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Abstract

We study oscillatory behavior of a class of second-order neutral differential equations relating oscillation of these equations to existence of positive solutions to associated first-order neutral differential

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inequalities. Our assumptions allow applications to differential equations with both delayed and advanced arguments, and not only. New theorems complement and improve a number of results reported in the literature. Two illustrative examples are provided.

Key words: Oscillation, neutral differential equations, delayed arguments; advanced arguments; positive solutions; comparison.

AMS Subject Classification: 34K11

1 Introduction

In this paper, we study the oscillation of a class of functional differential equations

$$(r(t)[x(t)+p(t)x(\tau(t))]')' + q(t)x(\sigma(t)) = 0 \tag{1}$$

in the case

$$\lim_{t\to\infty} R(t) < \infty, \tag{2}$$

where $R(t) = \int_{t_0}^{t} (r(s))^{-1}\, ds$. The increasing interest in oscillatory properties of solutions to second-order neutral differential equations is motivated by their applications in the engineering and natural sciences. We refer the reader to the papers [1]-[15] and the references cited therein.

We assume that the following assumptions are satisfied:

(H_1) r, p, $q \in \mathrm{C}([t_0,\infty))$, $r(t) > 0$, $0 \le p(t) \le p_0 < \infty$, $q(t) > 0$;

(H_2) $\sigma \in \mathrm{C}^1([t_0,\infty))$, $\lim_{t\to\infty} \sigma(t) = \infty$;

(H_3) $\tau \in \mathrm{C}^1([t_0,\infty))$, $\tau'(t) \ge \tau_0 > 0$, $\tau \circ \sigma = \sigma \circ \tau$.

Throughout the paper, we use the notation $z(t) := x(t) + p(t)x(\tau(t))$. By a solution of equation (1), we mean a function $x \in \mathrm{C}([T_x,\infty))$, $T_x \ge t_0$, such that $rz' \in \mathrm{C}^1([T_x,\infty))$ and x satisfies (1) on $[T_x,\infty)$. We consider only those solutions of (1) which satisfy $\sup\{|x(t)| : t \ge T\} > 0$ for all $T \ge T_x$ and assume that (1) possesses such solutions. A solution of (1) is called oscillatory if it does not have the largest zero on $[T_x,\infty)$, otherwise, it is called non-oscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

Very recently, Baculíková and Džurina [4] established several oscillation theorems for equation (1) via its comparison with associated first-order delay differential equations in the case when, in addition to (H_1)-(H_3), condition

$$\lim_{t\to\infty} R(t) = \infty \tag{3}$$

is satisfied. Assuming

$$\int_{t_0}^{\infty} r^{-1/\gamma}(t)\mathrm{d}t = \infty$$

instead of (3), Baculíková and Džurina [5] extended results of [4] to a nonlinear neutral differential equation

$$\left(r(t)\left[\left(x\left(t\right)+p\left(t\right)x\left(\tau\left(t\right)\right)\right)'\right]^{\gamma}\right)' + q(t)x^{\beta}(\sigma(t)) = 0,$$

where γ and β are ratios of positive odd integers. Using a similar idea, Baculíková et al. [6] studied the oscillation of an even-order neutral differential equation

$$\left(\left[\left(x\left(t\right)+p\left(t\right)x\left(\tau\left(t\right)\right)\right)^{(n-1)}\right]^{\gamma}\right)' + q(t)x^{\gamma}(\sigma(t)) = 0,$$

where $n \geq 2$ is an even number and $\gamma \geq 1$ is a ratio of two odd positive integers. Han et al. [10] employed Riccati transformation and integral averaging techniques to investigate the oscillation of a nonlinear differential equation

$$\left(r(t)\psi\left(x\left(t\right)\right)\left[\left(x\left(t\right)+p\left(t\right)x\left(\tau\left(t\right)\right)\right)'\right]^{\gamma}\right)' + q(t)f\left(x(\sigma(t))\right) = 0$$

in the case when (H_1)-(H_2) and $\int_{t_0}^{\infty} r^{-1/\gamma}(t)\mathrm{d}t < \infty$ hold, $\tau(t) \leq t$, $\sigma(t) \leq t$, $\sigma'(t) > 0$, and

$$\text{either} \qquad \sigma(t) \geq \tau(t), \qquad \text{or} \qquad \sigma(t) \leq \tau(t). \tag{4}$$

Li et al. [14] studied oscillation of equation (1) under the assumptions that (2) holds, τ and σ are strictly increasing, $p(t) > 1$, and (4) is satisfied. Li et al. analyzed oscillatory properties of equation (1) in the case where (H_1)-(H_3) hold, $\tau(t) \geq t$, and $\sigma(t) \geq t$. They derived a sufficient condition which ensures that solutions of equation (1) are either oscillatory or satisfy $\lim_{t\to\infty} x(t) = 0$ for the case where (2) holds and $0 \leq p(t) \leq p_1 < 1$ [15, Theorem 3.8]. Finally, we note that Hasanbulli and Rogovchenko [11] obtained new oscillation results for a nonlinear neutral differential equation

$$\left(r(t)[x(t) + p(t)x(t-\tau)]'\right)' + q(t)f(x(t), x(\sigma(t))) = 0$$

in the case where $0 \leq p(t) \leq 1$, $\sigma(t) \leq t$, $\sigma'(t) > 0$, and (3) holds.

We stress that results in [4, 5, 6, 11] cannot be applied to (1) in the case where (2) holds. The purpose of this paper is to extend a new method for the analysis of the oscillation of equation (1) via comparison principles suggested by Baculíková and Džurina [4] for the study of (1) under the assumption (2).

2 Main results

In what follows, all inequalities are assumed to hold eventually, that is, for all t large enough. We also use the notation $y(t) := -u(t) := r(t)z'(t)$, $Q(t) := \min\{q(t), q(\tau(t))\}$, and $\delta(t) := \int_t^\infty (r(s))^{-1} \mathrm{d}s$.

Theorem 1 *Assume that* (H_1)-(H_3) *and* (2) *hold, and let* t_1 *be large enough. Suppose that there exist two functions* $\eta, \xi \in \mathrm{C}([t_0, \infty))$ *such that* $\eta(t) \leq \sigma(t) \leq \xi(t)$ *and* $\lim_{t\to\infty} \eta(t) = \infty$. *If the first-order neutral differential inequalities*

$$\left(y(t) + \frac{p_0}{\tau_0} y(\tau(t)) \right)' + Q(t) \left(R(\eta(t)) - R(t_1) \right) y(\eta(t)) \leq 0 \tag{5}$$

and

$$\left(w(t) + \frac{p_0}{\tau_0} w(\tau(t)) \right)' - Q(t)\delta(\xi(t)) w(\xi(t)) \geq 0 \tag{6}$$

have no positive solutions, equation (1) *is oscillatory.*

Proof. Assume that (1) has a non-oscillatory solution $x(t)$ on $[t_0, \infty)$. Without loss of generality, we can suppose that $x(t)$ is eventually positive. As in the proof of [4, Theorem 1], one arrives at the inequality

$$(r(t)z'(t))' + \frac{p_0}{\tau_0} (r(\tau(t))z'(\tau(t)))' + Q(t)z(\sigma(t)) \leq 0. \tag{7}$$

Equation (1) yields that, for some t_1 large enough and for all $t \geq t_1$, either

$$z'(t) > 0, \quad (r(t)z'(t))' < 0, \tag{8}$$

or

$$z'(t) < 0, \quad (r(t)z'(t))' < 0. \tag{9}$$

Assume first that (8) holds. Inequality (7) and the fact that $\eta(t) \leq \sigma(t)$ yield

$$(r(t)z'(t))' + \frac{p_0}{\tau_0}(r(\tau(t))z'(\tau(t)))' + Q(t)z(\eta(t)) \leq 0.$$

It follows from condition (8) that

$$z(t) \geq \int_{t_1}^{t} \frac{r(s)z'(s)}{r(s)} \mathrm{d}s \geq r(t)z'(t) \int_{t_1}^{t} \frac{1}{r(s)} \mathrm{d}s,$$

for all sufficiently large t. Therefore, $y(t)$ is a positive solution of inequality (5), which contradicts our assumption that this inequality has no positive solutions.

Consider now the second case. It follows from (9) that

$$z'(s) \leq \frac{r(t)z'(t)}{r(s)} \qquad \text{for all } s \geq t,$$

which, upon integration, leads to

$$z(l) \leq z(t) + r(t)z'(t) \int_{t}^{l} \frac{1}{r(s)} \mathrm{d}s.$$

Passing to the limit as $l \to \infty$, we conclude that

$$z(t) + r(t)z'(t) \int_{t}^{\infty} \frac{1}{r(s)} \mathrm{d}s \geq 0.$$

Therefore,

$$z(t) \geq -r(t)z'(t)\delta(t). \tag{10}$$

It follows from (7) and condition $\sigma(t) \leq \xi(t)$ that

$$(r(t)z'(t))' + \frac{p_0}{\tau_0}(r(\tau(t))z'(\tau(t)))' + Q(t)z(\xi(t)) \leq 0. \tag{11}$$

Then, $y(t) < 0$ and, by virtue of (10) and (11),

$$\left(y(t) + \frac{p_0}{\tau_0} y(\tau(t))\right)' - Q(t)\delta(\xi(t))y(\xi(t)) \leq 0.$$

Writing the latter inequality in the form

$$-\left(y(t) + \frac{p_0}{\tau_0} y(\tau(t))\right)' - Q(t)\delta(\xi(t))\left(-y(\xi(t))\right) \geq 0,$$

we deduce that $u(t)$ is a positive solution of the inequality (6), which, according to our assumption, has no positive solutions. Therefore, equation (1) is oscillatory. ■

Under additional conditions on the coefficients of (1), one can deduce from Theorem 1 a number of oscillation criteria applicable to different classes of equations. In what follows, the notation τ^{-1} stands for the inverse of the function τ.

Theorem 2 *Assume that* (H_1)-(H_3) *and* (2) *hold, and let* t_1 *be large enough. Suppose that there exist two functions* $\eta, \xi \in C([t_0, \infty))$ *such that* $\eta(t) \leq \sigma(t) \leq \xi(t)$ *and* $\lim_{t\to\infty} \eta(t) = \infty$. *Assume also that*

$$\tau(t) \geq t. \tag{12}$$

If the first-order differential inequalities

$$h'(t) + \frac{\tau_0}{\tau_0 + p_0} Q(t) \left(R(\eta(t)) - R(t_1)\right) h(\eta(t)) \leq 0 \tag{13}$$

and

$$f'(t) - \frac{\tau_0}{\tau_0 + p_0} Q(t) \delta(\xi(t)) f(\tau^{-1}(\xi(t))) \geq 0 \tag{14}$$

have no positive solutions, equation (1) *is oscillatory.*

Proof. Suppose that (1) has a non-oscillatory solution $x(t)$ defined on $[t_0, \infty)$. As above, we can assume that $x(t)$ is eventually positive. Following the same lines as in [4, Theorem 2], we conclude that the inequality (13) has a positive solution, which contradiction our assumption. On the other hand, we have shown in Theorem 1 that the function $u(t)$ is positive, increasing, and satisfies (6). It follows from (12) that

$$w(t) \leq u(\tau(t)) \left(1 + \frac{p_0}{\tau_0}\right), \tag{15}$$

where

$$w(t) := u(t) + \frac{p_0}{\tau_0} u(\tau(t)). \tag{16}$$

Substituting (15) into (6), we conclude that $w(t)$ is a positive solution of (14), which contradicts the fact that this inequality does not have positive solutions. The proof is complete. ■

Combining Theorem 2 with the oscillation criteria presented in Ladde et al. [12, Theorems 2.1.1 and 2.4.1], we obtain the following result.

Corollary 3 *Assume that conditions* (H_1)-(H_3), (2) *and* (12) *are satisfied. Suppose further that there exist two functions* $\eta, \xi \in \mathrm{C}([t_0, \infty))$ *such that* $\eta(t) < t$, $\xi(t) > \tau(t)$, $\eta(t) \leq \sigma(t) \leq \xi(t)$, *and* $\lim_{t\to\infty} \eta(t) = \infty$. *If, for all sufficiently large* $t_1 \geq t_0$,

$$\liminf_{t\to\infty} \int_{\eta(t)}^{t} Q(s)\left(R(\eta(s)) - R(t_1)\right) \mathrm{d}s > \frac{\tau_0 + p_0}{\tau_0 \mathrm{e}} \tag{17}$$

and

$$\liminf_{t\to\infty} \int_{t}^{\tau^{-1}(\xi(t))} Q(s)\delta(\xi(s))\mathrm{d}s > \frac{\tau_0 + p_0}{\tau_0 \mathrm{e}}, \tag{18}$$

equation (1) *is oscillatory.*

Proof. By [12, Theorem 2.1.1], assumption (17) ensures that the differential inequality (13) has no positive solutions. On the other hand, by [12, Theorem 2.4.1], condition (18) guarantees that the differential inequality (14) has no positive solutions. Application of Theorem 2 yields the result. ∎

Theorem 4 *Assume that conditions* (H_1)-(H_3) *and* (2) *are satisfied, and let* t_1 *be large enough. Suppose that there exist two functions* $\eta, \xi \in \mathrm{C}([t_0, \infty))$ *such that* $\eta(t) \leq \sigma(t) \leq \xi(t)$ *and* $\lim_{t\to\infty} \eta(t) = \infty$. *Finally, assume that*

$$\tau(t) \leq t. \tag{19}$$

If the first-order differential inequalities

$$h'(t) + \frac{\tau_0}{\tau_0 + p_0} Q(t)\left(R(\eta(t)) - R(t_1)\right) h(\tau^{-1}(\eta(t))) \leq 0 \tag{20}$$

and

$$f'(t) - \frac{\tau_0}{\tau_0 + p_0} Q(t)\delta(\xi(t)) f(\xi(t)) \geq 0 \tag{21}$$

have no positive solutions, equation (1) *is oscillatory.*

Proof. Assume again that equation (1) has on $[t_0, \infty)$ a non-oscillatory solution $x(t)$ which is eventually positive. Along the same lines as in [4, Theorem 3], we deduce that the inequality (20) has a positive solution, which contradicts our assumption. On the other hand, it has been established in

Theorem 1 that the function $y(t)$ is positive, increasing, and satisfies the inequality (6). By virtue of (19), the inequality

$$w(t) \leq u(t)\left(1+\frac{p_0}{\tau_0}\right)$$

holds for the function $w(t)$ defined by (16). Substituting this into (6), we conclude that $w(t)$ is a positive solution of (21), which contradicts our assumption. The proof is complete. ■

Combining Theorem 4 with results in Ladde et al. [12, Theorems 2.1.1 and 2.4.1], we obtain the following oscillation criterion.

Corollary 5 *Assume that conditions* (H_1)-(H_3), (2) *and* (19) *are satisfied. Suppose further that there exist two functions* $\eta, \xi \in \mathrm{C}([t_0,\infty))$ *such that* $\eta(t) < \tau(t)$, $\xi(t) > t$, $\eta(t) \leq \sigma(t) \leq \xi(t)$, *and* $\lim_{t\to\infty}\eta(t) = \infty$. *If, for all sufficiently large* $t_1 \geq t_0$,

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\eta(t))}^{t} Q(s)\left(R(\eta(s)) - R(t_1)\right) \mathrm{d}s > \frac{\tau_0 + p_0}{\tau_0 \mathrm{e}} \tag{22}$$

and

$$\liminf_{t\to\infty} \int_{t}^{\xi(t)} Q(s)\delta(\xi(s))\mathrm{d}s > \frac{\tau_0 + p_0}{\tau_0 \mathrm{e}}, \tag{23}$$

equation (1) *is oscillatory.*

Proof. By virtue of [12, Theorem 2.1.1], condition (22) ensures that differential inequality (20) has no positive solutions. On the other hand, it follows from [12, Theorem 2.4.1] that condition (23) guarantees that the differential inequality (21) has no positive solutions. Application of Theorem 4 completes the proof. ■

3 Examples and discussion

The following two examples illustrate applications of theoretical results in the previous section.

Example 6 *For* $t \geq 1$, *consider a second-order neutral differential equation*

$$\left(t^2[x(t) + p_0 x(\alpha t)]'\right)' + q_0 x(\beta t) = 0, \tag{24}$$

where α, p_0, q_0, and β are positive constants, $\beta > \alpha > 1$. Let $\eta(t) = t/\alpha$ and $\xi(t) = \beta t$. Application of Corollary 3 yields that all solutions of equation (24) are oscillatory provided that

$$\frac{q_0}{\beta}\ln\frac{\beta}{\alpha} > \frac{\alpha + p_0}{\alpha \mathrm{e}}.$$

Example 7 *For $t \geq 1$, consider a second-order neutral delay differential equation*

$$\left(\mathrm{e}^t\left[x(t) + \frac{1}{2}x\left(t - \frac{\pi}{4}\right)\right]'\right)' + 12\sqrt{65}\mathrm{e}^t x\left(t - \frac{1}{8}\arcsin\frac{\sqrt{65}}{65}\right) = 0. \quad (25)$$

Let $\eta(t) = t - \pi/2$ and $\xi(t) = t + \pi/4$. Taking into account that $R(t) = \mathrm{e}^{-1} - \mathrm{e}^{-t}$, $Q(t) = 12\sqrt{65}\mathrm{e}^{t-\pi/4}$, $\delta(t) = \mathrm{e}^{-t}$, and $\tau_0 = 1$, we conclude that

$$\liminf_{t\to\infty}\int_{\tau^{-1}(\eta(t))}^{t} Q(s)\left(R(\eta(s)) - R(t_1)\right)\mathrm{d}s = \liminf_{t\to\infty}\int_{t-\pi/4}^{t} 12\sqrt{65}\mathrm{e}^{s-\pi/4}\left(\mathrm{e}^{-t_1} - \mathrm{e}^{-s+\pi/2}\right)\mathrm{d}s > \frac{3}{2\mathrm{e}}$$

and

$$\liminf_{t\to\infty}\int_{t}^{\xi(t)} Q(s)\delta(\xi(s))\mathrm{d}s = \liminf_{t\to\infty}\int_{t}^{t+\pi/4} 12\sqrt{65}\mathrm{e}^{s-\pi/4}\mathrm{e}^{-s-\pi/4}\mathrm{d}s > \frac{3}{2\mathrm{e}}.$$

By Corollary 5, all solutions to equation (25) oscillate. As a matter of fact, one such solution is $x(t) = \sin 8t$.

Remark 8 *Oscillation criteria established in this paper for equation (1) complement, on one hand, theorems reported by Baculíková and Džurina [4] and Hasanbulli and Rogovchenko [11] because we use assumption (2) rather than (3) and, on the other hand, those by Li et al. [14] since our criteria can be applied to the case where $0 \leq p(t) \leq p_0 < \infty$.*

Remark 9 *Using methods different from those exploited in [10, 11, 14, 15], we improve results of Han et al. [10] and Li et al. [15, Theorem 3.8] by removing assumptions* $\sigma(t) \le \tau(t) \le t$ *or* $t \ge \sigma(t) \ge \tau(t)$ *imposed in the cited papers and providing sufficient conditions which ensure that* all solutions *of* (1) *are oscillatory.*

Remark 10 *As mentioned by Baculíková and Džurina [4, Remark 4], assumptions* (H_2) *and* (H_3) *on functional arguments do not specify whether* $\tau(t)$ *is a delayed or advanced argument. Therefore* $\sigma(t)$ *can be a delayed argument and* $\sigma(t) - t$ *can even oscillate. However, to achieve such flexibility, we are forced to require, as in [4, Remark 4], monotonicity of* τ *and that* $\tau \circ \sigma = \sigma \circ \tau$*. The question regarding the study of oscillatory properties of equation* (1) *with other methods that do not require assumption* (H_3) *remains open at the moment.*

Remark 11 *Note that by using the inequalities*

$$x_1{}^{\alpha} + x_2{}^{\alpha} \ge (x_1 + x_2)^{\alpha}, \qquad \text{for } 0 < \alpha \le 1 \text{ and all } x_1, x_2 \in [0, \infty),$$

and

$$x_1{}^{\alpha} + x_2{}^{\alpha} \ge \frac{1}{2^{\alpha-1}}(x_1 + x_2)^{\alpha}, \qquad \text{for } \alpha \ge 1 \text{ and all } x_1, x_2 \in [0, \infty),$$

results reported in this paper can be extended to a second-order half-linear neutral differential equation

$$\left(r(t)\left([x(t) + p(t)x(\tau(t))]'\right)^{\alpha}\right)' + q(t)x^{\alpha}(\sigma(t)) = 0,$$

where $\alpha > 0$ *is a ratio of odd positive integers.*

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