AIP Physics of Fluids

# Effect of a finite external heat transfer coefficient on the Darcy-Bénard instability in a vertical porous cylinder

A. Barletta and L. Storesletten

Citation: Phys. Fluids 25, 044101 (2013); doi: 10.1063/1.4799253
View online: http://dx.doi.org/10.1063/1.4799253
View Table of Contents: http://pof.aip.org/resource/1/PHFLE6/v25/i4
Published by the AIP Publishing LLC.



PHYSICS OF FLUIDS **25**, 044101 (2013)

# Effect of a finite external heat transfer coefficient on the Darcy-Bénard instability in a vertical porous cylinder

A. Barletta[1,a)] and L. Storesletten[2,b)]
[1]*Department of Industrial Engineering, Alma Mater Studiorum Università di Bologna, Viale Risorgimento 2, 40136 Bologna, Italy*
[2]*Department of Mathematics, University of Agder, Postboks 422, 4604 Kristiansand, Norway*

(Received 25 January 2013; accepted 14 March 2013; published online 9 April 2013)

The onset of thermal convection in a vertical porous cylinder is studied by considering the heating from below and the cooling from above as caused by external forced convection processes. These processes are parametrised through a finite Biot number, and hence through third-kind, or Robin, temperature conditions imposed on the lower and upper boundaries of the cylinder. Both the horizontal plane boundaries and the cylindrical sidewall are assumed to be impermeable; the sidewall is modelled as a thermally insulated boundary. The linear stability analysis is carried out by studying separable normal modes, and the principle of exchange of stabilities is proved. It is shown that the Biot number does not affect the ordering of the instability modes that, when the radius-to-height aspect ratio increases, are displayed in sequence at the onset of convection. On the other hand, the Biot number plays a central role in determining the transition aspect ratios from one mode to its follower. In the limit of a vanishingly small Biot number, just the first (non-axisymmetric) mode is displayed at the onset of convection, for every value of the aspect ratio. © *2013 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4799253]

## I. INTRODUCTION

The studies on buoyancy-driven instability in a saturated porous medium heated from below are quite abundant in the literature of the last decades. Interest for these studies is motivated by the role that thermoconvective instability may play in many applications ranging from the analysis of geothermal systems to the strategies for inhibiting the dispersion of liquid pollutants in the soil. Most of the investigations dealing with the so-called Darcy-Bénard instability[1–3] and the many variants of this classical problem, based on different assignments of the boundary conditions and on the modelling of the momentum transfer in the porous medium,[3–6] are relative to plane layers as well as to rectangular cavities or channels.

Starting from the pioneering papers by Wooding,[7] Zebib,[8] and Bories and Deltour[9] some authors focused their research on the thermoconvective instability in a vertical porous cylinder. The elemental problem is based on the assumption that the horizontal boundaries are impermeable and isothermal, while the cylindrical sidewall is thermally insulated.[7–9] However, the same analysis has also been carried out by modelling the sidewall as perfectly conducting,[9,10] viz., by assuming that the temperature disturbance is subject to a Dirichlet condition, instead of a Neumann condition, at the vertical boundary wall. More recently, Nygård and Tyvand[11] studied the case where the sidewall is partially conducting and partially penetrative, so that third-kind boundary conditions for the temperature and for the radial velocity component were assumed. A variant of the problem investigated by Wooding,[7] Zebib,[8] and Bories and Deltour[9] is the study carried out by Wang,[12] where the lower horizontal wall is considered as isoflux (Neumann temperature condition), while

a)antonio.barletta@unibo.it
b)leiv.storesletten@uia.no

 

the upper wall is assumed to be isothermal. We mention that Wang[13] further extended this analysis, by considering also the case where the top boundary of the porous cylinder is permeable and at constant pressure. Another important variant of the stability analyses in a vertical porous cylinder is the case where there exists not only an external sidewall, but also an internal coaxial boundary wall. This case, where the porous medium is confined between two coaxial cylindrical walls, was analysed by Bau and Torrance[14] and, more recently, by Bringedal *et al.*[15] In particular, Bringedal *et al.*[15] compared the two cases where both sidewalls are either thermally insulated or perfectly conducting. By assuming a periodic change of the gravitational acceleration, Govender[16] proposed an interesting alternative approach to the linear stability analysis of the basic rest state in a vertical cylinder with adiabatic sidewall. The onset of the instability in a vertical porous cylinder saturated by a viscoelastic fluid was studied by Zhang *et al.*[17] including the investigation of the oscillatory modes of instability. The oscillatory instability has been studied also by Kuznetsov and Nield[18] with reference to the onset of the double-diffusive convection in a vertical porous cylinder. These authors considered a vertically heterogeneous medium and the presence of an imposed vertical through flow.

The possibility that the heating from below, or the cooling from above, is caused by a forced convection flow over the horizontal bounding wall has been regarded by Kubitschek and Weidman[19,20] as well as by Barletta and Storesletten.[21] These authors modelled the heating/cooling processes over the horizontal boundaries by means of third-kind boundary conditions, depending on the Biot number, for the temperature field. In particular, the study carried out by Kubitschek and Weidman[20] refers to a vertical porous cylinder, where the lower boundary is subject to a third-kind temperature condition, while the upper boundary is considered as isothermal (Dirichlet temperature condition). A problem similar to that investigated by Kubitschek and Weidman[20] will be studied in the present paper. More precisely, we will assume third-kind temperature conditions on both horizontal boundaries.

The analysis presented in our paper is dissimilar from that of Kubitschek and Weidman.[20] In fact, imposing Robin's conditions on both horizontal boundaries implies that, when the Biot number is very small, our present analysis will approach that of a cylinder with isoflux horizontal boundaries, while the analysis described in the paper by Kubitschek and Weidman[20] approaches that of a cylinder with a lower isoflux boundary and an upper isothermal boundary. The former case displays onset conditions of the instability at vanishingly small wave numbers, while the latter case is basically a variant of the Darcy-Bénard problem where the critical wave number is nonvanishing. This evidence implies that the behaviour of the two stability problems at small Biot numbers is deeply different.

Our study will be carried out by proving first that the principle of exchange of stabilities holds, and then by studying the neutral stability condition for the stationary modes. This is another very important difference between our analysis and the previous ones. Kubitschek and Weidman[20] limited their investigation to stationary disturbances, thus avoiding the proof of the principle of exchange of stabilities. The same choice was made by Kubitschek and Weidman[19] as well as by Barletta and Storesletten.[21] Thus, the approach followed in our present paper will be to consider the most general time-dependent and three-dimensional normal modes, and then proving that only the stationary modes are allowed at neutral stability. The normal modes of perturbation that are most dangerous at the onset of convection will be determined on varying the aspect ratio of the vertical cylinder, viz., the radius-to-height ratio.

## II. GOVERNING EQUATIONS

We aim to study the onset of convection in a fluid saturated porous medium contained in a vertical cylinder of height $\bar{H}$ and radius $\bar{a}$. Cylindrical coordinates $(\bar{r}, \theta, \bar{z})$ are employed, with the $\bar{z}$ axis in the vertical direction, as shown in Fig. 1. The components of the seepage velocity $\bar{\boldsymbol{u}}$ are denoted as $(\bar{u}, \bar{v}, \bar{w})$, respectively. The vertical cylinder wall, $\bar{r} = \bar{a}$, as well as the lower and upper surfaces, $\bar{z} = 0$ and $\bar{z} = \bar{H}$, respectively, are assumed to be impermeable. We assume that the effect of viscous dissipation is negligible and that the solid and fluid phases are in local thermal equilibrium. By employing the Oberbeck-Boussinesq approximation, we can express the local balance equations for mass, momentum and energy as

$$\bar{\nabla} \cdot \bar{\boldsymbol{u}} = 0, \tag{1a}$$

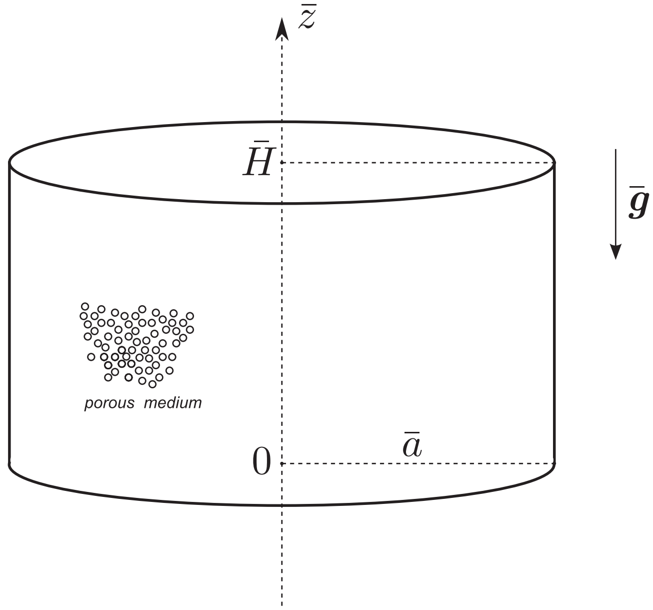


FIG. 1. A sketch of the fluid saturated porous cylinder.

$$\frac{\bar{\mu}}{\bar{K}}\bar{\boldsymbol{u}} = -\bar{\boldsymbol{\nabla}}\,\bar{p} + \bar{\rho}\bar{g}\bar{\beta}(\bar{T} - \bar{T}_0)\boldsymbol{e}_z, \tag{1b}$$

$$\sigma\,\frac{\partial \bar{T}}{\partial \bar{t}} + \bar{\boldsymbol{u}}\cdot\bar{\boldsymbol{\nabla}}\bar{T} = \bar{\alpha}\bar{\nabla}^2\bar{T}. \tag{1c}$$

Here, we are using an overline to indicate the dimensional quantities. In Eq. (1), $\bar{p}$ is the difference between the pressure and the hydrostatic pressure, $\bar{\mu}$ is the dynamic viscosity of the fluid, $\bar{K}$ is the permeability, $\bar{\rho}$ is the fluid density at the reference temperature $\bar{T}_0$, $\bar{\beta}$ is the thermal expansion coefficient, $\boldsymbol{e}_z$ is the unit vector along the vertical $\bar{z}$ axis, $\bar{\boldsymbol{g}} = -\bar{g}\,\boldsymbol{e}_z$ is the gravitational acceleration with modulus $\bar{g}$, $\bar{T}$ is the temperature field, $\bar{t}$ the time, $\bar{\alpha}$ is the average thermal diffusivity, and $\sigma$ is the ratio between the average volumetric heat capacity of the saturated porous medium and the volumetric heat capacity of the fluid.

The vertical cylinder wall, $\bar{r} = \bar{a}$, is assumed to be adiabatic, while heat transfer to external fluid environments occurs through the horizontal boundary walls. The lower boundary, $\bar{z} = 0$, is subject to forced convection heat transfer from an underlying fluid such that the temperature of the outer flow is $\bar{T}_0 + \Delta\bar{T}/2$, with a constant temperature difference $\Delta\bar{T} > 0$. On the other hand, the upper boundary, $\bar{z} = \bar{H}$, exchanges heat by convection with an external fluid such that the temperature of the outer flow is $\bar{T}_0 - \Delta\bar{T}/2$. The same external heat transfer coefficient, $\bar{h}$, is assumed at both the lower wall and the upper wall. Thus, the boundary conditions can be expressed as

$$\bar{u} = 0, \quad \frac{\partial \bar{T}}{\partial \bar{r}} = 0, \qquad \text{on} \quad \bar{r} = \bar{a}, \quad 0 < \bar{z} < \bar{H}, \tag{2a}$$

$$\bar{w} = 0, \quad \bar{k}\,\frac{\partial \bar{T}}{\partial \bar{z}} = \bar{h}\left(\bar{T} - \bar{T}_0 - \frac{\Delta\bar{T}}{2}\right), \qquad \text{on} \quad \bar{z} = 0, \quad \bar{r} < \bar{a}, \tag{2b}$$

$$\bar{w} = 0, \quad -\bar{k}\,\frac{\partial \bar{T}}{\partial \bar{z}} = \bar{h}\left(\bar{T} - \bar{T}_0 + \frac{\Delta\bar{T}}{2}\right), \quad \text{on} \quad \bar{z} = \bar{H}, \quad \bar{r} < \bar{a}, \tag{2c}$$

where $\bar{k}$ is the average thermal conductivity of the porous medium. The boundary conditions for the temperature field, Eqs. (2b) and (2c), are third-kind, or Robin, boundary conditions. These conditions, employed also by Barletta and Storesletten[21] express Newton's cooling law, namely, the boundary heat transfer from a wall to an external fluid environment.

## III. DIMENSIONLESS EQUATIONS

The governing equations and the boundary conditions can be reformulated in terms of dimensionless quantities (with no overline) defined as

$$(\bar{r}, \bar{z}) = (r, z)\bar{H}, \qquad (\bar{u}, \bar{v}, \bar{w}) = (u, v, w)\frac{\bar{\alpha}}{\bar{H}}, \qquad \bar{\nabla} = \frac{1}{\bar{H}}\nabla,$$
$$\bar{T} = \bar{T}_0 + T\,\Delta\bar{T}, \qquad \bar{p} = p\,\frac{\bar{\mu}\bar{\alpha}}{\bar{K}}, \qquad \bar{t} = t\,\frac{\sigma\bar{H}^2}{\bar{\alpha}}, \qquad \bar{a} = a\,\bar{H}. \tag{3}$$

Thus, Eqs. (1) can be rewritten in a dimensionless form as

$$\nabla \cdot \boldsymbol{u} = 0, \tag{4a}$$

$$\boldsymbol{u} = -\nabla p + R\,T\,\boldsymbol{e}_z, \tag{4b}$$

$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \nabla^2 T, \tag{4c}$$

while the boundary conditions (2) can be rewritten as

$$u = 0, \quad \frac{\partial T}{\partial r} = 0, \qquad \text{on} \quad r = a, \quad 0 < z < 1, \tag{5a}$$

$$w = 0, \quad \frac{\partial T}{\partial z} = B\left(T - \frac{1}{2}\right), \qquad \text{on} \quad z = 0, \quad r < a, \tag{5b}$$

$$w = 0, \quad \frac{\partial T}{\partial z} = -B\left(T + \frac{1}{2}\right), \qquad \text{on} \quad z = 1, \quad r < a. \tag{5c}$$

In Eqs. (4) and (5), the Darcy-Rayleigh number, $R$, and the Biot number, $B$, are given by

$$R = \frac{\bar{\rho}\bar{g}\bar{\beta}\bar{K}\bar{H}\Delta\bar{T}}{\bar{\mu}\bar{\alpha}}, \qquad B = \frac{\bar{h}\bar{H}}{\bar{k}}. \tag{6}$$

The special cases of isothermal horizontal boundaries and of adiabatic horizontal boundaries are represented by Eqs. (5) with the limits $B \to \infty$ and $B \to 0$, respectively.

## IV. BASIC SOLUTION

A stationary solution of Eqs. (4) and (5), corresponding to a zero velocity field, exists and is given by

$$u_b = 0, \qquad v_b = 0, \qquad w_b = 0, \qquad T_b = \frac{B}{B+2}\left(\frac{1}{2} - z\right),$$
$$p_b = p_0 + \frac{B\,R}{2(B+2)}\left(z - z^2\right), \tag{7}$$

where "$b$" stands for basic solution and $p_0$ is a constant. Since $p$ appears in the governing equations only through its derivatives, this field is determined only up to an arbitrary additive constant. Thus, in Eq. (7) we can choose $p_0$ arbitrarily.

## V. LINEAR DISTURBANCES

We perturb the basic solution, Eq. (7), with small-amplitude disturbances defined by

$$\boldsymbol{u} = \varepsilon\boldsymbol{U}, \qquad p = p_b + \varepsilon P, \qquad T = T_b + \varepsilon\Theta, \tag{8}$$

where $\varepsilon$ is a small perturbation parameter, while $\boldsymbol{U}$, $P$, and $\Theta$ are the dimensionless velocity, pressure, and temperature disturbances, respectively.

Following the usual linear analysis, we substitute Eq. (8) into Eqs. (4) and (5), and neglect terms of order $\varepsilon^2$. Thus we obtain the governing equations for the disturbances

$$\nabla \cdot \boldsymbol{U} = 0, \tag{9a}$$

$$\boldsymbol{U} = -\nabla P + R\,\Theta\,\boldsymbol{e}_z, \tag{9b}$$

$$\frac{\partial \Theta}{\partial t} - \frac{B}{B+2}\,W = \nabla^2 \Theta, \tag{9c}$$

subject to the boundary conditions

$$U = 0, \qquad \frac{\partial \Theta}{\partial r} = 0, \qquad \text{on} \quad r = a, \quad 0 < z < 1, \tag{10a}$$

$$W = 0, \qquad \frac{\partial \Theta}{\partial z} = B\,\Theta, \qquad \text{on} \quad z = 0, \quad r < a, \tag{10b}$$

$$W = 0, \qquad \frac{\partial \Theta}{\partial z} = -B\,\Theta, \qquad \text{on} \quad z = 1, \quad r < a. \tag{10c}$$

## VI. NORMAL MODES

We eliminate the pressure $P$ and the vertical velocity $W$ in Eqs. (9), so that we obtain the following equation:

$$\frac{\partial}{\partial t}\,\nabla^2 \Theta = \nabla^4 \Theta + \frac{B}{B+2}\,R\,\nabla_\perp^2 \Theta, \tag{11}$$

with the boundary conditions

$$\frac{\partial}{\partial r}\nabla^2 \Theta = 0, \quad \frac{\partial \Theta}{\partial r} = 0 \qquad \text{on} \quad r = a, \quad 0 < z < 1, \tag{12a}$$

$$\nabla^2 \Theta = \frac{\partial \Theta}{\partial t}, \quad \frac{\partial \Theta}{\partial z} = B\Theta \qquad \text{on} \quad z = 0, \quad r < a, \tag{12b}$$

$$\nabla^2 \Theta = \frac{\partial \Theta}{\partial t}, \quad \frac{\partial \Theta}{\partial z} = -B\Theta \qquad \text{on} \quad z = 1, \quad r < a. \tag{12c}$$

In the above, $\nabla^4 \equiv \nabla^2(\nabla^2)$, where $\nabla^2$ is the Laplacian operator in cylindrical coordinates and $\nabla_\perp^2 \equiv \nabla^2 - \partial^2/\partial z^2$ is the horizontal Laplacian operator.

The normal modes satisfying Eq. (11) and the sidewall boundary conditions are given by

$$\Theta(r, \theta, z, t) = J_n(\lambda r) \cos(n\theta)\, F(z)\, e^{\gamma t}, \qquad n = 0, 1, 2, \dots, \tag{13}$$

where $J_n$ is Bessel function of the first kind and order $n$, while $\lambda$ is a real parameter and $\gamma$ is, generally speaking, a complex parameter such that $\mathfrak{Re}(\gamma)$ yields the growth rate of the normal mode disturbance. Stability corresponds to $\mathfrak{Re}(\gamma) < 0$, while instability corresponds to $\mathfrak{Re}(\gamma) > 0$. The onset of the thermoconvective instability is defined by the neutral stability condition, $\mathfrak{Re}(\gamma) = 0$.

In order to satisfy Eq. (12a), $\lambda$ must be such that $\lambda\, a$ is a nonzero root of the transcendental equation

$$J_n'(\zeta) = 0, \qquad \zeta \equiv \lambda\, a. \tag{14}$$

The nonzero values of $\zeta$ that satisfy Eq. (14) form a strictly monotonically increasing sequence $\{\zeta_{n,\,m}\,|\, n = 0, 1, 2, 3, \dots;\, m = 1, 2, 3, \dots\}$, provided that the pairs $(n, m)$ are properly ordered. The first sixteen terms of this sequence are reported in Table I and the appropriate ordering of the pairs $(n, m)$ is shown. Therefore, each separable normal mode defined by Eq. (13) is labeled by the pair of integers $(n, m)$, where $n$ is the azimuthal mode number and $m$ is the radial mode number. The modes with $n = 0$ are two-dimensional, as $\Theta$ is independent of the azimuthal angle, while all modes with $n > 0$ are three-dimensional.

TABLE I. Roots of Eq. (14), $\zeta \equiv \lambda\, a$, in increasing order.

| $(n, m)$ | $\zeta_{n,\, m}$ |
|---|---|
| (1, 1) | 1.84118378134 |
| (2, 1) | 3.05423692823 |
| (0, 1) | 3.83170597021 |
| (3, 1) | 4.20118894121 |
| (4, 1) | 5.31755312608 |
| (1, 2) | 5.33144277353 |
| (5, 1) | 6.41561637570 |
| (2, 2) | 6.70613319416 |
| (0, 2) | 7.01558666982 |
| (6, 1) | 7.50126614468 |
| (3, 2) | 8.01523659838 |
| (1, 3) | 8.53631636635 |
| (7, 1) | 8.57783648971 |
| (4, 2) | 9.28239628524 |
| (8, 1) | 9.64742165200 |
| (2, 3) | 9.96946782309 |

Substitution of Eq. (13) into Eq. (11) yields a fourth-order ordinary differential equation, namely,

$$F'''' - \left(2\lambda^2 + \gamma\right) F'' + \lambda^2 \left(\lambda^2 + \gamma - Q^2\right) F = 0, \qquad (15)$$

where we introduced the dimensionless parameter, $Q^2$, with the meaning of a modified Darcy-Rayleigh number, $\mathcal{R}$,

$$Q^2 = \frac{B}{B+2}\, R, \qquad \mathcal{R} = Q^2. \qquad (16)$$

The boundary conditions are

$$\begin{aligned} &F''(0) - \left(\lambda^2 + \gamma\right) F(0) = 0, \qquad F'(0) - B\,F(0) = 0, \\ &F''(1) - \left(\lambda^2 + \gamma\right) F(1) = 0, \qquad F'(1) + B\,F(1) = 0. \end{aligned} \qquad (17)$$

### A. Exchange of stabilities

Equations (15) and (17) define an eigenvalue problem. An important feature is that solutions are allowed only if $\mathfrak{Im}(\gamma) = 0$. In other words, the principle of exchange of stabilities holds. The following proof is based on the integral method used for the first time, with reference to the classical Rayleigh-Bénard problem of a clear fluid, by Pellew and Southwell.[22]

We rewrite Eq. (15) as a system of two second order differential equations, by introducing

$$G \equiv F'' - \left(\lambda^2 + \gamma\right) F, \qquad (18)$$

namely,

$$\begin{cases} F'' - \left(\lambda^2 + \gamma\right) F - G = 0, \\ G'' - \lambda^2 G - \lambda^2 Q^2 F = 0. \end{cases} \qquad (19)$$

Equations (19), on account of Eqs. (17) and (18), are endowed with the boundary conditions

$$\begin{aligned} &G(0) = 0, \qquad F'(0) - B\,F(0) = 0, \\ &G(1) = 0, \qquad F'(1) + B\,F(1) = 0. \end{aligned} \qquad (20)$$

We multiply the first Eq. (19) by the complex conjugate of $F$, denoted by $F^*$, and, by taking into account Eq. (20), we integrate by parts with respect to $z$ over the interval $0 < z < 1$, so that we obtain

$$-B\left(|F(1)|^2 + |F(0)|^2\right) - \int_0^1 |F'|^2\, dz - \left(\lambda^2 + \gamma\right)\int_0^1 |F|^2\, dz = \int_0^1 F^* G\, dz. \tag{21}$$

We now multiply the second Eq. (19) by the complex conjugate of $G$, namely $G^*$, and carry out an integration by parts with respect to $z$ over the interval $0 < z < 1$, by taking into account Eq. (20). Thus, we may write

$$-\int_0^1 |G'|^2\, dz - \lambda^2 \int_0^1 |G|^2\, dz = \lambda^2 Q^2 \int_0^1 G^* F\, dz. \tag{22}$$

On account of Eq. (22), the integral of $G^*F$ is real valued and, hence, it coincides with the integral of $F^*G$. We can eliminate the latter integral in Eq. (21), by employing Eq. (22), so that we may write

$$-\int_0^1 |G'|^2\, dz - \lambda^2 \int_0^1 |G|^2\, dz + \lambda^2 Q^2 \left[ B\left(|F(1)|^2 + |F(0)|^2\right) + \int_0^1 |F'|^2\, dz + \left(\lambda^2 + \gamma\right)\int_0^1 |F|^2\, dz \right] = 0. \tag{23}$$

Equation (23) is satisfied if both the real part and the imaginary part of its left hand side vanish. In particular, the imaginary part vanishes if and only if

$$\mathfrak{Im}(\gamma) \int_0^1 |F|^2\, dz = 0. \tag{24}$$

This condition implies that either $\mathfrak{Im}(\gamma) = 0$ or $F$ is identically vanishing in the whole interval $0 < z < 1$. The second option is to be excluded since it would entail that the perturbation of the basic flow is zero. Therefore, we conclude that $\mathfrak{Im}(\gamma) = 0$, so that the exchange of stabilities holds.

### B. Analytical solution for neutral stability

We now focus our attention on the condition of neutral stability. The principle of exchange of stabilities, $\mathfrak{Im}(\gamma) = 0$, together with the definition of neutral stability, $\mathfrak{Re}(\gamma) = 0$, implies that hereafter we can set $\gamma = 0$ and thus investigate stationary normal modes, namely, those given by Eq. (13) with $\gamma = 0$.

The characteristic equation associated with the differential equation (15), with $\gamma = 0$, is expressed as

$$\eta^4 - 2\lambda^2\eta^2 + \lambda^2(\lambda^2 - Q^2) = 0. \tag{25}$$

Equation (25) has four real or imaginary roots, $\eta = \pm\, \eta_-$ and $\eta = \pm\, \eta_+$, where

$$\eta_-^2 = \lambda\,(\lambda - Q)\,, \qquad \eta_+^2 = \lambda\,(\lambda + Q)\,. \tag{26}$$

The general solution of Eq. (15) can thus be expressed as

$$F(z) = c_1\, e^{\eta_- z} + c_2\, e^{-\eta_- z} + c_3\, e^{\eta_+ z} + c_4\, e^{-\eta_+ z}. \tag{27}$$

Substitution of Eq. (27) into Eq. (17) yields an algebraic system of four linear and homogeneous equations,

$$\mathbb{M} \cdot \mathbb{c} = 0, \tag{28}$$

where

$$\mathbb{M} = \begin{bmatrix} \eta_-^2 - \lambda^2 & \eta_-^2 - \lambda^2 & \eta_+^2 - \lambda^2 & \eta_+^2 - \lambda^2 \\ \eta_- - B & -\eta_- - B & \eta_+ - B & -\eta_+ - B \\ (\eta_-^2 - \lambda^2)e^{\eta_-} & (\eta_-^2 - \lambda^2)e^{-\eta_-} & (\eta_+^2 - \lambda^2)e^{\eta_+} & (\eta_+^2 - \lambda^2)e^{-\eta_+} \\ (\eta_- + B)e^{\eta_-} & (-\eta_- + B)e^{-\eta_-} & (\eta_+ + B)e^{\eta_+} & (-\eta_+ + B)e^{-\eta_+} \end{bmatrix}, \tag{29}$$

and $\mathbb{c}$ is the column vector with components $(c_1, c_2, c_3, c_4)$. The linear homogeneous system Eq. (28) admits nontrivial solutions if and only if the determinant of $\mathbb{M}$ is zero. Therefore, the neutral stability condition is given by

$$\det \mathbb{M} = 0. \tag{30}$$

On account of Eq. (29), Eq. (30) can be expressed as

$$\left[\left(2B^2 + \lambda^2\right)\sinh\eta_+ + 2B\eta_+\cosh\eta_+\right]\sinh\eta_- \\ + \left(2B\eta_-\sinh\eta_+ + \eta_-\eta_+\cosh\eta_+\right)\cosh\eta_- - \eta_-\eta_+ = 0. \tag{31}$$

## VII. NEUTRAL STABILITY CURVES

Equation (31), through Eqs. (16) and (26), defines implicitly the functional relationship $R = R(\lambda, B)$, or $\mathcal{R} = \mathcal{R}(\lambda, B)$. By means of function $\mathcal{R}(\lambda, B)$, one may plot the neutral stability curves in the plane $(\lambda, \mathcal{R})$, for every prescribed value of $B$. The absolute minimum of $\mathcal{R}(\lambda, B)$, for a given value of $B$, yields the critical values, $(\lambda_c, \mathcal{R}_c)$, for the onset of the instability. An important point is that the geometry of the vertical sidewall does not have any influence on Eq. (31), on the function $\mathcal{R}(\lambda, B)$ and, hence, on the critical values, $(\lambda_c, \mathcal{R}_c)$. Thus, Eq. (31), function $\mathcal{R}(\lambda, B)$ and, the critical values $(\lambda_c, \mathcal{R}_c)$ for a given $B$ are perfectly identical to those obtained, for a rectangular box in Barletta and Storesletta,[21] so that we refer the reader to this previous study for any further detail. On the other hand, the geometry of the vertical sidewall plays an essential role when the allowed values of $\lambda$ have to be determined. In the present study, these values are $\lambda = \zeta/a$, where $\zeta$ is one of the roots of Eq. (14), namely, $\{\zeta_{n,m} \,|\, n = 0, 1, 2, 3, \ldots ; m = 1, 2, \ldots\}$, and the first sixteen terms are reported in Table I. Therefore, for a prescribed aspect ratio $a$, the functional relationship $\mathcal{R} = \mathcal{R}(\lambda, B)$, together with Eq. (14), yields the neutral stability function $\mathcal{R}(n, m, a, B)$. Instability occurs when $\mathcal{R} > \mathcal{R}(n, m, a, B)$, while linear stability corresponds to $\mathcal{R} < \mathcal{R}(n, m, a, B)$.

Plots of either $R(n, m, a, B)$ or $\mathcal{R}(n, m, a, B)$, for the most unstable modes $(n, m)$ of neutral stability are reported in Figs. 2 and 3, with different prescribed values of $B$. Function $R(n, m, a, B)$ is just $\mathcal{R}(n, m, a, B)$ multiplied by the factor $(B + 2)/B$, as it can be inferred from Eq. (16). Thus, plots of $\mathcal{R}(n, m, a, B)$ are preferred for small values of $B$ (Fig. 3) as $R(n, m, a, B)$ tends to infinity when $B \to 0$. This feature has been commented on by Barletta and Storesletten.[21] Tracing the limit of $R(n, m, a, B)$ when $B \to 0$ means that we investigate the special case where the horizontal boundaries, $z = 0, 1$, are thermally insulated. On the other hand, tracing the limit of $\mathcal{R}(n, m, a, B)$ when $B \to 0$ has the physical meaning of studying the special case where the horizontal boundaries are kept at a uniform heat flux, thus producing a vertically uniform temperature gradient in the basic state.[21] There is a good reason why we preferred to describe the condition of neutral stability by plotting $R(n, m, a, B)$ with larger values of $B$ (Fig. 2) and $\mathcal{R}(n, m, a, B)$ for smaller values of $B$ (Fig. 3). The Darcy-Rayleigh number $R$ is based on the temperature difference $\Delta\bar{T}$ between the external fluid environments. The modified Darcy-Rayleigh number $\mathcal{R}$ is based on the temperature difference $[B/(B + 2)]\Delta\bar{T}$ between the horizontal boundaries, $z = 0, 1$. As a consequence, chasing the neutral stability function $R(n, m, a, B)$ as $B$ decreases from infinity captures the stabilizing effect of a smaller and smaller Biot number (Fig. 2). Indeed, decreasing $B$ means, for a prescribed $\Delta\bar{T}$, decreasing the temperature difference between the horizontal boundaries, $z = 0, 1$.

In each frame of Fig. 2, the horizontal dotted lines denote the critical values $R_c$, evaluated as the minimum of function $R(\lambda, B)$, with $\lambda \in (0, +\infty)$, corresponding to the prescribed $B$. The values of $R_c$ have been computed and reported in a table contained in the paper by Barletta and Storesletten.[21] Similarly, in Fig. 3, the horizontal dotted lines correspond to the critical values of $\mathcal{R}$ for each assigned Biot number.

By increasing monotonically the aspect ratio $a$, the ordering of the $(n, m)$-modes that lead to the onset of convection is independent of the prescribed Biot number and coincides with the ordering of the sequence $\{\zeta_{n,m} \,|\, n = 0, 1, 2, 3, \ldots ; m = 1, 2, \ldots\}$ displayed, for the first sixteen terms, in

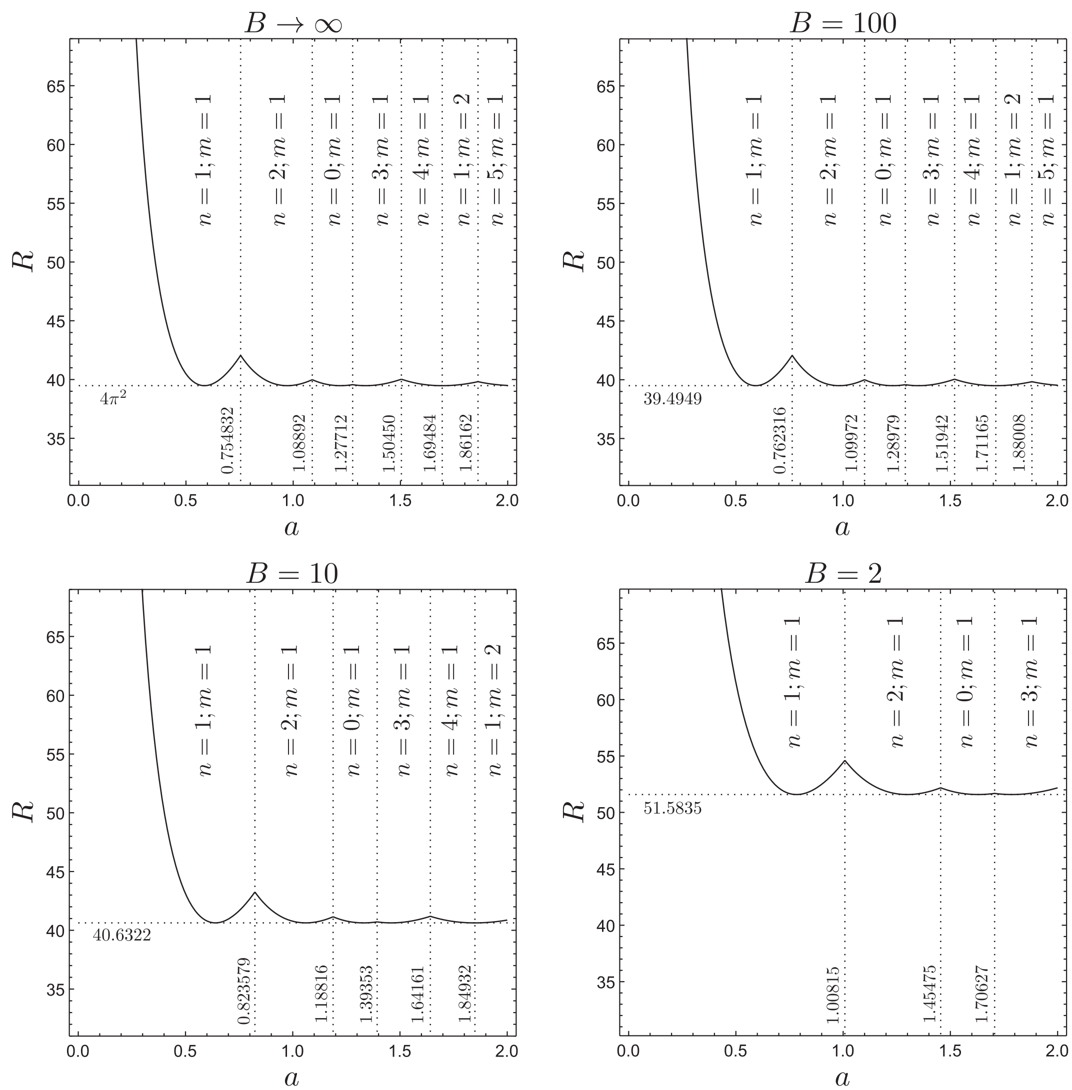


FIG. 2. Neutral stability: plots of $R$ versus $a$ for the most unstable modes, each frame is for a different value of $B$. The horizontal dotted line is for $R = R_c$; the vertical dotted lines bound the regions where each specified $(n, m)$-mode is preferred.

Table I. In fact, in Fig. 2, we report the plots of functions

$$R(n, m, a, B) = R\left(\frac{\zeta_{n,m}}{a}, B\right) \qquad n = 0, 1, 2, 3, \ldots;\ m = 1, 2, \ldots, \tag{32}$$

where $\zeta_{n,\,m}$ are the numerical values reported in Table I (first sixteen terms). The ordering of the functions $R(n, m, a, B)$ relies only on the independent variable $\zeta_{n,\,m}/a$ and, as a consequence, on the ordering of the sequence $\{\zeta_{n,\,m} \,|\, n = 0, 1, 2, 3, \ldots ; m = 1, 2, \ldots\}$. A similar reasoning can be drawn, with reference to Fig. 3, by considering functions $\mathcal{R}(n, m, a, B)$, instead of $R(n, m, a, B)$.

We can compare the top-left frame of Fig. 2 (the limiting case of isothermal horizontal boundaries, viz., $B \to \infty$) with the corresponding diagram reported in Zebib.[8] As it can be seen, the agreement is very good, but we mention that Zebib did not include the mode (5, 1) as the most unstable when $1.86162 < a < 2.08788$.[8]

For a given Biot number, the transition from any preferred $(n, m)$-mode to the next one is for a specific aspect ratio that depends on the value of $B$. For instance, Figs. 2 and 3 show that the most dangerous mode at low values of $a$ is $n = 1$, $m = 1$. This mode is superseded by the mode $n = 2$, $m = 1$ when $a = 0.754832$, if we consider the limit $B \to \infty$. If $B$ is finite and gradually decreases

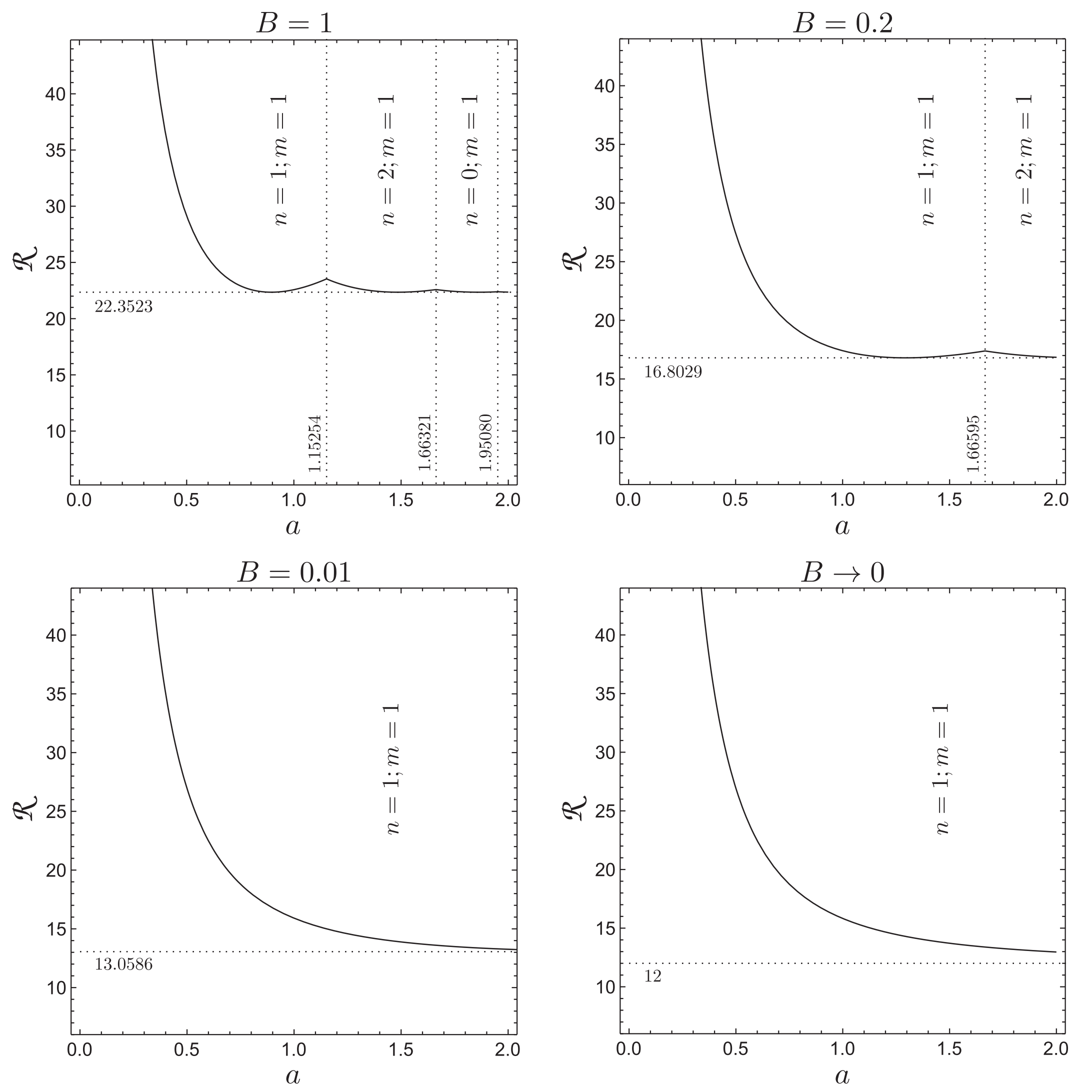


FIG. 3. Neutral stability: plots of $\mathcal{R}$ versus $a$ for the most unstable modes, each frame is for a different value of $B$. The horizontal dotted line is for $\mathcal{R} = \mathcal{R}_c$; the vertical dotted lines bound the regions where each specified $(n, m)$-mode is preferred.

from very large values, this transition aspect ratio $a$ increases monotonically, as it is recognized on inspecting Figs. 2 and 3. When $B \to 0$, all the transition values of $a$ tend to infinity. The reason is that $\mathcal{R}(\lambda, B)$ becomes a monotonic increasing function of $\lambda$ when $B \to 0$, as it is shown for instance in Barletta and Storesletten.[21] This behaviour implies that, when $B \to 0$, only the mode $n = 1$, $m = 1$ arises at the onset of convection, for every value of $a$.

## A. Instability patterns

The preferred normal modes at the onset of the thermoconvective instability depend on both $a$ and $B$. Figure 4 displays the regions in the parametric plane $(a, B)$ where each $(n, m)$-mode is selected. These regions form curved slices in the parametric plane, that tend to become very narrow as the order of the $(n, m)$-mode increases. In particular, we note that the transition from one mode to its follower is strongly sensitive to the value of $a$, especially when $B$ is large. This behaviour suggests an increasing complexity in the selected convection patterns as the vertical cylinder becomes more and more shallow.

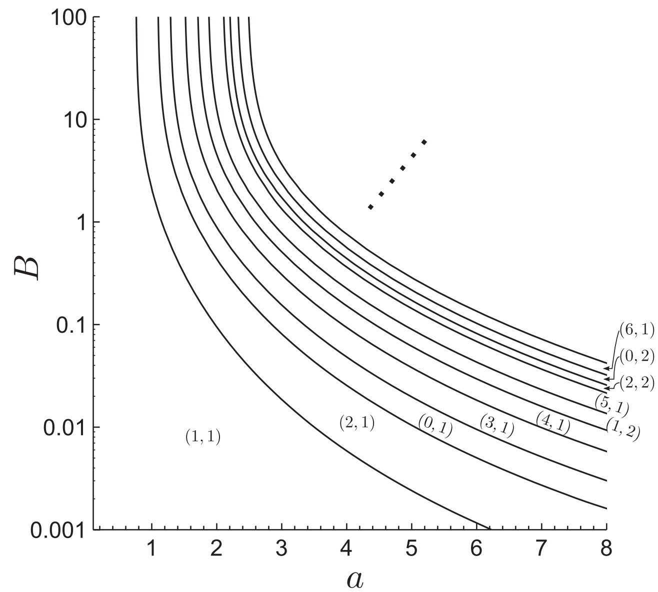


FIG. 4. Selected modes, $(n, m)$, at onset of convection in the plane $(a, B)$; the solid lines bound the regions where each specified mode is preferred.

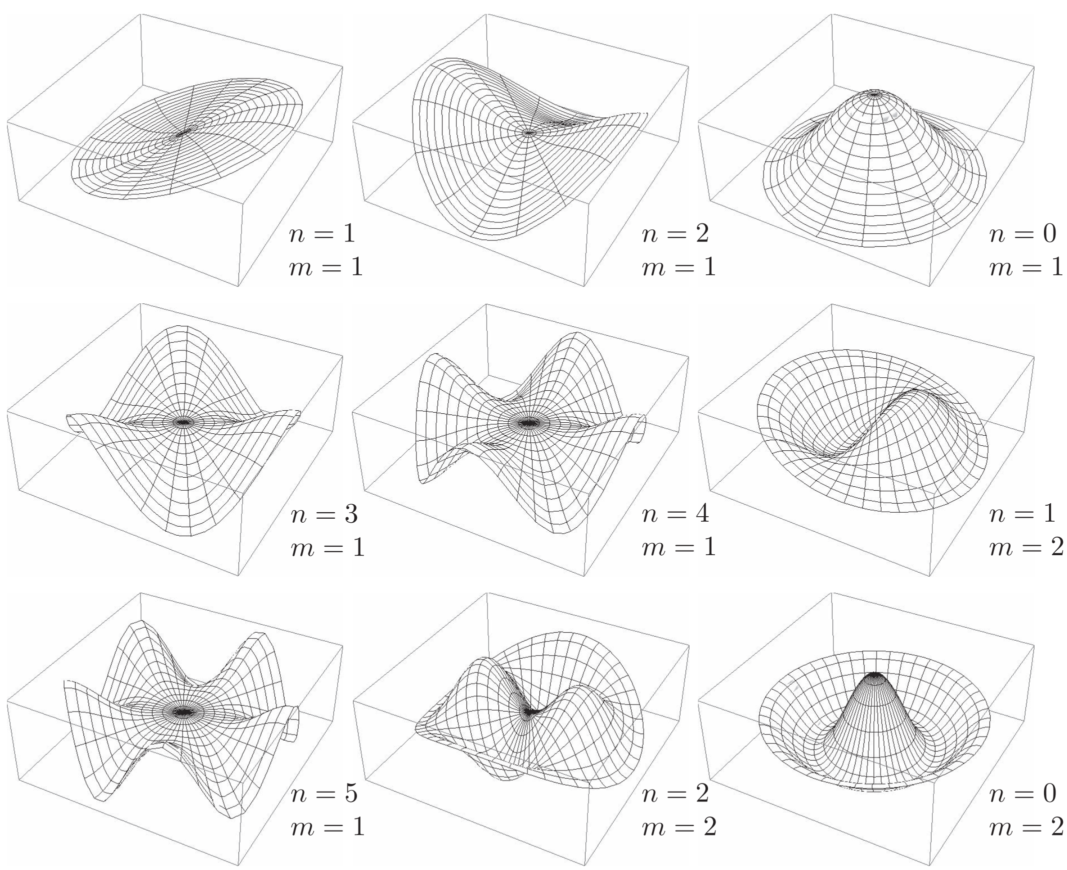


FIG. 5. Qualitative plots of the time-independent normal modes $\Theta(r, \theta, z)$, given by Eq. (13), as functions of $(r, \theta)$ at a fixed plane $z =$ constant, with $0 < z < 1$, for the pairs $(n, m)$ ordered according to Table I.

Qualitative sketches of the normal modes, Eq. (13), corresponding to different pairs $(n, m)$ are drawn in Fig. 5. This figure shows the normal modes,

$$J_n(\lambda r)\cos(n\theta)\,F(z),$$

regarded as functions of $(r/a, \theta)$ at a fixed $z$ such that $0 < z < 1$. Except for an overall scale factor, $F(z)$, these functions of $(r/a, \theta)$ do not depend either on $a$ or on $B$.

## VIII. CONCLUSIONS

A study of the onset of thermal instability in a vertical porous cylinder with a finite height has been carried out. The radius-to-height aspect ratio, $a$, is considered as arbitrary. The sidewall is modelled as impermeable and thermally insulated. The plane horizontal boundaries are assumed to be impermeable. The upper and lower boundaries are cooled and heated, respectively, by a forced convection process to an external environment, parameterized by a Biot number $B$. Separable solutions of the linearised governing equations for the disturbances are considered, and the exchange of stability is proved to hold for them. Then, the analysis has been focused on the stationary normal modes. The partial differential problem for the temperature disturbances has been transformed into an ordinary differential eigenvalue problem. The solution of this eigenvalue problem has been determined analytically, thus obtaining an implicit neutral stability condition either for the Darcy-Rayleigh number, $R$, or for the alternative governing parameter, $\mathcal{R} = BR/(B+2)$. From this neutral stability condition, the main results obtained are the following:

- The sidewall confinement influences only the selection of the allowed values for the radial wavenumber, $\lambda$, but not the functional relationship $R(\lambda, B)$ at the onset of convection. Thus, function $R(\lambda, B)$ is exactly the same as that obtained, for a rectangular sidewall, by Barletta and Storesletten.[21]
- The allowed values of the radial wavenumber, $\lambda$, form a discrete sequence that, in increasing order, determines the type of the normal modes selected at the onset of convection for increasing aspect ratios, $a$, of the vertical cylinder.
- In the limit $B \to \infty$, corresponding to the case where both the horizontal boundaries are isothermal, our results agree with those found by Zebib[8] with respect to both the neutral stability condition and the selected modes of instability.
- In the limiting case $B \to 0$, physically relevant to model the case where the lower (upper) horizontal boundary is subject to an incoming (outgoing) uniform heat flux, only the lowest non-axisymmetric mode of instability, defined by the modal numbers $n = 1$, $m = 1$, is selected for every possible aspect ratio $a$. The peculiarity of this limiting case relies on $\mathcal{R}$ being a monotonic increasing function of $\lambda$ at neutral stability.

## ACKNOWLEDGMENTS

This work was financially supported by Italian government, MIUR Grant No. PRIN–2009KSSKL3.

1 C. W. Horton and F. T. Rogers, "Convection currents in a porous medium," J. Appl. Phys. **16**, 367–370 (1945).
2 E. R. Lapwood, "Convection of a fluid in a porous medium," Proc. Cambridge Philos. Soc. **44**, 508–521 (1948).
3 D. A. S. Rees, "The stability of Darcy–Bénard convection," in *Handbook of Porous Media*, edited by K. Vafai and H. A. Hadim (CRC Press, New York, 2000), Chap. 12, pp. 521–558.
4 P. A. Tyvand, "Onset of Rayleigh–Bénard convection in porous bodies," in *Transport Phenomena in Porous Media II*, edited by D. B. Ingham and I. Pop (Pergamon, New York, 2002), Chap. 4, pp. 82–112.
5 D. A. Nield and A. Bejan, *Convection in Porous Media*, 3rd ed. (Springer, New York, 2006).
6 A. Barletta, "Thermal instabilities in a fluid saturated porous medium," in *Heat Transfer in Multi–Phase Materials*, edited by A. Öchsner and G. E. Murch (Springer, New York, 2011), pp. 381–414.
7 R. A. Wooding, "The stability of a viscous liquid in a vertical tube containing porous material," Proc. R. Soc. London, Ser. A **252**, 120–134 (1959).
8 A. Zebib, "Onset of natural convection in a cylinder of water saturated porous media," Phys. Fluids **21**, 699–700 (1978).
9 S. Bories and A. Deltour, "Influence des conditions aux limites sur la convection naturelle dans un volume poreux cylindrique," Int. J. Heat Mass Transfer **23**, 765–771 (1980).

[10] K. B. Haugen and P. A. Tyvand, "Onset of thermal convection in a vertical porous cylinder with conducting wall," Phys. Fluids **15**, 2661–2667 (2003).
[11] H. S. Nygård and P. A. Tyvand, "Onset of thermal convection in a vertical porous cylinder with a partly conducting and partly penetrative cylinder wall," Transp. Porous Media **86**, 229–241 (2011).
[12] C. Y. Wang, "Onset of natural convection in a fluid-saturated porous medium inside a cylindrical enclosure bottom heated by constant flux," Int. Commun. Heat Mass Transfer **25**, 593–598 (1998).
[13] C. Y. Wang, "Thermo-convective stability of a fluid-saturated porous medium inside a cylindrical enclosure – Permeable top constant flux heating," Mech. Res. Commun. **26**, 603–608 (1999).
[14] H. H. Bau and K. E. Torrance, "Onset of convection in a permeable medium between vertical coaxial cylinders," Phys. Fluids **24**, 382–385 (1981).
[15] C. Bringedal, I. Berre, J. M. Nordbotten, and D. A. S. Rees, "Linear and nonlinear convection in porous media between coaxial cylinders," Phys. Fluids **23**, 094109 (2011).
[16] S. Govender, "Stability of gravity driven convection in a cylindrical porous layer subjected to vibration," Transp. Porous Media **63**, 489–502 (2006).
[17] Z. Zhang, C. Fu, W. Tan, and C. Y. Wang, "Onset of oscillatory convection in a porous cylinder saturated with a viscoelastic fluid," Phys. Fluids **19**, 098104 (2007).
[18] A. V. Kuznetsov and D. A. Nield, "The onset of double-diffusive convection in a vertical cylinder occupied by a heterogeneous porous medium with vertical throughflow," Transp. Porous Media **95**, 327–336 (2012).
[19] J. P. Kubitschek and P. D. Weidman, "Stability of a fluid–saturated porous medium heated from below by forced convection," Int. J. Heat Mass Transfer **46**, 3697–3705 (2003).
[20] J. P. Kubitschek and P. D. Weidman, "Stability of a fluid-saturated porous medium contained in a vertical cylinder heated from below by forced convection," Heat Mass Transfer **42**, 789–794 (2006).
[21] A. Barletta and L. Storesletten, "Onset of convection in a porous rectangular channel with external heat transfer to upper and lower fluid environments," Transp. Porous Media **94**, 659–681 (2012).
[22] A. Pellew and R. V. Southwell, "On maintained convective motion in a fluid heated from below," Proc. R. Soc. London, Ser. A **176**, 312–343 (1940).