ such that*

$$f_\lambda = \begin{cases} 1 & , x \in \lambda \\ 0 & , x \notin \lambda. \end{cases}$$

Take $x \in [0,1]$, we have $f_{\lambda_0}(x) = 1$ where $\lambda_0 = \{x\}$. Then for every $\lambda \geq \lambda_0$ we have $x \in \lambda$ and therefore $f_\lambda(x) = 1$. This means that the net f_λ converges to the constant function 1 pointwise. Define the function

$$w(f_\lambda) = \int_0^1 f_\lambda(t)dt.$$

For each $\lambda \in \Lambda$ we have $w(f_\lambda) = \int_0^1 f_\lambda(t)dt = 0$, since f_λ is zero almost everywhere in $[0,1]$. On the other hand $w(1) = \int_0^1 1 \cdot dt = 1$. Now we have

$$w(f_\lambda) = 0 \neq 1 = w(1),$$

Therefore f_λ doesn't converge weakly to 1.

When it comes to sequences, it turns out that it is much more easy to make weak convergence and pointwise convergence agree with each other. All we need is boundedness of the sequence and then Riesz representation theorem together with Lebesgue dominated convergence theorem. Here we present the form of the Riesz representation theorem that we will use.

Theorem 5.3. *Let K be a compact Hausdorff space and let $w \in C(K)^*$. Then there exists a unique Radon measure μ_w on K such that*

$$w(f) = \int_K f d\mu_w \quad , \forall f \in C(K).$$

This result is really important because it tells us how members of $C(K)^*$ work on $C(K)$.

Proposition 5.4. *Let* $(f_n)_{n\in\mathbb{N}} \subset C(K)$ *then the following are equivalent*

(i) f_n *converges weakly to* f

(ii) f_n *converges pointwise to* f *and* $\sup_{n\in\mathbb{N}} \|f_n\| < \infty$

Proof. $\underline{(1) \Rightarrow (ii)}$
If f_n converges weakly to f, from the previous proposition it follows that f_n converges to f pointwise. From pointwise convergence we can write

$$\sup_{n\in\mathbb{N}} |f_n(x)| < \infty \quad ,\forall x \in K.$$

Now, applying the Uniform Boundedness Principle we have

$$\sup_{n\in\mathbb{N}} \|f_n\| < \infty.$$

$\underline{(ii) \Rightarrow (i)}$ Assume the condition in (ii) holds. Let $w \in C(K)^*$, then by Riesz representation theorem, there exists a unique Radon measure μ_w on K such that

$$w(f) = \int\limits_K f d\mu_w \quad ,\forall f \in C(K).$$

The functions f_n are dominated by the constant function $l : K \to \mathbb{R}$ with $l(x) = \sup_{n\in\mathbb{N}} \|f_n\|$ i.e.,

$$|f_n(x)| \leq l(x) \quad ,\forall x \in K.$$

Now applying Lebesgue dominated convergence theorem we get

$$w(f_n) = \int_K f_n d\mu_w \to \int_f d\mu_w = w(f) \quad , w \in C(K)^*,$$

and therefore f_n converges to f weakly.

$\square$

After looking into the previous proposition we might have a natural question: Why doesn't the same result apply to nets? The answer is

simple. Because Lebesgue dominated convergence theorem does not work for nets.
In the following example we will show that weak convergence and pointwise convergence are equivalent only for bounded sequences lying in $C(K)$ i.e., we cannot relax the boundedness criteria.

Example 4. *Take $K = [0,1]$ and the sequence of continuous functions*

$$f_n(x) = n^2 x e^{-nx} \quad n \in \mathbb{N}.$$

Using l'Hopital's rule we have

$$\begin{aligned}
\lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} n^2 x e^{-nx} \\
&= \lim_{n\to\infty} \frac{n^2}{e^{nx}} \\
&= \lim_{n\to\infty} \frac{2nx}{xe^{nx}} \\
&= 2 \lim_{n\to\infty} \frac{1}{e^{nx}} \\
&= 0,
\end{aligned}$$

which means that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges pointwise to $f = 0$. Define on K the norm $\|\cdot\|$ by

$$\begin{aligned}
\|f_n\| &= \lim_{x\to 0} f_n(x) \\
&= \lim_{x\to 0} \frac{n^2 x}{e^{nx}} \\
&= \lim_{x\to 0} \frac{n^2}{ne^{nx}} \\
&= \lim_{x\to 0} \frac{n}{e^{nx}} \\
&= n \lim_{x\to 0} \frac{1}{e^{nx}} \\
&= n.
\end{aligned}$$

Therefore

$$\sup_{n\in\mathbb{N}} \|f_n\| \geq \lim_{n\to\infty} \|f_n\| \\ = \infty$$

so f_n is not norm-bounded. Let's check now the weak convergence. Define $w : C(K) \to \mathbb{R}$ by

$$w(f) = \int_0^1 f(x)dx.$$

We have

$$w(f_n) = \int_0^1 n^2 x e^{-nx} dx = 1 - \frac{1}{e^n} - \frac{n}{e^n},$$

and so

$$\lim_{n\to\infty} w(f_n) = 1 \neq 0 = w(f).$$

Hence, f_n does not converge weakly to f.

6 A generalization of the Ebrlein-Šmulian theorem

Finally, in this section we are going to give the proof of a generalization of the Ebelein-Šmulian theorem. The sources which are used are [7] and [8]. We will need two important results before preceding with the proof.

Proposition 6.1. *Let X be a normed space. Then there is a compact Huasdorff space K such that X is isometrically isomorphic to a subspace of $C(K)$. If X is a Banach space, then X is isometrically isomorphic to a closed subspace of $C(K)$.*

Proof. Let $K = B_{X^*}$ equipped with the weak* topology of X^*. Define $T : X \to C(K)$ by the formula $\big(T(x)\big)(x^*) = x^*(x)$. T takes values in $C(K)$ and it is linear. We also can write

$$\|x\| = \sup\{|x^*x| : x^* \in B_{X^*}\} = \sup\{|\big(T(x)\big)(x^*)| : x^* \in K\} = \|Tx\|_\infty,$$

so T is an isometric isomorphism from X into $C(K)$. If X is a Banach space, then so is $T(X)$, and therefore $T(X)$ is closed in $C(K)$. □

Lemma 6.2. *If R and S are Hausdorff spaces, R is regular and $f : R \to S$ is injective and continuous. Suppose $D \subset R$ is dense and $f(\overline{C})$ is compact for every subset C of D. Then R is compact and f is a homeomorphism.*

Proof. Suppose that $U = \{U_i : i \in I\}$ is an open cover of R that has no finite subcover. Let $V = \{V_j : j \in J\}$ be any open cover of R such that for any V_j there exists U_i so that $\overline{V_j} \subseteq U_i$. Such a cover V exists. Suppose the contrary. Then for any open cover $V = \{V_j : j \in J\}$, $\exists\ V_j$ such that for any U_i we have $\overline{V_j} \not\subset U_i$ i.e., $\exists x \in \overline{V_j}$ with $x \in {U_i}^c, \forall i$ which is an contradiction. Since $\overline{D} = R$ then for any finite $R \subset J$ we have

$$D \setminus (\bigcup_{j\in A} \overline{V_j}) = D \cap (\bigcup_{j\in A} \overline{V_j})^C = D \cup (\bigcap_{j\in A} \overline{V_j}^C) \neq \emptyset,$$

which means that exists $r_A \in D \setminus (\bigcup_{j\in A} \overline{V_j}), \forall A \quad$ finite.
Let

$$E = \{r_A : A \subseteq J \quad \text{and A is finite}\}.$$

By construction E has no cluster point. Let s be a cluster point of $f(E)$. We know that $f(\overline{E})$ is compact. Choose $r \in \overline{E}$ such that $f(r) = s$. Since s is a cluster point of $f(E)$ then for all $\alpha \in J$

$$f(r) = s \in \overline{\{f(r_\lambda) : \alpha \leq \lambda\}}.$$

Since f is injective we have

$$r \in \overline{\{r_\lambda : \alpha \leq \lambda\}},$$

which means that r is a cluster point of the net E, this contradiction proves that R is compact. Since R is compact, S is Hausdorff and f is a continuous injection by Lemma 4 f is a homeomorphism. $\square$

Now we are ready to prove the generalized version of the Eberlein-Šmulian theorem:

Theorem 6.3 (Eberlein-Šmulian, Grothendieck). *Let $C(K)$ be the Banach algebra of continuous complex valued functions on the compact space K. Suppose that $T \subseteq C(K)$ is a norm bounded and relatively countably compact in the weak*. Then T is relatively weakly compact and every $f \in \overline{T}^{weak}$ is the weak limit of a sequence in T.*

Proof. Without loss of generality we can assume that T is a subset of the unit ball of $C(K)$. Take $R = \overline{T}^{weak}, D = T$, S equal to the pointwise closure of T and f the identity. If we prove that for any $C \subseteq T$ we have $(f\overline{C})$ is weakly compact then we can apply lemma above. Let h be any cluster point of $f(\overline{C})$ in the topology of pointwise convergence and let Y_0 be the finite dimensional subspace of the bounded complex valued functions on K spanned by $\{h, 1_K\}$. Choose a countable subset

$$\{k_{0,i} : i \in \mathbb{N}\}$$

of K that norms Y_0. Since h is a cluster point of C in the topology of pointwise convergence then there exists $x_1 \in C$ such that

$$|h(k_{0,i}) - x_1(k_{0,i})| < 1, \quad i \leq 1.$$

In general, let Y_{n-1} be the space spanned by all polynomials in

$$\{h, 1_K, x_1, \ldots, x_{n-1}, \overline{x_1}, \ldots, \overline{x_{n-1}}\}$$

of degree no more than $n-1$ and choose

$$\{k_{n-1,i} : i \in \mathbb{N}\} \subseteq K$$

that norms Y_{n-1}. Similarly we choose $x_n \in C$ such that

$$|h(k_{l,i}) - x_n(k_{l,i})| < \frac{1}{n}$$

for all $i \leq n$ and $l \leq n$. We have now constructed the sequence $\{x_n : n \in \mathbb{N}\} \subseteq T$. Since T is relatively countably compact in the topology of pointwise convergence then $\{x_n\}$ has a cluster point y. This means that any neighbourhood of y in the topology of pointwise convergence contains at least one element x_n for some $n \in \mathbb{N}$ .i.e., $\forall n \in \mathbb{N}$, $\exists x_n$ with

$$|x_n(k_{l,i}) - y(k_{l,i})| < \frac{1}{n} \qquad , i \leq n \quad \text{and} \quad l \leq n.$$

Now we can write

$$\begin{aligned} |h(k_{l,i}) - y(k_{l,i})| &\leq |h(k_{l,i}) - x_n(k_{l,i})| + |x_n(k_{l,i}) - y(k_{l,i})| \\ &< \frac{1}{n} + \frac{1}{n} \\ &= \frac{2}{n}, \end{aligned}$$

which implies that $h(k_{l,i}) = y(k_{l,i})$ for all l and i. The countable set $\{k_{l,i} : l, i \in \mathbb{N}\}$ norms the uniform closure of the algebra Y of bounded functions on K generated by $\{h, 1_K, x_1, \ldots, x_2, \ldots\}$. By the Stone-Weierstrass and the Gelfand-Naimark theorems we conclude that y is in Y. Thus

$$\sup_{k\in K} |h(k) - y(k)| = \sup_{k,l} |h(k_{l,i}) - y(k_{l,i})| = 0,$$

thus $h = y \in C(K)$ and therefore $f(\overline{C})$ is weakly compact. Now, applying the above Lemma we conclude that $\overline{T}^{weak} = R$ is weakly compact. Moreover, we have shown that h is continuous and that $\{x_n : n \in \mathbb{N}\}$ has only one continuous cluster point $h = y$. Since every subsequence of $\{x_n : n \in \mathbb{N}\}$ has a continuous cluster point y, then $\{x_n\}$ should converge pointwise on K to h. Since $\{x_n\} \subset T$ which is bounded, by Proposition 5.4 $\{x_n\}$ converges weakly to h. The proof is completed. □

Proposition 6.1 tells us that every Banach space X is isometrically isomorphic with $C(K)$ where $K = B_{X^*}$ equipped with the weak* topology. Let us argue that actually the above theorem is a generalization of the Eberlein-Šmulian Theorem.

First, let us prove that it generalizes $\underline{(CC) \Rightarrow (C)}$. Take a norm bounded set $A \in X$ which is relatively weakly countably compact. From Proposition 6.1 there exists a set $C \subseteq B_{X^*}$ such that $A = T(C) \subseteq C(B_{X^*})$ i.e., $A \subset C(B_{X^*})$. Since A is relatively weakly countably compact, this means that every sequence $(x_n) \subset A$ has a subnet (x_α) that converges weakly and therefore pointwise ,by Proposition 5.2. Thus $A \subseteq C(B_{X^*})$ is relatively countably compact in the topology of pointwise convergence. By Proposition 6.3 it follows that A is relatively weakly compact.

Now let us prove that Proposition 6.3 generalizes $\underline{(CC) \Rightarrow (SC)}$. Continuing the same reasoning as above, for the set $A \subset C(B_{X^*})$ bounded and relatively countably compact the Proposition 6.3 assures that for every $f \in \overline{A}^{weak}$ is the weak limit of a sequence in A. Take a sequence $(x_n) \subset A$, we want to prove that this sequence has a convergent subsequence (x_{nk}). For each fixed n there exists a sequence (y_{nk}) with $\lim_{k\to\infty} y_{nk} = x_n$. Now we can write

$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \lim_{n\to\infty} x_{nk}$. Which means that (x_n) has a convergent subsequence.

7 Some applications to Tauberian operator theory

In this section we will show some applications of Eberlein-Šmulian theorem in the theory of Tauberian operators as shown in [6].

Definition 19. *Let X and Y be Banach spaces. A function $T \in B(X,Y)$ is Tauberian if $T^{**-1}(Y) = X$ i.e. $g \in X^{**}$,$T^{**}g \in Y$ imply $g \in X$.*

It is immediate that a Tuberian operator has the property

(N): $g \in X^{**}$,$T^{**}g = 0$ imply $g \in X$.

If we denote the null space of T with NT, then as it is shown in Theorem 2 of [5] (N) implies

(R) : NT is reflexive.

Tauberian operators are, in a sense, opposite to weakly compact operators since $T \in B(X,Y)$ is weakly compact if and only if the range of T^{**}, $RT^{**} \subset Y$. Thus the set of the Tauberian operators lies in the complement of a closed subspace of $B(X,Y)$[6]. Firstly we will give a criterion for property (N).

Theorem 7.1. *Let X and Y be Banach spaces, $T \in B(X,Y)$. Then T has property (N) if and only if every bounded sequence $(x_n) \in X$ with $Tx_n \to 0$ has a weakly convergent subsequence.*

Proof. $(\Rightarrow)$. Let z be a weak* cluster point of $(x_n) \in X$. We have

$$T^{**}(x_n) = T(x_n) \to 0$$

in norm, thus $T^{**}(z) = 0$. By assumption we have $z \in X$ and therefore (x_n) is relatively weakly compact, hence by the Eberlein-Šmulian Theorem relatively weakly sequentially compact. This allows us to extract a convergent subsequence from (x_n) which converges weakly to some element $x \in X$.

($\Leftarrow$). Suppose $g \in X^{**}$ and $T^{**}g = 0$ we want to prove that $g \in X$. Without loss of generality we may assume that $\|g\| = 1$. By Goldstine's Theorem there exists a net $(x_\alpha)_{\alpha \in A}$ in B_X the unit ball of X, with $x_\alpha \to g$ weak*. Then

$$Tx_\alpha = T^{**}x_\alpha \to T^{**}g = 0,$$

which means that $Tx_\alpha \to 0$ weakly. Thus if C is any convex subset of X such that $x_\alpha \in C$ eventually i.e. 0 is in the weak closure and therefore, by Mazur's Theorem in the norm closure of TC. For each $\alpha \in A$ let C_α be the convex hull of

$$\{x_\beta : \beta \geq \alpha\}.$$

Each C_α contains a sequence (c_α^n) with $\|Tc_\alpha^n\| \to 0$. By assumption (c_α^n) has a subsequence converging weakly to some c_α, therefore $Tc_\alpha = 0$. Look at

$$\{c_\alpha : \alpha \in A\},$$

this is a relatively weakly sequentially compact set since for every sequence $(y_n) \subset c_\alpha$ we have $T(y_n) = 0$. By the Eberlein-Šmulian Theorem $\{c_\alpha : \alpha \in A\}$ is weakly compact. Thus c_α has a weak cluster point and therefore a weak* cluster point c. Since $c_\alpha \to g$ weak* and c is a weak* cluster point of c_α then $g = c \in X$. Which is what we wanted to prove. $\square$

During the proof of the next application we will need the following result

Theorem 7.2. *Let X and Y be Banach spaces and $T \in B(X,Y)$. The following are equivalent:*

(i) *T is Tauberian*

(ii) *T has property (N) and $T(B_X)$ is closed*

(iii) *T has the property (N) and the closure of $T(B_X)$ is included in the range of T.*

Since the main purpose of this section is to show the usefulness of Eberlein-Š Theorem we omit the proof which can be found in [5].

Theorem 7.3. *Let X, Y be Banach spaces and $T \in B(X,Y)$. The following are equivalent:*

(i) *T is Tauberian.*

(ii) *For every bounded set $B \subset X$ such that TB is relatively weakly compact, B is relatively weakly compact.*

(iii) *For every bounded set $B \subset X$ such that TB is relatively compact, B is relatively weakly compact.*

Proof. *(i) ⇒ (ii)*
Let T be Tauberian, $B \subset X$ bounded, TB relatively weakly compact. We want to prove that B is compact. Take a net $(x_\alpha)_{\alpha \in A}$ in B. We can see (x_α) as a bounded net in X^{**}. Thus it has a weak* convergent subnet which we may assume to be (x_α) itself. We can write

$$x_\alpha \xrightarrow{\text{weak*}} g, \quad \text{for some g} \in X^{**} \Rightarrow T^{**}x_\alpha \xrightarrow{\text{weak*}} T^{**}g.$$

Since TB is relatively weakly compact (x_α) has a convergent subnet, which we may assume again to be (x_α) itself such that

$$Tx_\alpha \xrightarrow{\text{weak}} y \quad \text{for some} \quad y \in Y.$$

Now we have

$$\begin{aligned} T^{**}g &= weak^* \lim T^{**} \\ &= weak^* \lim Tx_\alpha \\ &= y. \end{aligned}$$

Since T is Tauberian and $T^{**}g = y \in Y$ we conclude that $g \in X$. Thus B is relatively weakly compact.

$\underline{(ii) \Rightarrow (iii)}$

Assume that TB is relatively norm compact. This means that its norm closure is norm compact therefore weak compact. Hence TB is relatively weakly compact. Now (ii) assures us that B is relatively weakly compact.

$\underline{(iii) \Rightarrow (i)}$

According to Theorem [7.2], to prove that T is Tauberian it is enough to prove that $\overline{TB_X}$ is included in the range of T and that T has property (N). Take $y \in \overline{TB_X}$ and choose a sequence (x_n) in B_X with $Tx_n \to y$. Since the convergent sequence (Tx_n) can be seen as a compact set, by hypothesis (x_n) has a subsequence (x_{nk}) wich converges to some $x \in X$. Then, $x \in B_X$ and we have

$$(x_{nk}) \to x \Rightarrow T(x_{nk}) \to Tx.$$

Since $Tx_n \to y$ and $T(x_{nk}) \to Tx$, then $Tx = y$. Let's show now that property (N) holds. Take a bounded sequence in X with $Tx_n \to 0$ in Y. This means that the set $\{Tx_n : n \in \mathbb{N}\}$ is compact. By hypothesis $\{x_n : n \in \mathbb{N}\}$ is realtively weakly compact, hence by the Eberlein-Šmulian Theorem relatively weakly sequentially compact. Therefore it has a weakly convergent subsequence. According to Theorem [7.1] T has property (N). Proof is completed.

$\square$

Summary

In the first chapter we introduce the concept of compactness(C), countable compactness(CC) and sequential compactness(SC). In general topological spaces we have the relationship: $(C) \Rightarrow (CC) \Leftarrow (SC)$. It is a well known fact that in metric spaces the three types of equivalence are equivalent. It turns out that this equivalence holds for the weak topology of a Banach space too.

In the second chapter we present basis of the weak and the weak* topologies. We prove that the weak topology of an infinite dimensional Banach space is never metrizable. Some basic properties of these topologies are proved. In particular, we prove that for convex sets weak and norm closure agree, X is weak* dense in X^{**} and, B_{X^*} is weak* compact. These theorems, known respectively as Mazur's Theorem, Goldstine's Theorem and Banach-Alaouglu's Theorem help creating key arguments throughout all thesis.

In the third chapter we explore bases and Schauder bases of Banach spaces. We prove that these concepts are equivalent. Here is presented the concept of a basic sequence too. There are developed some tests which show whether a sequence of vectors $(e_n)_{n=1}^{\infty} \subset X$ is basis (basic sequence). We provide methods for constructing basis (basic sequences) on Banach spaces. It turns out that every Banach space contains a basic sequence. The use of basic sequences is a key ingredient in the approach that Pelchynski follows to prove the Eberlein-Šmulian Theorem.

In the following chapter we present the proof due to Whitley in 1967. This proof is simple and only requires some basic results from weak and weak* topologies.

In chapter five we investigate the connection between the weak topology and the topology of pointwise convergence in $C(K)$ for K compact Hausdorff. We show that a sequence of bounded functionals $(f_n) \in C(K)$ converges pointwise if and only if it converges weakly.

In chapter six a proof of the Eberlein-Šmulian Theorem for the weak topology of $C(K)$ for K compact Hausdorff is given. Since every Banach space is isometrically isomorphic with a closed subspace of $C(K)$ where $K = (B_{X^*}, weak^*)$, then this version of the theorem is a generalization of the others proved in the previous chapters.

In the last chapter we show some applications of the Eberlein-Šmulian Thorem to the theory of Tauberian operators.

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