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# Mapping properties for the Bargmann transform on modulation spaces

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**Abstract** We investigate mapping properties for the Bargmann transform and prove that this transform is isometric and bijective from modulation spaces to convenient Banach spaces of analytic functions. We also present some consequences. For example we prove that the spectrum of the Harmonic oscillator is the same for all modulation spaces.

**Keywords** Bijectivity properties · Harmonic oscillator · Hermite functions · Berezin–Toeplitz operators

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## 0 Introduction

In [1], Bargmannn introduce a transform $\mathfrak{V}$ which is bijective and isometric from $L^2(\mathbf{R}^d)$ into the Hilbert space $A^2(\mathbf{C}^d)$ of all entire analytic functions $F$ on $\mathbf{C}^d$ such

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that $F \cdot e^{-|\cdot|^2/2} \in L^2(\mathbf{C}^d)$. (We use the usual notations for the usual function and distribution spaces. See, e.g. [30], and refer to Sect. 1 for specific definitions and other notations.) Furthermore, several important properties for $\mathfrak{V}$ were established. For example:

- the Hermite functions are mapped into the normalized analytical monomials. Furthermore, the latter set forms an orthonormal basis for $A^2(\mathbf{C}^d)$;
- the creation and annihilation operators, and harmonic oscillator on appropriate elements in $L^2$, are transformed by $\mathfrak{V}$ into simple operators;
- there is a convenient reproducing formula for elements in $A^2$.

In [2], Bargmann continued his work and discussed mapping properties for $\mathfrak{V}$ on more general spaces. For example, he proves that $\mathfrak{V}(\mathscr{S}')$, the image of $\mathscr{S}'$ under the Bargmann transform is given by the formula

$$\mathfrak{V}(\mathscr{S}') = \cup_{\omega \in \mathscr{P}} A^{\infty,\infty}_{(\omega)}, \tag{0.1}$$

Here $A^{p,q}_{(\omega)}(\mathbf{C}^d)$ is the set of all entire functions $F$ on $\mathbf{C}^d$ such that $F \cdot e^{-|\cdot|^2/2} \cdot \omega_0$ belongs to the mixed Lebesgue space $L^{p,q}(\mathbf{C}^d)$, for some appropriate modification $\omega_0$ of the weight function $\omega$.

The Bargmann transform has a great impact in several other fields within mathematics and its related sciences. For example, in [23], the Bargmann transform is slightly hidden in the analysis of spectral properties for localization operators on modulation spaces. We refer to [3,8,32] and the references therein for other important contribution to the theory of Bargmann transform.

Since the Bargmann transform appears in several situations, it is important to know its image of convenient and frequently used function spaces, for example Lebesgue, Sobolev or Besov spaces. However, except for the Hilbert space case, it seems to be a hard task, in a simple way, describing the image of the latter spaces under the Bargmann transform.

In this paper we focus on the image of the Bargmann transform for the *modulation spaces*. We recall that the (classical) modulation space $M^{p,q}$, $p, q \in [1, \infty]$, as introduced by Feichtinger in [10], consists of all tempered distributions whose short-time Fourier transforms (STFT) have finite mixed $L^{p,q}$ norm. The theory of modulation spaces was developed further and generalized in [11,13,14,18], where Feichtinger and Gröchenig established the theory of coorbit spaces. See also [12] for a review of the theory of modulation spaces. In particular, the modulation space $M^{p,q}_{(\omega)}$, where $\omega$ denotes a weight function on phase (or time–frequency shift) space, appears as the set of tempered (ultra-) distributions whose STFT belong to the weighted and mixed Lebesgue space $L^{p,q}_{(\omega)}$. Here the weight $\omega$ quantifies the degree of asymptotic decay and singularity of the distribution in $M^{p,q}_{(\omega)}$.

A major idea behind the design of these spaces was to find useful Banach spaces, which are defined in a way similar to Besov spaces, in the sense of replacing the dyadic decomposition on the Fourier transform side, characteristic to Besov spaces, with a *uniform* decomposition. From the construction of these spaces, it turns out that modulation spaces and Besov spaces in some sense are rather similar, and sharp embeddings between these spaces can be found in [36,37], which are improvements of certain embeddings in [17]. (See also [28,35] for verification of the sharpness.)

During the last 15 years many results have been proved which confirm the usefulness of the modulation spaces in time–frequency analysis, where they occur naturally. For example, in [11,19,21], it is shown that all modulation spaces admit reconstructible sequence space representations using Gabor frames.

Parallel to this development, modulation spaces have been incorporated into the calculus of pseudo-differential operators, which also involve Toeplitz operators. (See, e.g. [19,22,26,27,35–38,40] and the references therein.)

Finally we remark that modulation spaces have been applied to get sharp estimates for solutions to partial differential equations. Some examples are [31] for the Navier Stokes and a nonlinear heat equation, [25] for the nonlinear Schrödinger equation, [29] for the Klein–Gordon equation, and [28] for the KdV and Benjamin–Ono equations.

The Bargmann transform can easily be reformulated in terms of the short-time Fourier transform, with a particular Gauss function as window function. By such reformulation, and using the fundamental role of the short-time Fourier transform in the definition of modulation spaces, it easily follows that the Bargmann transform is continuous and injective from $M^{p,q}_{(\omega)}$ to $A^{p,q}_{(\omega)}$. Furthermore, by choosing the window function as a particular Gaussian function in the $M^{p,q}_{(\omega)}$ norm, it follows that $\mathfrak{V} : M^{p,q}_{(\omega)} \to A^{p,q}_{(\omega)}$ is isometric.

These facts and several other mapping properties for the Bargmann transform on modulation spaces were established and proved by Feichtinger, Gröchenig and Walnut in [13,15,18,24]. In fact, here they prove that the Bargmann transform from $M^{p,q}_{(\omega)}$ to $A^{p,q}_{(\omega)}$ is not only injective, but in fact *bijective*.

When proving the surjectivity of $\mathfrak{V}$, they use the arguments that the Bargmann-Fock representation of the Heisenberg group is unitarily equivalent to the Schrödinger representation with $\mathfrak{V}$ as the intertwining operator. Then they explain that the general intertwining theorem [13, Theorem 4.8] applied to the Schrödinger representation and the Bargmann-Fock representation implies that $\mathfrak{V}$ extends to a Banach space isomorphism from $M^{p,q}_{(\omega)}$ to $A^{p,q}_{(\omega)}$, and the asserted surjectivity follows.

In this context we remark that when applying [13, Theorem 4.8] it is necessary that each element in $A^{p,q}_{(\omega)}$ for $p, q \in [1, \infty]$ is the Bargmann transform of a tempered distribution. The latter fact follows if (0.1) is extended into

$$\mathfrak{V}(\mathscr{S}') = \cup_{\omega \in \mathscr{P}} A^{p,q}_{(\omega)}. \tag{0.1$'$}$$

We are not able to find any proof of (0.1)$'$ in the literature, nor any references in [13, 15,18,24]. On the other hand, it seems to be obvious that the authors in [13,15,18,24] are aware of such proof in the literature. Furthermore, during our communications with K.H. Gröchenig, we were able to prove that the reproducing kernel holds for each element in $A^{p,q}_{(\omega)}$, which implies that (0.1)$'$ follows from (0.1).

In this paper we take an alternative approach for proving this bijectivity, based on a direct proof of the fact that each element in $A^{p,q}_{(\omega)}$ is a Bargmann transform of a temperate distribution, by proving that (0.1)$'$ holds for all $p, q \in [1, \infty]$. The fact that the Bargmann transform is continuous and injective from $M^{p,q}_{(\omega)}$ to $A^{p,q}_{(\omega)}$ then shows that this tempered distribution must belong to $M^{p,q}_{(\omega)}$, and the result follows.

When proving (0.1)′ we first consider mapping properties on Hilbert spaces, defined by the harmonic oscillator. We prove that such Hilbert spaces are modulation spaces of the form $M^{2,2}_{(\omega)}$, when $\omega(x,\xi) = \sigma_N(x,\xi) = \langle x,\xi\rangle^N$ for some even number $N$. Here

$$\langle x\rangle = (1+|x|^2)^{1/2} \quad \text{and} \quad \langle x,\xi\rangle = (1+|x|^2+|\xi|^2)^{1/2}$$

as usual. Furthermore, we use the analysis in [1,2] to prove that $\mathfrak{V}$ maps $M^{2,2}_{(\sigma_N)}$ bijectively and isometrically onto $A^{2,2}_{(\sigma_N)}$. Since any fixed tempered distribution belongs to $M^{2,2}_{(\sigma_N)}$ provided $N$ is a large enough negative number, (0.1)′ follows in the case $p = q = 2$.

The generalization of (0.1)′ from the case $p = q = 2$ to general $p$ and $q$ is thereafter done in Sect. 2 by an argument of harmonic mean values, from which it follows that (0.1)′ also holds for $p = q = 1$. Since $A^{p,q}_{(\omega)} \subseteq A^{1,1}_{(\sigma_N)}$, by Hölder's inequality, provided $N$ is a large enough negative number, it follows that each element in $A^{p,q}_{(\omega)}$ is a Bargmann transform of a tempered distribution. The asserted bijectivity is now a consequence of the fact that $\mathfrak{V} : M^{p,q}_{(\omega)} \to A^{p,q}_{(\omega)}$ is continuous and injective.

Here we remark that in several key steps we only considered properties of harmonic functions which is a significantly broader class of functions compared to the set of entire functions. Therefore we expect that several steps can be carried over to other integral transforms, with ranges contained in the set of harmonic functions.

We also list some consequences of our results for different types of operators. For example, we carry over the bijectivity properties for Toeplitz operators on modulation spaces in [22] to similar situations for Berezin–Toeplitz operators acting on $A^{p,q}_{(\omega)}$ spaces (see also [23]). As a consequence of these investigations we establish exact spectral properties for the harmonic oscillator when acting on $A^{p,q}_{(\omega)}$ spaces.

The paper is organized as follows. In Sect. 1 we recall some facts for modulation spaces and the Bargmann transform. In Sect. 2 we prove the main result, i.e. that the Bargmann transform is bijective from $M^{p,q}_{(\omega)}$ to $A^{p,q}_{(\omega)}$. In fact, we prove a more general result involving general modulation spaces $M(\omega,\mathscr{B})$, parameterized with the weight function $\omega$ and the translation invariant BF-space $\mathscr{B}$. In Sect. 3 we present some consequences of the main result. Several of these consequences can be found in [13,15,18,24]. However in our approach, such known consequences enter the theory in different ways compared to [13,15,18,24].

Finally, in Appendix A we show how some key steps can be obtained in different ways. We remark that the proofs of some of the results here were obtained together with K.H. Gröchenig, and that these results imply that each element in $A^{p,q}_{(\omega)}$ is a Bargmann transform of a tempered distribution (cf. the discussions here above).

## 1 Preliminaries

In this section we give some definitions and recall some basic facts. The proofs are in general omitted.

## 1.1 Translation invariant BF-spaces

We start with presenting appropriate conditions on the involved weight functions. Assume that $\omega, v \in L^\infty_{loc}(\mathbf{R}^d)$ are positive functions. Then $\omega$ is called $v$-moderate if

$$\omega(x + y) \le C\omega(x)v(y) \tag{1.1}$$

for some constant $C$ which is independent of $x, y \in \mathbf{R}^d$. If $v$ in (1.1) can be chosen as a polynomial, then $\omega$ is called polynomially moderate. We let $\mathscr{P}(\mathbf{R}^d)$ be the set of all polynomially moderated functions on $\mathbf{R}^d$. We also let $\mathscr{P}_0(\mathbf{R}^d)$ be the set of all $\omega \in \mathscr{P}(\mathbf{R}^d) \cap C^\infty(\mathbf{R}^d)$ such that $(\partial^\alpha\omega)/\omega \in L^\infty$, for every multi-index $\alpha$. We remark that for each $\omega \in \mathscr{P}(\mathbf{R}^d)$, there is an element $\omega_0 \in \mathscr{P}_0(\mathbf{R}^d)$ which is equivalent to $\omega$, in the sense that for some constant $C$, it holds

$$C^{-1}\omega_0 \le \omega \le C\omega_0$$

(cf., e.g. [37,38]).

We say that $v$ is *submultiplicative* when (1.1) holds with $\omega = v$. Throughout we assume that the submultiplicative weights are even. Furthermore, $v$ and $v_j$ for $j \ge 0$, always stand for submultiplicative weights if nothing else is stated.

An important type of weight functions is

$$\sigma_s(x) \equiv \langle x \rangle^s = (1 + |x|^2)^{s/2}, \tag{1.2}$$

For each $\omega \in \mathscr{P}(\mathbf{R}^d)$ and $p \in [1, \infty]$, we let $L^p_{(\omega)}(\mathbf{R}^d)$ be the Banach space which consists of all $f \in L^1_{loc}(\mathbf{R}^d)$ such that $\|f\|_{L^p_{(\omega)}} \equiv \|f\,\omega\|_{L^p}$ is finite.

Next we recall the definition of translation invariant Banach function spaces (BF-spaces).

**Definition 1.1** Assume that $\mathscr{B}$ is a Banach space of complex-valued measurable functions on $\mathbf{R}^d$ and that $v \in \mathscr{P}(\mathbf{R}^d)$ is submultiplicative. Then $\mathscr{B}$ is called a *(translation) invariant BF-space on* $\mathbf{R}^d$ (with respect to $v$), if there is a constant $C$ such that the following conditions are fulfilled:

1. $\mathscr{S}(\mathbf{R}^d) \subseteq \mathscr{B} \subseteq \mathscr{S}'(\mathbf{R}^d)$ (continuous embeddings).
2. If $x \in \mathbf{R}^d$ and $f \in \mathscr{B}$, then $f(\cdot - x) \in \mathscr{B}$, and

$$\|f(\cdot - x)\|_{\mathscr{B}} \le Cv(x)\|f\|_{\mathscr{B}}. \tag{1.3}$$

3. if $f, g \in L^1_{loc}(\mathbf{R}^d)$ satisfy $g \in \mathscr{B}$ and $|f| \le |g|$ almost everywhere, then $f \in \mathscr{B}$ and

$$\|f\|_{\mathscr{B}} \le C\|g\|_{\mathscr{B}}.$$

4. the map $(f, \varphi) \mapsto f * \varphi$ is continuous from $\mathscr{B} \times C_0^\infty(\mathbf{R}^d)$ to $\mathscr{B}$ and satisfies $\|f * \varphi\|_{\mathscr{B}} \le C\|f\|_{\mathscr{B}}\|\varphi\|_{L^1_{(v)}}$, for every $f \in \mathscr{B}$ and $\varphi \in C_0^\infty(\mathbf{R}^d)$.

*Remark 1.2* Assume that $\mathscr{B}$ is a translation invariant BF-space. If $f \in \mathscr{B}$ and $h \in L^\infty$, then it follows from (3) in Definition 1.1 that $f \cdot h \in \mathscr{B}$ and

$$\|f \cdot h\|_{\mathscr{B}} \le C\|f\|_{\mathscr{B}}\|h\|_{L^\infty}. \tag{1.4}$$

*Remark 1.3* Assume $\omega_0, v, v_0 \in \mathscr{P}(\mathbf{R}^d)$ are such that $v$ and $v_0$ are submultiplicative, $\omega_0$ is $v_0$-moderate, and assume that $\mathscr{B}$ is a translation-invariant BF-space on $\mathbf{R}^d$ with respect to $v$. Also let $\mathscr{B}(\omega_0)$ be the Banach space which consists of all $f \in L^1_{loc}(\mathbf{R}^d)$ such that $\|f\|_{\mathscr{B}(\omega_0)} \equiv \|f\,\omega_0\|_{\mathscr{B}}$ is finite. Then $\mathscr{B}(\omega_0)$ is a translation invariant BF-space with respect to $v_0 v$.

*Remark 1.4* Let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^d$ with respect to $v \in \mathscr{P}(\mathbf{R}^d)$. Then it follows that the map $(f, g) \mapsto f * g$ from $\mathscr{B} \times C_0^\infty(\mathbf{R}^d)$ to $\mathscr{B}$ extends uniquely to a continuous mapping from $\mathscr{B} \times L^1_{(v)}(\mathbf{R}^d)$ to $\mathscr{B}$. In fact, if $f \in \mathscr{B}$ and $\varphi \in C_0^\infty(\mathbf{R}^d)$, then Minkowski's inequality gives

$$\begin{aligned}\|f * \varphi\|_{\mathscr{B}} &= \left\| \int f(\cdot - y)\varphi(y)\, dy \right\|_{\mathscr{B}} \\ &\le \int \|f(\cdot - y)\|_{\mathscr{B}} |\varphi(y)|\, dy \le C \int \|f\|_{\mathscr{B}} |\varphi(y)v(y)|\, dy = C\|f\|_{\mathscr{B}}\|\varphi\|_{L^1_{(v)}}.\end{aligned}$$

The assertion is now a consequence of the fact that $C_0^\infty$ is dense in $L^1_{(v)}$.

*Remark 1.5* Let $\mathscr{B}$ be an invariant BF-space. Then it is easy to find Sobolev type spaces which are continuously embedded in $\mathscr{B}$. In fact, for each $p \in [1, \infty)$ and integer $N \ge 0$, let $Q_N^p(\mathbf{R}^d)$ be the set of all $f \in L^p(\mathbf{R}^d)$ such that $\|f\|_{Q_N^p} < \infty$, where

$$\|f\|_{Q_N^p} \equiv \sum_{|\alpha+\beta| \le N} \|x^\alpha D^\beta f\|_{L^p}.$$

Then for each $p$ fixed, the topology for $\mathscr{S}(\mathbf{R}^d)$ can be defined by the semi-norms $f \mapsto \|f\|_{Q_N^p}$, for $N = 0, 1, \ldots$ Since $\mathscr{S}$ is continuously embedded in the Banach space $\mathscr{B}$, it now follows that

$$\|f\|_{\mathscr{B}} \le C_{N,p}\|f\|_{Q_N^p}$$

for some constants $C$ and $N$ which are independent of $f \in \mathscr{S}$. Consequently, if in addition $p < \infty$, then $Q_N^p(\mathbf{R}^d) \subseteq \mathscr{B}$, since $\mathscr{S}$ is dense in $Q_N^p$. This proves the assertion.

The following proposition shows that even stronger embeddings compared to $Q_N^p(\mathbf{R}^d) \subseteq \mathscr{B}$ in Remark 1.5 hold when $p = \infty$. Here, we set $L_N^p = L_{(\omega)}^p$ when $\omega(x) = \langle x \rangle^N$.

**Proposition 1.6** *Let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^d$, and let $\omega \in \mathscr{P}(\mathbf{R}^d)$. Then there is a large number $N$ such that*

$$L_N^\infty(\mathbf{R}^d) \subseteq \mathscr{B}(\omega) \subseteq L_{-N}^1(\mathbf{R}^d).$$

*Proof* We may assume that $\omega = 1$ in view of Remark 1.3. First we note that $\|\langle\,\cdot\,\rangle^{-N}\|_{\mathscr{B}} < \infty$, provided $N$ is large enough. In fact, since $\mathscr{S}$ is continuously embedded in $\mathscr{B}$, it follows that

$$\|f\|_{\mathscr{B}} \le C \sum_{|\beta| \le N} \|\langle\,\cdot\,\rangle^N (\partial^\beta f)\|_{L^\infty}, \tag{1.5}$$

when $f \in \mathscr{S}$, for some choices of constants $C$ and $N$, and the assertion now follows since the right-hand side of (1.5) is finite when $f(x) = \langle x\rangle^{-N}$.

It follows from Definition 1.1 (2) that

$$\langle x\rangle^{-d-1}\|f\|_{\mathscr{B}} \ge C\langle x\rangle^{-N}\|f(\,\cdot\, - x)\|_{\mathscr{B}}, \tag{1.6}$$

for some constants $C$ and $N$. By integrating (1.6) and using Minkowski's inequality we get

$$\begin{aligned}
\|f\|_{\mathscr{B}} &\ge C_1 \int \langle x\rangle^{-N}\|f(\,\cdot\, - x)\|_{\mathscr{B}}\,dx \\
&\ge C_2 \left\| \int \langle x\rangle^{-N}|f(\,\cdot\, - x)|\,dx \right\|_{\mathscr{B}} = C_2 \left\| \int \langle x - \,\cdot\,\rangle^{-N}|f(x)|\,dx \right\|_{\mathscr{B}} \\
&\ge C_3 \left\| \int \langle x\rangle^{-N}|f(x)|\,dx\,\langle\,\cdot\,\rangle^{-N} \right\|_{\mathscr{B}} = C\|f\|_{L^1_{-N}},
\end{aligned}$$

for some positive constants $C_1, \dots, C_3$, where $C = C_3\|\langle\,\cdot\,\rangle^{-N}\|_{\mathscr{B}} < \infty$. This proves $\mathscr{B} \subseteq L^1_{-N}$.

It remains to prove $L_N^\infty \subseteq \mathscr{B}$. Let $N$ be as in the first part of the proof. By straightforward computations and using Remark 1.2 we get

$$\|f\|_{\mathscr{B}} = \|(f\langle\,\cdot\,\rangle^N)\langle\,\cdot\,\rangle^{-N}\|_{\mathscr{B}} \le C_1\|\langle\,\cdot\,\rangle^{-N}\|_{\mathscr{B}}\|f\|_{L_N^\infty} = C\|f\|_{L_N^\infty},$$

for some constant $C_1$, where $C = C_1\|\langle\,\cdot\,\rangle^{-N}\|_{\mathscr{B}} < \infty$. Hence $L_N^\infty \subseteq \mathscr{B}$, and the result follows. □

## 1.2 The short-time Fourier transform and Toeplitz operators

Before giving the definition of the short-time Fourier transform we recall some properties for the (usual) Fourier transform. The Fourier transform $\mathscr{F}$ is the linear and continuous mapping on $\mathscr{S}'(\mathbf{R}^d)$ which takes the form

$$(\mathscr{F} f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-d/2} \int\limits_{\mathbf{R}^d} f(x) e^{-i\langle x,\xi\rangle}\, dx$$

when $f \in L^1(\mathbf{R}^d)$. Here $\langle \cdot\, , \,\cdot \rangle$ denotes the usual scalar product on $\mathbf{R}^d$. The map $\mathscr{F}$ is a homeomorphism on $\mathscr{S}'(\mathbf{R}^d)$ which restricts to a homeomorphism on $\mathscr{S}(\mathbf{R}^d)$ and to a unitary operator on $L^2(\mathbf{R}^d)$.

Let $\phi \in \mathscr{S}(\mathbf{R}^d)\backslash 0$ be fixed. For every $f \in \mathscr{S}'(\mathbf{R}^d)$, the *short-time Fourier transform* $V_\phi f$ is the distribution on $\mathbf{R}^{2d}$ defined by the formula

$$(V_\phi f)(x, \xi) = \mathscr{F}(f\, \overline{\phi(\cdot\, - x)})(\xi). \tag{1.7}$$

We note that the right-hand side defines an element in $\mathscr{S}'(\mathbf{R}^{2d}) \cap C^\infty(\mathbf{R}^{2d})$. We also note that if $f \in L^q_{(\omega)}$ for some $\omega \in \mathscr{P}(\mathbf{R}^d)$, then $V_\phi f$ takes the form

$$V_\phi f(x, \xi) = (2\pi)^{-d/2} \int\limits_{\mathbf{R}^d} f(y)\overline{\phi(y - x)} e^{-i\langle y,\xi\rangle}\, dy. \tag{1.7$'$}$$

Next we consider Toeplitz operators, also known as localization operators. If $a \in \mathscr{S}(\mathbf{R}^{2d})$ and $\phi \in \mathscr{S}(\mathbf{R}^d)\backslash 0$ are fixed, then the *Toeplitz operator* $\mathrm{Tp}(a) = \mathrm{Tp}_\phi(a)$ is the linear and continuous operator on $\mathscr{S}(\mathbf{R}^d)$, defined by the formula

$$(\mathrm{Tp}_\phi(a) f, g)_{L^2(\mathbf{R}^d)} = (a\, V_\phi f, V_\phi g)_{L^2(\mathbf{R}^{2d})}. \tag{1.8}$$

There are several characterizations of Toeplitz operators and several ways to extend the definition of such operators (see, e.g. [22] and the references therein). For example, the definition of $\mathrm{Tp}_\phi(a)$ is uniquely extendable to every $a \in \mathscr{S}'(\mathbf{R}^{2d})$, and then $\mathrm{Tp}_\phi(a)$ is still continuous on $\mathscr{S}(\mathbf{R}^d)$, and uniquely extendable to a continuous operator on $\mathscr{S}'(\mathbf{R}^d)$.

Toeplitz operators arise in pseudo-differential calculus [16,33], in the theory of quantization (Berezin quantization [5]), and in signal processing [9] (under the name of time–frequency localization operators or STFT multipliers).

### 1.3 Modulation spaces

We shall now discuss modulation spaces and recall some basic properties. We start with the following definition.

**Definition 1.7** Let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, and let $\phi \in \mathscr{S}(\mathbf{R}^d)\backslash 0$. Then the *modulation space* $M(\omega, \mathscr{B})$ consists of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that

$$\| f \|_{M(\omega,\mathscr{B})} \equiv \| V_\phi f\, \omega \|_{\mathscr{B}} < \infty. \tag{1.9}$$

If $\omega = 1$, then the notation $M(\mathscr{B})$ is used instead of $M(\omega, \mathscr{B})$.

We remark that the modulation space $M(\omega, \mathscr{B})$ in Definition 1.7 is independent of the choice of $\phi \in \mathscr{S}(\mathbf{R}^d)\backslash 0$, and different choices of $\phi$ give rise to equivalent norms.

An important case concerns when $\mathscr{B}$ is a mixed-norm space of Lebesgue type. More precisely, let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $p, q \in [1, \infty]$, and let $L^{p,q}(\mathbf{R}^{2d})$ be the Banach space which consists of all $F \in L^1_{loc}(\mathbf{R}^{2d})$ such that

$$\|F\|_{L^{p,q}} \equiv \left( \int\limits_{\mathbf{R}^d} \left( \int\limits_{\mathbf{R}^d} |F(x,\xi)|^p \, dx \right)^{q/p} d\xi \right)^{1/q} < \infty.$$

(with obvious modifications when $p = \infty$ or $q = \infty$). Also let $L^{p,q}_*(\mathbf{R}^{2d})$ be the set of all $F \in L^1_{loc}(\mathbf{R}^{2d})$ such that

$$\|F\|_{L^{p,q}_*} \equiv \left( \int\limits_{\mathbf{R}^d} \left( \int\limits_{\mathbf{R}^d} |F(x,\xi)|^q \, d\xi \right)^{p/q} dx \right)^{1/p} < \infty.$$

Then the space $M(\omega, L^{p,q}(\mathbf{R}^{2d}))$ is the usual modulation space $M^{p,q}_{(\omega)}(\mathbf{R}^d)$, and $M(\omega, L^{p,q}_*(\mathbf{R}^{2d}))$ is the space $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ which is related to certain types of classical Wiener amalgam spaces. For convenience we use the notation $M^p_{(\omega)}$ instead of $M^{p,p}_{(\omega)} = W^{p,p}_{(\omega)}$, and we set $M^{p,q}_s = M^{p,q}_{(\sigma_s)}$ and $M^p_s = M^p_{(\sigma_s)}$, where $\sigma_s$ is given by (1.2). Furthermore, for $\omega = 1$ we set

$$M(\mathscr{B}) = M(\omega, \mathscr{B}), \quad M^{p,q} = M^{p,q}_{(\omega)}, \quad W^{p,q} = W^{p,q}_{(\omega)}, \quad \text{and} \quad M^p = M^p_{(\omega)}.$$

Here we recall that

$$\sigma_s(x,\xi) = \langle x, \xi \rangle^s = (1 + |x|^2 + |\xi|^2)^{s/2}.$$

In the following proposition we recall some facts about modulation spaces. We omit the proof, since the result can be found in [10,13,14,19,38].

**Proposition 1.8** *Let $p, q, p_j, q_j \in [1, \infty]$, $\omega, \omega_j, v, v_0 \in \mathscr{P}(\mathbf{R}^{2d})$ for $j = 1, 2$ be such that $\omega$ is $v$-moderate, and let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$ with respect to $v_0$. Then the following is true:*

(1) *if $\phi \in M^1_{(v_0 v)}(\mathbf{R}^d)\backslash 0$, then $f \in M(\omega, \mathscr{B})$ if and only if (1.9) holds, i.e. $M(\omega, \mathscr{B})$ is independent of the choice of $\phi$. Moreover, $M(\omega, \mathscr{B})$ is a Banach space under the norm in (1.9), and different choices of $\phi$ give rise to equivalent norms;*

(2) *if $p_1 \le p_2$, $q_1 \le q_2$ and $\omega_2 \le C\omega_1$ for some constant $C$, then*

$$\mathscr{S}(\mathbf{R}^d) \subseteq M^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^d) \subseteq M^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^d) \subseteq \mathscr{S}'(\mathbf{R}^d),$$
$$M^1_{(v_0 v)}(\mathbf{R}^d) \subseteq M(\omega, \mathscr{B}) \subseteq M^\infty_{(1/(v_0 v))}(\mathbf{R}^d);$$

(3) *the sesqui-linear form* $(\,\cdot\,,\,\cdot\,)_{L^2}$ *on* $\mathscr{S}(\mathbf{R}^d)$ *extends to a continuous map from* $M^{p,q}_{(\omega)}(\mathbf{R}^d)\times M^{p',q'}_{(1/\omega)}(\mathbf{R}^d)$ *to* $\mathbf{C}$. *This extension is unique, except when* $p=q'\in\{1,\infty\}$. *On the other hand, if* $\|a\|=\sup|(a,b)_{L^2}|$, *where the supremum is taken over all* $b\in M^{p',q'}_{(1/\omega)}(\mathbf{R}^d)$ *such that* $\|b\|_{M^{p',q'}_{(1/\omega)}}\le 1$, *then* $\|\cdot\|$ *and* $\|\cdot\|_{M^{p,q}_{(\omega)}}$ *are equivalent norms;*

(4) *if* $p,q<\infty$, *then* $\mathscr{S}(\mathbf{R}^d)$ *is dense in* $M^{p,q}_{(\omega)}(\mathbf{R}^d)$, *and the dual space of* $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ *can be identified with* $M^{p',q'}_{(1/\omega)}(\mathbf{R}^d)$, *through the form* $(\,\cdot\,,\,\cdot\,)_{L^2}$. *Moreover,* $\mathscr{S}(\mathbf{R}^d)$ *is weakly dense in* $M^{\infty}_{(\omega)}(\mathbf{R}^d)$.

The following proposition is now a consequence of Remark 1.3 (5) in [40] and Proposition 1.8 (2).

**Proposition 1.9** *Let* $\mathscr{B}$ *be a translation invariant BF-space on* $\mathbf{R}^{2d}$ *and let* $\omega_j$ *for* $j\in J$ *be a family of elements in* $\mathscr{P}(\mathbf{R}^{2d})$ *such that for each* $s\ge 0$, *there is a constant* $C>0$, *and* $j_1,j_2\in J$ *such that*

$$\omega_{j_1}(x,\xi)\le C\langle x,\xi\rangle^{-s}\quad\text{and}\quad C^{-1}\langle x,\xi\rangle^{s}\le\omega_{j_2}(x,\xi).$$

*Then*

$$\cup_{j\in J}M(\omega_j,\mathscr{B})=\mathscr{S}'(\mathbf{R}^d)\quad\text{and}\quad\cap_{j\in J}M(\omega_j,\mathscr{B})=\mathscr{S}(\mathbf{R}^d).$$

## 1.4 The Bargmann transform

We shall now consider the Bargmann transform which is defined by the formula

$$(\mathfrak{V}f)(z)=\pi^{-d/4}\int_{\mathbf{R}^d}\exp\left(-\frac{1}{2}(\langle z,z\rangle+|y|^2)+2^{1/2}\langle z,y\rangle\right)f(y)\,dy,\tag{1.10}$$

when $f\in L^2(\mathbf{R}^d)$. We note that if $f\in L^2(\mathbf{R}^d)$, then the Bargmann transform $\mathfrak{V}f$ of $f$ is the entire function on $\mathbf{C}^d$, given by

$$(\mathfrak{V}f)(z)=\int\mathfrak{A}_d(z,y)f(y)\,dy,$$

or

$$(\mathfrak{V}f)(z)=\langle f,\mathfrak{A}_d(z,\,\cdot\,)\rangle,\tag{1.11}$$

where the Bargmann kernel $\mathfrak{A}_d$ is given by

$$\mathfrak{A}_d(z,y)=\pi^{-d/4}\exp\left(-\frac{1}{2}(\langle z,z\rangle+|y|^2)+2^{1/2}\langle z,y\rangle\right).$$

Here

$$\langle z, w\rangle = \sum_{j=1}^{d} z_j w_j, \quad \text{when } z = (z_1, \dots, z_d) \in \mathbf{C}^d \quad \text{and} \quad w = (w_1, \dots, w_d) \in \mathbf{C}^d,$$

and $\langle \,\cdot\, , \,\cdot\, \rangle$ denotes the duality between elements in $\mathscr{S}(\mathbf{R}^d)$ and $\mathscr{S}'(\mathbf{R}^d)$. We note that the right-hand side in (1.11) makes sense when $f \in \mathscr{S}'(\mathbf{R}^d)$ and defines an element in $A(\mathbf{C}^d)$, since $y \mapsto \mathfrak{A}_d(z, y)$ can be interpreted as an element in $\mathscr{S}(\mathbf{R}^d)$ with values in $A(\mathbf{C}^d)$. Here and in what follows, $A(\mathbf{C}^d)$ denotes the set of all entire functions on $\mathbf{C}^d$.

From now on we assume that $\phi$ in (1.7), (1.7)$'$ and (1.9) is given by

$$\phi(x) = \pi^{-d/4} e^{-|x|^2/2}, \tag{1.12}$$

if nothing else is stated. Then it follows that the Bargmann transform can be expressed in terms of the short-time Fourier transform $f \mapsto V_\phi f$. More precisely, for such choice of $\phi$, it follows by straight-forward computations that

$$\begin{aligned}(\mathfrak{V} f)(z) = (\mathfrak{V} f)(x + i\xi) &= e^{(|x|^2+|\xi|^2)/2} e^{-i\langle x,\xi\rangle} V_\phi f(2^{1/2}x, -2^{1/2}\xi) \\ &= e^{(|x|^2+|\xi|^2)/2} e^{-i\langle x,\xi\rangle} (S^{-1}(V_\phi f))(x, \xi),\end{aligned} \tag{1.13}$$

or equivalently,

$$\begin{aligned}V_\phi f(x, \xi) &= e^{-(|x|^2+|\xi|^2)/4} e^{-i\langle x,\xi\rangle/2} (\mathfrak{V} f)(2^{-1/2}x, -2^{-1/2}\xi). \\ &= e^{-i\langle x,\xi\rangle/2} S(e^{-|\,\cdot\,|^2/2}(\mathfrak{V} f))(x, \xi).\end{aligned} \tag{1.14}$$

Here $S$ is the dilation operator given by

$$(SF)(x, \xi) = F(2^{-1/2}x, -2^{-1/2}\xi). \tag{1.15}$$

For future references we observe that (1.13) and (1.14) can be formulated into

$$\mathfrak{V} = U_{\mathfrak{V}} \circ V_\phi, \quad \text{and} \quad U_{\mathfrak{V}}^{-1} \circ \mathfrak{V} = V_\phi,$$

where $U_{\mathfrak{V}}$ is the linear, continuous and bijective operator on $\mathscr{D}'(\mathbf{R}^{2d}) = \mathscr{D}'(\mathbf{C}^d)$, given by

$$(U_{\mathfrak{V}} F)(x, \xi) = e^{(|x|^2+|\xi|^2)/2} e^{-i\langle x,\xi\rangle} F(2^{1/2}x, -2^{1/2}\xi). \tag{1.16}$$

We are now prepared to make the following definition.

**Definition 1.10** Let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ and let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d} = \mathbf{C}^d$.

1. The space $\mathscr{B}_{\mathfrak{V}}(\omega)$ is the modified weighted $\mathscr{B}$-space which consists of all $F \in L^1_{loc}(\mathbf{R}^{2d}) = L^1_{loc}(\mathbf{C}^d)$ such that

$$\|F\|_{\mathscr{B}_{\mathfrak{V}}(\omega)} \equiv \|(S(Fe^{-|\cdot|^2/2}))\omega\|_{\mathscr{B}} < \infty.$$

Here $S$ is the dilation operator given by (1.15);
2. The space, $A(\omega, \mathscr{B})$ consists of all $F \in A(\mathbf{C}^d) \cap \mathscr{B}_{\mathfrak{V}}(\omega)$ with topology inherited from $\mathscr{B}_{\mathfrak{V}}(\omega)$;
3. The space $A_0(\omega, \mathscr{B})$ is given by

$$A_0(\omega, \mathscr{B}) \equiv \{(\mathfrak{V} f);\; f \in M(\omega, \mathscr{B})\},$$

and is equipped with the norm $\|F\|_{A_0(\omega,\mathscr{B})} \equiv \|f\|_{M(\omega,\mathscr{B})}$, when $F = \mathfrak{V} f$.

The following result shows that the norm in $A_0(\omega, \mathcal{B})$ is well-defined.

**Proposition 1.11** *Let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\mathscr{B}$ be an invariant BF-space on $\mathbf{R}^{2d}$ and let $\phi$ be as in (1.12). Then $A_0(\omega, \mathscr{B}) \subseteq A(\omega, \mathscr{B})$, and the map $\mathfrak{V}$ is an isometric injection from $M(\omega, \mathscr{B})$ to $A(\omega, \mathscr{B})$.*

*Proof* The result is an immediate consequence of (1.13), (1.14) and Definition 1.10. □

We employ the same notational conventions for the spaces of type $A$ and $A_0$ as we do for the modulation spaces. In the case $\omega = 1$ and $\mathscr{B} = L^2$, it follows from [1] that Proposition 1.11 holds, and the inclusion is replaced by equality. That is, we have $A_0^2 = A^2$ which is called the Bargmann-Foch space, or just the Foch space. In the next section we improve the latter property and show that for any choice of $\omega \in \mathscr{P}$ and every translation invariant BF-space $\mathscr{B}$, we have $A_0(\omega, \mathscr{B}) = A(\omega, \mathscr{B})$.

## 2 Mapping results for the Bargmann transform on modulation spaces

In this section we prove that $A_0(\omega, \mathscr{B})$ is equal to $A(\omega, \mathscr{B})$ for every choice of $\omega$ and $\mathscr{B}$. That is, we have the following.

**Theorem 2.1** *Let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$ and let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $A_0(\omega, \mathscr{B}) = A(\omega, \mathscr{B})$, and the map $f \mapsto \mathfrak{V} f$ from $M(\omega, \mathscr{B})$ to $A(\omega, \mathscr{B})$ is isometric and bijective.*

We need some preparations for the proof, and start with some remarks on the images of $\mathscr{S}(\mathbf{R}^d)$ and $\mathscr{S}'(\mathbf{R}^d)$ under the Bargmann transform. We denote these images by $A_{\mathscr{S}}(\mathbf{C}^d)$ and $A'_{\mathscr{S}}(\mathbf{C}^d)$ respectively, i.e.

$$A_{\mathscr{S}}(\mathbf{C}^d) \equiv \{\mathfrak{V} f;\; f \in \mathscr{S}(\mathbf{R}^d)\} \quad \text{and} \quad A'_{\mathscr{S}}(\mathbf{C}^d) \equiv \{\mathfrak{V} f;\; f \in \mathscr{S}'(\mathbf{R}^d)\}.$$

As a consequence of (1.13) and Propositions 1.9 and 1.11, the inclusion

$$A'_{\mathscr{S}}(\mathbf{C}^d) \subseteq \{F \in A(\mathbf{C}^d);\; \|Fe^{-|\cdot|^2/2}\sigma_{-N}\|_{L^p} < \infty \quad \text{for some } N \geq 0\} \tag{2.1}$$

holds. We recall that in [2] it is proved that (2.1) holds with equality when $p = \infty$. An essential part of our investigations concerns to prove that equality is attained in (2.1) for each $p \in [1, \infty]$.

2.1 The image of the harmonic oscillator on $M^2_{2N}$.

Next we discuss mapping properties for a modified harmonic oscillator on modulation spaces of the form $M^2_{2N}(\mathbf{R}^d)$, when $N$ is an integer. The operator we have in mind is given by

$$H \equiv |x|^2 - \Delta + d + 1, \tag{2.2}$$

and we show that they are bijective between appropriate modulation spaces. Since Hermite functions constitute an orthonormal basis for $L^2 = M^2$ and are eigenfunctions to the harmonic oscillator, we shall combine these facts to prove that dilations of such functions constitute an orthonormal basis for $M^2_{2N}$, for every integer $N$.

We recall that if $\phi$ is given by (1.12) and

$$a(x, \xi) = \sigma_2(x, \xi) = |x|^2 + |\xi|^2 + 1,$$

then $H = \operatorname{Tp}_\phi(a)$ (cf., e.g. Section 3 in [39]). Let $\mathscr{B}$ be a translation invariant BF-space and $\omega \in \mathscr{P}(\mathbf{R}^{2d})$. By Theorem 3.1 in [22] it now follows that $H = \operatorname{Tp}(a) = \operatorname{Tp}(\sigma_2)$ is a continuous isomorphism from $M(\sigma_2\omega, \mathscr{B})$ to $M(\omega, \mathscr{B})$. Since this holds for any weight $\omega$, it follows by induction and Banach's theorem that the following is true.

**Proposition 2.2** *Let $N$ be an integer, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ and let $\mathscr{B}$ be an invariant BF-space. Then $H^N$ on $\mathscr{S}'(\mathbf{R}^d)$ restricts to a continuous isomorphism from $M(\sigma_{2N}\omega, \mathscr{B})$ to $M(\omega, \mathscr{B})$. In particular, the set*

$$\{f \in \mathscr{S}'(\mathbf{R}^d);\ H^N f \in L^2(\mathbf{R}^d)\}$$

*is equal to $M^2_{2N}(\mathbf{R}^d)$, and the norm $f \mapsto \|H^N f\|_{L^2}$ is equivalent to $\|f\|_{M^2_{2N}}$.*

We remark that the second part of Proposition 2.2 is proved in [6]. From now on we assume that the norm and scalar product of $M^2_{2N}(\mathbf{R}^d)$ are given by

$$\|f\|_{M^2_{2N}} \equiv \|H^N f\|_{L^2} \quad \text{and} \quad (f, g)_{M^2_{2N}} \equiv (H^N f, H^N g)_{L^2}$$

respectively. Then it follows from Proposition 2.2 that $(e_j)_{j\in J}$ is an orthonormal basis for $M^2_{2N}$ if and only if $(H^N e_j)_{j\in J}$ is an orthonormal basis for $L^2$. In the following we use this fact to find an appropriate orthonormal basis for $M^2_{2N}(\mathbf{R}^d)$ in terms of Hermite functions.

More precisely, we recall that the Hermite function $h_\alpha$ with respect to the multi-index $\alpha \in \mathbf{N}^d$ is defined by

$$h_\alpha(x) = \pi^{-d/4}(-1)^{|\alpha|}(2^{|\alpha|}\alpha!)^{-1/2}e^{|x|^2/2}(\partial^\alpha e^{-|x|^2}).$$

The set $(h_\alpha)_{\alpha\in\mathbf{N}^d}$ is an orthonormal basis for $L^2$, and it follows from the definitions that $h_\alpha$ is an eigenvector of $H$ with eigenvalue $2|\alpha| + 2d + 1$ for every $\alpha$, i.e. $Hh_\alpha = (2|\alpha| + 2d + 1)h_\alpha$ (cf., e.g. [34]). The following result is now an immediate consequence of these observations.

**Lemma 2.3** *Let $N$ be an integer. Then*

$$\{(2|\alpha| + 2d + 1)^{-N} h_\alpha\}_{\alpha\in\mathbf{N}^d}$$

*is an orthonormal basis for $M^2_{2N}(\mathbf{R}^d)$.*

## 2.2 Mapping properties of $\mathfrak{V}$ on $M^2_{2N}$

We shall now prove $A^2_{0,N} = A^2_N$ when $N$ is a non-zero even integer. Important parts of these investigations are based upon the series representation of analytic functions, using the fact that every $F \in A(\mathbf{C}^d)$ is equal to its Taylor series, i.e.

$$F(z) = \sum_{\alpha\in\mathbf{N}^d} a_\alpha \frac{z^\alpha}{(\alpha!)^{1/2}}, \qquad a_\alpha = \frac{(\partial^\alpha F)(0)}{(\alpha!)^{1/2}}. \tag{2.3}$$

We also recall the result from [1] that $A^2_0(\mathbf{C}^d) = A^2(\mathbf{C}^d)$, and that $F \in A^2(\mathbf{C}^d)$, if and only if the coefficients in (2.3) satisfy

$$\|(a_\alpha)_{\alpha\in\mathbf{N}^d}\|_{l^2} = \sum_{\alpha\in\mathbf{N}^d} |a_\alpha|^2 < \infty.$$

Furthermore, $F = \mathfrak{V}f \in A^2(\mathbf{C}^d)$ if and only if $f \in L^2(\mathbf{R}^d)$ satisfies

$$f(x) = \sum_{\alpha\in\mathbf{N}^d} a_\alpha h_\alpha(x), \tag{2.4}$$

i.e. $f$ inherites the coefficients from $F$, and, since $\mathfrak{V}$ is isometric,

$$\|F\|_{A^2} = \|f\|_{L^2} = \|(a_\alpha)_{\alpha\in\mathbf{N}^d}\|_{l^2}. \tag{2.5}$$

We now have the following result.

**Proposition 2.4** *Let $N$ be an integer. Then the following is true:*

1. $A_0(\sigma_{2N}, L^2(\mathbf{R}^d))$ *consists of all $F \in A(\mathbf{C}^d)$ with expansion given by* (2.3)*, where*

$$\|F\| \equiv \|(a_\alpha \langle\alpha\rangle^N)_{\alpha\in\mathbf{N}^d}\|_{l^2} < \infty. \tag{2.6}$$

   *Furthermore, $\|\cdot\|$ and $\|\cdot\|_{A(\sigma_{2N},L^2)}$ are equivalent norms;*
2. $A(\sigma_{2N}, L^2(\mathbf{R}^d)) = A_0(\sigma_{2N}, L^2(\mathbf{R}^d))$.

For the proof we recall that

$$(\mathfrak{V}h_\alpha)(z) = \frac{z^\alpha}{(\alpha!)^{1/2}} \tag{2.7}$$

(cf. [1]), and we let $\mathscr{S}_0(\mathbf{R}^d)$ be the set of all sums in (2.4) such that $a_\alpha = 0$ except for finite numbers of $\alpha$.

*Proof* (1) First we consider the case when $F \in P(\mathbf{C}^d)$, and we let $a_\alpha$ be as in (2.3). Then it follows from (2.7) that $F$ is equal to $\mathfrak{V}f$, where $f \in \mathscr{S}_0(\mathbf{R}^d)$ is given by the finite sum (2.4). By (1.14), Proposition 2.2, Lemma 2.3, and (2.5) it follows that

$$C^{-1}\|F\|_{A(\sigma_{2N},L^2)} \le \|((2|\alpha| + 2d + 1)^N a_\alpha)_\alpha\|_{l^2} \le C\|F\|_{A(\sigma_{2N},L^2)}, \tag{2.8}$$

for some constant $C$ which is independent of $F \in P(\mathbf{C}^d)$. Since $\mathscr{S}_0$ is dense in $M^2_{2N}$, it follows that (2.8) holds for each $F \in A_0(\sigma_{2N}, L^2)$ when $a_\alpha$ is given by (2.3). This proves (1).

In order to prove (2) we recall that $\mathfrak{V} : M^2_{2N} \mapsto A_0(\sigma_{2N}, L^2)$ is a bijective isometry in view of Proposition 1.11. Hence Lemma 2.3 together with (2.7) show that

$$\left\{(2|\alpha| + 2d + 1)^{-N} \frac{z^\alpha}{(\alpha!)^{1/2}}\right\} \tag{2.9}$$

is an orthonormal basis for $A_0(\sigma_{2N}, L^2)$. By Proposition 1.11 it follows that $\|F_0\|_{A_0(\sigma_{2N},L^2)} = \|F_0\|_{A(\sigma_{2N},L^2)}$ when $F_0 \in A_0(\sigma_{2N}, L^2)$. Hence $A_0(\sigma_{2N}, L^2)$ is a closed subspace of $A(\sigma_{2N}, L^2)$. Consequently, we have the unique decomposition

$$A(\sigma_{2N}, L^2) = A_0(\sigma_{2N}, L^2) \oplus (A_0(\sigma_{2N}, L^2))^\perp,$$

and it follows that (2.9) is an orthonormal sequence in $A(\sigma_{2N}, L^2)$. The fact that every $F \in A(\sigma_{2N}, L^2)$ has a Taylor expansion now implies that (2.9) is an orthonormal basis for $A(\sigma_{2N}, L^2)$. Hence $(A_0(\sigma_{2N}, L^2))^\perp = \{0\}$ and the result follows. □

**Corollary 2.5** *There is equality in* (2.1) *in case* $p = 2$, *i.e.*

$$\{\mathfrak{V}f;\ f \in \mathscr{S}'(\mathbf{R}^d)\} = \{F \in A(\mathbf{C}^d);\ \|Fe^{-|\cdot|^2/2}\sigma_{-N}\|_{L^2} < \infty \ \text{for some } N \ge 0\}.$$

*Proof* The result follows from Proposition 2.4 and the fact that

$$\cup_{N\in\mathbf{Z}} M^2_N(\mathbf{R}^d) = \mathscr{S}'(\mathbf{R}^d).$$

□

### 2.3 Mapping properties of $\mathfrak{V}$ on $\mathscr{S}'$, and proof of the main theorem

We shall now consider the relation (2.1) and prove that we indeed have equality when $1 \le p \le 2$. In order to do this we need the following lemma. Here we let $B_r(z)$ denote the open ball in $\mathbf{C}^d$ with radius $r$ and center at $z \in \mathbf{C}^d$.

**Lemma 2.6** *There is a family $(B_j)_{j\in J}$ of open balls $B_j$ such that the following conditions are fulfilled:*

1. $\complement B_4(0) \subseteq \cup B_j$;
2. $B_j = B_{r_j}(z_j)$ *for some* $r_j$ *and* $z_j$ *such that* $|z_j| \ge 4$, $r_j \le 1/|z_j|$;
3. *there is a finite bound on the number of overlapping balls* $B_{4r_j}(z_j)$.

*Proof* Let $k \ge 4$ and let $N$ be a large integer, and consider the spheres

$$S_{k,l} = \{z \in \mathbf{C}^d;\, |z| = k + l/kN\}, \quad l = 0, \dots, kN-1.$$

On each sphere $S_{k,l}$, choose a finite number of points $z_j$ in such way that for any two closest points $z$ and $w$ the distance between them is $1/2k \le |z-w| \le 1/(k+1)$. It is easily seen that such a sequence $(z_j)$ exists when $N$ is chosen large enough. The result now follows if we choose $B_j = B_{r_j}(z_j)$ with $r_j = 1/(k+1)$. □

We now have the following result.

**Proposition 2.7** *Let $p \in [1,2]$ be fixed. Then there is equality in* (2.1).

*Proof* Let $\Omega_p$ be the set on the right-hand side of (2.1). In view of Corollary 2.5, it suffices to prove that $\Omega_p$ is independent of $p$. First assume that $p_1 \le p_2$, and let $r \in [1,\infty]$ be such that $1/p_2 + 1/r = 1/p_1$. Then it follows from Hölder's inequality that

$$\begin{aligned}\|Fe^{-|\cdot|^2/2}\langle\cdot\rangle^{-N-d-1}\|_{L^{p_1}} &= \|(Fe^{-|\cdot|^2/2}\langle\cdot\rangle^{-N})\langle\cdot\rangle^{-d-1}\|_{L^{p_1}}\\ &\le C\|Fe^{-|\cdot|^2/2}\langle\cdot\rangle^{-N}\|_{L^{p_2}},\end{aligned}$$

where $C = \|\langle\cdot\rangle^{-d-1}\|_{L^r} < \infty$. This proves that $\Omega_{p_2} \subseteq \Omega_{p_1}$.

The result therefore follows if we prove that $\Omega_1 \subseteq \Omega_2$. Assume that $F \in \Omega_1$. It suffices to prove that

$$\int\limits_{|z|\ge 4} |F(z)\langle z\rangle^{-N} e^{-|z|^2/2}|^2\, d\lambda(z) < \infty, \tag{2.10}$$

for some $N \ge 0$. Here and in what follows, $d\lambda(z)$ denotes the Lebesgue measure on $\mathbf{C}^d$.

Since $F \in A(\mathbf{C}^d)$, the mean-value property for harmonic functions gives

$$F(z) = C|z|^{2d} \int\limits_{|w|\le 1/|z|} F(z+w)\, d\lambda(w),$$

where $1/C$ is the volume of the $d$-dimensional unit ball. Since

$$C^{-1}e^{-|z|^2} \le e^{-|z+w|^2} \le Ce^{-|z|^2}, \quad C^{-1}\langle z\rangle \le \langle z+w\rangle \le C\langle z\rangle \quad \text{and} \quad \langle z\rangle \le C|z|$$

for some constant $C > 0$, when $|w| \le 1/|z|$ and $|z| \ge 3$, we get

$$\begin{aligned}
&\int\limits_{|z|\ge 4} |F(z)\langle z\rangle^{-N} e^{-|z|^2/2}|^2\, d\lambda(z) \\
&\le C_1 \int\limits_{|z|\ge 4} \left( \int\limits_{|w|\le 1/|z|} |F(z+w)|\, d\lambda(w) \langle z\rangle^{-N+2d} e^{-|z|^2/2} \right)^2 d\lambda(z) \\
&\le C_2 \int\limits_{|z|\ge 4} \left( \int\limits_{|w|\le 1/|z|} |F(z+w)\langle z+w\rangle^{-N+d} e^{-|z+w|^2/2}|\, d\lambda(w) \right)^2 \langle z\rangle^{2d} d\lambda(z) \\
&= C_2 \int\limits_{|z|\ge 4} \left( \int\limits_{|w-z|\le 1/|z|} |F(w)\langle w\rangle^{-N+d} e^{-|w|^2/2}|\, d\lambda(w) \right)^2 \langle z\rangle^{2d} d\lambda(z). \qquad (2.11)
\end{aligned}$$

Now let $B_j$ be as in Lemma 2.6. Then Lemma 2.6 (1) gives that the integral on the right-hand side of (2.11) is estimated from above by

$$C \sum_{j\in J} \int_{B_j} \left( \int\limits_{|w-z|\le 1/|z|} |F(w)\langle w\rangle^{-N+d} e^{-|w|^2/2}|\, d\lambda(w) \right)^2 \langle z\rangle^{2d}\, d\lambda(z).$$

Since $|w - z_j| \le 4/|z_j|$ when $|w - z| \le 1/|z|$ and $z \in B_j$, the last integral can be estimated by

$$\begin{aligned}
&C_1 \sum_{j\in J} \int_{B_j} \left( \int\limits_{|w-z_j|\le 4/|z_j|} |F(w)\langle w\rangle^{-N+d} e^{-|w|^2/2}|\, d\lambda(w) \right)^2 \langle z\rangle^{2d}\, d\lambda(z) \\
&\le C_2 \sum_{j\in J} \left( \int\limits_{w\in B_{4r_j}(z_j)} |F(w)\langle w\rangle^{-N+d} e^{-|w|^2/2}|\, d\lambda(w) \right)^2
\end{aligned}$$

$$
\begin{aligned}
&\leq C_2 \left( \sum_{j\in J} \int_{w\in B_{4r_j}(z_j)} |F(w)\langle w\rangle^{-N+d} e^{-|w|^2/2}|\, d\lambda(w) \right)^2 \\
&\leq C_3 \left( \int_{\mathbf{C}^d} |F(w)\langle w\rangle^{-N+d} e^{-|w|^2/2}|\, d\lambda(w) \right)^2,
\end{aligned}
$$

for some constants $C_1, \dots, C_3$. Here the first inequality follows from the fact that $\int_{B_j} \langle z\rangle^{2d}\, d\lambda(z) \leq C$ for some constant $C$ which is independent of $j$ by the property (2) in Lemma 2.6, and the last two inequalities follow from the fact that there is a finite number of of overlapping balls $B_{4r_j}(z_j)$ by (3) in Lemma 2.6. Summing up we have proved that

$$
\left( \int_{|z|\geq 4} |F(z)\langle z\rangle^{-N} e^{-|z|^2/2}|^2\, d\lambda(z) \right)^{1/2} \leq C \| F\langle\,\cdot\,\rangle^{-N+d} e^{-|\,\cdot\,|/2}\|_{L^1},
$$

for some constant $C$. The proof is complete. $\square$

*Proof of Theorem* 2.1 By Proposition 1.11 it follows that the map $f \mapsto \mathfrak{V} f$ is an isometric injection from $M(\omega, \mathscr{B})$ to $A(\omega, \mathscr{B})$. We have to show that this mapping is surjective.

Therefore assume that $F \in A(\omega, \mathscr{B})$. By Propositions 1.6, 2.4 and 2.7, there is an element $f \in \mathscr{S}'(\mathbf{R}^d)$ such that $F = \mathfrak{V} f$. We have

$$
\| f\|_{M(\omega,\mathscr{B})} = \|\mathfrak{V} f\|_{A(\omega,\mathscr{B})} = \|F\|_{A(\omega,\mathscr{B})} < \infty.
$$

Hence, $f \in M(\omega, \mathscr{B})$, and the result follows. The proof is complete. $\square$

## 3 Some consequences

In this section we present some results which are straight-forward consequences of Theorem 2.1 and well-known properties for modulation spaces. Most of these results can be found in [15,18,19,24].

We start with introducing some notations. We set

$$
A^{p,q}_{(\omega)}(\mathbf{C}^d) = A(\omega, L^{p,q}(\mathbf{R}^{2d})) \quad \text{and} \quad A^p_{(\omega)} = A^{p,p}_{(\omega)},
$$

when $\omega \in \mathscr{P}(\mathbf{C}^d)$ and $p, q \in [1, \infty]$. We also set

$$
A^{p,q} = A^{p,q}_{(\omega)} \quad \text{and} \quad A^p = A^p_{(\omega)} \quad \text{when } \omega = 1.
$$

Let

$$d\mu(w) = \pi^{-d} e^{-|w|^2} d\lambda(w),$$

where $d\lambda(z)$ is the Lebesgue measure on $\mathbf{C}^d$. We recall from [1,2] that the standard scalar product on $A^2(\mathbf{C}^d)$ is given by

$$(F, G)_{A^2} \equiv \int_{\mathbf{C}^d} F(w)\overline{G(w)}\, d\mu(w). \tag{3.1}$$

Furthermore, there is a convenient reproducing kernel on $A'_{\mathscr{S}}(\mathbf{C}^d)$, given by the formula

$$F(z) = \int_{\mathbf{C}^d} e^{(z,w)} F(w)\, d\mu(w), \quad F \in A'_{\mathscr{S}}(\mathbf{C}^d), \tag{3.2}$$

where $(\,\cdot\,,\,\cdot\,)$ is the scalar product on $\mathbf{C}^d$ (cf. [1,2]). For future references we observe that (3.2) is the same as

$$F(z) = \pi^{-d} \langle F \cdot e^{(z,\,\cdot\,)}, e^{-|\,\cdot\,|^2} \rangle, \quad F \in A'_{\mathscr{S}}(\mathbf{C}^d), \tag{3.2$'$}$$

## 3.1 Embedding and duality properties

We shall now discuss embedding properties. The following result follows immediately from Proposition 1.8 (2) and Theorem 2.1.

**Proposition 3.1** *Let $p_j, q_j \in [1, \infty]$, $\omega, \omega_j, v, v_0 \in \mathscr{P}(\mathbf{R}^{2d})$ for $j = 1, 2$ be such that $p_1 \le p_2$, $q_1 \le q_2$, $\omega$ is $v$-moderate, and $\omega_2 \le C\omega_1$ for some constant $C$. Also let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$ with respect to $v_0$. Then*

$$A_{\mathscr{S}}(\mathbf{C}^d) \subseteq A^{p_1,q_1}_{(\omega_1)}(\mathbf{C}^d) \subseteq A^{p_2,q_2}_{(\omega_2)}(\mathbf{C}^d) \subseteq A'_{\mathscr{S}}(\mathbf{C}^d),$$
$$A^1_{(v_0 v)}(\mathbf{C}^d) \subseteq A(\omega, \mathscr{B}) \subseteq A^\infty_{(1/(v_0 v))}(\mathbf{C}^d).$$

**Proposition 3.2** *Let $\omega \in \mathscr{P}(\mathbf{C}^d)$. Then $P(\mathbf{C}^d)$ is dense in $A^{p,q}_{(\omega)}(\mathbf{C}^d)$ when $1 \le p, q < \infty$.*

*Proof* Recall that $\mathscr{S}_0(\mathbf{R}^d)$, the set of finite linear combinations of the Hermite functions, is dense in $\mathscr{S}(\mathbf{R}^d)$ (cf. [34, Theorem V.13]). Hence the result follows immediately from Proposition 1.8, Theorem 2.1, and the fact that $\mathfrak{V}(\mathscr{S}_0(\mathbf{R}^d)) = P(\mathbf{C}^d)$.□

**Proposition 3.3** *Let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ and $p, q \in [1, \infty]$. Then the form (3.1) on $P(\mathbf{C}^d)$ extends to a continuous sesquilinear form on $A^{p,q}_{(\omega)}(\mathbf{C}^d) \times A^{p',q'}_{(1/\omega)}(\mathbf{C}^d)$, and*

$$|(F, G)_{A^2}| \le \|F\|_{A^{p,q}_{(\omega)}} \|G\|_{A^{p',q'}_{(1/\omega)}}. \tag{3.3}$$

*This extension is unique, except when $p = q' \in \{1, \infty\}$.*

*Moreover, let*

$$\|F\| \equiv \sup |(F, G)_{A^2}|, \tag{3.4}$$

*where the supremum is taken over all* $G \in P(\mathbf{C}^d)$ *(or* $G \in A^{p',q'}_{(1/\omega)}(\mathbf{C}^d)$*) such that* $\|G\|_{A^{p',q'}_{(1/\omega)}} \le 1$*. Then* $\|\cdot\|$ *and* $\|\cdot\|_{A^{p,q}_{(\omega)}}$ *are equivalent norms on* $A^{p,q}_{(\omega)}(\mathbf{C}^d)$*.*

*Proof* The extension assertions and the inequality (3.3) are immediate consequences of Proposition 1.8 (3), Theorem 2.1 and Hölder's inequality.

Let

$$\Omega_M = \{g \in M^{p',q'}_{(1/\omega)}(\mathbf{R}^d);\ \|g\|_{M^{p',q'}_{(1/\omega)}} \le 1\},$$
$$\Omega_A = \{G \in A^{p',q'}_{(1/\omega)}(\mathbf{C}^d);\ \|G\|_{A^{p',q'}_{(1/\omega)}} \le 1\}.$$

For any $F \in A^{p,q}_{(\omega)}$ there is a unique $f \in M^{p,q}_{(\omega)}$ such that $\mathfrak{V}f = F$. By Proposition 1.8 (3) and Theorem 2.1 we get

$$\begin{aligned}\|F\|_{A^{p,q}_{(\omega)}} = \|f\|_{M^{p,q}_{(\omega)}} &\le C \sup_{g \in \mathscr{S} \cap \Omega_M} |(f, g)_{L^2}| \\ &\le C \sup_{g \in \Omega_M} |(f, g)_{L^2}| = C \sup_{G \in \Omega_A} |(F, G)_{A^2}| \le C\|F\|_{A^{p,q}_{(\omega)}},\end{aligned}$$

for some constant $C$, where the first inequality follows from Proposition 1.8 (3) and the last one from (3.3). Since any $g \in \mathscr{S}$ can be approximated by its truncated Hermite expansion, the supremum over $\mathscr{S}$ may be substituted for a supremum over the finite Hermite expansions. These, in turn, are the inverse images of the polynomials in $P(\mathbf{C}^d)$ which proves the last statement. □

*Remark 3.4* We note that the integral in (3.1) is well-defined when $F \in A^{p,q}_{(\omega)}(\mathbf{C}^d)$, $G \in A^{p',q'}_{(1/\omega)}(\mathbf{C}^d)$, $\omega \in \mathscr{P}(\mathbf{C}^d)$ and $p, q \in [1, \infty]$. Also in the case $p = q' \in \{1, \infty\}$, we take this integral as the definition of $(F, G)_{A^2}$, and we remark that the extension of the form $(\,\cdot\,,\,\cdot\,)_{A^2}$ on $P(\mathbf{C}^d)$ to $A^{p,q}_{(\omega)}(\mathbf{C}^d) \times A^{p',q'}_{(1/\omega)}(\mathbf{C}^d)$ is unique also in this case, if in addition narrow convergence is imposed (cf. Definition 3.13 and Proposition 3.14 below).

**Proposition 3.5** *Let* $\omega \in \mathscr{P}(\mathbf{C}^d)$ *and* $1 \le p, q < \infty$*. Then the dual of* $A^{p,q}_{(\omega)}(\mathbf{C}^d)$ *can be identified with* $A^{p',q'}_{(1/\omega)}(\mathbf{C}^d)$ *through the form* $(\,\cdot\,,\,\cdot\,)_{A^2}$*. Moreover,* $P(\mathbf{C}^d)$ *is weakly dense in* $A^{\infty}_{(\omega)}(\mathbf{C}^d)$*.*

*Proof* The result is an immediate consequence of Proposition 1.8 (4), Theorem 2.1, and the fact that $\mathscr{S}_0$ is dense in $\mathscr{S}$. □

3.2 Reproducing kernel and Berezin–Toeplitz operators

For general $F \in L^2(d\mu)$, it is proved in [1] that the right-hand sides of (3.2) and (3.2)$'$ defines an orthonormal projection $\Pi_A$ of elements in $L^2(d\mu)$ onto $A^2(\mathbf{C}^d)$. We recall that $A^2$ is the image of $L^2$ under the Bargmann transform. In what follows we address equivalent projections where the Bargmann transform is replaced by the short-time Fourier transform. We use these relations to extend $\Pi_A$ to more general spaces of distributions.

When dealing with the short time Fourier transform, it is convenient to consider the twisted convolution $\widehat{*}$ on $L^1(\mathbf{R}^{2d})$, which is defined by the formula

$$(F \widehat{*} G)(x, \xi) = (2\pi)^{-d/2} \iint F(x-y, \xi-\eta) G(y, \eta) e^{-i\langle x-y, \eta\rangle}\, dy\, d\eta.$$

(cf., e.g. [13,19].) By straight-forward computations it follows that $\widehat{*}$ restricts to a continuous multiplication on $\mathscr{S}(\mathbf{R}^{2d})$. Furthermore, the map $(F, G) \mapsto F \widehat{*} G$ from $\mathscr{S}(\mathbf{R}^{2d}) \times \mathscr{S}(\mathbf{R}^{2d})$ to $\mathscr{S}(\mathbf{R}^{2d})$ extends uniquely to continuous mappings from $\mathscr{S}'(\mathbf{R}^{2d}) \times \mathscr{S}(\mathbf{R}^{2d})$ and $\mathscr{S}(\mathbf{R}^{2d}) \times \mathscr{S}'(\mathbf{R}^{2d})$ to $\mathscr{S}'(\mathbf{R}^{2d}) \cap C^\infty(\mathbf{R}^{2d})$.

*Remark 3.6* By Fourier's inversion formula, it follows that

$$(V_{\phi_1} f) \widehat{*} (V_{\phi_2} \phi_3) = (\phi_3, \phi_1)_{L^2(\mathbf{R}^d)} \cdot V_{\phi_2} f \tag{3.5}$$

for every $f \in \mathscr{S}'(\mathbf{R}^d)$ and every $\phi_j \in \mathscr{S}(\mathbf{R}^d)$. The relation (3.5) is used in [13,19] to prove the following properties:

1. The modulation spaces are independent of the choice of window functions [cf. Proposition 1.8 (1)];
2. Let $\phi \in \mathscr{S}(\mathbf{R}^d)$ satisfy $\|\phi\|_{L^2} = 1$, and let $\Pi$ be the mapping on $\mathscr{S}'(\mathbf{R}^{2d})$, given by

$$\Pi F \equiv F \widehat{*} (V_\phi \phi). \tag{3.6}$$

Also let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, and set

$$V_\phi(\Sigma) \equiv \{V_\phi f\, ;\ f \in \Sigma\},$$

when $\Sigma \subseteq \mathscr{S}'(\mathbf{R}^d)$. Then

$$\Pi: \mathscr{S}(\mathbf{R}^{2d}) \to V_\phi(\mathscr{S}(\mathbf{R}^d)) \subseteq \mathscr{S}(\mathbf{R}^{2d}) \tag{3.7}$$

$$\Pi: \mathscr{S}'(\mathbf{R}^{2d}) \to V_\phi(\mathscr{S}'(\mathbf{R}^d)) \subseteq \mathscr{S}'(\mathbf{R}^{2d}) \tag{3.8}$$

$$\Pi: \mathscr{B}(\omega) \to V_\phi(M(\omega, \mathscr{B})) \tag{3.9}$$

are continuous projections. Furthermore, if in addition $\phi$ is given by (1.12), then it follows by straight-forward computations that $\Pi$ is self-adjoint on $L^2(\mathbf{R}^{2d})$. Hence, for such a choice of $\phi$ it follows that $\Pi$ is an orthonormal projection from $L^2(\mathbf{R}^{2d})$ to $V_\phi(L^2(\mathbf{R}^d))$.

Now we recall that the orthonormal projection $\Pi_A$ of $L^2(d\mu)$ onto $A^2(\mathbf{C}^d)$ is given by the right-hand sides of the reproducing formulas (3.2) and (3.2)$'$, i.e.

$$(\Pi_A F)(z) = \int_{\mathbf{C}^d} e^{(z,w)} F(w)\, d\mu(w), \quad F \in L^2(d\mu). \tag{3.10}$$

We extend the definition of $\Pi_A$ to the set

$$\mathfrak{S}'(\mathbf{C}^d) \equiv \{F \in \mathscr{D}'(\mathbf{C}^d);\ Fe^{-|\cdot|^2/2} \in \mathscr{S}'(\mathbf{C}^d)\},$$

by the formula

$$(\Pi_A F)(z) = \pi^{-d}\langle F \cdot e^{(z,\cdot)}, e^{-|\cdot|^2}\rangle, \quad F \in \mathfrak{S}'(\mathbf{C}^d), \tag{3.10$'$}$$

and we note that (3.10)$'$ agree with (3.10) when $F \in L^2(d\mu)$.

We note that the set $\mathfrak{S}'(\mathbf{C}^d)$ is equal to $U_{\mathfrak{V}}(\mathscr{S}'(\mathbf{R}^{2d}))$, where $U_{\mathfrak{V}}$ is given by (1.16). Furthermore, by letting $\phi_j(x) = \phi(x) = \pi^{-d/4}e^{-|x|^2/2}$, the reproducing formulas (3.2) and (3.2)$'$ are straight-forward consequence of (1.14) and (3.5). From these computations it also follows that $\Pi_A$ is the conjugation of $\Pi$ in (3.6) by $U_{\mathfrak{V}}$, i.e.

$$\Pi_A = U_{\mathfrak{V}} \circ \Pi \circ U_{\mathfrak{V}}^{-1}. \tag{3.11}$$

The following result is now an immediate consequence of these observations, Theorem 2.1 and (3.7)–(3.9). Here we let

$$\mathfrak{S}(\mathbf{C}^d) \equiv \{F \in \mathscr{D}'(\mathbf{C}^d);\ Fe^{-|\cdot|^2/2} \in \mathscr{S}(\mathbf{C}^d)\},$$

which is the same as $U_{\mathfrak{V}}(\mathscr{S}(\mathbf{R}^{2d}))$.

**Proposition 3.7** *Let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$, and let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then the following hold:*

1. *$\Pi_A$ is a continuous projection from $\mathfrak{S}'(\mathbf{C}^d)$ to $A'_{\mathscr{S}}(\mathbf{C}^d)$;*
2. *$\Pi_A$ restricts to a continuous projection from $\mathscr{B}_{\mathfrak{V}}(\omega)$ to $A(\omega, \mathscr{B})$;*
3. *$\Pi_A$ restricts to a continuous projection from $\mathfrak{S}(\mathbf{C}^d)$ to $A_{\mathscr{S}}(\mathbf{C}^d)$.*

Next we consider Toeplitz operators in the context of the Bargmann transform. It follows from (1.8) that if $a \in \mathscr{S}'(\mathbf{R}^{2d})$ and $f, \phi \in \mathscr{S}(\mathbf{R}^d)$, then

$$(V_\phi \circ \mathrm{Tp}_\phi(a))f = \Pi(a \cdot F_0), \quad \text{where } F_0 = V_\phi f. \tag{3.12}$$

The close relation between the short-time Fourier transform and the Bargmann transform motivates the following definition.

**Definition 3.8** Let $a \in \mathscr{S}'(\mathbf{C}^d)$, and let $S$ be as in (1.15). Then the Berezin–Toeplitz operator $\mathrm{T}_{\mathfrak{V}}(a)$ is the continuous operator on $A'_{\mathscr{S}}(\mathbf{C}^d)$, given by the formula

$$\mathrm{T}_{\mathfrak{V}}(a)F = \Pi_A((S^{-1}a)F).$$

It follows from (3.11) and (3.12) that

$$\mathrm{T}_{\mathfrak{V}}(a) \circ \mathfrak{V} = \mathfrak{V} \circ \mathrm{Tp}(a).$$

The following result is now an immediate consequence of the latter property and [22, Theorem 3.1]. We recall that $\mathscr{P}_0$ consists of all smooth elements $\omega$ in $\mathscr{P}$ such that $(\partial^\alpha \omega)/\omega \in L^\infty$.

**Proposition 3.9** *Let $\omega \in \mathscr{P}(\mathbf{C}^d)$, $\omega_0 \in \mathscr{P}_0(\mathbf{C}^d)$, and let $\mathscr{B}$ be a translation invariant BF-space. Then $T_{\mathfrak{V}}(\omega_0)$ is continuous and bijective from $A(\omega, \mathscr{B})$ to $A(\omega/\omega_0, \mathscr{B})$.*

### 3.3 Mapping properties of Harmonic oscillator on modulation spaces

We shall now show how our investigations can be used to get spectral properties of harmonic oscillator. For each $t \in \mathbf{C}$, we let the $t$-harmonic oscillator be defined by

$$H_t \equiv (|x|^2 - \Delta + 2t - d)/2, \tag{3.13}$$

and we observe that $2H_t$ agrees with (2.2) when $t = d + 1/2$. By Subsection 3.e in [1] it follows that

$$\mathfrak{V}(H_t f) = 2\left(\sum_{j=1}^{d} z_j \frac{\partial F}{\partial z_j}\right) + t \cdot F, \quad F = \mathfrak{V} f \in A'_{\mathscr{S}}(\mathbf{C}^d), \tag{3.14}$$

which implies that if $F \in A'_{\mathscr{S}}(\mathbf{C}^d)$ is given by

$$F(z) = (\mathfrak{V} f)(z) = \sum_\alpha a_\alpha \frac{z^\alpha}{(\alpha!)^{1/2}}, \tag{3.15}$$

then

$$\mathfrak{V}(H_t^N f)(z) = \sum_\alpha a_\alpha \frac{(|\alpha| + t)^N z^\alpha}{(\alpha!)^{1/2}},$$

as $N \geq 0$ is an integer.

For the harmonic oscillator we now have the following result.

**Theorem 3.10** *Let $t \in \mathbf{C}$ and $N \in \mathbf{Z}_+$ be fixed, $\omega \in \mathscr{P}(\mathbf{R}^d)$, and let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$. Then the following conditions are equivalent:*

1. *$H_t^N$ is continuous and bijective on $\mathscr{S}(\mathbf{R}^d)$;*
2. *$H_t^N$ is continuous and bijective on $\mathscr{S}'(\mathbf{R}^d)$;*

3. *$H_t^N$ is continuous and bijective from $L^2(\mathbf{R}^d)$ to $M^2_{-2N}(\mathbf{R}^d)$;*
4. *$H_t^N$ is continuous and bijective from $M(\omega, \mathscr{B})$ to $M(\omega/\sigma_{2N}, \mathscr{B})$;*
5. *$t \notin \{-n;\ n \in \mathbf{N}\}$.*

*Furthermore, if* (5) *is fulfilled, then* (1)–(4) *hold for each $N \in \mathbf{Z}$.*

By (3.14) and Theorem 2.1 it follows that Theorem 3.10 is equivalent to the following proposition.

**Proposition 3.11** *Let $t \in \mathbf{C}$ and $N \in \mathbf{Z}_+$ be fixed, $\omega \in \mathscr{P}(\mathbf{R}^d)$, and let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$. Also let $T$ be the operator*

$$T : F \mapsto \left( \sum_{j=1}^{d} z_j \frac{\partial F}{\partial z_j} \right) + tF.$$

*Then the following conditions are equivalent:*

1. *$T^N$ is continuous and bijective on $A_{\mathscr{S}}(\mathbf{C}^d)$;*
2. *$T^N$ is continuous and bijective on $A'_{\mathscr{S}}(\mathbf{C}^d)$;*
3. *$T^N$ is continuous and bijective from $A^2(\mathbf{C}^d)$ to $A^2_{-2N}(\mathbf{C}^d)$;*
4. *$T^N$ is continuous and bijective from $A(\omega, \mathscr{B})$ to $A(\omega/\sigma_{2N}, \mathscr{B})$;*
5. *$t \notin \{-n\,;\ n \in \mathbf{N}\}$.*

*Proof of Theorem* 3.10 *and Proposition* 3.11 The results follow if we prove that each one of (1)–(4) implies (5) in Proposition 3.11, that (5) implies (3) in Proposition 3.11, and that (3) $\Rightarrow$ (4), (4) $\Rightarrow$ (1) and (4) $\Rightarrow$ (2) in Theorem 3.10.

First we assume that (5) in Proposition 3.11 does not hold. Then $T^N z^\alpha = 0$ when $|\alpha| = -t \in \mathbf{N}$. This implies that (1)–(4) in Proposition 3.11 do not hold. It remains to prove that (5) $\Rightarrow$ (3) in Proposition 3.11, and that (3) $\Rightarrow$ (4), (4) $\Rightarrow$ (1) and (4) $\Rightarrow$ (2) in Theorem 3.10.

Therefore, assume that (5) in Proposition 3.11 holds, and let $S$ be the operator on $A'_{\mathscr{S}}(\mathbf{C}^d)$ which maps $F$ in (3.15) into

$$\sum_\alpha a_\alpha \frac{(2|\alpha| + 2d + 1)^N}{(|\alpha| + t)^N} \frac{z^\alpha}{(\alpha!)^{1/2}}.$$

Then $S \circ T^N = \mathfrak{V} \circ H^N \circ \mathfrak{V}^{-1}$, where $H$ is given by (2.2). Furthermore, it follows from the assumptions on $t$ that $S$ is continuous and bijective on each $A^2_{2N_0}$ for every integer $N_0$. The assertion (3) in Proposition 3.11 and Theorem 3.10 are now consequences of Proposition 2.2, and (4) in Theorem 3.10 follows from [22, Theorem 3.1] and Theorem 2.1.

Finally, (1) and (2) in Theorem 3.10 now follows from (4) and the relations

$$\mathscr{S}'(\mathbf{R}^d) = \cup_{\omega \in \mathscr{P}} M(\omega, \mathscr{B}) \quad \text{and} \quad \mathscr{S}(\mathbf{R}^d) = \cap_{\omega \in \mathscr{P}} M(\omega, \mathscr{B}).$$

The proof is complete. $\square$

*Remark 3.12* Let $N \geq 0$ be an integer, $t \in \mathbf{C} \setminus \{-d - 2n\,;\ n \in \mathbf{N}\}$, $m, s \in \mathbf{R}$, and let $\mathrm{Sh}_1^m(\mathbf{R}^{2d})$ be the Shubin-class of all smooth symbols $a$ on $\mathbf{R}^{2d}$ which satisfy

$$|\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq C_{\alpha,\beta} \langle x, \xi \rangle^{m-|\alpha|-|\beta|},$$

for some constants $C_{\alpha,\beta}$ which only depend on $a, \alpha$ and $\beta$. Also let the pseudo-differential operator $\mathrm{Op}_s(b)$ with symbol $b \in \mathscr{S}'(\mathbf{R}^{2d})$ be defined in the usual way (cf., e.g. [22]). Then it follows that $H_t^N = \mathrm{Op}_s(a)$ for some $a \in \mathrm{Sh}_1^{2N}(\mathbf{R}^{2d})$. Furthermore, since $\mathrm{Sh}_1^0(\mathbf{R}^{2d})$ is a Wiener algebra (cf. [4,7]), the proof of [22, Theorem 2.1] in combination with Theorem 3.10 show that the inverse $H_t^{-N}$ of $H_t^N$ is equal to $\mathrm{Op}_s(b)$ for some $b \in \mathrm{Sh}_1^{-2N}(\mathbf{R}^{2d})$.

### 3.4 The narrow convergence

We shall now discuss the narrow convergence for modulation spaces and discuss the corresponding concept in context of generalized Bargmann-Foch spaces.

The main reason why introducing the narrow convergence in context of Bargmann-Foch spaces is to improve the possibilities for approximating elements in $A^{p,q}_{(\omega)}(\mathbf{C}^d)$ with elements in $P(\mathbf{C}^d)$. In terms of norm convergence, Proposition 3.2 does not guarantee that such approximations are possible when $p = \infty$ or $q = \infty$. In the case $p = q' \notin \{1, \infty\}$, the situation is usually handled by using weak$^*$-topology, if necessary. However, the remaining case $p = q' \in \{1, \infty\}$ may be critical since $P(\mathbf{C}^d)$ is neither dense in $A^{\infty,1}_{(\omega)}(\mathbf{C}^d)$ nor in $A^{1,\infty}_{(\omega)}(\mathbf{C}^d)$. Here we shall see that such problems may be avoided by using an analogy of the narrow convergence from [38] for the $A^{p,q}_{(\omega)}$ spaces.

First we define the narrow convergence of such spaces.

**Definition 3.13** Let $\omega \in \mathscr{P}(\mathbf{C}^d)$, $S$ be as in (1.15), $p, q \in [1, \infty]$ and let $F_j, F \in A^{p,q}_{(\omega)}(\mathbf{C}^d)$, $j \geq 1$. Then $F_j$ is said to converge narrowly to $F$ as $j$ turns to infinity whenever

1. $F_j \to F$ in $A'_{\mathscr{S}}(\mathbf{C}^d)$ as $j \to \infty$;
2. if

$$H_j(\xi) = \left( \int_{\mathbf{R}^d} |F_j(z) e^{-|z|^2/2} (S^{-1}\omega)(z)|^p \, dx \right)^{1/p},$$

$$H(\xi) = \left( \int_{\mathbf{R}^d} |F(z) e^{-|z|^2/2} (S^{-1}\omega)(z)|^p \, dx \right)^{1/p},$$

with $z = x + i\xi$ and $x, \xi \in \mathbf{R}^d$, then $H_j \to H$ in $L^q(\mathbf{R}^d)$.

The following proposition justifies the definition of narrow convergence.

**Proposition 3.14** *Let $\omega \in \mathscr{P}(\mathbf{C}^d)$ and let $G \in A^{1,\infty}_{(1/\omega)}(\mathbf{C}^d)$. Then the following hold:*

1. *$P(\mathbf{C}^d)$ is dense in $A^{\infty,1}_{(\omega)}(\mathbf{C}^d)$ with respect to narrow convergence;*
2. *if $F_j \in A^{\infty,1}_{(\omega)}(\mathbf{C}^d)$ converges narrowly to $F \in A^{\infty,1}_{(\omega)}(\mathbf{C}^d)$ as $j \to \infty$, then $(F_j, G) \to (F, G)$ as $j \to \infty$.*

*Proof* The result follows immediately from Proposition 1.10 and Lemma 1.11 in [38], Theorem 2.1 and the fact that $P(\mathbf{C}^d)$ is dense in $A_{\mathscr{S}}(\mathbf{C}^d)$. □

## Appendix A

In this appendix we present the announced alternative approach, where the key result Proposition A.1 is obtained after several discussions with K.H. Gröchenig (cf. [20]). In a way similar as in Sect. 2, we are especially interested of the sets

$$\begin{aligned} \Omega^0_{t,p} &= \{F \in A(\mathbf{C}^d);\ e^{-s|\cdot|^2}F \in L^p(\mathbf{C}^d), \quad \text{for some } s < t\} \\ \Omega_{t,p} &= \{F \in A(\mathbf{C}^d);\ \sigma_{-N}e^{-t|\cdot|^2}F \in L^p(\mathbf{C}^d), \quad \text{for some } N \geq 0\}, \end{aligned} \tag{A.1}$$

which are related to (2.1). Here $t > 0$ is fixed and $p \in [1, \infty]$, and we note that $\Omega^0_{t,p} \subseteq \Omega_{t,p}$. We have now the following result which is one of the key steps when proving Theorem 2.1.

**Proposition A.1** *Let $t > 0$. Then $\Omega^0_{t,p}$ and $\Omega_{t,p}$ are independent of $p \in [1, \infty]$.*

For the proof we need the following lemma.

**Lemma A.2** *Let $\Pi_A$ be as in (3.10). If $F \in \Omega_{1,1}$, then $\Pi_A F = F$.*

*Proof* Let $F \in \Omega_{1,1}$, and set

$$a_\alpha = \frac{\partial^\alpha F(0)}{\alpha!}. \tag{A.2}$$

It is obvious that $F_1 \equiv \Pi_A F$ is well-defined and defines an entire function on $\mathbf{C}^d$. We have to prove that $F_1 = F$. Since both $F$ and $F_1$ are entire functions it suffices to prove

$$\partial^\alpha F_1(0) = \partial^\alpha F(0), \tag{A.3}$$

for every multi-index $\alpha$.

By the assumptions of $\Omega_{1,1}$ we may replace the order of integration and differentiation when applying derivatives on the right-hand side of (3.10). Hence, if $\Delta_d \equiv$

$[0, \infty)^d$ and $I_d \equiv [0, 2\pi]^d$, we get

$$\begin{aligned}
\partial^\alpha F_1(0) &= \pi^{-d} \int\limits_{\mathbf{C}^d} \overline{w}^\alpha F(w) e^{-|w|^2}\, d\lambda(w) \\
&= \pi^{-d} \int\limits_{\Delta_d} \left( \int\limits_{I_d} r^\alpha e^{-i\langle \theta, \alpha \rangle} F(r_1 e^{i\theta_1}, \dots, r_d e^{i\theta_d}) r_1 \cdots r_d e^{-|r|^2}\, d\theta \right) dr \\
&= \pi^{-d} \int\limits_{\Delta_d} r^\alpha r_1 \cdots r_d e^{-|r|^2} J_\alpha(r)\, dr, \qquad \text{(A.4)}
\end{aligned}$$

where

$$J_\alpha(r) = \int\limits_{I_d} e^{-i\langle \theta, \alpha \rangle} F(r_1 e^{i\theta_1}, \dots, r_d e^{i\theta_d})\, d\theta \qquad \text{(A.5)}$$

Here we have taken $w = (r_1 e^{i\theta_1}, \dots, r_d e^{i\theta_d})$, with $r = (r_1, \dots, r_d) \in \Delta_d$ and $\theta = (\theta_1, \dots, \theta_d)$, as new variables of integration.

We shall evaluate $J_\alpha(r)$. Since $F$ is an entire function, the function

$$\mathbf{R}^d \ni \theta \mapsto F(t_1 e^{i\theta_1}, \dots, t_d e^{i\theta_d}) = \sum_\beta a_\beta r^\beta e^{i\langle \theta, \beta \rangle} \qquad \text{(A.6)}$$

is a smooth and periodic for every $r \in \Delta_d$. This implies that

$$\sum_\beta |a_\beta r^\beta| < \infty \qquad \text{(A.7)}$$

for every $r \in \Delta_d$.

By (A.5) and (A.6) we get

$$J_\alpha(r) = \int\limits_{I_d} \left( \sum_\beta a_\beta r^\beta e^{i\langle \theta, \beta - \alpha \rangle} \right) d\theta.$$

It now follows from (A.7) that we may interchange the order of summation and integration. This gives

$$J_\alpha(r) = \sum_\beta a_\beta r^\beta \int\limits_{I_d} e^{i\langle \theta, \beta - \alpha \rangle}\, d\theta = (2\pi)^d a_\alpha r^\alpha. \qquad \text{(A.8)}$$

By inserting (A.8) into (A.4) and taking $u_j = r_j^2$ as new variables of integration, (A.2) gives

$$\partial^\alpha F_1(0) = 2^d a_\alpha \int_{\Delta_d} r^{2\alpha} r_1 \cdots r_d e^{-|r|^2}\, dr$$
$$= a_\alpha \int_{\Delta_d} u^\alpha e^{-(u_1+\cdots+u_d)}\, du = a_\alpha \alpha! = \partial^\alpha F(0).$$

The proof is complete. □

*Proof of Proposition* A.1 We only prove that $\Omega_{t,p}$ is independent of $p \in [1, \infty]$. The assertion for $\Omega^0_{t,p}$ follows by similar arguments and is left for the reader. We may assume that $t = 1/2$. Then the result follows if we prove

$$\Omega_{t,q} \subseteq \Omega_{t,p} \quad \text{and} \quad \Omega_{t,1} \subseteq \Omega_{t,\infty}, \quad \text{when } p \le q \text{ and } t = 1/2. \tag{A.9}$$

First we assume that $F \in \Omega_{1/2,q}$. Then $\sigma_{-N} e^{-|\cdot|^2/2} F \in L^q$ for some $N \ge 0$. Hence, if $r \in [1, \infty]$ satisfies $1/p = 1/q + 1/r$, Hölder's inequality gives

$$\|\sigma_{-(N+2d+1)} e^{-|\cdot|^2/2} F\|_{L^p} \le C \|\sigma_{-N} e^{-|\cdot|^2/2} F\|_{L^q},$$

where $C = \|\sigma_{-2d-1}\|_{L^r} < \infty$. This gives the first inclusion in (A.9).

In order to prove $\Omega_{1/2,1} \subseteq \Omega_{1/2,\infty}$, we assume that $F \in \Omega_{1/2,1}$. Then

$$F(z) = \pi^{-d} \int_{\mathbf{C}^d} F(w) e^{(z,w)-|w|^2}\, d\lambda(w)$$

by Lemma A.2. This gives

$$|\sigma_{-N}(z) e^{-|z|^2/2} F(z)| \le C_1 \int_{\mathbf{C}^d} \sigma_{-N}(z) |F(w)|\, |e^{-|z|^2/2+(z,w)-|w|^2}|\, d\lambda(w)$$
$$\le C_1 \int_{\mathbf{C}^d} \langle z\rangle^{-N} e^{-|w|^2/2} |F(w)|\, e^{-|z-w|^2/2}\, d\lambda(w)$$
$$\le C_2 \int_{\mathbf{C}^d} (\langle w\rangle^{-N} e^{-|w|^2/2} |F(w)|)\, (\langle z-w\rangle^N e^{-|z-w|^2/2})\, d\lambda(w),$$

for some constants $C_1$ and $C_2$. By applying the supremum norm, Hölder's inequality now gives

$$\|\sigma_{-N} e^{-|\cdot|^2/2} F\|_{L^\infty} \le C \|\sigma_{-N} e^{-|\cdot|^2/2} F\|_{L^1},$$

where

$$C = C_2 \|\sigma_N e^{-|\cdot|^2/2}\|_{L^\infty} < \infty.$$

Hence $\Omega_{1/2,1} \subseteq \Omega_{1/2,\infty}$, and the proof is complete. □

*Remark A.3* Proposition A.1 seems to be less technical comparing to Proposition 2.7. On the other hand, the proof of Proposition 2.7 can be used in more general situations where the analyticity assumptions are relaxed. Furthermore, both Proposition A.1 or the proof of Proposition 2.7 can be used to extend Theorem 2.1, to involve mapping properties of the Bargmann transform on modulation spaces in context of Gelfand-Shilov spaces. These and other questions will be investigated in future papers.

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