Learning and Teaching Functions and the Transition from Lower Secondary to Upper Secondary School

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Preface

Functions is one of my favourite topics. My interest in this increased towards the end of my teacher education at the University of Stavanger (UiS) and led to me taking a master's degree in complex analysis at the Norwegian University of Science and Technology (NTNU) under the supervision of Professor Kristian Seip.

My first teaching experience was in lower secondary school as school experience during my teacher education in Stavanger. As part of the master programme I also gained some valuable experience by working as a substitute teacher in various lower secondary schools in Trondheim. When I finished my master's degree, I worked for several years in upper secondary school.

These experiences added to my interest in the teaching and learning of functions as well as to my growing awareness of general issues involved in the transition between lower and upper secondary schools. Back then, I had a relatively fresh memory of being a student at both these levels. All this led me to hypothesize that the cultures of lower and upper secondary schools are governed by different traditions and beliefs which can, in turn, influence students' learning. My main hypothesis was that the teaching cultures differ and that this is mainly due to the different backgrounds of the teachers.

I presumed that teachers in lower secondary schools usually have a background in integrated teacher education, while those in upper secondary tend to have a subject-specific university background. As a consequence of this, I expected typical "traditional" teaching methods, such as blackboard lessons and individual work with textbook tasks, in upper secondary, while at lower secondary I expected more "experimental" teaching forms, such as group work, problem solving, ICT and interdisciplinary projects.

This then was the background for the PhD project description, initially guided and supervised by Tine Wedege (NTNU), which has culminated in this research.

I am very grateful to Sør-Trøndelag University College (HiST), where I am now employed, for giving me the opportunity to conduct this research. It has been externally funded by The Research Council of Norway (NFR) as a part of the Teaching Better Mathematics project (TBM) led from the University of Agder (UiA) where I am formally registered as a PhD student.

I want to express my gratitude to all the students who allowed me to interview them and to observe them during their last year in lower secondary school and their first in upper secondary. Thanks also to all their teachers for giving me access to their classrooms.

My deepest gratitude goes to my main supervisor Anne Berit Fuglestad (UiA) and to my co-supervisor Frode Rønning (HiST – now employed at NTNU) for their support throughout this whole process. Both have demonstrated a sincere interest in the project through constructive and valuable feedback in a thoroughly professional manner. Thanks also to all my wonderful colleagues at HiST for helpful discussions and feedback and to the staff at UiA for valuable advice throughout the PhD programme. Many thanks also to my fellow PhD students for motivating discussions and contributions. I am especially grateful to Tine Wedege (NTNU) who was willing to spend considerable time helping me to establish a solid platform for the project before I became an "official" PhD student. I also want to thank my good colleague Sandra Foldvik for helping me with the English.

My gratitude also goes to my brother Sverre Nilsen, my childhood friends Geir Atle Helland, Jan Erik Helland, Kjell Tjora and Øyvind Rott and to my newer – but no less important - friends Sigve Hovda, Trine Myhre, Morten Wiig Bjorland, Lena Kvalevåg, Thor Pedersen, Knut Husdal, Zenon Taushanis, Rune Åkre, Anne Løberg-Dahl, Solbjørg Bandlien and all the other acquaintances – close or more "peripheral". Thank you all for being such great social and academic motivators! Moreover, heartfelt gratitude goes to my girlfriend Silje Lilly Nitter Meberg – you have been a priceless support!

Finally, I want to express my gratitude to my parents, Anders and Anne Sofie Nilsen, for motivating me and for raising me with love, in a home where knowledge and education have always been encouraged and highly valued.

-Life is about transition, which includes change, growth, learning and exploring[1]

Hans Kristian Nilsen
Trondheim, Norway
October, 2013

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[1] From Leslie A. Gallardo's (2006) *Hurricane Katrina In re: Our day with the cross*, p. 39

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1 Introduction and research questions

1.1 Personal motivation

From my previous years as a student I will emphasize four phases which I consider as personal encounters with transitions. The first phase consists of nine years at obligatory public school. Between the 6th and the 7th grade, I experienced my first institutional transition in terms of the transition between primary and lower secondary school (Section 2.2). After finishing my last year at lower secondary, I continued general study programmes at upper secondary. It was during these years that I really decided that I wanted to become a teacher, so in the four subsequent years I attended the teacher education programme at Stavanger University College (now University of Stavanger). During these years my motivation for mathematics really started to grow, and I decided to continue my education in pure mathematics at university. This was a rather unusual choice, primarily due to a certain mismatch between the type of mathematics courses provided in teacher education and the type of mathematics courses accepted in order to start the master programme in pure mathematics. I resolved this issue by attending several mathematical courses in addition to those in teacher education, and was eventually accepted.

These experiences, especially the teaching at upper secondary, teacher education and university made me realise that mathematics was not really "just mathematics". Priorities related to content, mediation and assessments varied. In retrospect, I have reflected on these different "worlds" of mathematics, and what seems clear to me is that each of these different learning environments has its own culture, constituted by the teachers and their interactions with students. After graduation, I worked as a mathematics teacher in several schools, mainly at upper secondary level. During the periods of practice in teacher education, and while working as a substitute teacher while doing my master thesis, I also taught mathematics in lower secondary schools. During these periods I noticed several differences, especially related to the teaching culture, and began to develop some related hypotheses.

To be able to see and compare the teaching of the actual mathematical content, I found that the topic of functions was a central topic relevant in both lower and upper secondary school. Functions also have the potential for being expanded to possible prospective transition studies, for example by taking the universities/university colleges into account.

1.2 This study as a part of the TBM project

This PhD study is a part of the project Teaching Better Mathematics (TBM) led by the University of Agder in the period 2007-2010.

Teaching Better Mathematics primarily aimed to develop better understanding of, and competency in, mathematics for pupils in schools. This entailed encouragement and exploration of better teaching methods and approaches to achieve that aim. Although this was the overarching aim of the project, TBM involved five Norwegian universities/university colleges each of which focused on different aspects. Sør-Trøndelag University College, where I am employed, focused on "Learning of mathematics through activities and communication". The local project description contained a list of suggestions of possible research areas, one of which was "transition between levels".

At first glance, the connection between my area of research (transition between levels) and the overarching aims of the TBM project might seem tenuous. However, in line with the local project description (which mentions transition between levels as a possible area of research), I will argue that a good transition study is useful, also in the context of the overarching aims of TBM. For the development of better teaching methods in mathematics, the overarching aims of the project confirm the need for a better understanding of such teaching methods. As a part of this understanding, a status-quo survey could be of great importance as it allows already existing competencies as well as possibilities for improvements to be identified. Transition studies might especially contribute in this sense, as they incorporate a potential for comparison since they involve comparing already existing methods of teaching and their corresponding learning outcomes. An analysis of the methods practised in different institutions, will hopefully in turn both contribute to and enrich the task of improving them.

1.3 Initial hypotheses

My hypotheses before conducting this research mainly had its source in my personal experiences as a student and a teacher in both lower secondary and upper secondary education. I hypothesized that teachers at upper secondary tend to have a university background consisting of a subject-specific teacher education, while teachers at lower secondary schools usually have background from integrated teacher education. I assumed that this would influence the actual teaching. Further, I assumed that these differences would be realised in terms of different teaching methods and different approaches to the mathematical content. A main reason for this assumption is that the teaching methods that the teachers themselves are exposed to at university and in integrated teacher education might differ. My impression is that university education is for

the most part dominated by lectures which focus on mathematics as a discipline. Often didactics and pedagogy feature in the students' schedule only in the final years of the study. On the other hand, integrated teacher education aims to intertwine pedagogical knowledge and mathematical knowledge more holistically throughout the whole programme. Based on my own experiences, students in general teacher education are confronted with methods that promote "active learning", while at university level learning is more passive. I assumed that this might lead to a teaching culture at upper secondary which favoured lectures at the blackboard and individual tasks from textbooks. On the other hand, at lower secondary, I expected more group work, problem solving and interdisciplinary projects. My initial, underlying hypotheses before carrying out the research was:

– *Mathematics teaching at lower and upper secondary school is different. Teaching at upper secondary is dominated by traditional teaching methods, while teaching at lower secondary consists of more practical and experimental approaches.*
– *These differences will affect students' learning in such a way that students depending on variation and practical approaches will experience a loss of motivation and learning outcome at upper secondary. Motivated students, used to work independently, will keep or might increase their motivation and learning outcome at upper secondary.*

My aim is that by focusing on the *transition* between these phases of schooling, possible differences related to classroom practices and teaching culture have the potential of being exposed. By following a group of students in the transition from lower to upper secondary school, comparative lenses hopefully address relevant aspects related to learning and teaching mathematics at both institutions.

1.4 Research questions

In the light of these personal experiences and initial hypotheses, I now pose the following research questions:

1) *How do students' conceptions of functions develop from 10th grade at lower secondary school to 11th grade at upper secondary school?*
   a. How do students, indirectly or directly, express their conception of functions at lower secondary and upper secondary school?
   b. How do students, indirectly or directly, express their conception of the gradient of a function at lower secondary and upper secondary school?

c. How do students in upper secondary vocational programmes relate the gradient of a function to proportional magnitudes?
d. How do students in upper secondary general studies programme relate the gradient of a function to the concept of differentiation?

2) ***How is the topic of functions mediated at selected lower secondary schools compared to selected upper secondary schools?***
a. How is the concept of functions presented in lower secondary compared to upper secondary school, general studies programme?
b. How is the gradient of a function presented in lower secondary compared to upper secondary school, general studies programme?
c. How are gradients related to proportional magnitudes in upper secondary, vocational studies programme?
d. How are gradients related to differentiation in upper secondary, general studies programme?

3) ***What is the relation between research question 1) and research question 2) at lower secondary and at upper secondary levels?***
a. What is the relation between teaching and students' reasoning in lower secondary?
b. What is the relation between teaching and students' reasoning in upper secondary?
c. What characterises the differences between lower and upper secondary school, illuminated through classroom mathematical practices and sociomathematical norms?

1.5 An overview of the research

In this study, I mainly use qualitative methods. I was convinced that carrying out qualitative research in terms of observations in the classrooms and interviews with students and teachers would provide me with more in-depth information and richer empirical data than quantitative methods. Patton (2002) writes: "Qualitative findings in evaluation illuminate the people behind the numbers and put faces on the statistics" (p. 10). Aspects like "recognition" and "a more complete story" are also often mentioned as powerful dimensions of the qualitative approach. My main argument for a qualitative approach is that conversations and interviews open up for possibilities of going more in depth, especially related to students' reasoning in mathematics. At the same time, conversations and interviews make it possible to focus on

particular phenomena which might have been hard to predict or detect through, for example, a questionnaire.

Briefly summarized, I followed a group of twelve students in the transition from lower to upper secondary school in the period 2007/2008 – 2008/2009. I focused on students' learning and the teaching they received related to the topic of functions. I therefore have a two-fold aim: to investigate the development of students' reasoning related to functions (learning) and the mediation and presentation of this topic in lower secondary and upper secondary schools (teaching).

In the initial phase of the project, for the sake of diversity in my material, I decided to focus on five lower secondary schools. In the first part of the research, which took place while the students were in their last year of lower-secondary, students were informed about the project and its intentions, and I decided to include all who volunteered to take part, in total 33 students. This made it possible for me to include some criteria for selecting the group I wanted to follow up in upper secondary, in total 12 students. I included students from both the vocational and the general studies programme. While the students in lower secondary attended five different schools and different classes, the twelve students in upper secondary were distributed over six schools and ten classes. Circumstances like inadequate empirical material, combined with the need for narrowing the focus, eventually resulted in a group of eight students from four upper secondary schools and eight classes. This will be dealt with more thoroughly in the methodology chapter (Chapter 6).

The first research question concerning students' development of functions was operationalized through conversations and interviews with the students. In lower secondary, the conversations (unstructured interviews) took place in the classroom and were normally related to students' work with tasks provided by the teacher. Semi-structured interviews were conducted subsequent to the observation period. Since the upper secondary environment was new to the students, I did not want to run the risk of putting them in an uncomfortable situation, so for these ethical reasons I chose not to have conversations with the students in the classroom there. But with more time available for the interviews in upper secondary, I was able to include more tasks and questions related to students' reasoning. As the first research question suggests, my focus was on students' understanding of the function concept and gradients. For the students in the upper secondary general studies programme, what I found to be of particular interest was the relation between their understanding of the gradient of linear functions in lower secondary and the topic of differentiation at upper secondary. For the students in the vocational programmes, my focus was on the relation between gradients

and proportional magnitudes as differentiation was not a topic on the curriculum for these students.

The second research question relates to teaching, and to approach this question my main source of information was through classroom observations and teacher interviews. I aimed to observe all the lessons related to the topic of functions, in both lower and upper secondary schools. Due to practical circumstances, this was not possible, but I did observe some lessons in the topic at every involved school. All the teacher interviews in both lower and upper secondary were conducted as planned.

The third research question concerns the relation between research questions 1 and 2 and the comparison between lower and upper secondary.

1.6 Theoretical positioning

Methodologically, this research belongs within the interpretative paradigm (Mertens, 2005) which implies a multiple constructed reality with the interpretative and value-bound nature of findings and mediated actions as the units of analysis.

I take the ontological position where I consider mathematics as a cultural historical developed set of rules, notations and signs, born out of certain needs within a given set of practices (Pozzi, Noss, & Hoyles, 1998). To succeed in the field of mathematics is then being able to act and participate within this given culture (of mathematics) through mediation in terms of language and communication (Cole, 1985; Lerman, 2000).

My overarching theoretical position is rooted within the socio-cultural (or cultural-historical) perspective, as conceived of by Vygotsky. My main arguments for this theoretical perspective are the important role of mediation and instruction in this study. In addition, semiotic approaches, consistent with such a perspective, provide me with suitable analytical tools for analysing both teaching sequences and students' reasoning.

Consistent with socio-cultural theories, in terms of their emphasis on communication, language and the use of signs, semiotic models have evolved. Such models have also been modified, developed and applied within the field of mathematics education (Presmeg, 2005; Steinbring, 2005) with a view to pinpoint how mediation contributes to students' conceptual development in mathematics. Related to the first two research questions, I have applied Steinbring's epistemological triangle as the main analytical tool. In particular, I find this model to be useful in analysing teaching sequences, since the role of mediation is made visible

by emphasizing the interplay between the "reference context" and "the sign".

1.7 The structure of the thesis

Subsequent to the introduction I provide an overview of the Norwegian educational context (Chapter 2). This covers the educational system from primary school to upper secondary school. The curricula related to functions and gradients relevant for lower secondary and for the different programmes at upper secondary will also be discussed. The overview includes Waldorf Schools, since one such school was involved in the research at lower secondary level (School A). I will include a section concerning teacher education, as there are several educational choices which can qualify for the teaching profession. In Chapter 3, the historical background of the function concept and gradients is presented. Chapter 4 deals with the theoretical framework applied in this study. This includes the underpinning socio-cultural theory and the application of Steinbring's epistemological triangle as an analytical tool. I will also discuss important aspects like concept formation and mathematical representations. In Chapter 5 I present a selected overview of the literature which I consider relevant for this study. Chapter 6 concerns methodological issues such as the research paradigm, the research design, methods of data collection and the data analysis strategy.

Chapter 7 is the first part of the analysis. In this chapter I have chosen to present four students, one from four of the lower secondary schools involved. The chronological aspects are emphasized in these analyses and presentations to make it easier for the reader to grasp the impression of the transition and the actual shift of context. This chapter is also intended to justify and provide a basis for the analytical categories, briefly presented in Chapter 6 and applied in Chapter 8. In Chapter 8, more general analysis is provided, based on the categories presented in Chapter 6 and developed in Chapter 7. The analysis in Chapter 8 is divided into three parts, in accordance with the three research questions. In Chapter 9, I summarize the findings and present some conclusions and final remarks.

Chapter	Title	Content
1	Introduction and research questions	- Background and motivation - Presentation of the research questions - Overview and structure of the thesis
2	Norwegian educational context	- The Norwegian school system - Relevant curricula - Teacher education

3	Historical development	- The historical development of the function concept and the derivative.
4	Theoretical background	- Overarching theory (the socio-cultural perspective) - Semiotic models and Steinbring's epistemological triangle - Concept formation - Conceptual understanding - Mathematical representations - sociomathematical norms
5	A selected overview of relevant literature	- Literature on teaching and learning functions, gradients, derivatives and proportional magnitudes - Literature on transition studies
6	Methodology	- Research paradigm - Research design - Data analysis strategy - Validity and trustworthiness - Analytical categories
7	A chronological presentation of four participating students: Otto, Matt, Thea and Olga	- Analysis and presentation of four students (one form each involved lower secondary school) emphasising chronology and the development of some of the analytical categories
8	Further analysis	- General analysis, drawing on categories developed in Chapter 7 and presented in Chapter 6 - This chapter is divided into three parts in accordance with the research questions
9	Summary and conclusions	- Summary, conclusions and final remarks

Table 1.1. The structure of the thesis.

2 The Norwegian educational context

2.1 The Norwegian school system and mathematics

The Norwegian school system can be described as a unitary system which emphasises that all pupils should have the same legal educational rights. To some extent, this implies access to the same curriculum and content independent of a school's geographical location. In 1997 the compulsory school was extended from nine to ten years, which means that pupils now start first grade when they are six years old. The first seven years of obligatory schooling correspond to "primary school" and grades eight to ten to "lower secondary". These schools may not necessarily be separate institutions, but in areas with dense population this is normally the case.

The obligatory part of the school system does not include upper secondary. However, "Reform 94" (introduced in 1994, relevant for upper secondary education) legally entitles every student to attend upper secondary school. Since 1994, there have been two subsequent curriculum reforms, the first, L97 (applying to primary and lower secondary school, valid from 1997) and the current one, LK06, also known as the *Knowledge Promotion* (Utdanningsdirektoratet, 2006, 2010). LK06 applies to the first 13 years of education, including upper secondary level.

In upper secondary there are two main study programmes; the general studies and the vocational studies programme. The vocational programme is orientated towards practical professions, educating, for example future carpenters, plumbers and electricians, while the general studies programme aims to prepare students for tertiary education. The curricula for these programmes differ, with general studies considered to be more theoretical also with regard to mathematics. Both the vocational and the general studies programme are included in this research. Obligatory mathematics in vocational programmes, "1YP", is the same for all vocational programmes. However, textbooks sometimes adjust the content to suit a specific vocational programme. Such adaptations are primarily apparent in the phrasing of examples and tasks. Publishers choose different solutions; some publish only one 1YP textbook for all the vocational programmes while others publish "1YP for carpenters", "1YP for electricians" and so forth. I encountered examples of both during my observations.

In the general studies programme, students can choose between two mathematics courses: 1P and 1T, where 1T is considered to be the more theoretical. Choice of 1P or 1T determines which of the mathematics courses can be chosen in the two subsequent years in upper secondary.

For students attending upper secondary school, general studies, a minimum of five hours of mathematics per week in the first year, and three hours per week in the second year is required. This means that a student attending 1P or 1T as a minimum has to continue attending either 2P or 2T (2T is not possible from 1P) the second year at upper secondary, as these both amounts to three hours per week. For students choosing an in-depth study module in mathematics, 1P can be followed by S1 and S2 respectively in the second and the third year, while 1T can be followed by R1 and R2 or S1 and S2.

In turn, these courses affect later possibilities at universities/ university colleges where some require R1 and/or R2. Initially, both 1P and 1T were included in this research, but due to inadequate data from observations and interviews for 1P, combined with a need to narrow the focus of the analysis, the 1P students were eventually omitted.

Private schools, which have to be approved of by the government, constitute an alternative to the public school system. Usually these are established by religious or ideological foundations and are intended to offer an alternative to the public system. Some private schools also offer an alternative or revised curriculum. According to a survey from 2005, 2% of the students in the primary and lower secondary school and 5% of the students in upper secondary attended some kind of private school (Reisegg & Askheim, 2013). At lower secondary level, one Waldorf School was included in this research (School A). Waldorf Schools, based on the ideas from Rudolf Steiner and anthroposophy, constitute the majority of the private schools in Norway. The first Waldorf School was founded in Germany in 1919. The name *Waldorf* stems from Waldorf-Astoria, the name of a cigarette factory where Rudolf Steiner held a speech for the workers including the topic of education and educational rights (Weisser, 1996). It is beyond the limitation of the thesis and the relevance for this study to account for the content of Steiner's anthroposophy in general, but one of his core ideas when it comes to education was to orchestrate teaching and learning in line with the progressive development he found common to all children (Wiesser, 1996, p. 20). In accordance with this view, Steiner advocated for the role of what he called "artistic teaching" which entailed a focus on aesthetic approaches like music, motions and arts especially during the primary years of education (Wiesser, 1996, p. 48). Steiner also stressed the importance of artistic teaching for the sake of treating children in a holistic manner. From Steiner's perspective this holistic view entailed promoting the relations between the intellect, the emotions and the will (Weisser, 1996, p 51). For example, related to mathematics as a subject, mastering body movements like walking, running and jumping, climbing and balance are regarded as "pre-steps" towards the development of

mathematical thinking. Similarly, rhythms are emphasized as means for developing number sense (Steinerskolene i Norge, p. 91).

2.2 The transitions in the Norwegian school system

In Norway, the transition between different phases of schooling, particularly in relation to the learning and teaching of mathematics, is an area where little research has been carried out. Most international research in this area relates to the transition from upper secondary school to higher education, often called the secondary-tertiary transition (Gueudet, 2008; Guzmán, Hodgeson, Robert, & Villani, 1998; Stadler, 2009). In Norway lower secondary and upper secondary school are separate institutions, with only few exceptions (e. g. Waldorf Schools). The figure below gives an overview of the transitions in the Norwegian school system.
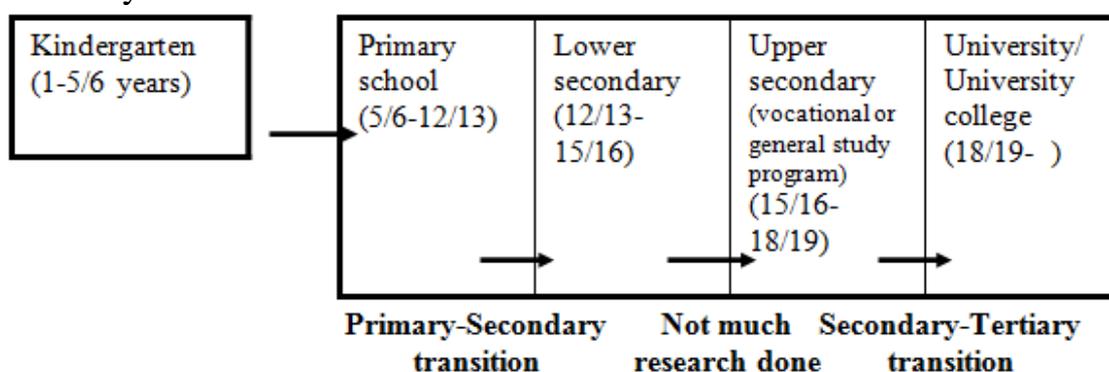
Figure 2.1. Transitions in the Norwegian school system

Transition from primary school to lower secondary does not necessarily involve a shift of institutions as some schools as these (especially in rural areas) could be combined in one institution. In the international literature, the transition from primary to (lower) secondary is often denoted as the "primary-secondary" transition. Similarly "secondary-tertiary" denotes the transition from (upper) secondary to university/university college. No such corresponding term exist in the literature, regarding the transition from lower to upper secondary education.

   Reform 94, which legally entitled all 16 to 19 year olds to upper secondary education, led in turn to increased political focus on upper secondary education. Now it is the transition between lower and upper secondary which is raising political concern as reports document alarming dropout rates of 30 % from upper secondary education (Chaudhary, 2011). These statistics have stimulated discussion of core issues such as gearing teaching more towards the individual student, less theoretical and more vocational programmes and in-service training for teachers.

That many students experience the transition from lower to upper secondary as problematic is confirmed by the relatively high number of dropouts. However, hypotheses which can account for these numbers are not easy to test and constitute only underlying preliminary thoughts for my study.

2.3 Functions, gradients and proportionality in the intended curriculum

In accordance with my research questions, I will now present those parts of the curricula which deal with the topics functions, gradients and proportionality for the four main contexts which are the object of this study, lower secondary (public school), lower secondary (Waldorf School), upper secondary – vocational studies programme and upper secondary – general studies programme, 1T version.

The mathematics which is taught can be considered on three different levels (Flanders, 1994; Handal & Herrington, 2003). The top level is the *intended curriculum* as represented by the National Curriculum. Included in this term are also curriculum guides and the way textbook reflects the national curricula. At the local level, how schools and teachers try to implement the various curricula and the actual teaching constitutes the *implemented* level. Thirdly, the eventual students' learning outcome is the *attained* level (Flanders, 1994). Most of the analysis deals with the implemented and the attained level, as these are most prominent in my data and most relevant for my research questions.

In addition, this section will provide a short account of the intended curriculum as it appears in LK06. Since Waldorf Schools have their own officially approved curriculum, the relevant parts of this curriculum will be presented separately.

2.3.1 The National Curriculum for public schools

In LK06 (Utdanningsdirektoratet, 2010), "functions" is one of five main areas in mathematics specified as competence goals on completion of 10th grade.

Functions:

The pupil should be able to

- prepare, on paper and digitally, functions that describe numerical relationships and practical situations, interpret them and convert between various representations of functions, such as graphs, tables, formulas and text
- identify and exploit characteristics of proportional, inversely proportional, linear and simple square[2] functions, and provide examples of situations which can be described using these functions (p. 7)

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[2] Quadratic (functions)

As can be seen from this excerpt, the content of this part of the curriculum includes functions as numerical relationships, specific situations, various representations, characteristics and applications of proportional and inversely proportional functions, as well as linear functions and quadratic functions.

Students in upper secondary can chose between two study programmes, vocational studies and general studies. The vocational programme prepares students for a practical profession, while general studies prepare students for higher education. Within general studies there are two versions of mathematics, 1P, considered to be the more practical and 1T, considered to be the more theoretical. This means that the curriculum for the first year in upper secondary is divided in three, one version for vocational studies plus the two versions for general studies. The 1T version has the following aims:

Functions:

The pupil should be able to

- elaborate on the concept of functions and draw graphs by analysing the function concept
- calculate zero, intersection and average rate of change, find approximate values for instantaneous rates of change and provide some practical interpretations of these aspects
- elaborate on the definition of the derivative, use the definition to deduce a rule for the derivative of polynomial functions and use this rule to discuss functions
- make and interpret functions that describe practical questions, analyse empirical functions and find expressions for an approximate linear function
- use digital aids to discuss and elaborate on polynomial functions, rational functions, exponential functions and power functions (pp. 7-8)

Notice that differentiation is included in the aims for these students. Valid for the 1P version is:

Functions

The pupil should be able to

- examine functions which describe practical situations by determining the intersection, zero, minimum or maximum and gradient, and to interpret the practical value of the results
- convert between different representations of functions
- elaborate on the concept of linear growth, demonstrate the progress of such growth and use this in practical examples, including digitally (p. 8)

As one can see, the content of the 1P version is considerably reduced compared to 1T, and the part containing the derivative is left out.

In the vocational studies curriculum, there is no separate area called "Functions"; in fact, the concept of function is not mentioned at all. On the other hand, the sub-paragraphs "Numbers and Algebra" include the following:

Numbers and algebra
The pupil should be able to

- make estimates of answers, calculate practical tasks, with and without technical aids, and assess how reasonable the results are
- interpret, process, assess and discuss the mathematical content of written, oral and graphic presentations
- interpret and use formulas that apply to day-to-day life, working life and the education programme area
- calculate using proportions, percentages, percentage points and growth factors
- deal with proportional and inversely proportional magnitudes in practical contexts. (p. 8)

The second bullet point, "graphic representation" can be related to functions, and in the fifth, "proportional magnitudes" can be regarded as a special case of linear functions. Also "growth factors" in the fourth bullet point is relevant to the topic of functions. Although these formulations do not impose an explicit link to the concept of functions, the possibility of making the connections exists. One could also argue that this is a natural link, since the aims in LK06 after 10th grade to some extent presuppose a certain familiarity with the concept of functions.

2.3.2 The intended curriculum as reflected in textbooks

In this section I will provide an overview, where I briefly consider topics relevant to my research questions, and how these were dealt with in textbooks used at the schools involved in my study. Related to my research questions, I will in the following shortly present how the textbooks deal with the definition of the function concept, gradients and proportional magnitudes. It is important to underline that these presentations will not be extensive, and mostly deal only with definitions. These definitions are presented primarily for the sake of providing an overview and a basis for references, since some of these will be referred to in the analysis. (One should bear in mind that School A, the Waldorf School, did not use any textbooks). In the case of gradients and differentiation I have reproduced central illustrations as they appear in the textbooks. The reason for including these is the explanatory potential that these figures offer, supplementary to the text.

Textbooks at lower secondary

My periods of observations took place in 2008/2009, two years after the implementation of the new national curriculum, LK06. Despite of this, all the textbooks used in lower secondary were written in accordance with the previous National Curriculum, L97. During the interviews, most teachers expressed the awareness of this. Some teachers stated that because some parts of the textbooks were outdated, they often copied material from other books and handed out to the students. In addition, tasks provided for the students were also sometimes taken from other

sources. Nevertheless, related to functions, all the schools observed used the textbooks rather directly. The textbooks defined the function concept in the following terms:

Functions as defined in textbook applied at School B
> In everyday situations we often run into to magnitudes which have a certain relation. The media often uses graphs to illustrate how this relation is. Sometimes it is so that one value of x only gives one value of y. Then we call this relation a function (Martinsen, Oldervoll & Pedersen, 1999, p. 184, my translation)

Functions as defined in textbook applied at School C
> When for each value of a magnitude it corresponds a specific value of another magnitude, we call this relation a *function* (Gulbrandsen & Melhus, 2002, p. 50, my translation)

Functions as defined in textbook applied at School D
> An expression where y is connected to the variable, x, is called a *function.* For every function one can draw a diagram, either by hand or a computer (Bakke & Bakke, 1999, p. 354, my translation)

From the first two quotations one notices that the concept of variables is not explicitly being used, and in the third quotation variable is used only to denote the independent variable, x. Instead the first two textbooks make use of other words in the descriptions, like "magnitudes" and "values". In the two first quotations, the uniqueness property is described through pinpointing that there is only one y-value for each x-value. The descriptions in the two first quotations differ in the sense that the letter y is only used in the first, while the second just refers to "a specific value of another magnitude". One should also notice that in the third quotation, no attempt is made to elaborate on the uniqueness property.

In the three textbooks listed above, gradients were also dealt with. For reasons mentioned above this will be limited only to the introduction of the concept.

Gradients presented in textbook used at School B
> In the 9th grade we learned that a linear function can be written as $y = ax + b$ … We call the number a the gradient of the line. The number tells how much y increases or decreases when x increases by 1 (Martinsen, Oldervoll, & Pedersen, 1999, p. 187, my translation)
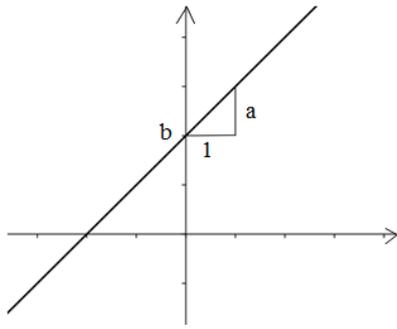
Figure 2.2. Illustration of the gradient as presented in the textbook used in School B (Adapted from Martinsen, Oldervoll, & Pedersen, 1999, p. 187)

Gradients presented in textbook used at School C
In this textbook no explicit definition is provided. It is worth noticing though, that the first section in the chapter entitled "graphs and functions" is given the headline "repetition". Still, in the margin of this section it says "Remember that for a linear function $y = ax + b$, a indicates the gradient and b indicates the intersection point with the *y*-axis" (Gulbrandsen & Melhus, 2002, p. 51, my translation). This sentence is accompanied by the figure below, which indicates a similar approach as provided in the textbook applied in School B (above):
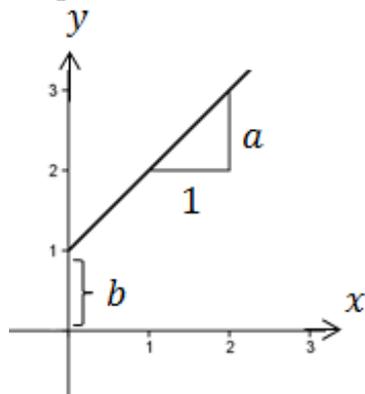
Figure 2.3. Illustration of the gradient as presented in the textbook used in School C (Adapted from Gulbrandsen & Melhus, 2002, p. 51)

Gradients presented in textbook used at School D
Initially, in the section called "linear functions" it is stated that "one can write linear functions as the formula $y = ax + b$, where a and b are numbers" (Bakke & Bakke, 1999, p. 358, my translation). In the wake of this, the textbook presents examples of how to plot points and draw straight lines in a coordinate system. Two pages later a similar figure as in the textbook used at School C is provided, but in this case combined with some short explanations:
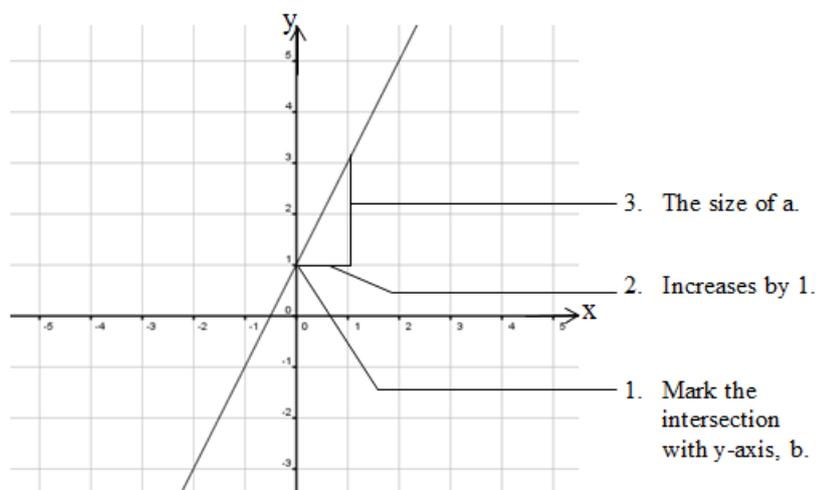
Figure 2.4. Illustration of the gradient as presented in the textbook used in School D (Adapted and translated from Bakke & Bakke, 1999, p. 360)

The three remarks (read from the bottom and up) respectively says "mark the intersection with the y-axis, b" (b alludes to the b in the general linear expression $y = ax + b$), "increases by 1" and "the magnitude of a" (a alludes to the a in $y = ax + b$).

From the excerpts of the three textbooks above one similarity in the approach to the concept of gradients is apparent in terms of the emphasis on the correspondence between movements in the x and y directions. In some different phrasings it is pointed out that the gradient can be understood as the increasing of y as x increases by 1. This is similar to what I denote as the *one-unit-right-a-up/down* strategy (6.4.4). None of the textbooks extends this definition or in other ways elaborate on this, for example in terms of developing this approach towards change in the y-direction divided by change in the x-direction.

Textbooks at upper secondary

At upper secondary school, all the textbooks were written in accordance with the prevailing curriculum, LK06. Like I described in Section 2.3.1, explicitly dealing with functions is not a part of the National Curriculum at upper secondary, vocational studies. Below is a presentation of how the textbooks used at upper secondary, general studies, define the function concept:

Functions as defined in textbook used at School 2b

> When each value of x gives one specific value of y, we say that y is a *function* of x. (Heir, Erstad, Borgan, Engeseth, & Moe, 2009, p. 102, my translation)

Functions as defined in textbook used at School 3
> Mathematical definition: Let y be a variable which is connected with a variable x, following a specific rule. If each value of x gives one specific value of y, y is a function of x. We say that y is the dependent variable and x is the independent variable. (Andersen, Jasper, Natvig, & Aadne, 2006, p. 200)

Functions as defined in textbook used at School 4
> y is a function of x if each possible value of x gives exactly one value of y. (Oldervoll, Orskaug, Vaaje, Hanisch, & Hals, 2009, p. 90)

The uniqueness property is dealt with in all these definitions and in the second quotation, the textbook also makes use of the concepts independent and dependent variable.

The relevance and importance of these quotations could be discussed, but my impression was that these definitions primarily were treated by teachers only by referring to these quotations, or reading them out loud. This seemed to be the case at both lower and upper secondary. This was either done by encouraging the students to read these definitions on their own, or by the teacher reading the definitions out loud.

In line with my research questions, one of my foci at upper secondary, general studies is the relation between gradients and the topic of differentiation (research question 2d). In an attempt to illuminate how this was dealt with in the different textbooks I will now present some excerpts relevant to this issue:

Differentiation as presented in textbook used at School 2b
In the textbook used in School 2b, the concept of differentiation followed the section about instantaneous growth rate, and is introduced and presented as follows.
> Usually we call the instantaneous growth rate at a point, the derivative at this point. We write the derivative of a function f at a point where $x = 2$ like $f'(2)$, and we read this as 'the derivative of f where $x = 2$' or 'f-derivative of 2'. We write the derivative of an arbitrary x-value as $f'(x)$.
> The derivative of a function f for a specific x-value is the gradient of the tangent at the point which has this as its x-value. (Heir et al., 2009, p. 250, my translation)

Three pages later Δx and Δy are introduced in the section called "To deduce the derivative by applying the definition". Here, the following figure and the function $f(x) = x^2$ serve as a basis for the elaborations:
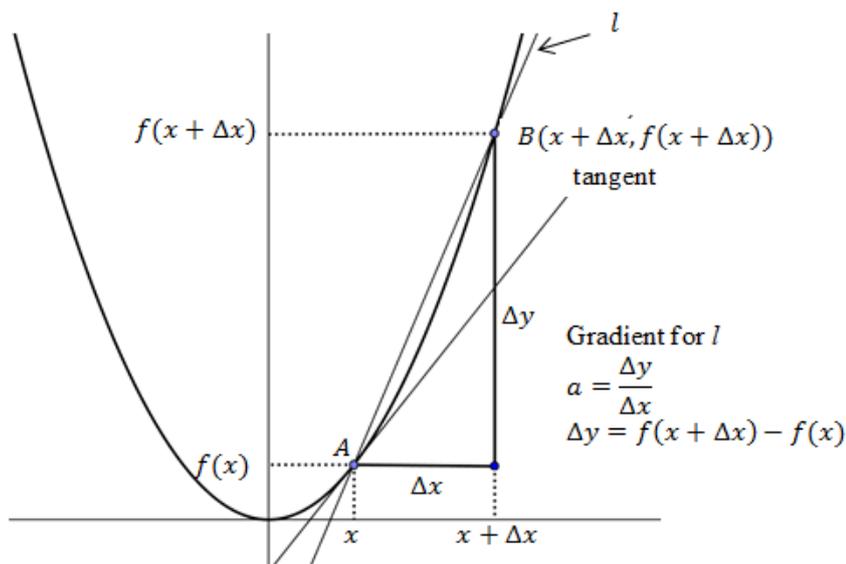
Figure 2.5. Illustration related to differentiation as presented in the textbook used in School 2b (Adapted and translated from Heir et al., 2009, p. 254)

The explanations continue:

> We let A be a point on the graph with x as its first-coordinate. A then has the coordinates $(x, f(x)) = (x, x^2)$. We draw a line through A which intersects the graph in a point B. The line l is then a "construction line" which we use to find the gradient of the tangent at A. ... The gradient of the line l is therefore
> $$\frac{\Delta y}{\Delta x} = \frac{2x \cdot (\Delta x) + (\Delta x)^2}{\Delta x} = \frac{(\Delta x)(2x + \Delta x)}{\Delta x} = 2x + \Delta x$$
> ...We imagine that the point B moves along the graph towards the point A. This is the same as letting Δx approach zero. When B approaches A, the line l will approach the gradient of the tangent. That means that the gradient of l is approaching $f'(x)$ when Δx is approaching zero... $f'(x)$ is the value which $\frac{\Delta y}{\Delta x}$ approaches when Δx approaches zero. (Heir et al., 2009, pp. 254-255, my translation)

Differentiation as presented in textbook used at School 3

Also in this textbook the section about differentiation follows the section of instantaneous growth rate:

> The instantaneous growth rate in a point we call the *derivative.* Definition: The derivative of a function f given by $f(x)$ at a point on the graph, is the gradient to the tangent at the point...On the figure below we have drawn a part of the graph of a quadratic function f. The figure shows that Δy is the difference between the function values $f(k + \Delta x)$ and $f(k)$. (Andersen et al., 2006, p. 231, my translation)

Subsequently, the following illustration is provided:
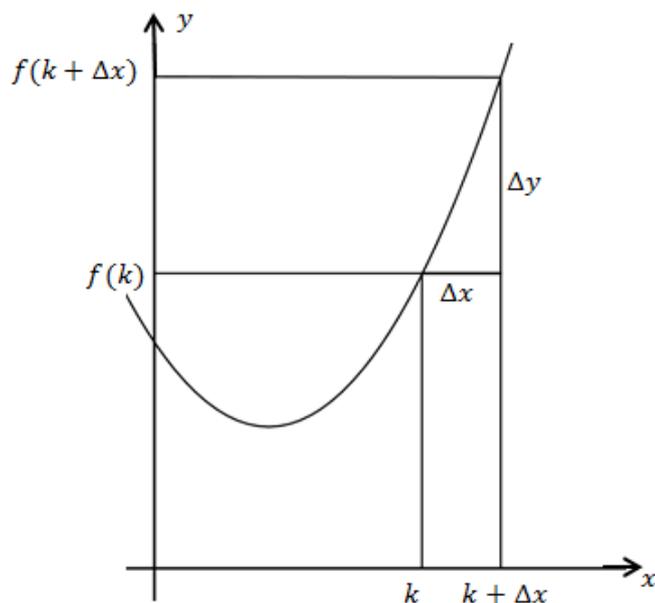
Figure 2.6. Illustration related to differentiation as presented in the textbook used in School 3 (Adapted from Andersen et al., 2006, p. 231)

The illustration is followed by these elaborations:

> If we let Δx approach zero, then $\frac{\Delta y}{\Delta x}$ will approach a value for the instantaneous growth rate at the point where $x = k$.
> Definition:
> $f'(k) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(k+\Delta x)-f(k)}{\Delta x}$,
> lim is an abbreviation for limit[3] which means grense[4]
> *Thus we find the limit of $\frac{\Delta y}{\Delta x}$ when Δx approaches zero*
> (Andersen et al., p. 231, my translation)

Differentiation as presented in textbook used at School 4
Similar to the two previous textbooks, also in this case, the chapter dealing with differentiation succeeds the section about instantaneous growth rate and the chapter of mathematical models and growth rate. The section where the differentiation concept is introduced is titled "growth rate as limit" and starts by considering the function $f(x) = x^2 - 2x + 4$. Δx is being defined as $(2 + h) - 2 = h$ and Δy as $f(2 + h) - f(2)$. The elaborations which follow consider $\frac{\Delta y}{\Delta x} = \frac{f(2+h)-f(2)}{h}$ as h approaches zero. Towards the end of the section, a general definition is provided:

> The derivative of a function f at $x = a$ is given by

---

[3] This should be *limes*
[4] «Grense» is the Norwegian word for limit.

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

$f'(a)$ gives the growth rate in the point $x = a$ and in addition the gradient of the tangent in the point $(a, f(a))$. (Oldervoll, Orskaug, Vaaje, Hanisch, & Hals, 2009, p. 212, my translation).

The argument above is illustrated by the figure below:
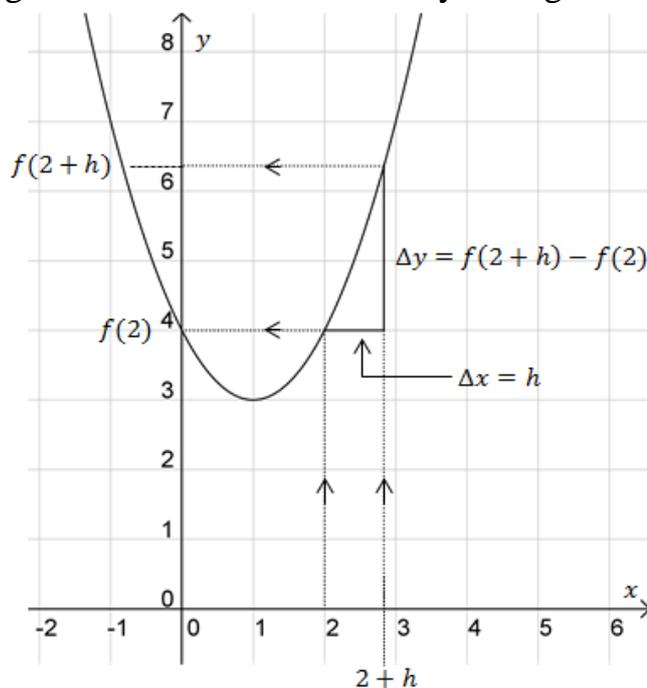
Figure 2.7. Illustration related to differentiation as presented in the textbook used in School 4 (Adapted from Oldervoll, Orskaug, Vaaje, Hanisch, & Hals, 2009, p. 211)

The connection between instantaneous growth rate and the derivative is made explicit in all these three examples, and the illustrations provided share essential similarities. Although the link between growth rates and derivatives is emphasized in each of the textbooks, the textbook used at School 3 (Andersen et al., 2006) is not making use of tangents and secants, neither in the illustrations nor the explanations. The textbook used at School 4 (Oldervoll, Orskaug, Vaaje, Hanisch, & Hals, 2009) mentions the tangent in the text, but it is absent in the illustration. In the textbook at School 4, Δx and Δy are dealt with in a slightly different manner then in the other two cases, as h is introduced to "replace" Δx in the general expression of the derivative.

Finally, in line with research question 2c, I will briefly present how the topic of proportional magnitudes was presented and defined in the actual textbooks of the vocational studies programme involved in this study.

Proportionality as presented in textbook used at School 1

The textbook starts by giving an example of renting a car which costs 600 kroner per day.

> A doubling of the renting period leads to doubling of the costs. Tripling the renting period leads to tripling the costs. We say that the costs are proportional to the renting period, or that the costs period and the renting period are proportional magnitudes (Bue, Engeseth, Solvik, Heir, & Pedersen, 2006, p. 44, my translation)

Subsequently, the textbook provides an example with valuta and Norwegian and Swedish kroner:

> [T]he amount of Norwegian kroner = 0.87 · the amount of Swedish kroner. We let *y* equal the amount of Norwegian kroner when we buy *x* Swedish kroner. We then have $y = 0.87x$…The factor 0.87 which is multiplied by *x*, is called the *constant of proportionality*…When *y* and *x* are proportional magnitudes, we can write $y = kx$ where the number *k* is called the *constant of proportionality*…When the ratio between two variable magnitudes *y* and *x* is constant, then *y* and *x* are proportional (Bue et al., 2006, pp. 45-46, my translation)

Furthermore, an example dealing with costs and the number of kilograms of moose meat is provided and displayed as a linear graph, concluding that "[a] graph which displays the relation between proportional magnitudes will always be a straight line passing through the origin" (p. 47).

Proportionality as presented in textbook used at School 2a

In this textbook one starts by exemplifying proportionality through the prize per kilogram of apples and the number of kilograms bought. This is converted into a table with three rows, *M* (kilograms) *P* (costs) and *P/M* (the ratio).

> The ratio of the costs *P* and the amount *M* is the same for all corresponding values of *P* and *M*. We observe that $\frac{P}{M} = 15 \, kr/kg$. The number 15 we call the *constant of proportionality*. In this case the constant of proportionality is the same as the price per kilogram of apples. Two magnitudes *x* and *y* are proportional if it is a fixed ratio *a* of all the corresponding values of *y* and *x*. $\frac{y}{x} = a$. The fixed ratio *a* is the *constant of proportionality* (Oldervoll, Orskaug, Vaaje, & Hanisch, 2009, p. 77, my translation).

Both the excerpts above illustrate that if the ratio between two magnitudes are constant, they have to be proportional. The actual ratio is defined as being the constant of proportionality. This is done in quite similar terms in both these two textbooks. In addition, the textbook used at School 1 (Bue et al., 2006) also provides a graphical interpretation of the relation between proportional magnitudes, and emphasizes that the corresponding graph of proportional magnitudes always pass through the origin.

2.3.3 The National Curriculum for Waldorf Schools (lower secondary)

Waldorf Schools constitute an alternative to public schools in Norway, both at primary, lower secondary and upper secondary levels. In the end of Section 2.1, I briefly pointed to some of the underlying pedagogical ideas related to Waldorf Schools. Since neither of the two students who participated in this research chose to continue at the Waldorf School after finishing lower secondary, the description of the intended curriculum will be limited to that for lower secondary school.

The mathematics curriculum for Waldorf Schools is divided into two main areas – "Arithmetic and Algebra" and "Geometry". Unlike LK06, this curriculum has specific goals for each grade. The topic of functions after the tenth grade falls partly under "Arithmetic and Algebra" and partly under "Geometry". "Proportionality, straight line in the coordinate system" is listed under "Arithmetic and Algebra" and "Conic sections geometry" is listed under "Geometry". As regards conic sections, there are no explicit connections to different types of functions mentioned, although such connections were prominent during my observation period.

The more detailed descriptions in the curriculum do not fully explain what the focus should be in work with linear functions. The curriculum does, however, mention that supplementary content, such as the study of diagrams combined with students' previous knowledge of maps, leads to further experiments in coordinate systems. In particular, proportionality should be emphasized (Steinerskolene i Norge, 2004, p.104, my translation).

2.4 Teacher education

As there are several educational paths to the teaching profession, one could assume (as stated in my initial hypotheses in 1.3) that these differences in teachers' educational background might have some influence on the teaching in the classroom. I will therefore briefly present an overview of these different paths, as they appear in Norway.

In general terms there are two main roads to becoming a teacher in primary and in lower secondary school. On the one hand there are teachers who choose a 4-year integrated study programme offered at universities/university colleges (GLU). This study programme is divided in two: Teacher education aiming towards grades 1 through 7 or teacher education aiming towards grades 5 through 10. Students have to choose between one of these programmes. This is due to the overlap of grades 5-7, which in practical terms means that both teachers from the grade 1-7 programme and teachers from the grade 5-10 programme are permitted to teach in grades 5-7. Until 2010, teacher education in Norway was not

separated in this manner but offered a 4-year study programme which qualified for teaching in all grades from 1 to 10 (ALU). This involved teaching in all grades from 1 to 10. It should be mentioned that from 2009 it was decided on a political level that teaching the subjects of English, Norwegian or mathematics at lower secondary requires one year fulltime study (60 sp) of the actual subject to be taught.

On the other hand there is subject-specific teacher education, where students study various school subjects at university and, at some point in the programme, complete one year of "practical pedagogical education" (PPU). This corresponds to the English "Postgraduate Certificate in Education" (PGCE). This practical pedagogical education prepares students for teaching by offering both theoretical background and practical training. Traditionally, this is how most teachers in upper secondary school enter the profession, but it also qualifies for teaching from grade 5. It should be said that the subject-specific teacher education in recent years tend to develop towards more and more integrated programmes.

An overview of teacher education in Norway could be summarized through the following figure:
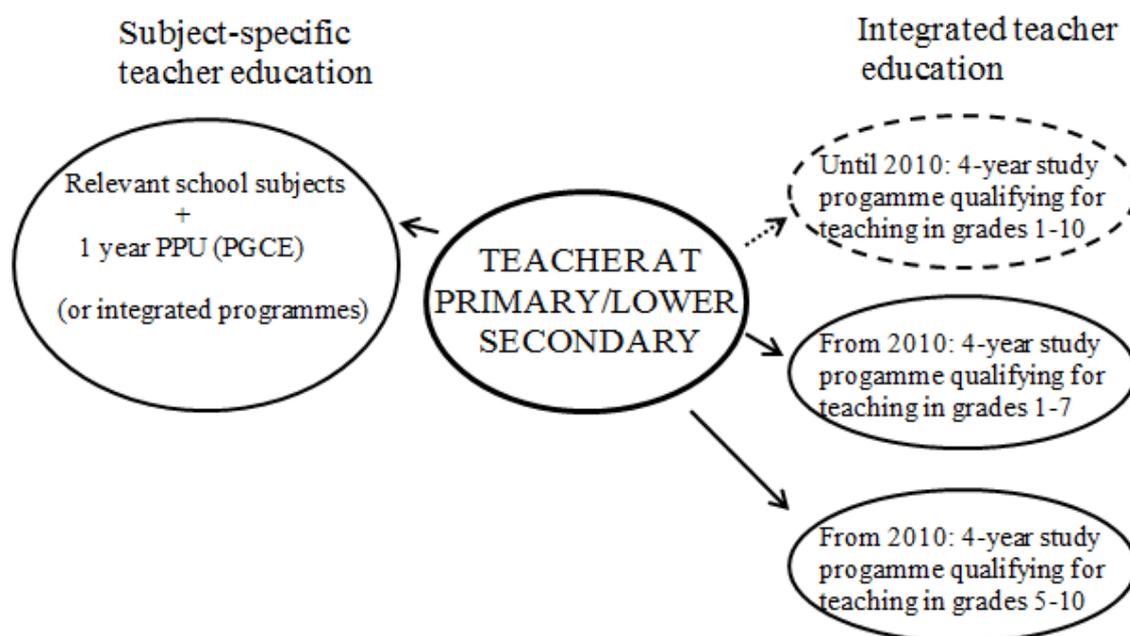
Figure 2.8. Teacher education in Norway

3 Historical development of functions and differentiation

This section provides a brief overview of the historical development of the function concept, the concept of differentiation and proportional magnitudes. These concepts are of particular interest with regard to my research questions. This should not be understood as a defence of the generic idea that concept formation and students' development of scientific concepts are identical to the historical evolution of the corresponding concepts. However, I consider mathematics to be a cultural, historically developed set of rules, notations and signs, born out of certain needs within a given set of practices (Pozzi et al., 1998). In the wake of this, the process by which the student becomes part of this enculturation is essential. By being able to participate in the mathematical community in terms of appropriating this cultural, historically developed content, students come to understand mathematics, in a Vygotskian sense.

3.1 Historical development of the function concept

Inspired by and rooted in ancient physical and geometrical problems of determining the areas of regions bounded by curves, Leibniz and Newton, in different ways, contributed to the creation of the foundations of calculus. Mathematicians at that time were not in general concerned with the function concept itself, but focused on curves defined only as a relation between variables (Katz, 2004). Such curves could take many forms, such as ellipses and circles, which fall outside today's function concept. Exemplified by the case of velocity and Oresme's representation of motion in the middle ages (Atkinson, 2002), an implicit awareness of functions as we know the concept today is claimed to have been present in physics also in earlier times, through the principle of causality. Causality means that every effect has its cause, a principle applied, for example, by Babylonian astronomers who studied the movement of celestial bodies by regarding their positions as a function of time (Thompson, 1991). However, a definition of the concept was not really formally established until Euler's work in the 18th century.

> [H]is predecessors had considered the differential calculus as bound up with geometry, but Euler made the subject a formal theory of functions which had no need to revert to diagrams or geometrical conceptions. (Boyer, 1949, p. 243)

Consequently, at the beginning of calculus, no specific need existed for formalizing a theory of functions outside the realm of geometry. At this point, for example various curves related to physical phenomena in astronomy in terms of circles and ellipses, frequently occurred. Hence, there were no fundamental formal distinctions between these types of

curves and what we today know as functions, in terms of the one/many-to-one principle or the property of uniqueness. For Euler, the concept of functions was not primarily conceived of as a quantity dependent on variables, but more in terms of

> an analytic expression in constants and variables which could be represented by simple symbols. Functionality was a matter of formal representation, rather than a conceptual recognition of a relationship. (Boyer, 1949, p. 243)

Since calculus had, for centuries, been considered an instrument for "dealing with relationships between quantities involved in geometrical problems" (Boyer, 1949, p. 271), Euler and Lagrange in their eagerness to establish the calculus on the formalism of the function concept, represented the exception to this rule. In 1748, Euler's definition took the following form: "A function of a variable quantity is an analytical expression composed in any way whatever from that variable and numbers and constant quantity" (Burton, 2003, p. 571).

Euler also introduced the notations of *f* and parenthesis for a function, in terms of *f(x)*. His definition gradually evolved, and somewhat later he developed the analytical expression by introducing the following definition: "If, therefore, *x* denotes a variable quantity, then all quantities which depend upon *x* in any way or are determined by it are called functions of it" (Burton, 2003, p. 572).

Fourier went further, by defining functions as different parts of a curve. This entailed a broader function concept compared to preceding definitions:

> The function *f(x)* represents a succession of values or ordinates each of which is arbitrary....We do not suppose these ordinates to be subject to a common law; they succeed each other in any manner whatever, and each of them is given as if it were a single quantity. (Burton, 2003, p 572)

What the previous definitions lacked were the concepts of independent and dependent variables. Another essential property of functions missing in prevailing definitions was the property of uniqueness. The uniqueness property means that for each value of the independent variable there exists one and only one value of the dependent variable. Finally, in 1837 Dirichlet came up with a definition which to a large extent still prevails in the mathematical community:

> y is a function of the variable x, defined on the interval a < x < b, if to every value of the variable x in this interval there corresponds a definite value of the variable y. Also, it is irrelevant in what way the correspondence is established. (Burton, 2003, p. 572)

In modern times, methods of defining functions have been offered from set-theory. Here, functions can be regarded as *morphisms*, i.e. mappings between a domain and its codomain, where every element in the domain corresponds to one, and only one element in the codomain. If the domain of the function is denoted *X* and the codomain *Y*, the

corresponding morphism can be written as $f: X \rightarrow Y$. Inspired by set-theory, Bourbaki gave the following definition of a function in 1939:

> Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and a variable element y of F is called a *functional relation* in y if, for all $x \in E$, there exists a unique $y \in F$ which is in the given relation with x. We give the name of *function* to the operation which in this way associates with every element $x \in E$ the element $y \in F$ which is in the given relation with x; y is said to be the *value* of the function at the element x, and the function is said to be *determined* by the given functional relation. Two equivalent functional relations determine the *same* function. (Bottazzine, 1986 as cited by Kleiner, 1989, p. 299)

Compared, for example, to the limitations of Euler's definition which entails the necessity of a function expression, Dirichlet and Bourbaki's definitions gained new ground. The relation between the independent and dependent variable and the property of uniqueness are emphasized. In my study, variants of these prevailing mathematical definitions occurred in different simplified versions in textbooks used in both lower and upper secondary school (Section 2.3.2).

This brief outline gives rise to some relevant questions related to functions and school mathematics. For example, why do we *need* the uniqueness property in definitions, and why should this property be emphasized in teaching? Pragmatically, as a result of the uniqueness property, calculus becomes more manageable since, for example, derivatives and integrals become uniquely determined. As a consequence of uniqueness, both differentiation and integration become well-defined mathematical operations. Also, as previously mentioned, causality and the idea of determinism constitute a historical link to the uniqueness property. From this perspective, observations in *time* and the preference for only "one answer" for each value of the independent variable clearly have potential in teaching for justifying the formal definition of functions.

3.2 The historical development of differentiation

In dealing with polynomial functions, and the corresponding tangents to their graphs, Fermat constructed the difference $f(A + E) - f(A)$ and divided this difference by E to obtain the quantity

$$\frac{f(A + E) - f(A)}{E}.$$

He then sets $E = 0$,

$$\left. \frac{f(A + E) - f(A)}{E} \right|_{E=0},$$

and computes the result (Edwards, 1979, p. 190).

If $E > 0$ the expression would have corresponded to the gradient of a secant to the graphical representation of f, passing through the points

$(A, f(A))$ and $(A + E, f(A + E))$. But when $E = 0$, as in the equation above, the expression equals the gradient of the tangent to the point $(A, f(A))$ on the graphical representation of the corresponding function. Although this corresponds to the derivative of a function, Fermat did not name this quantity. It was Fermat's translation of Kepler's observation "that the increment of a function becomes vanishingly small in the neighbourhood of an ordinary maximum or minimum value" (Eves, 1990, p. 390) which led to the above equation.

In the 17th century Isaac Barrow contributed to the development of the differentiation concept by introducing the *differential triangle*.
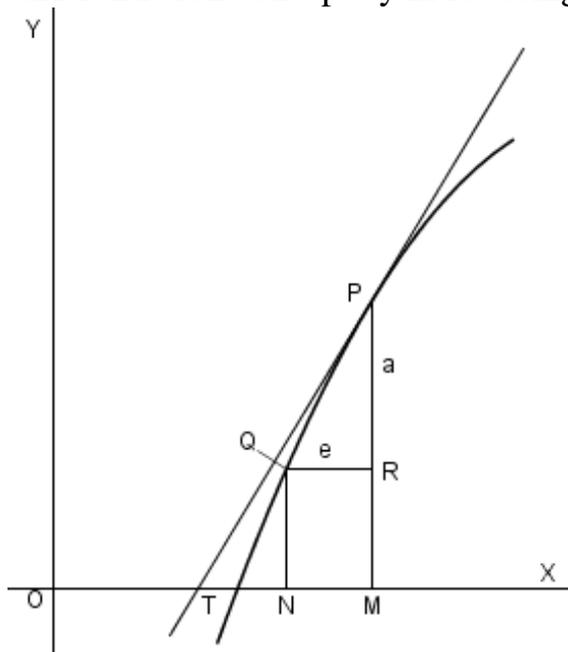
Figure 3.1. The differential triangle (Adapted from Eves, 1990, p. 395)

Figure 3.1 illustrates Barrow's argument, that if Q is a neighbouring point to P on the curve, then the approximate triangle PQR becomes infinitely small and hence "very nearly similar" (Eves, 1990, p. 395) to the triangle PTM. Then the equation

$$\frac{RP}{QR} = \frac{MP}{TM}$$

is satisfied (Eves, 1990).

Further, Barrow is also credited with the important discovery of the so-called *fundamental theorem of calculus* which establishes the "inverse relationship between tangent and area problems" (Edwards, 1979, p. 190), or in more modern terms, the "full generality that differentiation and integration are inverse operations" (Eves, 1990, p. 296).

The gradient of a function is central to my research questions and strongly relates to the derivative concept, the roots of which are briefly presented above. The expression *gradient of a function* usually refers implicitly to the representation forms graphs, function expression and

situation. I will use the concept *gradient* throughout this thesis, as I think it is the best translation of the Norwegian word "stigningstall" which is invariably used in Norwegian textbooks and which was what I observed being used in teaching situations.

Currently, the gradient of a function (in general terms) is given by

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

This is particularly relevant for students in upper secondary general studies. In the case of linear functions, because $\Delta y / \Delta x$ is constant, we no longer need to take the limit of the gradients of the corresponding secants through the points $(x, f(x))$ and $(x + \Delta x, f(x + \Delta x))$ as $\Delta x \to 0$. This is because the gradient is then defined by

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

and the secant now coincides with the line itself. A more common 'textbook version' of this, valid for linear functions, includes two fixed points (x_1, y_1) and (x_2, y_2):

$$a = \frac{y_2 - y_1}{x_2 - x_1}, \text{ where } a \text{ represents the gradient.}$$

Finally in this section, I will only briefly point to a historical aspect from Euclid, related to research question 2c) and proportional magnitudes. The definition of proportional magnitudes is relevant for both lower secondary and upper secondary vocational studies. "Let magnitudes which have the same ratio be called *proportional*" (Joyce, 1996: Euclid's "Elements", book 5, definition 6).

The *ratio* is the quotient which emerges when dividing two magnitudes. In the applied textbooks, the ratio of proportional magnitudes is often denoted as the "proportionality constant", which in turn can be conceived of as gradients of linear functions with constant term equal to zero.

4 Theoretical background

In this chapter I present a theoretical basis for the subsequent analysis in chapters 7 and 8. The first section of this chapter includes elaborations of my position within the sociocultural perspective. Concept formation, semiotics and the application of Steinbring's epistemological triangle as an analytical tool will be discussed in subsequent sections. My approach to sociomathematical norms and the terms conceptual and procedural knowledge will be clarified. Relevant here are mathematical representations, functions as boundary objects and conceptual knowledge.

4.1 The socio-cultural perspective

4.1.1 A brief introduction

Lev Vygotsky, the founder of what is known as the socio-cultural (or cultural-historical) theory of learning, does not distinguish between social and individual aspects of learning. In contrast to other learning theories such as constructivism (Bruner, 1997; Jaworski, 1994), which focus on the individual cognitive construction of knowledge, there is no separation between the social and the individual when it comes to the essence of learning. In fact, a dialectic focus between the social and individual aspect is what characterises the socio-cultural epistemology.

> [T]he most significant moment in the course of intellectual development, which gives birth to the purely human forms of practical and abstract intelligence, occurs when speech and practical activity, two previously completely independent lines of development, converge. (Vygotsky, 1978, p. 24)

According to this view, speech and language, arise from social settings, and are directly linked to the individual's development of higher mental functions.

> The greatest change in children's capacity to use language as a problem-solving tool takes place somewhat later in their development, when social speech…is *turned inward*…language thus takes on an *intrapersonal function* in addition to its interpersonal use. (Vygotsky, 1978, p. 27)

Important concepts related to this view are *interpersonal* and *intrapersonal* processes. While interpersonal processes describe ways in which an individual can mediate his/her conception of the world, intrapersonal processes describe the individual's conception of the world (Vygotsky, 1978). The idea that mind and society are almost literally intertwined constitutes the main characteristic of the socio-cultural perspective. Hence, social interactions such as peer collaboration and instructions play a vital role for individual development as these social interactions bear the potential of becoming *internalized*. Social interactions can also potentially reveal new areas of knowledge which it

is unlikely that the child could have reached alone. Vygotsky describes such areas as *the zone of proximal development*:

> [T]he distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under guidance or in collaboration with more capable peers. (Vygotsky, 1978, p. 86)

As can be seen from this short introduction, communication in its many forms is of considerable importance for the individual's learning possibilities.

4.1.2 Mediation

The notion of *mediation* is embedded in my research questions, and demands clarification and further elaboration.

Even though mediation "runs throughout the writings of Lev Semënovich Vygotsky" (Wertsch, 2007, p. 178), Vygotsky does not examine the concept of mediation in detail by explicitly defining the concept, but he points out in various ways, for example by referring to Hegel, that a mediated activity consists of the interplay between *signs* and *tools*.
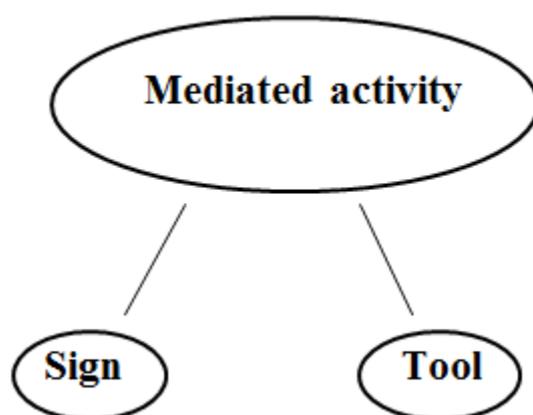
Figure 4.1: Mediated activity (Adapted from Vygotsky, 1978, p. 54)

In this study I draw on Vygotsky's (1997b) rather broad definition of signs: "[E]very conditioned stimulus created artificially by man that is a means of mastering behaviour – that of another or one's own – is a sign" (p. 54). The influence of Pavlov's stimuli-response work is apparent in this quotation; the parallel is also explicitly drawn by Vygotsky himself. This definition entails not only all kinds of written language, symbols and notations, but also actual spoken language and speech as "speech is a sign for the communication between consciousnesses" (Vygotsky, 1997a, p. 137). Vygotsky mostly exemplifies *tools* by working tools and instruments. On several occasions he compares tools and signs and points out that in terms of this analogy the conception of signs is coherent with what he calls psychological tools.

The following can serve as examples of psychological tools, and their complex systems: language; various systems for counting; mnemonic techniques; algebraic symbol systems; works of art; writing; schemes, diagrams, maps, and mechanical drawings; all sorts of conventional signs; and so on. (Vygotsky, 1981, p. 137)

The quotation above contains examples of what Vygotsky describes as systems of psychological tools. The quotation also demonstrates the equivalence of Vygotsky's notions *psychological tools* and *signs*. Hand-written material produced by students, their computer work, their answers and arguments during interviews and conversations, all related to learning mathematics, in this case specifically functions, are all examples of signs related to various tools. Signs and tools can jointly be termed *mediating means* and from now on mediation will be understood and defined as all kinds of interplay between signs, tools and human beings.

4.1.3 Artefacts, tools and signs

Artefacts, tools and *signs* are key notions in the socio-cultural theory of learning (Säljö, 2000). It is therefore appropriate to define these concepts and explain how they are interpreted in this study. In the previous section I gave an account of the Vygotskyian perspective of tools and signs, and in this section I will briefly discuss how these relate to artefacts. Säljö (2000) points out that "physical tools are included in the culture – *artefacts* – forming the whole of our everyday life…the development of material resources goes hand in hand with the development of ideas and intellectual knowledge" (p. 29, my translation).

Utilities which may be used in teaching and learning mathematics (e.g. chalk and blackboards, textbooks, calculators, computer software etc.) are examples of *artefacts*. When these artefacts are actually used, I will denote these as *tools*. In the course of my research I have observed several lessons where the teacher mediates mathematical content using chalk and blackboard. Students typically worked individually, solving tasks from textbooks, mostly using a calculator, a pencil, a ruler and a notebook. On a few occasions, teachers used computers and computer software, especially for graphing functions. Sometimes in these lessons, students also used computers (and this software) while working on exercises. The above examples constitute the use of typical tools in mathematics teaching in this study.

Less obvious materialized utilities generated by written and spoken form are sometimes called tools in the literature (Tall, 1991) and in Section 4.1.2 I described how Vygotsky applied the term psychological tools. As I draw on Vygotsky in describing mediation as the interplay between tools and signs, I find it appropriate to separate the notions of tools and signs. Only physical, materialized utilities, such as the various

instruments already mentioned will be called tools. For example, a pencil, a computer, a textbook and a blackboard are tools, while algebraic symbol systems, writing, schemes, diagrams and maps are signs. In the particular case of functions, various representations of functions are important examples of signs.

Students working with representations of the function concept in 10th and 11th grade are an essential part of this research.

4.1.4 Justifications

Summarized, related to students' learning, I will be working within the framework of the socio-cultural theory of learning. There are several reasons for drawing on this framework and I will briefly point out some of the main ones. In my view, mediation is a relevant issue in this research as my observations include teaching situations where teachers act as mediators of mathematical content in the classroom. Students' explanations and reasoning during interviews as well as observations can also be understood in terms of mediation. The importance of language and mediation in social-cultural learning theory entails the possibility of powerful frameworks for operationalizing the process of mediation for example by semiotics (Section 4.2) and semiotic models. Further, semiotic models and semiotic chaining (Section 4.2.3) to some extent solve the issue of "development" as development may be understood through the construction of semiotic chains. In my view, semiotic chains offer an alternative to constructivist concept formation models, as concept formation is closely linked to the acquisition of "new" concepts through different "reference contexts" (Steinbring, 2005, 2006).

In addition to the justifications already presented, a theoretical perspective inevitably reflects a researcher's personal convictions about the very nature of learning as well as the ontological and epistemological considerations which these convictions generate. And the underlying ideas of the sociocultural perspective match most of my convictions in the discourse related to teaching and learning.

4.2 Semiotics

In a broader perspective, the discussion about signs and mediation belongs to the field of *semiotics*. Semiotics has several subdomains including the semiotics of mathematics. In very general terms semiotics can be defined as the "science of signs". However, due to pluralism within this discipline, this simple definition is not accepted by everyone (Nöth, 1990). It is beyond the range of this study to provide an extensive overview of the history and the many branches of semiotics, but for the sake of establishing the sources of the analytical tools which I employ, I will outline a few historical roots.

Most of the scientific community considers Charles Sanders Peirce (1839-1914) to be the founder of modern semiotics (Nöth, 1990). To appreciate some of his main ideas it is worth looking at some of his philosophical roots. Like, for example, Aristotle and Kant, Peirce developed universal categories as a basis for his phenomenology. He argued for three such categories which he called *firstness, secondness* and *thirdness* (Nöth, 1990). Firstness is "the mode of being of that which is such as it is, positively and without reference to anything else" (Nöth, 1990, p. 41). One example of a phenomenon belonging to this category is an "unreflected feeling" with no qualities other than its own immediacy and independence. Secondness is a relation of a first to a second, for example in terms of comparison and experience in time and space. An example of this is the experience and realization of physical forces and cause and effect. Thirdness considers the relation between the second and the third. Communication, signs and mediation are important phenomena belonging to the thirdness category. From this perspective, as a phenomenon of thirdness, Peirce formed his *semiosis*, namely a triadic model which serves to illustrate the process "in which the sign has a cognitive effect on its interpreter" (Nöth, 1990, p. 42). This triadic model of sign consists of the *representamen*, the *object* and the *interpretant*. The representamen constitutes the observable sign, while the object is what the sign represents. The interpretant corresponds to the meaning of the sign and has been defined as the outcome created in the mind of the interpreter (Nöth, 1990). In the context of mediation (4.1.2) this outcome may be understood as a result of the mediation between signs and objects.

Charles W. Morris described semiosis as a process of semiotic mediation. "A sign is used with respect to some goal if it is produced by an interpreter as a means of attaining that goal; a sign that is used is thus a means-object" (Morris, 1946 as cited by Nöth, 1990, p. 52). Here it seems natural to point to the striking resemblance to the quote from Vygotsky (1997b) presented in Section 4.1.2: "[E]very conditioned stimulus created artificially by man that is a means of mastering behavior – that of another or one's own – is a sign" (p. 54). Morris' triadic model notations differ from those of Peirce as Morris suggested *sign vehicle, designatum* and the *interpretant* to denote the three components of his model. "*S* is a sign of *D* for *I* to the degree that *I* takes account of *D* in virtue of the presence of *S*" (Morris, 1938, p. 4). As I see it, this terminology even more clearly displays the role of mediation, especially through the notation sign *vehicle*. I find Morris' definition of semiotic mediation to be of particular interest since it stresses that the "sign is produced with respect to some goal" which in my view is precisely the key issue in teachers' mediation in the mathematics classroom. Morris

takes two concrete examples to illustrate the relations between the interpretant (I), the designatum (D) and the sign vehicle (S).

> A dog responds by the type of behavior (*I*) involved in the hunting of chipmunks (*D*) to a certain sound (*S*); a traveler prepares himself to deal appropriately (*I*) with the geographical region (*D*) in virtue of the letter (*S*) received from a friend. (Morris, 1938, p. 3)

Through this example the sign vehicle has the role of being a mediator between the interpretant and the designatum. In the first example the mediating role of the sign vehicle is a certain sound, telling the dog about the presence of chipmunks and hence, how to act. In the second example a letter plays the role of a sign vehicle by mediating relevant information of the geographical location (designatum) to the traveler (interpretant).

At last, the triadic model of Frege should be mentioned as this serves as important background for the section about Steinbring's epistemological triangles (Section 4.2.2). In Frege's triangle the vertices consist of *sign* (Zeichen), *sense* (Sinn) and *meaning* (Bedeutung)[5] (Frege, 1980b; Sowa, 2000; Steinbring, 2005). The sign is to be understood as a label for the existing objective idea, or the meaning. One of Frege's central ideas is that all such meanings exist independently beforehand (Steinbring, 2005, p. 23). According to Frege, a sign can take many forms but still represent the same objective idea, as in the case of mathematical expressions (Frege, 1980a, p. 22). "Sense" represents the subjective interpretation and differs from the objective meaning. Frege (1980b) writes: "If what a sign means is an object perceivable by the senses, my idea of it is an internal image, arising from memories of sense impressions" (p. 59).

This brief outline of the development of triadic semiotic models provides the background for the implementation of triadic models in my study. However, it should be mentioned that *dyadic* semiotic models have evolved in juxtaposition to the triadic ones (Saussure, 1959; Walkerdine, 1988). In these models signs are regarded as a dyad consisting of two components; "the signified" and "the signifier". The object (signified) is represented through a certain symbol (signifier), and the sign is constituted of both, taken together.

4.2.1 Semiotics and concept formation

By basing my argumentation on the Vygotskian understanding of mediating through signs and tools, I approach concept formation from a semiotic perspective. Presmeg (2005) applies a triadic semiotic model and uses the same terms as Peirce, namely *representamen*, *object* and *interpretant*. One can regard the representamen as a symbol or an idiom.

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[5] I included the original German notation in parentheses since literature is not always consistently translated. The English translation I use is taken from Steinbring (2005).

One example could be the linear expression $y = 2x + 3$. Classifying this expression (sign) in terms of "being a function", an "algebraic expression" or a "linear equation" are all ways to describe the mathematical *object*. Interpreting this sign, by acting on it through different representations, for example by drawing a straight line intersecting the y-axis at -3, making a value table or performing algebraic manipulations are all acts of the *interpretant*. In turn, such acts contribute to the individual's concept formation and meaning making.

> This interpretant involves meaning making: it is the result of trying to make sense of the relationship of the other two components, the object and the representamen. It is important to note that the entire first sign with its three components constitutes the second object, and the entire second sign constitutes the third object, which thus include both the first and the second signs. Each object may thus be thought of as the reification of the processes in the previous sign. (Presmeg, 2005, p. 107)
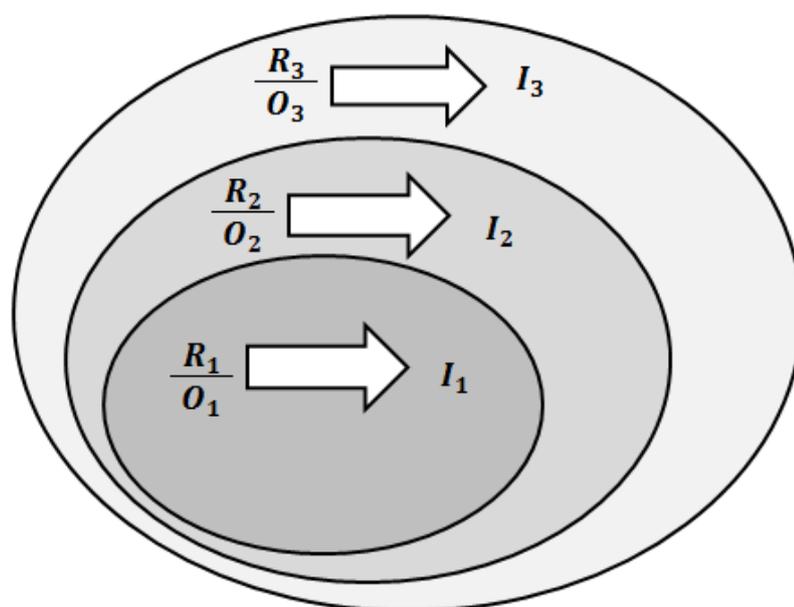
Figure 4.2. A representation of a nested chaining of three signs (Adapted from Presmeg, 2005, p. 107)

The figure above illustrates concept formation as understood by Presmeg (2005). R is the representamen, O is the object and I is the interpretant. These three components together constitute the sign. This inner ellipse represents a certain object (O_1) its representamen (R_1) and the intrepretant (I_1) interpreting the relation between these two (O_1 and R_1). Related to my study, this could for example be the mathematical object of *gradients,* its representamen a (as in function expression $= ax + b$) interpreted for example by the interpretant to be "the change in the y-direction divided by the change in the x-direction", related to a linear

graphical representation. In the second circle a new mathematical object (O_2) and a corresponding representamen (R_2) are available for the interpreter (I_2) but in a such way that this object directly requires a knowledge of the previous (the inner circle). More precisely, O_2 is constituted by the whole inner circle and could be expressed in terms of $O_2 = \{R_1, O_1, I_1\}$. The mathematical object could for example be the concept of differentiation, where the interpretation of this concept builds on the previous (the case of gradients). In similar terms the outer circle could be the case of linear operators, where differentiation might serve as a pre-step towards that generalisation. This process of hierarchical concept formation is what Presmeg (2005) calls *semiotic nesting.*

Presmeg's model is also in some sense a dynamic model. In the cyclic nature of this process, when students through communication and mediating activities (in a Vygotskian sense) slightly change their interpretation of a given representamen and its corresponding object, this will "also inform the creation of this new object" (Presmeg, 2005, p. 107).

The role of students' personal interpretations in developing mathematical concepts is prominent in most of Vygotsky's work. Vygotsky distinguishes between *everyday concept*s, concepts as we might use them in our everyday language and *scientific concepts* as defined and used, for example, in science and scientific research (Vygotsky, 1987). In the possible transition from everyday concept to scientific concept, Vygotsky emphasises the importance of instruction:

> Conscious instruction of the pupil in new concepts (i.e. new forms of the word) is not only possible but may actually be the source for a higher form of development of the child's own concepts, particularly those that have developed prior to conscious instruction! (Vygotsky, 1987, p. 172)

4.2.2 The application of Steinbring's epistemological triangle as an analytical tool

Steinbring (2005, 2006) offers an alternative triadic model. Like those of Presmeg and Peirce, this model consists of three components but in Steinbring's model they are called *reference context/object*, *sign* and *concept*. Important semiotic roots of what Steinbring denotes as the *epistemological triangle* are found in the work of Frege (see Section 4.2) and Ogden and Richards (1930).

Ogden and Richards (1930) developed the ideas of Frege and his concepts of "sign", "sense" and "meaning" (Section 4.2) represented by a new semiotic triangle.
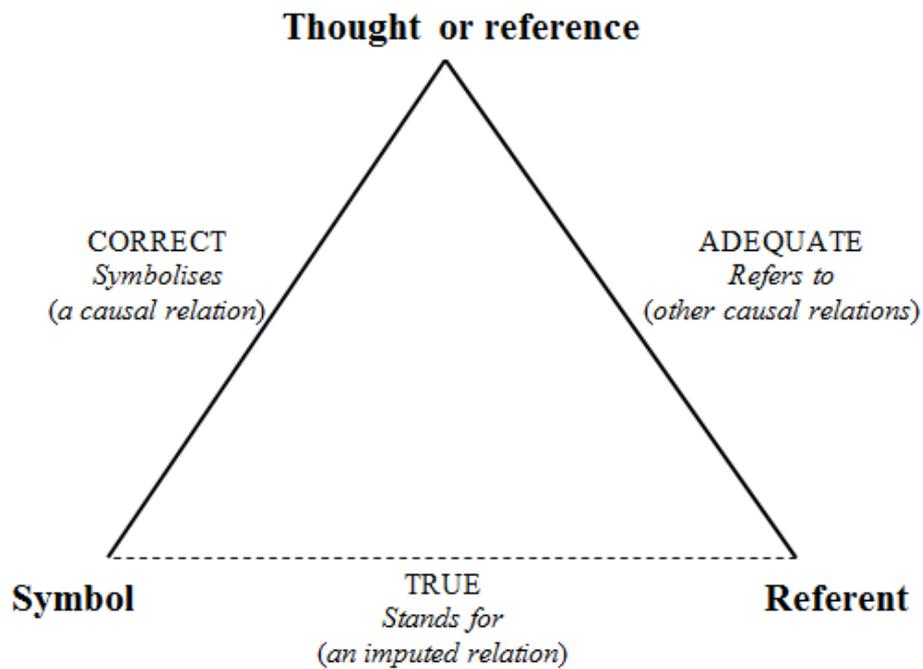
Figure 4.3: The triangle of meaning (Adapted from Ogden & Richards, 1930, p. 11)

Unlike Frege, Ogden and Richards do not claim the independent existence of meaning. "It is Thought (or, as we shall usually say, *reference*) which is directed and organized, and it is also Thought which is recorded and communicated" (Ogden & Richards, 1930, p. 9). By "directed and organized" Ogden and Richards suggest that meaning in this sense can change, for example through scientific progress or human development. It is not a static or independent phenomenon, as it is for Frege. Steinbring points out that "the relation between symbol and referent is not given in a pre-fixed manner but is of an indirect nature, and thus this relation has to be constructed in an agreed way" (Steinbring, 2005, p. 24). In this sense, I consider the model consistent with a cultural-historical evolution of meaning and the Vygotskian perspective. To a large extent, Steinbring builds on the ideas of Ogden and Richard, but develops them further:

> [I]n contrast to the triangle of meaning by Ogden and Richards, the constructions of relations between "sign/symbol" and "object/reference context" over the "concept" does not lead to final, unequivocal definitions, but is understood as a complex relationship. As explained before, the connections between the corners of the triangle are not explicitly defined and unchangeable…In the course of further development of mathematical knowledge, the interpretation of the sign system with matching reference contexts will change. (Steinbring, 2005, p. 24)

Compared to Vygotsky's framework, as I interpret it, signs understood in Steinbring's sense strongly correlate with Vygotsky's view on child development and the role of external signs. Vygotsky (1987) writes

with gradual accumulation of naive mental experience, the child reaches the stage of external sign and external operations. Here, the child solves the internal mental task on the basis of the external sign. (p. 115)

I have applied Steinbring's model as my main analytical tool for two main reasons. First, in accordance with a socio-cultural perspective on learning, this model emphasizes the interaction between the reference context and the sign, so that mediation becomes an essential part of students' concept formation. It is therefore suitable for analyzing the actual teaching/mediation going on in the classroom. Secondly, I see this as a useful way of analyzing students' conceptual development, as it is possible to see conceptual development as a semiotic chain consisting of linked epistemological triangles. The figure below depicts Steinbring's epistemological triangle:
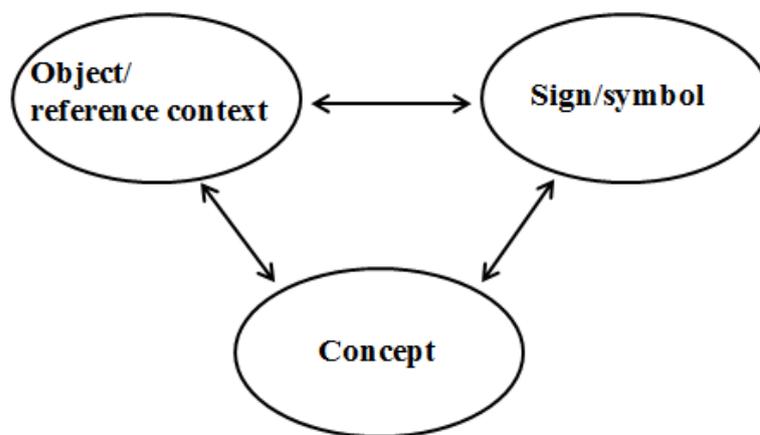
Figure 4.4: The epistemological triangle (Adapted from Steinbring, 2005, p. 22).

Steinbring's main idea is that mathematical signs do not have a meaning of their own, and therefore meaning has to be "produced by students or teacher by establishing mediation between signs/symbols and a suitable reference context" (Steinbring, 2005, p. 22). In this sense, two functions can be associated with mathematical signs:

1) A semiotic function: the role of the mathematical sign as "something which stands for something else".
2) An epistemological function: the role of the mathematical sign in the context of the epistemological interpretation of mathematical knowledge.
   (Steinbring, 2005, p. 21)

The "object/reference context" in Steinbring's epistemological triangle represents what the sign/symbol may refer to. For Presmeg, meaning is rooted in the interpretations of the "interpretant" based on the given relation between the "object" and the "representamen". In Steinbring's model the epistemologically grounded mediation between the object/reference context and the sign/symbol is emphasized. At the same time, this mediation with its epistemological possibilities and constraints also allows for the construction of "new and more general mathematical

knowledge" (Steinbring, 2005, p. 22). In my study, examples of this are the symbols $\Delta y / \Delta x$ and their reference to "the gradient of a linear function". The concept is to be mediated through establishing a relation between the symbols and the reference context. Steinbring (2005) points out that "in order to obtain meaning, mathematical sign systems require suitable reference contexts" (p. 21). Even though choice of suitable reference contexts will lead to certain characteristics in the mediation determined by the relation to the sign/symbol, Steinbring claims that due to mathematical epistemological conditions, this mediation is not entirely subjective. This is because meaning obtained through such mediations rests on certain epistemological conditions of mathematical knowledge and the intrinsic relations between them which in turn secure some objectivity with respect to the meaning of the concept. In the case of $\Delta y / \Delta x$ and "the gradient of a linear function" the epistemological mathematical restrictions lie in the given relation between gradient and the relation between a given "change in the y-direction divided by its corresponding change in the x-direction". One possibility for mediated meaning in this particular case could be discussions about "steepness" and corresponding visualizations.

4.2.3 Semiotic chains

In Section 4.2.1 I argued that semiotic nesting can be regarded as a tool for investigating students' concept formation. The core idea of semiotic nesting is that mathematical concepts relate to one another in terms of building on prior mathematical concepts. This process could also be referred to as *semiotic chaining*, and in this section I will elaborate on this further as I will argue for the link between semiotic chaining and the notion of *development* (applied in my research questions). The notion of development must obviously be expanded and clarified. How can development be operationalized and measured in a study such as this?

Within the field of semiotics, one way of investigating students' development is through the study of semiotic chains. Building on ideas mainly from Farrugia (2007), Maracci and Mariotti (2009) and Presmeg (2005), I will define a semiotic chain as an *iterative movement between two signs*. The core idea of semiotic chains as these are applied in my study is to identify how students and teachers mediate meaning of mathematical signs by linking these signs to prior (or other) mathematical signs. This is also the way I conceive of the notion of development in this study. By operationalizing the notion of development in this way, students' arguments and reasoning can be studied in detail, with focus on certain key concepts used in their explanations. In this study, students' understanding of concepts such as functions, gradients and differentiation could be analyzed through their choice of phrasing and articulations when they are asked to explain

certain tasks or concepts. These phrasings might have been adopted from teacher explanations and/or discussions with peer students. Thus semiotic chains also open for the possibility of suggesting certain links between teaching and learning. Looking at students' reasoning and teaching sequences through the lenses of semiotic chains is therefore one way of describing and operationalising students' understanding and their development with regard to mathematical concepts. Also, in my view, some suggestions concerning the relations between teaching and learning can be made by attempting to identify similarities between students' reasoning and the semiotic chains provided from different teaching sequences.

The construction of semiotic chains is not limited to one type of semiotic models, such as the triadic models discussed in preceding sections. In principle, all semiotic models can be composed in such a way that they constitute semiotic chains.

As clarified and argued for in Section 4.2.2, in my analysis I will mainly apply Steinbring's epistemological triangle. It is also possible to construct semiotic chains within this framework as Steinbring (2005) writes: "Furthermore, one can accordingly draw up a sequence of epistemological triangles for the interaction, or a sequence of learning steps to reflect the development of interpretations made by the subject" (p. 23). Farrugia (2007) draws on the Steinbring model to construct such semiotic chains to interpret meanings for multiplication and division. In the figure below, I have modified Farrugia's model to illustrate an example from my study, involving the case of differentiation (Figure 4.5).

In her model, Farrugia (2007) replaced Steinbring's original notion of "concept" with "meaning". "This is because while Steinbring had considered number relationships, I wished to consider words that denoted a variety of notions" (p. 1202). In my case I draw on Farrugia's argument and replace "concept" by "meaning" for many of the same reasons. Gradients, for example, can be represented by a (as in $y = ax + b$), $\Delta y/\Delta x$ or by percentage (as in road signs). In addition, in my opinion, "meaning" more clearly points at possible "subjective aims" for the actual observed lesson. I also find "meaning" more flexible than "concept" in the sense of being able to accommodate other approaches more readily.

By the example of Figure 4.5, I also want to point out another aspect, which might cause some confusion. It might not always be evident what categorizes as "meaning" and what categorizes as "reference context" and/or "signs". I my study I did not always find this distinction to be obvious.
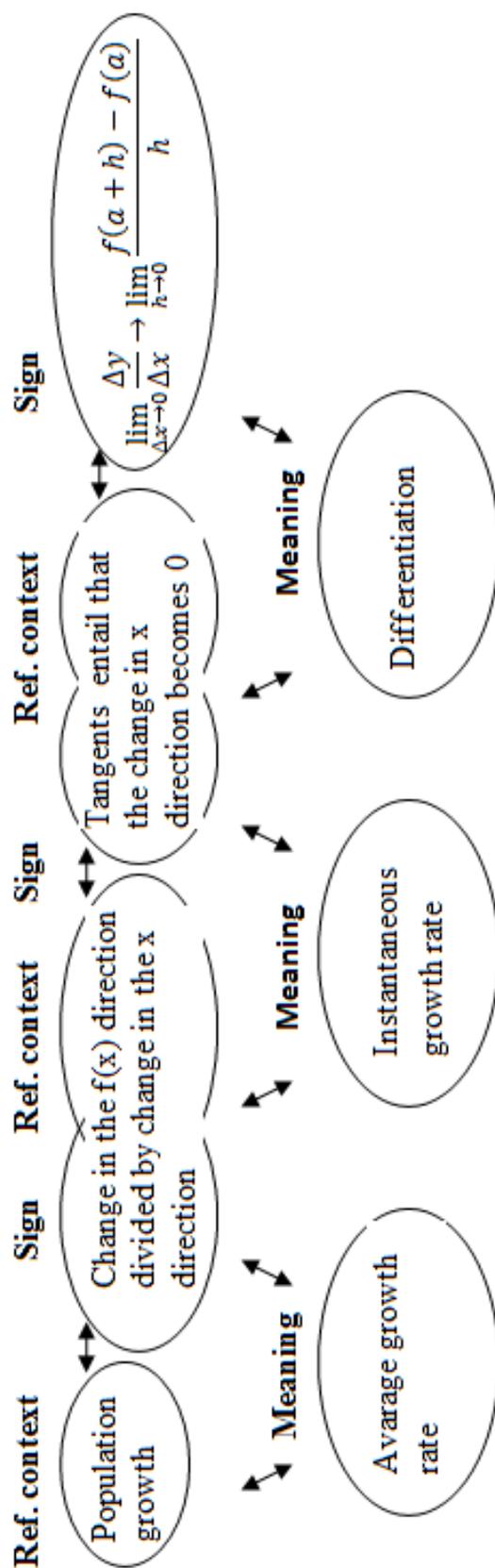
Figure 4.5. Semiotic chaining for differentiation (Based on Farrugia, 2007, p. 1205)

One could argue for "average growth rate" as being the sign, and that "change in the $f(x)$ direction divided by the change in the x-direction" was the meaning as mediated by the teacher. The reason for doing the opposite is to separate, in a consistent manner, what the teacher (or

student) chooses to be the reference context in their mediation process. I regard meaning more as the mathematical topic of the lesson, which is to be explained and mediated to the students through the use of certain examples, drawings and tasks. In other cases, when teachers actively underline a mathematical concept (for example gradients) and explicitly point to the relation between this concept and a certain phenomenon related to this concept (or one might say "topic") this (gradients) would be identified as sign. Meaning would then be the underlying mathematical relations identified, and often implicitly evoked through the teacher's explanations. For example, in the case of gradients, different approaches were offered. At the Waldorf School, the teacher tried to emphasize the relation between gradients in function expressions and the steepness measured in percent as in road signs. The other schools mainly focused on the relation between the gradient and what I denote as the *one-unit-right-a-up/down* strategy (Section 6.4.4).

Ways of considering semiotic chains are also paralleled and theoretically supported in the work of Vygotsky.

> [T]he process of concept formation came to be understood as *a complex process involving the movement of thinking through the pyramid of concepts*, a process involving constant movement from the general to the particular and from the particular to the general. (Vygotsky, 1987, p.162)

Here Vygotsky suggests a movement from the particular towards the general, and vice versa. The parallel to semiotic chains is present by regarding the process of concept formation as a pyramid. Vygotsky's work also suggests a similar two-direction development from "everyday concepts" to "scientific concepts" and vice versa. For the sake of simplicity I refer to these respectively as "bottom-up" and "top-down" chains.

> [T]he level of development of scientific concepts forms a zone of proximal possibilities for the development of everyday concepts. The scientific concept blazes the trail for the everyday concepts. (Vygotsky, 1987, p. 169)

Several such chains can be identified in my study, and I will return to these in the analysis.

Scientific concepts

Function expressions
and the formal
definition of a function

⇧

Finding expressions

⇧

Everyday concepts

Boxes «transforming»
numbers into other
numbers
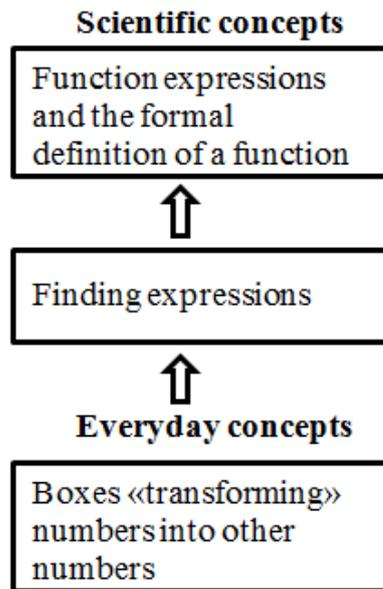
Figure 4.6. Example of a "Bottom-up" chain

Similarly, there are also examples of chains moving in the other direction in my observations. The gradient was often introduced by the teacher identifying (sometimes by asking the students) the a in the expression $y = ax + b$ as the gradient. Then a series of explanations and examples were provided to explain the characteristics of the gradient, often ending up with the "one-unit-right-a-up/down" strategy (Section 6.7). This series of explanations is a typical example of a chain moving in the other direction, from the scientific concepts towards everyday concepts.

Everyday concepts

Moving one-unit-right-
and-a-up/down

⇧

The «steepness» of the
corresponding linear
function

⇧

Scientific concept

The gradient of a linear
function, as a in
$y = ax + b$
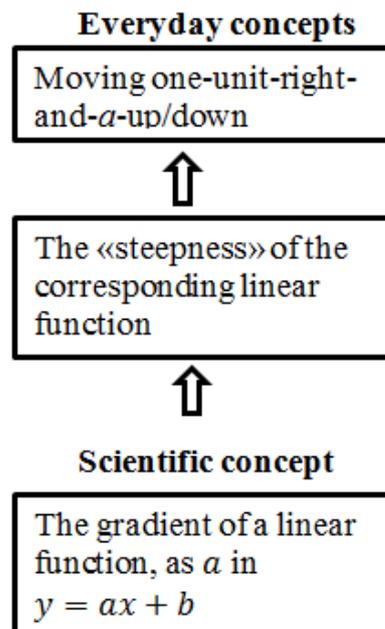
Figure 4.7. Example of a "Top-down" chain

To summarize, the construction of semiotic chains as an operational component in students' development finds support in the work of Vygotsky. I call these *Vygotskyan chains*. The two types are illustrated in Figure 4.6 and 4.7.

4.2.4 Representations

In Section 4.1.3 I mentioned how different representation forms could be understood as being signs. In the following, I present crucial ideas related to mathematical representations and representations of functions in particular. Duval (1999, 2005) is central for grasping the idea of representations in mathematics in general and in my study, I use these ideas to serve as an overarching framework for understanding mathematical representations as a phenomenon. Janvier (1978) also considers a framework for representing functions in particular and his elaborations are of particular interest as his main focus is on different representations of functions. In this section, I start therefore with some general considerations on representations based on the ideas of Duval (1999, 2006) followed by some function-specific theory based on the elaborations of Janvier (1978).

4.2.4.1 Duval and representations

Systems of representations and the transitions between them are of enormous importance in all fields of mathematics, including function related topics. As my research questions indicate, function related topics such as slopes/gradients, differentiation and proportional magnitudes are all part of this study, and representation and transitions between them are as relevant for these topics as they are for any mathematical concept. For example, different representations of functions are ways of making the function concept accessible for our minds even though functions, which these represent, are not external physical objects:

> In other fields of knowledge, semiotic representations are images or descriptions about some phenomena of the real external world, to which we can gain a perceptual and instrumental access without these representations. In mathematics it is not the case. (Duval, 1999, p. 4)

By this ontological statement about the nature of mathematics, Duval points to a central aspect of mathematics and how it differs from other subjects: namely the representations. Not only do different representations help us to understand mathematics – they are unavoidable and absolutely necessary to access mathematical objects in the first place. Duval (1999) points to two historical sources relevant to the development of mathematical representations: language and image. From written language came algebraic notations and writings and formal languages. From imagery came the construction of two and three dimensional figures, curves and graphs. Highly relevant to my study, Duval exemplifies how students struggle with connecting different

representations. He uses the example of two linear functions represented through algebraic and graphical representations and claims that most students are actually unable to connect them.

> All these repeated observations show that semiotic representations constitute an irreducible aspect of mathematical knowledge and that wanting to subordinate them to concepts leads to false issues in learning….Each semiotic register of representation has a specific way of working, of which the students must become aware. (Duval, 1999, p. 7)

Duval hereby points to the importance of being aware that the subordinating of representations to mathematical concepts is a non-trivial issue. According to Duval, the transition between different representations can be divided in two categories: transformation within the same *register* of representations and transformation from one register to another. Same register in this sense means that the representation form is the same, but expressed or depicted in different ways (for example different ways of presenting an algebraic expression). Different register means that the corresponding representation forms are different (for example the transformation from an equation into a Cartesian graph). "Only students who can perform register change do not confuse a mathematical object with its representation and they can transfer their mathematical knowledge to other contexts" (Duval, 1999, p. 9). An example of this within the topic of functions is to draw a graph corresponding to the function expression $y - 2x = 5$. For some students it would be easier to draw the graph of this function if it were given another representation within the same *register*, e.g. $y = 2x + 5$. Duval's claim implies that only students who are able to switch between such registers are able to separate the concept of function from its representations. Hence, what is relevant to the thinking process "in any mathematical activity is to focus on the level of semiotic representation systems and not on the particular representation produced" (Duval, 2006, p. 110). With reference to Frege, Duval states that "it is only at this level that the basic property of semiotic representation and its significance for mathematics can be grasped: the fact that they can be exchanged one for another, *while keeping the same denotation*" (Duval, 2006, p.110). Another important point is the idea that "a mark cannot function as a sign outside of the semiotic system in which its meaning takes on value in opposition to other signs within that system" (p.110).

4.2.4.2 The Janvier representations
Duval's (1999, 2005) claim that mathematical objects are only accessible through representations also involve functions and the function concept. In the case of functions, Janvier (1978) identifies four representation forms: situations, tables, graphs and formulae. In his work, he discusses

the importance of translation processes in order to move between these different representations of functions.

TO FROM	Situations, verbal description	Tables	Graphs	Formulae
Situations, verbal description		Measuring	Sketching	Modelling
Tables	Reading		Plotting	Fitting
Graphs	Interpretation	Reading off		Curve fitting
Formulae	Parameter recognition	Computing	Sketching	

Figure 4.8. Translation skills (Adapted from Janvier, 1978, Section 3.2).

Even though "verbal" related to situation is emphasized in the table, Janvier stresses that situations can take many forms: "Actually, means to describe or simply create situations are numerous. One can think of diagrams, pictures, photographs, films, model works, simulation devices and obviously experiments." (Janvier, 1978, Section 3.4).

In addition to providing an overview of the different representations, this table also pinpoints the different "translation skills" required in order to be able to move from one representation to another.

4.3 Sociomathematical norms and classroom mathematical practices

In order to elaborate on various aspects related in particular to research question three, I frame some of the analysis within the concept of *sociomathematical norms* (Yackel & Cobb, 1996). I use sociomathematical norm as a concept to describe "what becomes mathematically normative in a classroom" (Yackel & Cobb, 1996, p. 460). Transition between lower and upper secondary school involves changes in mathematical content and teaching methods. Content-related issues such as mathematical tasks and the explanatory examples provided by the teacher are important factors to study. In my study, these constituted the major part of the observable teaching sequences, both in lower and upper secondary school. Moreover, these tasks and examples

evoked certain types of language and communication. Teaching sequences often entailed certain kind of mathematical language, which was different in the two settings, lower and upper secondary. Sometimes this was made explicit, and in the case of functions the teachers in upper secondary in various ways emphasized the transition from writing $y =$ … to the notation of $f(x) =$… Correspondingly, for gradients when approaching the topic of differentiation, notation in terms of $\Delta y / \Delta x$ was introduced. Even though there are instances where different types of acceptable mathematical language were made explicit by the teacher, most of the time differences were implicitly embedded in the examples and tasks. One of many such examples is from a lesson in School A, where the teacher used loci and road signs as reference context for gradients. At most of the other schools, gradients were illustrated by right triangles, with a horizontal side of length "one" and a vertical side, of length "a" (Section 2.3.2). In addition to the examples provided by the teachers (mostly on the blackboard), tasks and examples from the textbooks also provide a source for comparing different approaches and priorities.

I consider sociomathematical norms a suitable framework for describing differences (and similarities) in the actual teaching observed in the various schools and in the two phases of schooling (lower and upper secondary). Sociomathematical norms which apply to the actual teaching, mediation and communication observable in the classroom, as well as to the reflections and reasoning behind these possible changes reveal themselves primarily through interviews with both students and teachers. Sociomathematical norms are in turn "constrained by the current goals, beliefs, suppositions, and assumption of the classroom participants" (p. 460). Yackel and Cobb (1996) look at sociomathematical norms in three areas: mathematical difference, mathematical sophistication and acceptable mathematical explanation or justification. The first two areas deal with "issues concerning what counts as different, sophisticated and elegant solutions" (p. 461). In the setting of sociomathematical norms, the focus is on how these issues are "taken-as-shared" (p. 461) in the mathematics classroom. This is also the case for "acceptable mathematical explanations or justifications" (p. 461) which strongly relate to types of tasks and solution-methods expected both *from* and *by* the students.

In principle, sociomathematical norms as defined by Yackel and Cobb only involve normative aspects which characterize mathematics as a subject. However, these norms can be studied in different settings. For example, Yackel, Rasmussen and King (2000) focused on the interaction between the instructor and the students in an undergraduate mathematics course at university level, while Tatis and Koleza (2008) focused on the

sociomathematical norms established between students through collaborative problem solving. Whole-class discussions were the setting for Lopez and Allal's (2007) longitudinal study. Hershkowitz and Schwarz (1999) conducted a study in middle school where they argued that sociomathematical norms do not arise from interactions only, but also from non-verbal actions like computer manipulations. The latter is important in my study, as "what counts as evidence for a phenomenon" (Hershkowitz & Schwarz, 1999, p. 164) is given through activities in a more general perspective. Similarly, I argue that mathematical tasks also entail certain sociomathematical norms, depending on the nature of the task as different types of examples provided by teachers and textbooks, often serve as "solution manuals" for students.

In my comparison between lower and upper secondary school, and the teaching and learning of functions, gradients and differentiation I ground my analyses in both sociomathematical norms and *classroom mathematical practices* as described and defined by Cobb, Stephan, McClain and Gravemeijer (2001). While sociomathematical norms "are concerned with the evolving criteria for mathematical activity and discourse, they [classroom mathematical practices] are not specific to any particular mathematical idea" (p. 126). Classroom mathematical practices "focus on the taken-as-shared ways of reasoning, arguing, and symbolizing established while discussing particular ideas" (p. 126). In my empirical data, these two aspects are intertwined and hard to separate, so at this point I find it most suitable to treat sociomathematical norms and classroom mathematical practices as a whole. For example, in a teaching sequence where the symbol $f(x)$ denotes a function expression, the teacher could approach this symbol in various ways. Not all these various approaches are important factors in the formation of a particular mathematical idea, but at the same time they are indicators of prevailing sociomathematical norms, in terms of which explanations count as acceptable or not.

From the perspective of sociomathematical norms and classroom mathematical practices, I focus on three slightly different aspects regarding the transition (from lower to upper secondary school):

    1) Mathematical language, notations and symbols
    2) Mathematical explanations and justifications
    3) Mathematical tasks.

1) emphasizes the relevance of certain notations observable in this study, such as the use of $f(x) = \cdots$ instead of $y = \ldots$, and the introduction of the set of symbols related to slopes and derivatives. Almost every lesson I observed consisted of two separate parts, where the teacher explained new content at the blackboard in the first part. Teacher explanations and

the nature of these correspond to 2). Tasks and activities provided for the students, and the nature of these tasks correspond to 3).

The perspective of sociomathematical norms and classroom mathematical practices makes it possible to operationalize concepts like passive and active learning, and even though my study is not interaction research where discussions about developing these norms were part of the interviews, these interviews do reveal some attitudes about how teachers and students think these norms "ought to be". On the basis of a developmental research project in a first-grade classroom, McClain and Cobb (2001) argue for the value of explicit discussions involving sociomathematical norms, and claim that such discussions can raise teacher awareness and improve mathematics teaching and learning.

4.4 Conceptual and procedural knowledge

Students' reasoning is central in the analysis and discussions which follow. In particular, students' reasoning is prominent in research question 1. In Section 4.2.2 and 4.2.3, I presented a semiotic tool for analyzing teaching sequences and students' reasoning. As already argued for, semiotic chains in particular, allow essential parts of students' concept formation and conceptual development to be pinpointed. For concept formation and conceptual development, the focus is mainly on the mediation between the reference context and sign/symbols as emphasized and described through the prescribed models. However, in my view, dimensions related to the very nature of students' reasoning are not so easily conveyed by these models alone. These semiotic models primarily offer fruitful ways to characterize how mediation constitutes meaning. But in principle, mediation of meaning always entails a subjective dimension related to teachers and/or students. In an attempt to describe the nature and the quality of the mediated meaning observed in various situations, and questions concerning the *type* of knowledge which students demonstrate through their arguments and reasoning, I find it necessary to make use of some additional terminology. I draw on Hiebert and Lefevre's (1986) terms *procedural* and *conceptual* knowledge.

> Conceptual knowledge is characterized most clearly as knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. […]. In fact, a unit of conceptual knowledge cannot be an isolated piece of information; by definition it is a part of conceptual knowledge only if the holder recognizes its relationship to other pieces of information. (Hiebert & Lefevre,1986, p. 4)

In other words, conceptual knowledge should never be regarded as a stand-alone piece of information. This applies to, for example, students' potential for communicating how various aspects of the function concept

are related. Conceptual knowledge is therefore more than referring to certain rules, algorithms or facts that are learned by heart. Related to my research, conceptual knowledge of functions might imply an ability to relate different representation forms, and the awareness of functions as a mathematical concept transcending the representations. Procedural knowledge is defined in the following way:

> Procedural knowledge, as we define it here, is made up of two distinct parts. One part is composed of the formal language, or symbol representation system, of mathematics. The other part consists of the algorithms, or rules, for completing mathematical tasks. (Hiebert & Lefevre,1986, p. 6)

The first part of this definition relates only to "surface features, not to a knowledge of meaning" (p. 6). The second part usually constitutes a step-by-step approach by following certain instructions related to a specific type of mathematical task. Examples from this study are where students are able to apply the rules of differentiation, without being able to express why these rules work or what differentiation really is about.

Hiebert and Lefevre (1986) claim that: "in some theories of learning and development, the distinction [between conceptual and procedural knowledge] occupies center stage. Although the types of knowledge identified from theory to theory are not identical, there is much overlap" (p. 1). This suggests that conceptual and procedural knowledge could be adaptable to different theories and should not be confused with terms like "understanding" and the process of *how* knowledge is created. For this reason, I consider conceptual and procedural knowledge to be legitimate concepts also within my sociocultural perspective of learning.

I use Hiebert and Lefevre mainly as support for labeling students' knowledge as revealed in interviews and through observations in a rather descriptive manner. I am aware that the concepts procedural and conceptual knowledge most frequently occur in constructivist theories and are therefore normally not associated with sociocultural theories. One of the main reasons for this may be certain associations to constructivist concept formation in which procedural and conceptual are seen as different hierarchical levels of understanding. Because of this, I emphasize that such possible interpretation is *not* in line with my application of these concepts. Hiebert and Lefevre (1986) do not explicitly tie their concepts to such hierarchical levels, nor to a specific theory of learning. Rather, they argue that their distinction between procedural and conceptual knowledge does not belong to one specific theoretical platform but can be applied from several perspectives. In line with this view, I allow myself to maintain my sociocultural approach also when dealing with procedural and conceptual knowledge in the analysis of students' reasoning.

4.5 Functions, gradients and proportionality as boundary objects

In a comparative study such as this, it is an advantage that the mathematical content in focus is regarded as relevant to each of the institutions involved. In the national curriculum, *Knowledge promotion* (LK06), presented in Section 2.3, one can observe that functions play a prominent role at both lower and upper secondary school. However, to recapitulate from the discussions in Chapter 2, the curricula for upper secondary vocational studies do not list functions as a separate topic, even though it can be argued that functions are implicitly included in several of the other major topics. One concept which can appropriately identify a certain object (in this case functions) which is relevant for every institution involved in a comparative study is the one introduced by Star and Griesemer (1989), namely *boundary objects*.

> Boundary objects are objects that are both plastic enough to adapt to local needs and constraints of several parties employing them, yet robust enough to maintain a common identity across sites. (Star & Griesemer, 1989, p. 46)

Star and Griesemer (1989) use boundary objects as an analytical concept "which both inhabit several intersecting worlds and satisfy the informational requirements of each of them" (p. 393). Although Star and Griesemer used it in their study of scientific cooperation at a zoological museum (Berkeley's Museum of Vertebrate Zoology) boundary object is a concept which has been adapted to several scientific domains, including educational studies.

From a sociocultural perspective, activities such as learning and teaching closely relate to the field of practice and it makes little sense to analyse or discuss activities detached from their environment. In this sense, analysing and comparing how an academic discipline is practised within different institutions has an intrinsic value and closely relates to the study of the outcome of students' learning.

Akkerman and Bakker (2011) discuss how boundary objects can be encountered in educational studies by providing a broad analysis of boundary object's many aspects. They conclude that "boundary crossing and boundary objects urges us to look at learning across and between multiple social worlds and thus expands education research beyond the study of learning within single domains and practices" (p. 150). Wenger (2000) discusses the notion of "boundary" as he points out that "the term boundary often has negative connotations because it conveys limitations and lack of access" (p. 232). But instead of maintaining such a view, Wenger argues that the notion of boundary offers possibilities in terms of shared practice and for exploration. Understood in this manner, boundary objects should be seen as "support between different practices" (p. 236).

Conceiving the concept of functions as a boundary object provides justification for the mathematical focus of this study. Hopefully, it establishes common ground for teachers and researchers interested in developing mathematics teaching and the related transition between lower and upper secondary school.

5 A selection of relevant literature

5.1 Relevant studies

In this section I provide a review of selected literature related to issues relevant to my study. I performed keyword-based searches with corresponding findings in five major international journals of mathematics: Journal for Research in Mathematics Education, Educational Studies in Mathematics, Journal of Mathematics Teacher Education, Mathematics Thinking and Learning and ZDM. Based on keywords like function(s), graph(s), representation(s), slope(s), gradient(s), proportional, derivative(s) and transition I read through several abstracts. This process resulted in a selection of articles which I consider highly relevant to this study. Students' reasoning related to functions, slopes, proportional magnitudes and derivatives were relevant criteria for selecting one set of articles. Studies concerned with mathematics in the transition from one phase of schooling to another make up the other set. Because it was difficult to find any studies which focused on both transition issues and functions, I divide the literature review in two: 1) Studies concerning the teaching and learning of functions, slopes/gradients and differentiation, and 2) The concept of "transition" and recent transition studies. In addition to the articles in the five major journals mentioned above, relevant articles from other journals are included in the overview but are not dealt with so systematically.

I first briefly mention here how and why I think these studies relate to my research. Since my analysis and results have not yet been presented, it seems appropriate to postpone more detailed discussion and my conclusion to the end of the thesis. I have organized the presentation of selected literature in two main sections. The first section considers literature dealing with the teaching and learning of topics relevant for my study. I begin the first section with the treatment of the function concept. Several aspects are involved, linked to the complexity of the function concept and students' reasoning. Subsequently, I focus on different representation forms and students' difficulties in linking these representations. Towards the end of the first section I focus on literature which considers gradients, slopes and differentiation. In the second section I deal with literature involving transition. I start the second section by including some general considerations related to the transition issue as a phenomenon, before moving on to literature which deals with specific transition studies.

5.1.1 Studies concerning teaching and learning of functions, slopes/gradients and differentiation

The focus of several of these selected studies relates to students' understanding of the concept of function itself (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dreyfus & Eisenberg, 1982; Sajka, 2006; Williams, 1998).

Dreyfus and Eisenberg (1982) argue for the complexity of the function concept by pointing to three decisive aspects:

> 1) The relatively large number of associated sub-concepts.
> 2) The way functions can be used to "tie together seemingly unrelated subjects, for example geometry and algebra.
> 3) The same function can be represented in a number of different settings (e.g., as a table, arrow diagram, graph, formula or by verbal description). (p. 361)

They further claim that all these aspects contribute to difficulties students experience when it comes to learning the function concept. Dreyfus and Eisenberg illustrate the challenges which must be overcome in their "function block" (Figure 5.1) where the x-axis contains "various settings", the y-axis different "concepts" and the z-axis "a taxonomic scale of levels of abstraction and generalization". They also point out that the axes themselves are multidimensional. The idea behind Dreyfus and Eisenberg's structuring of different aspects of the function concept as a coordinate system is related to their view of learning, in terms of the notions "vertical and horizontal transfer". Vertical transfer "contains components of generalisation and abstraction" (p. 362) while horizontal transfer "is the process of taking a concept from one setting and applying the same concept in a different setting" (p.362). Accordingly, the x-axis represents processes defined as horizontal transfer and mainly corresponds to the third point in the quotation above. The z-axis represents vertical transfer, in accordance with the second point. The first point under the "three decisive aspects" relates to the y-axis. The learning of new associated concepts differs from movements along the x- and z-axis, as learning new concepts "cannot in general be expected to occur without an external stimulus" (p. 364).
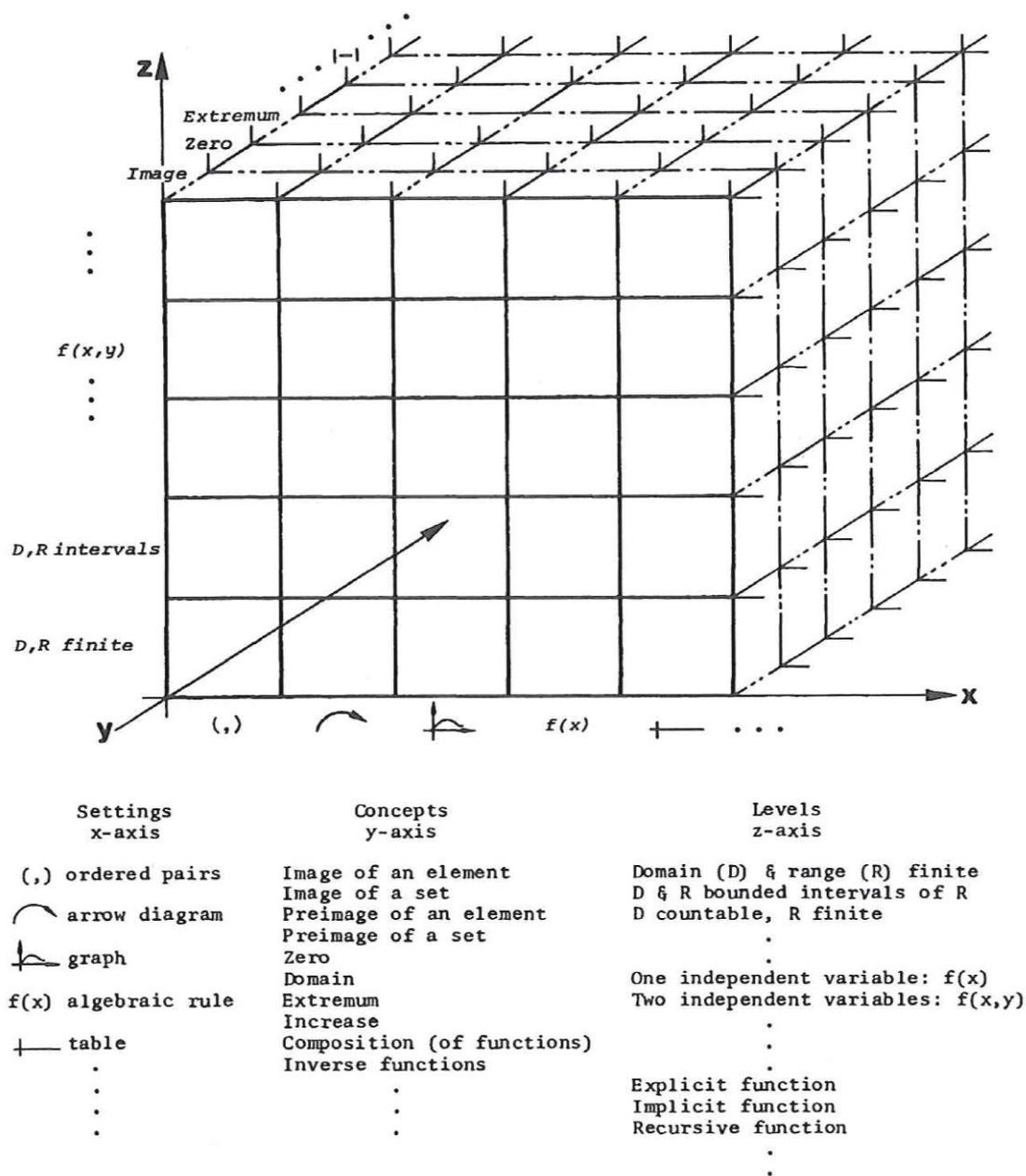
Figure 5.1. The function block (Reproduced from Dreyfus & Eisenberg, 1982, p. 365)

Some of this complexity is also discussed by Sajka (2003) who points out that, for example, "f(x) can represent both the name of a function and the value of the function f" (p. 230). In line with Dreyfus and Eisenberg (1982) she emphasizes that the function concept is related to a series of other concepts such as variables, coordinates and graphs. In her qualitative study, which is basically a case study based on interviews with one secondary school student (Kasia), she especially focuses on the understanding of symbols used in functional notation. She identifies three factors which contribute to and influence Kasia's understanding of the symbols used in functional notation:

1) The intrinsic ambiguities of the mathematical notation;
2) The restricted context in which some symbols occur in teaching, and a limited choice of mathematical tasks at school;
3) Kaisa's idiosyncratic interpretation of school mathematical tasks. (Sajka, 2003, p. 246)

Sajka discusses how the conception of functions is developed in this case, as function is initially seen as "the first signal for beginning a certain procedure, known to the pupil, of solving the problem (p. 250)". At a later stage "the symbol f means to her 'the beginning of the equation that means the formula of the function'" (p. 250). It is further argued that for this student there exists an indistinguishable connection between the concept of function and the concept of the formula of a function. This relates to the category labelled as "functions through representations" in my study and I will later argue that the close relation between students' conception of functions and their conception of a certain representation of functions is an important finding (Section 8.1 and 9.1.1).

Related to definitions of functions, in a study of 12th grade students' conception of a mathematical definition, Zaslavsky and Shir (2005) conclude that the students "employed, to a large extent, example-based reasoning, mainly as a vehicle to refine their understanding of the more subtle defined concepts" (p. 338). This is interesting as it seems as if the students in Zaslavsky and Shir's study are aware of the difference between examples and representations and the requirements of a mathematical definition.

During my observations at one particular school, a concept map was sometimes applied (mainly by the teacher at the blackboard) in the introduction to the topic of functions. A study conducted by Williams (1998) examined the value of such concept maps as instruments for assessment of conceptual understanding of functions. The study was based on the drawings of 28 calculus students produced subsequent to basic instructions of concept maps (in general). The hierarchy in the maps (which varied considerably between students) showed, for example, that letters were used for variables and that some students were "listing x, y and z as concepts" (p. 417). Williams describes "the algorithmic nature", which relates to what I categorize as "functions through examples" and "functions through representations". Function expressions (serving as examples) and statements such as "function can be graphed" are examples of this.

Breidenbach, Dubinsky, Hawks and Nichols (1992) conducted a study which they claim shows that college students "do not have much of an understanding of the function concept" (p. 247). In their study (equivalent to my study) the students were asked "what is a function?" and the answers were systematised into four categories: prefunction,

action, process and unknown. *Prefunction* indicated that "students do not have very much of a function concept at all" (p. 252). *Action* contained "responses that emphasized the act of substituting numbers for variables and calculating to get a number, but did not refer to any overall process" (p. 252). The *process* category involved responses where "the input, transformation, and output were present, integrated and fairly general" (p. 252). *Unknown* was used for responses which fell outside any of these three categories. Breidenbach et al. (1992) discovered that as many as 40 % of the 62 students involved matched the "prefunction" category, while 24, 14 and 21 % fell under the "action", "process" and "unknown" categories respectively.

Among studies concerning students' understanding of function as a concept, is also Blomhøj's (1997) study of Danish students in the ninth grade. By posing the question "what can you say about how x relates to y?" given the expression "y = x + 5" four categories were identified. One of these categories was responses which indicated that "x is five bigger than y". Excerpts from interviews with the students show that many of these responses contain contradictions between verbally expressing that x is five times bigger than y, but at the same time providing tables which clearly illustrate that y is 5 more than x. Blomhøj provides several interpretations of this phenomenon, one of which is of particular interest, namely "the possibility that the students see the expression y = x+5 as a recipe of a function machine, which changes the numbers put into the machine" (Blomhøj, 1997, p. 24). In that sense x is put into the machine and *changes* into another number. These observations lead Blomhøj to assume that one consequence of a one-sided focus on function machines in school could be the misleading conclusion that the independent variable *changes* its value and is transformed into the dependent variable. This is interesting since several of the schools involved in my study only introduced functions through function machines.

The literature presented above mainly deals with general conceptions of the function concept. I will now present some studies which focus in particular on representations of functions.

In a recent study conducted by Font, Bolite and Acevedo (2010) the research area was related to Cartesian graphs and metaphors used in teaching at high school level. Several metaphors were identified in the teaching sequences, but I will focus on four of them which are relevant to my study: "object metaphor", "orientation metaphors", "fictive motion", and "interaction of metaphors". An object metaphor is defined as "a grounding metaphor that maps the object image schema in mathematics. This image schema is experientially grounded in our physical and social interactions with our own bodies and with other discrete entities in the world." (p. 138). This may be exemplified by

utterances such as "find the function" and "you mustn't confuse the function with its graphical representation" (p. 139). "These expressions suggest the grounding ontological metaphor 'mathematical objects are physical objects'". (p. 139). An orientation metaphor could be "do not structure one concept in terms of another, but rather organize entire systems of concepts with respect to each other" (p. 143). Typical examples of these are the word "up" instead of "y > 0" and "down" instead of "y < 0". Fictive motion "suggest[s] to students that they should understand the graph as a path that one walks along, or a line which one follows" (p.144). In my study, this was of particular interest at one lower secondary school, where linear functions were consistently dealt with as loci. Interaction of metaphors is understood as a combined and "flexible use of metaphorical expressions" (p. 146). An example provided for this is of a teacher explaining a vertical asymptote as a path with a certain direction. Indeed all these metaphors are also highly relevant in my analysis related to the teaching sequences. Font et al. (2010) conclude that these metaphors are important in teaching functions, but at the same time sometimes unconsciously used by the teachers.

Kaldrimidou and Moroglou (2009) conducted a study of 190 grade 12 students, related to functions and representations. The research questions were: "a) does the way students conceive a function depend on its representation and b) are the procedures used by the students related to their conceptions about the notion of function?" (p. 265). The students were given tasks involving the representation forms function expressions, tables and graphs. The responses were organized in three categories (geometrical conception, algebraic conception and functional conception) for each of the tasks. The findings suggest that

> [I]n the algebraic and the numerical context the majority project an algebraic conception, focusing their description on the values of the independent variable. However, when the function is represented graphically, the majority of the students express a geometrical conception especially in the case of the more familiar function. (p. 269)

In my findings, I do not pose an equivalent research question, but the tasks provided for the students and the different representations involved in students' reasoning certainly constitute different types of reasoning which are important to take into considerations in further analysis. Moreover, Kaldrimidou and Moroglou (2009) conclude that "procedures used by the students are not related to the mode of representation; while conceptions appear to be influenced by the representational context" (p. 271).

Even (1998) also focused on functions and students' flexibility in using different representations in the article "Factors involved in linking representations of functions". The study involved 152 secondary

students. Pointing to the complexity in moving from one representation to another, Even found that a global understanding of the graphic representation provided the students with a "better and more powerful understanding of the relationships between graphic and symbolic representations" (p. 120). On the other hand, findings contradicted the expectation that a global understanding of functions automatically would entail a better understanding of the meaning of graphs and functions in general. On the basis of these findings, Even suggests that a pointwise approach to functions "was helpful in monitoring naive and/or immature interpretations" (p. 120) and corresponds to "the way procedural knowledge can help in monitoring naïve conceptual knowledge" (p. 120).

As regards the close connection between students' conception of function and of representations it is interesting to note that some researchers argue for this also from the perspective of educational neuroscience (Thomas, Wilson, Corballis, Lim, & Yoon, 2010). They claim that different representation forms such as algebraic expressions and graphs and the transitions between them are located as brain activity at specific areas in the brain. Further, they argue that since these brain areas are the same as for number and arithmetic calculations, "this suggests that a focus of instruction on number sense and spatial cognition is critical not only for mastery of number, but also for more advanced mathematical concepts. It is important when teaching function, especially algebra, to continue to link the topic of number" (p. 616). In conclusion they admit however, that even if this brain activity plays a role both in representing mathematical functions independent of their external format and in translating between different formats of functions, "the exact nature of the role remains to be determined" (p. 617).

In the literature summary above, some challenges in the work on different representation forms and the transition between them have been pointed out. Related to what Janvier (1978) labelled as the representation form "situations", the question of transferring the function concept to new contexts emerges. Michelsen (2006) discusses this issue, regarding functions as a modelling tool claiming that "it is still difficult for teachers of mathematics and teachers of other subjects to see the use of mathematics in other subjects" (p. 269). According to Michelsen, functions as a modelling tool could solve some of these issues as "a focus on model and modelling avoids the problems of transfer and domain specifity" (p. 278). I chose to include also this quote since I observed that modelling aspects, especially at lower secondary school (related to "situation" in Janvier's (1978) table) were inadequate or omitted most of the time.

The references above deal mainly with the function concept and different representations. As my study especially focuses on gradients/slopes and proportional magnitudes (for the students in the vocational programme) and differentiation (for the students in the general programme), I will now draw attention to some studies related to this.

There are several approaches to the topic of slopes and gradients which also can be dealt with by different representations. One of these representations is additive structures (Walter & Gerson, 2007). Initially Walter and Gerson (2007) point out that the conception of slopes is intuitively present independent of the topic of functions. For example "slanty" and "steep" are concepts in many students' vocabulary. In my observations, these pre-concepts were used to a certain extent in teaching in several schools, even though one specific method like the one which I categorize as the *one-unit-right-a-up/down* strategy dominated most of the teaching sequences. Walter and Gerson (2007) show how this approach can be developed, supplemented by other approaches to the concept of slope. They present an alternative approach based on additive structures.
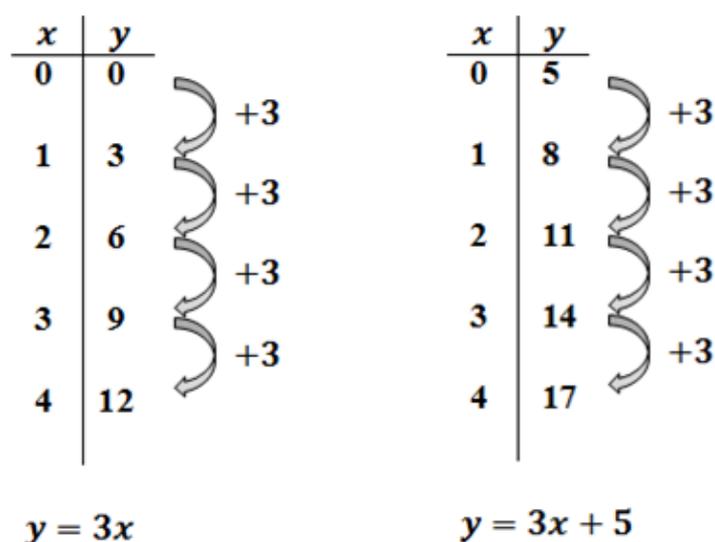
Figure 5.2. Created tables of data with discovered equation.
(Adapted from Walter & Gerson, 2007, p. 213)

Although inquiry-based teacher development is the focus of Walter and Gerson's study, the outcome of working with slopes as additive structures suggests that such an approach, which works for both representations such as tables and graphs, is advantageous.

Walter and Gerson suggest a flexible, dynamic approach which can be applied in different situations. In my case, an appropriate question

when working with algorithmic approaches to slopes like the "one-unit-right-a-up/down" is whether this method is flexible and dynamic enough to be applied in different contexts. Zaslavsky, Sela and Leron (2002) investigated students' reasoning when scale is changed. 11th grade students were given two different tasks involving finding the slope when scale was altered. The findings suggest that methods involved were "clogged by automatism" (p. 138). This term stems from Freudenthal (1983), and basically means that an activity is mastered "so perfectly well" (p. 469) that questions involving how and why are no longer asked. This caused several students to fail the task.

In a study from 1983 of students' conception of differentiation, 110 students participated (Orton, 1983). These students consisted of two groups; one group of 60 students in the age range 16-18 from four schools, the other of 50 students in the age range 18-22 who were training to become teachers of mathematics. Fully in line with some of the findings from my study, Orton (1983) points to the fact that students often are introduced to differentiation as a rule to be applied without much attempt to reveal the reasons for and justifications to the procedure. Applying to the whole group of students at both school and college, Orton concludes that "the symbols of differentiation and the approach to differentiation were clearly badly understood by the students" (p. 244).

Students' conception of the derivative is a central aspect of my study, especially with respect to students in general studies programmes. In a study involving five high school students, Hähkiöniemi (2008) argues that students' conception of the derivative is restricted to a graphical context such as the slope of a tangent and the rate of change, while knowledge of the more formal definition has almost vanished. Related findings are provided by Bardelle (2009) in a large-scale study involving 123 Italian science freshman students attending an introductory mathematics course. Focusing on the link between properties of differentiation to graphical representations, she suggests that "behaviours like those …denote that these students cannot link their knowledge of the derivative of functions to the figural properties of the graphs. Their answers explicitly show lack of coordination of different semiotic systems" (p. 110).

For the students in lower secondary and those attending upper secondary vocational programmes differentiation is not a topic. On the other hand, slopes and gradients, proportional magnitudes and proportional reasoning are important parts of the curricula for these students. Modestou and Gagatsis (2010) conducted a study involving analogical, proportional and non-proportional situations, focusing on students from grade seven to nine. The findings and discussions of this

study are very consistent with my own observations and findings in the topic (Chapter 9), as the dominance of "routines and automatized procedures" (p. 51) seems effective in achieving high achievement scores on certain type of tasks even though they claim that this does "not represent pupils' real abilities in solving proportional tasks" (p. 51).

5.1.2 The concept of transition and recent transition studies

In this section I will focus on the transition issue. For reasons discussed in Section 1.3, it has been difficult to find any published studies focusing on exactly the same transition and topic as in my study, so this section will deal with transition related to mathematical issues in a broader perspective.

Gueudet (2008) discusses various perspectives on transition, focusing on the secondary-tertiary transition. Transition is not an unproblematic concept. For example, the question of when transition happens is problematic. In the popular literary book "Outliers: The story of success", Gladwell (2009) claims to have discovered some of the reasons why particular students succeed better than others in school. Among the many reasons are what Gladwell claims to happen between one grade and another. He especially notices that student activities during summer holidays, for example attending summer camps, have a great influence on students' learning in the coming semester. This is consistent with Gueudet's remark that "transition certainly happens also outside of the period starting at the end of secondary school and finishing at the end of the universities". Gueudet (2008) also refers to other aspects which are important to take into account when discussing transition such as bridging courses and bridging projects, which all aim to ease students' experience of the actual transition. In Norway typical transition projects between lower and upper secondary school are carried out towards the end of the 10th grade at lower secondary and take the form of school visits and activities resulting from collaboration between teachers and principals at both institutions. There are no national guidelines related to such bridging activities so how this is done, and *if* it is done at all, depends on the local school authorities.

Other aspects, closely related to the curriculum, identified by Gueudet are "students' difficulties and teaching designs" (p. 239). As expressed in my research questions, students reasoning (and thus, difficulties) and teaching will also be main focus areas in my study. The institutional perspective, focusing on transition as "a shift between two institutional cultures" (p. 245), is of course also important, but in my case there will be no explicit elaboration of these issues as my research questions do not justify or invite an extensive account of these aspects. However, it is important to stress that institutional aspects are not omitted, but are dealt with in close relation to observable situations in the

classroom and analysed through the framework of sociomathematical norms. To illustrate another of the many aspects of transition, the role of classroom environment (Athanasiou & Philippou, 2008) should also be mentioned as a part of the institutional perspective, even though this will not be treated in my study.

The importance of students as mathematics learners in the transition is also central to my research questions. Abreu, Bishop and Presmeg (2001) problematize transitions from the learners' perspective as they see that transitions as movements between practices require "theorisation of both the social environment and the individual learner as dynamic entities" (p. 11). Within a socio-cultural perspective, Abreu et al. (2001) also provide an alternative to what they claim is the "common use of the concept transition in the traditional developmental psychology stage theories" (p. 11) as they are interested in "transitions as bi- or multi-directional trajectories" (p. 11). By the term "multidirectional trajectories" Abreu et al. suggest that different forms of mathematical knowledge and understanding can co-exist. This means that if old knowledge is replaced, this is not necessarily caused by stage development, but rather by a shift in what particular groups count as legitimate knowledge. In connection with teaching, learning and the relation between them, it is important to be aware of the complexity of transition issues is in this study. In my case, neither lower secondary nor upper secondary school should be conceived of as homogenous groups of students. For example, the observable internal differences between lower secondary schools were quite large and maybe as significant as the observable differences between lower and upper secondary. "Multidirectional trajectories" could be understood both at an individual and a group level, where "different forms of mathematical knowledge and understanding can co-exist and that replacements when they occur are not necessarily based on a scale of development, but can instead be the result of what particular social groups count as legitimate knowledge" (p. 12). Referring to Beach (1999), Abreu et al. (2001) emphasize four categories of transition:

1. Lateral transition – occur when an individual moves between two historically related activities in a single direction, such as moving from school to work. Participation in one activity is replaced by participation in another activity in a lateral transition.
2. Collateral transition – involve individuals' relatively simultaneous participation in two or more historically related activities, such as daily movements from school to home.
3. Encompassing transition – occur within the boundaries of a social activity that is itself changing, and is often where an individual is adapting to existing or changing circumstances in order to continue participation within the bounds of the activity.

4. Mediational transition – occur within educational activities that project or simulate involvement in an activity yet to be fully experienced.

(Abreu et al., 2001, p. 14)

All these four categories are important aspects of transition and involve change in context through time. Taking this as a model to describe transition, the focus of my research questions is mostly related to 1) and 3) on the list above. The transition from one institution to another is a distinct shift between "historical related activities", in accordance with what Abreu et al. call "lateral transition". At the same time, both within and between these two institutions, it is interesting to see how "encompassing transitions" (which involve change in activities themselves) unfold in the mathematics classroom.

Especially from the students' perspective, transition between contexts as "shift in meaning" is an essential aspect. Meaning entails the making of connections (Presmeg, 2002). A central point in Presmeg's study is the analysis of how such connections are made by the students at upper secondary, as functions, slopes, and proportional magnitudes are topics which they should be familiar with from lower secondary.

Guzmán, Hodgson, Robert and Villani (1998) identify three sources of difficulties related to mathematics which students might encounter in the move from secondary to tertiary education.

i) Difficulties linked to the way teachers present mathematics at the university level and to the organization of the classroom

ii) Difficulties coming from changes in the mathematical ways of thinking at the higher level

iii) Difficulties arising from the lack of appropriate tools to learn mathematics

(Guzmán et al., 1998, p. 748)

Although this study and the three sources above are linked to the secondary-tertiary transition, these are all general sources of difficulty which are certainly also applicable to other transitions such as the one from lower to upper secondary. In addition, but still intertwined with these sources, Guzmán et al. (1998) discuss "sociological and cultural difficulties" (p. 755) and "didactical difficulties" (p. 756). In the discussion of didactical difficulties, Guzmán et al. point to the sudden change in teaching methods at universities compared to those which students are used to from secondary school. Listed as possible reasons for these differences and the rupture experienced by several students are "lack of pedagogical and didactical abilities", "lack of innovative teaching methods" and "lack of feedback procedures" (pp. 757-758). Guzmán et al. (1998) suggest measures which could prevent or ease these experiences involving a better dialogue between the two phases of schooling. These involve orientation activities and change of context so that the tertiary courses are closer to the secondary teaching style

especially in the first year. In addition, faculty resources, increased focus on the individual student, culture changes and dialogue between teachers and students are mentioned. Also related to the secondary-tertiary transition, Stadler (2009) conducted a study in Sweden, focusing on the students' perspective. Based on observations and interviews with the students during the first year at university, Stadler investigated students' view of mathematics and mathematical knowledge, what it means to learn mathematics and how learning mathematics should happen. She identified three categories: "mathematical learning objects" which refers to students' experiences of the aim of the course, "mathematical resources" which relates to the actions or objects applied in learning mathematics and "the student as learning actor" which relates to students' actions, intentions and beliefs. Stadler summarizes her findings and their implications for teaching in three parts: 1) The actual transition implies an inconsistency between mathematics as a learning object and the mathematical resources. 2) The transition causes a reorganisation of the learning objects in mathematics and 3) Learning mathematics at university requires more mathematical resources (p. 219). Stadler's study provides a way of analysing the transition individually, based on the experiences of the individual students, which may not necessarily match the view of an "outside" observer.

Focusing on the transition from primary to secondary school, Fernández, Figueiras, Deulofeu and Martínez (2011) suggest how the concept of "horizon content knowledge" can promote a "general awareness of the previous and the forthcoming, and requires an overview of students' mathematical education so that it can be applied to the mathematics taught in the classroom" (p. 5). To achieve a smooth transition, they stress the importance making teachers aware of this through teacher training programmes.

Although it was hard to find any published studies involving transitions which corresponds to the Norwegian lower- upper secondary transition, one study from England conducted by Hernandez-Martinez, Williams, Black, Pampka, Wake and Davis (2011) matches. In this study the authors focus on what they call the "transition from school to college mathematics". This can be compared to the lower-upper secondary transition in my study. Mainly based on interviews, Hernandez-Martinez et al. identify three categories essential to the students' responses to transition: i) the social dimension, ii) coherence of curriculum and pedagogy (in mathematics) and iii) individual information-progression in mathematics for AS[6] (pp. 124-127). The first category, the social dimension, with "students' sense of belonging to the new institution" (p.

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[6] AS is an abbreviation for the specific mathematics course, *advanced subsidiary* (first year).

120). The second category implies an "awareness of the 'gap' between practices on either side of the transition" (p. 120), while the third focuses on the ability of an institution "to become aware of, and take account of, the individual history and progression of each learner" (p. 120). In connection with my research, the second and third categories are particularly relevant. Hernandez-Martinez et al. (2011) conclude that a surprisingly large number of students regard mathematics at upper secondary as "new" or at least "too new". "It is apparently ironic that – for mathematics – the troubles seem largely to arise exactly from mathematics being 'all new' or at any rate too new (for some)" (Hernandez-Martinez et al., 2011, p. 128).

The literature presented in Section 5.1.1 and 5.1.2 points to several important issues related to various parts of my study. Teaching and learning functions and related topics such as gradients, slopes and derivatives are recurring issues throughout my study and I draw on the literature presented in 5.1.1 when discussing my findings in the concluding part of my thesis. Transition studies and the issue of transition presented in 5.1.2, relate to the very nature of my study which concerns the transition from lower to upper secondary school. The relation between my findings and the main ideas in the literature presented above are discussed in the concluding parts of the thesis.

6 Methodology

In chapter 4, I described my theoretical perspective and my analytical framework. In this chapter, I present and discuss methodological issues such as the research paradigm, research design, methods of data collection, data management, analysis strategy, validity and ethical issues. I locate my study within a suitable research paradigm with the aim of strengthening reliability. This will be done within well-elaborated, historically developed frames. Coherence between the choice of methodology and the applied theoretical framework is needed to ensure the production of meaningful results (Goodchild, 2001; Lincoln & Guba, 1985).

6.1 Research paradigm

In this section I discuss how my research is located within the methodological landscape. I do this by pointing to some important and general methodological considerations and challenges which are relevant for my study as well as for most qualitative studies. Furthermore, I relate relevant methodological issues to my study in particular.

Before starting, some terms need to be defined. A research paradigm normally postulates its ontology, epistemology and a methodology (Mertens, 2005). Briefly, *ontology* can be defined as the nature of being and the assumptions underlying one's worldview (Corbin & Strauss, 2008, p. 5). One such assumption could be certain phenomena essential to the research, such as that consciousness is imposed from an external social reality, or that it arises from within the individual (Cohen, Manion, & Morrison, 2007, p. 7). The latter is also a very relevant question about the ontological foundations of the socio-cultural perspective on learning. Within the socio-cultural perspective, as elaborated in Section 4.1, the individual is not separated from social reality. In constructivist theories of learning (Bruner, 1997; Jaworski, 1994) on the other hand, such a division is very clearly made. This has led to some interesting philosophical debates. From a constructivist perspective, for example, Paul Cobb (1994) raises the question "where is the mind?" (Cobb, 1994).

Epistemology concerns the bases of knowledge and deals with questions related to the nature of knowledge and how it can be communicated to others (Cohen, et al., 2007, p. 7). What should count as acceptable knowledge is a relevant epistemological question in this sense (Bryman, 2004, p. 693).

Methodological questions deals with how knowledge can be obtained and how the knower can acquire the desired knowledge and understanding (Mertens, 2005, p. 8) .

When constructing a suitable research paradigm, one should be aware of the diversity of overlapping terminologies in different scientific fields such as philosophy, psychology, sociology and education. The overlap is partly due to the fact that concepts have slightly different meanings depending on which scientific field they are applied to, but also because they are interpreted differently by researchers within the same scientific field (Adhami, Johnson, & Shayer, 1998; Weaver, 1996). Hence, paradigms may be referred to by different labels and Mertens (2005) presents a table of such labels. For example, the *interpretive* paradigm may also be labeled the constructivist, naturalistic, phenomenological, hermeneutic, ethnographic or simply qualitative paradigm (Mertens, 2005, p. 8). Due to the possible confusion this may cause, I will elaborate briefly on some of these labels before explicitly defining the ontology, epistemology and methodology underlying my research. This gives me the opportunity to clarify how I perceive the nuances and distinguish between them while at the same time providing more detail on how my research fits into this methodological landscape.

My study is a *qualitative study*. Qualitative research is described by Bryman (2004) as research which "usually emphasizes words rather that quantification in the collection and analysis of data" (p. 697). When comparing qualitative and quantitative methods, Hardy and Bryman (2004) point to differences in the nature of the accumulated collected data. In quantitative research, the data material tends to be measurable variables, subject to for example frequency tables and measurement of central tendencies. In qualitative research the collected data often consists of interview transcripts, field notes, texts and documents which are elaborated through systematic coding and descriptive analyses (pp. 4-5). The process of interpreting empirical data also suggests that qualitative research entails a subjective dimension. This has been pointed out by many including Cohen et al. (2007) who describe the qualitative paradigm as an antagonism to positivism and the scientific methodology. Subjectivity is linked both to the research object itself and to the researcher: "The imposition of external form and structure is resisted, since this reflects the viewpoint of the observer as opposed to that of the actor directly involved" (Cohen et al., 2007, p. 21).

Patton (2002) writes about different "qualitative traditions" (p. 79) to show that the history of qualitative research entails a great deal of variety and that there is currently a multitude of different qualitative approaches. Thus "qualitative" is a broad term which encompasses other more specifically defined paradigms such as the naturalistic paradigm as defined by Lincoln and Guba (1985) and "grounded theory" as described by Glaser and Strauss (1967). These paradigms operate with a certain defined set of axioms which form the basis for their ontology and

epistemology while no specific set of axioms is to be found for "qualitative research" in general. I interpret this to mean that one can think of "qualitative paradigm" as an umbrella concept which covers several, more specific paradigms. Corbin and Strauss (2008) define qualitative analysis as "a process of examining and interpreting data in order to elicit meaning, gain understanding, and develop empirical knowledge" (p.1).

There is a link between the formulation "examining and interpreting data" in the above definition from Corbin and Strauss (2008) and the label "interpretive paradigm". In addition to being antagonistic to the positivistic view, this also points to the subjective nature of experience.

> As we have seen, the central endeavor in the context of the interpretive paradigm is to understand the subjective nature of human experience. To retain the integrity of the phenomena being investigated, efforts are made to get inside the person and to understand from within. (Cohen et al., 2007, p. 21)

One can deduce from this that the interpretive paradigm is part of a wide-ranging tradition which includes several approaches and foci within the paradigm itself.

Another broad term within the qualitative paradigm is the ethnographic paradigm. This term is rooted within sociology and was introduced by Harold Garfinkel in 1967 who was studying jurors and "decided that the deliberation matters of the jurors, or for that matter of any group, constituted an 'ethnomethodology'" (Patton, 2002, p. 110). Hence, ethnomethodology seems to focus on a certain group within the society. This "group focus" is even clearer in the related term "ethnography" from anthropology, where the focus on a specific culture within a group of people often underpins the research (Patton, 2002). In a very broad sense, if one regards the students and teachers in this research as belonging to different cultures (the "culture in lower secondary" and "the culture in upper secondary") one can view the transition with ethnographic lenses. Although I observed and experienced diversity between the various lower secondary and upper secondary schools in my research, I do not find it appropriate to call this an ethnographic study.

My research has characteristics of what can be described as both a qualitative and interpretive paradigm. It is qualitative according to Corbin and Strauss' (2008) rather broad definition, in the sense that certain data are collected, analyzed and interpreted in an attempt to gain understanding. The interpretive dimension is also apparent in terms of emphasis on a subjective nature of the individuals involved in the research (students and teachers) and the subjective nature of myself as an interpreting researcher. I position myself in the *interpretive paradigm within the traditions of the qualitative paradigm*, where the qualitative paradigm overlaps but extends the interpretive paradigm. Of course, this

is not a very accurate description of my view of the world since it still involves a rather broad categorization. To determine my methodological position more clearly, I provide more systematic elaboration in accordance with what Mertens (2005) describes as the three aspects of a research paradigm; its ontology, its epistemology and its methodology.

6.1.1 Ontology

Ontology is understood as "the being", and the ontological question can be phrased as "What is the nature of reality?" (Mertens, 2005, p. 8). To examine the deep philosophical roots of this question is beyond the limitation of this thesis, but some assumptions must be made. In line with the socio-cultural theoretical perspective, (Section 4.1) I maintain the multiple constructed nature of social phenomena. My research involves human beings acting within a social environment and investigates teaching and students' reasoning and experiences related to functions as a topic. In this reality the "constituted system of activity" (Packer & Goicoechea, 2000, p. 229) formed by both the students and the teachers is part of this multiple constructed reality, engaging within a certain context. Lincoln and Guba (1985) recognise some underpinning assumptions for what they called the "naturalistic paradigm". In total they provide five such axioms:

1) Realities are multiple, constructed, and holistic.
2) Knower and known are interactive, inseparable.
3) Only time- and context bound working hypotheses (idiographic statements) are possible.
4) All entities are in a state of mutual simultaneous shaping, so that it is impossible to distinguish causes from effects.
5) Inquiry is value-bound.

<div align="center">(Lincoln & Guba, 1985, p. 37)</div>

Of these five axioms, axiom one and four are of prominent ontological significance. As I have already pointed out, I fully subscribe to this first axiom. The fourth, however, is rather challenging. This is primarily because it would be difficult to suggest any teaching improvements or measures for how students' reasoning and teaching might relate better, without at the same time alluding to certain "causes and effects". This issue is not an explicit part of this study, but could arise in terms of possible follow-up discussions. Possible implications should be presented very carefully and modestly since many factors are involved in shaping the multiple and holistic realities. On one hand, my empirical data rarely points to indisputable causalities about, for example, the relation between teaching and learning. On the other hand, it is unavoidable that possible causes and effects are implied, but these are put forward by assumptions instead of certainty and definite conclusions. However, as is the case with several major theories, I think it is possible to focus on specific elements without contradicting the belief of a

holistic reality. It is important to stress that when narrowing down the focus and when making suggestions related to causes and effects, there is a risk of oversimplifying if the focus is not explicit.

In their qualitative paradigm, Corbin and Strauss (2008) list in total 16 "assumptions" (pp. 6-8) which are described as resting on "Pragmatists and Interactionist philosophies" (p. 6). Their understanding of pragmatism is mostly based on the work of philosophers like Dewey and Mead. Thus, pragmatism cannot account for the nature of an idea without involving a certain human perspective, but radical relativism is avoided through the assumption that social knowledge is cumulative as it "provides the basis for the evolution of thought and society". This position is much in line with the sociocultural perspective presented in chapter 4, where I describe my view of mathematics as cultural-historical developed knowledge. The first two ontological assumptions of Corbin and Strauss (2008) are:

> Assumption 1. The external world is a symbolic representation, a "symbolic universe." Both this and the interior worlds are created and recreated through interaction. In effect, there is no divide between external or interior world.
> Assumption 2. Meaning (symbols) are aspects of interaction, and are related to others within systems of meaning (symbols). Interactions generate new meanings…as well as alter and maintain old ones. (p. 6)

I maintain that this first assumption is in line with the socio-cultural perspective, as the existence of an external world has little meaning unless the external world is undertaken action and internalized by human beings. In both these assumptions the role of symbols is central. In that sense an appropriate approach for studying how students make sense of mathematics, is through semiotics as described in Chapter 4. Further, Corbin and Strauss' axioms do not explicitly omit the possibility of investigating causes and effects (as in the case of Lincoln and Guba) although the complexity of related issues is emphasised in assumption 15:

> Assumption 15. A major set of conditions for actors' perspectives, and thus their interactions, is their memberships in social worlds and subworlds. In contemporary societies, these memberships are often complex, overlapping, contrasting, conflicting, and not always apparent to other interactants (Corbin & Strauss, 2008, p.7).

Although it might be possible to interpret the consequences of these ontological assumptions in different ways, I think my arguments above show that these can also be applied in this research without causing any ontological contradictions.

6.1.2 Epistemology

According to Mertens (2005) the epistemological question asks: "What is the nature of knowledge and the relationship between the knower and the would-be-known?" (p. 8). This question points to the strong link

between ontological and epistemological issues since the nature of knowledge depends on the nature of what one has knowledge *of*. In this study, mathematics is viewed as socioculturally developed knowledge which enables individuals to engage and participate in the cultural-historically developed field of mathematics. This entails an emphasis on communication and mediation through the use of symbols. Adapted and adjusted to this study, "symbols" can be replaced by "signs" because in Steinbring's terms, symbols are a subset of signs. The ontological position as in the first assumption of Corbin and Strauss suggests a broad definition of symbols. "The nature of knowledge" in this view is therefore not something "static" outside the participant. Dewey emphasized this point by the epistemological claim:

> Insofar, we have the earnest of a possibility of human experience, in all its phases, in which ideas and meanings will be prized and will be continuously generated and used. But they will be integral with the course of experience itself, not imported from the external source of a reality beyond. (Dewey 1929, p. 138, as cited in Corbin & Strauss, 2008, p. 4)

These experiences evolve through interactions, negotiations and shared perspectives. "Assumption 6: Courses of interactions arise out of shared perspectives, and when not shared, if action/interaction is to proceed, perspectives must be negotiated" (Corbin & Strauss, 2008, p.7).

Further, in a qualitative study such as this one, the link between me as a researcher and the "would-be-known" (my research questions) is characterized by certain values, and also, therefore, by selections and priorities based on these values. Personal interpretations of what might be important in accordance with my framework is a part of this. These epistemological assumptions are also made explicit through the second and the fifth axiom of the naturalistic paradigm mentioned in the previous section. I return to this issue in the section dealing with methods and validity.

6.1.3 Methodology and unit of analysis

The third overarching question when conducting research is what Mertens (2005) calls the methodological question: "How can the knower go about obtaining the desired knowledge and understandings?" (p. 8). This is an important question which has puzzled me throughout my research. To begin with, a proper unit of analysis should be defined, as this influences both the sample size and the sampling strategies. Usually within the socio-cultural perspective, mediated action is the unit of analysis. In this study mediated actions in a broad sense, e.g. statements during interviews, explanations given by the teachers at the blackboard and students accomplishments of certain tasks, constitute the units of analysis. I try to answer my research questions by looking at the units of analysis in the light of my theoretical framework, in a hermeneutical

sense. Here I use hermeneutics in a broad sense, interpreting and analysing the empirical data through my predefined socio-cultural perspective entailing semiotics and Steinbring's epistemological triangles. "Hermeneutic theory argues that one can only interpret the meaning of something from some perspective, a certain standpoint, a praxis or a situational context" (Patton, 2002, p. 115). In this case, mediated actions through students' and teachers' use of signs, tools and artefacts are the basis on which further analysis and conclusions can be made. My own role and priorities, and how these mediated actions are collected and dealt with in this research is a recurring issue throughout this chapter.

The notion of grounded theory plays a central role in the discussion of the nature of predefined theoretical perspectives. This methodology was primarily developed by Glaser and Strauss in the sixties, and strongly emphasized emancipation from predefined theoretical lenses by arguing that the theoretical perspective itself should also be intrinsically developed from the existing empirical data. "[W]e address ourselves to the equally important enterprise of *how the discovery of theory from data – systematically obtained and analyzed in social research – can be furthered*" (Glaser & Strauss, 1967, p. 1). In the research community there are various discussions related to this issue, one of which concerns the extent to which qualitative research should be rooted within some framework.

> Knowledge and theory are used as if they were another informant. This is vital, for without this grounding in extant knowledge, pattern recognition would be limited to the obvious and the superficial depriving the analyst of the conceptual leverage from which to develop theory. Therefore, contrary to popular belief grounded theory research is not "atheoretical". (Goulding, 1998, p. 52)

A strict interpretation in terms of Glaser and Strauss' (1967) account of grounded theory would probably not fit the analysis strategy for this study as parts of theory construction are carried out prior to the analysis. On the other hand, according to Goulding (1998), I claim that grounded theory and my data analysis strategy have a common ground, as the tool of analysis applied can be regarded as "informant" for further illumination of the findings.

6.2 Research design

This study involves four different lower secondary schools which are labelled as School A, School B, School C and School D. School A is a Waldorf School, while Schools B-D are public schools. In total, I focused on 8 students, distributed in these schools as illustrated in Table 6.1. At upper secondary these students attended four different schools, but most of them attended different classes. These schools are labelled School 1-4, followed be a small letter (a-c) indicating the actual class.

Only one out of the eight students attended School 4, so no letter was needed in that case.

Pseudonyms are provided for each of the participants, and to make the analysis clearer and more readily understood I have used 3-letter names for the teachers in lower secondary, 4-letter names for the students and 5-letter names for the teachers at upper secondary school. Originally, 12 students took part in this study, distributed over five lower secondary schools and six upper secondary schools. However, during the process of data management I found that data material from the students in one lower secondary and one upper secondary school was insufficient and therefore inadequate for further elaboration. This was mainly a consequence of a very short period of observation at these schools. In another of the upper secondary schools, the topic of proportional magnitudes was deliberately omitted from teaching (justified by the actual teacher as "too difficult and irrelevant") and the outcome of my observations and interviews were also limited in this case. Reconsidering my material, I therefore found that it would be more beneficial to omit one of the lower secondary and two of the upper secondary schools from most of the analysis. This means that I will sometimes supplement observed phenomena with examples from these cases. It will be clear from the analysis when this is done.

As this study consists of multiple cases and extensive analyses of them, I consider this as a case study even though the number of cases involved exceeds what is normal in such studies. Bryman (2004) limits the definition of case studies to at most two or three cases (p. 691) while Patton (2002) states that "fieldwork, then, can be thought of as engaging in a series of multi-layered and nested case studies, often with intersecting and overlapping units of analysis" (p. 298).

Table 6.1 below shows the "final" situation, with the eight students and the corresponding schools, which form the basis for the subsequent analysis:

Lower secondary schools

School A (Waldorf School)	School B (Public school)	School C (Public school)	School D (Public school)
Students: Otto, Edna **Teacher:** Kim	**Students:** Lena, Olga **Teacher:** Oda	**Students:** Kent, Anna, Matt **Teachers:** Tim, Tom	**Student:** Thea **Teacher:** Roy

Upper secondary schools

School 1 (VS)	School 2	School 3[7] (GS – 1T)	School 4 (GS – 1T)
Class a **Student:** Otto **Teacher:** Bernt	Class a (VS) **Student:** Olga **Teacher:** Ronny	Class a **Student:** Kent* **Teacher:** Derek	**Student:** Thea **Teacher:** Kerry
Class b **Student:** Edna **Teacher:** Sonja	Class b (GS – 1T) **Student:** Lena **Teacher:** Tommy	Class b **Students:** Kent, Anna **Teacher:** Greta	
		Class c **Student:** Matt **Teacher:** Henry	

Table 6.1. Distribution of the 8 students who form the basis for further analysis. VS indicates "vocational studies programme" and GS indicates "general studies programme". 1P and 1T indicate respectively the 1P and 1T versions of mathematics in the general studies programme (see Chapter 2).

6.2.1 A longitudinal study

This study should be categorized as a longitudinal study as longitudinal research can be described as involving "a research design in which data are collected on a sample (of people, documents, etc.) on at least two occasions" (Bryman, 2004, p. 540). In accordance with Bryman's model for longitudinal studies (p. 47), Table 6.2 below gives an overview of the observations. The table shows the eight participating students, the dates and numbers of observations in their classrooms.

The number of students in each class who volunteered to participate varied between three and ten, so I decided to include all volunteers in lower secondary and then select twelve students at upper secondary. In total 33 students volunteered at lower secondary. This relatively large number made it possible for me to select criteria for subsequently selecting upper secondary pupils. These criteria ensured equal representation of gender and equal numbers from general and vocational studies. I also wanted there to be some diversity among the students' marks. All the criteria aimed to enrich the material making it diverse

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[7] In this school, students were grouped in classes according to their marks from lower secondary school. This is why these 4 students attend different classes even though all of them do 1T in the general studies program.

enough to be able to grasp several aspects and nuances. The consequence of this relatively large number of students was primarily apparent through limitations on the interviews and conversations with individual students. The short amount of time available for each of them forced me to be very selective in choice of questions during conversations and interviews. Reducing the number of students to twelve as they entered upper secondary, provide a better possibility for going into more depth.

Lower secondary					
SCHOOL A Otto, Edna	Ob 1 19.5.08	Ob 2 20.5. 08	Ob 3 21.5. 08	Ob 4 22.5.08	Ob 5 23.5. 08
SCHOOL B Olga, Lena	Ob 1 5.4.08	Ob 2 17.4.08	Ob 3 28.4.08	Ob 4 6.5.08	
SCHOOL C Kent, Anna, Matt	Ob 1 26.3.08		Ob 2 2.4.08		
SCHOOL E Thea	Ob 1 15.4.08	Ob 2 23.4.08	Ob 3 30.4.08	Ob 4 14.5.08	

Upper secondary				
SCHOOL 1a Otto	Ob 6 10.12.08			
SCHOOL 1b Edna	Ob 6 17.12.08			
SCHOOL 2a Olga	Ob 5 31.10.08			
SCHOOL 2b Lena	Ob 5 19.11.08			
SCHOOL 3a Kent[8]	Ob 2*[9] 17.10.08			
SCHOOL 3b Kent, Anna	Ob 3 12.3.09	Ob 4 26.3.09	Ob 5 28.4.09	Ob 6 30.4.09
SCHOOL 3c Matt	Ob 3 10.3.09		Ob 4 27.3.09	
SCHOOL 4 Thea	Ob 5 26.2.09		Ob 6 3.3.09	Ob 7 17.3.09

Table 6.2. An overview of the observations.

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[8] During the first year at upper secondary Kent shifted from class "School 5a" to "School 5b".

[9] Observation 2* indicates the observation made before Kent shifted to class "School 5b", which also was Anna's class.

6.3 Methods for data collection

6.3.1 Methods

To answer my research questions, I have used qualitative methods. It is my belief that a qualitative approach in this field will be useful for educators and teachers in both lower and upper secondary school. Despite the limitations, I hope that readers will recognize aspects of the elaborated analysis and thereby confirm that: "Case studies…will often be the preferred method of research because they may be epistemologically in harmony with the reader's experience and thus to that person a natural basis for generalization" (Stake, 1978, p. 5).

My main methods for data collection were observations and interviews. Related to the first research question, I did two types of interview, unstructured and semi-structured. I prefer to call the unstructured interviews *conversations*; they took place in the classroom the first few times I met the lower secondary students. Practical considerations made it hard to predict the students' activities prior to the observation. Even though my first research question served as a guideline for these conversations, the form and structure totally depended upon which mathematical tasks and activities the student experienced in particular lessons. For ethical reasons, I did not conduct such conversations in upper secondary school as this was a new setting for the students and I did not want to risk making them feel uncomfortable. The semi-structured interviews with the students after my period of observation were conducted both in lower and upper secondary school with a view to being able to answer research questions 2 and 3. The interview questions mainly draw on Kvale (2007) and Goldin (2000). I aimed to phrase the interview questions simply, and with possibilities for encouraging an open-ended dialogue.

> From a dynamic point of view the questions should contribute to a positive interaction – maintaining the conversation and motivate the interviewee to tell about his/her own experiences and emotions. These questions should be easy to understand, short and free of academic terminology. (Kvale, 2007, p. 77, my translation)

The interviews in lower secondary school were intended to provide an overview of students' conceptual understanding of functions, their view on different teaching methods and their attitudes towards mathematics and functions in particular. This mainly relates to research question one. The interview also contains task-based questions, accomplished in accordance with Goldin's (2000) four stages: 1. Posing the question, 2. Minimal heuristic suggestions, 3. The guided use of heuristic suggestions, 4. Exploratory, metacognitive questions (p. 523).

In addition to questions concerning students' experiences of the transition, similar task based questions were also posed in the more

extensive interviews with the students at upper secondary school. I will illustrate this through an example from one of these interviews: "Please tell me what you see here: $y = 2x - 3$". The first stage is posing the question and uses only non-guiding, follow-up questions. As some answers went in the direction of just reading the signs or perhaps "I don't understand this…" the second stage allows "minimal heuristic suggestions" such as "What is this?" or "What does this expression say?". At the third stage, heuristic suggestions can be made such as, "Can you draw this?", "What does this look like?" or "What numbers might x and y be?" At the fourth stage meta-cognitive questions are posed. I must admit that this was the most challenging stage, mostly because students tended to be reticent.

For the sake of comparison, some of the questions were identical in the lower secondary and upper secondary interviews. I found it valuable also to investigate students' reasoning in more familiar tasks, so some tasks were selected from the students' textbooks. The interview guide for the semi-structured interviews with students in lower secondary is given in appendix A, and the interview guide for the students at upper secondary is given in appendix B.

The more extensive interviews with the teachers lasted approximately 30-35 minutes. It was the same length for the teachers in lower and upper secondary school. The questions covered issues such as the teachers' views on teaching and their applications of various didactical methods related to the topic of functions (second research question).

To support my data from observations and interviews, I also obtained copies of students' handwritten work. Copies of locally elaborated working plans provided by the teacher, together with the textbooks in use and the national curricula also provided a richer material for answering research question number two.

6.3.2 Use of instruments

Due to some constraints, the use of instruments varied from school to school. The following table gives an overview of the instruments used:

Lower secondary		Upper secondary	
SCHOOL A Otto, Edna	Notes (the first lesson) Voice recorder (second - fourth lesson)	**SCHOOL 1a** Otto	Video camera (one lesson)
		SCHOOL 1b Edna	Video camera (one lesson)
SCHOOL B Lena, Olga	Notes (four lessons)	**SCHOOL 2a** Olga	Video camera (one lesson)
		SCHOOL 2b Lena	Voice recorder (one lesson)
SCHOOL C Kent, Anna, Matt	Video camera (three lessons)	**SCHOOL 3a** Kent[10]	Video camera (one lesson)
		SCHOOL 3b Kent, Anna	Video camera (four lessons)
		SCHOOL 3c Matt	Video camera (two lessons)
SCHOOL D Thea	Video camera (four lessons)	**SCHOOL 4** Thea	Voice recorder (three lessons) [Teacher interview: notes]

Table 6.3. The use of instruments in schools and classes.

Table 6.3 shows some variation, and except for one occasion where the video camera had a flat battery (upper secondary school 3b), the variation was due to restrictions determined by the teacher or the principal. Naturally, in the cases where I was allowed to use a video camera, I was able to use both a voice recorder and notes as supplements. In the interviews and conversations with the participating students I was always permitted to use voice recorder and video camera. I was allowed the use of voice recorder in all the teacher interviews.

6.3.3 Constraints
The instruments I was allowed to use in classrooms also varied (see Table 6.3). In some classrooms, I could use whatever I found suitable (e.g. video camera or voice recorder) and in other classrooms I was allowed only to take handwritten notes.

6.4 Data analysis strategy and data management

6.4.1 Transcriptions
During this study, a total of approximately 28 hours of video or voice recordings were made, 15 hours in the classrooms, 6 hours of the teacher interviews and 7 hours of student interviews. In addition, there are field notes and copies. The excerpts included in the thesis are presented both in Norwegian and in English translations.

I started the data management by listening to and by watching the recordings several times, and conducted a data reduction. This data

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[10] During the first year at upper secondary Kent moved from class "School 5a" and changed to "School 5b".

reduction consisted of writing down keywords and remarks related to the content of the recordings. As the work proceeded, I transcribed parts of this material, based on the keywords and comments made in the process of data reduction. I ended up by transcribing about 50 % of the collected material in detail, and in addition my notes related to the rest of my material was quite extensive. I did all of the transcriptions myself. I aimed to transcribe as authentically as possible, but occasionally in the teacher interviews, for practical reasons such as saving space, expletives which I consider of minimal semantic value are removed. For transcription keys, see appendix E.

6.4.2 Coding

To be able to describe certain phenomena, I found that categorizing the material was helpful. Not only did this help in describing the phenomena themselves, but it also made it easier to identify certain nuances and patterns. Different ways of presenting the definition of functions and the gradient (second research question) are apparent in teachers' mediation, but parallel to this, students' explanations and reasoning (first research question), share much of the same aspects. I will undertake a retrospective analysis of all observations and apply what Miles and Huberman (1994) label as pattern codes:

> A third class of codes, pattern codes, is even more inferential and explanatory [compared to descriptive codes]. A coded segment of field notes illustrates an emergent leitmotiv or pattern that you have discerned in local events and relationships. (Miles & Huberman, 1994, p. 57)

This inductive coding strategy emerges as an interplay between my theoretical framework and the empirical data, where "the analyst moves back and forth between the logical construction and the actual data in search for meaningful patterns" (Patton, 2002, p. 468).

6.4.3 Data analysis

In the process of data analysis I found it useful to classify the empirical data into different categories to better grasp the nuances and the underlying conceptions of the different approaches and elaborations provided mainly by the teachers, textbooks and students. This was a long and dynamic process as many questions and challenges appeared during this work. My aim was to have enough categories to cover the essentials of my material, but at the same time not so many that it became chaotic and difficult to grasp.

The categories are mainly developed as a support in answering my first two research questions. (Concerning the third, a more dialogical approach is employed). The categories are constructed in such way that the conceptual content of each one can be considered to apply to either a learning or a teaching context. All of the categories emerged solely from

my empirical data material, in line with the basic ideas of grounded theory.

6.4.4 Categorization

The following categories are related to research question 1a and 2a and are meant to grasp the main focus in learning and teaching related to the function concept and gradients. Each of the categories listed emerged from my empirical data, and I choose to present these categories prior to the analysis. This is done to make the reading of the analyses easier to follow, as these categories appear (in italics) and is referred throughout the analysis. This section could then serve as a reference list, where the exact definitions of the categories are provided, and could be looked up by the reader if needed.

Functions as loci

This category emerged from observations in lower secondary School A (the Waldorf School). In the lesson, curves were dealt with as loci and functions became a special case of these curves. One example is "draw a path that illustrates how to move in order to keep equidistant to two perpendicular walls" (the wall was meant to illustrate the coordinate system).

Functions by representations

Functions by representations were dealt with in one way or another in every school as e.g. the algebraic function expression, value table, graph and situation. But as these representations are all related to the very nature of the function concept (aspects of its definition), it is important to stress that this category is only applied in situations where representational forms are the main tool for explaining the function concept and replaced the formal approach to the concept. This category is for example used to describe teaching sequences where there was no meta-discussion on the function concepts. Hence by using *functions by representations* related to teaching, I refer to observations where emphasis on the function concept itself was given little or no priority and was mainly dealt with through representations.

Functions by examples

This category has a lot in common with the previous category, except that here concrete examples of representation forms are provided. When a specific function or a certain situation (for example hourly wage and total income) is mentioned, this category is used. For it to make sense, this category is not used to categorize teaching, since examples were always in some sense used as a reference context. In the case of students this category is primarily used if examples are a main point in their elaborations concerning the nature of the function concept.

Functions as a hidden structure

Functions as a hidden structure describes situations in teaching or students' explanations which indicate that a function is a mathematical object, superior to representation forms and examples.

Function machine

The function machine category was present in teaching sequences both in lower and upper secondary school. Characteristically, function machines apply different models or examples to illustrate the uniqueness property of functions, or the one/many-to-one principle. Common for these models is that a certain object is put into the model and a well-defined object comes out. These models may be provided by tasks intended to make the students discover the function expression on the basis of a series of input and output values, or they can be used as demonstration models in the introductory phase of functions. For a specific input value, the same output value emerges from the machine every time.

Formal definition

Textbooks often include more or less simplified versions of the function definitions. In some cases I observed teachers reading these definitions out loud, on other occasions teachers just referred to them so the students could read on their own. As I define this category, a formal definition in this sense is any attempt of defining the function concept, where the uniqueness property is included. This category is most frequently used related to the textbooks used.

Functions as co-variance

This last category is a more imprecise version of the previous one. In this case, it is emphasized by various terms that functions have to do with two variables that somehow relate to each other. This category was found both in students' arguments and in teaching sequences.

Related to research question 1b-1d and 2b-2d, gradients (or slopes) are important and are often dealt with as a subtopic of functions. Proportional magnitudes is also a related topic which is often treated as a sub-topic of functions in lower secondary, but as a rather isolated topic on its own in upper secondary, vocational studies programme. Gradients and slopes become essential in upper secondary, general studies programme – especially at the 1T version. Here gradients are related to average and instantaneous growth rate and this relation often constitutes the preface of differentiation. The categories below are based on work with gradients in lower secondary and upper secondary schools. The latter include work with proportional magnitudes in the vocational programmes and treatment of gradients and differentiation in the general studies programme.

Gradient as locus

As in the case of functions, this category was only present in School A. An example was "how to move in order to be twice as far from the x-axis as from the y-axis". The result was a function expression (in this case y = 2x) where these calculations were done in accordance with the "equilibrium-line", $y = x$.

Gradient measured in percent

Also this category was only observed in School A, and was closely connected to the previous category. The starting point was to relate gradients to road signs, where slopes are often measured in per cent. This percentage is calculated by dividing the change in the vertical direction by the change in the horizontal direction and multiplying by 100.

One-unit-right-a-up/down

This method or strategy was the most prominent strategy in lower secondary, but also in the early stages of upper secondary, general studies. It suggests that the gradient of a linear function is identified with a specific technique which involves starting from an arbitrary point on the graph and then first moving one unit to the right in a horizontal direction and then *a* units vertically ("up" or "down" depending on whether the gradient is positive or negative) until one meets the graph. A triangle is then formed by the lines of movements and the graph itself. The vertical distance covered is then equal to the gradient. If one moves upwards (in the positive direction) the gradient is positive and negative if movement if in the opposite direction. Local variants of this strategy were observed, but in most cases the students were advised to start this procedure at a point on the graph where the x and y values were integers.

Gradient as a diagonal movement

For some students it seemed difficult to link the gradient of a function expression to a graphical representation. Some seemed to have trouble decomposing the gradient into a vertical and horizontal part, as in the *one-unit-right-a-up/down* strategy and instead simply counted diagonally along the graph. Diagonals of the squares in background-grids of coordinate systems (on computer screen or notebooks) often constituted the counting units.

Delta y divided by delta x

This category differs from the previous one in that it refers to a change in the y-direction divided by a change in the x-direction. This implies a conception of gradients as "height divided by length" which is not necessarily automatically established from the previous category. Mathematically this follows from the *one-unit-right-a-up/down* category but as I will argue for in the analysis, students' *conceptions* of this do not necessarily follow. This category is only relevant for students doing the

general studies programme and is often related to growth rate and differentiation.

Gradient as the derivative

Differentiation as a topic is also only relevant for students doing general studies, and this category is the more general definition of gradients, also known as the derivative. The most common way of representing the corresponding mathematical definition is in terms of

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Proportionality constant

I decided to focus on the topic of proportional magnitudes, especially in upper secondary school, vocational studies mainly to investigate if and how this was related to gradients both in teaching and learning. It appeared that this link was rather problematic, as it was rarely observed in textbooks, teaching or students' arguments. Still, the proportional constant could be regarded as a "special case" of gradients of linear functions and the category is applied mainly in upper secondary vocational programmes.

6.5 Presentation of the analysis

I will present the analysis in Chapter 7 and 8. In Chapter 7 I will present a student-by-student analysis of one student from each of the four lower secondary schools and his or her transition to upper secondary. By choosing one student from each lower secondary school, I intend to capture any diversity related to aspects of both teaching and learning. Transition is an essential component in this research, and the continuous aspect of the transition is illuminated by first providing a chronological analysis of the individual student's experiences in lower secondary prior to those in upper secondary. In these analyses, I present essential characteristics of learning and teaching issues at the two phases of schooling which are related to the first and the second research questions. Through the analysis in Chapter 7, I identify certain phenomena so that this chapter will also serve as a basis for justification of the categories presented above (in 6.4.4) and discussed in Chapter 8. In Chapter 8, I discuss my findings in the light of the categories which emerged, and I will draw a more holistic picture by elaborating and comparing these categories.

ANALYSIS	
Chapter 7	**Chapter 8**
- One-by-one presentation and analysis for four students (one from each of the lower secondary schools), emphasizing chronology. - Presentation and analysis of the observations and interviews in the sequence in which the data were collected, emphasising chronology. - Localising certain phenomena through the use of terminology from the analysis categories.	- Presentation of my findings through the prescribed categories, in a holistic manner. - Examples, analysis and elaborations based on these categories

Table 6.4. Presentation of the forthcoming analysis

6.6 Validity and trustworthiness

There is a certain tendency in the methodological literature on some paradigms, such as, for example, the naturalistic paradigm of Lincoln and Guba (1985), that methodology should strive to expose the subjective nature of experiences through various methods like observations, in-depth interviews and what Lincoln and Guba (1985) call "thick descriptions".

Maxwell (1992) distinguishes between "descriptive validity", "interpretive validity" and "theoretical validity". Descriptive validity addresses questions concerning the trustworthiness of the researcher's descriptions. There should be no doubt that all the transcriptions applied in this research truly exist, also beyond the excerpts presented in the thesis. Certainly I do not claim a "God's eye view" (Putnam, 1990), and I recognize that in terms of being a researcher, I am a part of the realm in which I act, with no possibility of stepping outside it and providing some observer-independent account for what I experience. This clearly relates to the epistemological assumptions described earlier. In that sense, the excerpts I choose to include or omit depend on my interpretation in the light of my theoretical frameworks and my research questions.

Interpretative validity has to do with how the collected material is interpreted by me as a researcher. During this research I had the opportunity of participating at two conferences and two schools for doctoral candidates, and on these four occasions I presented and discussed a selection of my findings and interpretations. In addition to feedback from supervisors and colleagues, I consider the feedback and discussions on these occasions of great value. In the four cases in which

I was not permitted to use a voice recorder during the teacher interviews, I checked by sending the teachers an e-mail of my summary of the interview to avoid possible misinterpretations and misunderstandings.

Theoretical validity goes beyond concrete observation and involves a high degree of abstraction (Maxwell, 1992). Interpreting my findings in the light of my framework has been a very long process but has evolved mainly through conversations with my supervisors, my colleagues but also through participation at the conferences and doctoral schools mentioned above. I hope that my clarifications of the theoretical framework (Chapter 3) will further strengthen the theoretical validity of the research.

6.7 Ethical issues

As pointed out by Cohen, Manion and Morrison (2007), questions concerning validity and trustworthiness are also ethical issues. However, in this section I will focus on the rights and needs of the teachers and students involved in the study.

Anonymity of the schools, teachers and students involved is taken care of primarily by the use of pseudonyms as described in the earlier section about the transcriptions. Because of the students' age, written parental consent was required before the interviews and conversations took place. In accordance with existing legal requirements, the project was also reported and approved by the Norwegian Social Science Data Service (NSD).

As the students participated in the interviews during their spare time, I found it appropriate to thank them by way of a small gift voucher for a CD or a DVD. To avoid undesirable motivations for participation, this was not announced before students volunteered to take part.

During conversations and interviews with the students I ran into another ethical dilemma, especially in situations where students were engaged in mathematical tasks. Sometimes their argumentation and reasoning was valuable to me, even though it would not be accepted in, for example, a mathematical test. So when I encouraged this reasoning (by nodding my head, make noises such as "mhm" and so forth) I started to worry that I might be giving them the impression that they were on "the right track". I tried to resolve this issue by having an "informal" conversation with them after the interview where I gave them feedback and an informal evaluation which they, hopefully, learned from.

Schools are numerated with letters for the lower secondary schools/classes and numbers indexed by small letters for the upper secondary schools. The explicit information given related to school A, in terms of "revealing" that this is a Waldorf School was clarified with the teacher. I found it important to be explicit on this, as these schools have

their own curricula and constitute a distinct alternative to ordinary public schools.

7 A chronological presentation of four participating students: Otto, Olga, Matt, and Thea

In this chapter I will individually present four of the eight students who form the basis for the analysis. I will analyse their mediated actions in terms of statements made in interviews and conversations. Analyses related to their approaches to and strategies for working with mathematical tasks and activities will also be carried out. The purpose of these presentations and analysis is two-fold, firstly to prepare for and provide some concrete examples related to the more general analysis in the next chapter, and secondly, since transition concerns the shift from one institution to another, to provide more details of the individuals involved. The analyses are relevant for my research questions in terms of a chronological description and analysis of the individual student's situation in lower secondary and his (or her) new situation in upper secondary. In the following I will only focus on some of the observations and parts of the interviews which are relevant to my research questions and will include only some brief narrative summaries from parts which are not so directly linked to the research questions.

I have chosen to present one student from each of the four lower secondary schools involved, and since this study involves both the vocational and general studies programme, I have chosen to include two from each programme. There are two main reasons for selecting these four students. Primarily, after evaluating my empirical data, I found that the material collected from them was rich in essential and representative observations. Secondly, they represent both vocational and the general studies. The students at lower secondary level are as follows– *Otto* from School A (Waldorf School), *Olga* from School B, , *Matt* from School C, and *Thea* from School D, (see Table 6.1, Chapter 6).

7.1 The case of Otto – School A

I observed in total five lessons at School A (Waldorf School) where Otto was one of the two students in the class participating in the study.

7.1.1 Teaching at Lower Secondary - School A

Throughout the observed lessons, the teacher mostly lectured at the blackboard providing explanations in dialogue with the students. The forthcoming examples also show that by means of his explanations the teacher tried to emphasize the connection between functions and geometry. The students were given problems based on geometrical illustrations in terms of loci, and in some cases they were allowed to

spend some time on their own trying to solve them. The teacher did not explicitly use the term "loci" while teaching, but the students were given geometrical criteria for different paths, and were challenged to draw these. Otherwise these problems were discussed through dialogues between the teacher and the students without there being much time for individual work. The recurrent issue in these problems was how to move (which path to follow) in order to maintain an equal distance to certain objects. The starting point for these discussions was a problem presented during my first observation, where perpendicular lines were introduced in connection with an example concerning how to move in order to always have equal distance to two trees. Subsequent to this example, perpendicular walls were applied as a reference context for the coordinate system, when straight lines and their gradients were introduced.

The examples and problems provided became more and more advanced in order to show characteristics of straight lines such as slopes and constant terms, as well as to involve conic sections like the parabola and the hyperbola. The teacher briefly introduced the students to conic sections and in connection with this, the teacher also pointed to the fact that lines could be regarded as a "special case" of conic sections. After working on the conic-section problems for a while, the teacher returned to the case of straight lines and their characteristics. I found this part very important as this relates to research question two, and since it also turned out that it served as an introduction to the concept of functions.

No	Who	Translation	Original
7.1a	Kim (teacher)	In mathematics we make use of walls like these. They're not two walls, but what do we call them?	Nå er det slik at i matematikken så benytter man seg av disse murene. Det er ikke to murer, men hva er det vi kaller dem?
7.1b	Student[11]	y and x	y og x
7.1c	Kim	Yes, we call them y and x […]. So when we move like this, y equals x. Always. No matter where we are along this path, the distance to y and the distance to x is the same, right?	Vi kaller dem for y og x, ja. […]. Så når vi går på denne måten her så er y lik x. Bestandig. Uansett hvor vi befinner oss langs denne stien så vil avstanden til y og avstanden til x være lik, ikke sant?
7.1d	Kim	We say that the fact that y = x, that is what we call a function, while this drawing here, we call a graphical presentation.	Vi sier at det at y = x, det kaller vi for en funksjon, mens denne tegningen her kaller vi for en grafisk fremstilling.

[11] When I write "student" (and no names) in transcriptions, this indicates that the utterance is made by students who are not among the eight students I am focusing on in my study.

| 7.1e | Kim | The function illustrates the relation between two varying magnitudes. If we vary x, then we also have to vary y related to x. We say that y is a function of x. | Funksjonen viser sammenhengen mellom to varierende størrelser. Hvis vi varierer x så må også y variere i forhold til x. Vi sier at y er en funksjon av x. |

Excerpt 7.1

In terms of Steinbring's (2006), model this excerpt which considers the equidistant movement between the two walls can be understood as the point of origin for constituting the first reference context for a following series of arguments. Building on Steinbring's model in terms of semiotic chaining, there is a shift from describing the situation in terms of walls and movements between them to an algebraic representation of the movement in terms of x and y (7.1a – 7.1b). Further, as the link between paths of movements and a more formal mathematical description of the line presented in 7.1d was made through "y = x", the semiotic chain expands. In this step, the meaning of x and y established in 7.1b and 7.1c constitutes a new reference context for establishing the equation y = x in 7.1d. In 7.1d Kim also emphasizes the difference between functions and graphical representations. However, it seems that Kim, by being rather imprecise in this formulation in 7.1e implies that the fact in this particular case, namely that y equals x, constitutes the definition of a function. However, 7.1d indicates that he distinguishes between the algebraic expression (even though he is denoting the algebraic expression as "function") and the graphical expression. The more general formulation provided in 7.1e represents yet another similar step in this chain, namely the description of x and y as variables. It should also be pointed out that 7.1d and 7.1e is the only time during my observations in School A that the word "function" is used.

As pointed out in the methodology chapter, I organised my empirical data in different categories. It transpired that several categories were usually intertwined in the sense that teaching and learning in one particular class often seemed to involve several of the categories. My observations at School A constitute the basis for the analytical category *functions as loci,* but Kim's concluding remark in 7.1e also falls into the category *function as co-variance.*

The arguments provided by the teacher in Excerpt 7.1 are followed up by moving on to gradients. The task was to move in such a way that the distance to the x-axis is always twice the distance to the y-axis.

| 7.2a | Kim | How big are the y's compared to x here? | Hvor store er y i forhold til x her? |
| 7.2b | Student | y equals 2x | y er lik 2x |

7.2c	Kim	Exactly. y is twice the size of x. Now, have some of you have done this the opposite way, and how is it then, and what's the name of that curve? That's the next task. [The students are given some time to solve the task]. y is half of x, but what's the name of the curve?	Nettopp. y er dobbelt så stor som x. Nå er det noen av dere som har gjort dette omvendt og hvordan blir det da, og hva blir den kurven hetende? Dette blir neste oppgave. [Elevene blir gitt litt tid til å løse oppgaven]. y er halvparten av x, men hva blir navnet på kurven?
7.2d	Student	2y equals x	2y er lik x
7.2e	Kim	We write 2y = x [writes on the blackboard]. Can we write this in another way?	Vi skriver 2y = x [skriver på tavla]. Kan vi skrive det på en annen måte?
7.2f	Student	x = 2y	x = 2y
7.2g	Kim	But related to y equals…? y equals half of x, y is half of x. [Writes y = 1/2x on the blackboard]	Men i forhold til y er lik…? y er lik en halv x, y er halvparten av x. [Skriver y = 1/2x på tavla]

Excerpt 7.2

Also in this case the use of locus was employed, since the students were to move in such a way that the distance to the x-axis was always twice the distance to the y-axis. But at the same time x and y were perceived as variables and by expressing straight lines in mathematical terms, the teacher built on these concepts to introduce the gradient of a linear function. In the following excerpt the concept of gradient is also related to road signs. Previous to this, the teacher has talked about road signs, and how slope in those cases were measured in percent.

7.3a	Kim	Let's say if this had been a hill, how many percent would this have been? [Points to the line y = 1/2x]	La oss si hvis dette hadde vært en bakke, hvor mange prosent hadde dette vært? [Peker på linja y = 1/2x]
7.3b	Student	About 22 percent	Cirka 22 prosent
7.3c	Kim	Ok, so what does this percentage mean here?	Ok, så hva betyr denne prosenten her?
7.3d	Student	Incline	Stigning
7.3e	Kim	But what does the percentage mean?	Men hva betyr prosent?
7.3f	Student	Per hundred	Per hundre
7.3g	Kim	Yes, per hundred. So it actually means that when we look at the sign it means eight per hundred. […]. For each hundred meter you drive forward you have driven eight	Per hundre ja. Så det betyr altså når vi ser på skiltet så betyr det åtte per hundre. […] For hver hundre meter du kjører bortover så har du kjørt åtte meter oppover. Når du har kjørt hundre meter fra der du var med bilen din, så har du kommet åtte

| | | meters upwards. When you have driven a hundred meters from where your car was, you have come eight meters higher, do you understand? [Illustrates more accurately by making a drawing on the blackboard where the driving distance is decomposed in a horizontal and a vertical component. The horizontal component is 100 meters and the vertical is eight meters] | meter høyere opp, skjønner dere det? [Illustrerer mer nøyaktig ved å lage en figur på tavla hvor han dekomponerer den skrå vegstigningen inn i en horisontal og vertikal komponent. Den horistontale komponenten er 100 meter og den vertikale er åtte meter] |

Excerpt 7.3

One should note that in 7.3g the teacher was inaccurate in his description as he equates the distance the car has driven with the horizontal component of this driving distance. The link between the slope measured in percent and the mathematical expression for the straight line in 7.2g is an interesting link, as this serves as yet another reference context related to gradients. It might seem natural to see Steinbring's model as a chain moving towards more abstract or "general" concepts, but in this example, "road signs" and "slope measured in percent" follow immediately *after* the introduction of the sign "$y = 2x$". The path from the initial problem of locus to the concept of functions and the concept of gradients is illustrated in figure 7.1below. Figure 7.2 displays how the gradient is linked to road signs and slope measured in percent.
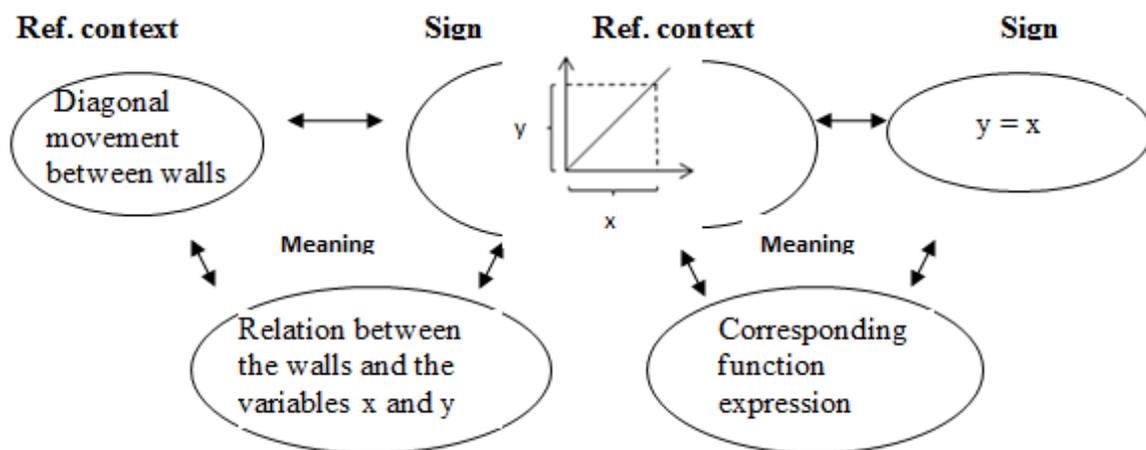
Figure 7.1 From locus to function expression
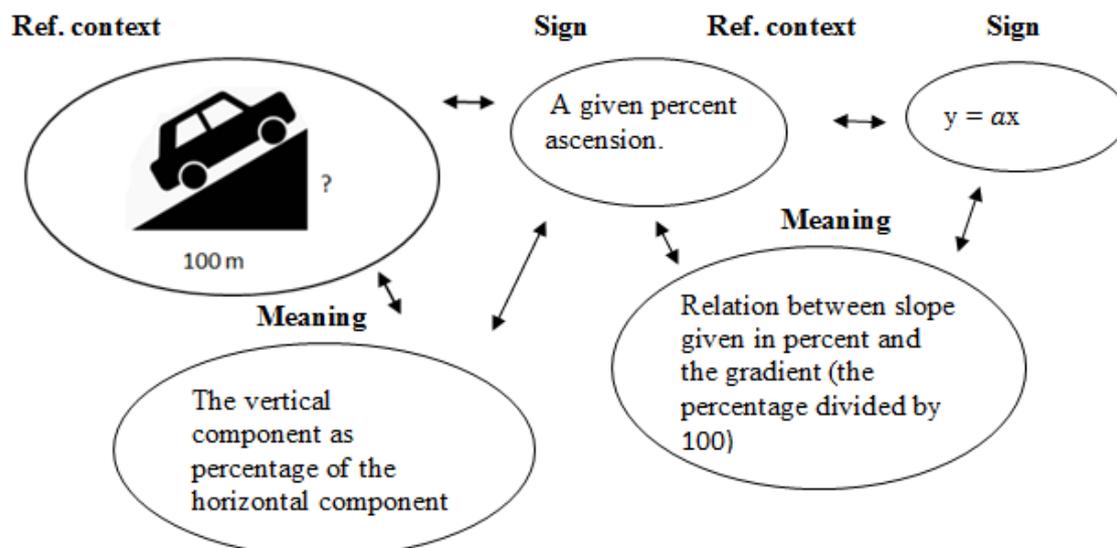
Figure 7.2 From slopes in road signs to gradients.

This excerpt from the interview with Kim after my period of observations illustrates how the geometrical aspects of mathematics consciously underpin his teaching of functions.

7.4a	Interviewer	I get the impression that it's strongly emphasized, these relations between different fields of mathematics, when it comes to functions?	Jeg har et inntrykk av at det vektlegges sterkt, dette med å vise sammenhenger i ulike felt av matematikken, med tanke på funksjonslære?
7.4b	Kim	Also in that case, geometry comes in. So, actually it arises from geometry, because loci are, in reality, a field in geometry. One starts there and identifies paths or roads to follow to maintain certain conditions […]. So, from the start, geometry and mathematics are united.	Også der kommer jo geometrien inn. Altså, det springer jo egentlig ut fra geometrien, i og med at dette med geometriske steder egentlig er et felt i geometrien. Man starter der og finner fem til stier eller veier man skal gå for å opprettholde visse forutsetninger […] Så geometri og matematikk er til å begynne med en enhet.

Excerpt 7.4

In line with the curriculum of Waldorf Schools (Section 2.3.2), the underpinning geometric approach which I observed in the topic of functions is clearly visible. As Kim stated in the interview, conic sections would normally be modelled in 3D by using Plasticine. Linear functions are introduced by the use of loci in terms of paths adjusted to certain distances from perpendicular walls. Functions are presented as co-variation between x and y, but due to requirements of uniqueness, this is incomplete if compared to the Dirichlet definition and similar definitions treated in Chapter 3. The way the gradient of a linear function

is presented in terms of loci and the link to percentages, differs from the formal calculus definition, but still serves to describe certain characteristics. In calculus, the gradient of linear functions is related to the distance in the vertical direction (y) divided by the corresponding distance in the horizontal direction (x). At School A, it is presented in two ways: in terms of loci and in terms of determining the percentage of a given slope. Related to the categories accounted for in Chapter 6, I have categorized Kim's approach to gradients as *gradient as loci* and *gradient measured in per cent.*

7.1.2 Tasks in Lower Secondary

In my relatively brief considerations of the students' tasks, I will draw mainly on some concepts developed by Stein, Grover and Henningsen (1996) where "context", "task features" and "cognitive demands" are included. The situated context of the tasks mainly considers the question of whether a task is situated in a "real-life" context or in the "abstract world" of mathematics. Task feature includes possible solution strategies, representations and communication requirements, while cognitive demands focus on whether memorization, procedures (with and without connections to concepts) or the "doing of mathematics" are involved in the solving process.

   In line with the ideology of Waldorf Schools, no textbooks were used at School A, and therefore individual or collective tasks were often provided directly by the teacher. In this case, the tasks given were related to conic sections, linear functions and gradients, and only one of them, related to research questions one and two, is of interest here. The task exemplified in the conversation with Otto (Excerpt 7.5) did not have a direct link to a real-life problem. Rather, it was a constructed task, using the mathematical world in terms of the coordinate system and linear functions as references. The task was also directly based on the instructions given by the teacher prior to the task, so it would have been possible to solve this task almost only through memorization. Although communicating with peers was allowed, this was not a condition for solving the task.

7.1.3 Conversation with Otto in Lower Secondary

While the students were working on their own to find the slope of the straight lines $y = 2x$ and $y = \frac{1}{2}x$, I was able to talk to Otto who was trying to calculate the slope of graphical representation of the function $y = 2x$.

7.5a	Interviewer	I wonder if you have found a gradient here [referring to the line y = 2x]?	Jeg lurer på om du har funnet noe stigningstall her [refererer til linja y = 2x] ?
7.5b	Otto	I've found that it's 200% on the upper one [y = 2x].	Jeg har funnet ut at det er 200% på den øverste [y = 2x].

		When it's one x [y = x] then it's 100% and on the upper when it's 2x we have to double. So there's nothing to calculate.	Når det er en x [y = x] så er det 100% og på den øverste når det er 2x må vi bare doble. Så det er ikke noe å regne.
7.5c	Interviewer	So you see this directly from the expressions?	Så du ser det direkte fra uttrykkene?
7.5d	Otto	Yes	Ja
7.5e	Interviewer	But when you say 200%, then it's usually 200% of something. So what is it 200% of?	Men når du sier 200%, så pleier det jo ofte være slik at det er 200% av et eller annet. Så hva er det 200% av?
7.5f	Otto	200 meters increment then, per 100 meters.	200 meter stigning, da, per 100 meter.
7.5g	Otto	[…] But that would really only mean that it's twice the upper one [y = x] because it's two x's. So then you just see it, right?	[…] Men det vil jo egentlig bare si at det er det dobbelte av den øverste [y = x] for det er jo to x'er. Så da ser du det jo, ikke sant?

Excerpt 7.5

Otto uses y = x as a reference point for finding the expression y = 2x. Probably because he assumes a linear relationship between the gradient and the percentage, this process of doubling seems so obvious to him that he does not consider this a calculation at all (7.5b). From 7.5b it is also clear that Otto relates the gradient of a function to the slope measured in percent, probably due to Kim's explanations in Excerpt 7.3. From Otto's argumentation in 7.5b, 7.5f and 7.5g it seems that he is able to make a connection between the sign (y = 2x) and the reference context (slopes measured in percent).

7.1.4 Interview with Otto at Lower Secondary

In my report from the interview conducted with the students in lower secondary school, I will mainly focus on the content related to research question 1a. As most of the material relevant to research question 3 is provided through the interviews with the students after entering upper secondary school, I will deal with that issue in the upper secondary section.

To investigate the development of the students' conception of functions, I asked them to explain what is meant by a function in mathematics. The same question was also posed in upper secondary, for the sake of comparison. Otto (like several of the other students) was very reticent at the start and had difficulties expressing himself at all about what is meant by a function. After encouraging him just to say whatever came into his head without worrying about whether his answers were correct or not, he said:

| 7.6 | Otto | Yes, then I would of course have known a bit about how it worked. With such a cross [the hand movements indicates that he is talking about the coordinate system], I would have made such points upwards and downwards [indicates the plotting of points with his right hand]. Then I would have shown that x is horizontal and that y is downwards and such. But how I would have explained it and taught someone about this, I'm really not sure. [...]. Then, with the little table you make and then make a line. | Ja, da ville jeg jo selvsagt ha vist litt hvordan det fungerte da. Med sånn kryss [håndbevegelsene signaliserer at han nå snakker om koordinatsystemet], så hadde jeg jo laget sånne punkter oppover, høyre, venstre og nedover [indikerer plottingen av selve punktene med høyre hånd]. Så ville jeg vist at x er vannrett og y er nedover og sånn. Men hvordan jeg skulle forklart det og lært det til noen andre er jeg ikke helt sikker på. [...] Så med den lille tabellen som man lager og så lage en strek. |

Excerpt 7.6

Otto's answer, which is categorized as *functions by representations,* explained what is meant by a function in terms of expressing it through some of its representations. His gestures refer to the coordinate system, followed by a description corresponding to the plotting of points. The line drawn at the end suggests a graph as the intended representation and the "little table" probably refers to the value table. There is no direct trace of the concept of variables (dependent and independent) or of a more formal definition in his explanations.

The next question was also posed in both lower and upper secondary schools for the sake of comparison. The students were shown the linear expression $y = 2x - 3$, and were asked to elaborate as much as possible on what they saw. In the presentation of these answers, I will focus especially on the function concept and the gradient in accordance with research questions 1a and 1b.

| 7.7 | Otto | Either this is an equation or it's a function. I think I have seen this before. But I don't remember how I calculated it. But if you have 2x then it should be twice as long as 1x. But I don't remember how I calculate it. | Enten er det en likning eller så er det en funksjon. Jeg synes jeg har sett denne før. Men jeg husker ikke hvordan jeg regnet den ut. Men hvis du har 2x da skulle den jo være dobbelt så lang som 1x. Men jeg husker ikke hvordan jeg regner ut. |

Excerpt 7.7

The question seemed to challenge Otto's conception about the relationship between an "equation" and a "function". The way he formulates it suggests that it has to be "either or" and not both. It is not

clear if the doubling technique expressed here is directly related to the doubling technique applied in Excerpt 7.5, but there seems to be a connection since he claimed that he did "not remember how to calculate this". At least this indicates that he regarded this as a familiar problem. When I attempted to go into more detail concerning the slope, he came up with this rationale:

7.8a	Interviewer	So what does the number 2 [in front of the x] tell about what this straight line looks like?	Så hva sier det 2-tallet [foran x'en] om hvordan denne rette linjen ser ut?
7.8b	Otto	That it has to go one step upwards.	At den må et hakk lenger opp.
7.8c	Interviewer	What do you mean by that?	Hva mener du med det?
7.8d	Otto	If there's one which goes like this, right through the cross in a way [points to illustrate a straight line through the origin, with a gradient around one], and it's 2x, then it has to go one step upwards.	Hvis det er en som går sånn, rett igjennom krysset på en måte [peker for å illustrere en rett linje gjennom origo, med tilnærmet stigningstall 1] , og det er 2x, så må den et hakk lenger opp.
7.8e	Interviewer	One step compared to what?	Et hakk opp i forhold til…?
7.8f	Otto	Wait…if there's only one x there in a way, for example there [plots a point in (-2,-4)] then it has to go one step upwards there [plots a point in (-1,-3) and draws a line between the two points]…but now I'm very unsure…	Vent, da…hvis det bare er en x der sånn, for eksempel der [plotter et punkt i (-2,-4)] så må den ett hakk lengre opp der [plotter så et punkt i (-1,-3) og trekker en linje mellom de to punktene]…men nå er jeg veldig usikker da…

    Excerpt 7.8

If Otto, in Excerpt 7.7, had the doubling technique of Excerpt 7.5 in mind, the argument in 7.8d shows that his line of argument has changed from two being the double of one to two being one more than one. Initially he tended towards sliding the graph one unit in the y-direction. In this case, the y-value increases by one while the gradient stays the same. As a result, "one step up" seemed to be his natural conclusion. Although he constantly expressed doubts, in 7.8f he explained the sliding by fixing a point, in this case (-2,-4) and "one step up" is interpreted as one step in a diagonal direction towards the point (-1,-3). His focus is on two single points on a straight line and "one step up" in this case is the diagonal movement from (-2,-4) to (-1,-3). He was then able to draw a line between the points to illustrate the expression, but this line has slope one. When comparing Otto's reasoning in Excerpt 7.5, 7.7 and 7.8, one observes that his arguments were not consistent. His reasoning in

Excerpt 7.5 falls in to the category *gradient measured in per cent*. While I am unsure about the overall interpretation of Excerpt 7.7, and of what Otto referred to by "twice as long", Excerpt 7.8 seems to be an example of the category *gradient as a diagonal movement"*.

7.1.5 Teaching in Upper Secondary – School 1a

In upper secondary school, Otto attended the vocational studies programme "media and communication". As indicated in Chapter 2, functions are not explicitly mentioned in the curriculum, so the observed lessons consist of teaching related to proportional magnitudes. In Otto's case I observed one lesson on this topic. The teacher, Bernt, introduced the topic using an example showing the connection between the number of items bought, the prize per item and the total cost.

7.9a	Bernt	So if we're going to explain what a proportional magnitude is, then we can continue with the apples that we were talking about. We buy 1 kg of apples that costs for example 15 kroner. How much will 2 kg cost?	Så hvis vi skal prøve å forklare hva en proporsjonal størrelse er, så kan vi fortsette med eplene vi snakket om. Vi kjøper 1 kg epler som koster for eksempel 15 kroner. Hva koster det da for 2 kg epler?
7.9b	Student	30	30
7.9c	Bernt	30. How much does 3 kg of apples cost? And then, how much does 6 kg of apples cost?	30. Hvor mye koster det da for 3 kg epler? Og hvor mye koster det da for 6 kg epler?

Excerpt 7.9

As I will show later, similar examples from other upper secondary vocational studies classes, form a typical pattern for how the principles of proportionality were introduced. Multiplicative structures were frequently applied to illustrate the relation between two proportional magnitudes. As in 7.9a, the operation of doubling usually constitutes the first step in a row of exemplifications.

The teacher continued with another example involving number of steps and total walking distance, and illustrated the results by means of a table and a graph. Then he moved towards a more general expression via a similar example where the hourly wage was 125 kroner per hour and wrote "$y = 125x$" on the blackboard.

7.10a	Bernt	What happens if we now divide by x on each side? [No response] Then the x vanishes [illustrates this by removing x from the right side and putting it in the denominator below y on the left side and $y/x = 125$ is	Hva skjer om vi nå deler med x på begge sider? [Ingen respons] Da forsvinner x'en [viser dette ved å fjerne x fra høyre side og setter den i nevneren under y på vensre side og det står nå $y/x = 125$ på tavla].

		now on the blackboard].	
7.10b	Bernt	That is the rule of proportional magnitudes. If y divided by x is one number, then there are proportional magnitudes. [This is exemplified by 10 working hours giving 1250 kroner in earnings]	Det der er regelen for proporsjonale størrelser. Hvis du har at y delt på x er ett tall, så er det proporsjonale størrelser. [Dette eksemplifiseres så ved 10 arbeidstimer og 1250 opptjente kroner]
7.10c	Bernt	If there are proportional magnitudes, then y divided by x, or m divided by n, or q divided by r, is a constant number.	Er det proporsjonale størrelser så er alltid y delt på x, eventuelt m delt på n eller q delt på r, et konstant tall.

Excerpt 7.10

A more general statement was made to conclude the lesson. The number 125 was replaced by "a", and the expression was reformulated into $y = ax$. Bernt concluded that "$y/x = a$" defines the property of proportional magnitudes and emphasized that a is the proportionality constant. The semiotic chains constructed through these instructions are visible primarily through the explicit link between the concrete examples and the more general and formal notations. One such movement is made in terms of the notation "$y = 125x$", where the connection between y as the total wage, and x, the number of working hours (the reference context) is made explicit. The link between the arithmetic operation of dividing both sides of the equation by x and working for 10 hours is also made (7.10a and 7.10b). The conclusions are drawn by pointing to the relation between the hourly wage of 125 kroner and the general term "a". Bernt's arguments for presenting proportional magnitudes in this manner are expressed in the following excerpt:

7.11a	Interviewer	I noticed something in connection with pro-portional magnitudes. Can you give some examples of how you did that?	Jeg observerte jo noe i forbindelse med propor-sjonalitet. Kan du gi noen eksempler på hvordan du gjorde dette da?
7.11b	Bernt	Yes. I try to make them see this in relation to something. Not only say that here is a straight line, ax + b, but in a way first try to make them think on their own. What is a straight line, and what does proportionality mean.	Ja. Jeg prøver å få dem til å se det i forhold til noe, da. Ikke bare si at her er en rett linje, ax + b, men på en måte prøve å få dem til å tenke litt selv først. Hva er det som er en rett linje, og hva betyr proporsjonalitet. At det er dobbelt, på en måte, og at

		That it is twice as much, in a way, and that it is a smooth and even increase, without necessarily providing the mathematical answers.	det er jevn stigning, uten nødvendigvis å gi de matematiske svarene.
7.11c	Interviewer	So if you think about proportionality and inverse proportionality as relevant for vocational studies, which examples would you use to illustrate this?	Så hvis du ser for deg proporsjonalitet og omvendt proporsjonalitet, relevant for yrkesfag, hvilke eksempler ville du ha brukt for å illustrerer dette?
7.11d	Bernt	Proportionality as such is pretty easy because much is proportional. A lot related to shops and costs.	Akkurat proporsjonalitet er jo ganske lett for det er veldig mye som er proporsjonalt. Mye i forhold til butikk og priser.

Excerpt 7.11

One should note that there was no attempt to make an explicit connection between the gradient and the proportionality constant, even though LK06 (Utdanningsdirektoratet, 2010) states that students after the 10th grade should be familiar with characteristics of linear functions. Similarly, the concept of functions is also not explicitly mentioned in this teaching sequence.

7.1.6 Tasks in upper secondary

The tasks given at School 1a were taken from the textbook, "Sinus" (Oldervoll, Orskaug, Vaaje, & Hanisch, 2009). In total four tasks were given to the students at the end of the observed lesson, and all of these were to some extent rooted in a real-life context. The pattern was similar for all four tasks – first a table was presented showing the relation between two proportional magnitudes (price per kilogram and total price, hourly wage and the total wage, time and the distance of lightning). In each case the students were asked to investigate whether the magnitudes were proportional or not, and if they were, to find the proportional constant.

Task 3.66

The table shows the time t passing, from the moment you are observing lightning until you hear the thunder, and the distance, x, to the lightning measured in meters, for some values of t.

Time t (s)	1	2	3	4	5
Distance x (m)	340	680	1020	1360	1700

a) Investigate if t and x are proportional magnitudes, and if so, find the possible proportionality constant.
b) One day, when a thunderstorm occurs, it takes 6.5 s from Maria observes the lightning until she can hear the thunder.
How far away from the lightning is Maria?

Figure 7.3. Example of task related to proportional magnitudes in School 1a (Oldervoll, Orskaug, Vaaje, & Hanisch, 2009, p. 79, my translation)

This task illustrates the pattern of all these four tasks; a two-row table which serves to show the relation between two magnitudes (in this case time and distance, based on lightning and thunder). In part a) the students are asked to check if the magnitudes are proportional, and if so, to find the proportionality constant. In b) they are asked about the distance from the lightning, if the sound of the thunder is heard 6.5 seconds after the lightning appears. Memorization based on Bernt's instructions combined with simple calculations and procedures is the most obvious strategy for solving part a) of the tasks and the need for communication is minimal. In b) the context is not so close to the standard procedures provided in the textbook, and the students are challenged more on their conceptual understanding of ratio.

7.1.7 Interview with Otto in Upper Secondary

One might claim that since functions are not an explicit part of the curriculum for vocational studies, it is natural that no explicit link to functions was made in the teaching sequence.

7.12a	Otto	A graph. I would say that that's the increase in something that happens. If somebody works for that long and receives that much salary. Then you can see how much salary there is related to how much you earn, for example.	En graf. Det vil jeg si er veksten på noe som skjer da. Hvis det er noen som jobber så og så lenge og får så og så mye lønn. Så kan man se hvor mye lønn det er i forhold til hvor mye du tjener, for eksempel.
7.12b	Interviewer	Yes. And that in a way is an example of a function.	Ja. Og det er jo for så vidt et eksempel på en funksjon det.
7.12c	Otto	Is it?	Er det?

Excerpt 7.12

Otto had no response when he was asked to explain what is meant by a function. So, in an attempt to adapt the question to his current situation, I asked him if he could explain what was meant by a graph. Otto immediately related graph to "something that grows", and gave the example of wages and work. He also pointed to a relation between magnitudes, in this case the relation between time and wage, but also between the wage and "what you earn". These may seem like synonyms, but in this case "what you earn", should most likely be interpreted in terms of hourly wage. Disregarding the substitution of "functions" by "graph" in 7.12a, this example falls into the category *functions by examples*. Contrary to his response in lower secondary school, Otto now came up with this concrete example. This probably had to do with my rephrasing of the question, but from this answer one can detect rudimentary traces of variables represented by time (hours) and total wage. These traces were hard to find in his lower secondary explanations (Excerpt 7.6). So, in this case the question concerning development in reasoning (research question one) through the possible emergence of semiotic chains becomes rather problematic since functions and gradients are no longer an explicit part of the teaching.

Even though linear expressions including the constant term (different from zero) were not present in upper secondary vocational studies, for the sake of comparison I showed Otto the same expression as in lower secondary, $y = 2x + 3$, and asked him to elaborate on it. In accordance with research question one, the focus here is on the "2x" part.

7.13a	Otto	It must be that y, that is the answer, and then something is unknown plus three becomes y. That's an equation.	Det må vel være at y, det er svaret, også er det noe ukjent som pluss tre blir y da. Det er en likning.
7.13b	Interviewer	Would it have been possible to draw this like a sort of graph or a representation?	Hadde det gått an å tegne dette som en slags graf eller en fremstilling?
7.13c	Otto	Yes, I suppose so.	Ja, det vil jeg tro.
7.13d	Interviewer	Do you have any idea how?	Har du noen ide om hvordan?
7.13e	Otto	[Sketching a coordinate system and plots the point (2,1)] but plus three, I don't quite know what that means.	[Skisserer et koordinatsystem og merker av punktet (2,1)] men pluss tre, det vet ikke helt hva vil si.
7.13f	Interviewer	Can you say it once more – you went one upwards on the y-axis and two forward on the x-axis, why did you say you did that?	Kan du ta det en gang til – du gikk en opp på y aksen og to bort på x aksen, hvorfor gjorde du det, sa du?
7.13g	Otto	Yes, because there are points up on both sides, and there's one step upwards	Jo, fordi det går punkter oppover på begge sidene, også er det ett hakk opp fordi at det er

		because there's one y.	en y.
7.13h	Interviewer	How do you think this graph would have looked if you'd drawn it?	Hvordan tror du den grafen hadde sett ut om du hadde tegnet den?
7.13i	Otto	Something a bit like this [sketches a straight line through the origin and (2,1)]	Noe cirka sånt [tegner en rett linje gjennom origo og (2,1)].

Excerpt 7.13

7.13a is an example of what I previously called indirect reasoning related to the one/many-to-one principle – Otto indicated that "y" is *the* answer, and that this value corresponds to "something" added by three.

This time his strategy was also slightly different than in lower secondary as he related the coefficients of x and y directly to a point in the coordinate system. The straight line was drawn with the origin as the other reference point. The similarities between this and his argument in lower secondary are still apparent in terms of regarding the points in the coordinate system as a direct result of the coefficients in the given expression. The x having the coefficient two, and y one, led Otto to believe that these values somehow could be plotted directly into the coordinate system. The line is drawn by picking the origin as the other point needed. In lower secondary, he assumed that the first point could be chosen at random. This argument is hard to categorize as anything that has to do with gradients at all, since it seems more like a plotting technique. Nonetheless, Otto's conception of gradients can be summarized as inconsistent, moving from *gradient measured in per cent* and *gradients as diagonal movements* in lower secondary to a more *non-mathematical* conception at upper secondary. Non-mathematical is basically the category which refers to lack of argumentations and justifications, for example students who do not answer or explicitly state that they do not know.

7.1.8 Otto's experience of the transition

Otto's background from the Waldorf School, in some ways made him a special case in this study. As mentioned earlier, Waldorf Schools have their own curriculum based on the philosophy of Rudolf Steiner, so they could be expected to be rather different from ordinary public schools.

7.14a	Interviewer	How do you experience the teaching methods, if you compare them?	Hvordan opplever du undervisningsmetodene, hvis du sammenligner dem?
7.14b	Otto	It's quite similar, actually. At the Waldorf School the teacher also explained very thoroughly on the blackboard, then you had to get out the book and	Det er faktisk ganske likt. På Steinerskolen så var det jo også slik at læreren sto å skrev og forklarte veldig nøye på tavla, så var det bare å bla opp boka å begynne da. Så det er ganske likt

		start. So it is quite similar as here. But actually it's only the math I think is quite similar.	som her. Men det er faktisk bare matten som jeg synes er ganske lik.

Excerpt 7.14

Through the period of observation, teacher-controlled lessons in terms of instructions at the blackboard were prominent in both the Waldorf and this upper secondary school (School 1a). This impression was also shared by Otto (Excerpt 7.14). In spite of Otto's familiarity with the teaching methods, what he experienced as lack of individual follow-up of students troubled him.

7.15	Otto	Here it is do it like this and that and then just get out the book and do it. At the Waldorf School we got more help and guidance all the time.	Her er det bare sånn og sånn gjør du det og da er det bare å bla opp boka å gjøre det. Mens på Steinerskolen så fikk vi jo mer hjelp og veiledning hele tida.

Excerpt 7.15

In this statement, Otto refers to two things, namely that there were no textbooks in the Waldorf School while there were in the upper secondary school, and that there was a lack of individual follow-up at upper secondary. In the lower secondary interview, Otto was basically positive to the fact that Waldorf Schools do not use textbooks, but this background seemed to make it hard for him to adapt to the new situation in upper secondary where textbooks played a significant role in the teaching.

7.2 The case of Olga – School B

Olga was one of two participants from lower secondary School B.

7.2.1 Teaching in Lower Secondary – School B

In total, I observed three lessons at this school. The teacher, Oda, started the first lesson by handing out some introductory tasks to the students. They were given approximately five minutes to work with these, before there was a discussion. I will focus on two of the questions in the handout: "What does the gradient of a straight line tell us?" and "We think of x and y as two numbers; in that case, what does it mean that y is a function of x?"

During the discussion of the first question, three students raised their hands. One suggested "steep" and the other two suggested "increase". Oda rephrased the question into "how do we measure increment?" and since there was no response, she answered the question herself: "One out, and how much does the y–axis increase or decrease". She elaborates further on this by using the example $y = 2x + 3$.

7.16a	Oda (teacher)	The trick is here [draws the figure on the whiteboard].	Trikset er her [tegner figur på tavla]
7.16b	Student	If x increases by one, it's two up	Hvis x øker med én så er det to opp
7.16c	Oda	You can draw triangles like these or use a table.	Dere kan bruke sånne trekanter eller bruke tabell

Excerpt 7.16

This short dialogue between Oda and the student provides an example of the *one-unit-right-a-up/down* strategy for finding the gradient of a linear function. Oda then turned to the example $y = -2x + 4$ and illustrated this on the whiteboard as well, this time emphasizing the downward movement. The epistemological triangle related to this corresponds to the left part of Figure 7.4, where horizontal and vertical movements constitute a link between the visual steepness and the gradient. However, in this case Oda also included an example with a negative gradient. Both these examples were illustrated on the whiteboard including the triangles formed by the lines of movement and the graph itself. It is these triangles which are referred to in 7.16c.

The second question concerning the definition of a function was also answered by one of the students:

7.17a	Student	One value of x is one value of y	Én x-verdi er én y-verdi
7.17b	Oda	Could two x values have the same y value?	Kan to x-verdier ha same y-verdi?
7.17c	Student	No	Nei
7.17d	Oda	Yes, they can actually	Jo, Det kan de faktisk

Excerpt 7.17

The student's statement in 7.17a was challenged by Oda, as she immediately problematized the student's suggestion, probably with the intention of illuminating the difference between the one/many-to-one principle and the one-to-many property (invalid in the case of functions). This question is rather tricky and the student's quick conclusion in 7.17c might suggest that the uniqueness property of a function was somehow familiar even though the one/many-to-one and one-to-many properties were confused, for example related to their brief treatment of simple quadratic functions. Oda elaborated on this by using the two previous function expressions to calculate the y values for x = 1 in the first expression and x = 4 in the second.

7.18	Oda	It's always the case that one x value corresponds to one y value. Read page 184 in the textbook.	Det er alltid sånn at én x-verdi svarer til én y-verdi. Les side 184 i læreboka.

Excerpt 7.18

By means of this introductory discussion the function concept was made explicit, and a formal definition was provided, partly by Oda's own

statement in 7.18 and partly by referring to the textbook. The textbook provides the following definition:

> In everyday life, we often run into magnitudes that are somehow related. The media often use graphs to show the nature of this relation. Sometimes this is in such a way that each x value only gives one y value. Then the relation is called a function. (Martinsen, Oldervoll, & Pedersen, 1999, p. 184, my translation)

I consider Oda's explanation and her reference to the textbook to constitute a *formal definition*. It is worth noticing that Oda's comment in 7.17d was absent in her exemplifications as most of the functions dealt with were linear functions and hence one-to-one. This emphasis on one-to-one functions might also influence students' understanding and their ability to cope with the exact content of the formal definition of functions. When linear functions are regarded as *prototypes* (Presmeg, 1992) of functions they might at the same time serve as *distractors* (Nesher & Teubal, 1975) of fully illuminating the uniqueness property, as many-to-one examples of functions rarely are treated.

The next lesson was related to the topic of proportional magnitudes. Also in this lesson, Oda chose to let the students read for themselves from the textbook and work with tasks from the textbook. This was followed by some short discussions. Based on their readings and a task displaying the relation between weight and prize the students were challenged to explain the application of two different methods for finding whether these magnitudes were proportional or not.

7.19a	Student	Weight divided by price gives the same answer, and you can also draw a line which goes through the origin.	Vekt delt på pris gir same svar, og vi kan også tegne en linje som går igjennom origo.
7.19b	Oda	In principle this is correct, but we usually take the price divided by weight	I prinsippet er dette riktig, men vi pleier å ta prisen delt på vekta.

Excerpt 7.19

Oda made a table on the board, showing that the values for y/x in the table became the same in each case (78.5 kroner per kilogram). The question of the second solution to the problem (straight line through the origin) was not further dealt with in this task. But briefly at the end of the lesson in connection with similar task, Oda explained that if the magnitudes plotted into a coordinate system do not result in a straight line passing through the origin, then they are not proportional. The relations between these two methods of testing if magnitudes are proportional or not, were not accounted for.

The relation between proportional magnitudes and gradients of linear functions was not made explicit neither by the teacher *nor* in the textbook. According to my observations, the teachers through some form of discussion (as in 7.19) seemed to define the standard method for

deciding if two magnitudes were proportional or not. These arguments led to a constant, in this textbook denoted simply as "k" (and in other textbooks also called the "proportionality constant").
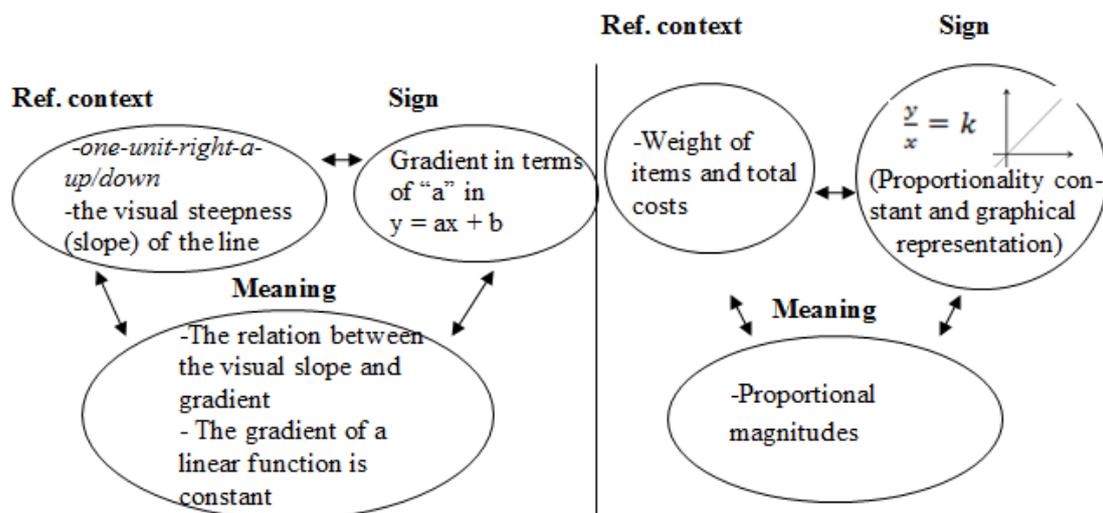
Figure 7.4 Epistemological triangle related to gradients (left) and to proportional magnitudes (right)

The epistemological triangles above give an overview of mediated concepts in this and similar lessons, related to gradients and proportional magnitudes. One should note that these topics were presented without explicit connections, for example by drawing parallels between the proportionality constant and the gradient of linear functions. Such attempts of possible parallels were not present neither in the textbook nor in the teachers' explanations.

7.2.2 Conversations with Olga in Lower Secondary

I had the opportunity of talking to Olga while she was trying to solve a textbook task where students were asked to draw the graph of the functions: a) y = x + 2, b) y = x − 3, c) y = 4x + 1 and d) y = 3x − 3 (Martinsen, Oldervoll, Pedersen, & Enger, 1999, p. 137) into the same coordinate system.

| 7.20a | Olga | No, actually, in the first one, then [thinking]. It only says sort of that y=x+2. And then I don't quite understand how to do it because it was, sort of, only x and the number two. So I took a quick look at the answer how it should end, or what it should look like in the end. | Nei, altså, på den første så [tenker]. Det står liksom bare y=x+2. Og da forstår jeg ikke helt hvordan jeg skulle gjøre det for det var liksom bare x og et to-tall. Så da så jeg litt i fasiten hvordan det skulle ende, eller hvordan det skulle se ut til slutt. |

7.20b	Interviewer	I see, mhm.	Sånn ja, mhm.
7.20c	Olga	Then I thought that the a [the function in part a) of the task] cuts through both in two at y and in two at x – actually minus two at the x. So then I, sort of, just had to see – then it was, sort of, maybe, just two plus two – then the slope became one. Then it sort of match quite well with how it should be.	Da tenkte jeg at a'en [funksjonen i deloppgave a)] skjærer da over både to på y og to på x – altså minus to på x'en. Så da måtte jeg liksom bare se – da var det kanskje to pluss to liksom – da ble stigningen en. Da passet det liksom ganske bra med sånn som det skulle være.
7.20d	Interviewer	I see, mhm. Why why did you say that the slope was one in that case?	Sånn ja, mhm. Hvorfor var det du sa at stigningen var én der?
7.20e	Olga	Because we move one out and a bit upwards, sort of, so it cuts through sort of, then…	For da går vi én ut også går vi litt opp liksom, så skjærer den liksom da så…

Excerpt 7.20

Based on 7.20a, Olga did have some difficulties with visualising the graphical representation of the function expression $y = x + 2$, so her strategy was to look at the answers at the end of the book and then try to generate meaning from the relation between the expression and the depicted graph in the solutions. She noticed that the graph intersects 2 on the y axis and -2 on the x axis and then, by applying the *one-unit-right-a-up/down* strategy 7.20c implies that she was able to see that the gradient should be one. This probably means that Olga mastered the transition from the graphical representation to the function expression, but had difficulties with going in the opposite direction.

7.2.3 Teaching in Upper Secondary – School 2a

I observed one lesson in Olga's upper secondary school related to the topic of proportional magnitudes. As in the case of Otto, the teacher Ronny, exemplified proportional magnitudes by relating the total price and the number of items bought.

7.21a	Ronny (Teacher)	[Based on the textbook]. What does a krone ice cream cost?	[Tar utgangspunkt i boka]. Hva koster en krone-is?
7.21b	Student	17 kroner.	17 kroner.
7.21c	Ronny	17 kroner, let's take that as a starting point.	17 kroner, vi tar utgangspunkt i det.
7.21d	Student	Isn't it 13?	Er det ikke 13 da?
7.21e	Ronny	No, we take 17 as a basis. Can someone here calculate	Nei, vi tar utgangspunkt i 17. Er det noen som tar i hodet

		in their head how much it is for two ice creams?	hvor mye det blir for to is?
7.21f	Student	34.	34.
7.21g	Ronny	And for three ice creams?	Og for tre is?
7.21h	Student	51.	51.

Excerpt 7.21

Agreement on costs based on the price of one ice cream (Krone-is) was followed by a series of examples of multiplicative structures, which also constituted the basis for this example.

7.22a	Ronny	But I'm going to relate this to something called functions.	Men jeg skal koble dette til noe som heter funksjoner.
7.22b	Ronny	[…]x – that could be the number of – what did we start with? – ice creams. One ice cream, two ice creams, three ice creams, four ice creams, five ice creams [marks the x-axis]. I could have replaced ice creams with almost anything. And I could have replaced the one and the two and the three and the four with almost anything.	[…]x – det kan være antall – hva var det vi begynte med? – is. En is, to is, tre is, fire is, fem is [merker av på x – aksen]. Jeg kunne ha erstattet is med nesten hva som helst. Og jeg kunne ha erstattet den eneren og toeren og treeren og fireren med nesten hva som helst.
7.22c	Ronny	y – that is the price. And here somebody said – wow – one ice cream went for 17 korner. Two ice creams, was it 34 we said?	y – det er prisen. Og her var det allerede noen som sa at – oi – en is gikk for 17 kroner. To iser – var det 34 vi sa?

Excerpt 7.22

In this teaching sequence there was a certain shift. Until this point, proportional magnitudes had only been discussed using examples (reference context) but here Ronny used the example to introduce variables (sign) to represent the amount and prize. Ronny's attempt to approach a generalization is visible in 7.22b. He states that the number of ice creams could be replaced by *almost* anything, so in this case possible restrictions on the variable x were not explicitly discussed. By the statement: "y – it is the price" (7.22c), the dependent variable was restricted to be the price. This excerpt is also interesting because Ronny explicitly mentioned the link between proportional magnitudes and functions (7.22a). It should be repeated that no such link was explicitly made in any of the textbooks used in any of the schools.

After some elaborations on the structure of the coordinate system, Ronny returned to the ice cream example, and made a table with three rows, respectively "x", "y" and "y/x" which he filled out for different x

values. By establishing the fact that y/x equals 17 in each column, the situation ended up at this stage with the situation illustrated in the right part of Figure 7.4.

7.23a	Teacher	Yes, it becomes 17, do you understand?	Ja, det blir 17, skjønner dere.
7.23b	Teacher	And that 17, we put in front of the x [writes 17 in front of the x, resulting in y=17x]. So here we have– if we want to use a multiplication sign as well it is ok, but we don't need it. y = 17x, that's the function which applies for buying krone ice creams.	Og den 17'en, den setter vi foran x'en [skriver 17 foran x, slik at det står y = 17x]. Så her har vi – hvis vi skal spandere et gangetegn også så er det ålreit det, men vi trenger ikke noe. y = 17x, det er den funksjonen som gjelder for kjøp av krone-is.

Excerpt 7.23

To link the situation with a function expression, the proportionality constant was used as a main source for further discussions. The transition from y/x = 17 to y = 17x was not emphasized. Instead this is reduced to a procedural manoeuvre, with a certain replacement resulting in the number 17 "in front of x" and the possible "insertion of a multiplication sign" (7.23b).

Subsequent to this, the concept of proportionality constant was introduced again by using the number 17 as an example, but also in this case no explicit link was made between this number and the gradient.

7.2.4 Tasks in Upper Secondary

There was a mix of real-life and mathematically situated tasks, but most of the tasks could be solved by using prescribed procedures. In the case of proportional magnitudes, these mainly consisted of different tables with two rows of magnitudes which were to be checked for proportionality.

The types of tasks related to proportional magnitudes offered in the textbooks in this case were very similar to those for Otto, and to avoid repeated arguments, I will not go into more detail about these tasks in this section.

7.2.5 Interview with Olga in upper secondary

One of the tasks I gave Olga during the interview from her textbook was to find out if certain magnitudes were proportional. The magnitudes were listed in two rows. Olga worked on these numbers by calculating the ratio of the numbers listed in the two rows, for each pair of corresponding numbers. She found that all the ratios were the same (1.5) except for the last pair of numbers (which was 1.6). Henceforth she concluded that in this case, the magnitudes were not proportional.

7.24a	Interviewer	Why isn't this proportional?	Hvorfor er ikke dette proporsjonalt?
7.24b	Olga	Because everything must be the same number when you divide.	Fordi alle sammen må være det samme tallet når du deler.
7.24c	Interviewer	Mhm. Why is that so, do you think, that it has to be the same number when you divide, to be porportional magnitudes?	Mhm. Hvorfor er det slik at det må være det samme tallet når du deler for at det skal være proporsjonalt, tror du?
7.24d	Olga	I'm not sure about that, I don't remember.	Det er jeg ikke helt sikker på. Jeg husker ikke det der.
7.24e	Interviewer	Do you remember something that characterises a proportional line, in relation to other lines which are not proportional, if you drew it?	Husker du noe om hva som kjennetegner en proporsjonal linje i forhold til andre linjer som ikke er proporsjonale, hvis du skulle tegnet det?
7.24f	Olga	No.	Nei.
7.24g	Interviewer	If I'm allowed to draw two lines here now, one going like this and one going like this [draws one line going through the origin and one that does not]. Which one of these lines do you think is proportional?	Hvis jeg får tegne to linjer her nå, en som går sånn og en som går sånn [tegner en som går gjennom origo og en som ikke gjør det]. Hvorfor en av disse linjene tror du er proporsjonal?
7.24h	Olga	This one, it seems like [points to the one going through the origin].	Den, virker det som [peker på den som går gjennom origo].

Excerpt 7.24

Olga's answers in 7.24b and 7.24h indicate that she was able to use the two proportionality tests which have been mediated, but when confronted with questions concerning why these methods work, she did not have any suggestions. This can be understood from many angles. The teaching she had received and the textbook she used seemed to focus on the procedures of proportionality tests and the more conceptual aspects of proportionality were more or less absent , except for a couple of tasks at the end of the textbooks, marked as "extra difficult".

7.25a	Interviewer	How would you explain what a function is?	Hvordan vil du forklare hva er funksjon er for noe?
7.25b	Olga	Mm. That was a bit difficult. Maybe the way the numbers are, in relation to each other.	Mm. Den ble litt vanskelig. Kanskje måten tallene er i forhold til hverandre.

Excerpt 7.25

In the case of Olga, Excerpt 7.22 shows that the function concept was explicitly mentioned by Ronny, by suggesting a certain link to proportional magnitudes. On the other hand, the nature of this connection is not quite clear. The function concept was not explicitly defined, but it was linked to the coordinate system in terms of a linear function going through the origin and to the tables which mainly consisted of illustrating the proportionality constant by calculating the ratio between corresponding numbers. "The way numbers are related to each other" (7.25b) may suggest a link to both these situations, and suggests a weak example of *functions as co-variance*. Olga's statement did not pinpoint co-variation in terms of explicitly emphasizing that "one variable depends on the other" but insinuated a more general relation which in strict terms could include all numbers and possible relations between them. So Olga's explanation is far from precise and certainly would match relations which are not functions.

7.26a	Interviewer	Can you say something about what you see here? [shows the expression y = 2x – 3]	Kan du si litt om det du ser her? [viser uttrykket y = 2x -3]
7.26b	Olga	Either it's a graph or it's an equation or it's both.	Enten så er det graf eller så er det en likning eller så er det begge deler.
7.26c	Interviewer	Yes, if we take the first suggestion, why is it a graph?	Ja, hvis vi tar den første, hvorfor er det en graf?
7.26d	Olga	Because we have an y.	Fordi vi har en y.
7.26e	Interviewer	Mhm.	Mhm.
7.26f	Olga	And it's one of the lines that we draw on the graph.	Og det er en av linjene som vi tegner på grafen.
7.26g	Interviewer	Why is it a line?	Og hvorfor er det en linje?
7.26h	Olga	Because it has x and y in it.	Fordi den har x og y i seg.
7.26i	Interviewer	Mhm. Is it also a function, do you think?	Mhm. Er det en funksjon også tror du?
7.26j	Olga	Yes, I think so.	Ja. Det tror jeg.
7.26k	Interviewer	Why?	Hvorfor det?
7.26l	Olga	I don't know, I just feel that it's right that it could be a graph.	Vet ikke, jeg bare føler at det stemmer med at det kan være en graf.

Excerpt 7.26

Above Olga was shown the function expression $y = 2x + 3$. From 7.26b and 7.26l Olga seemed to associate the expression to a graph and an equation, but did not mention anything about functions. As a response to the question if this is a function, her statement "I just feel that it's right

that it could be a graph" (7.26l) emphasizes this equivalence even further and makes this an example of *functions by representations*. Her justifications in 7.26d point back to the function expression. This could mean that she recognized the expression by the way it was written (in terms of "y = …") and that her later experiences told her that this was, therefore, an example of a function. But it could also indicate a more general conception that whenever there is a y on the left side of the expression, one has to do with a function, regardless of the rest of the expression. Further, her answer in 7.26h indicates that an x also ought to be present in the expression.

7.2.6 Olga's experience of the transition

The following excerpt displays Olga's experience of the difference between mathematics teaching in lower and upper secondary.

7.27a	Interviewer	Mhm. Are there any differences in the way that mathematics is presented?	Mhm. Er det noen forskjeller i måten matematikken blir formidlet på?
7.27b	Olga	Yes, the teachers are totally different.	Ja, lærerne er helt forskjellige.
7.27c	Interviewer	I see. So in what way do they differ?	Akkurat. Så hvilken måte er de forskjellige på?
7.27d	Olga	The teacher we have now is a bit more committed. Also he demonstrates a bit more on the blackboard and gives us some more tasks. Then we get to work a bit more on it instead of just calculating with numbers and such. We, sort of, get to try the lines and build our own tasks and things like that.	Han som vi har nå er litt mer engasjert. Så viser han litt mer på tavla ved tegning og gir oss litt mer oppgaver. Så vi får gjøre litt mer med det istedenfor bare å regne med tall og sånn. Vi får liksom prøvd linjene og bygge egne oppgaver og sånn alt sånt.

Excerpt 7.27

In 7.27b and 7.27d Olga pointed to differences between her teachers, but in addition she also described more "active learning" as a main difference. By stating her impression that they get to "do more" (7.27d), she indicated that the classroom practices at lower secondary consisted of solving tasks more passively in terms of "just calculating with numbers" (7.27d).

7.28a	Interviewer	Can you say something more about the tasks that you get now?	Kan du si litt mer om akkurat de oppgavene dere får nå?
7.28b	Olga	We have – I will show you [shows me some perspective drawings,	Vi har - jeg skal vise deg [viser meg noen perspektivtegninger med forsvinningspunkt]. Dette

| | | with vanishing points]. This is about measures and such. We've already had some of these tasks and it has been much easier to follow, you become a bit more interested when you get things like that, compared to the teacher just speaking at the blackboard. And afterwards you get loads of tasks which are, sort of, just numbers and such. Much more cunning. | handler om litt sånn mål og sånn. Vi har fått en del sånne oppgaver og det har vært mye enklere å være med, man blir litt mer engasjert selv, når du får sånt i forhold til at læreren bare står på tavla og prater. Og etterpå så får du en drøss med oppgaver som er liksom bare tall alt mulig sånt. Mye snedigere. |

Excerpt 7.28

In this excerpt Olga exemplified the activities at upper secondary with an exercise related to drawing and the existence of a vanishing point. She said that mathematics teaching in upper secondary, to a greater degree than in lower secondary, supported her creative interests, such as drawing. Her impression of the mathematics classroom practices in lower secondary was again amplified by her reference to earlier experiences from lower secondary in terms of "demonstrations at the blackboard" and a "large number of tasks".

7.3 The case of Matt – School C

Matt was one out of three participating students at lower secondary school, School C.

7.3.1 Teaching in Lower Secondary - School C

During the two observations in lower secondary, School C, I observed the introduction to the topic of functions and linear functions. Some basic aspects, such as the coordinate system, how to draw a graph, the gradient and constant term of linear functions were emphasised. The introductory lesson was dominated by properties of the coordinate system and how to plot points. The equation, "$y = x + 1$", was written on the blackboard as a starting point for further elaborations. With respect to my first research question, no explicit discussion about the concept of function itself was apparent during observation time. The two teachers involved in the class (Tim and Tom) only used the word functions twice, to denote different expressions, like the one above. On the other hand, the expressions were related to equations with two unknowns:

| 7.29a | Tim (Teacher) | And we have two unknowns, we have one y and we have one x in our | Og vi har to ukjente, vi har en y og vi har en x i denne likningen vår her. Et |

		equation here. An expression with letters. First I think Tom should tell a mathematical story to explain how this can be applied in a practical sense and you've been doing storytelling in mathematics since second and third grade.	bokstavuttrykk. Først så tror jeg Tom skal ta en liten regnefortelling å forklare hvordan dette kan brukes rent praktisk, og regnefortellinger har dere gjort helt nede i andre og tredje klassen.

Excerpt 7.29

It is worth noticing how the symbols x and y are referred to as "unknowns" and not "variables".

The gradient was dealt with in terms of *one-unit-right-and-a-up/down*, a strategy common in the excerpts from School B (previous section). In this class, the following example illustrates the situation:

7.30a	Tim	Then we had another notion, and that was gradients. There were some of you who said what the gradient is, does anyone see? How much does it increase when we go one out and one upwards? [Marks this on the graph by a small triangle, bounded by the horizontal line from about (0,1) to (1,1) and the vertical line from (1,1) to (1,2) and the graph]. It's not necessarily one out, it was a bit stupid to put it that way, but anyway. [Asks a student who has raised his hand].	Så hadde vi et annet begrep, og det var stigningstall. Det var noen som sa hva stigningstallet er, er det noen som ser det? Hvor mye stiger det når vi går en ut og en opp? [Markerer dette på grafen ved en liten trekant avgrenset av det horisontale linjestykket fra omtrent (0,1) til (1,1) og det vertikale linjestykket fra (1,1) til (1,2) og grafen]. Det er ikke nødvendigvis en ut, det var litt dumt å si det sånn. Men, men. [Spør en elev som har rukket opp hånden].
7.30b	Student	It then increases by one upwards.	Det stiger med en opp da.
7.30c	Tim	Yes, it does. So the gradient becomes one.	Ja, det gjør det. Så stigningstallet blir en.

Excerpt 7.30

The semiotic chains visible through this instruction mainly consist of linking the horizontal and vertical movements (demonstrated on the blackboard) to the gradient like in Figure 7.4. The second question posed by Tim in 7.30a, followed by the illustration of a corresponding right-angled triangle serves to establish the link.

7.3.2 Tasks in lower secondary

At School C, students were provided with a variety of tasks. During the first of my two periods of observation, the students were given the

opportunity to choose between working with tasks on the computer or working with tasks in a pamphlet (copied from other textbooks put together by the teachers). In the next lesson, the students alternated. There were eight tasks in the pamphlet related to the topic of functions, the first five of which had no "real-life" context but aimed to make the students familiar with different representations such as making a value table, plotting points and drawing straight lines in the coordinate system. The sixth and the seventh tasks at the end were about proportional magnitudes in different real-life contexts, specifically the relations between price per item and total price and hourly wage and total wage. The last task (coded "yellow sign", which meant that it was considered more difficult) dealt with interpreting a (non-linear) graph which showed the water level in a tank at various times of the day. The water level decreased as water was used, and increased when the tank was filled up.

The first five tasks build to a great extent on memorization and procedures, which means that it is possible to solve them simply by applying the method shown on the blackboard earlier. The last three were more challenging in the sense that the students had to convert an everyday problem into the mathematical language of functions. Even though cooperation is not explicitly required in any of the eight tasks, the last one (task eight) constitutes a new problem (a non-linear graph) which may have invited some peer discussion and cooperation.

Figure 7.5 Example of an interactive task at School C (Reproduced with permission, from Cappelen Damm, 2008a, translations added by the author).

The students working on the computer were given two different websites to choose between. One was the website of "Sinus" (Cappelen Damm,

2008a) which contained different interactive tasks related to linear functions. These tasks fell into two main categories: finding the constant term and the gradient of a given a linear function depicted on the screen and drawing the linear function based on its given expression. The task shown above (Figure 7.5) was the same as the one Matt worked with in the next section, and represents the template for a series of similar tasks. A graph was depicted and the students were asked to fill in the gradient and the constant term (in the boxes to the right). Then they had to check their answer by clicking "check answer" on the bottom of the page, and get response immediately. In neither of these cases did a real-life context frame the tasks, and even though interactive participation constitutes an alternative way of working, it is hard to see that these tasks demand other qualities than memorization and application of procedures given on the blackboard prior to this individual work.

The other website was a digital resource site for mathematics teachers in both lower and upper secondary school (Cappelen Damm, 2007). This site mainly provides teacher guidance for textbooks and interactive tasks for students. On this occasion the function machine was introduced, and based on observations of various input and output values, the students were asked to find the explicit expression for the corresponding function.
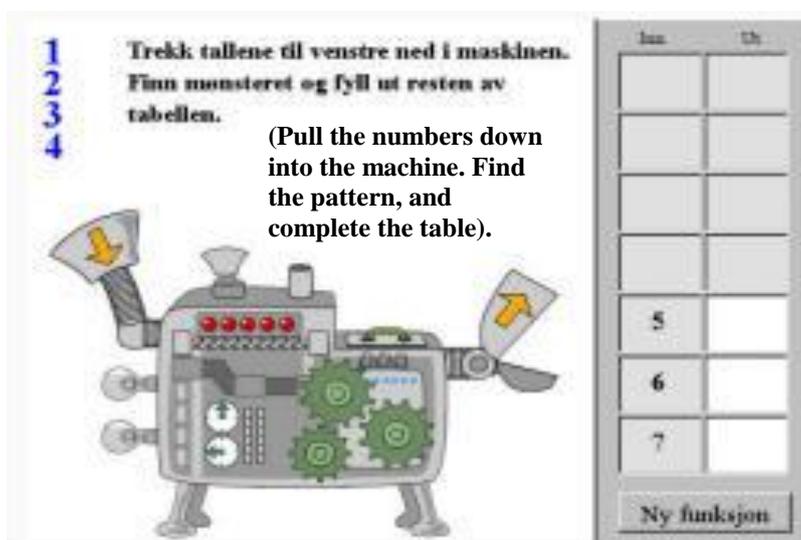
Figure 7.6 The interactive function machine at School C (Reproduced with permission, from Cappelen Damm, 2007, translations added by the author)

Figure 7.6 illustrates the principle of the function machine – certain input and output values were provided and the students were asked to "find the pattern". Unlike the previous tasks, this task is hard to solve by memorization or simply by applying certain algorithms, as the numbers and corresponding function expressions varies. This falls into what Stein

et al. (1996) call "doing mathematics", involving mathematical thinking and reasoning.

The variety of tasks represented during these observations is also something remarked on by the teacher Tim in his subsequent reflections.

| 7.31 | Tim | I think that I've had good response on using the PC, so a lot thought that they got to see, sort of, moving the values and that it was interactive in such a way that you see that the equation changes when you move the line. It seemed that this was a bit of an eye-opener for some. | Jeg tror kanskje at jeg har fått bra tilbakemeldinger på det med å bruke PC da, så mange synes at de fikk sett liksom dette med å flytte verdiene og at det var så interaktivt at du ser at likningen forandrer seg når du flytter linja. Det virket det som om var litt a-ha opplevelse for noen. |

Excerpt 7.31

7.3.3 Conversations with Matt in Lower Secondary

Given a choice at the start of the lesson, Matt chose to work on the computer with the tasks from the "Sinus" websites. He worked on the expression $y = 2x + 1$ (Figure 7.5), which was represented as a graph on the screen, but the expression itself was not shown. His task was to find the constant term and the gradient. As regards research question one, this excerpt shows his reasoning for finding the gradient.

7.32a	Matt	And then the gradient, then it's logical that it's three.	Og så stigningstallet, da er det logisk at det er tre da.
7.32b	Interviewer	Ok, in that case, how are you thinking?	Ok, hvordan tenker du da?
7.32c	Matt	Because it increases to three, and then one has moved one there [points one unit along the y-axis, and in the point (1,3) at the line].	For det stiger til tre, og så har man gått en der [peker en bortover langs y aksen, og i punktet (1,3) på linja].
7.32d	Interviewer	Ok…	Ok…
7.32e	Matt	Let's see [Matt clicks the "check answer" icon and gets wrong answer]. No…	Skal vi se da [Matt trykker på "sjekk svar" ikonet og får feilsvar]. Nei…

Excerpt 7.32

Matt's reasoning is clearly influenced by the *one-unit-right-a-up/down* strategy but his starting point was the origin even though the graph did not intersect there. So even if his last part, of moving upwards until he hit the graph is done in accordance with this method, the answer was wrong as he found out when he clicked the "check answer" button. Matt's reasoning is quite consistent in this subsequent example, where the graph shown on the screen (Figure 7.5) corresponded to the equation $y = 3x - 1$. Parallel to the previous example, Matt quickly started to go

one step to the right, with the origin as his starting point and then concluded that the gradient should be two, as the vertical line meets the graph at (1,2).

When Matt tried to find the slope in 7.32c, the sign to interpret is the function visually depicted on the computer screen. The gesture of his hand movement from the x-axis, parallel to the y-axis until he hit the graph itself, constitutes the explicit link between the concrete visual figure (sign) and his understanding of the notion gradient (meaning). The gradient of this function is two, so in this case Matt's strategy does not match the mathematical definition.

7.3.4 Interview with Matt in Lower Secondary

During the interview, Matt said that he could not explain what was meant by a function in mathematics. But in previous observations he had dealt with the principles of the one-to-one property and the independent and dependent variables in an indirect manner while working with different tasks. So his short account first came to the surface when the question was rephrased as "what comes to your mind when you hear the word functions?"

7.33	Matt	I think of a graph like this. I also think of numbers like these, two point four, for example. I have no idea why, but that's what I think of now.	Jeg tenker på en sånn graf. Også tenker jeg på sånne tall, to komma fire for eksempel. Aner ikke hvorfor men jeg tenker nå på det.

Excerpt 7.33

He associates functions with a graph and points in the coordinate system, which makes this an example of *functions through representations*. He made no attempt at any formal definition and neither was the concept of variables prominent in his explanations.

7.34a	Interviewer	What was it that you recognized?	Hva var det du kjente igjen?
7.34b	Matt	I recognized y equals. Also two x minus three. So minus three probably represents the constant term.	Jeg kjente igjen y er lik. Også to x minus tre. Så minus tre står vel for konstantleddet.

Excerpt 7.34

7.35a	Interviewer	What [is it that] represents the gradient?	Hva [er det] som står for stigningstallet?
7.35b	Matt	Two x.	To x.
7.35c	Interviewer	What can you say about this, here?	Hva kan du si om det, her?
7.35d	Matt	It means that it goes two upwards before it intersects with the next…yes.	Det vil si at den går opp to før den skjærer neste…ja.

Excerpt 7.35

When shown the expression y = 2x – 3, he rather quickly associated the expression with a function explaining that he recognized it from the tasks they had been working on. The reason for this recognition, he said, was the part in the expression, "y = 2x - 3".Matt (in Excerpt 7.35) was probably recalling the teacher's explanations on the blackboard and as in Excerpt 7.34, his strategy for finding the gradient of a linear function was related to the *one-unit-right-a-up/down* strategy.

Matt's answers in 7.35b and 7.35d show that he made a link between the sign "2x" and the gradient of a function. The inclusion of the variable "x" in his answer in 7.35b indicates that the link between the actual sign and meaning was not fully established. His explanation in 7.35d might indicate that his concept of gradient was limited to the procedure involving these horizontal and vertical movements. It is worth noticing that in 7.35d he was simply describing the vertical part of the movement. This could be a result of his previous problems (exemplified in Excerpt 7.32) with finding the right place to start when he had to move in the horizontal direction. Unfortunately it is not clear from the interview what Matt meant by "before it intersects the next" but it is possible that he is referring to the vertical line of a triangle similar to the one drawn by the teacher Tim in 7.30a.

7.3.5 Teaching in Upper Secondary school – School 3c

In upper secondary Matt took general studies and the 1T version of mathematics. At this school they divided the mathematics students into three groups, based on the students' marks from lower secondary school. This system of three groups applied to both the 1T and the 1P versions. Matt belonged to the group of the students with lowest marks from lower secondary. The mathematics classes are henceforth just denoted as "groups" at this school and Matt's group is called the "level 3" group.

I observed two lessons with Matt's group in upper secondary. Unfortunately, the three students I decided to follow in this upper secondary school all attended different 1T groups. The two lessons I observed in this class therefore only covered the introduction to the topic of functions. One of the first points that the teacher, Henry, made was about the use f(x) instead of y.

| 7.36a | Henry (Teacher) | That way of writing [alluding to y = …] is in a sense typical for equations. When we move over to functions, we replace that one [points to y, in the expression y = 2x + 2] and then we write [writes f(x) = 2x + 1 on the blackboard]. The reason for this is to | Den skrivemåten der [henspeiler på y =…], den er på en måte typisk for likninger. Når man skal over på funksjoner, så bytter man ut den der [peker på y, i uttrykket y = 2x + 1]] og så skriver man [skriver f(x) = 2x + 1 på tavla]. Grunnen til at man gjør det, det er for å få inn dette begrepet |

		make use of the concept of function, because the concept of function is that you in a way have a variable x [points at the blackboard]. So you put the x into an expression, and then you get a result [points to f(x)]. That means, in a sense, that the x which is the variable is treated inside the function, and something comes out.	funksjon, for begrepet funksjon er jo at man på en måte har en variabel x [peker på tavla]. Så putter man x inn i et uttrykk, så får man ut et resultat [peker på f(x)]. Det betyr på en måte at den x'en som er variabelen den blir behandlet inni funksjonen, så kommer det ut et eller annet.
7.36b	Henry	One can in a way draw this. [Henry draws a figure on the blackboard, consisting of two parallel horizontal lines, extended at each end]. An x comes in there and something comes out [draws arrows indicating that the x-value goes through this horizontal funnel]. And then it is 2x + 1 which treats the x [writes 2x + 1 above the tilted funnel]. In, and then it eventually comes out.	Man kan på en måte tegne det. [Henry tegner en figur på tavla, bestående av to parallelle horisontale streker, som er utvidet i hver av endene]. Det kommer en x inn der og så kommer det et eller annet ut [tegner piler som indikerer at x verdien går gjennom denne horisontale trakta]. Og så er det da 2x + 1 som behandler den x'en [skriver 2x + 1 på oversiden av den liggende trakta]. Inn, så kommer den ut etter hvert.

Excerpt 7.36

In 7.36a, Henry introduced the notation f(x) instead of y which was a new notation for most of the students, as lower secondary schools most often seem to use y. The last sentence of 7.36a shows that he also related the independent and dependent variables to this notation, although he did not mention these concepts explicitly. This introduction differs from the one given in Matt's lower secondary school, in that functions as a concept is explicitly and illustrated both by the descriptions of variables in 7.36a and by the function machine described in 7.36b and adapted in Figure 7.7, below.
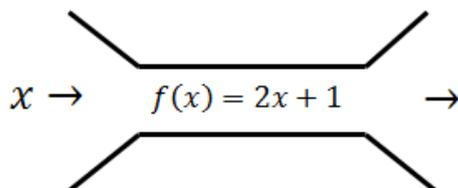

$$x \rightarrow \quad f(x) = 2x + 1 \quad \rightarrow$$

Figure 7.7 Henry's function machine

7.36b places Henry's approach to the function concept in the category *function machine.* Even though function machines are often used to illustrate the property of uniqueness, this is not explicitly mentioned in Henry's elaborations. "One x comes in and something comes out" is somewhat close, but it is not evident that "something" has to be the same every time a given x is put in.

To teach gradients, Henry started by pointing out some main principles concerning the positive and negative gradient.

7.37	Henry	It is about how steep the graph is. If it is – thus, if we have a graph going straight ahead like this [moves his hand horizontally along the coordinate system on the blackboard] then you have no increase at all. The a equals zero. If it goes upwards like this [illustrates with a hand movement] the a is positive. But if it goes downwards [illustrates with a hand movement] then it is negative.	Det er hvor bratt grafen er. Om den er – altså hvis vi har en graf som går rett bortover sånn [viser en horisontal håndbevegelse langs koordinatsystemet på tavla] da har du ikke noen stigning i det hele tatt. Da er a'en lik null. Går den oppover sånn [viser ved håndbevegelse] så er a'en positiv. Mens går den nedover [viser med håndbevegelse] så er den negativ.

Excerpt 7.37

This explanation illustrates a relation between different signs. The sign "a" (as in $f(x) = ax + b$) is linked to gestures in terms of hand movements, and these gestures also count as a kind of sign. The process of further describing the strategy of finding the gradient of a linear function is first done in accordance with the *one-unit-right-a-up/down* strategy.

7.38a	Henry	What does this 2 mean? [Points to the 2 in y=2x+2] How much does it increase?	Hva betyr dette totallet her da?[Peker på 2 tallet i y=2x+2] For hvor mye stiger den?
7.38b	Student	It increases by two for each step forward.	Den stiger med to for hver bortover.
7.38c	Henry	That's right. For each step forward along the x-axis, it increases by two. So, actually, one can draw this without making a table. You can go one forward in that direction [starts in (0,2)] and then one goes…	Det stemmer. For hver gang man går et skritt bortover på x aksen, så stiger den to. Så egentlig kan man må tegne det der uten å lage tabell. Dere kan gå en bortover der [starter ved (0,2)] også går man…
7.38d	Students	Two upwards.	To oppover.
7.38e	Henry	Two upwards, then one ends up there [illustrates on	To oppover. Da kommer man dit [viser på tavla]. Og så

		the blackboard]. And then we draw the line [draws the line with a ruler]. If I get this right is not easy to tell, because I have not got squared paper.	trekker vi linja [trekker linja ved linjal]. Om jeg får det rett er ikke så godt å si for jeg har ikke ruteark.
7.38f	Henry	What about the other one, what does one do then? [Refers to y=-2x+2].	Med den andre da, hva gjør man da? [Sikter til y=-2x+2]
7.38g	Students	Then you get minus. It goes one, and then the other direction.	Da får du minus. Den går en, også andre veien.
7.38h	Henry	One, in that direction [draws a line towards the left from (0,2)].	En, den veien [tegner en linje mot venstre fra (0,2)].
7.38i	Students	And then two upwards.	Også to opp.
7.38j	Henry	You always move towards the right [corrects the line and draws it in the opposite direction from the same point (0,2)]. One to the right and then, when it is negative then you go?	Du går bestandig mot høyre [retter opp linja, og tegner den på ny i motsatt retning, fra samme punktet (0,2)]. En mot høyre også når det er negativt så går du da?
7.38k	Student	Two downwards.	To nedover.
7.38l	Henry	Two downwards. And then you arrive here. [Shows in the coordinate system and draws the line].	To nedover. Og da kommer du dit. [Viser i koordinatsystemet og trekker linja].

Excerpt 7.38

7.38a – 7.38e show the introduction of gradients following the *one-unit-right-a-up/down* strategy, similar to the introduction in Matt's lower secondary school, School C. In 7.38g-7.38l the negative gradient is introduced and since the strategy of the first example is no longer valid, there was some confusion about how to adjust the previous strategy. The students in 7.38g suggested moving one to the left instead of one to the right as they probably interpreted the minus sign as moving in the opposite horizontal direction. This in fact would be equally good, as one to the left and two up yields the same gradient as one to the right and two down. Without any further discussion, Henry simply stated "you always have to move to the right" 7.38j). As the lesson proceeded, a "new" definition of gradients was introduced:

7.39a	Henry	And the gradient as – and we'll now gradually introduce some new notations [writes $a = \Delta y/\Delta x$]. You might as well see them now, because	Og stigningstallet som – og etter hvert nå så skal vi innføre noen nye betegnelser [skriver $a = \Delta y/\Delta x$]. Dere kan like godt se dem nå, for vi er nødt til å komme borti det uansett.

		we'll have to deal with them anyhow.	
7.39b	Henry	So, delta x and delta y these are two terms which mean that one has a tiny – one can add a tiny bit to the y-axis [points at Δy] and then one adds a tiny bit to the x-axis [points to Δx]. One does not say how much is added, but one adds a tiny bit.	Altså delta x og delta y det er to begrep som betyr at man har en liten – man kan legge til litte grann på y aksen [peker på Δy] også legger man til litte grann på x aksen [peker på Δx]. Man sier ikke hvor mye man legger til men man legger til en liten bit.
7.39c	Henry	So that if one, in a way, stays in a specific position, let us say there, then one add a tiny bit which could be there or there [illustrates along the x-axis]. Thus, the added part, delta x. Then, if one stands there and adds something upwards, then it is delta y [illustrates along the y-axis]	Så hvis man på en måte står en eller annen plass, la oss si at man står der da, så legger man til en liten bit, og det kan være dit og det kan være dit [viser langs x aksen]. Altså tillegget delta x. Også hvis man står der og legger til et tillegg oppover dit så blir det delta y [viser langs y aksen].

Excerpt 7.39

Probably in preparation for differentiation, Δx and Δy were introduced, and $\Delta y/\Delta x$ constituted this "new" definition of gradients (7.39a). In 7.39c these signs are defined and explained through corresponding illustrations in the coordinate system. Although it might have been expected that Henry made an explicit link to Excerpt 7.38, he made no such link and it is unclear whether the students were able to link the *one-unit-right-a-up/down* to $\Delta y/\Delta x$. The semiotic chain linking the gradient to $\Delta y/\Delta x$ is most apparent in Henry's attempt to link Δy and Δx to distances in the coordinate system, in 7.39b.

7.3.6 Tasks in upper secondary

The tasks given to the students during my observations were all from the textbook "Giga" (Andersen, Jasper, Natvig, & Aadne, 2006) and consisted of eleven tasks. The content of the tasks is directly linked to Henry's instructions on the blackboard, and involves different representations of linear functions.

5 Make a table with x- and y-values and draw the straight lines in the same coordinate system.

a) $y = 2x + 3$ b) $y = x + 3$ c) $y = -2x + 3$

Which point on the y-axis do all the three straight lines pass through?

Figure 7.8 Example of a typical textbook-task at School 3c (Andersen et al., 2006, p. 202, my translation)

In the task above the students are to make a table, and draw the lines. After the lines have been drawn the students are asked to observe where the lines intersect the y-axis. It should be noticed that these tasks do not apply the f(x) notation. None of the eleven tasks are situated in real-life contexts and all of them are easily solvable through memorization and the use of procedures. From Henry's point of view this seemed to be an intended choice, as he stressed the importance of these kinds of tasks, especially for the students attending the level 3 group.

| 7.40 | Henry | The word drilling-tasks are something I think is very important. Because the students have a twisted idea that if they manage one task they know it. And we know, as teachers that this is not the case. Even though they master one task they have to drill about 10, 15, 20 times before it, sort of sticks. So when they get this in a test, they remember it. And the weakness of this [points to the textbook] is that it contains almost no drill tasks. | Ordet drilloppgaver er noe som jeg synes er veldig viktig. Fordi at elevene har en forskrudd oppfatning at om de har fått til en oppgave så kan de det. Og det vet jo vi som er lærere at det stemmer ikke. For om de har fått til en oppgave så må de altså drille en 10, 15, 20 ganger før det liksom sitter. Så når de får det på en prøve, så husker de det. Og svakheten med dette her [peker på læreboka] er at det finnes nesten ikke drilloppgaver. |

Excerpt 7.40

7.3.7 Interview with Matt in Upper Secondary

In the light of this approach to the concept of functions, apparently different from that in lower secondary in terms of the explanations in Excerpt 7.36, it is interesting to see how this might affect Matt's understanding of the function concept.

| 7.41 | Matt | It is…no if I were to have explained it like this, it is something that shows what a graph should look like. | Det er… nei hvis jeg skulle ha forklart det sånn så er det noe som viser hvordan en graf skal se ut. |

Excerpt 7.41

At first glance, this explanation is not very different from the one given in lower secondary. However, in terms of pointing to "something that shows…", Matt suggests a kind of "hidden structure" or a mathematical object that "lies behind" the visual representation. As indicated in Excerpt 7.33 and 7.34, there was no trace of this in the interview conducted in lower secondary. When challenged on this "something" he becomes rather vague and concludes that this "something" determines the graph without coming any closer to a possible definition. A possible guess could be that this "something" points to the algebraic expression, but there are few utterances to support this hypothesis.

7.42	Matt	It is, in a way, how the graph is created. But I don't know the real definition.	Det er jo liksom hvordan grafen blir da. Men jeg vet ikke noe ordentlig definisjon på det.

Excerpt 7.42

Since no further elaboration of this "something" is provided during the interview it is hard to establish what he really means. His emphasis on "graphs" seemed to constitute an example of *functions by representation,* and this "something" might allude to the function expression.

Confronted with the expression y = 2x − 3, he immediately started to describe the characteristics of its corresponding graph, but he was confused as he did not remember what was the constant term and what was the gradient. Nor did he solve the problem when he was given time to study the expression.

Even though I did not make any observation related to the teaching of derivatives in Matt's class, this is a topic the class had been working with before this interview was conducted. So I asked Matt to find the derivative of the expression $f(x) = x^2 - 3x - 2$, and he quickly came up with the answer $f'(x) = 2x - 3$, and wrote this on a piece of paper. He justified his calculation by referring to 'simple rules of differentiation'. When he was asked what differentiation is about, this is what he said.

7.43	Matt	That I don't know, I only know how to do it. I've never learned why.	Det vet jeg ikke, jeg vet bare hvordan jeg gjør det. Jeg har aldri lært meg hvorfor.

Excerpt 7.43

He also claimed that they have not worked a lot on these things, which corresponds with Henry's statement about teaching derivation in this level 3 group:

7.44	Henry	Differentiation per se, is perhaps the most difficult thing they have dealt with. But it went very well. 9 out of 10, or should I say 11 out of 12, grasped it very easily.	Derivasjon er i og for seg kanskje det vanskeligste de har holdt på med. Men det også gikk veldig bra. 9 av 10 eller skal vi si 11 av 12 tok det der veldig lett. Jeg er ikke sikker på at alle

| | | I'm not sure that everyone understands the theory behind differentiation, but everybody can apply it. They can find the derivatives because there are such clear, manageable rules. But I didn't include what the other groups got, namely the theory behind differentiation and how to use the definition to arrive at finding derivatives. I didn't do that. Because none of my students would have been able to do that. So I went directly to the rules of differentiation and used them directly and it went surprisingly well, actually. | sammen skjønner teorien bak derivasjon, men alle sammen får til å bruke det. De får til å derivere, for det er så klare og greie regler. Men jeg tok ikke med det som kanskje de andre gruppene tok med, nemlig teorien bak derivasjon og utredningen av derivasjon og bruke definisjonen til å komme frem til derivasjon. Det gjorde ikke jeg. For det ville ikke mine elever klart. Så jeg gikk direkte på derivasjonsreglene og brukte de direkte og det gikk forbausende bra altså. |

Excerpt 7.44

In this excerpt, Henry justifies his reasons for not introducing "the theory behind differentiation" to his group of (low performing) students. In his view none of his students "would be able to do that", and he evaluated the lesson as being successful when "11 out of 12" students were able to apply only the differentiation rules.

7.3.8 Matt's experience of the transition

As I pointed out earlier, there is a marked contrast between the relatively rich variety of tasks in Matt's lower secondary school compared to those in upper secondary. When Matt was asked how he experienced possible differences, he did not seem to share this impression.

| 7.45 | Matt | Yes, actually it was like that most of the time in lower secondary as well. Teaching from the blackboard. It is only once a year that we work on PCs, but it's the same in both. | Ja, det var egentlig mest av det på ungdomsskolen også. Tavleundervisning. Det er jo en gang i året da vi driver på med PC da, men det er jo like ens på begge. |

Excerpt 7.45

This could mean that working with computers was not very common, and in that case my observations at lower secondary were an exception. Neither was the content of the provided tasks something that he noticed as being different. When describing the difference between lower and upper secondary, Matt emphasized that the students' responsibility for their own learning had increased in terms of less (controlled) homework. As with Otto, he felt that there was more individual follow-up in lower secondary school.

7.4 The case of Thea – School D

Thea was the only participant from lower secondary School D.

7.4.1 Teaching at Lower Secondary – School D

I observed four lessons in Thea's class in lower secondary, School D. In the first of these four lessons the teacher, Roy, introduced the topic of functions by drawing a mind map on the blackboard with the word "functions" in the centre. He invited the students to contribute what they associated with the concept, and wrote these associations on the blackboard as they were suggested. Suggestions which were eventually written were: "how things work", "coordinate system", "equations", "diagram", "graph", "lines" and "curves". After this introduction Roy introduced the concept of function by using an example of cars driving into a tunnel and emerging from the tunnel with a new colour.

7.46a	Roy (Teacher)	Ok the first example is – what in a way is going to illustrate a function – it is with a tunnel. We have a green car which drives into the tunnel her [illustrates on the blackboard] and when it comes out of the tunnel we have a blue car [illustrates this by using a coloured chalk and arrows]. Ok?	Ok første eksempelet det er – som på en måte skal illustrere en funksjon – det er med en tunnel. Vi har en grønn bil som kjører inn i tunnelen her [illustrerer på tavla] og når den kommer ut av tunnelen så har vi en blå bil [illustrerer dette med fargekritt og piler]. Ok?
7.46b	Roy	Then we have the second car, it is a red car. It drives into the tunnel and when it comes out, it is…?	Så har vi bil to, det er en rød bil. Den kjører inn i tunnelen og når den kommer ut så er den…?
7.46c	Student	Green.	Grønn.
7.46d	Roy	Green. Or?	Grønn. Eller?
7.46e	Student	Blue.	Blå.
7.46f	Roy	Blue. [Illustrates by the coloured chalk]. But then, a blue car drives into the tunnel. Which colour is it when it comes out?	Blå. [Illustrerer med fargekritt]. Men så kjører det en blå bil inn da. Hvilken farge har den når den kommer ut da?
7.46g	Student	[…] So everything that comes out becomes blue?	[…]Så alt som kommer ut der blir blått?
7.46h	Roy	Everthing that comes out is blue, yes. Good observation. But everything that goes in, that is not blue, actually. And then the question is, what is the function of the tunnel?	Alt som kommer ut der er blått, ja. Bra observert. Men alt som går inn, det er jo ikke blått da. Og da er spørsmålet, hva er da tunnelen sin funksjon?

7.46i	Student	To transform things to blue.	Gjør om ting til blått.
7.46j	Roy	Yes, it paints the cars blue, actually. There's a painting shop inside here [smiles]. The function of the tunnel: To paint the cars blue.	Ja, den maler altså bilene blå. Det er lakkeringsverksted inni her [smiler]. Funksjonen til tunnelen: Male bilene blå.

Excerpt 7.46

This "colour-changing tunnel" example, 7.46h shows that one of Roy's aims was to illustrate the property of uniqueness (the one/many-to-one principle). If one car drove into the tunnel, then the same car, but with another colour, came out in the other end. The new colour of the car is the same every time. The "colour-changing tunnel" served as a *function machine* in this example, and it acts on its domain (the cars) by painting them blue (7.46i and 7.46j). It is worth noticing that Roy did not start by introducing this "painting-cars-blue"-function explicitly, but by first describing what happens to one specific car and inviting the students to guess what would happen to the next one. The conclusion in 7.46j, "The function: to paint cars blue" is interesting because it does not represent an obvious link to the type of explicit function expressions that the students were about to work with in their textbooks. Subsequent to this example Roy provided another example where he drew some boxes on the blackboard and dropped different numbers into each of them, and the students were asked to look for patterns (Figure 7.9).
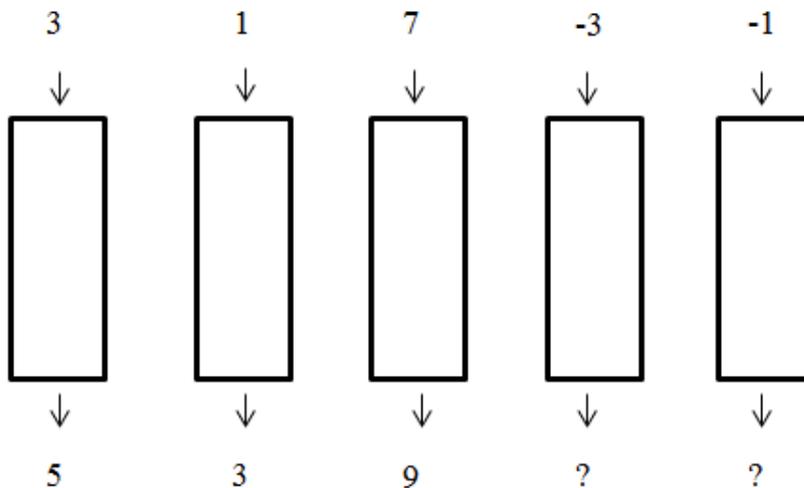
Figure 7.9 Roy's box example

He revealed that dropping the input values 3, 1 and 7 into the boxes resulted respectively in the output values 5, 3 and 9. He then asked the students what will happen to the input values -3 and -1, and they responded by suggesting the output values -1 and 1. Roy then established agreement that the value two was added to these numbers. By explicitly telling the class that the input values are denoted by x and the boxes by y, he then shifted his reference context from numbers and patterns to the

operation of adding 2. It is worth noticing that it was the boxes which Roy (the teacher) denoted by y and not the output values. Hence, in the notation $y = x + 2$, y would have a double role, as it denotes both the function and the dependent variable. In this activity though, y denoted only the function.

7.47a	Roy	Then, how to find a link between y and x here then and what we found happened inside the box?	Å finne en sammenheng mellom y og x her da og det vi fant ut som skjedde i boksen?
7.47b	Student[12]	Added 2.	Plusset på 2.
7.47c	Roy	Yes, added 2.	Plusset på 2 ja.
7.47d	Student	Yes, but then, in a way, the box becomes 2. Thus, yes, 2. It adds 2.	Ja men da blir boksen 2 da på en måte. Altså, ja, 2. Den plusser jo på 2.
7.47e	Roy	Yes, you are definitely on to something. Thea?	Ja, du er absolutt inne på noe. Thea?
7.47f	Thea	x plus y equals plus 2.	x pluss y er lik pluss 2.
7.47g	Roy	x + y equals 2 [writes at the blackboard].	x + y er lik 2 [skriver på tavla].
7.47h	Thea	Plus 2.	Pluss 2.
7.47i	Roy	Yes, 2. Yes. Very close. What was that, when we were working with functions? What was it that always came first? Did one write x equals or y equals?	Ja, 2. Ja. Veldig nært. Hva var det, da vi holdt på med funksjoner? Hva var det som alltid sto først? Sto det x er lik, eller y er lik?
7.47j	Student	Equal was first.	Er lik sto først.
7.47k	Roy	There must be something on each side of the equal sign, right?	Det må stå noe på hver side av likhetstegnet, må det ikke det?
7.47l	Student	2 equals x plus y.	2 er lik x pluss y.
7.47m	Roy	Actually that is the same – we only turn this around.	Det blir jo egentlig det samme – da snur vi jo bare på dette her.
7.47n	Student	x + y are called 2 when y is 2.	x + y kalles for 2 når y er 2.
7.47o	Thea	Then, y equals x plus two.	y er lik x pluss 2 da.
7.47p	Roy	[Crosses out x + y = 2 and writes y = x + 2]. y equals x plus 2. Very good. [Frames this in red]. What can we learn from this? We can make a function	[Krysser over x + y = 2 og skriver y = x + 2]. y er lik x pluss 2. Kjempebra. [Rammer dette inn i rødt]. Hva er det vi kan lære av dette da? Vi kan lage et funksjonsuttrykk [peker]

[12] Like in 7.1, "student" (and no names) in transcriptions, indicates that the utterance is made by students who are not among the eight students I am focusing on in my study. In this case *not* Thea.

| | | expression [points] and we are going to work a great deal with that. Much of this you know already [writes "function expression" beside the expression on the blackboard]. | og det skal vi jobbe en del med. Mye av dette kan dere [skriver "funksjonsuttrykk" ved siden av uttrykket på tavla]. |

Excerpt 7.47

The task here was to find the function expression which described the "addition by two" in the example given on the blackboard. In 7.47f, Thea suggested that this expression might be "$x + y = + 2$". Thea's comment in 7.47h might suggest that she thought of "+" (in +2) as an operator, and not a sign. The nature of the following discussion between the teacher and the students (7.47i-7.47p) changed somewhat, as the teacher no longer related the discussion directly to the example, but to prior knowledge of rules and common notions which students were expected to have. This finally led to the expression, "$y = x + 2$". 7.47i indicates that they have been working with this topic before (probably the previous year). Building on this, Roy used the next lesson to draw linear graphs in the coordinate system as examples of linear functions and ended up with the general expression $y = ax + b$.

By applying Steinbring's model and semiotic chaining, one can study how the teacher aimed towards development of the function concept, based on different reference contexts.
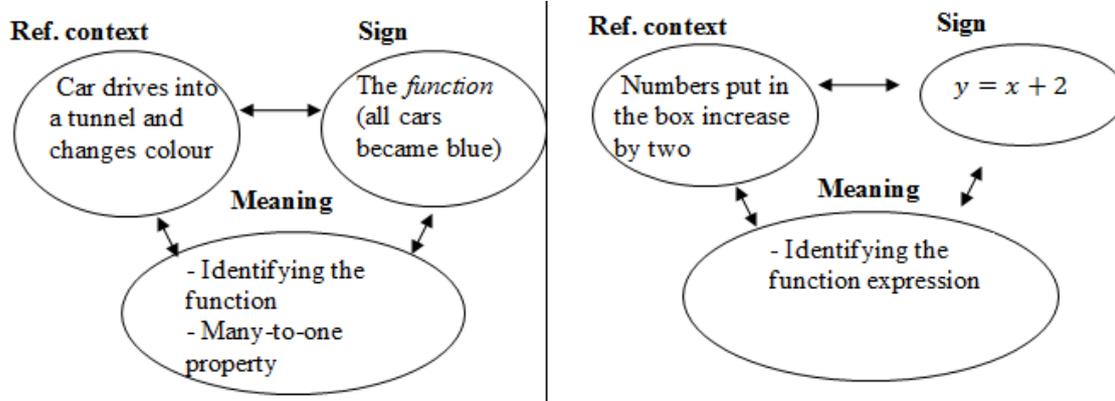
Figure 7.10 The epistemological triangle in the two introductory examples given in the first lesson

When Roy approached gradients he used the graphs of $y = 2x$ and $y = x + 1$ as starting points for further discussion.

| 7.48a | Roy | One. There's really 1 in front here [points to x in y = x + 1]. So, is there any difference between these graphs here, between that one and that one [y = x + 1 | En. Det står 1 foran her egentlig [peker på x i y = x + 1]. Er det noen forskjell på grafene her da, på den og den [y = x + 1 og y = 2x]? |

		and y = 2x]?	
7.48b	Student	One is more like slanting.	Den ene er mer sånn skrå.
7.48c	Roy	More like slanting?	Mer sånn skrå?
7.48d	Student	More like a hill.	Mer sånn bakke.
7.48e	Roy	More steep?	Mer bratt?
7.48f	Student	Yes.	Ja.

Excerpt 7.48

A connection between the steepness of the graph and the gradient is now established. Through suggestions from the students as they use words like "more slanting" (7.48b) and "more like a hill" (7.48d) the teacher's suggestion in 7.48e was "more steep". The discussion now moved forward as Roy tried to establish a more precise relation between the gradient and the visual steepness:

7.49a	Roy	How do we find the gradient? The gradient is found by going – we can start at any point at the graph, the gradient is the same all the way since we have got a straight line. We go one unit to the right [illustrates on the black-board by using y=x-2 and (0,2) as a basis]. There, you see, from zero to one and then we go upwards until we hit the graph, one. [Marks the height 1]. And that re-mains the same no matter where we do this [shows this by choosing another point, higher up, as a basis]. If we do this from there, we go one step to the right and then it becomes one there as well. Then that's what the gradient is [points to the height in the triangle (Figure 8.4) and marks the height in both triangles].	Hvordan finner vi stignings-tallet? Stigningstallet finner vi ved å gå – vi kan ta hvilket som helst punkt på grafen, stigningstallet er det samme hele veien siden vi har en rett linje. Vi går altså en enhet til høyre her [viser på tavla med utgangspunkt i y=x-2 og (0,-2)]. Der, ikke sant, null til en også går vi opp til vi treffer grafen, en. [Markerer høyden 1]. Og det blir det samme uansett hvor vi gjør det hen [viser for et punkt lengre oppe]. Gjør vi det der, går vi en til høyre så blir det en der også. Så det er det der som er stigningstallet [peker på høyden i trekanten (figur 8.4) og markerer høyden i begge trekantene].
7.49b	Roy	Then, if we do the same for this more steep one, y equals 2x. Then I go one unit to the right and then I have to go two in the upwards direction until I hit the graph again, right [illustrates on the black-board]. Finding a point on the graph, one to-wards the right	Hvis vi gjør det samme på den som er brattere her da, y er lik 2x. Da tar jeg en enhet til høyre og da må jeg to opp før jeg treffer grafen igjen, ikke sant [viser på tavla]. Finner et punkt på grafen, en til høyre også opp til du treffer grafen igjen [peker med linjal]. Her måtte jeg to

		and then upwards until you hit the graph again [points by using a ruler]. Here I had to go two units upwards, right, one, two. Thus, two [points at the number 2 in 2x].	enheter opp, ikke sant, en, to. Altså to [peker på 2 tallet i 2x].

Excerpt 7.49

The method introduced for finding the gradient, or to establish an explicit connection between the gradient and the visual steepness is also in this situation through the *one-unit-right-and-a-up/down* strategy. One should notice that Roy in 7.49a emphasized that one "can start anywhere on the graph as the slope is the same everywhere, because we have a straight line". This is valid for linear functions since their gradients are always constant. On the other hand this is not transferable to non-linear functions of the type that students meet in upper secondary general studies, dealing with differentiation. At the end of the lesson this was further generalised as Roy pointed to the fact that the "a" in "$y = ax + b$" in general equals the gradient of a linear function. A corresponding semiotic chain of signs for the concept of gradients can be illustrated as in the figure below.
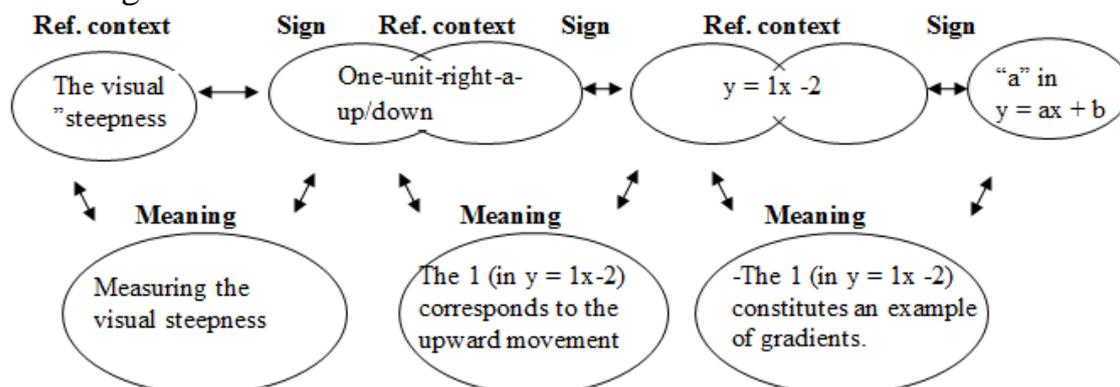
Figure 7.11 Gradients - the semiotic chain related to the teaching sequence

7.4.2 Tasks in Lower Secondary

The teacher consistently divided students' tasks into three categories (A, B and C – A considered to be the easiest and C the most difficult). He stated in the interview that this was his main method of differentiation. All the tasks were from the textbook "Grunntall 10" (Bakke & Bakke, 1999) which in itself did not categorize tasks according to difficulty. In general, the first section in the chapter about functions used real-life situations, interpreting graphs and diagrams based on various practical situations involving "water supply", "hourly wage", "time-distance" and so forth. Actually each of the tasks in this first section of the chapter had a real-life context. But in the two subsequent sections called "funksjoner (functions)" and "lineære funksjoner (linear functions)" this gradually changed to a mix of real-life context tasks and non-contextualized tasks

in the section about "functions", and then to tasks situated only in the realm of mathematics (non-contextual) in the section about "linear functions". I will illustrate this development with some examples.

8.9 The cheetah is the world's fastest animal. It is a predator which is hunting for example antelopes. The diagram below illustrates such a hunt.
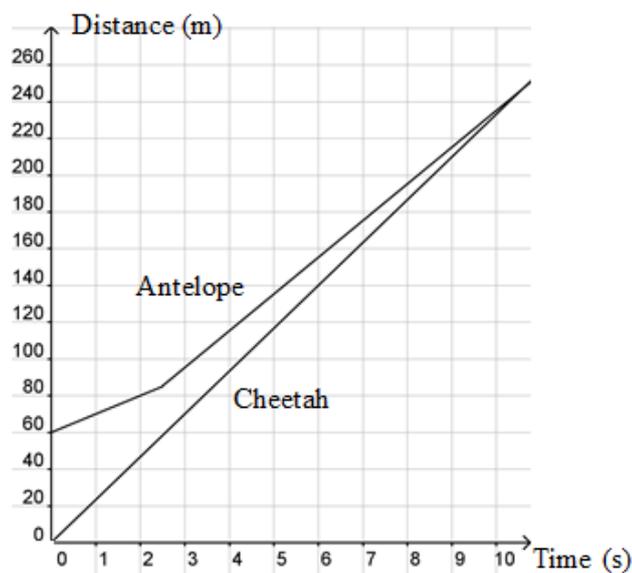
a) Describe briefly how this hunting developed.
b) Make three answerable questions, by reading the diagram.
c) Read your description for another student, and listen to the other student's description. Answer each other's questions.

Figure 7.12 Example of a real-life context task, with non-procedural solution strategies (Adapted and translated from Bakke & Bakke, 1999, p. 350)

This time-distance diagram illustrates the case of an antelope being hunted by a cheetah. In a) the students were asked to describe how the hunt developed. In b) they were to come up with three questions and make use of the diagram to provide the answers. In c) they had to read the description in a) and work with a peer posing and answering the questions in b). I consider this to be an example of a non-procedural task, as it calls for a rather sophisticated interpretation of graphical representations. Unlike these real life situated tasks, the pattern seemed to change radically in the section about linear functions:

8.35   Draw the graphs of the functions without making a value table:

a) $y = 2x - 4$ b) $y = x + 1$ c) $y = -x + 3$

8.36   Draw the graphs of the functions without making a value table:

a) $y = \frac{x}{2} - 1$ b) $y = -\frac{x}{3} + 4$ c) $y = -\frac{3x}{4} + \frac{3}{2}$

8.37   **a)** What is b in this function?
b) What is a in this function?
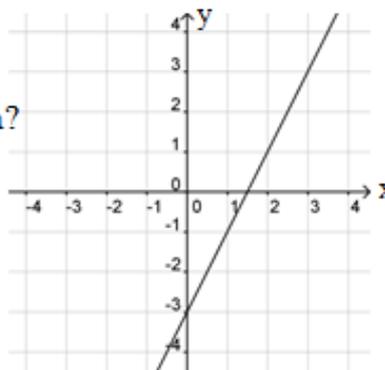**c)** Which function match this graph?

Figure 7.13 Example of a mathematical context task, with procedural solution strategies (Adapted and translated from Bakke & Bakke, 1999, p. 361)

The first two of these tasks (8.35 and 8.36) presents different linear expressions and the students are asked to draw corresponding graphs without using a value table. In task 8.37 a graph is depicted and the students are asked to find b and a (as in y = ax + b) and the function expression. In figure 7.13 one notices that there is no longer a real-life context, and the solution strategies demand only simple procedures, compared to the previous example. Almost all of these tasks can be solved by applying the *one-unit-up-a-up/down* strategy and by remembering that the constant term is where the graph intersects the y-axis.

7.4.3 Conversations with Thea in Lower Secondary

In total I had three conversations with Thea as she was working on tasks from the textbook, during the last part of the lessons. In the first conversation, Thea was working on a task based on the following context:

> The distance from Røros to Moss - a total of 440 kilometers. Kjell Arne starts driving form Moss at 12:30 and drove 120 kilometers at a constant speed for the first two hours. Then he rested half an hour before he continued and drove at an average speed of 70 kilometres per hour. He arrived at Røros at 20:00. Ellen drove in the opposite direction and started from Røros at 12:00. She drove 200 kilometers during the first three hours before resting for 45 minutes. She continued driving at an average speed of 64 kilometers per hour (Bakke & Bakke, 1999, p. 351, my translation).

7.50a	Thea	He starts at 12.30. So then I, sort of, start by 13.30 in that direction [points to the x-axis]	Han starter kl. 12.30. Så da starter jeg liksom med 12.30 i den retningen [peker på x aksen]
7.50b	Interviewer	Mhm…	Mhm…
7.50c	Thea	Then he drives 120 kilometers. So, after two hours, that means that he has driven at 60 kilometers an hour, since it is two hours and he has driven in 120 kilometers […]. Then he rested for half an hour, so then he has reached 15.00. Then he drove for three hours at 70 kilometers an hour, and then he has reached 18.00. Then it says he arrived – so then you only have to draw a line.	Så kjører han i 120kilometer. Så etter to timer da, det vil si at han har kjørt 60 kilometer i timen, siden det er to timer og han har kjørt 120 kilometers […] Så tok han en halvtime pause, da har han kommet til kl. 15.00. Så kjørte han i tre timer med 70 kilometer i timen, da har han kommet til kl. 18.00. Så står det at han er framme kl. 20.00 – så da er det bare å trekke en strek

Excerpt 7.50

In 7.50c, Thea applied the information about distance and time to calculate the velocity. By dividing 120 km by 2 hours, she found the velocity to be 60 km/h, and throughout her deliberations she was plotting the information given in the task into the diagram. In this example it seemed evident for Thea that a certain amount of time travelled implied one specific distance travelled when driving at constant speed, and hence the "one/many-to-one" principle is applied even though this is made explicit neither during the teaching nor in the conversations. Later in this conversation, Thea was asked about the connection between this task and functions:

7.51a	Interviewer	Which connection do you see between what you have done here and the word function?	Hvilken sammenheng ser du mellom det som du har gjort her og ordet funksjoner?
7.51b	Thea	In a way it gives a picture [puts her palm over the diagram]. Instead of just being numbers and such, it provides a picture of how you can figure things out. If I had only been told that those two drove like this or that, and I wasn't going to figure something out, and then they had asked when they did they meet – that would have been difficult to	Det gir på en måte et bilde da [legger håndflaten over diagrammet]. Istedenfor at det bare er tall og sånn, så gir det på en måte et bilde over hvordan du kan finne ut ting. Hvis jeg bare hadde fått opplysninger om at de to kjørte sånn og sånn, og jeg ikke skulle gjøre noe ut av det, også hadde de spurt når møtes de – da hadde det blitt vanskelig å si, for da måtte jeg tegnet veldig mye og sånne ting, men dette gir

		say, because then I would have had to draw a lot and things like that, but this in a way provides a picture which makes it a bit easier. Gives it a good function.	på en måte et bilde som gjør det litt lettere da. Gir det en god funksjon.
7.51c	Interviewer	So you associate the fact that you get a picture with the word function?	Så du assosierer det at du får se et bilde til ordet funksjon?
7.51d	Thea	Yes, then, in a way, it gives me a visual experience.	Ja, på en måte at det gir meg en visuell opplevelse da.

Excerpt 7.51

Thea here indicated that functions, represented in terms of graphs and diagrams serve as a visual support for providing relevant information.

In the next conversation, one task was to draw a graph corresponding to the expression $y = 2x - 3$ (which, incidentally, is the same function which I used during the semi-structured interviews).

7.52	Thea	And the 2x tells that when I go one out in one direction and go upwards [points] then there should be two in between.	Og den 2x'en forteller at når jeg da går ut en vei og går opp [peker] så skal det være to i mellom.

Excerpt 7.52

One observes that the technique provided by the teacher, Roy, was adapted and even though Thea's description was rather imprecise. Her supplemental hand movements showed that she moved towards the right, and then upwards until she hit the graph as in the *on-unit-right-a-up/down* strategy.

7.4.4 Interview with Thea in Lower Secondary

When questioned about what she understood by the concept of functions, Thea once again emphasized the visual representation of a function and its corresponding graph.

7.53a	Interviewer	What do you understand by the word function?	Hva forstår du med ordet funksjon?
7.53b	Thea	I understand that it is a way of illustrating something graphic. That, in a way, it illustrates a simple way of explaining some things. Instead of arranging a lot of this and that and doing this and that, they have put it into one simple thing. To make it a little kind of easier to understand.	Jeg forstår at det er en måte å se noe på grafisk. At det viser på en måte en forenklet måte å forklare noen ting på. Istedenfor å sette opp masse sånn og sånn skal du gjøre og sånn og sånn, så har de satt det inn i en enkelt ting. For å gjøre det litt sånn lettere å forstå det.

Excerpt 7.53

Once again, the graphical representations of functions were emphasized as being able to simplify a mathematical problem. Even though Thea already, in an indirect manner, had dealt with central aspects of the function concept, I challenged her to define functions in more general terms. I asked her what she would have written if she had to write a definition of functions in, for example, a dictionary.

| 7.54 | Thea | Then it is, probably, hm, the range of variation. There you are able to illustrate the variation between things, graphically. | Det er nå vel, hm, variasjonsbredden da. Der du får vist opp variasjon mellom ting grafisk. |

Excerpt 7.54

After hesitating a little she is explicit on the co-variation between "things" and names this the "range of variation[13]". But even in this definition she held on to the graphic representation as a part of the very definition of functions.

When shown the expression y = 2x − 3, Thea was making an account of the constant term, by explaining that it corresponded to the intersection point between the graph and the y-axis. Her explanation of the gradient was in line with her answer in Excerpt 7.52.

7.4.5 Teaching in Upper Secondary – School 4

In upper secondary, Thea attended the general studies programme and the 1T version of mathematics. I observed in total six lessons in School 4, but unfortunately I did not get the opportunity to observe the introductory lessons related to functions. The topic of my observations at this school is therefore related to research question 1a and 1d, and to 2b and 2d.

In the first lesson *growth rate* was the topic. Using a mathematical model based on the number of inhabitants in the county where the school is situated, the discussions were about how to find the average annual growth rate related to the number of inhabitants. These discussions were the starting point for the topic of the next lesson, dealing with *linear regression*.

In lesson three, the teaching moved in the direction of instantaneous growth rate, and the following task was discussed:

The height of a tree measured in centimeters t years after the seeds germinated, is given by the function $h(t) = -\frac{1}{30}t^3 + \frac{5}{2}t^2, t \in [0,50]$.

Find the growth rate of the tree after : a) 10 years, b) 30 years c) 40 years

(Oldervoll, Orskaug, Vaaje, Hanisch & Hals, 2009, p. 200, my translation)

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[13] The (mathematical) correct translation of the Norwegian word "variasjonsbredde" is "range of distribution" but some of Theas reasoning related to "variation" ("variasjon" in Norwegian) would then be lost in the translation. "Variasjonsbredde" is therefore translated literally into "range of variation".

| 7.55 | Kerry (Teacher) | Then I should find the growth rate exactly at that point, and then I would have to find out the direction of the graph exactly in that point. In such a way that I placed the ruler so that it has exactly the same direction as the graph in the area where the point is located. And that's how it turned out for me. I must try to get a somewhat smooth curve. Then I calculated delta h by using those values which I find there, on the axis. Then I found delta t, which is the distance between the points at the x-axis, or the t-axis, though. It should be t there, on the horizontal axis. Then I had to divide delta h by delta t. Then I got 40,7 but I noticed it says 40 in the answers. But we have to consider the answers only as guidelines when we do tasks like these. | Også skulle jeg finne veksthastighet akkurat i det punktet, og da måtte jeg finne ut hva retningen til grafen var akkurat i det punktet. Sånn at jeg la linjalen sånn at den har samme retning som grafen i området der punktet er. Og da ble den sånn for meg. Jeg må jo prøve å få den buen litt jevn, da. Også regner jeg ut delta h, ved å ta de verdiene som jeg leser av borte på aksen der. Også fant jeg delta t, som er avstanden mellom punktene på x aksen, eller på t aksen da. Det skal stå t der, på første aksen. Så skulle jeg ha delta h delt på delta t. Da fikk jeg 40,7, men jeg ser i fasiten så står det 40. Men vi må ta fasiten som veiledende når vi gjør sånne oppgaver. |

Excerpt 7.55

This function is said to represent the height h of a tree measured in centimetres, after t years. Instantaneous growth rate was presented as finding the gradient of a function at one specific point (in this case after 10 years). To do so, the teacher used a ruler and a line was drawn so that the line constituted the tangent to the graph (Kerry did not explicitly use the term "tangent"). Δh and Δt were introduced to find the slope of this tangent. Subsequent to the average growth rate, the instantaneous growth rate and the notation of delta, the teacher now moved on to the topic of differentiation. Kerry built on students' prior experiences with instantaneous growth rate, and used an example from the textbook, the function $f(x) = x^2 - 2x + 4$. With the help of the illustrations, pupils were to find the instantaneous growth rate at $x = 2$. Kerry wrote $\Delta x = (2 + h) - 2$ and $\Delta y = f(2 + h) - f(2)$. Subsequently, *limes* (denoted as "lim") was introduced to denote the limit when $h \to 0$. (See also Section 3.2 for discussion related to limits and differentiation). The following figure recaptures parts of Kerry's illustration:
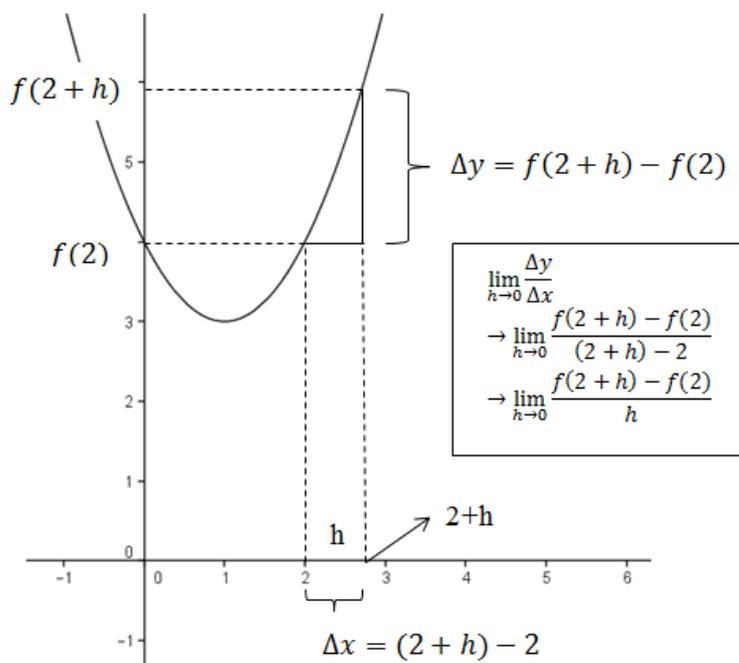
Figure 7.14 Reproduction of Kerry's illustration related to her explanations about the derivative if the function $f(x) = x^2 - 2x + 4$, for $x = 2$.

| 7.56a | Kerry | Last time we started with a point, and then we were going to find the growth rate at that point. And the first thing we did was, that if we had an h here [points to the x-axis] and found the point corresponding – or increased x by h and found the average growth rate between those two points, then that becomes delta y divided by delta x, and delta z equaled h and delta y was found when we put 2 + h into the function. And 2. And that was right – we used the point 2 as an example. And then, when we made h less, this point moved downwards and coincided with that point at which we started. And then we said that if we let the h approach zero, we would then find the growth rate at exactly that point. | Sist så startet vi med et punkt, også skulle vi finne vekstfarten i det punktet. Og det første vi gjorde var at hvis vi hadde en h her [peker på x aksen] og fant igjen det punktet som svarte til – eller øket x med h og fant det gjennomsnittlige stigningstallet i mellom de to punktene, så er det delta y delt på delta x, og delta x var lik h og delta y den fant vi ved å sette inn 2 + h i funksjonen. Og 2. Og da var det jo – vi brukte jo punktet 2 som et eksempel. Også var det når vi gjorde den h'en mindre at det punktet her flyttet seg nedover og falt sammen med det punktet, som vi startet med. Og det var da vi sa at hvis vi lot den h'en gå mot null, så ville vi finne vekstfarten akkurat i punktet. |
| 7.56b | Kerry | And that was written by limes, lim h approach zero. | Og det skrev vi jo med den limes, lim h går mot null. Og hva var |

		And what did we call the growth rate?	det vi kalte den vekstfarten?
7.56c	Student	Differentiation	Derivasjon.
7.56d	Kerry	Yes, the derivative. And that was written by using the apostrophe here. The derivative when x equals 2 for that function.	Ja, den deriverte. Og det var den vi skrev med sånn apostrof her. Den deriverte når x var lik 2 for den funksjonen.

Excerpt 7.56

Naturally, this introduction of new symbols, notations and meaning (understood in terms of Steinbring's terminology) led to some confusion and questions from the students. "Should this [x+h] be put into the function instead of numbers?", "Is it always the case that the h should approach zero?", "Should we always do this [the elaborations involving finding the limit] first, or can we just calculate?" and "Why did the h in x+h disappear?". These (and similar) questions and comments might indicate a gap between what Kerry intended by introducing new signs and terminology and the students' understanding of these signs. Kerry's decision to introduce the "h", in addition to the Δx, entailed yet another symbol for the students to deal with. Figure 7.15 displays a semiotic chain based on the teaching sequences observed at School 4.

Sign

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \to \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Ref. context

Tangents entail that the change in x direction becomes 0

Meaning

Differentiation

Sign

Ref. context

Change in the f(x) direction divided by change in the x direction

Meaning

Instantaneous growth rate

Sign

Ref. context

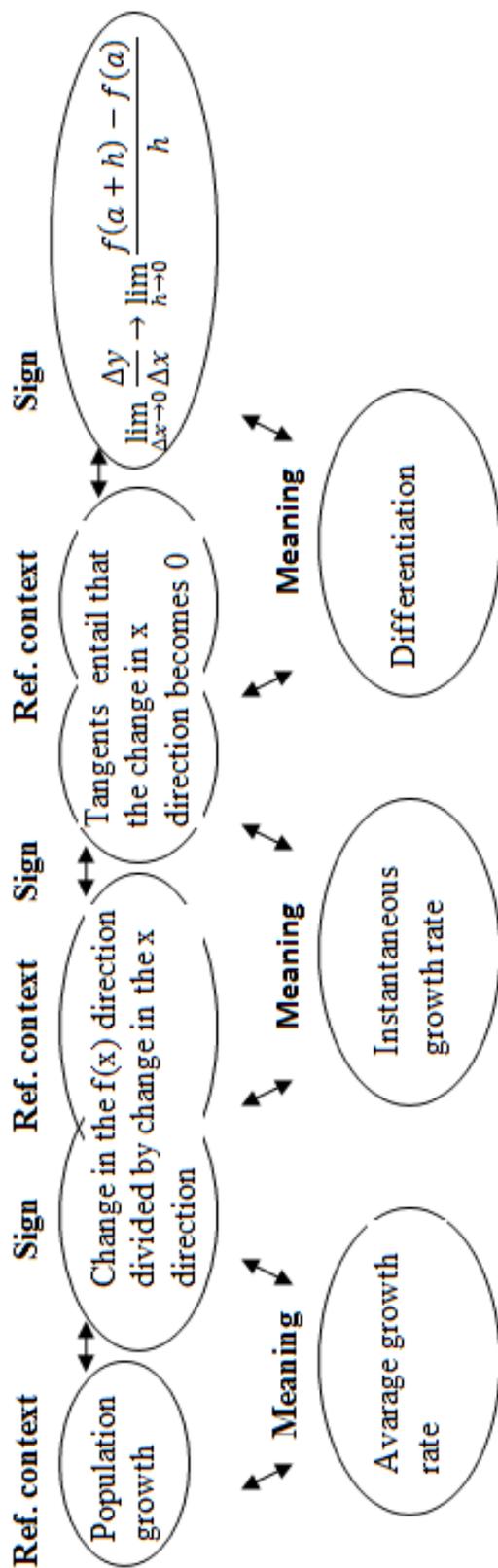Population growth

Meaning

Avarage growth rate

Figure 7.15 Semiotic chaining in the actual teaching sequences at School 4.

7.4.6 Tasks in Upper Secondary

As in the previous cases, the tasks provided during the period of my observations were all taken from the the seventh and eight chapters of the pupils' textbook "Matematiske modellar og vekstfart (mathematical

models and growth rate)" and "Derivasjon (differentiation)". The tasks in the seventh chapter are mainly related to real- life situations, and the example in the previous section, about the growth of a tree, constitutes one such example. Even though that task is framed within a real life context, it could be solved by the use of certain acquired procedures. In the chapter about differentiation the real-life contexts are absent, and tasks with procedural nature dominate.

EXAMPLE
Find $f'(x)$ when f is given by
a) $f(x) = 4x^2$ b) $f(x) = 2x^3 + 3x^2$ c) $f(x) = x^2 + 3x + 2$

Solution:
a) $f'(x) = (4x^2)' = 4 \cdot (x^2)' = 4 \cdot 2x = \underline{8x}$
b) $f'(x) = (2x^3)' + (3x^2)' = 2 \cdot (x^3)' + 3 \cdot (x^2)'$
$\qquad = 2 \cdot 3x^2 + 3 \cdot 2x = \underline{6x^2 + 6x}$
c) $f'(x) = (x^2)' + (3x)' + 2' = 2x + 3 + 0 = \underline{2x + 3}$

Task 8.40
Find the derivatives of these functions:
a) $f(x) = x^2 - 2x + 5$ \qquad b) $g(x) = 3x^2 + 5x - 2$
c) $h(x) = 2x^3 - 2x^2 + 5x - 1$ \qquad d) $s(t) = t^4 + 5t^2 + 5t - 3$

Task 8.41
Differentiate these expressions:
a) $-2x^2 + 4$ \qquad b) $3x^2 - 2x$
c) $5x^3 + 7x - 1$ \qquad d) $-3t^3 + 3t^2 + t - 1$

Figure 7.16 Examples of procedural, mathematical context tasks (Adapted and translated from Oldervoll, Orskaug, Vaaje, Hanisch, & Hals, 2009, p. 219)

Figure 7.16 is an example of procedural types of task framed only within a mathematical context. The example provided in the beginning demonstrates how certain types of tasks can be solved by applying differentiation rules. It was typical of this textbook that the procedural steps were included as a template prior to the tasks provided for the students.

7.4.7 Interview with Thea in Upper Secondary

In the interview in lower secondary, Thea emphasised two aspects of the function concept, namely *functions as co-variance* (Excerpt 7.54) and *functions as representations* (Excerpt 7.51 and 7.54), the latter primarily by emphasizing graphical representations. In upper secondary she elaborated on functions in the following way:

7.57a	Thea	When I hear the word function, I automatically think of a coordinate system which illustrates a model for an event. For example a road trip, how fast one drives, rests or when you throw a stone, or the size of a population increasing or decreasing. And then I think function, then I think of a fact, or a model which illustrates an event in time.	Når jeg hører ordet funksjon, så tenker jeg automatisk på koordinatsystem som viser en modell over en hendelse. For eksempel da en biltur, hvor fort den har gått, pauser eller når du kaster en stein, eller folketallet som har gått opp og ned. Også tenker jeg funksjon da, da tenker jeg en fakta, eller en modell som viser en hendelse eller en handling over tid.
7.57b	Interviewer	I see. So if that was written, for example in an encyclopedia, next to the word function, then you would agree?	Akkurat. Så hvis det hadde stått for eksempel i et leksikon, ved siden av ordet funksjon, så hadde du vært enig i det?
7.57c	Thea	Yes.	Ja.

Excerpt 7.57

From this explanation one observes that Thea preserves her main suggestions from lower secondary (Excerpt 7.51, 7.53 and 7.54). In 7.57a, she still referred to visual representations and exemplified these by a car trip, a stone being thrown and annual population change. She generalized her examples by categorizing them as "models that display events over a period of time". Since she did not explicitly mention the principle of uniqueness, I drew an ellipse in a coordinate system and asked her whether she thought this was a function or not, and she suggested that this actually was a function. Although the copies of her handwritten material suggest that she was aware that for each value picked from the domain, there is just one corresponding function value, she was not able to apply this to the example with the ellipse.

Her thoughts about the shift from consistently using "y = …" in lower secondary to the use of "f(x) = …" in upper secondary was that this was primarily for convenience. She states that it is "easier to follow, and easier to depict functions in the same coordinate system". Further, I asked if she thought there are any mathematical aspects of the function concept which become clearer when one writes f(x):

7.58a	Thea	I don't think that we have talked a lot about that.	Det tror jeg ikke vi har snakket så mye om.
7.58b	Interviewer	Have you seen this notation before? [write f(2)].	Har du sett denne betegnelsen her før? [skriver f(2)].
7.58c	Thea	Yes.	Ja.
7.58d	Interviewer	What does it tell?	Hva den angir?

7.58e	Thea	It is a function where x is 2.	Det er en funksjon av x som 2.
7.58f	Interviewer	Yes. Can you write it like that, in this this case [refers to the form y=… and points at the function y = 2x + 3].	Ja. Kan du skrive det her [sikter til skrivemåten y=… og peker på fuksjonen y = 2x + 3]
7.58g	Thea	No. Here [f(x)] you can have more varied – here [f(x)] you can choose for yourself, it is in a way another problem [task] besides being a function. Here you get f of x, if you take for example x equals 2, then you get 2 times 2 plus 3. So here [f(x)] you can get an answer, but here [y] it's only a linear function.	Nei. Her kan [f(x)] du få mer variert – her [f(x)] kan du velge selv, det er på en måte et stykke i tillegg til at det er en funksjon. Her kan du få f av x da, hvis du setter for eksempel x som 2, så får du 2 ganger 2 pluss 3. Så her [f(x)] kan du få et svar, mens her [y] det på en måte bare en lineær funksjon.

Excerpt 7.58

The example f(2) started rather interesting reasoning. Her answer in 7.58e suggests that she was familiar with this notation and in 7.58g she was anticipating my point and reflected on the possibilities embedded in this notation. The last sentence in 7.58g invites further analysis. It is possible to interpret her suggestions such that the explicit use of "x" in f(x) emphasizes a kind of "freedom of choice" related to the x values. She said that "you can choose [the x-values] for yourself", indicating that this might not seem quite as obvious in the case of "y = …". She calls f(2) a "problem" (task), probably because she thought that f(2) was linked to a procedure and required some kind of answer. In the case of "y =…" it is "in a way only a linear function" she states.

When it came to the gradient in linear expressions of the form $y = ax + b,$ Thea still used the *one-unit-right-a-up/down* strategy. Thinking back on some of her notes from lower secondary I asked her about the gradient in the expression $y = \frac{x}{2} - 2$ to see if the fraction affected her strategy. The excerpt below demonstrates that Thea used the *one-unit-right-a-up-down* strategy even in the case of fractional gradients.

7.59a	Thea	Because it was one out.	For det var en ut.
7.59b	Interviewer	Ok, yes.	Ok, ja.
7.59c	Thea	So a half upwards. And since it is divided by two, then it will be a half, and since there's only an x then there's only one two	Også en halv opp. Og siden det er todeler så vil det jo bli en halv, og siden det bare er en x så blir det jo bare en todel på en måte. Og det er jo det samme

		division then it just becomes a half, in a way. And that the same as one half – you only go one out and a half upwards.	som en halv – tar bare en ut også en halv opp.

Excerpt 7.59

Her reasoning in 7.59c only involved ways of determining the movement in the upward direction and none reflections were present related to her choice of strategy itself. During the interview, she explicitly stated that her strategy for finding the gradient of a linear function is exactly the same as in lower secondary.

Moving over to the topic of differentiation, I asked her to differentiate the function $f(x) = x^2 + 2x + 3$

7.60a	Interviewer	If you had to find the derivative of this function?	Hvis du skulle ha derivert denne funksjonen?
7.60b	Thea	Then x^2 would be 2x and 2x would be 2, then it would be zero. So f of the derivative will be 2x+2.	Da vil x^2 bli 2x og 2x blir 2 også blir den null. Så f av den deriverte vil bli 2x+2.
7.60c	Interviewer	I see. What does the derivative tell, the one you found there?	Akkurat. Hva forteller den deriverte, som du har funnet der?
7.60d	Thea	Eh, I don't know, , the function then [laughs]. I was absent when we learned about differentiation so I don't know anything about what differentiation means. I only know how the find derivatives.	Eh, jeg vet ikke, funksjonen da [ler]. Jeg var borte da når vi lærte om derivasjon så jeg vet ikke hva derivasjon sier noe om. Jeg vet bare hvordan jeg deriverer det.

Excerpt 7.60

According to Thea she was absent during some of the lessons related to differentiation, but still she learned the "technique" (differentiation rules).

7.4.8 Thea's experience of the transition

When comparing lower secondary and upper secondary, Thea primarily remarked that she thought that her learning outcome has increased. She elaborated on this by explaining that she thought that they had spent too much time on each topic at lower secondary.

7.61	Thea	[…]. What's more, I think it's more theoretical, though, more that we do solve tasks, we don't deal a lot with trivial questions about this	[…] Ellers så synes jeg at det er mer teoretisk da, mer at vi regner, vi går ikke så mye inn på bare overfladiske spørsmål om ditt og datt, og vi går over på

		and that, and we change to new topics the whole time, so that we learn more.	nytt tema hele tiden, sånn at vi lærer mye mer.

Excerpt 7.61

Further, her statement below also suggested a noticeable change when it comes to her perception of the mathematical knowledge of their teacher:

7.62	Thea	[…] But there's a big difference in their competence in mathematics. I easily see that the teacher we have now possesses more knowledge about several aspects of mathematics than the teacher in lower secondary, at least as I feel about it.	[…] Men det er veldig forskjell på kompetansen de har i matematikk. Jeg merker fint at hun vi har nå har mye mer kunnskap til flere ting og inn på matematikkfaget, enn hva læreren på ungdomsskolen hadde, føler jeg da.

Excerpt 7.62

She used the fact that mathematics in upper secondary is more advanced to justify her claim concerning their teacher's knowledge. As regards the teaching methods, Thea did not notice any particular change:

7.63	Thea	Yes, not the method that in a way stays the same, that is going through content on the blackboard, theory first followed by questions and then they help us while we solve tasks.	Ja, ikke metoden, den blir på en måte noe av det samme, at det blir gjennomgang på tavla og at teorien først også er det spørsmål også går dem rundt og hjelper til når vi gjør oppgaver.

Excerpt 7.63

Thea seemed to be pleased with the teaching offered in her upper secondary school. She also found mathematics in upper secondary to be more advanced. Compared to lower secondary she experienced that there was less time spent on each of the topics. Explanations on the blackboard and task solving are common practice in both lower and upper secondary school, in Thea's experience. Further, she does not express any objections to these methods themselves and her marks are high in both lower and upper secondary school. At the interview in lower secondary she also stated that she "tends to learn new stuff very easily".

The case of Thea completes this chapter about the four selected students and their attendance at lower and upper secondary school. The chronological presentations and the detailed analyses and provided in this chapter form a basis for the more general accounts in the next chapter.

8 Further analysis

The aim of this chapter is to elaborate on the findings from the study by explicitly systematizing, structuring and comparing the findings for all eight student participants. This will be done in accordance with the categories that emerged from the data management, and partly presented in Chapter 7. The process of developing the categories for the first two research questions is presented in the methodology chapter (Chapter 6), and Chapter 7 provides some of the basis for the categories identified. Based on my material and by systematizing the findings through the developed categories I aim to identify certain phenomena and characteristics of the learning and teaching of functions and gradients in the actual transition. I will also briefly outline more generally students' experiences of the lower-upper secondary transition with respect to the learning and teaching of mathematics.

8.1 The concept of functions

8.1.1 Teaching (research question 2a)

All the categories accounted for in Chapter 6, emerged from my empirical data so that I aim to cover the main characteristics of teaching and learning collected through a series of interviews and observations. Related to teaching and learning functions, the category *functions by representations* was, of course, involved to some extent in most parts of my material and in all schools. However, as explained in Section 6.4.4, I mainly apply this category to situations where it constituted the main approach to the function concept. For example, in Matt's case (7.3), no explicit discussion of the function concept itself took place and *functions by representations* was pervasive in the teaching which I observed. Mostly representations in terms of function expressions, graphs and value tables dominated teaching in all the five lower secondary schools involved in this study. Still one could claim that some of the tasks and examples provided dealt with the uniqueness property (one/many-to-one) in an indirect manner by, for example, calculating different function values.

No	Who	Translation	Original
8.1a	Tim (teacher)	[Draws a table consisting of two rows (x and y) and inserts in the values -1, 0, 1, 2 for x]. We choose values of x. If I now write an equation here [writes y = x +1 on the blackboard]	[Tegner en tabell bestående av to rader (x og y) og setter inn verdiene -1, 0, 1, 2 for x]. Vi velger oss verdier for x. Hvis jeg nå setter opp en likning her [skriver y = x + 1 på tavla].
8.1b	Tim	[…]. So if we insert minus one – minus one plus one,	[…]. Så hvis vi setter inn minus en – minus en pluss en, hva blir

		what will that be?	det?
8.1c	Students	Zero	Null
8.1d	Tim	That's zero, right [writes 0 as y in the table on the blackboard].	Det blir null, ja [skriver 0 for y i tabellen på tavla]

Excerpt 8.1

This example shows how the uniqueness property was indirectly dealt with in School C. A given x value put into the function expression resulted in one y value. Corresponding situations were also present in most observations, but the concept of variables (independent and dependent) and the one/many-to-one property were seldom mentioned explicitly by the teachers. In similar demonstrations, the representations of functions in the introductory phase were characterized by moving from the function expression to graphs through a value table, as shown in Figure 8.1. This can be described as "computing" and "plotting" in Janvier's (1978) table.

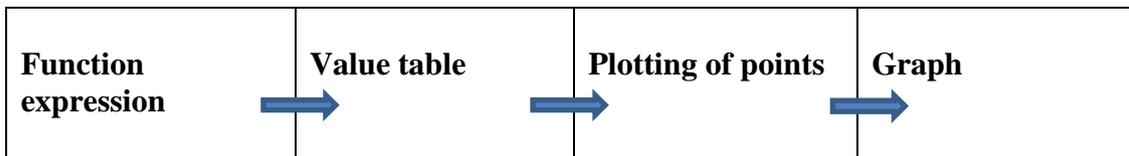| Function expression	→	Value table	→	Plotting of points	→	Graph

Figure 8.1 Preferred path in the introductory phase.

Even though one might argue that the property of uniqueness, as well as independent and dependent variables, are implicitly dealt with in the demonstrations and tasks provided for the students, neither of these concepts are made explicit. As accounted for in Section 4.2.3, semiotics, understood in a Vygotskian sense, emphasizes the correlation between language and conceptual understanding. I therefore find it appropriate to wonder whether the omission of such key mathematical concepts in the classroom dialogues could create obstacles for the students' conceptual development. The definition of the function concept is essential to students at a later educational stage as they are introduced to for example calculus and computer programming. And by omitting explicit discussions related to the definition of the function concept, essential properties of functions might not be clear to the students.

Quite different approaches to the function concept were observed in all the four lower secondary schools involved. Perhaps the *functions as loci* at School A was the one that stood out most. The function concept was not discussed explicitly in this case either, except from the relatively short, and one might claim imprecise, comment in Excerpt 7.1 (Chapter 7). However, representations in the form of graphs, primarily with different loci, dominated most of the illustrations. The representations with different loci in the way they were presented at this school did not,

in some instances, meet the uniqueness property. On some occasions these curves became circles and ellipses and hence they did not coincide with the function definition.

My observations from School B (Section 7.2) and School D (Section 7.4), show that the function concept was treated more explicitly in these two schools. At School B functions were introduced by the use of a mind map, two concrete *function machines* and by *formal definition*, the latter by reading the definition from the textbook. The teacher in School D problematized the one/many-to-one versus the one-to-many properties. In addition she referred to the definition in the textbook.

The category *formal definitions* mainly arose from the presentations offered in the various textbooks. In lower secondary School B and School D, these definitions were dealt with either by reading this for the students (School B) or by referring to it, as in school D. To some extent this was also the case in upper secondary:

8.2a	Tommy (teacher)	Why can we say that the curve of the graph is a function? [Refers to an illustration in the textbook, and no response comes from the classroom]	Hvorfor kan vi si at kurven til grafen er en funksjon? [Henviser til en illustrasjon i tekstboka, og ingen respons lyder fra klasserommet]
8.2b	Tommy	Can you tell what is meant by a function, in simple terms, as it's written in this textbook?	Kan dere si hva som menes med en funksjon på en enkel måte, slik som det oppgis i boka her?
8.2c	Tommy	No, we are not going to spend a lot of time on this, but if you browse towards the end of the book, you find that each value of x should correspond to one and only one value of y.	Nei, vi skal ikke bruke så mye tid på dette, men hvis dere blar noen sider bak i boka så finner dere at hver enkel x verdi skal tilsvare en og bare en y verdi.

Excerpt 8.2

Excerpt 8.2 shows how the formal definition of functions was dealt with at School 2b, upper secondary. The question posed by the teacher in 8.2a was from a question in the textbook, given to the students as homework. In 8.2c the teacher quickly summarizes the formal definition in the textbook and no further examples or problems were provided. Tommy's statement "No, we are not going to spend a lot of time on this" indicates that the uniqueness property was not regarded as a particularly important part of the topic of functions.

In lower secondary School D and in upper secondary School 3c, there were examples of *function machines*. The model used in the case of Matt (Figure 7.7) differs from the car-painting example, and the examples with the boxes provided in lower secondary School D (Section 7.4.1).

The uniqueness property was not explicitly mentioned in School 3c but Henry's statement "one x comes in there and then something comes out" (Excerpt 7.36b) might count as an oblique reference.

Several of my observations in upper secondary verified the common shift from y to f(x) in the function expression. Recapitulating from Section 7.3.5, Henry in School 3c elaborated on this notation in the following terms:

| 8.3 | Henry (teacher) | That way of writing is in a way typical for equations. When we move over to functions, we replace that one [points to y, in the expression y = 2x + 2] and then one writes [writes f(x) = 2x + 1 on the blackboard]. The reason for this is to instill the concept of function, because the concept of function is that one has in a way a variable x [points at the blackboard]. So one puts the x into an expression, and then one gets a result [points to f(x)]. That means, in a way, that the x which is the variable is treated inside the function, and something comes out. | Den skrivemåten der, den er på en måte typisk for likninger [sikter til y = 2x + 1]. Når man skal over på funksjoner, så bytter man ut den der [peker på y] og så skriver man [skriver f(x) = 2x + 1 på tavla]. Grunnen til at man gjør det, det er for å få inn dette begrepet funksjon, for begrepet funksjon er jo at man på en måte har en variabel x [peker på tavla]. Så putter man x inn i et uttrykk, så får man ut et resultat [peker på f(x)]. Det betyr på en måte at den x'en som er variabelen den blir behandlet inni funksjonen, så kommer det ut et eller annet. |

Excerpt 8.3

Here an explanation in terms of *function as co-variation* is offered. The uniqueness property is not explicitly emphasized, but it is clear from Henry's statements that what "comes out" of the machine depends on the input value. The statement "one puts x into an expression and one gets a result" could be interpreted as meaning that f(x) is a more convenient way of denoting functions as it is a better way of promoting the idea of co-variation. The point was amplified by pointing to the f(x) and the relation between this and "the result".

8.1.2 Learning (research question 1a)

As illustrated in the previous section, observations of teaching related to functions suggest that the function concept was dealt with in different ways at the different schools. The emphasis on central aspects of the function concept (like the uniqueness property and the independent and dependent variable) varied from being omitted (School C) to being discussed more fully (School D). During the semi-structured interviews conducted at the end of my observations in each of the schools, the students were asked to elaborate on the function concept. It should be emphasized that the time these interviews took place varied, and was not

necessarily carried out immediately after the topic of functions was treated. The summary below displays a condensed version of each of the students' explicit elaborations on lower and upper secondary level. It is of interest to present the students separately to provide an overview and to form a basis for approaching the more general discussions and analysis.

Student	Lower secondary	Upper secondary
Lena (GS)	Then I think coordinate system. Linear functions, I think. y equals x to the second plus one. And to find the relation between x and y, for example, weight and money. You can calculate it fast and draw it.	It is to relations, in a way. That are dependent on each other. A function could be y = 5x − 2. If you have to find, for example, how much fuel the car uses per mile, then you can draw a graph illustrating it. The relation between two things that increase and decrease.
Kent (GS)	Functions are actually only lines in a coordinate system that show measurements.	A function, that is a line in a graph that shows the y and x values in relation to each other.
Anna (GS)	One, or two upwards something, actually? y and x. And such graphical solutions, and such displays and such − y and x. We calculate and such.	Honestly, I don't know.
Matt (GS)	Think of a graph, and such. And then I think of such numbers that are displayed, such as two point four, for example.	What shows what a graph should look like. Yes, isn't a function such f of x equals ax plus b, in a way? It is like how the graph becomes, then.
Thea (GS)	It is a way of visualizing something graphically. It shows, in a way, in a simplified manner, how to explain some things. Instead of arranging a lot − so and so done should do − one has put it into one simple thing, to make it easier to understand. One gets to show variation between things, graphically.	When I hear the word function, I think automatically of a coordinate system that shows a model of an event. For example, a car trip, how fast it went, pauses − or when you throw a stone, or the population which has increased or decreased. Then I think facts, or a model which shows an event through time.
Otto (VS)	There is a lot of	I would say that is the

	multiplication and division for each problem and a kind of table beside… and a lot of answers which one has to change into other problems and so on. But when it comes to functions, then I more visualize that one needs such a compass to calculate degrees and such things.	growth on something that happens. If someone is working for such and such long time and gets such and such much salary. Then one could see how much salary it is, related to how much you earn, for example.
Edna (VS)	I'm not sure. No… I'm like just used to having to learn it in a way, then it's just good to be finish with it.	No, functions actually… I attend mechanics, right, so functions there are very important to us. Things should be in function. But in maths, I don't know. Function – is it that we get a problem and solve it, in a way…?
Olga (VS)	If you are to display – that is – usually you draw a cross, and then there's the minus-side, the plus-side, and it's like the numbers are going like increases and such. Then we are to find points and numbers which should fit into that system. Then it should like be displayed so that it becomes like lines or curves and such things.	The way numbers are arranged in relation to each other.

Table 8.2 Condensed version of students' explicit elaborations of the function concept. GS indicates that the student attended the general studies and VS indicates vocational studies

It should be emphasised that Otto, Edna and Olga all attended the vocational studies programme in upper secondary. The word "functions" is not explicitly used in the curricula for these programmes and especially Edna's *non-mathematical* approach to the concept could be understood in the light of this. Probably due to the absence of functions from the curriculum, the concept of functions was only explicitly dealt with in the vocational study programmes.

Many of the students' accounts in lower secondary contain examples of *functions by associations*, as in the cases of Kent, Anna, Matt, Otto and Olga. Each of these students associated functions with some type of

graphical representation. Kent stated that he associated functions with "a coordinate system that shows measurements" and Anna with "graphical solutions". Matt mentioned plotting of points in addition to graphs, while Olga accounted for the coordinate system, plotting of points and graphical representations of linear functions. Otto was associating functions with "calculating degrees", probably referring to the "steepness" of graphical representations of linear functions. (In one teaching sequence the teacher at School A was referring to the angle between a straight line and the x-axis when talking about steepness). In the cases of these five students, little was said to explain the function concept itself, but associations to the concept were mentioned in the various ways presented.

In Lena's case, a specific example is given in terms of "*weight and money*", which makes this an example of *functions by examples*, but at the same time she mentioned the function expression "y equals to x to the second" and she stated "you can calculate it fast and draw it". This categorizes as *functions by representations.* By suggesting a "relation between x and y" a third category *functions as co-variance* is involved. Thea emphasizes both *functions as representations* and *functions as co-variance* by her statement "one gets to show variation between things, graphically".

In upper secondary Thea and Otto's accounts are examples of *functions by examples* and *functions by representations* since their elaborations on the concept are primarily linked to specific examples or representations. Thea suggested three examples in terms of driving and speed, throwing a stone and population growth. In addition she referred to functions as being models (corresponding to "situations" in the Janvier (1978) table) and graphical representations. Work and salary were mentioned by Otto. Lena constituted an example of all the three categories mentioned above, as the driving-fuel example correspond to *functions through examples*, her "$y = 5x - 2$" and "you can draw a graph illustrating it" belongs to *functions as representations* and her last statement "the relation between two things that increase and decrease" illustrates *functions as co-variance.* Kent stated that a graph "shows the y and x values in relation to each other", while Olga in more general terms states "the way numbers are arranged in relation to each other". Both Kent and Olga's utterances classifies as *functions as co- variance.*

Student	Lower secondary	Upper secondary
Lena (GS)	-*Functions by examples* -*Functions as co-variance* -*Functions as representations*	-*Functions through examples* -*Functions as co-variance* -*Functions as representations*
Kent (GS)	-*Functions by associations* -*Functions by representations*	-*Functions as co-variance* -*Functions by representations*

Anna (GS)	*-Functions by associations* *-Functions by representations*	*-No answer*
Matt (GS)	*-Functions by representations*	*-Functions by representations* *-Functions by examples*
Thea (GS)	*-Functions by representations* *-Functions as co-variation*	*-Functions by representations* *-Functions by examples*
Otto (VS)	*-Functions by associations*	*-Functions by examples*
Edna (VS)	*-No answer*	*-Non-mathematical*
Olga (VS)	*-Functions by associations* *-Functions by representations*	*-Functions as co-variance*

Table 8.3 An overview of students' accounts based on the categories described in Chapter 6. (GS = general studies, VS = vocational studies)

Of course, these interviews are not sufficient to be able to draw firm conclusions about these students' conceptions of the function concept. As accounted for also using the four cases in Chapter 7, most of the students implicitly dealt with the uniqueness property of functions, for example, in connection with different tasks. One such example is illustrated below.

8.4a	Interviewer	The hourly wages, how did you find that? [Refers to a task in a pamphlet, where the wages for a given number of working hours was illustrated]	Timelønna, hvordan fant du den? [Sikter til en oppgave i arbeidshefte, hvor lønnen gitt ved antall arbeidstimer er illustrert]
8.4b	Matt	Then I observed that it increased – after one hour he had earned 40 kroner [illustrates this by pointing with a pencil at «1 hour» on the horizontal axis, moves the pencil vertically until it meets the graph and then moves in the horizontal direction and points to "40 kroner" on the y-axis].	Da så jeg at den steg – når det har gått en time så har han tjent 40 kroner [viser dette ved å peke med blyanten på "1 time" på den horisontale aksen, føre blyanten vertikalt opp til grafen for så å bevege den i horisontal retning og peker på "40 kroner" på y aksen].

Excerpt 8.4

As indicated in the previous section, my observations suggest that the shift from "y =…" to "f(x) =…" characterized the teaching in upper secondary, general studies. For the students in the general studies program, I therefore included some questions about this new notation in the interviews.

8.5a	Interviewer	Why can it be ok to write f(x) and g(x) […] instead of, for example, y and z […]?	Hvorfor kan det være greit å skrive f(x) og g(x) […] istedenfor å for eksempel skrive y og z […]?
8.5b	Thea	Yes, because if you have y	Ja, fordi at hvis du har y og z da,

		and z, then it might be stated in two different ones. You might get one up there and one down there. It is harder to put them together in a sense. If you have f(x) and g(x) you could put them in the same [coordinate system] without any problems. And you can, in a way, illustrate a much neater graph, I think.	så blir det gjerne oppgitt i to forskjellige. Du får gjerne en oppå der og en nedpå der. Det blir vanskeligere å få satt det sammen liksom. Hvis du har det sånn som f(x) og g(x) så kan du putte dem inn i det samme [koordinatsystemet] uten problemer. Og du kan på en måte vise en mye mer oversiktlig graf, synes jeg da.
8.5c	Interviewer	But why do you think it would have been problematic, I didn't quite get it, to represent them in the same coordinate system even if it says y and z?	Men hvorfor synes du det hadde vært problematisk, jeg fikk ikke helt med det, å fremstille dem i samme koordinatsystem selv om det hadde stått y og z?
8.5d	Thea	Because they are much more similar. So, now you in a sense get one for this and one for that, but still they are different because you give them different names, which is very clear.	For de er mye likere. Altså nå får du på en måte en for den og en for den, men de er allikevel forskjellig for du har forskjellig benevning på dem, som er veldig tydelig.
8.5e	Interviewer	[…]But are there any mathematical properties which one brings out by writing it like this [writes f(x)=2x+3 og y=2x+3]?	[…]Men er det noen matematiske egenskaper man får frem ved å skrive det sånn [skriver f(x)=2x+3 og y=2x+3]?
8.5f	Thea	I don't think we have talked a lot about that.	Det tror jeg ikke vi har snakket så mye om.

Excerpt 8.5

The interviews revealed that most of the students did not have any opinions about why the f(x) was used instead of y in upper secondary. The mathematical implications of this notation, in terms of clarifying the relation between the independent and the dependent variable, did not come to the surface in any of the interviews. Thea was actually the only student who tried to explain her thoughts concerning this, and in 8.5b and 8.5d she stresses what she understands as convenient aspects of the f(x) notation. Her statement in 8.5b, "if you have f(x) and g(x) you could put them in the same [coordinate system] without any problems", could imply the idea that for y =… and z =… one would need an x-y coordinate system for the first one and an x-z system for the second one. At least her reasoning suggests that the notations f(x) and g(x) have to do with a more distinct representation of the dependent variable. On the other, her statements in 8.5d suggest that her main point is the visual

effect of the applied symbols. I consider this to be further confirmed in 8.5f.

8.1.3 Possible relations between teaching and learning (research questions 3a and 3b)

In the following analysis I presuppose that there are correlations between teaching and learning which in turn influence students' understanding of the function concept. Tables 8.2 and 8.3 show that none of the students expressed the meaning of the function concept in terms which completely met a formal mathematical definition. As described in the previous chapter, I observed that this formal definition was made explicit only in School C, where the teacher quoted the textbook. The uniqueness property of functions was indirectly dealt with in one form or another in most schools by means of various types of function machines. However, based on the students' own statements, the outcome of these examples seems to reduce the uniqueness property to *functions as co-variance*. Even if several of the examples provided by the teachers entailed the uniqueness property, it was only made explicit on rare occasions. On these occasions this was mediated primarily through teacher explanations and not through tasks or student activities.

In lower secondary, the utterances of Thea and Otto (Table 8.2) are worth noticing. Thea states

> [I]t is a way of visualizing something, graphically. It shows, in a way, in a simplified manner, how to explain some things. Instead of arranging a lot – so and so one should do – one has put it into one simple thing, to make it easier to understand. One gets to show variation between things, graphically. (Table 6.2)

As demonstrated by the task related to Excerpt 7.50, Section 7.4.3, Thea had experienced various graphical representations of functions. Also, she usually selected her tasks from the selections marked by the teacher as "tasks with high level of difficulties". In a sense, she also had experience with various cases of mathematical models based on real-life situations. These experiences could account for her explanations of the function concept quoted above.

When interviewed in lower secondary school, Otto (in School A) made the following point:

> [W]hen it comes to functions, then I more visualize that one needs such a compass to calculate degrees and such things. (Table 8.2)

From observing the teaching at this school, I noticed that the teacher approached the topic of functions via the category *functions as loci*. This way of dealing with visual representations of functions made the curves themselves represent concrete objects, as, for example, paths and roads. In the activities related to the introduction of gradients, the linear graphs represented roads in a literal sense, as a slope or steepness. Hence, one way of visualizing the steepness of these roads was to look at the angle between the graph (road) and the x-axis. Therefore, the visualizing Otto

mentions in Table 8.2 and the need for a compass "to calculate degrees and such things" might relate to some extent to these activities and illustrations and similar teaching sequences.

8.2 Gradients

8.2.1 Teaching (research questions 2b- 2d)

Except for Otto in School A, the accounts in Chapter 7 show that teaching in lower secondary related to gradients was dominated by various versions of the category *one-unit-right-a-up/down*. To recapitulate, this category suggests that the gradient of a linear function is identified with a specific technique where one starts from an arbitrary point on the graph and then first moves one unit to the right in a horizontal direction followed by a vertical movement ("up" or "down" depending on whether the gradient is positive or negative) until one meets the graph. Then, one counts/measures how far up one has moved to get "a". Linear functions with suitable constants are often taken as a starting point for these demonstrations. In School B the method was demonstrated by the function given by the expression y = 2x + 3, in School C y = x + 1 was applied while y = x − 2 and y = 2x was used at School D. The *one-unit-right-a-up/down* strategy often results in triangles which are meant to provide a visual picture of the strategy.
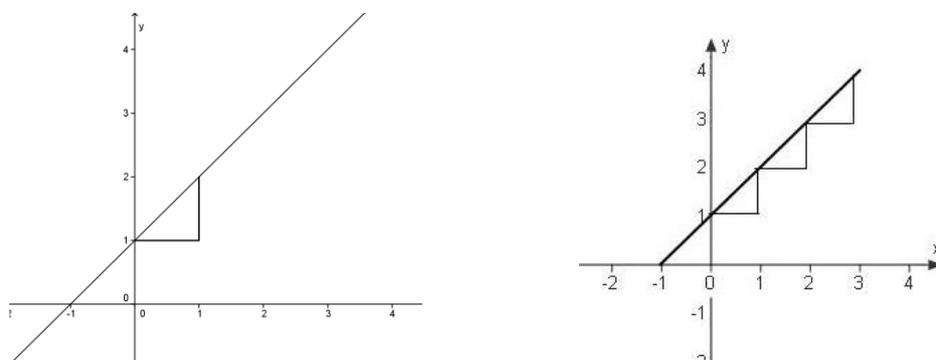

Figure 8.4 Typical visualisations of the *one-unit-right-a-up/down* strategy here represented by reconstructions of the figures demonstrated in School C

The reconstructions above are examples of how these triangles are applied. On this occasion the teacher started the procedure from the intersection point on the y-axis where these were points with integer values for both x and y. The extended staircase version of the triangle is meant to illustrate that the gradient is independent of the starting point. The typical epistemological triangle associated with several similar examples can be summarized in the diagram below:
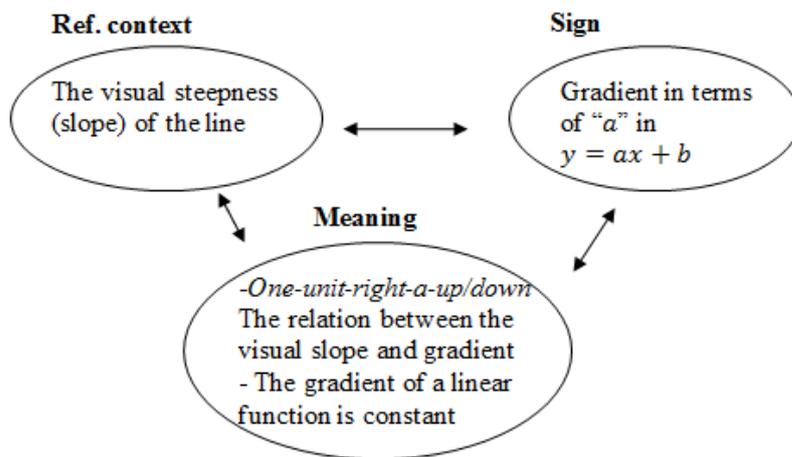
Figure 8.5 An epistemological triangle related to teaching gradients in lower secondary school.

With the geometric approach in terms of loci, School A, is in a sense a special case with respect to teaching gradients. In the upper secondary schools 3b and 3c, (general studies), the *one-unit-right-a-up/down* strategy seemed to some extent to be applied, both in relation to linear functions and in the initial stage of differentiation. There was a sudden shift, especially in connection with differentiation at both School 3b and School 3c and also at School 5.

The Excerpts 7.39 and 7.56 (Chapter 7) illustrate the introduction of the mathematical symbols Δx and Δy which were probably unfamiliar to most of the students. In 7.39b and 7.39c, Henry points to "adding a small distance" to illustrate some conceptual aspects with the deltas, obviously intending to prepare the students for the topic of differentiation. From these observations it emerges that the Δy/Δx category is primarily related to the growth rate of functions by expressing the gradient of suitably constructed secants related to the actual function. But at the same time, this new notation also implies the conception of gradients as a certain distance in the y–direction divided by a certain distance in x-direction. For linear graphs the gradient is constant so the height measured in the *a-up/down* part (of the *one-unit-right-a-up/down* strategy) divided by one (the *one-unit-right* part) will give the same number for any $\Delta y/\Delta x$. On the other hand, this might not necessarily be evident to the students, as suggested when they dealt with fractional gradients (like in the case of Kent in Excerpt 8.7, Section 8.2.2).
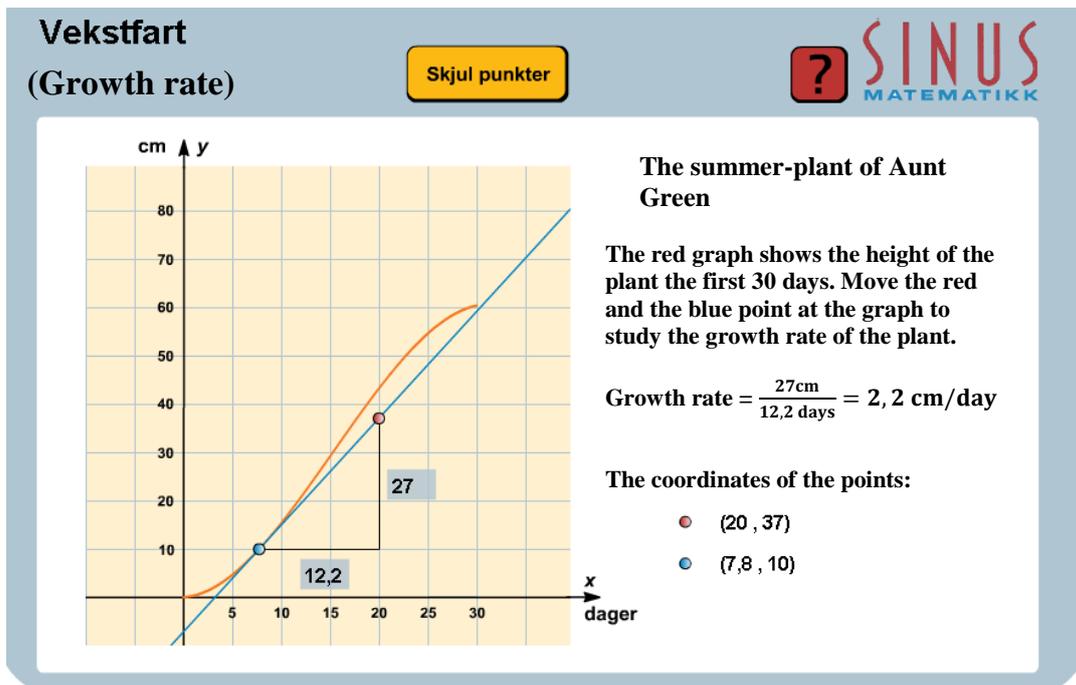
Figure 8.6 The interactive model related to growth rate applied at School 3b (Reproduced with permission, from Cappelen Damm, 2008b, translations added by the author)

By letting Δx approach zero, the secant approaches the tangent, which in turns leads to the concept of differentiation. This principle is illustrated in Figure 8.6 from School 3b, where by moving the red and blue dots the students had to determine the instantaneous growth of a plant after certain periods of time.
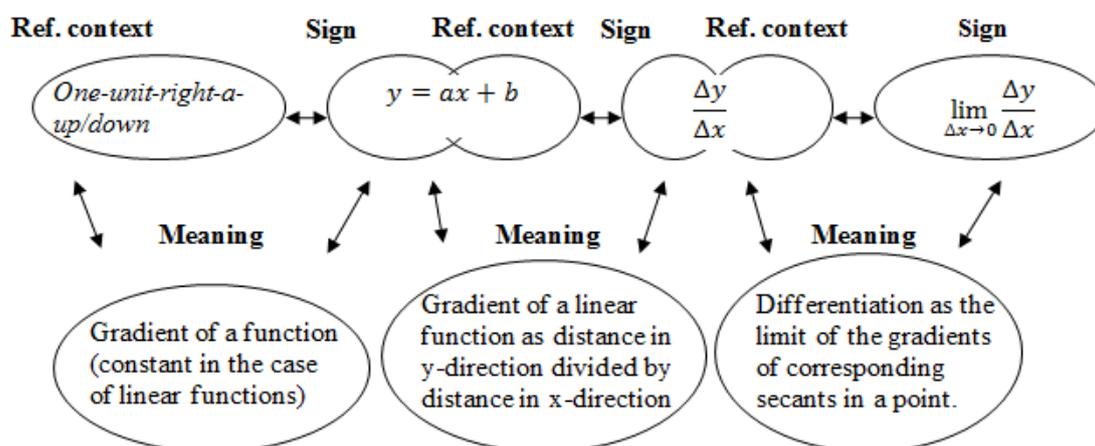

Figure 8.7. Semiotic chain, describing the teaching of gradients in upper secondary, general studies

The third and fourth parts of this semiotic chain constitute what one might call the expansion of the gradient concept in upper secondary apparent in observations in School 3b, School 3c and School 4.

In upper secondary, vocational studies programme, I focused upon the topic of "proportional magnitudes" since teaching in this topic, in my view, could potentially build on different aspects of both functions and gradients which were familiar to the students from lower secondary. As illustrated in the cases of Otto and Olga (Chapter 7), even though the category *proportionality constant* was essential, the only observable links to functions and gradients were the graphical illustrations of linear functions, intersecting the origin. The possible interpretation of proportional constants as special cases of gradients of linear functions was not pinpointed in teaching. This was not only the case in upper secondary, vocational studies, but also in lower secondary. Even though textbooks used in School B, School C and School D treated proportional magnitudes as a sub-topic of functions, no explicit link was made between the proportionality constant and the gradient.

8.2.2 Learning (research questions 1b-1d)

In the previous section I argued that the most dominant approach to gradients in lower secondary was the *one-unit-right-a-up/down* strategy. In the interviews with the students, this method also characterized students' answers. In the semi-structured interviews, all the students were asked to elaborate on the "2x part" of the function expression $y = 2x - 3$.

| 8.6 | Anna | That I start in minus three, right, and that it increases by two for each x and such. | At den starter i minus tre, er det ikke det, og at den stiger med to for hver x og sånn |

Excerpt 8.6

Even though some of the other students who applied the *one-unit-right-a-up/down* strategy preferred to choose a starting point where the value of y was positive, the excerpt above is typical for the students' attempts to explain the number '2' in the function expression. Anna's formulation "increases by two for each x" is interesting as this is quite a precise mathematical description which goes beyond just referring to the strategy in terms of going "one step to the right…".

Excerpt 7.32 (Chapter 7) illustrates the situation where Matt had the idea that the starting point in this strategy was at the origin, seemingly independent of the function expression. Kent, who also attended School C, seemed more aware of the starting point when he applied the method. It is interesting to see how he used the strategy, also when the gradient was a fraction. In this task the function expression was given, and by moving a pointer on the screen, the straight line was placed in accordance with the expression. In this case the expression given was $y = \frac{3}{2}x - 2$.

| 8.7 | Kent | Let's see […]. Three halves equal one and a half – so it goes one and a half upwards. One and a half, because it goes upwards from minus two [counts one and a half in the y direction, starting from (0,-2)] Let's see ["check answer"] – it was correct. | Skal vi se […]. Tre todeler er en og en halv – så den går en og en halv opp. En og en halv fordi den går oppover fra minus to [Starter fra (0,-2) og teller en og en halv i y retningen]. Skal vi se [klikker på «sjekk svar»] – det var riktig. |

Excerpt 8.7

In this case, by converting the fraction into decimals, he used the strategy in the same way as for whole numbers. Apparently, this strategy worked well in Kent's case, as he was able to adapt the method to new situations, as in the case of fractions.

In connection with the example of Otto, I briefly gave an example of what I categorized as *gradient as a diagonal movement.* Another example of this is found in the excerpt below where another student in School C, Tony[14], was working with an interactive task (as described in 7.3.2). Tony's task was to use the mouse to adjust a straight line depicted on the computer screen, to fit the prescriptions of the constant term being two and the gradient being three.

8.8a	Interviewer	So you have to find the gradient…	Så skal du finne stigningstallet…
8.8b	Tony	Yes.	Ja.
8.8c	Interviewer	It is 3.	Det blir 3.
8.8d	Tony	Yes. Let's see…[moves the blue marker diagonally along the grid. He counts three such diagonals, so that the gradient then becomes 1].	Ja. Skal vi se… [flytter den blå markøren langs rutenettet, diagonalt i forhold til hver rute. Han teller tre slike diagonaler, slik at stigningstallet dermed blir 1].
8.8e	Interviewer	What is your reasoning now?	Hvordan tenker du nå?
8.8f	Tony	Forward, diagonally, like that [shows that he counts three diagonals by moving the cursor with the mouse].	Sånn på skrått fremover sånn [viser at han teller tre diagonaler ved hjelp av musepekeren].

Excerpt 8.8

For some students, such as Otto and Tony, the visual steepness related to gradients seems to be associated with some kind of diagonal movements. In 8.8d and 8.8f it seemed difficult for Tony to decompose the linear function into a vertical and a horizontal component, necessary to apply the prescribed *one-unit-right-a-up/down* strategy. Instead he counted diagonally by applying the diagonals in the squares of the background

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[14] Tony is not listed among the eight involved students in this study (table 6.1) due to insufficient empirical data from upper secondary.

grid as counting units. When he was moving the marker on the graph on the computer screen in this sense, this kept the value of the gradient of linear graphs of the type $y = x + b$ unchanged, throughout the whole process of Tony's diagonal counting.

In upper secondary interviews, most of the students in the general studies programme still applied the *one-unit-right-a-up/down* strategy when elaborating on gradients.

At the end of my observation period in upper secondary, general studies, the students were also asked to elaborate on the topic of differentiation. As illustrated in Chapter 7, it seemed that the main intention of introducing $\Delta y / \Delta x$ at this stage was to prepare the students for the topic of differentiation. Hence, one important reason for also focussing on differentiation in these interviews was to investigate if and how the students related this to their previous knowledge of gradients.

8.9a	Interviewer	Can you tell me a bit about what you have understood by the concept of differentiation, now in upper secondary?	Kan du si litt om hvordan du har forstått begrepet derivasjon, nå på videregående?
8.9b	Anna	I think differentiation was dealt with too fast, so I haven't really grasped it yet.	Derivasjon synes jeg at vi gikk igjennom for fort, så det har jeg ikke skjønt enda.
8.9c	Interviewer	So if you were given a task where you had to find the derivative of a function like this, x squared, plus three x, minus two [also writes this on a paper, simultaneously]. Would you have been able to do it?	Så hvis du hadde fått i oppgave og derivert en funksjon som så slik ut, x i andre, pluss tre x minus to [skriver også samtidig dette på et papir]. Hadde du visst hvordan du skulle ha gjort?
8.9d	Anna	[She quickly writes "$2x + 3$" on the paper].	[Hun skriver kjapt "$2x + 3$" på papiret].
8.9e	Interviewer	Not bad. How do we write the derivative?	Ikke verst. Hvordan skriver vi den deriverte?
8.9f	Anna	[Writes f'(x) on the paper].	[Skriver f'(x) på papiret].
8.9g	Interviewer	That was very quick. What does the derivative tell us?	Ja det var jo kjemperaskt. Hva forteller den deriverte?
8.9h	Anna	That's what I don't know. It was, sort of – we didn't have that many lessons…	Det er det jeg ikke vet. Det ble liksom – vi hadde ikke så mange timene…

Excerpt 8.9

This excerpt turned out to be rather typical for the general studies students' accounts of the derivative in the interviews. Technically, there were few problems, and the calculations went relatively fast. Still, none

of the students were able to elaborate on the derivative by, for example, relating it to gradients or growth rate.

8.2.3 Possible relations between teaching and learning (Research questions 3a and 3b)

The introductory phase related to gradients in the case of linear functions in lower secondary school was characterized by the *one-unit-right-a-up/down* strategy. This is apparent, as indicated, from several conversations and interviews with students. The relation between the examples and instructions provided by the teacher and the way that the students apply this strategy in solving tasks can be illustrated in, for example, Excerpt 7.30 and 7.32. In 7.30 the teacher (Tim) introduced the strategy by taking the linear expression y = x + 1 as an example. By taking the point (0,1) as the starting point, he used chalk and drew a line with length one in the horizontal direction and (in this case) one unit in the vertical direction, and then the line met the graph. At the same time he asked his students "how much does the function increase when we go one to the side and one up?" Other examples with various gradients were also demonstrated in similar terms. In 7.32, Matt tried to apply the same strategy but failed as a result of starting the procedure at the origin instead of starting at a point on the corresponding graph. This exemplifies students' attempts to adapt the *one-unit-right-a-up/down* strategy as mediated through the teachers' explanations and instructions. At the same time it also shows that in some cases this does not become more than a procedure, algorithmic in nature which in turn could prevent some students from using more flexible methods.

The case of Otto and School A constituted an exception to this prevailing strategy, in terms of the teachers' attempt to connect gradients to loci, and further, to slopes measured in percent, like in road signs. In Excerpt 7.3 the teacher (Kim) alluded to the relations between the gradient in the linear expressions and slopes by asking "let's say this had been a hill, how many percent would this have been? [Points to the line y=1/2x]". Otto's reasoning in Excerpt 7.5 shows traces of a similar type of reasoning. This suggests that the teacher's strategy is adapted and applied also in this case.

For the two students Matt (Section 7.3) and Thea (Section 7.4) who attended the general studies programme, a sudden shift occurred when the teachers approached the topic of differentiation. In Matt's case this is shown in Excerpt 7.39, where the teacher (Henry) introduces the notions Δy and Δx to define gradients (of linear functions) in terms of $\Delta y/\Delta x$. In the case of Thea I did not observe a similar introduction, but Excerpt 7.56 shows the presence of similar notations. Due to the lack of activities involving applications of this new conception of gradients on linear functions, there is nothing in my empirical data to determine whether

students at this point would have applied this "new" way of conceiving gradients when working with similar tasks as in lower secondary. Still, based on the interview subsequent to my observations in upper secondary, none of students referred to this new definition of gradients. Thea still found the *one-unit-right-a-up/down* strategy to be the most convenient way to define the gradient when she was confronted with the linear expression $y = ax + b$. While this does not invalidate her possible understanding of the meaning of $\Delta y/\Delta x$, it still suggests that this is the most convenient way "at hand" for her to explain this. Hence, one could argue that the preceding and dominant mediation of gradients in terms of the *one-unit-right-a-up/down* strategy had influenced her choice of words.

8.3 The transition from lower secondary to upper secondary (research question 3c)

The transition from lower secondary to upper secondary school is a complex and wide-ranging issue and contains facets which can be approached from several perspectives. In the following I aim to discuss only aspects relevant to my first two research questions. This means that the discussion will focus directly on the teaching and learning of functions, gradients and differentiation. More general aspects, such as social issues, students' and teachers' overall impressions, students' and teachers' beliefs and so forth, will not be objects of discussion. A detailed discussion concerning these aspects would be beyond the scope of this thesis.

Probably the most striking difference between the teaching of functions in upper secondary, general studies programme and lower secondary is the consistent use of "$y = \ldots$" in lower secondary and "$f(x) = \ldots$" in upper secondary to denote function expressions. Excerpt 5.37 shows how Matt's teacher, Henry, in School 3c introduces this concept:

| 8.10 | Henry (Teacher) | That way of writing is in a sense typical for equations. When we move over to functions, we replace that one [points to y, in the expression y = 2x + 2] and then one writes [writes f(x) = 2x + 1 at the blackboard]. The reason for this is to instill the concept of function, because the concept of function is that one in a sense has a variable x [points at the blackboard]. | Den skrivemåten der, den er på en måte typisk for likninger. Når man skal over på funksjoner, så bytter man ut den der [peker på y, i uttrykket y = 2x + 1]] og så skriver man [skriver f(x) = 2x + 1 på tavla]. Grunnen til at man gjør det, det er for å få inn dette begrepet funksjon, for begrepet funksjon er jo at man på en måte har en variabel x [peker på tavla]. Så putter man x inn i et uttrykk, så får man ut et resultat |

		So one puts the x into an expression, and then one gets a result [points to f(x)]. That means, in a sense, that the x which is the variable is treated inside the function, and something comes out.	[peker på f(x)]. Det betyr på en måte at den x'en som er variabelen den blir behandlet inni funksjonen, så kommer det ut et eller annet.

Excerpt 8.10

Henry emphasized that the f(x) notation is applied to "instill the concept of function". By pointing to the "x" and "f(x)" he visualizes that the (independent) variable x is contained in the symbolic notation f(x) in terms of being the "x" within the parenthesis. At lower secondary, Matt was not exposed to this notation. In the introductory lesson, involving linear functions, the concept of variables was not discussed either. The following excerpt from lower secondary displays Tim's introduction to the linear function "$y = x + 1$".

8.11	Tim (Teacher)	And we have two unknowns, we have one y and we have one x in our equation, here. An expression with letters.	Og vi har to ukjente, vi har en y og vi har en x i denne likningen vår her. Et bokstavuttrykk.

Excerpt 8.11

Tim's explanation contains the terms "equation" and "expression involving letters" but "functions" seems to be avoided, even though the word functions was mentioned twice before this example. The contrast between Henry's explanation and Tim's is also striking in that Henry states that the use of "y" is typical for equations while "f(x)" is typical for functions. Henry's statement implies that equations and functions are different mathematical objects. Conventional use of the terms "equations" and "function expressions" normally implies a slight difference in the significance of terms, since functions deal with the relation between variables and equations deal with determining unknown values. In addition, equations (in general) are not subject to the uniqueness property. Also at lower secondary, I discussed how Tim's statements could undermine the contextual nuances of these concepts by referring to equations while talking about linear functions (Excerpt 8.11). As the sign $y = x + 1$ could refer to both an equation and a function, both these concepts are of course legitimate, but if $y = x + 1$ is contextualized as a function, *x* and *y* automatically play the role as "variables", while they ought to be considered only as "unknown values" if the sign is referred to solely as an equation.

As regards the teaching of gradients, the *one-unit-right-a-up/down* strategy prevailed in all lower secondary schools with the exception of School A. I have discussed, in the case of Matt, how conception of gradients through this strategy alone might for some students simply become a procedure which lacked the desired flexibility. In upper

secondary, the teaching of this strategy seemed to continue until the introductory phase of differentiation, where $\Delta y/\Delta x$ was introduced. The case of Thea and the interview conducted at the very end of her first year in upper secondary showed that even after finishing the curriculum involving derivatives, she still preferred to explain gradients through the *one-unit-right-a-up/down* strategy.

Transition from lower secondary to upper secondary for Otto and Olga differs from that experienced by Matt and Thea as Otto and Olga attended the vocational studies in upper secondary. This entails that the topic of functions is no longer an explicit part of their curriculum, hence also neither gradients nor differentiation. Still, functions and gradients are implicitly included in the topic of proportional magnitudes. In Otto's case the examples provided by the teacher in lower secondary school as well as the student activities were characterized by *functions as loci* and *gradients measured in per cent.* A short reminder is provided below.

8.12a	Interviewer (me)	I wonder if you have found a gradient here [referring to the line y = 2x]?	Jeg lurer på om du har funnet noe stigningstall her [refererer til linja y = 2x] ?
8.12b	Otto	I have found that it is 200% on the upper one [y = 2x]. When it is one x [y = x] then it is 100% and on the upper when it is 2x we have to double. So there's nothing to calculate.	Jeg har funnet ut at det er 200% på den øverste [y = 2x]. Når det er en x [y = x] så er det 100% og på den øverste når det er 2x må vi bare doble. Så det er ikke noe å regne.

Excerpt 8.12

In this example from lower secondary (analyzed in the previous chapter) "$y = 2x$"serves as an example of a linear function. In upper secondary the example of "an hourly wage of 125 kroner per hour", and the corresponding expression "$y = 125x$" served to illustrate an example of proportional magnitudes and this teaching sequence followed:

8.13a	Bernt	What happens if we now divide by x on each side? [No response] Then the x vanishes [illustrates this by removing x from the right side and putting it in the denominator below y on the left side and y/x = 125 is now illustrated on the blackboard].	Hva skjer om vi nå deler med x på begge sider? [Ingen respons] Da forsvinner x'en [viser dette ved å fjerne x fra høyre side og setter den i nevneren under y på høyre side og det står nå y/x = 125 på tavla].
8.13b	Bernt	That is the rule of proportional magnitudes. If you have that y divided by x is one number, then there are	Det der er regelen for proporsjonale størrelser. Hvis du har at y delt på x er et tall, så er det proporsjonale størrelser.

		proportional magnitudes. [This is exemplified by 10 working hours and earning 1250 kroner]	[Dette eksemplifiseres så ved 10 arbeidstimer og 1250 opptjente kroner]
8.13c	Bernt	If there are proportional magnitudes, then y divided by x, or m divided by n, or q divided by r, is a constant number.	Er det proporsjonale størrelser så er alltid y delt på x, eventuelt m delt på n eller q delt på r, et konstant tall.

Excerpt 8.13

The excerpt above illustrates how proportional magnitudes were dealt with in Otto's upper secondary class. It is clear that x represents the number of hours, 125 is the hourly wage and y represents the total income. In the previous example, from lower secondary, the graphical representation itself (the line) represents a physical road and x and y represents the lengths of respectively the horizontal and vertical component. The different contexts in these two examples and the methods applied in the corresponding activities differ considerably, and in the interview Otto reflections indicate that he does not see these two examples as related at all. In general its seems like even though he understood the lower secondary approaches at the time, he felt it was very hard for him to draw on any of these in upper secondary as the mathematical content was usually presented in a different way.

9 Summary and conclusions

In this final chapter I summarize the analyses and findings from the two previous chapters. I discuss and compare my findings with research dealing with similar issues. Finally, in some concluding remarks, I point to some possible consequences of the results and suggest some possible implications.

To a great extent, the transition between lower secondary and upper secondary pervades this research as I have followed a group of eight students from their last year at lower secondary (10th grade) to their first year at upper secondary (11th grade). This transition also involved a shift of institution for each of the students. As outlined in the literature overview Gueudet (2008) identifies different aspects and perspectives for investigating educational transition issues, and various approaches are discussed in Section 5.1.2.

My focus in this study has been on the mathematical content in terms of the teaching and learning of functions, gradients, differentiation and proportional magnitudes. The topic of functions has served as a boundary object (Akkerman & Bakker, 2011; Star & Griesemer, 1989; Wenger, 2000) between lower and upper secondary schools, as separate and different institutions (Section 4.5). The four parts of my first research question focus on students' reasoning at both lower and upper secondary school and the four parts of the second research question focus on the teaching. The first two parts of my third research question deal with the relation between teaching and learning at both lower and upper secondary school. The third and last part of this research question concerns the comparison of teaching and learning aspects at the two phases of schooling, by focusing on the mathematical content (see Section 1.4 for a complete presentation of my research questions). In the following I will provide an overview of my findings. I start with an overview based on the different topics involved in the research questions.

9.1 Summary of the topics

In Chapters 7 and 8, I have illuminated and discussed findings related to my research questions. Through a detailed analysis of four cases (Chapter 7) I have highlighted some aspects related to the teaching and learning of the function concept, gradients, proportional magnitudes and differentiation. Both students attending the vocational studies and students attending the general studies were deliberately chosen in order to grasp the complexity and diversity which characterise the transition from lower to upper secondary schools in Norway.

The analyses in Chapter 7 were further elaborated and more widely discussed in Chapter 8. As my research questions naturally address four mathematical topics, I find it convenient to organise the summary by focusing on each of these.

1) The concept of functions
2) Gradients
3) Proportional magnitudes
4) Differentiation

9.1.1 The concept of functions

Chapters 7 and 8 suggest that there were differences related to the teaching of functions in the lower secondary schools involved. These differences primarily related to types of functions, and the use of symbolism (for example y=… versus f(x)=…) . On the other hand a proper treatment of the very definition of functions was absent in both lower and upper secondary. Extensive use of representations emphasizing especially graphs, expressions and value tables was apparent in most schools. At the same time, deficient and sometimes absent explicit treatment of the function concept and aspects that had to do with its definition seemed common to most of the teaching sequences observed.

For example in School C, there were no traces of explicit discussion concerning the function concept during my period of observation in the classroom. The introductory lesson seemed to focus on function expressions, which to a certain degree seemed familiar to the students from prior work. In the lessons which I observed at this school, functions were almost solely discussed and demonstrated through representations. That being said, one of the interactive tasks, the function machine in Figure 7.6, implicitly dealt with the uniqueness property.

In School D the uniqueness property of functions was implicitly dealt with through the use of two different examples of function machines. Except for the question posed by the teacher in School B ("which of these are not functions?"), aspects of the uniqueness property were not explicitly discussed in mathematics classrooms. What also seemed common to these cases was the absence of the mathematical concepts independent and dependent variables. Not only were these absent in the dialogues that took place in the classroom, but they were also missing in the definition and treatment of functions in the textbook, as referred to in Chapter 2.

In upper secondary the f(x) notation was introduced by means of various justifications and arguments by the various teachers. Questions from students like "do we have to use f(x), can't we just use y if we want to?" and "why do we have to write f(x)?" were observed, for example, in School 4. In school 3c, the Excerpt 7.36 shows that the teacher Henry

approached the introduction of f(x) by stating "the x, which is the variable is dealt with in the function, and then something comes out". The uniqueness property of functions did not seem to be Henry's main concern and the f(x) notation as a more explicit way of displaying the relationship between the independent and the dependent variable was not explicitly present in his explanation. Students' difficulties related to this were also observed in tasks where values of x and f(x) were represented in pair of coordinates of the form (x, f(x)).

My impression was that the teaching of functions at upper secondary, general studies and lower secondary primarily differed in terms of the examples applied in the different representation forms. For example, it seemed common that the type of functions expanded from including only linear functions in lower secondary to involve polynomial functions in upper secondary. There are also certain shared shifts in notations and mathematical symbols, such as f(x) instead of y. Even if the use of mathematical concepts (such as the uniqueness property, independent and dependent variables) seemed rather arbitrary both in lower and upper secondary classrooms, the concept of variables is more extensively treated in the upper secondary textbooks. On the basis of the interviews and tasks it is hard to trace or establish any development related to students' conceptions of the function concept.

In Chapter 5, I have presented literature and studies which point to the many challenges related to teaching and learning functions. For example, one might discuss the details related to Dreyfus and Eisenberg's (1982) "The function block" (Chapter 5, Figure 5.1) but it illustrates that teaching and learning functions is a broad task, calling for sophisticated, conceptual models. Sajka (2003) points to some of the "intrinsic ambiguities of the mathematical notation" (p. 246) for example that "f(x) can represent both the name of a function and the value of the function f" (p. 230). The most striking relevance of these "intrinsic ambiguities" might be tied to the way functions were dealt with in my observations in upper secondary school, where f(x) was introduced mainly as a renaming of y which had been used previously. At first glance and for some students this shift did not seem to entail more than a shift of conventional notation, illustrated by students asking "do we *have* to write f(x) or could we just continue to write y?" It is my view that the f(x) notation can offer several advantages related to students' conception of functions, if it is used in teaching. For example, as Sajka (2003) points out, the f(x) notation allows us to write pairs of coordinates on the form (x, f(x)), so for example (2, f(2)) becomes one way of expressing a pair of coordinates if the exact value of the corresponding dependent variable is unknown. Hence, Henry's explanations and illustrations of the relation between dependent and independent variables could have been

developed by encouraging student participation in activities involving more examples like these.

One aspect related to independent and dependent variables and the observed use of variants of function machines should be pointed out. As mentioned previously, function machines were observed in both lower and upper secondary school. One of Blomhøj's (1997) conclusions from a study involving Danish students in ninth grade, was that some students tend to "see the expression y = x + 5 as a recipe of a function machine, which changes the numbers put into the machine" (Blomhøj, 1997, p. 24, my translation). This suggests that an extensive and uncritical use of function machines might lead to the misconception that the independent variable is transformed or *changed into* the dependent variable.

In turn, the considerations above also relate to mathematical representations in general. As Duval (1999) points out, the only way to access mathematical objects is through their representation forms. Based on student interviews summarized in Table 8.2 (Chapter 8), it can be claimed that the qualitative difference between ontology and epistemology related to the function concept did not seem to be a trivial issue for the students. The interviews included few attempts to separate functions as mathematical objects with certain definitions and relations to other concepts on one hand, and the different representations of these objects on the other. Instead, Table 8.2 shows that equivalences are drawn between definitions and representations. These equivalences may also be nourished by the way teachers dealt with the function concept in the classroom. Similar relations have been identified in the studies of Kaldrimidou and Moroglou (2009) who conclude that "conceptions appear to be influenced by the representational context" (p. 271). Font et al. (2010) also identify some crucial contributions on this issue. In their studies of metaphors and Cartesian graphs, they identified different types of metaphors used by teachers in the mathematics classroom. The "object metaphor" is particularly interesting here. Object metaphors are "object image schema in mathematics" (p. 138) which in turn suggest that graphs are physical manifestations of the objects (functions). Utterings like "what does the function look like?" and "draw the function" are examples of such object metaphors. These could enforce an understanding that mathematical objects (like functions) are equivalent and on the same ontological level as their representations (for example graphs). But these representation forms are all mediating means, which belong to the realm of epistemology, as they play the role of making the function concept conceivable and accessible for further treatment.

Based on my observations, I find it reasonable to argue for certain links between conceptual understanding of functions as mathematical objects on one hand, and the understanding of the different

representation forms on the other. This also, in my view, entails a hierarchy of representation forms as being subordinated to the concept of function as a mathematical object. I find the importance of a hierarchy of mathematical concepts to be supported by, for example, Duval (1999). He separates mathematical representations into two categories: transformations within the same "register" (for example different algebraic expressions) and transformations between different registers (for example from algebraic expression to graph). Duval (1999) claims that "only students who can perform register change do not confuse a mathematical object with its representation" (p. 9).

In their study, Breidenbach et al. (1992) asked students "what is a function?" The students' answers were categorized in four groups "prefunction", "action", "process" and "unknown". As accounted for, I posed that very same question to the students in both lower and upper secondary school, as a part of the individual interviews. Although my categories, as explained in Chapter 6, emerged from a holistic analysis of my empirical data, and through a different theoretical perspective, our findings share important similarities. As many as 40 % of the students studied in Breidenbach et al. (1992) fell into the "prefunctions" category, where "prefunctions" were defined as "students do not have very much of a function concept at all" (p. 252). 24 % fell into the "action" category i.e. those with "responses that emphasized the act of substituting numbers for variables and calculating to get a number, but did not refer to any overall process" (p. 252).

Finally, an interesting remark arises from the different representations of functions observable in the case of Otto, in School A. As I have mentioned earlier this was a Waldorf School, and the approach to the concept of functions here through loci differed from those I observed in the other schools. These loci resulted in different "paths", visually identical to corresponding graphical representations of linear functions, quadratic functions (parabolas) and rational functions (hyperbolas). Students constructed these paths based on the characteristics of the loci. Loci in this sense differ radically from the traditional plotting of points and the drawing of corresponding graphs. As drawn by the students, these loci do not have any obvious parallels to independent and dependent variables, or to variables at all. Loci, represented by paths drawn in the coordinate system relate to the context in a very different, and one might say more "physical" sense than, say, the linear graph of the general expression $y = ax + b$ for some given values of a and b. Janvier (1978) representations as I see it fail to grasp locus as a representation form as observed at School A. This could call for an interesting expansion of the four representation forms suggested by Janvier, with a fifth representation (locus). One might argue that loci

could be seen as a kind of "situation", but situations do not usually capture the element of figurative construction baked into the concept of loci. Even though I find the consequences and the elaborations of this possible expansion to be outside the limitations of this thesis, I find it worth mentioning as a potential for prospective investigations and elaborations.

To summarize, several aspects of my findings related to functions could be related to other studies and similar findings. What might stand out as unique to this research is the representation form *functions-as-loci*, as mediated in School A. This is not among the representations found in any of the other schools, and neither to be found in the Janvier table. The rather sudden introduction of the notion $f(x)$ at upper secondary should also be pointed out, and that none of the students expressed the mathematical potential of applying this notion instead of just writing y as they were used to from lower secondary. From Table 8.2 (Chapter 8) one notices that even though students use different phrasings and examples at lower and upper secondary, their utterings show no immediate signs of a conceptual development when it comes to how they conceive of functions as mathematical objects.

9.1.2 Gradients

The prevailing approach to the concept of gradients in lower secondary seemed to be the *one-unit-right-a-up/down* strategy. Excerpt 7.32 (Chapter 7) demonstrates an example where Matt always seemed to interpret the origin as being the starting point of the procedure. In Matt's case, gradients were treated solely like a procedure so when he got wrong answer he seemed unable to adjust his methods. Kent, who applied this strategy more correctly (Excerpt 8.7, Chapter 8), was also able to apply the method to fractions by turning the fraction into a decimal number, which in turn provided a basis for his vertical counting manoeuvre.

Still I would claim that this strategy has limitations, even in Kent's case. In addition to the *one-unit-right-a-up/down* strategy, a more flexible approach to the concept of gradients could have been mediated. This could have been carried out by introducing an additional approach, for example in terms of "height divided by length", ("height" and "length" being the perpendicular sides in the triangles in Figure 8.4, Chapter 8). Then Matt could have been provided with an opportunity to test his answers, and Kent would have been able to handle fractions which are not easily converted into a decimal number.

Even in this case it is worth noticing School A, which constituted a counter example in the teaching of gradients. In the case of Otto (as illustrated through Excerpt 7.2 and 7.3, Chapter 7) gradients were related to slopes measured in percent, by using road signs as a reference context.

Almost as in the case of loci, the graphical representations of this "steepness" understood in this sense, literally became a physical drawing (profile) of the road itself. Even though Otto had some difficulties explaining why "percent" entered the picture, in terms of explaining what was the percentage of what, the examples provided by the teachers suggested that the percentage should be calculated on the basis of the ratio height (Δy) divided by length (Δx). In the excerpts presented in 7.3 the corresponding gradient was found by dividing that percentage by 100.

One might regard this "road sign approach" as an attempt to build on students' prior knowledge and experiences related to slopes. Walter and Gerson (2007) points out that conceptions of slopes are intuitively present before the topic of functions, as a part of students' vocabulary. Examples of this might be the use of words like "steep" or "slanty". A certain use of such concepts was present in most schools, in the classroom dialogues in the introductory phase of the concept of gradients. However, with the exception of School A, this consistently ended in a rather inflexible application of the *one-unit-right-a-up/down* strategy. The examples of Matt and Kent illustrate yet another example. Zaslavsky, Sela and Leron (2002) investigate students' reasoning related to gradients when the scale is changed and their findings suggest that students' strategies are "clogged by automatism" (p. 138). This term stems from Freudenthal, and points to the "blind" use of certain strategies without asking how and why these strategies work. The parallel to Matt and Kent's use of the "one-unit-right-a-up/down" strategy can be drawn as also in these cases this method lacks the required flexibility and proper evaluation of its suitability.

As an alternative approach to gradients Walter and Gerson (2007) suggest a model based on additive structures (Chapter 5, Figure 5.2). One obvious advantage of such an approach is that it applies to several representation forms, for example value tables and graphs. Based on my own reflections and considerations, I would suggest yet another approach which I found to be missing during my observations. At an early stage, already in lower secondary, I suggest that the *one-unit-right-a-up/down* strategy could be complemented by *height-divided-by-length* (movement in y-direction divided by movement in the x-direction). This could be accomplished by expanding the triangles used to illustrate the *one-unit-right-a-up/down* strategy, so that the baseline no longer *had* to be one. At the same time as such an approach could offer more flexible methods, it also prepares the ground for the "$\Delta y / \Delta x$"-approach offered in upper secondary, general studies.

Summarized, what stood out as the most striking resemblance for all my observations at lower secondary (School A being the only exception)

were my findings related to the extensive use of the *one-unit-right-a-up/down* strategy as the only approach to gradients. This was also the case at upper secondary, prior to the introduction of differentiation at the general study programme.

9.1.3 Proportional magnitudes

The work related to proportional magnitudes in upper secondary, vocational programmes, did not display any noticeable diversity. The case of Olga (Excerpt 7.23, Chapter 7) shows how certain links to the topic of functions were created, but as in the case of Otto and Edna, no connection between the proportionality constant and the gradient was made explicit in teaching, during my observations. Neither was this the case in the textbooks which were used. As mentioned in Chapter 8, this was the case not only in upper secondary, but also in the various lower secondary schools. Even though proportional magnitudes were treated in the same chapter as functions, my impression from the teaching and the actual textbook section was that proportional magnitudes were more or less dealt with as an independent topic. There were no explicitly expressed links between a series of concepts which could have been connected. Examples of potential links are proportionality constants and gradients on one hand, and proportional magnitudes and linear functions on the other. Concerning the latter, the link to linear functions was not explicitly pointed out even when graphical representations of proportional magnitudes were applied.

My empirical data on proportionality from lower secondary (primarily textbooks) and upper secondary (observations in the case of Otto and Edna) mainly reveal two common methods/procedures for checking if pairs of magnitudes are proportional. The first method consists of dividing the corresponding magnitudes and checking if the same number appears for each pair of magnitudes (the proportionality constant). The second method is to check if the graph through the plotted points is linear and if it intersects the origin. Even though teaching sequences (as in the case of Edna) emphasise practical implications of proportional magnitudes like "doubling the number of apples means doubling the price", tasks and activities are mainly reduced to checking methods and procedures. This is consistent with Modestou and Gagatsis' (2010) study, focusing on students from grade seven to nine. They conclude that the topic of proportional magnitudes is dominated by "routines and automatic procedures" (p. 51) and that this domination does "not represent pupils' real abilities in solving proportional tasks" (p. 51).

Recapitulated, practical examples of proportional magnitudes were richly provided at both lower secondary and upper secondary school (vocational studies). However the potential of explicitly creating links

between this topic and the topic of (linear) functions was not visible neither in teaching nor in textbooks. It should be underlined that functions is not a separate topic at the vocational studies at upper secondary, like being the case at lower secondary. In addition, the tasks were dominated by examples containing different numbers where the students were to test if these were proportional or not. This required only to check if they got the same ratio for each corresponding pair of numbers.

9.1.4 Differentiation

I observed that the continued use and development of the *one-unit-right-a-up/down* strategy seemed to be common in upper secondary, general studies. The distinctive 'shift' in terms of introducing $\Delta y/\Delta x$ was done by the teachers late in the process, only few lessons prior to the topic of differentiation.
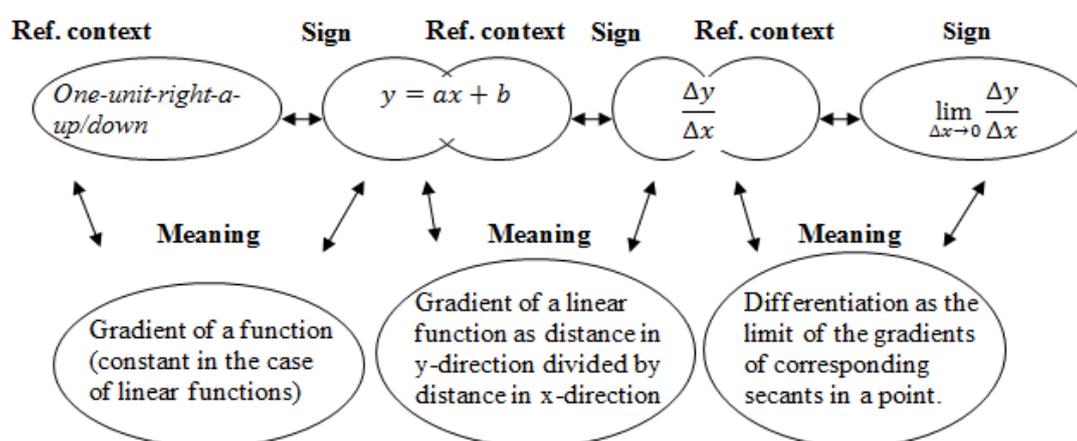

Figure 9.1. Semiotic chain, describing the teaching of gradients at upper secondary

I also observed that the intended mediated meaning related to the transition from the "gradient" to the notation "$\Delta y/\Delta x$" constitutes a possible source of enriching the students' understanding of gradients. Still, on the basis of my observations and interviews, it seems that this potential was strongly limited by the time spent on this transition, as this was dealt with only in one or at most two lessons.

In the interviews at the end of my period of observations, neither Matt nor Kent had revised their explanation of defining the gradient in terms of *one-step-right-a-up/down*. It is also important to remark that neither of them (nor any of the four students from the general studies programme, 1T version) were able to account for the theoretical foundations of differentiation in terms of explicitly relating differentiation to the concept of gradients. There may, of course, be several reasons for this. In Matt's class it was obvious that his group had not even been provided with the possibility, as Henry stated that he

regarded this to be too difficult for these students. Other reasons might be that the students noticed that the tasks/activities and the design of mathematical tests, which aimed to measure this competence, were almost totally omitted. Students' "tactical learning" , the learning needed to succeed in terms of getting high marks, especially related to learning just the rules of differentiation are accounted for also in other studies such as Guzmán, Hodgson, Robert and Villani (1998). But my main hypothesis is that the students' difficulties with the concept of differentiation are to a certain degree caused by the lack of a more rich and flexible concept of gradients. The rather stereotypic use of *one-unit-right-a-up/down* in lower secondary seemed to be rooted in the students' way of conceiving of the concept of gradients, even subsequent to the work done on differentiation. The transition from the *one-unit-right-a-up/down* strategy to the relatively abstract notions of $\Delta y / \Delta x$, for example by regarding gradients as "height divided by length" did not seem to be prioritized either in terms of teaching, tasks or tests.

The focus on rules and procedures related to the topic of differentiation is known from other studies as well. Orton's (1983) study of in total 110 students, 60 in the age range 16-18 at four schools and 50 in the age range 18-22 who were training to become teachers, concluded that for the whole group of students "the symbols of differentiation and the approach to differentiation were badly understood" (p. 244). Fully in line with the findings in my study, Orton (1983) points out that the reasons for this seemed to be rooted in the introduction of differentiation as a rule without any proper attempt to reveal the reasons for and justifications for the procedure. Hähkiöniemi (2008) and Bardelle (2009) are examples of other studies which produced similar conclusions.

Summarized, students' treatment of differentiation as a topic were characterized by carrying out procedures and differentiation rules, as found in other studies. What might be of some concern, related to my study is that an attempt of providing the students with a more conceptual understanding consciously was omitted by the teacher in the group consisting of low-performing students at School 3c. In this group of students the teacher advocated for procedural approaches to be sufficient.

9.2 Transition issues - summary

The summary provided in 9.1 addresses topics all of which could be conceived of as boundary objects (Star & Griesemer, 1989, see also Section 4.6) between lower and upper secondary school. In Section 4.3 I presented and discussed three different aspects in line with sociomathematical norms and classroom mathematical practices: 1) Mathematical language, notations and symbols, 2) Mathematical

explanations and justifications and 3) Mathematical tasks. Table 9.2 gives an overview of the different topics at lower and upper secondary, related to mathematical content of my research questions. Further, it is an attempt to highlight what I find to be the most crucial points in the transition in terms of different approaches provided, and the thematic expansion offered in upper secondary.

Topic	Lower secondary	Upper secondary
Functions	Notation: $y =...$ Types: Linear functions	Notation: $f(x) =...$ Types: Polynomial functions Rational functions Exponential functions
Gradients	Average growth rate One-unit-right-a-up/down	Average growth rate Instantaneous growth rate One-unit-right-a-up/down $\Delta y/\Delta x$
Proportionality	No noticeable differences in content nor presentation	
Differentiation	None	Related to gradients ($\Delta y/\Delta x$) and instantaneous growth rate. Characterized by rules and procedures

Table 9.2. Overview of the major differences observed in lower and upper secondary school.

9.2.1 Mathematical language, notations and symbols

In lower secondary, mainly linear functions were discussed. All the examples provided by both textbooks and teachers denoted the function expression by $y =...$ At upper secondary, general studies, polynomial, rational and exponential functions were introduced and functions were denoted as $f(x)...$ In Section 8.1 and 9.1.1 I discussed some aspects related to this introduction, and how this was conceived by the students.

As accounted for in 8.2 and 9.1.2, in all upper secondary schools general studies courses included in my study gradients were treated in the same manner as in the lower secondary schools. That is, through the application of the *one-unit-right-a-up/down* strategy. However, at the initial phase of differentiation the focus shifted towards "change in the y-direction divided by change in the x-direction" and the $\Delta y/\Delta x$ approach. Further, the topic of differentiation entailed a number of new notations and mathematical concepts like limits and $\lim_{\Delta x \to 0} \Delta y/\Delta x$. Section 7.4.5 illustrates the case at School 4, where Δx was replaced by h during the teachers' explanations.

9.2.2 Mathematical explanations and justifications

In the case of functions at lower secondary I have presented various examples of function machines, used by the teachers as a way of introducing and presenting the concept of functions. For example at School D the car-painting example and boxes with different input and output numbers were used (Section 7.4.1, Excerpt 7.46). Function machines were also observed at upper secondary, general studies, at School 3c (Section 7.3.5, Excerpt 7.36). The function machine used at upper secondary School 3c differs from those at lower secondary in terms of only having the purpose of serving as an explanation related to the concept of variables. The function machines presented at School D to some extent entailed student activity since the students were asked to guess different output values. In the car-painting example, the students were challenged to predict the colours of the cars and in the examples with the boxes the students were challenged to seek for patterns as they were asked to suggest the output value of certain input values.

Gradients were presented and explained through variants of the *one-unit-right-a-up/down* strategy in every lower secondary school except for School A (the Waldorf School) where the teacher approached gradients and slopes through *gradient measured in percent* and road signs. The *one-unit-right-a-up/down* was also the prevailing approach in upper secondary, general studies, prior to the topic of differentiation. As a step towards approaching the topic of differentiation, examples involving growth rates were used both School 3b and School 4. In School 2b, an interactive, web-based example with the growth of a plant was applied (Section 8.2.1) while at School 4 population growth in the local government was discussed. In both these cases, the teachers started by discussing the average growth rate, and moved towards the instantaneous growth rate. In the case of School 2, the transition from average to instantaneous growth rate could easily be illustrated through moving two points on the graph. A line intersected the graph of the growth of the plant in two different points. This line represented the average growth rate. As one of the points was moved towards the other one until they coincided, the line became a tangent which illustrated the corresponding instantaneous growth rate. At School 4 the transition from average to instantaneous growth rates was carried out by the teacher using a transparency and a ruler.

Teaching sequences related to proportional magnitudes were observed both at lower and upper secondary school (Section 7.2, Excerpt 7.19, Section 7.1.5 and Section 7.2.3). The use of prices and costs was common to all these examples and in School 2a (Section 7.2.3) one observes how the relation between the number of krone ice-creams and total costs is illustrated like an additive structure like in Figure 5.2,

Section 5.1.1). In various terms, the teachers all explained that the ratio between two corresponding numbers had to be the same for each pair of corresponding sequences of numbers, and that this ratio is called the proportionality constant.

9.2.3 Mathematical tasks

Related to the topic of functions at lower secondary, tasks and activities were dominated by moving between different representation forms as found in the Janvier (1978) table. The most typical pattern of such movements is illustrated in Figure 8.1 (Section 8.1.1). From my observations and from the textbooks used it seems that moving from function expression to tables, from tables to graphs and from graphs to function expressions dominated at lower secondary. School A stood out as an exception, by illustrating *functions as loci,* a category and a representation form not found in the Janvier table. At School A, student activity basically consisted of drawing graphs in terms of loci based on constructed practical descriptions provided by the teacher. Example of this was different problems in terms of "How to move, such as…?". At School C, Section 7.3.2 shows that interactive web-based software was used by the students when working with linear functions. With the exception of School A, tasks from textbooks and handouts dominated the observed student activity both at lower and upper secondary school.

9.3 Implications

In this final section I will consider some possible implications of this study. Based on the analyses and summary, implications for teaching will be suggested, and I will focus on aspects relevant for both lower and upper secondary school. I will also briefly focus on implications for further research by pointing to some possible and potential research areas which might arise in the wake of this study.

9.3.1 Implications for teaching

One aspect that struck me while conducting this research was the immediate personal need to gain an overview of the curriculum for both of these levels of schooling. Obviously, the *intended*, National Curriculum (LK06) played an important role but also the *implemented* curriculum in terms of the actual teaching and the student activities in use. During this research it became more and more evident to me that students' education is a continuum, in which lower and upper secondary schools are influential parts. Teaching in upper secondary should therefore not be seen as independent of the teaching in lower secondary and vice versa. In this sense, I think every teacher should be encouraged to study the teaching of topics both *prior* and *subsequent* to the grades or institution where she/he teaches. This view is in line with what Ball, Thames and Phelps (2008) in their model denote as "horizon content

knowledge" in terms of promoting "awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403).

As regards functions, findings from this study show several challenging aspects which relate to both lower and upper secondary school. Among these are the understanding of the notions dependent and independent variables and the differences and nuances between functions as a mathematical object and their various representations. As confirmed in the study of Font et al. (2010), for example teachers' wording related to the "object metaphor" (Section 5.1.1 and 7.1.1) might unintentionally contribute to this confusion. By being aware of formulations and consciously adjusting these towards emphasizing the difference between the mathematical objects (functions) and their representation forms one might help students towards better conceptual understanding.

The interplay between the dependent and independent variables could be dealt with in more varied and practical terms. Young students could be offered different approaches to strengthen their conception of the independent variable as the variable which "they are free to choose" while the dependent variable depends on the chosen ones. Examples to illustrate this in the case of linear functions might be practical exercises like dropping balls from different heights (independent variable) and the measurements of the corresponding rebound heights (dependent variable).

One might also instigate a discussion about the use of notations, such as $y = \ldots$ in lower secondary versus $f(x) = \ldots$ in upper secondary. My impression is that the $f(x)$ notation is avoided in lower secondary, due to its apparent "complexity" even though one might argue that it could contribute to a more sophisticated understanding of the relations between the independent and dependent variables. The x in parenthesis in the notation f(x) more clearly illustrates that x is the independent variable, which together with the function is determining the value of the dependent variable f(x).

Another point raised in earlier discussions is related to gradients and the *one-unit-right-a-up/down* strategy which dominates the teaching in lower secondary and to some extent is maintained throughout the initial phase of the first year in upper secondary, general studies. I suggest the application of more flexible approaches, for example in terms of expanding the triangles (fig 8.4, Chapter 8) used as a support for illustrating gradients, so that the baseline of such triangles could also take on values different from one. In this way one could move towards the $\Delta y/\Delta x$ (the approach needed for differentiation) more smoothly, for example in terms of "height divided by length". On the other hand, the

students would also be provided with an alternative, and sometimes more suitable method for coping with fractional gradients (Nilsen, 2012).

Related to gradients and differentiation in upper secondary, general studies, are the concepts of average and instantaneous growth rate. Treatment of these concepts as a preparation for subsequent elaborations on derivatives was observed in both School 3c and School 4. In lower secondary, my general impression was that focusing only on linear functions entailed that the students got little experience with instantaneous growth rates in particular. In the lower secondary textbooks, few activities covered this concept even though it could easily have been used on any non-linear curve, for example in terms of drawing approximate tangents using a ruler. One possible effect of emphasizing this type of growth rate could be that the students would become to some extent familiar with this, prior to the formalization provided in upper secondary. These sophisticated notations and illustrations and new signs such as *lim* and Δ, are in themselves a challenge for many students. If this is combined with an insufficient understanding of underlying aspects such as instantaneous growth rates, some students might never achieve a conceptual understanding of differentiation. One final point I would like to make concerns organizing students in different groups based on presupposed or expected achievements and abilities. School 3c and the case of Matt constitute a relevant example in this study. The interviews and observations revealed that such organization affects teaching and the way mathematical topics are treated and mediated in the different groups. For example, in the interview the teacher Henry (School 3c) explicitly states that his students (who at the time were participating in the low-preforming group) only get to learn the differentiation rules because the justifications and the reasoning behind these rules were too difficult for them. This raises a series of issues and touches on a fundamental debate involving contrasting educational values, educational policies and perspectives and theories of teaching and learning. In my view, it is legitimate to ask whether these students, whose opportunities to learn are restricted to the application of rules, actually have learned anything about differentiation. If they have, building on Hiebert and Lefevre (1986), this would qualify only as some procedural knowledge. While I find discussions involving issues of principles related to segregating students based on individual performance to be very engaging, they are outside the limitations of this study. For further reading involving more fundamental aspects of this, see Botten, Daland and Dalvang (2008), who partly draw on Ollerton (2003).

9.3.2 Implications for further research

It is difficult to generalize from a small scale, qualitative study such as this. It might therefore be worthwhile looking at some of the findings from a more large scale perspective involving quantitative methods. I think the most prominent and intriguing aspects relevant for possible large-scale research are the teaching and learning of specific mathematical topics. Some of my findings involve possible obstacles to students' understanding of functions, gradients, proportional magnitudes and differentiation. A survey of learning and teaching aspects related to these topics from a quantitative perspective providing well-documented results, would provide the basis for a plan of action. Such plan of action, focusing on promoting students' learning could remove possible obstacles related to the transition from lower secondary to upper secondary mathematics, possibly turning them into affordances.

This study primarily focuses on the teaching and learning of mathematical topics and the way these are dealt with in lower and upper secondary schools. In a holistic picture, transition between institutions and students being exposed to different mathematical content, different teaching methods, different textbooks and differences in the environment in general, certainly also contain aspects which belong in the affective domain. During the interviews and conversations with students, conducted in both lower and upper secondary school, several aspects involving feelings, beliefs and individual opinions about the many sides of transition appeared. Also in conversations and interviews with the teachers various aspects of teachers' beliefs and practices were identified (Nilsen, 2009a, 2009b). To avoid treating these complex issues superficially and in order to maintain a consistent framework for the thesis, I have omitted these findings from the discussions presented here. Still, I believe that these issues belong in the broader picture and that they would be worthwhile studying at a later stage.

In trying to understand transition issues, teachers on both institutional levels play a major role. Conducting research on (or with) teachers, aiming to improve the transition from lower secondary to upper secondary school (or between educational institutions in general) emerges, therefore, as highly relevant. Possible relations between teachers' beliefs and educational background on the one hand and the actual teaching on the other is an example of something else which would be worth looking into further. Similarly, one could investigate and compare the nature and frequency of different teaching methods used by the mathematics teachers in lower and upper secondary school.

Several schools offer transition programmes where students in their last year in lower secondary visit upper secondary schools to gain some preparatory experience. The way this is accomplished varies

considerably between the schools, but in some cases, these arrangements, have resulted in various types of collaboration projects between lower and upper secondary teachers. Potential research focusing on the possible outcomes of such collaborative arrangements obviously relates to several aspects of the transition issue, including the teaching and learning of mathematics.

9.4 Closing remarks

This study provides insights into teaching and learning issues related to functions, gradients, differentiation and proportional magnitudes. By considering functions as a boundary object between lower and upper secondary school, the thesis offers an analysis of the situation in lower and upper secondary school. Challenges primarily related to the attained curriculum have been illuminated and discussed through the presentations and analyses in Chapter 7 and Chapter 8 and implications have been suggested in Chapter 9.

In terms of being a qualitative study, findings should not be conceived of as an attempt of painting a general picture of the actual transition. Instead it should be understood as a "detailed and in-depth description so that others can decide the extent to which findings…are generalizable to another situation" (Cohen et al., 2007, p. 137). Teacher educators, teachers, prospective teachers and policy makers all constitute possible target groups of readers who could benefit from studying the findings of my research.

10 References

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11 Appendices[15]

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[15] Appendices A, B, C and D are translated into English by the author.

Appendix A: Interview guide for students at lower secondary

1) What do you think of mathematics as a subject?

2) What do you think of functions as a topic?

3) Can you explain what is meant by a function in mathematics?
   If you were put to the task of writing just a few lines in an encyclopedia – how would you define functions?

4) Can you tell me as much as possible related to what you see here:
$$y = 2x - 3$$

   [If not mentioned by the student, ask specifically about gradients (the number 2) and the constant term (the number -3)]

5) Can you tell me about a specific episode, if any, where you really learned something about functions?

6) What mark are you likely to achieve in mathematics, do you think?

7) What thoughts and expectations do you have about related to mathematics teaching the following year, at upper secondary?

Appendix B: Interview guide for students at upper secondary

1) How do you experience mathematics teaching at upper secondary compared to lower secondary?

   The case of functions (Discuss the various aspects raised by the students)

2) How would you evaluate your understanding of functions (or proportional magnitudes if vocational studies) now at upper secondary compared to lower secondary?

   (Discuss various aspects raised by the students. Provide the student with copies of handwritten material and/or textbooks from lower secondary to refresh memories, if needed)

3) If you evaluate the textbooks applied at upper secondary and lower secondary, related to the topic of functions, what will you emphasize as the main similarities/differences? (Give the student some time to skim through the textbooks brought).

4) Can you explain what is meant by a function in mathematics?

   If you were put to the task of writing just a few lines in an encyclopedia – how would you define functions?

5) Can you tell me as much as possible related to what you see here:
$$y = 2x - 3$$

   [If not mentioned by the student, ask specifically about gradients (the number 2) and the constant term (the number -3)]

6) a) What mark did you achieve in mathematics at lower secondary?

   b) What mark are you likely to achieve in mathematics this semester (or year), do you think?

Additional aspects to discuss with the students at upper secondary, general studies:

   - The concept of variables
   - The uniqueness property
   - The notation $f(x)$ vs. y
   - Differentiation (provide some simple tasks)
   - Solving some suitable tasks from the textbook used

Additional aspects to discuss with the students at upper secondary, vocational studies:

   - Relevance to the actual study programme (for example carpenters, media & communication and so forth)
   - Reading and interpreting graphs
   - Proportional magnitudes (provide some simple tasks)
   - Solving some suitable tasks from the textbook used

Appendix C: Interview guide for teachers at lower secondary

1) Could you refer to some episodes from your teaching, related to functions as a topic, which you felt were successful?
   Why?

2) What do you think characterizes good teaching of mathematics?

3) What do you think are the main differences between mathematics teaching at lower secondary compared to upper secondary school?

4) Have you studied the National curriculum in mathematics (vocational and/or general studies) for upper secondary?
   -Why/why not?
   -If «partially» yes: To what extent and have you studied it, and why (if so) is this relevant for your teaching?

5) Related to teaching mathematics, how often are applying the following teaching methods:

   -Go through new content at the blackboard (or by the use of other resources)

   -Going through homework?

   -Students solving tasks individually
        -Textbook/ICT/other

   -ICT

   -Group work

   -Interdisciplinary projects

   -Excursions

   -Outdoor activities

-Other

6) What is your educational background?

7) Which remarks do you have (if any) regarding the National curriculum at lower secondary?
   -Level of difficulty
   -Relevance
   -Sequence
   -Volume
        -If too extensive: What would you reduce, and why?
        -If deficient: What would you add, and why?

Appendix D: Interview guide for teachers at upper secondary

1) Could you refer to some episodes from your teaching, related to functions as a topic (proportional magnitudes for teachers at vocational studies) which you felt were successful?
   Why?

2) What do you think characterizes good teaching of mathematics?

3) What do you think are the main differences between mathematics teaching at lower secondary compared to upper secondary school?

4) Have you studied the National curriculum in mathematics applying to lower secondary school?
   -Why/why not?
   -If «partially» yes: To what extent and have you studied it, and why (if so) is this relevant for your teaching.

5) Related to teaching mathematics, how often are applying the following teaching methods:

   -Go through new content at the blackboard (or by the use of other resources)

   -Going through homework?

   -Students solving tasks individually
        -Textbook/ICT/other

   -ICT

   -Group work

   -Interdisciplinary projects

   -Excursions

   -Outdoor activities

   -Other

6) What is your educational background?

7) Which remarks do you have (if any) regarding the National curriculum at upper secondary?
   -Level of difficulty
   -Relevance
   -Sequence
   -Volume
        -If too extensive: What would you reduce, and why?
        -If deficient: What would you add, and why?

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Anmodning om tillatelse til videoopptak i klassene

Høsten 2007 startet jeg som doktorgradsstipendiat ved Høgskolen i Sør-Trøndelag, avdeling for lærer- og tolkeutdanning. Prosjektet er en del av et større prosjekt "Teaching Better Mathematics" som ledes fra Universitetet i Agder. Et av hovedmålene med prosjektet er å bedre matematikkundervisningen i grunnskolen. I min avhandling ønsker jeg å se nærmere på overgangen fra grunnskole til videregående med fokus på undervisning og læring av matematikk. Forskningen baseres på et nært samarbeid med praksisfeltet blant annet bestående av noen utvalgte grunnskoleklasser på 10. trinn. Det er viktig at forskningen baseres på så godt dokumenterte data som mulig, og derfor ber jeg om tillatelse fra dere til å kunne benytte lyd og videoopptak i samtaler og intervjuer med elevene.

Opptakene vil kun bli hørt av meg, min veileder og eventuelt andre i forskningsøyemed. Materiale som skrives eller på annen måte presenteres for andre vil ikke være mulig å spore tilbake til enkeltindivider ettersom involverte personer vil bli anonymisert. Etter at den aktuelle studien er sluttført vil innsamlede data bli slettet.

Hvis noen vil vite mer om dette, eller hva det innsamlede materialet skal brukes til, så er det bare å ta kontakt med meg på telefon eller e-post (se øverst for detaljer).

Forutsetningen for tillatelsen er at alt innsamlet materiale blir behandlet med respekt og blir anonymisert så langt råd er, og at prosjektet ellers følger gjeldende retningslinjer for personvern. Prosjektet er også rapportert til Norsk Samfunnsvitenskapelig Datatjeneste (NSD). Det er naturligvis helt frivillig å delta og man kan til enhver tid trekke seg fra deltakelse uten å måtte oppgi noen grunn til det.

Jeg håper dere synes dette er interessant og viktig, og at dere er villige til å la deres barn være med på det. Jeg ber foreldre/foresatte om å fylle ut svarslippen på neste side om hvorvidt dere gir eller ikke gir tillatelse til videoopptak i klassen.

På forhånd takk!

Vennlig hilsen

Hans Kristian Nilsen

…	Pause, not exceeding 3 seconds
[silence]	Pause in at least 3 seconds
[Text in brackets]	Account of nonverbal action, comment on utterance or added words
[…]	Omitted utterances
[indecipherable]	One or more words omitted because not possible to decipher