UNIVERSITY OF AGDER

Tenth grade pupils' engagement with the use of brackets in Algebra

A case study in a school in Albania

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This Master's Thesis is carried out as a part of the education at the University of Agder and is therefore approved as a part of this education. However, this does not imply that the University answers for the methods that are used or the conclusions that are drawn.

University of Agder, 2012

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Acknowledgement

I had the luck to be in front of a choice, which seemed as challenging as it is valuable for the direction of my career. Although the three years studied pure mathematics, I chose to accept the challenge of studying mathematics education, taking up the opportunity given by the collaborative relations between University of Shkodra “Luigj Gurakuqi” and University of Agder.

In front of me, a new life and a new way of living was opened for two years. The reality of living in Kristiansand and, studying and being a master student in University of Agder was much better than I expected. I felt the worm relationships and the help that several people gave to me during these two years.

Considering what I perceived from being a teacher for several months while I was following the master program in mathematics and informatics education, I decided to study directly several difficulties that pupils have while they operate with brackets, and indirectly the comparison between the reality in Albanian classrooms and the reality showed by studies and researches. And this latest will be important for me while I will be teaching in Albanian context.

I would like to thank the head-master of the school that I conducted my data collection, the teacher who followed me for two weeks, and the pupils who participated in this study. They made possible the collection of data for my study.

The administration’s employees, the academic employees, and my fellow students have shown willingness to help me and to contribute in my education. I must mention the support that my two Albanian friends gave me every day during my staying in Norway.

Special thanks go for my supervisor Claire Vaugelade Berg, who supported me for more than one year. I deeply appreciate her professionalism, advices, and courage that gave me as a mentor and as a friend. She has been close to me and has supported me during my ups and downs while I was designing and writing my master thesis, by feeding the belief that I can.

This work goes as dedication to my family for all their efforts and support until now; and to my fiancée, Alban Smajli. Warm thanks go to him for his support and encouragement during my work, and staying two years far away from each-other.

Thank you!

Kristiansand, June 2012

Dorela Kraja

Abstract

Since algebra is considered as a language, it has its syntax, semantics, and pragmatics. In the third linguistic level (pragmatics) stand the relations between the elements of language and the users. Since brackets are elements of algebra language they have their syntax, which involves some rules that lead transformations of algebraic expressions even with brackets; and brackets have their semantic aspect, which is related to tackling of algebraic expressions into brackets such as mathematical objects. On the other side, the consideration of the syntactic aspect of algebraic expressions leads to instrumental understanding of the use of brackets; and the consideration of semantic aspect of algebraic expressions leads towards relational understanding of the use of brackets. Based on relational understanding of the use of brackets, it is possible to identify cases of using brackets, to discriminate them, to generalize each of them, and finally to achieve the synthesis concerning the use of brackets in algebraic expressions.

In terms of this theoretical framework I have designed my study and I tried to give answer to my two research questions:

1. What kind of mistakes pupils in grade 10, involved in my study, do?

In order to achieve an overview of mistakes concerning the use of brackets that Albanian pupils in tenth grade, involved in my study, do I conducted test for thirty pupils. After I checked all the tests, I selected four pupils to be interviewed about their test's answers and to present in depth analysis of the answers of two of them. I conducted interviews about the test (both, test and the interview are called pre-test) and some teaching lessons in order to affect pupils' performances during the task-based-interview (post-test).

Form data analysis, it seems that some pupils show evidence of improvement in their performance during the post-test comparing to the pre-test. Thereby I want to address the following research question, which is composed by two parts:

2. a) What kind of improvement is showed in performance during the post-test?
2. b) What kind of causes can be identified concerning the improvement?

In general, the pupils showed that they know how to operate with brackets in algebraic expressions. However, mostly of them had an instrumental learning concerning the use of brackets, and other pupils had done mistakes in their answers concerning the use of brackets and tackling of algebraic expressions. In terms of the two selected pupils' performance during all my data collection, it is noticed an improvement of their mathematical thinking concerning the use of brackets in algebraic expressions.

Contents

1 Introduction

In this section I present the theme of my study and the rationale for it. Then I present my education and its finalization (until now) with master thesis. Also in this chapter, a description for “the environment” where data are collected is involved, also for the education system in Albania, and for chapters included in my dissertation.

This study is a master thesis in mathematics education program. Its topic has an algebraic character and it is related to the usage of brackets especially in algebraic expressions. I chose this topic because algebra is a cornerstone for mathematics and an important basis for achieving success in other parts of the mathematics components. Mathematics is so wide, and algebra too. I had to focus only on one algebraic element and to organize my thoughts and to orientate them towards clarification of ideas. It has been so interesting and difficult to pass through some issues included one to another, that embody mathematics and its parts, such as: algebra, algebraic errors, errors with the usage of brackets, errors with brackets in algebraic expressions, misconceptions and misunderstanding of brackets and the process of bracketing in algebraic expressions.

The topic of my study is the understanding of brackets and pupils’ process of bracketing. It is needed to emphasize the terms that I use in my study, since while I was reading previous studies I encountered different names for the same signs, which depends by authors, or something else. In my study I refer with word “brackets” to signs ‘( )’, “square brackets” to ‘[ ]’, and “braces” to ‘{ }’.

In this part of mathematics in school, pupils have different misconceptions and incomprehension linked with brackets. I thought, understanding and usage in the right way of brackets will help pupils for completing their knowledge in different parts of mathematics, such as: trigonometry (sin(x+Π), cos(2Π-x)); calculus, for example: functions (f(x+a), f[g(x)]), limit, derivation, integration, etc.; geometry.

Despite of all these and referring to mathematics Albanian textbooks, there is no lesson that is dedicated to fully understanding of brackets and to the basic use of them. I am mentioning only the basic use of brackets, which means the principle of the use of brackets considering as mathematics object what is involved into brackets, since the use of brackets is widespread and useful in different branches of mathematics.

Throughout my education, I have had commitment from my parents: to help me with my homework and learning during elementary school and then to choose a high school, in which the pupils and their education were its epicenter. So, during my first years of education I had each mathematics lesson explained twice, once by my teacher and once by my father (this time more in details), until my mathematical bases were strongly built. After this, I was always looking for a “why and because”. And in higher classes, I have always been surprised why some pupils bore misconceptions from earlier classes, and I thought: “It’s time for new things”. This was one of the standpoints of seeing pupils’ difficulties.

After I finished the upper secondary level, in a private school, I have chosen to study mathematics for three years, in the public university in my hometown. Also, I followed a master degree in mathematics and informatics education. I have had an experience as a student-teacher for five months during my practices for the master program in mathematics and informatics education in my hometown. It was a valuable experience for me because I was confronted for the first time with a real class, where I had the teacher's role. There I saw from another standpoint, not as a pupil but as a teacher, all kinds of difficulties that pupils could have.

Now, I am following the master program in math education in University of Agder, for two years. It has been a very interesting and valuable experience. It was a fortunate but also a challenge for me. During the first year of my studies for math education I have followed these courses: The Development of Mathematics (MA-414), Working Methods in Mathematics Education (MA-413), Modern Technology in Mathematics Education (MA-411), Research Methods in Mathematics Education (MA-410), Learning and Teaching Mathematics (MA-404); and two pedagogical courses: Education in Norway and Comparative Education (PED-216), and School practice and Report (PED-223). All these were very interesting, valuable and helpful for my research in the second year of my studies and for my professional future. And, the finalization of this stage of my education is this research study.

During my data collection I had the opportunity for having another standpoint of seeing and understanding misconceptions of pupils. In this case I had the role of "researcher" and I wanted to go in depth of causes and consequences of misunderstandings.

In my dissertation, I am going to have a resultant standpoint based on previous ones. And, this should be based on the intention to understand and to help pupils as they face different challenges in algebra. In this study my aim is to analyze pupils' errors made on a set of brackets operations as: brackets expansion, factorization etc.; and to achieve a general description about the algebraic mistakes that Albanian pupils in 10^{th} grade ,involved in this study, do and to relate these with results from previous research studies.

I want to find a way to classify the errors of pupils and to see the causes of those errors. From my experience as a student-teacher I can see that some of those errors are caused by the fragility of their mathematical background, some others are caused from unclear algebraic conceptions (especially about brackets) or other reasons (carelessness, negligence).

In the end of my study, I would like to give answer to the following two research questions, the first research question address all pupils in the class I observed, and through the second research question I focus on only two selected participants:

1. What kind of mistakes pupils in grade 10, involved in my study, do?

Form data analysis, it seems that some pupils show evidence of improvement in their performance during the post-test comparing to the pre-test. Thereby I want to address the following research question, which is composed by two parts:

2. a) What kind of improvement is showed in performance during the post-test?
2. b) What kind of causes can be identified concerning the improvement?

In order to answer these research questions I conducted my data collection in the following way.

Firstly I observed in a class in upper secondary level for one week, and the next part of my data collection passed through these steps:

1. Pre-test
    a. Test for the whole class
    b. Interview with four pupils
2. Teaching lessons
3. Post-test (Task-based-interview for the four pupils)

From the first research question I expect an answer that could give me a general view of misconceptions and errors that selected pupils in Albania do. For this, I have to make an analysis of all the tests collected in my class.

The answer to the second research question would be a comparison between the pre-test and post-test for each of four pupils. Also, I would like to involve in this part the description of the process of understanding the concepts of brackets and bracketing, for two pupils. Among all pupils in my class I chose four to be interviewed and among these latest I chose two pupils' performances to be analyzed. And for this I have to study in depth the interviews of these two pupils. The second research question required an answer that is particularly conditioned. My intention is to know if there is something which caused in one way the difference in pupils' performance, since I noticed improvement in their performance during the post-test. My goal is to seek evidence to a link between my teaching activities and pupils' performances during the post-test.

Figure 1.1. The map of Albania

I see this research as a case study, whose purpose is to understand mathematical thinking of some Albanian pupils during they are working with brackets, and I collected data in only one class of one school in the northwest city of Albania. Albania is a country in Southeast Europe. It is bordered to the northwest by Montenegro, to the northeast by Kosovo, to the east by Republic of Macedonia and to the south and south east by Greece. It has a coast on the Adriatic Sea to the west and on the Ionian Sea to the southwest. Albania is a parliamentary democracy with transition economy, and the capital city is Tirana.

To have a picture of the system of education in Albania, I am writing briefly some words about that, according to the official website of Ministry of Education and Science. Education in the Republic of Albania is public and private. Public education is secular. The New Structure of the Education System in Albania consists of:

- Preschool education
- Basic education (elementary, lower secondary cycle, special education)
- Secondary education (upper secondary cycle , vocational education, social - cultural education)
- Higher education
- Adult education

Pre-school education in the Republic of Albania is public and non-public. It consists of nursery schools (crèches) and kindergarten. Nursery schools are for children between the ages of zero (0) and three (3), and kindergartens are for children between the ages of three (3) and six (6). Compulsory basic education starts at the age of six (6) and lasts no less than nine (9) years. Basic education consists of two levels: the elementary level (grades I-V) and the lower secondary level (grades VI-IX). Secondary education in the Republic of Albania is not compulsory and consists of Secondary schools (full-time and part-time) and Secondary Vocational Schools. Secondary education has duration of 3 years and terminates with the State Matura Exams.

The Albanian school system has changed in 1999 - 2000. The old system for many years had this structure: four years of Elementary level, four years of Lower Secondary level, and four years of Upper Secondary level.

The new secondary school structure and the new teaching curricula were implemented for the first time during the academic year 2009-2010. There are nine key fields included in these curricula: Arts; Physical education and sports; Foreign languages; Albanian language and literature; Career promotion and personal growth; Mathematics; Technology, IT & computers; Natural sciences; Social sciences.

Secondary school curriculum consists of the core curriculum and the elective curriculum. Secondary schools with a reduced time-table (part-time) are also available to adults. Studies in part-time secondary schools last four (4) years and upon completion students take the State Matura exams (SM). Teaching and curricula programs are approved by the Minister of Education and Science.

This thesis is separated in 10 chapters as the following: Introduction, Review of Literature, Theoretical Framework, Methodology and Data Collection, Analysis of Tests, Portrayal of Alba, Portrayal of Dea, Discussion and Conclusion, References, and Appendices.

In my dissertation I have involved a chapter on review research studies (Review of Literature) concerning algebraic errors on bracketing process and the use of brackets. In this chapter, I have introduced the composition of algebraic language and the role of brackets in algebra. Also I have introduced some results from other authors that indicate the existence of pupils' mistakes and misconceptions concerning the brackets' use.

Then, in the third chapter, I present my theoretical approach for compiling and adapting tasks of the test, teaching activities and task-based-interview, and for analyzing pupils' performances. Also in this chapter, I describe how this study is based on this framework.

In the fourth chapter I describe methods that I have chosen for my study and how these are conducted during data collection. I have used observation, test, task-based-interview, and I have designed and adapted tasks for test, teaching activities, and task-based-interview. Also, I will describe in detail why and how the data collection took place in a tenth grade class in upper secondary level. This will be followed by the description of the way of analyzing data.

The fifth chapter follows up with participants' mistakes extracted from tests' analysis. Considering these types of mistakes, I have achieved an overview concerning the way of brackets' tackling by pupils in tenth grade involved in my study.

Considering my theoretical approach, as in tests' analysis, I have analyzed performances of two pupils during the test and interview about the test, teaching activities, and task-based-interview, by introducing two portrayals for these two pupils called Alba and Dea.

Chapter 8, Discussion and Conclusions, contains a discussion between my study design, my main findings from tests and interviews conducted during data collection, my theoretical approach and conclusions from previous researchers. Also, I have looked back to my research questions, considering my expectations and my findings, in order to have answers for my two research questions. Following, limitations that are arise during my work, suggestions for improvement and further research, and my improvement as a researcher and teacher are involved in this chapter.

There is also the chapter about all references involved in my study, and the last chapter contains appendixes, which is divided into 5 parts: Test, Evaluating algebraic expressions, Interpreting multiple representations, Post-test, and The fieldwork permission by the participants.

2 Review of Literature

The use of brackets is important in mathematics and not only in algebra, in which I am focused on. In addition, pupils have different misconceptions, furthermore these latest are not few in pupils' mathematical background. I have seen this not only during my experience as a teacher but also I am informed by different authors, which have searched for and studied pupils' understanding of brackets and the brackets' usage. For this issue I mention authors, such as: Booth (1984), Ayres (1995, 2000), Kaur (1990), and Kieran (1979, 1989).

In my study, I used the term *signs* since Ducrot and Todorov (cited in Drouhard and Teppo, 2004, p. 231) define "signs" as *"entities that have, for a defined group of users, a particular form (the signifier) and a "meaning" (the signified)"*. And based on this determination and on my understanding, brackets have this signifier: '( )'; '[ ]'; '{ }' and the entity involved into these signs should be considered as a whole object.

This chapter involves two sections: Algebraic Language and Misconceptions.
In the first section, Algebraic Language, I have involved different authors who define mathematics as language, and algebra too, as subset of mathematics language. Linguistic levels of algebra as a language are mentioned and explained in this section; also the importance and the use of brackets in algebra language are tackled here. While in the second section called 'Misconceptions', some types of mistakes, misunderstandings and misconceptions are introduced, which I have noticed during my experience as a teacher and different authors have concluded about them in their studies.

The topic of my dissertation is pupils' understanding and tackling of brackets, if pupils tackle brackets as punctuation mark in order to contain certain data, or as part of algebraic language. In other words, does pupils' work with brackets come just from practical rules that are served to them, or is pupils' work based on reasoning and arguing the presence and the use of brackets? In my study, I relate these two types of tackling brackets with two types of understanding the use of brackets, which in accordance to Skemp (1976) are called instrumental and relational understanding.

As the beginning I would like to introduce two interesting examples from research literature related to the ways of tackling of brackets:

- Example 1

  It is taken from Drouhard and Teppo (2004), in which is introduced a non-understanding of brackets. Mathematical understanding might be symbolized and in Drouhard and Teppo's study (Drouhard and Teppo, 2004, p. 228), the answer of a 12 years-old pupil about explaining why '3(x+2)' and '3x+6' are equivalent, was: "Yeah … because they're rules … they are there so that you can follow them, so that everybody'll do the same thing", providing a rule-based interpretation. The pupil appears to be focused on the mastery of a set of manipulation rules. To him, symbols may be just carriers of practical rules, developed to obtain results.

- Example 2

  It is taken from the 4th grade's Albanian textbook (Daka and Malaj, 2010, p. 96), in which is emphasized only the syntax of brackets.

Task: Look at the rectangle, and fill in the blanks:

75 m

The perimeter = 210m

The length = 75m

The width = (210-(2*75m)):2 = (210-150m):2

=___m:2

=___m

And pupils have just to make two calculations leaded by expressions that are written in the book.

The aim of the second task, considering also the ability of fourth grade's pupils, is to get answer to the question by manipulating given numbers. In addition, it is different from making sense the expression written in the book and to understand *why* the length of the rectangle is calculated in that way, by emphasizing in the same time algebraic usage of brackets, which is related to separation and inclusion of mathematical objects.

While in the first example is required an explanation about using distributive law, which is related to the use of brackets, and the pupil's answer has been referred to syntactic aspect of symbols' manipulation, using rules without necessary understanding these. In front of this aspect stands semantic aspect that refers to the meaning which the symbols endorse. Still continuing in accordance to Berg (2009), and Demby (1997) syntactic aspect, which refers to the organization and transformation of symbols using rules without necessarily grasping why these are valid, could be seen related to Skemp's instrumental understanding (Skemp, 1976); while relational understanding is related to semantic (level of meaning) that refers to the getting of meaning from what these symbols represent.

2.1 Algebraic language

Mathematics is a language, according with several authors, such as: Laborde (cited in Drouhard and Teppo, 2004, p.231) and Drouhard (2009) which refer to 'mathematical language'; Surakkai (2008) which defines mathematical as a special language; and Drijvers, Goddijn, and Kindt (2011) which determines the language aspect of algebra, since according to Drouhard and Teppo (2004) algebraic language is a subset of mathematical language. In addition, I agree that mathematics is a language, since it has its own vocabulary, and we use it to construct statements based on some rules that are accepted and understood by everyone, with purpose that statements to be understandable by all.

According to Drouhard (2009, p. 3) 'mathematical language' is a written one, since mathematical semio-lingustic units are written texts. It is needed a careful distinguish of mathematical objects (procepts of Gray and Tall (1991, 1992)), like the number '20', and semio-lingustic *representation,* like the string of characters '20' made of a '2' and a '0', that occur mathematical units in written texts.

Following, "the uniqueness of mathematical objects, the way it deals with relations and the rules that constitute mathematics are very well exemplified in mathematical writing" (Sarukkai, 2008, p. 1) and "mathematical writing is not mere symbolic, it is an art of creating new symbols" (Sarukkai, 2008, p. 3). The mathematical alphabets correspond to larger linguistic expressions than the alphabet of natural languages, which correspond to some sounds.

In addition, Sarukkai (2008, p. 3) indicates an example that, during reading f(x) as a natural language text I can say ‘fx’, a combination of two alphabetic sounds. Strictly speaking, I should be reading it as ‘f, open bracket, x, close bracket’ but the brackets are silent. While as a mathematics writing I will communicate it as, ‘function of x’ or at least ‘f of x’. According to Sarukkai this constitutes a source of ambiguity for pupils in the context of teaching mathematics. What is written is not what is read and “the reading has to be much more than the symbol one sees” (Sarukkai, 2008, p. 3). In mathematics language, mathematical concepts would be also visually coded. So, Sarukkai (2008, p. 3) gives the example of sets, as collection by stating that “sets are denoted usually by the use of brackets; and brackets (as visual marks) enclose, collect and bring together elements”.

According to Guillerault and Laborde, cited in “The future of teaching and learning of algebra” (Drouhard and Teppo (2004)), “mathematical language” is composed of both, natural language and symbolic system; and symbolic system is broken down into symbolic writings and compound representations. Since algebraic language is a subset of mathematical language, according to Laborde (cited in Drouhard and Teppo, 2004, p. 231), algebraic language is characterized as a set composed of natural language, algebraic symbolic writings, and algebraic compound representation, which consist of symbolic writings, drawings, natural language for labels, graphs, etc.

Drouhard and Teppo (2004) describe the language components of the algebraic language using linguistic levels of analysis, such as syntax (the organization and transformation of symbols), semantics (the level of meaning), and pragmatics (the relation between signs and their users). The natural language component of algebra is, clearly, a language. What is not as obvious is that the symbolic writings component is also a language. Since, as with the natural language component of algebra, symbolic writings can be described using linguistic levels of analysis and linguistics concepts. So, the system of compound representations is not a language, since they are indeed complex entities and they have meaning but it does not fit with a particular characterization of a language.

By agreeing with Drouhard and Teppo’s study (Drouhard and Teppo’s, 2004) about linguistic levels of algebraic language, Shaban (2010, p. 3) categorizes elements of algebraic language under linguistic levels.

- **Numbers, symbols and operators** (basic elements).

  These signs form the ‘vocabulary’ of algebra and, the rules how to manipulate these signs constitute the syntax. Also, it is needed to know their meaning (semantics) in order to understand the substance of what they express.

- **Expressions, manipulations, rules and algorithms**.

  These are formed by combining the basic elements of algebra. Syntax is a set of normative rules which ensure that everyone manipulates “mathematical” [algebraic] symbols in the same way.

- Application in the form of **calculations, charts, models, theorems**.

  These forms are used to communicate and to apply mathematics [algebra] in different situations which could be realized through pragmatics. And essentially through these different application forms, the intern (who is practicing) will measure the effectiveness of mathematics [algebraic] language.

This classification shows that algebra is structured in the same way as other languages concerning three linguistic levels such as: syntax, semantics and pragmatics. Rowland (2002) says that pragmatic meaning is an important tool in fulfilling the interactional function of language. And I think that in this 'chain', misconceptions and misunderstandings are established and developed since one can intervene.

I will put in front of pragmatic meaning the metalinguistic awareness according to MacGregor and Price (1999, p. 451), concerning their operation in the level of abstraction: "*Metalinguistic awareness enables the language user to reflect on the structural and functional features of text as an object, to make choices about how to communicate information, and to manipulate perceived units of language".*

I consider that metalinguistic awareness in ordinary language has its equivalent components in algebraic language since analyzing structures, making choices about representations and manipulating expressions are intrinsic to algebra. So, in accordance to MacGregor and Price (1999, p. 452) and concerning the context of algebra, there are two components of metalinguistic awareness in algebra language:

1. *Symbol awareness* includes knowing that numerals, letters, and other mathematical signs can be treated as symbols detached from real-world referents. It follows that symbols can be manipulated to rearrange or simplify an algebraic expression, regardless of their original referents. Another aspect of symbol awareness is to know that groups of symbols can be used as basic meaning-units. For example, $(x + 2)$ can be considered as a single quantity for the purposes of algebraic manipulation. Also Gray and Tall (1991, 1992), is tackled 'x+2' as *a single quantity* (object, product), which will be developed further in the next chapter.

2. *Syntax awareness* includes recognition of well-formedness in algebraic expressions. For example, knowing that $2x = 10$ implies $x = 5$ is well formed, whereas $2x = 10 = 5$ is not well formed) and ability to make judgments about how syntactic structure controls both meaning and making of inferences (e.g., knowing that if $a – b = x$ is a true statement, then it is not generally true that $b – a = x$.

Algebraic language is used to express algebraic ideas in a way that is detached from the initial, concrete problem. In this sense, abstraction takes place and "it incorporates essentially mathematical concepts" (Esty, 1992, p.32). In accordance to Drijvers, Goddijn, and Kindt (2011) it would be going too far to say that algebra is a language, but algebra does have a powerful language. He views algebra as a system of symbolic representations, where (algebra) uses its own standardized set of signs, symbols and rules about how you can write something; algebra seems to have its own grammar and syntax. This makes it possible to formulate algebraic ideas unequivocally and compactly.

In this symbolic language, variables are simply signs or symbols that can be manipulated with well-established rules, and that do not refer to a specific context-bound meaning. In accordance to Drijvers, Goddijn, and Kindt's view (Drijvers et al., 2011), I could also mention Esty (1992, p. 31) that calls 'Mathematics' the language with which are expressed mathematical results. This is because it has its own grammar, syntax, vocabulary, word order, synonyms, negations, conventions, abbreviations, sentence structure, and paragraph structure, like other languages. However, on the other hand, mathematics (as language) has certain features unparalleled in other languages, such as representations (for example, theorems expressed with 'x' and also apply to 'b' and to '2x-5'). Since algebra has its own syntax and semantics, "x+5x is not equivalent to (x+5)x" (Esty, 1992, p. 35) and brackets "play an important role in algebra 'sentences'" (Drijvers et al., 2011, p. 17).

It has been a little difficult to find articles related to brackets' tackling by pupils not connected directly with order of operations. According to Welder (2012, p. 256) brackets should be tackled as essential element of mathematical notation in arithmetic and algebra, and he emphasizes Linchevski's (1995) results that, algebra requires pupils to have a much more flexible understanding of brackets. This is because Linchevski (1995) states that: *"In arithmetic, the student views brackets as static, a 'do-it-first' signal, which does not leave room for any further consideration" (p. 116)*. In this line of tackling brackets, I introduce also, Ocken (2007, p. 9), which says that "order of operations rules show where to insert parentheses in arithmetic expressions".

In addition, considering Ocken's (2007, p. 55) article, I will list some possibilities of using brackets in algebra:

- Around the result of an algebra operation.
- Around an expression that is substituted for
    - a variable in a larger expression;
    - a variable in an algebra law;
    - a function argument;
    - a function value.

I would like to summarize all these cases of using brackets in one, since everything involved into brackets should be tackled as a mathematical object (product), and we should use brackets when we consider an algebraic expression as an mathematical object regardless of situations or conditions. I used the term 'algebraic expression' since in accordance to Ocken (2007) *"an expression involves numbers, variables, brackets, and algebra operations. Basic types of expressions are integers, variables, monomials, polynomials, and so forth"* (p. 3). This is because perceiving a *symbol* (or group of symbols, or an expression) as a *concept* (*object*) and not only as a *process* since according to Crowley, Thomas, and Tall (1994, p. 2), the *concept* allows pupils to think about it and to manipulate it mentally, and not only to do mathematics that is enabled by the *process*.

2.2 Misconceptions

I have started this section mentioning some typical errors related to bad-use or non-use of brackets that are embedded in my mind during being a pupil and then a teacher. Considering these examples, I tried to learn from them when I was a pupil and to emphasize and to pay attention to them while I was a teacher, with purpose to clarify the importance of the brackets' use. Based on my experience as a teacher I see these examples as typical for pupils' mistakes.

Now I will introduce some examples concerning non-correct use of brackets and lost-use of brackets associated by some comments. These are found at this site:
http://tutorial.math.lamar.edu/Extras/CommonErrors/CommonMathErrors.aspx

1. $(3x)^2=3x^2$

In this case only the quantity immediately to the left of the exponent has gotten the exponent. So, it is squared only the 'x' while the '3' is not squared. But the correct case should be $(3x)^2=(3)^2(x)^2=9x^2$, since the brackets is immediately to the left of exponent. And, this signifies that everything inside the brackets should be squared. Brackets are required in this case to make sure that is needed to be squared the whole "thing" (mathematics object, '4x') and not just 'x'. In this line, based on this reasoning, I will mention another example:

2. $(-5)^2=-(5)(5)=-25$.

In this case the whole object '-5' (the negative number) involved into brackets should be squared, as: $(-5)^2=(-5)(-5)=25$. These errors come from decision that brackets are not needed and from non-understanding of brackets usage.
Another example is demonstrated by this task:

3. Subtract '2x+5' from 'x^2+3x-6'. And the answer of some pupils is:
   $x^2+3x-6-2x+5=x^2+x-1$.

This is a lose-case of use brackets and only '2x' is subtracted. Since it is subtracted a polynomial it is needed to make sure to subtract the whole polynomial, and the only way is to put brackets around it, $x^2+3x-6-(2x+5)=x^2+3x-6-2x-5=x^2+x-11$.

Another type of algebraic errors related to the use of brackets is from the use of distribution property:

4. $2(x^2-5)=2x^2-5$

In this case the '2' is multiplied only with the first term ignoring the second term. And I think, this is most common when the second term is a number. If the second term contains variables pupils would remember to do the distribution correctly more than not.

5. $5(2x+3)^2=(10x+15)^2$

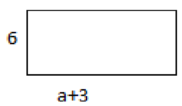
The first operation related to '2x+3' is exponentiation and then distribution, since '2x+3' is included into brackets and this bracket is squared and multiplied by '5'. '$(10x+15)^2$' is the answer for $[5(2x+5)]^2$, which means that even '5' and '2x+3' are squared, or (10x+15) is squared.

I do not know the real reason of some following errors but I think that some pupils consider a(b+c)=ab+bc, and they think that everything work like this:

6. $(a+b)^2=a^2+b^2$

I will mention another lost-use of brackets, and now in relation to geometrical part.

7. Write the expression for the area of rectangle:



A=6a+3

Here is required to multiply two sides of rectangle (since that means to find the area of a rectangle) and one side is a number and the next one is a polynomial (algebraic expression). And the only way to make sure it is correct is to put polynomial into brackets. In this case the correct answer is: A=6(a+3).

These types of mistakes and misunderstanding are encountered in pupils' performances in several grades while I have been present. Some of them are also mentioned by several authors that I have involved in this section of 'Review of Literature'. And I will relate the very last example with Kaur's results and evidences of the non-use of brackets in her study (Kaur, 1990). In Figure 2.1, the proportion of pupils *"who did the exercise were either ignorant of the use of brackets or chose to ignore the use of brackets mainly because they considered them unnecessary"* (p. 35) is shown.

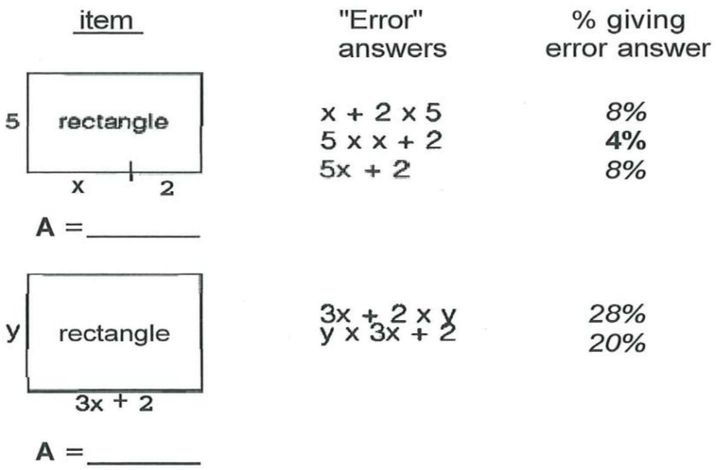


Figure 2.1. Non-correct answers and Proportions

Also Booth (1984, p. 53-55) concludes that a large proportion of pupils appear to regard brackets as irrelevant in algebraic expressions and they consider these expressions with and without brackets to be equivalent. Also of interest is the fairly high proportion of pupils who exclude brackets statements from their answers. From Booth's research (Booth, 1984) it is interesting to see how pupils in the top ability mathematics group also appear to ignore the need of brackets, and not only pupils in lower ability mathematics groups. On the other hand, the results from teaching experiments in this study concluded that pupils in middle ability mathematics groups know about brackets but they do not use them since they do not consider their use necessary. And, after a series of thirteen interviews of pupils who have been selected by their teachers as being particularly able at mathematics, these pupils appear to be familiar with the bracket notation but consider their use to be largely optimal.

Ayres (2000) identifies in his study, which is an analysis of errors made on a set of brackets expansions problems, that more errors were made during the expansions of the second bracket compare with the first bracket and more errors are made during the second operation compare with the first one within each bracket. In this study, 229 pupils take part and they are in grade eight (average age of 13.4 years) and grade nine (average age of 14.2 years). They are required to solve a set of eight problems presented in Table 2.2.

Q1	5(3+4x)+3(-x-3)	Q2	2(5x+2)+4(-3+4x)
Q3	-3(1+3x)-7(-2+x)	Q4	-4(4x-8)-3(2x+6)
Q5	-7(-1-2x)+8(2x+3)	Q6	-2(-2x+6)-3(9-x)
Q7	2(-3x-5)+3(4-3x)	Q8	5(-3+5x)-2(-4x-7)

Table 2.2. The problem set used in the study

Each task contains four operations that lead to the expansion of brackets, each of two pre-multipliers should multiply both two terms included into respectively next bracket. From the analysis of pupils' answers, 25.7% of all errors are from forgetting to include the number before (the pre-multiplier) during calculation of one from four operations.

Errors, made by pupils while they are expending linear algebraic brackets, are classified by Ayres (1995, p. 39) as *slips* and *bugs*. Slips are errors of a careless nature and they are caused by difficulties experienced within working memory. On the other hand, bugs are errors of a procedural nature involving an incorrect routine in an otherwise correct method.

Another type of mistakes related to the wrong meaning attached to brackets is appeared in Kaur's study (Kaur, 1990, p. 36), in which several pupils "have the misconception that brackets indicate multiplication". In Figure 2.3, some pupils' possible error answers are given:

item	"Error" answers	% giving error answer
$(2x - y) + y$	$2xy - y^2$	12%
$3x - (2x + y)$	$-6x + 3xy$	8%
	$-5x - y$	4%
$(x+y) + (x-y)$	$x^2 - xy + xy - y^2$	16%
	$x^2 - y^2$	20%
$(3x+2y) - (x-2y)$	$2x - 6xy - 2xy - 4y$	8%
	$5xy - 2xy$	4%
	$3x^2 - 2xy - 6xy - 4y^2$	16%

Figure 2.3. Non-correct answers and proportions

Another evidence of problems with the use of brackets in the rank of pupils, are results of Kieran's study (Kieran, 1979). This study is conducted in "junior high school level" (12-14 years old) and six pupils are involved in solving problems related to the order of operations and the use of brackets. Kieran concludes that none of participants used brackets in constructing their arithmetic identities even though they all had worked in the past, even in elementary school, with bracketed expressions. For two pupils, bracketed operations in arithmetic identities were not only to be done first but also to appear first; and the opposite was another pupil which did not see the need of bracketing the first operation in numerical statements since there was the one he was calculating first in any case.

Kieran (1979, p. 128) concludes that these errors would not be from forgetting the rules, mislearning them originally or remembering them incorrectly by pupils. According to Welder (2012, p. 257) *"alarmingly, these junior high school students simply did not see a need for the rules presented within the order of operations".*

Kieran (1979) suggests a more compelling approach to the topic of bracketing, emphasizing the creating in the pupil's mind a need for the notation of brackets firstly. And this is illustrated by an example: taking '2·5=10' and subsequently replacing '5' by '4+1' yields '2·4+1=10' and the left side must still have the value '10'. So, if the pupil, whether he/she evaluates by his own left-to-right method or by the standard ordering conventions, will come to see that the only way to maintain this arithmetic identity is by bracketing, hence '2·(4+1)=10' (p. 133). So, Kieran (1979) suggests that pupils need to develop their intuition for using brackets before they can learn the rules related to the order of operations or the process of bracketing. And I will relate what she suggests to Welder (2012), who states:

> *Kieran's work highlights a need for more analysis, in addition to the traditional computation, of numerical equations in the elementary and middle school curricula. Furthermore, Kieran's suggestion requires bracket introduction to occur in mathematics curricula long before formal algebra. This notion is supported by Linchevski (1995), who also believes that students' conceptions of brackets need to be expanded during their study of arithmetic prior to algebra. (p. 257)*

Since the use of brackets in algebra should be based on relational understanding (Skemp, 1976) rooting the relation between mathematical objects and brackets, I agree with authors introduced in the previous quotation. I think that understanding of the concept of brackets in arithmetic would be helpful for pupils to understand the concept of brackets in algebra because concretization (as arithmetic is for algebra) helps in understanding concepts. In addition, for if "relational understanding" means knowing which rule to use and why, and how it works (Skemp, 1976) firstly pupils must be aware of the necessity of the bracket's use rule.

I think that, one way of addressing algebraic difficulties concerning tackling of algebraic expressions (polynomials) as mathematical objects is the non-use and bad-use of brackets. This is because pupils do not see an algebraic expression as a product without having certain values for variables or parameters involved in it. And the only way of considering an algebraic expression for several pupils is as a process, which is based on operations and items with whom they will operate. This ambiguity of algebraic expressions (mathematical symbols) is tackled by Gray and Tall (1991, 1992). On the other hand this is occurred by Kieran (1989, p.45) in her study since only one student (from all participants in her study) solved equation '4(2r+1)+7=35' for '2r+1' and all the others solved it for variable 'r', and then they found '2r+1'.

Considering brackets as part of algebra, which is a symbolic language and has its "alphabet", syntax and semantics, different types of mistakes, misunderstandings, and misconceptions that are indicated by my experience and previous studies of several authors, are expected to be encountered in pupils' performances concerning different types of tackling brackets. And these ways of tackling brackets would be related to algebraic expressions' non-tackling as objects, and to not being able to discriminate that way of tackling to the correct one.

It is needed to mention the difficulty of finding adequate literature concerning the topic, which is pupils' tackling of brackets in algebra. In other words, I am focused on the ways how pupils tackle algebraic expressions involved into brackets, and on the use of brackets involving algebraic expressions if they should be tackled as mathematical objects (products). In my dissertation I do not emphasize the order of operations and it was difficult to find previous studies, which emphasize the meaning and the use of brackets as part of algebraic language and separately from the order of operations.

In terms of looking back in previous literature as in this chapter, previous work of several authors is involved in the next chapter. In chapter 3, Theoretical Framework, theoretical basis of different authors are involved, in which I am based to establish my theoretical approach relevant for the topic and the aim of my study.

3 Theoretical Framework

This chapter presents the theoretical framework on which my study is based on, and it is divided into two parts: in the first part, I introduce my theoretical approach and in the second part, I explain how these issues are related to my study. The first part contains an overview of the framework that I have used in my study concerning some theoretical basis of different authors. And, the appropriateness of theoretical basis in my study is described in the second part. The first part is called "Theoretical approach" and includes three sections: Mathematics, Learning, and Teaching; which have influence in the process of understanding the use of brackets. Section 3.1.2, Learning/ Learners, involves the subsection "Mathematical understandings", in which I am concentrating on Skemp's instrumental and relational understanding (Skemp, 1976).

3.1 Theoretical approach

While engaging in reading relevant research literature for this study, I realized that different authors discussed issues related to the nature of subject matter (mathematics), to the learning of this subject matter and to the complexity of organizing teaching activity in order to obtain rich learning opportunities for pupils. These three aspects are present in the different authors' writings and therefore I choose to organize this chapter on theoretical perspective according to three strands: mathematics, learning of mathematics, and teaching of mathematics. This organization is in accordance with Swan's model (2005) in which the following two processes are involved, learning and teaching mathematics, and mathematics itself (Figure 3.1). This model emphasizes the interconnected nature of the subject matter, mathematics, and confronts pupils' difficulties through careful explanation of teacher rather to attempt to avoid them.

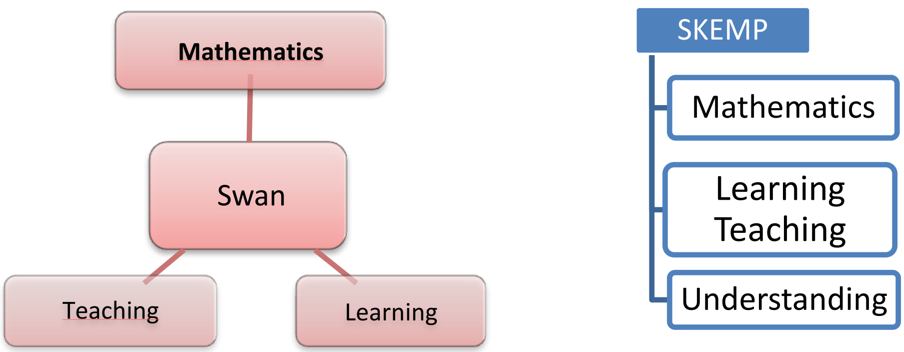


Figure 3.1. Swan's and Skemp's model of doing mathematics

I have designed two diagrams (Figure 3.1), with purpose to present my understanding about Swan's model and Skemp's theoretical basis. In these two diagrams I can find the same elements, such as: mathematics, and the processes of learning and teaching, but the structures of diagrams are different.

My understanding on Skemp's model (Skemp, 1976) is that the elements are organized in a hierarchical structure, in which the four involved entities are distributed in three levels. It starts from *understanding*, which is the first level and the base of *learning* and *teaching*, considering both of them in the second level of the hierarchy. Pupils develop their mathematical knowledge based on their understanding and through the process of learning, also helped by teaching, which is based on teachers' understanding. So, *mathematics* is the third level that includes the previous ones. As I mentioned above, this hierarchical structure is different, in the organization perspective, from Swan's model, which I consider as an interconnection between three elements: mathematics, learning, and teaching.

I think that, not everything taught by teacher become knowledge for pupils, it is only a part of what is said from teacher that becomes owned by pupils. It depends from teacher, pupils, and the social environment. And the amount of knowledge transferable from teacher to pupils depends mostly from the work of the teacher. According to Swan (2005, p. 5) teachers should have a much more pro-active role than simply to present tasks to learners, and to expect learners to explore and discover ideas for themselves.

3.1.1 Mathematics

There are different definitions and different viewpoints concerning the nature of mathematical knowledge. Swan (2005) considers mathematics as *"an interconnected body of ideas and reasoning processes"* (p. 5). In my study I am focused on understanding the usage of brackets and on the use of brackets in a meaningful way. I agree with Swan (2005) since I consider the understanding of bracketing as a process based on reasoning and supported by other mathematics concepts that will be introduced and developed later in this chapter.

While Skemp (1976) proposes two conceptions of mathematics account for sharp differences in classroom practices and emphases: 'relational mathematics' and 'instrumental mathematics'. Referring to Skemp's (1976, p. 13) example, I can confront instrumental and relational mathematics with two activities presented in the example below: when somebody goes to stay in a new town, firstly he learns several particular routes. He learns how to get between his house and his university, between the university and his friend's house, between his house and downtown. So, he learns a number of fixed plans, based on particular starting locations and particular goal locations. But when he begins to explore the town with no particular starting location and goal location, the construction of a cognitive map of the town is his purpose. According to Skemp (1976, p. 14), instrumental knowledge of mathematics is knowledge of "*a set of 'fixed' plans"* for performing mathematical tasks (step-by-step procedure), whereas relational knowledge of mathematics is characterized by the possession of "*conceptual structures"* that enable the teacher/pupil to construct "*several plans"* for performing a given task. It seems like two effectively different subjects being taught under the same name, 'mathematics'.

I agree with Skemp on this point because from my experience as a teacher, I have seen evidence of both these two mathematics categories during several mathematics lessons. There are pupils, who want to catch just a set of rules (as fixed plans) to operate with and they do not care *why* they are using them. But other pupils want to operate in a meaningful way and using what they know to achieve a solution for new situations. Related to the topic of my study, this is in accordance with opening brackets considering several template rules and formulas, without knowing their origin or proof. For example, most of pupils are able to memorize the formula $(a+b)^2=a^2+2ab+b^2$, and they do not know that its validation (proof) is linked with multiplication of the algebraic expression (a+b) with itself, which is the multiplication of two algebraic expressions included into brackets considering distributive law. And these pupils, in new situations where the task does not ask explicitly the formulae $(a+b)^2=a^2+2ab+b^2$ or in similar situations but after a period of time, instead of that formulae they use $(a+b)^2=a^2+b^2$ and they do not consider its validity.

In mathematics categories, as Skemp (1976) calls them relational and instrumental mathematics, are involved 'a set of fixed plans', 'conceptual structures', and 'several plans' that could compose the *"interconnected body of ideas"* of Swan (2005, p. 5). And, I think that ‘reasoning processes’ are involved into Skemp’s relational mathematics, because reasoning processes are needed to make the construction of ‘several plans’ starting from ‘conceptual structures’. And in my study, I stress the *meaningful* use of brackets and reasoning for the process of bracketing.

Mathematics has its own language, which involves the symbolic system. Drijvers et al. (2011) explains that algebra is considered as a system of symbolic representations and uses its own standardized set of signs, symbols and rules about how one can write something; algebra seems to have its own grammar and syntax and furthermore according to Berg (2009) algebra seems to have its syntax and semantic. In terms of syntactic and semantic aspect of algebra, Demby (2007, p. 66) states that semantic aspect of an algebraic expression refers to *“its meaning (based on the meaning of a variable as symbol of a number)”*. And syntactic aspect refers to *“the way the expressions can be manipulated according to formal rules, regardless of the meaning of the symbols”* (Demby, 1997, p. 66).

So, regarding some rules, symbols are appropriated and internalized. In addition, in accordance to Berg (2009), Demby (1997), and Drijvers et al. (2011) and referring to the above example, pupils consider only the syntactic aspect of the formulae (algebraic identity), which is $(a+b)^2=a^2+2ab+b^2$, and not the semantic one that means $(a+b)^2=(a+b)(a+b)$. If pupils will base on both semantic and syntactic aspect of symbols, and even though they will forget the formulae they could achieve a correct result.

Gray and Tall (1992) emphasize the distinguishing between the meanings of symbols in mathematics. A symbol might be understood as a mathematical object (product), a thing that can be manipulated in mind. Another possible understanding of a symbol might be a procedure to be executed. The symbol which evokes either a process or a product (object) of that process is called ‘procept’. Such a symbol stands dually for both a process and a concept.

“We define a procept to be a combined mental object consisting of a process, a concept produced by that process, and a symbol which may be used to denote either or both.” (Gray & Tall, 1992, p. 2)

So, it is not just the relation of two ideas, or the giving of a meaning to the process or concept. In accordance to Gray and Tall (1992, p. 7), *“it is the ability to give meaning to a process in a flexible way that allows process and concept to be interchanged at will, often without any distinction being made between the two”*.

For example, an algebraic expression such as ‘2x+7’ is a procept that stands dually for the process “add two times x to seven” and the algebraic expression which can be manipulated mentally as an object on its own. In addition, the idea of procept is relevant for my topic since ‘2x+7’ should be considered as an object in algebraic expression ‘2(2x+7)-5’ that means to look for a product of expression involved in brackets and then to continue with other operations. And, obtaining the product of ‘2x+7’ is related to the process that means this algebraic expression.

According to Gray and Tall (1991), this ambiguity between the process and the concept is at the root of pupils’ difficulties concerning mathematical thinking. The proceptual known fact, that is the ability of knowing a fact as a procept, would be involved into relational mathematics by virtue of its rich inner structure which maybe decomposed and recomposed to produce derived facts; and should be distinguished from a rote learned fact, that is a synonym of knowledge of a set of fixed plans. For some pupils it is difficult to recognize the duality: process, concept and to understand an algebraic expression as a procept.

Each understanding act has *"an object which the pupil has to notice and identify as an object of his or her understanding for any conscious thinking on it to start at all"* (Sierpinska, 1994, p.41). All the objects in real life could be understanding objects, but these latest one involve much more. Even though the *"whiteness"* could not be considered as an object because it does not exist separated from things that are white, Sierpinska states that it can be regarded as an object of our thinking, our understanding. "*In this sense mathematical abstract concepts could be objects for us"* (Sierpinska, 1994, p. 31), and a concretization of mathematical abstract concept could be linked to the idea of procept according to Gray and Tall (1991, 1992). So in this line, for some pupils, the abstractness and the difficulties stand in tackling of algebraic expression as a product, not as a process. Because they think that an algebraic expression presents just an order of operations that should be executed in relation to numbers, parameters or variables involved in that algebraic expression. They do not think about the product of that expression without having numerical values for parameters and variables.

I will to mention some of the possible objects of understanding in mathematics: concepts, relations between concepts, problems, arguments, methods, mathematical formalism, mathematical symbolism, and mathematical representations such as diagrams, graphs etc., and texts. Also judgments (theorems, conjectures, etc.), reasonings (proofs, explanations) can be regarded as objects. However, according to Sierpinska (1994, p. 32), especially in mathematics, objects are being often only constructed in acts of understanding.

Concerning the objects of understanding involved in my study, these could be: the concept of brackets and the symbol of brackets, the process of bracketing, and the process of factorizing; multiplication, division, subtraction and addition of two algebraic expressions; tackling algebraic expressions as products (objects).

On the other side, it is difficult to capture the object of understanding, and Sierpinska (1994, p. 32) states that:

> *"The person may not be able to say what it is that he or she intends to understand. He or she only understands that may lead to some clarification and identification of this object. But still it seems that without a feeling of there being 'something' to understand it is difficult to speak about any act of understanding to have occurred at all."*

Since the identification of the object of understanding is important for the process of understanding, also understanding itself supports subject of understanding to identify its object, in case when it is not clear. And the support mentioned above has its basis in meaningful learning.

3.1.2 Learning / Learners

Learners are the subject in the process of understanding a concept according to Sierpinska (1994). Related to my study the subject in the process of understanding are pupils in grade ten in an Albanian school. It is important to clarify the distinction between what we, as teachers, want the pupils to learn, and what exactly they learn; because according to Sierpinska (1994), it happens that in the transition between teacher to pupils the understanding object changes its identity, and everything taught by teacher is not becoming into knowledge for pupils.

I want to introduce an example about this change of the object of the process of understanding. During one of my teaching activities in my data collection, I wanted to present and discuss about the validity of one algebraic statement: $(2x)^2=2x^2$. My aim was to stress the role of the brackets in one algebraic expression by having as an object of the process of understanding the presence (role) of brackets. I and my pupils discussed about the algebraic expressions that are in the left and the right side, but on the other hand one pupil intervened in the discussion and said: “I transformed it in an equation and I solved it getting the solution x=0.” So, the pupil’s aim was only to find an answer for this task. We did not have the same object of understanding.

In addition, one of the components of the process of understanding concepts, the basis of understanding, depends on the background of learners. As Sierpinska (1994) states, there are four possible categories of basis of understanding, but here I present only those two which are relevant for my study: representations, and mental models. Representations include mental images and conceptual representations, and these are created on the basis of previous processes of understanding; and I could base on them to notice misconceptions about the use of brackets.

So, from what is mentioned above, we may identify some components that are important for the process of understanding. The person who understands is the “understanding subject”. This person intends to understand the “object of understanding” and this person has some thoughts that we call them “the basis of understanding”. Of course, there are the operations of the mind that links the object of understanding with its basis, and according to Sierpinska (1994, pp. 56-60) there are four mental operations. It starts with *identification* of the object of understanding, and *discriminating* this object to another one; these two are followed by the identification of the object of understanding as a particular case of another object of understanding, which is called *generalization*; and finally is *synthesis* that connect generalizations to each-other. Mental models encompass our knowing, understanding and reasoning, which could be related to correcting misunderstandings.

Below I will introduce an example related to the usage of brackets, which is adapted from Kind (2011), as a means to illustrate the explanation of four mental operations in Sierpinska’s (1994) book.

Task: Put brackets in the left side of following algebraic statement with the purpose to save the equal sign.

a) $x+x^2+3\cdot x+4= x^2+4x+4$ {answer: $x+x^2+(3\cdot x)+4= x^2+4x+4$}
b) $x+x^2+3\cdot x+4= x^2+4x+12$ {answer: $x+x^2+3\cdot(x+4)= x^2+4x+12$}

Identification is the main mental operation in the act of understanding, in the level of *“discovery or recognition”* (Sierpinska, 1994, p. 56). To identify the object of my understanding means, to discover that it is isolated (hidden) in the background of my consciousness, and to recognize it as something that I intend to understand. It consists in a re-organization of the field of consciousness so that some objects that were in the background are now in the foreground.
If I will concretize this in the given task, I would mention the identification of the crucial place of brackets in saving mathematics identity in an algebraic statement.

Discrimination between two objects *“is an identification of two objects as different objects”* (Sierpinska, 1994, p. 57). Discrimination is the identification of differences between objects. There are several degrees of discrimination: The first one is just the perception that the two objects are two and not one. The second degree of discrimination *“is that when two objects are compared with one another with respect to certain sensible circumstances, contingent to the objects themselves”* (Sierpinska, 1994, p. 58). The third degree of discrimination and higher one is *"when two general ideas are compared from the point of view of abstract relations"* (Sierpinska, 1994, p. 58).

Related to the example, the first object is '3x' involved into brackets as an addend in the left side of the first mathematical statement. Another object is 'x+4' involved into brackets as a factor multiplied by '3'. In these two previous sentences are introduced the first and the second degree of discrimination of two cases where are used brackets, because there are introduced *two* objects and their *situations* in which they are used in. The first brackets contain only one term but the second ones two terms. From opening the first brackets the algebraic expression in the left side of the first mathematical statement will, its product not change, also the product will be the same with that if there will not be brackets. On the other side, from opening the brackets in the left side of the second mathematical statement, the product will not be the same with that if there will be no brackets. In this way two objects are discriminated concerning the third degree of discrimination.

Generalization is the third operation of the mind that can be defined as an identification of one situation (which is the object of understanding) as *"a particular case of another situation"* (Sierpinska, 1994, p. 58). It is, seeing an object as a particular case of a situation. So, generalization may be considered derived from the notion of identification that could be seemed more fundamental. *"But, in understanding mathematics, generalization has a particular role to play"* (Sierpinska, 1994, p. 59). In addition, it is important to note that the operation of generalization has its object. We generalize something, for example: a concept, a problem, a mathematical situation, and we have to identify this 'something' as an object.

We can consider the second possibility of putting the brackets (case b) as a particular case of operations with brackets, there is a number as one factor and the other factor is involved into brackets; the first factor should multiply both terms of the second factor, and this condition is derived from the presence of brackets. It is different from the first mathematical statement that mean, the term involved into brackets, which is preceded and followed by the sign of addition could be independent from brackets.

Synthesis is the last mental operation and is considered as the search for a common link, a unifying principle, a similitude between several generalizations and their grasp as a whole (a certain system) on this basis. It is the search of a common link between generalizations. For example, synthesis happens during a mathematical proof when we capture the idea of this proof followed step by step. And so, "*the proof becomes a whole; it is no more just a set of isolated logical moves from one statement to another*" (Sierpinska, 1994, p. 60).
Related to the above example, all terms involved into brackets should be tackled as a mathematical object and the whole one should undergo operations that are related to these brackets.

To unify, reduce, generalize and synthesize, there must be something in one's mind that one can unify, reduce, generalize and synthesize. According to Sierpinska (1994, p. 32) people feel that have understood something when they have order and harmony in their thoughts, so it is like feeling that the essence of an idea has been captured. In other words, according to Swan (2005, p. 82), understanding could be referred to the French word *"comprendre"* that means to capture the meaning.

Now I want to introduce Skemp and his theoretical basis about understanding, since I think that understanding stands in the roots of the process of learning. In Skemp's theoretical approach (Skemp, 1976) the term "schema" is essential and occurs many times, so I want to introduce briefly what it is and why Skemp (1976) uses the idea of "schema". For Skemp

(1979, p. 219), a schema is "*a conceptual structure existing in its own right, independently of action*". And according to him (1979, p. 148), *"to understand a concept, group of concepts, or symbols is to connect it with an appropriate schema"* and *"our conceptual structures are a major factor of our progress"* (1979, p. 113). I have stopped here the further developing of the idea of "schema" since I am not focused on creating conceptual structures and on the way they effect on understanding the concept of brackets, and this idea is not close related to my topic.

Skemp (1976) is focusing on understanding a concept or a symbol, and it means to assimilate a concept or a symbol into a suitable schema, in other words understanding is the creation of a connection between ideas, facts or procedures that are generally accepted. In the same line of reasoning about the process of understanding, as operation (it is relates to the verb, to assimilate), could be introduced Swan's definition about the process of understanding. In his book Swan (2005, p. 82) states that when someone understands something it becomes part of his/her, and then he/she owns it.

3.1.2.1 Mathematical Understandings

Skemp's categorization of understanding (Skemp, 1976) is fundamental for my study because it is based on different abilities for implementing and using knowledge that the subject of the process of understanding possesses. In my study I have tackled misconceptions and misunderstandings that particular pupils have related to the process of bracketing and the concept of brackets. From all activities that pupils have participated such as: test, teaching lessons, and task-based-interview, the meaningful knowledge is involved and the meaningful learning is required. Meaningful learning means that the learned knowledge (let us say a fact) is fully understood by the subject that owned it and he or she knows how that specific fact relates to other stored facts (stored in his/her brain). And, in the opposite there is the rote learning based not on full understanding. Considering types of mathematics understanding, I want to know also how much rote learning is involved and to make meaningful learning or learning with understanding more present for the participants of my study. I emphasize learning with meaning because it would lead pupils towards exploration and discovering ideas by themselves, which is important in Swan's model (Swan, 2005).

Instrumental understanding
Instrumental understanding is the ability to apply an appropriate remembered rule to the solution of a problem without knowing why the rule works. So, in this situation it is known *"how"* but not *"why"*. The rule or the procedure used can be applied only for a limited number of tasks and the mental structures (schemas) built through the instrumental understanding cannot be easily modified because the method or formulae is memorized as it is required for the first time. Skemp (1976, p. 14) quoted that instrumental learning *"consists of an increasing number of fixed plans, by which pupils can find their way from particular starting points (the data) to required finishing points (the answer to the questions)"*. This type of learning leads to instrumental mathematics.

Since I consider concepts, algorithms (processes) and proofs as components of mathematics, there are two ways that one can understand these components, instrumental and relational. On the other hand, I consider task's solution by pupils as a proof that contains compiling algorithms based on concepts. If someone can recall an algorithm and is capable of executing it, I can say he/she has an instrumental understanding of algorithm; an individual *"can state the definition of the concept, is aware of the important theorems associated with a concept, and can apply those theorems in specific instances since he/she has instrumental understanding of the concept"* (Weber, 2002, p. 1). According to Weber (2002), someone achieves an instrumental proof if he/she primarily uses definitions (concepts) and logical manipulations (algorithms) without referring to his/ her intuitive conceptual understanding (Figure 3.2).

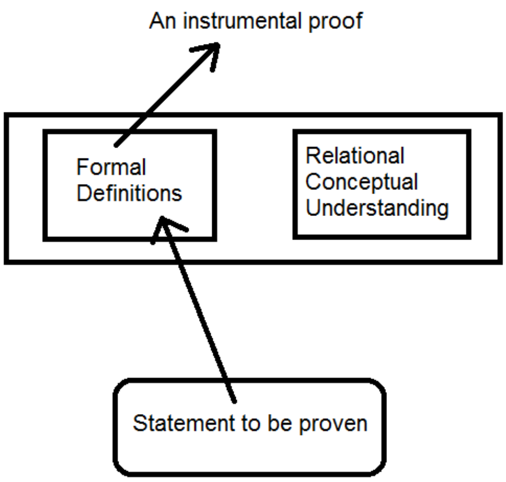


Figure 3.2. Instrumental proof

I illustrate this type of proofing within the context of usage brackets, which is the topic of my study. It is required to notice and to justify if mathematical identity is saved passing through these two steps: 6(x+2)-20=22

[6(x+2)-20]:2=11

An instrumental proof might be: the mathematical identity is saved because even the right side and the left one of the first algebraic statement are divided by '2', to divide the left side by '2' means *to include it into square brackets*. I emphasize here rote, formal and instrumental justification.

Relational understanding

Relational understanding is the ability to deduce specific rules or procedures from more general mathematical relationship. So, one knows *"how"* and *"why"* (Skemp, 1976, p. 38). The goal of relational understanding is the construction of relational schemas that means to make a connection between newly encountered concepts and the appropriate (relational) schemas. So, during this process the schema itself has undergone further development. Also, another goal may be the deduction (to deduce) of specific methods for particular problems, or specific rules for classes of tasks. In addition, another kind of goal is to improve existing schemas by reflecting on them in order to make existing schemas more cohesive and better organized, and also more effective for the first and second kind of goal. Relational understanding requires to pupils to choose, to change and to apply data, formulae and principles in new situations. In addition, in front of what is said above I could introduce definitions about learning according to Swan (2006). Learning could be the changing in fluency of performance (or behavior) and emphasizes the value of repetitive practice with feedback; but also it could be the development of conceptual understandings, also reflection, cognitive conflict and discussion. As Skemp (1976) quoted: *"learning relational mathematics consists of building up a conceptual structure (schema) from which its possessor can (in principle) produce an unlimited number of plans for getting from any starting point within his schema to any finishing point"* (pp. 14-15).

Since I consider pupils' solutions to mathematics tasks as evidence (proof) that contains compiling algorithms based on concepts, considering instrumental understanding, I introduce also relational understanding of a concept, algorithm, and proof. According to Skemp (1987) an individual has a relational understanding of an algorithm if he knows the purpose of the algorithm and why it works; and according to Weber (2002) an individual has relational understanding of a concept if he/ she understands *"the informal notion this concept was created to exhibit, why the definition is a rigorous demonstration of this intuitive notion, and why the theorems associated with this concept are true"* (p. 2). So, an individual achieves a relational proof if he/ she uses his/ her intuitive understanding of a concept as a basis for constructing a formal argument, as it is showed in Figure 3.3:

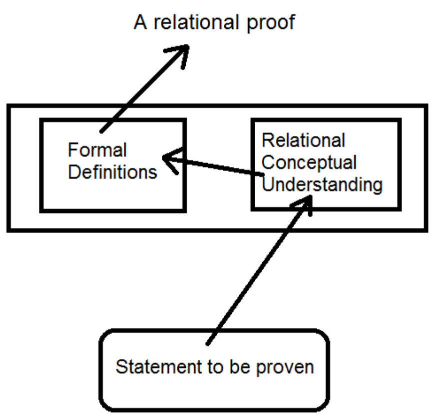


Figure 3.3. Relational proof

The relational proof of the example tackled in instrumental understanding section would be: algebraic expression of the first algebraic statement is included into square brackets in the second algebraic statement with purpose its product to be divided by '2'. Also '22' is divided by '2' because the result is '11', and the mathematic identity is saved.

3.1.3 Teaching

It is obvious that pupils must attend to their object of understanding, and they must be motivated by some interesting and meaningful questions, states Sierpinska (1994); and it is less obvious for the teacher what to do, what activities to design, in order to draw the students' attention, to motivate them, and to engage them into the activity of understanding. The understanding that students develop will depend on the kind of 'didactical contract' that will establish itself between the teacher and the pupils in the given classroom situation.

According to Swan (2005, p. 5), teaching is exploring meanings and connections through non-linear dialogue between teacher and learners, presenting problems before offering explanations, and making misunderstandings explicit and offering learning opportunities from them. And non-linear dialogue between teacher and pupils means to give opportunity to all pupils to present their ideas and thoughts, even the wrong ones, and not only when pupils are asked by the teacher and they have a linear dialogue. This would involve all the pupils in the activity that is happening during the lesson and they will exchange their thoughts correcting the wrong ideas and enforcing the right ones; and everyone is thinking in her/his own and showing thoughts and ideas to others. In other words, everyone is participating in discussion, and according to Swan (2005, p. 162), *"discussion leads pupils toward learning"*.

As Swan (2006) states the main task for teacher is to help students interpret mathematical presentations and to develop meanings for the concepts and relations that they are intended to relate, so a teacher should help pupils to create and to sharp a conceptual framework, and *"all this involves the development of links and multiple perspectives"* (Swan, 2006, p. 81).

I consider Swan's ideas (Swan, 2005, 2006) concerning teaching of mathematics as related to Skemp's relational understanding, presented above; and according to Skemp (1976) one has relational mathematical understanding if he/she makes links between his/ her knowledge and any new situation, building up a conceptual structure (schema), or developing existing ones.

According to Skemp (1976, p. 4) there are two kinds of mis-matches in the relation teacher-pupil, which can occur:

1. *Pupils whose goal is to understand instrumentally, taught by a teacher who wants them to understand relationally.*
2. *The other way about.*

The first case will cause fewer difficulties to the pupils, though it will be frustrating to the teacher. These pupils will not want to know detailed explanations, but some kinds of rules to get the answer. So, they catch these rules and ignore the rest; they will get wrong if teacher does not require just fitting a rule.

In the other case, where pupils are trying to understand relationally but the teaching makes this impossible, can be a more damaging one for these pupils. According to Skemp (1976), for some teachers who teach according to instrumental understanding, it might have same advantages, as:

1. Relational understanding would take too long to achieve, and to be able to use a particular technique is all pupils need.
2. Some topics are much easier to understand instrumentally than relationally.
3. The rewards are more immediate and apparent. It is sufficient for who want a page with right answers, quickly and easily.
4. It would be more adaptive to new tasks.
5. It is easier to remember.
6. Relational knowledge can be effective as a goal in itself.

However, a correct answer, formal knowing of a concept, an imitative algorithm do not lead toward learning with understanding. Teacher's role is more than presenting knowledge and to require it from pupils, as a reproduction in imitative way. Regardless referred advantages of teaching in accordance to instrumental understanding, it does not invite pupils to single a concept (or another object of understanding) out and bring it to the forefront of attention (identify); notice similarities and differences between this concept and other similar ones (discriminate); identify general properties of the concept in particular cases of it (generalize) and begins to perceive a unifying principle (synthesize) (Sierpinska, 1994). In my opinion, this is the way to lead towards relational understanding of a concept and that should be emphasized more during teaching mathematics.

3.2 How are these issues related to my study?

In this section I present the theoretical framework as the foundation of my dissertation, and I explained how it is built and why these theoretical perspectives are relevant for my dissertation. And to justify this relevance, I will take start from Swan's model (Swan, 2005) that is the collaborative organization of activities in class, which is different from what I saw during observation in my data collection.

While there are misconceptions and misunderstandings, I think “it is time for new things”, that would change the flow of thoughts of participants in this study, hoping for a better output.

Inspired by Swan’s model (Swan, 2005) and Skemp’s model (Skemp, 1976) of making mathematics and in order to be as valid for my topic and goal, I have divided my theoretical approach in three sections:

- Mathematics (subject matter)
- Learning (learners)
- Teaching (teachers)

In understanding a mathematical concept, in my case the concept of brackets and the process of bracketing, I have aligned the impact of learning by pupils and teaching by teachers (educators). In my dissertation, I am not interested on the way how teachers organize their teaching, but I am interested on the way of understanding and learning of pupils. The aim of my study is to look for pupils’ misconceptions and misunderstandings concerning the use of brackets. I do not exclude the possibility that misconceptions and misunderstandings may come from the way of teaching and teacher’s aims (objectives) concerning the organization of their teaching. The pupils are the subject of my dissertation since I discuss about pupils’ performances.

Brackets are part of algebraic language, which is estimated as symbolic language. Being a language gives algebra the possibility to have its own rules (syntax) and meanings (semantics) (Berg, 2009; Demby, 1997; Drijvers et al., 2011) for its components (elements) such as brackets are. The syntax of algebra enables forming of monomials, polynomials, algebraic expressions and statements by using mathematical objects (numbers, variables, parameters, algebraic operations). The important elements of algebra are brackets, which enable grouping of elements. On the other side, there is semantics that enables the interpretation of strings and other elements of algebraic expressions.

I think, in an algebraic expression, such as ‘2(x+5)’, the syntax of brackets is that even the ‘x’ and the ‘5’ should be multiplied by the ‘2’. On the other hand, the opposite, if I should multiply the ‘2’ with the polynomial ‘x+5’, I have to use brackets around the ‘x+5’. I think this is all what syntactic aspect involves, concerning expanding brackets and the reason of using brackets. In addition, all this is based only on “mechanical processes” which are a set of fixed plans.

Taking inspiration from Kieran’s (2000) study, I will concretize more syntactic and semantics aspect of the use of brackets, introducing this example:

Task: Find the value of ‘x+5’ concerning the algebraic statement, 2(x+5)-4=10.

Considering the syntactic aspect, one may find the value of ‘x’ firstly, which means to solve the equation for x, and then to add five to the value of x. This solution has followed this way:

$$\boxed{2(x+5)-4=10} \rightarrow x \rightarrow \boxed{x+5}$$

So, this syntactic aspect’s consideration is related to knowing a set of fixed plans for performing mathematical tasks as step-by-step procedure, which is based on particular starting points to particular goal points. This is called by Skemp (1976) as instrumental learning. During this type of learning and considering of syntactic aspect of algebraic expressions (Berg, 2009; Demby, 1997; Drijvers et al., 2011), it is not intended about tackling the ‘x+5’ as an object (Gray and Tall, 1991, 1992), a product, which should be multiplied by the ‘2’.

Otherwise, the processes of multiplying the ‘2’ with both the ‘x’ and the ‘5’ (as in syntactic aspect) become understandable, reasonable and meaningfully. This is the semantic aspect of considering an algebraic expression that is involved into brackets. Also, on the other hand, if it is necessary that an algebraic expression, such as ‘x+5’, to be tackled as a mathematical object (which means I am interesting on its product, value) it should be involved into brackets. And then, I can put it in any algebraic statement in any situation. I think these are the semantic aspect that the use of brackets involves in. So, in this case is considered and involved the concept of procept (Gray and Tall, 1991, 1992).

Considering again the algebraic statement, ‘2(x+5)-4=10’, and the demand to find the value of ‘x+5’ considering semantic aspect, the equation might be solved directly for ‘x+5’. Based on the semantics of ‘(x+5)’, this latest could be tackled as a new variable, with purpose to show that only its product has importance. So, this way of solving the equation for ‘x+5’ has passed through one step:

$$\boxed{2(x+5)-4=10} \longrightarrow \boxed{x+5}$$

In this case, it is emphasized the construction of several plans using what is known, in accordance to new situations that could be raised. And this is what I think Skemp (1976) has called relational learning concerning the use of brackets.

Faced with semantic aspect (Berg, 2009; Demby, 1997; Drijvers et al., 2011), tackling algebraic expressions as procept (Gray and Tall, 1991, 1992), and relational understanding of use of brackets in algebraic expressions, it is possible to identify different cases when the use of brackets is needed. And then, one can discriminate them emphasizing different and common properties for each brackets’ use. Based on these it is possible to generalize the use of brackets for each particular case. In this way the synthesis (Sierpinska 1994) concerning the use of brackets in algebraic expressions could be achieved.

I tried to introduce this way of tackling algebraic expressions and the use of brackets to my participants by developing some teaching lessons by my own. And I think the most relevant way of doing that is by developing Swan’s activities (Swan, 2005, 2006), which are based on non-linear dialogues between pupils and between teacher and pupils. Since pupils discuss to each-other they can introduce their understandings and misunderstandings, their thoughts and ideas, exchanging them and maybe even correcting them. Another reason of choosing these activities is the involvement of multiple perspectives and representations considering tackling of algebraic expressions in these activities (for more detail see section 4.2.2) and this leads toward discriminating cases of tackling algebraic expressions. It is important to emphasize that during these activities I was not interesting on correct results, but on argumentation of pupils’ answers since they were confronting to new situations, as these activities’ tasks were.

On this basis, I will describe in the next chapter the methods used during my data collection, in order to get answer for my research questions.

4 Methodology and Data Collection

In this chapter I explain in detail how I engaged in data collection with the purpose of collecting relevant data for my study. This chapter contains the following five sections:

- goal and methods,
- design of tasks and teaching activities,
- analysis of data collection,
- ethical considerations,
- implementation of methods.

The very last section involves three following parts: context, participants and data collection.

4.1 Goal and methods

The goal of my master's thesis is to explore pupils' misconceptions and misunderstandings related to the usage of brackets, and I have generated two research questions:

1. What kind of mistakes pupils in grade 10, involved in my study, do?

Form data analysis, it seems that some pupils show evidence of improvement in their performance during the post-test comparing to the pre-test. Thereby I want to address the following research question, which is composed by two parts:

2. a) What kind of improvement is showed in performance during the post-test?
2. b) What kind of causes can be identified concerning the improvement?

I would like to have a general view about the preparation of tenth grade pupils involved in my study, in relation to the usage of brackets and the process of bracketing, by emphasizing their misconceptions and misunderstandings. With purpose to achieve this general view of pupils' preparation, I thought to observe during several mathematics lessons and to test pupils' tackling of brackets and the process of bracketing by setting a written test for all pupils. But, only from these two methods I could not have access to notice and to discuss about misconceptions and misunderstandings of participants. Because it is related to capturing of notions and understanding of concepts and processes, and these cognitive levels could not be expressed just in several tasks solution.

So we, my supervisor and I, discussed the possibility to interview pupils with purpose to allow them to extend and clarify their answers that were presented in written test. On the other hand, by interviewing I can be provided with additional information that could be of relevance for my study. It would be impossible to interview thirty pupils that will participate in the test, because such an inquiry means too much work for one year master's work. So, I decide to select four pupils for interview, and for this selection I will focus only on written-test and the criteria for selection will be developed in subsection 4.5.2.

Since I was looking for relevant literature for my study, I was impressed by Swan's book (Swan, 2006) "Collaborative Learning in Mathematics" that my supervisor recommended it to me. I would like to introduce Swan's methods to my participants and to implement them during mathematics lessons in Albanian school. I found Swan's methods really interesting because they help pupils to adopt more active approaches towards learning. I select the three most relevant methods for my topic from Swan's methods to implement and to concretize them in my data collection. This will be developed in section 4.2.

However, I am curious to know if there would be any change in pupils' understanding of brackets and tackling of brackets after several teaching lessons related to the usage of brackets. So, I want to check one more time pupils' preparation in the end of the chapter, by using task-based-interview. I choose this method because I need to interact directly with four selected pupils since the moment of introducing the tasks, then with their thoughts and solutions, and finally with the acceptance or not of my suggestions.

My research method is ethnography, according to Bryman (2008, p. 402-403), since I am immersed in a tenth class setting in an Albanian school for two weeks. During this period of time I have made observations of participants' work during mathematics lessons' activities, I have listened to and I am engaged in conversations during my teaching lessons, I have interviewed four particular pupils, I have collected documents as: tests and group works of pupils (will be extended further in one of next sections), and I am writing about all aspects as mentioned above. As an ethnographer, I was a complete observer during the first week of my data collection because I did not interact with pupils (Bryman, 2008, p. 410). In addition I was a participant-as-observer since I organized and leaded three teaching lessons, which shows also the character of my research as a design study. In accordance to Wood and Berry (cited in Berg, 2009, p. 78) the design study consists of five steps:

1. *A physical or theoretical artifact or product is created.*
2. *The product is tested implemented, reflected upon and revised through cycles of iterations.*
3. *The multiple models and theories are called upon in the design and revision of products.*
4. *Design research of this nature is situated soundly in the contextual setting of the mathematics teachers' day-to-day environment, but results should be shareable and generalizeable across a broader scope.*
5. *The teacher educator/researcher is an interventionist rather than a participant observer in a collaborative, reflective relationship with the teacher(s) as the professional development model evolves and is tested and revised.*

So, I have adapted my teaching activities from Swan (2006), which are both created and tested by him, and I associated my theoretical approach to these activities and situated to the context of my study. But, on the other hand, even though I have compiled tasks and adapted activities for teaching activities, also for the test and the task-based-interview, the aim of my research is not designing and developing particular mathematical tasks and teaching activities. It is the highlighting of pupils' understanding about the use of brackets and to have different outcomes from task-based-interview compared to the pre-test outcomes. In this point I would like to mention Jaworski citation (taken from Berg, 2009, p. 94):

> *However, design research talks particularly of a product emerging from the design research process, and sometimes it is hard, in a teaching development context, to identify what is the product of this developmental process. We might therefore talk rather of developmental research, where the tools of development form the basis of what is studied and the outcomes of the research process constitute a combination of development and of better understandings of the developmental process and its use of tools.*

In my study the "tools of development" are tasks involved in activities conducted during my teaching lessons, and the interview during the pre-test. I think so, since tasks are adapted from other authors and the interview in pre-test has been organized on the basis of dialogue concerning tasks involved in. Concerning the "outcomes of the research process" I consider the conclusions from analysis of the post-test's performance of two participants.

My master work is a case study because it is associated with a particular class in tenth grade in an Albanian school. According to Bryman (2008, p. 53), case study has relevant qualitative methods such as: participant observation and unstructured interviewing, because they are viewed as helpful to achieve a detailed examination for the case. However, I chose to implement semi-structured interview in my study because of the fact that it would allow me to lead the discussion about what I am interesting in, and during the interview I could have a schedule to guide the interview. I will interview four pupils, and according to Bryman (2008, p. 440) it gives me the possibility to compare the cases to a certain extend. I think it is an adequate method comparing to structured and unstructured interview, because structured interview will give me a strong comparison between cases (pupils' answers) and this interview will not be able to answer my second research question, which waits for the answer that will emerge by analyzing of interviews. And the unstructured interview may take different directions and the comparing of cases would be difficult.

As I introduced in chapter 1, four pupils are interviewed twice: interview about their answers during the test, and during the post-test. The first interview is a semi-structured interview based on what they have written during the test. The second interview is a task-based-interview, which would be a variant of semi-structured interview because I have compiled tasks and some questions by myself before doing the interview with the purpose to guide the interview and to extend it depending on pupil's answers. The task-based-interview would help me to notice (as much as I can) pupil' mathematical thinking while they are solving given tasks that contain the usage and tackling of brackets. Another advantage of the use of task-based-interview is that it allows me to go deeper in pupil's reasoning and I can act in the moment.

Based on Goldin's (2000) description about task-based-interview, the post-test involved the subject (the problem solver, which is one of four pupils selected) and me (interviewer) interacting in relation to some tasks introduced to the subject (pupil) by me in a replanted way, but also having possibility of branching sequences of questions taking into account my research purpose. In my case my second research question is composed by two parts, such as: a) What kind of improvement is showed in performance during the post-test?; and b) What kind of causes can be identified concerning the improvement? In other words my goal is to investigate how much participants have understood the usage of brackets and what is the nature of their mathematics understanding for brackets in comparison with their situation at the beginning of the chapter related to algebraic expressions in their mathematical textbook. All interviews are video recorded for later analysis. This is a paper-and-pencil task-based-interview that gives me possibility to focus my attention more directly on the pupils' processes of addressing mathematical tasks. I have followed principles and techniques described by Goldin (Goldin, 2000, p. 539-544) to design and to implement (to execute) the task-based-interview. Now, I will bring ten points he mentioned, and briefly I will explain how I intended to achieve them.

1. *Design task-based-interviews two address advance research questions.* I have a clear goal for my conducting interviews, and in the interview situation, I present tasks that are designed as a means to achieve this goal.

2. *Choose tasks that are accessible to the subjects.* I have designed tasks before conducting observation and knowing the academic level of pupils, but I based the

design on mathematics school book and adequate previous literature related to the algebraic understanding of 14-16 years old pupils.

3. *Choose tasks that embody rich representational structures.* I have designed three exercises considering the encountering of pupils to the challenge, this is because the first two of three exercises, have a different presentation (the way of organizing data) from the previous exercises viewed by pupils before, even though in these cases is required the same knowledge. The first type of exercise is adapted from Kindt (2011), 'operation trees' since he states that it is a way *"to visualize more composed computations and reverse questions"* (p. 154).

4. *Develop explicitly described interviews and establish criteria for major contingencies.* During the interview I do not want to provide much help for pupil to solve the tasks. I will try to give the feedback only when it is necessary, and to adopt his/ her answer just saying "ok". My questions will be of the type: "How do you think to solve it?", "Why do you do it in this way?", "Can you be clearer?" etc. On the other hand, I will give help if the pupil will be stuck or will go in one direction that it is not important for my topic. I will tend to help the pupil through questions that are directly related to the answer I am expecting, or through examples that can introduce explicitly the idea I am looking for. On the second exercise, I expect pupils to have some difficulties because it is a new task for them even though it will be introduced before the interview, during one of my teaching lessons. And, in this part I will give some help to the pupil reminding firstly some basic concepts related to this task. The structure of the interview, the order of exercises, and their content are the same for each interview.

5. *Encourage free problem solving.* This point will be applicable in my interviews because students will have time to think by themselves and to express it before my feedback, or suggestions.

6. *Maximize interaction with the external learning environment.* During the interviews, pupils will have only: pencil, eraser, and paper. It might be a little different from external learning environment but they will have the basic and necessary tools.

7. *Decide what will be recorded and record as much of it as possible.* I plan to film all the interviews and the camera will be focused on the paper where the pupil is writing. Also I will use an audio recorder except of the video recorder.

8. *Train the clinician and pilot-test the interview.* I am conducting the interviews by myself, and I am based on Goldin's (2000, p. 539-544) criteria concerning compiling and conducting task-based-interview. And, the period of time that I was in Albania for data collection was limited, I had no possibilities to test the interview.

9. *Design to be alert to new or unforeseen possibilities.* During the interview I will manage to focus on both, listening to what the pupil says and noticing what he/she writes on the sheet of paper (oral responses and written ones). During this time, I will be open to different new ideas that pupil could have.

10. *Compromise when appropriate*. I should consider this point during the whole interview to evaluate situations to achieve the best for my data collection.

To collect data I need for my analysis I will start by observing classroom to be known with the organization of lessons and participants, and based on pupils' test I will show their performance related to my topic. Among four pupils that are interviewed, I will focus on only two pupils, which had improvement in their performances during the post-test, in order to analyze in detail their performance throughout the data collection including, test and interview about written test, participation in my teaching activities, and task-based interview. The way I designed and conducted these methods is introduced in respectively two next sections: Design of tasks and teaching activities, and Implementation of methods.

4.2 Design of tasks and teaching activities

When I was designing tasks for the pre-test and the post-test, and tasks for activities for three teaching lessons, I relied on four authors' theoretical basis as mentioned in the section related to my theoretical approach (Skemp (1976), Gray & Tall (1991, 1992), Swan (2005, 2006), Sierpinska (1994)), I take ideas from these researchers such as: relational and instrumental understanding, the procept, identification, discrimination, generalization and synthesis, and taking in consideration my research questions. In general, my goal is to create assignments that take into account the use of brackets with the purpose to notice pupils' misconceptions and their understandings related to brackets, and to develop their understanding.

I expect to answer the first research question referring to the written-tests, so I tend to compile tasks which, I assume would require instrumental and relational understanding for the use of brackets and tackling of algebraic expressions as objects. And, I will highlight mistakes of pupils in these points and the first interview will take place related to them.

The design of my teaching activities is completely based on Swan's approaches. I think it would be completely different from ordinary activities during mathematics teaching and learning in Albanian classes. In these classes, transmission approaches are conducted during mathematics lessons. Learners adapt passive learning strategies, without creativity and teachers do not stimulate pupils to explore. Teachers only question learners in order to lead them in a particular direction or to check if they are following the taught procedure.

My aim for implementing these activities is to engage pupils in discussion and to offer opportunities for explaining their ideas, also for learning in a different and meaningful way. Although perhaps there would be no change in pupils' performance (answers) during the post-test, or it would be any change from teaching activities developed by me just because it will be something new introduced to the pupils, the purpose of conducting these activities evaluated before, is to offer pupils possibilities to make justifications, and argumentations based on reasoning.

I expect to get answer for the second research question from task-based-interview, and for this reason I have compiled its tasks in accordance to Skemp (1976), Sierpinska (1994), Gray and Tall (1991, 1992), Swan (2006) and Kindt (2011), Booth (1984), Kaur (1990). The last three authors are mentioned because they have used in their studies the same tasks as I will do. Tackling of algebraic expressions as *procept*, which requires *relational understanding* of processes and products from these processes; *the four mental operations* in order to understand the significant of brackets' usage; and a new representation of algebraic expressions referring sides of geometrical shapes, are involved in these tasks. In Table 4.1 are introduced authors, on which I am based to compile and to adapt three tasks.

Exercises	Authors
Nr. 1	Kindt, Gray and Tall, Sierpinska, Skemp
Nr. 2	Swan, Sierpinska, Gray and Tall, Skemp
Nr. 3	Swan, Booth, Kaur, Gray and Tall

Table 4.1. Authors and exercises

I have compiled the test, the teaching activities and the task-based-interview before knowing pupils' level but I had information about mathematics text book that participants use and I based my design on chapter 3, Expressions with Variables (Babamusta and Lulja, 2009), by including in my tasks a medium difficulty for pupils in tenth grade. This chapter will be developed for one school week (five lessons) and it corresponds to one week of my data collection.

4.2.1 Test

While I was compiling the test and deciding for the tasks, I had in mind what I have read from previous studies, which are involved in chapter 2 and chapter 3, and my experience (especially as a teacher); also I took in considerate the Albanian mathematics textbook for the tenth grade. So, from reading of the research literature I got some initial ideas and I evaluated them bringing in my mind my experience, if it would be acceptable for the chosen environment (Albanian classroom). And finally, I completed and fit them in accordance to mathematics textbook that participants learn with. Review literature and my experience helped me to be faced to misconceptions and errors that pupils do and I tried to compile tasks that would be a challenge for several pupils.

Test involves five types of exercises (see Appendix A).
The first one has nine tasks, each of them have multiple correct answers (*Multiple-choice* questions). I selected this type of exercise to challenge the assurance of pupils related to their concept of brackets and the process of bracketing, and in the same time I had in mind some possible mistakes and misconceptions. This exercise is meant to be solved by pupils using previous knowledge and it has similarities with tasks introduced in mathematics textbook, except involvement of alternatives. So these assignments are situations that the participants have seen before and, furthermore they require basic knowledge that pupils should have it from previous classes. Each of these tasks contains five alternatives as probable answers and only two or three of them are correct answers. Wrong answers might be achieved by using knowledge in an incorrect way.

I will introduce only two from nine tasks and I will show where alternatives come from:

Exercise 1, task 4:
$(x+4)(x-5)=\ldots$

a) x^2-20 b) $x^2+4x-5x-20$ c) x^2+20 d) x^2-x-20 e) $2x-1$

Exercise 1, task 5:
$x^2y-(xy)^2=\ldots$

a) $2x^2y$ b) $x^2y-x^2y^2$ c) 0 d) x^2y-xy^2 e) $x^2y(1-y)$

There are two kinds of relations between the correct answers and the wrong ones: one is the between correct answers, which are different representations of each-other and it is related to a deeper thinking about the solution; the second relation is between the correct answers and the wrong ones, because these wrong answers are results of aware errors, considering the misconceptions and errors that pupils, in general, do. These two relations are interesting because the first one shows if pupils know different presentations of algebraic expressions, for example: in the second task as presented above, alternatives b) and e) are correct, and e) is the answer obtained from factorization of 'x^2y' in b). In this line, from choosing pupils' incorrect answers and taking in considerate the origin of these answers, in one way they can show if there are misconceptions in their learning. For example: referring to Kaur (1990) that for some pupils ignore the presence of brackets, they can choose alternative d) in the second task above. Another misconception involved in the first task above is presented by alternative a), where everything in the first brackets should multiply each element in the second ones and not to multiply only the first two elements of two brackets with each-other and the second elements with each-other.

This part of the test is related to the usage of some basic algebraic rules, especially concerning the process of bracketing as, distributive law and factorizing, and also it is related to Skemp's instrumental understanding (Skemp, 1976). The first exercise does not require pupils to use what they know in different situations or to generate new procedures based on previous knowledge, but just to use what pupils know, in type of tasks that they are confronted with even before. And from this exercise, I expect to achieve "a picture" of possible mistakes that test participants have, especially in relation to tackling of brackets. Since tasks do not present new situations and require instrumental understanding of bracketing process, I think pupils can achieve good results in this exercise selecting at least one correct answer because, according to mathematics textbook, multiple-choice is new for them. Saying that, I am making explicit my own expectations as a researcher with basis in my experiences as a teacher.

I was thinking about an exercise to challenge pupils and to require an active use of brackets different from the passive use of brackets which is involved in the first exercise and furthermore the passive use of brackets is submitted more by Albanian mathematics textbook. The idea of active use of brackets was presented by Kindt (2011, p. 154) in one exercise, which I conducted as the second exercise of my test. It includes four tasks, and requires finding of the correct place of brackets, in the left side of algebraic statement, with condition to save the equal sign.

Exercise 2: Place parentheses in these expressions to the left of the equal sign to create equality.

a) $x+x^2+3\cdot x+4= x^2+4x+4$

c) $x+x^2+3\cdot x+4= x^3+4x+4$

b) $x+x^2+3\cdot x+4= x^2+4x+12$

d) $x+x^2+3\cdot x+4= x^3+5x^2+7x+12$

Similarly this exercise requires knowing of distributive law, which is related to the process of bracketing during multiplication of one term with brackets, or multiplication of two brackets, but this is different from the first exercise because these rules are used in different and new conditions. From my experience, some pupils know and use in correct way the distributive law: $a\cdot(b+c)=a\cdot b+a\cdot c$, but they do not know where to place brackets in the left side of the following algebraic statement: $a\cdot b+c=a\cdot b+a\cdot c$, to make equal the both sides. All difficulties concerning this type of exercise might be from the rote they and mechanical learning, and are related to Skemp's relational understanding (Skemp, 1976).

I think that, this is an interesting exercise because it requires the use of some basic knowledge in a new and challenging situation. Pupils who will solve this task might have relational understanding and know more about the use of brackets, but not just to expand them. And concerning my theoretical approach, pupils would achieve a relational proof solving this exercise; on the other hand this exercise gives the possibility to generate the four mental operations of understanding the importance of the use of brackets, as it is explained in section 3.1. I think this type of exercise might seem to pupils as difficulty to understand and to solve it, and I do not expect to have correct solutions from mostly of them. This is because the rote learning dominates during mathematics activities (teaching and learning) and this test will be developed in the first lesson of the chapter related to algebraic expressions in the textbook.

Continuing in the light of both, the first and the second exercise, in which the passive and the active use of brackets respectively dominate, I compiled the third exercise that includes five tasks, as the following examples:

Exercise 3, task a) and e):

a) $(-4x)(2x+8)= ? -32x$ e) $(x+?)(x+?)= x^2+6x+8$

Brackets are located and expansions are finished but they contain missing terms that pupils should complete them with purpose to make equity between the left and the right side. The third exercise is meant to be solved by pupils, knowing two following rules of multiplication of brackets (distributive law): $a(b+c)=ab+ac$, and $(a+b)(d+c)=ad+ac+bd+bc$.

This type of exercise emphasizes the consideration of algebraic expressions (polynomials) as objects and each element of them should be multiplied by each-other in order to achieve a multiplication of two expressions. Based on Skemp's theoretical framework, relational understanding requires the pupil to choose, to change and to apply the formulae in new situations; to pupils it is given the possibility to use their relational mathematics to solve these tasks.

The forth exercise includes five algebraic expressions and it requires to expand brackets and to execute operations. I am interesting on expanding brackets step by step because in the previous exercises answers were given not allowing pupils to write whatever and however they know related to the task. And, from solutions of these tasks, I expect to have a wider view of pupils' mathematical thinking concerning tackling of brackets, mathematical objects, operations with brackets, types of brackets. Here are some examples from **Exercise 4, task c), d), e):**

c) $(12x^3-x^2)-3x(2x+1)(2x-1)=$

d) $(\sqrt{2x}-2\sqrt{y})^2+(x+x^2+2)(5-x)=$

e) $-\{5x-(11y-3x)-[5y-(3x-6y):3]\}=$

From the first fourth exercises I expect to achieve a consideration about tackling algebraic expressions as objects (procept) by participants since each task involves algebraic expressions.

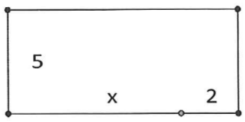
		Considerations
Exercise 1	9 tasks	Tackling algebraic expressions as a whole one based on their syntactic aspect
Exercise 2	4 tasks	Tackling algebraic expressions as objects (products) based on the active use of brackets
Exercise 3	5 tasks	Tackling algebraic expressions as objects (products) based on the passive use of brackets
Exercise 4	5 tasks	Considering algebraic expressions involved into brackets as a mathematical object and using brackets for algebraic expressions which should be tackled as a product (object)
Exercise 5	5 tasks	Considering algebraic expressions as objects even in new representations based on the active use of brackets

Table 4.2. Considerations of five exercises

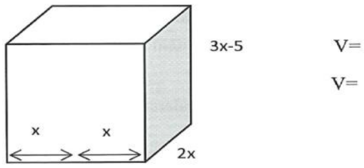
During interviews I expect to have pupils' justifications, for exercises, concerning their manipulations, decisions and results. This is because being able to explain the solution by basing it on mathematical foundations leads me somehow, to the type of pupil's understanding and learning.

But my demand to enhance meaningful learning during mathematics activities leads me to link tackling of algebraic expressions not only in "algebraic environment" (as in the first four exercises) but also in relation to other topics. I am also interesting on this type of exercise to "explore" if tackling of mathematical objects depends from the way of presenting. I involved as the fifth exercise of the test writing of algebraic expressions for area and volume of particular diagrams. This exercise is meant to check pupils' creativity and their assurance referring to the way of considering mathematical objects. I encountered this type of exercise in several previous studies as: Kaur (1990), Swan (2006), and Booth (1984). The fifth exercise contains four area diagrams and one volume diagram.

Exercise 5, task 2 and 5:



A=



The purpose of the last task is to encourage pupils to write down algebraic expression of cuboid volume and to extend that by working with variables multiplication and the distributive property.

As I described above (see Table 4.2) I designed all tasks on the basis of aware expectations and planed frames with purpose to help myself in analyzing pupils' performances and to know what I am looking for.

4.2.2 My own teaching 1, 2, 3

The reason why I decided before observing classroom to develop three (depending from the number of lessons involved in the chapter) lessons by myself, was to bring some new activities for pupils. From my experience as a teacher I know that the organization of mathematics lessons is similar almost every time, and I think it comes from the large number of pupils in the class, limited time of 45 minutes for a school hour, also from teaching orientation. I am based on Swan's 'collaborative' orientation (Swan, 2005) to bring an effective teaching for participants concerning understanding of the use of brackets. During these activities the following six aspects (Swan, 2006) are emphasized:

- Working in small groups with purpose to encourage critical, constructive discussion.
- Reasoning rather than 'answer getting' because is better to aim for depth than for superficial 'coverage' since pupils are more concerned with what they have 'done' than what they have 'learned'.
- Exposing and discussing common misconceptions that allow pupils to confront their thoughts and ideas.
- Using higher-order questions with purpose to promote explanation, application and syntheses; it is more effective than mere recall.
- Using rich, collaborative tasks to encourage discussion, creativity, justifying, and to encourage 'what if?' and 'what if not?' questions.
- Creating connections between topics with aim that teaching to be effective. And it is difficult for pupils to generalize and to transfer their learning to other topics and contexts.

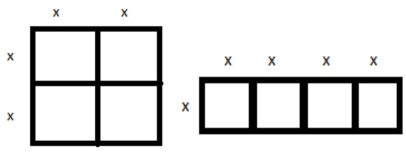
Among several activities suggested by Swan (2006) I am focused on three of them, which I think, could be relevant for understanding the use of brackets. In accordance to Swan's theoretical basis (Swan, 2005, 2006), which emphasizes discussion that offers explanations, reasoning, and making misunderstandings and misconceptions explicit I divided pupils in small groups and developed these activities. Also I organized class discussion with purpose to take part all pupils and I introduced what pupils said in blackboard for all.

Evaluating mathematical statements
There are ten statements and generalizations, and pupils should justify their answers, evaluate if each of them is always, sometimes or never true. In this activity pupils are asked to decide whether the statements are always, sometimes or never true and to give explanations for their decisions introducing examples or counter-examples to support or to refute the statements. Also, students may be invited to add conditions or otherwise to revise the statements so that they become 'always true'. I chose it because this type of activity develops pupils' capacity to explain, convince and prove. On the other hand, it is the explanation that makes explicit the reasoning that drives the solution or the proof forward and, to achieve explanations it is needed the mathematical foundations achieved in accordance to relational understanding where the pupil is aware about knowledge he is using.

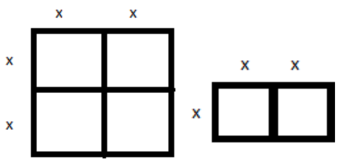
I have compiled ten algebraic generalizations in relation to the use of brackets (see Appendix B), for example:

1. $(2x)^2 = 4x^2$ 2. $(2x)^2 = 2x^2$ 3. $\frac{4x+6}{2x+3} = \frac{1}{2}$

The first one is always true since the product of '2x' is squared that means the algebraic expression '2x', involved into brackets, is tackled as an object. It will be acceptable and valuable representing two area diagrams which have areas respectively: $(2x)^2$ and $4x^2$.



The second mathematical statement is sometimes true and it is true only for value zero of x. In addition, even using geometric representations, area diagrams with respectively area $(2x)^2$ and $2x^2$ do not introduce the same surface:



I consider this couple of examples valuable to show the importance of tackling algebraic expression included in brackets as object and it helps pupils to make discrimination between two cases of tackling mathematical objects. The confrontation of algebraic context and geometric context of mathematics statement is effective in building bridges between ideas and representations that help pupils for a deep understanding and in a meaningful way.

While the third example is part of the group of statements that are never true. It requires factorization of '2x+3' and emphasizing condition that '2x+3' should be different from value zero (x≠- $\frac{3}{2}$) with purpose to simplify it in numerator and denominator. After these we get numerical statement that 2= $\frac{1}{2}$ which is never true.

In addition, participating in this activity *"encourage pupils to focus on common convictions concerning mathematical concepts"* (Swan, 2006, p. 146), and it emphasizes developing mathematical arguments and justifications or other ways of representing.

Interpreting multiple representations

During this activity pupils work together matching cards that show different representations of the same mathematical idea, in my case algebraic expressions are related to emphasizing the use of brackets. Pupils draw links between representations and develop new mental images, or develop existing ones concerning the concept of brackets and the process of bracketing. So, the concepts of bracketing and the concept of procept (Gray and Tall, 1991, 1992), as other mathematical concepts, have many representations: words, diagrams, algebraic symbols, tables; and pupils should sort cards that show different representations of mathematical objects into sets and, each set has equivalent meaning. I will show some cards from four representing ways (Swan, 2006, p.146), and for more show Appendix C:

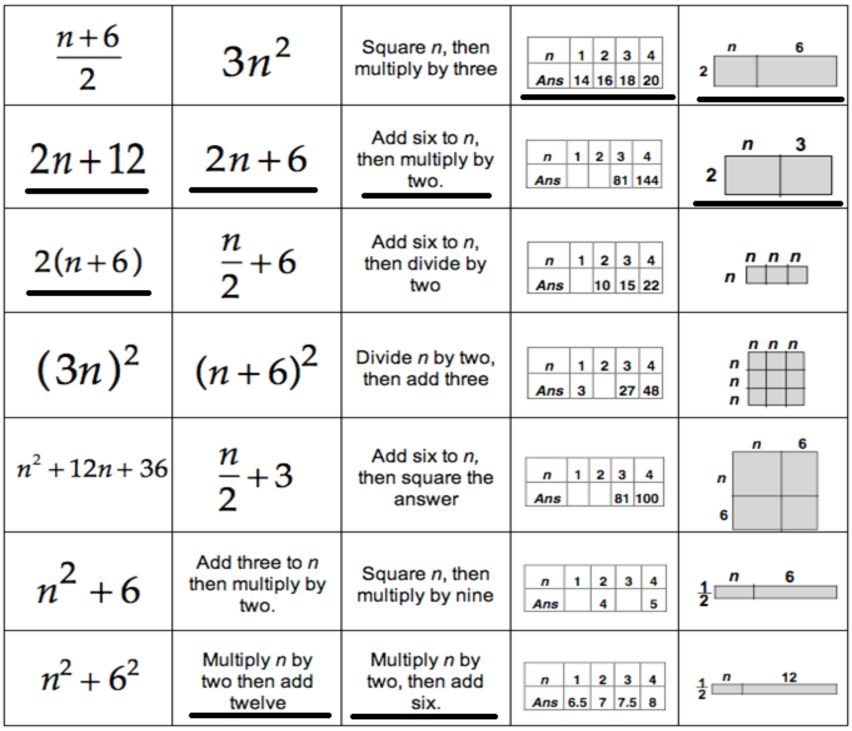


Figure 4.1. Cards of four types of presentation

Cards that are underlined would be interpreted in relation to each other. Matching the corresponding words and symbols cards focuses attention to the order of operation highlighting the use of brackets with purpose to construct an object since its product is needed primarily; following by matching words and symbols cards to area representation and tabular representation. Also there is the possibility to present differently one card in the same type of representation, 2(n+6) and 2n+12, which enables generalization (Sierpinska, 1994) of an event of brackets. On the other side, pupils are allowed and encouraged to revealed and to explicitly discusse common errors, as 2(n+6) = 2n+6, discriminating each set of cards, respectively for 2(n+6) and 2n+6, for each representation. Except written cards, there are blank cards and pupils are asked to construct them but *"the focus of the activity is on interpreting rather than the production of representations"* (Swan, 2006, p. 145).

This activity allows pupils to construct: meanings, links between the concepts, and links between different representations by discriminating carefully common conceptual obstacles (related to tackling of mathematical objects) and by explaining similarities and differences in meaning of the use of brackets and of tackling of mathematical objects. These cards sets are powerful ways of encouraging pupils to see mathematical ideas from a variety of perspectives and to link ideas together. These cards are adapted by Swan's ones and I have translated them in Albanian language.

Creating problems
I am focused on creating equations, in which pupils build up their own equations transforming both sides of an algebraic statement. I adapted this method from Swan (Swan, 2006) and I developed it further since I organize this activity for all pupils with purpose to involve everybody in listening, also in discussion, while according to Swan (2006) this activity is developed in pairs of pupils. We start to create equation (algebraic statement) from a particular value of the variable, for example '$x=2$'; it is followed by determination of an operation (+, -, *. /) and the object that participates in this instruction, for example 'add $3x$', and the condition is to save mathematical identity which could be achieved by operating equally in both sides of the equal sign, $x+3x=2+3x$; and so on.

From this activity pupils gain confidence with the notation used in equations, including the use of brackets with purpose to tackle an algebraic expression included in brackets as an object, and to involve into brackets an algebraic expression from which should be taken primarily the product (object). This activity develops pupils' abilities to create equivalent shifts, to explain their ideas, and to participate in discussion about common misconceptions related to understanding the importance of using brackets.

So, the aim of conducting these three activities is to help pupils to adapt more active approaches towards learning, by engaging in discussion with their ideas and justifications, and by working collaboratively to share their methods and results. My intention is to bring "something new" and non-routine in mathematics lessons putting in the center pupils and their understanding. The interconnected nature of these activities confronts conceptual difficulties through discussion and develops the challenging connected to collaborative orientations towards teaching. On the other side, *"learning is a collaborative activity in which learners are challenged and arrive at understanding through discussion"* (Swan, p. 162).

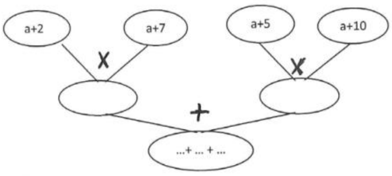
4.2.3 Post-test

I have compiled three exercises for the post-test, which is a task-based-interview, in order to achieve an answer for the second research question, which is composed by two parts: "*what kind of improvement is showed in performance during the post-test?"* and "*what kind of causes can be identified concerning the improvement?"*

I have compiled these tasks taking into account the purpose to notice pupils' understandings related to brackets and their misconceptions (since I expect to notice also in pre-test) and to make a comparison between participants' answers from the pre-test and the post-test. In addition, I have created assignments for interview based on theoretical approach of Skemp (1976), Swan (2005, 2006), Sierpinska (1994) and, Gray and Tall (1991, 1992).

Task-based-interview is composed by three types of exercises: operation trees, creating equations, and area diagrams. The first type of exercise is called "operation trees", it contains two tasks and according to Kindt (2011, p. 153-154) this type of exercise enables visualization of more composed computations and reverse questions and pupils have to decide where to start the filling of the trees; and it is more challenging than monotonous tasks which primarily lead to imitative behavior. And the last advantage is the reason why I decide to involve “operation trees” in task-based-interview, and I used it as a new way to represent two algebraic expressions connected to each other through an operation as: multiplication, division, addition, and subtraction. I think it is an interesting type of task and it is related to my topic since it requires tackling algebraic expressions as objects, which is a very important aspect of algebra according to Gray and Tall (1991, 1992), and it is achieved by including them into brackets.

Exercise 1, task 1:



Tackling algebraic expressions as mathematical objects and involving them into brackets, such as (a+2)(a+7), with purpose to multiply the two given algebraic expressions and to create conditions to use the distributive law. The identification of polynomials as *products* is followed by *identification* of the multiplication of polynomials as one case of the brackets’ use. Pupils can make *discrimination* between multiplication of polynomials and addition of polynomials in relation with the use of brackets with purpose to identify general properties (*generalization*) for each case and finally to perceive a unifying principle (*synthesis*) for the use of brackets (Sierpinska, 1994). Buy going through these steps the reason *why* the pupil is going to use brackets (Skemp, 1976) will dominate and he/she does not suffice only on distributing terms of polynomials according to distributive law. Based on the same principles is compiled another “operation tree”, see Appendix D.

Even though “Creating equations” is one of my teaching activities, I decided to involve creating equations examples in task-based-interview, with aim to see if my teaching activities were clear and understandable for pupils. This exercise contains two equations constructed through several steps, and pupils have to justify if mathematics identity is saved or not and if algebraic statements are equivalent from one step to the next one. And, saving the mathematics identity could be achieved by tackling algebraic expression of the same side in the previous step, as a *product* (object) even though in one previous step it should be tackled as a *process*. And in mostly cases to achieve this, is helpful and necessary the use of brackets.

Exercise 2, task 2

x=3

2x=6

2x-4=2

2x-4•2=4

2x-8=4

(2x-8):4=1

(2x-8):4+3x=1+3x

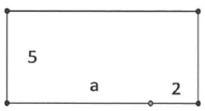
(2x-8):4+3x•3=3(1+3x)

(2x-8):4+9x+4=3(1+3x+4)

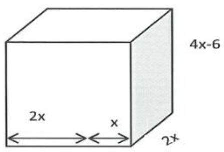
This example is one of two tasks of the second exercise and, there are three incorrect shifts (from the III step to the IV one; from VII to VIII; and from VIII to IX). It is necessary to mention that I have compiled this task considering mistakes and compiling the next step based on previous one even though it is not correct. This is because I did not give indications that there are mistakes through steps.

I got the idea for this type of exercise from Swan's activities (Swan, 2005, 2006) and I adapted it for task-based-interview. I expect to highlight tackling of *mathematical objects* and *discriminating* of operating with algebraic expressions in two cases: with and without brackets, correct and non-correct use of brackets. This will help pupils to capture the reason *why* the use of brackets is necessary.

The third exercise is about writing algebraic expressions for geometrical magnitudes (perimeter, area, volume) of several diagrams, it is the same with the fifth exercise of the test (pre-test), so there is the same purpose of implementing this type of exercise in task-based-interview as in the test. Another reason that I implemented it in task-based-interview is my attempt of having a concrete (explicit) comparison about pupils' performance in the first and in the last lesson of the chapter. There are five diagrams, for the first two is required algebraic expression for the perimeter and the area, for the third and the fourth diagram is required only the area and, the volume of the cuboid which is the last one. Here are two examples from **Exercise 3**:



P= A=



V=
V=

I have designed completely all these before confronting with pupils' level and the way in that class teaching of class. I have only mathematics textbook for tenth grade and teacher's permission to use her mathematics lessons for one week in one class (five school hours). I did not have possibility to test tasks and teaching activities in front of pupils before presenting them to participants.

4.3 Analysis of data collection

My data collection lasted two weeks and I collected a "data set" as the following one:

- Notes during classroom observation of four school hours
- Written tests of thirty pupils
- Interviews of four pupils (selected by me) about their test
- Written responses and posters developed by pupils during my teaching activities
- Task-based-interview of the same four pupils
- Pupils written responses during the interviews
- Field notes.

I was preparing to take notes in coherence to data collection but this proved to be difficult. During interviews I was focused on both written and orally pupils' responses, with attention to continue the interview with other questions, which were depend also from previous pupils' answers. And this prevented me from taking notes during interviews. And my field notes for this part consist mostly in some comments about what impressed me from pupil's answers, since I wrote them after our discussion with purpose to take them into account again during transcription and analysis. I have been in the same conditions also during my teaching lessons when I should organize and lead activities, and it was impossible to take notes but I have filmed everything that happened. While the field notes, taken during the classroom observations describe mostly the environment, the organization of class, the organization of teaching lessons, etc. This helps me to compile section 4.5.2.

My data analysis is organized in two plans:

- The analysis of test
- The analysis of interviews.

The analysis of tests consists on investigation of answers' correctness since there were written answers without justifications, but the analysis of participants' justifications, argumentations and proofs are parts of the interviews' analysis. Based on these I have achieved, a general view of pupils' mistakes and misconceptions related to tackling of brackets, standing within the frame that allow written tests; and I have achieved a portrayal for two from the four pupils interviewed, starting from:

1. His/her preparation (answers) in the pre-test (test and interview about the test)
2. His/her preparation (activeness) during my teaching activities
3. His/her preparation (answers) in the task-based-interview.

To achieve an overview of pupils' misconceptions and mistakes I have been interested on wrong answers and their density, and on the tasks in which they are occurred. This is because for each task I have my expectations related to my theoretical approach. For example:

$x^2y-xy^2=$	$x^2(y-y)^2$ $x^2(y-y^2)$	1 2
$x+x^2+3\cdot x+4= x^2+4x+4$	$[x+x^2+(3\cdot x)+4]= x^2+4x+4$ $(x+x^2+3\cdot x)+4= x^2+4x+4$ $x+x^2+(3\cdot x+4)= x^2+4x+4$ $x+x^2+3\cdot(x+4)= x^2+4x+4$ $x+(x^2+3\cdot x+4)= x^2+4x+4$	1 1 1 2 1

Task 6 in the first exercise and task 1 in the second exercise are introduced in the first column, possible pupils' wrong answers taken from the tests and their density are introduced respectively in the second and third column. I have formulated three helping questions in answering the first research question:

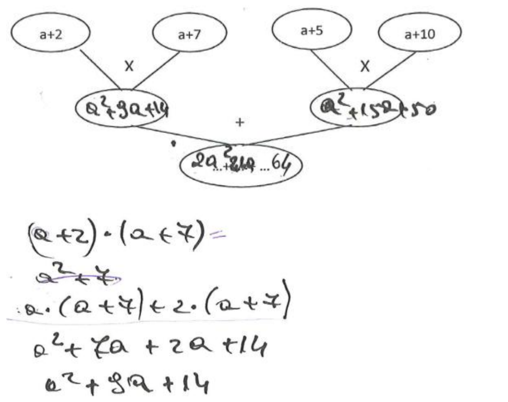
- How are mathematical objects tackled by the pupils?
- Which is more present: instrumental or relational understanding?
- Are there any Slips or Bugs?

This is only to organize my analysis of tests and to lead me during analysis, because I cannot achieve definitive answers about them since written tests do not give me more access than this. It is necessary to emphasize that these answers remain open.

Skemp's (relational and instrumental understanding (Skemp, 1976)), Sierpinska's (mental operations (Sierpinska, 1994)), Gray and Tall's (precept (Gray and Tall, 1991, 1992)) framework have been fundamental for analyzing of four interviews of two selected pupils among four pupils that were interviewed. I have listened and watched many times all interviews until I decided for several episodes (sequences) relevant for my topic and useful for answering my second research question. In addition, I have transcribed and translated in

English relevant parts from interviews with purpose to involve them into analysis; also I have associated these parts with photos from written responses of participants. I have not transcribed data collected during my teaching lessons because in these activities have participated the whole class. I am focused on some short dialogues between me and the two selected pupils with purpose to involve them in analysis. It is needed to mention that it was difficult to stop and to discuss with pupils that I really need to, in relation to my study (four selected pupils), because all pupils were involved in activities and they wanted help and instructions how to operate.

I decided to compile portrayals of participants to get an overview of his/her preparation during all my data collection. In the first section of pupil's portrayal I introduced pupil's answers for several tasks in written test, followed by my comments that were based on my perception. I tend to make clearer the reason of pupil's answers in test, asking him/her about justifications during the first interview. And below to this part I involve pupil's answers and justifications followed by my analysis in accordance to my theoretical approach. In the second section of pupil's portrayal I describe his/her participation and activation during my teaching lessons. And finally, in the last section I involve pupil's answers, justifications and argumentations related to tasks presented in that moment, also followed by my analysis in accordance to my theoretical approach. In this part I try to make a comparison between answers and justifications given in the first interview and these given in the second interview with an emphasis on tackling mathematical objects, mathematics understandings and mental operations. Here is an example of analysis one dialogue from task-based-interview:

Dialogue 11		
Names	Dialogue	Notes
D	Why are you using brackets?	Exercise 1/ Task 1 
A	Because I got the first one as a single number, also the second; and I should multiply these two.	
D	And what would happen if you don't use brackets?	
A	*Only '2' will be multiplied with 'a'.*	
	...	
D	What does 'a(a+7)+2(a+7)' mean?	
A	It means that each elements of the first bracket multiply the second bracket.	

"… I can see a creative justification by Alba for the use of brackets while she aims to tackle two algebraic expressions as *objects*. I would like to categorize this as *relational* proof because to calculate the product of two algebraic expressions Alba considers her understandings for mathematical objects and distributive law, and passes to formal definitions as it is the process of multiplication. From the last dialogue, it seems that Alba could *identify* one case of using brackets while she is multiplying two algebraic expressions, and she *discriminates* it with the case of multiplying these two algebraic expressions using no brackets. …"

I can mention choosing episodes relevant for my aim, achieving results from written tests and making conclusions for improvement of participants as steps that I was stuck during data analysis.

4.4 Ethical considerations

During conducting a research it is essential having in mind several ethical conditions and challenges in the same time. It is important to refrain to ethical conditions because, I try to focus on understanding of pupils extracting (looking for) their misconceptions and misunderstandings, and on the other hand the errors and non-achievement are private "settings" for everyone and I want to make public my research.

Taking permission to conduct my inquiry passes through some steps: firstly I have communicated with the head-master of the upper secondary school in Albania where I supposed to conduct my data collection; also I have communicated with the teacher who I presumed will help me to conduct my research's activities in her class; then I took a positive response that, my research was approved by "Norsk Samfunnsvitenskapelig Datatjeneste" (NSD, Norwegian data protection service) in Norway; and the last approval was by pupils (participants) which signed fieldwork permission for participating in test and interviews (see Appendix E).

According to Bryman (2008, p. 377-380) there are five criteria in establishing and assessing the quality of research: credibility, transferability, dependability, confirmability, and authenticity. In my research, I adhere to these criteria in this way:

Credibility- According to my understanding and my background, I try to select methods that will give me an adequate output for analysis and answers for my research questions. And my data analysis is based on my theoretical approach as well as the design of tasks and the implementation of interviews, leading me to the center of the circle, which involves my topic (understanding and tackling of brackets), research questions, theoretical approach, methods, data analysis, and results of my research.

Transferability- This is a case study and a qualitative research since I am studying several pupils' understanding, which means my results could not have a general validity. Furthermore, I can see possibilities for further research, maybe by considering new variables and by generalizing through social settings. And I believe that this research will be helpful for teachers even though the pupils are the subject of my research.

Dependability- As it is impossible to "freeze" social settings and circumstances where the research is conducted and the researcher is, also the study would be replicable and results of my study can be reproducible under the similar methodology. In addition, I would like to emphasize that I consider the possibility of improving the preparation of participants in my study since I developed interviews in level of conversation (giving and getting thoughts and ideas) and I conducted several new activities related to the importance of tackling brackets in correct way.

Confirmability- I made subjective decisions to determine my choices and how to implement them. In going in my research helps me to develop to be awareness of my personal values and preferences, and I tried not to be influenced even though it is difficult. So, I based myself on theoretical framework and previous studies during my data collection and, during writing my thesis I consider also my perceptions, understandings and interpretations; but I tend to be transparent in describing in details all my data collection and basing completely my data analysis on written tests and transcription of interviews.

Authenticity- I have video recorded all moments during my data collection, but the analysis focus on a "limited" part of material filmed. I have selected some relevant and important parts for analysis and these are related to my topic since the selected data help me in answering my research questions. On the other side I am aware that transcription of the interviews, even though I tried to be authentic, does not show the whole truth, and furthermore participants may be influenced by the circumstances, the situation of being the participant, and by the presence of camera.

In addition, I want to emphasize the confidentiality that is involved in this inquiry since during my interviews conversations related to personal the beliefs and characteristics of other pupils or to the teacher are not appearing, everything was related to tasks and their solutions. Also participants' names are replaced by pseudonymous not found in that class, and I do not give detailed information about the city where the school is and discerning information about the participants. But I cannot exclude the possibility that someone can recognize the participants, the school or the city where my study is conducted.

4.5 Implementation of methods

This section consists on introducing the context where my research is conducted, the participants which are involved in this research, and my data collection.

4.5.1 Context

I conducted my research in a private school at upper secondary level. This school has five hundred and eighteen (518) pupils, 34% are from countryside and 66% from urban area. These pupils are organized in eighteen classes, six classes for each grade: tenth, eleventh and twelfth grade. The average of number of pupils per class is twenty-eight pupils. In the tenth grade, there are three classes that have chosen as the first preference the advanced mathematics. All the classes in the tenth and eleventh grade, except English, have to choose another obligatory foreign language among Italian, French and German.

All Albanian teachers have a test (national exam) every five years, and they get some points. According to these points there are four categories of classification, the first one is the best and the forth category is the weakest. The Ministry of Education and Science organizes training seminars for teachers with aim the teachers' qualification. Except the state training, seminars about teacher's training are organized even in this school. Every year, two training seminars were organized with the support of foreign experts and some of this school's teachers. All teachers in this school (forty-six teachers) have the higher education (all levels), from who twenty have the first level of qualification, eight teachers have the second level of qualification, six have the third level of qualification, and twelve teachers have the fourth level of qualification.

The school is leaded by the headmaster and two sub-headmasters, and the Leading Council of the school is composed of seven members, including the headmaster, teachers and parents. The school environment is new one and according to contemporary standards since it is built last year. The building contains eighteen classes, one special-class where tests are conducted, one audio-visual meeting room, physics laboratory, chemistry laboratory, ICT laboratory, one auditorium, and environments for physical education.

The school days are from Monday to Friday and, during one school day six or seven school hours can be held; and pupils have two breaks, each from fifteen minutes, one after the second and one after the forth school hour. One school hour lasts forty-five minutes.

I conducted my research in the same school where I followed my upper secondary level (four grades) because I thought I will be welcome in that school more than in others. My advantage was that it is a context I know from before, and I know most of teachers who helped me allowing the four pupils to be all interviewed in the same day by leaving their other lessons.

4.5.2 Participants

In my study I am focusing only on one class in 10th grade. After I consulted with mathematics textbooks for each grade of upper secondary level, I saw the chapter “Expressions with variables” (Babamusta and Lulja, 2009) from tenth grade as more relevant for my topic. This is because this chapter involves basic knowledge, related to manipulations and operations with algebraic expressions, even obtained from lower grades. One of the sub-headmasters (she has the first level of qualification) teaches mathematics in all tenth grade classes. It was impossible for me to follow some mathematics lessons in all tenth grade classes with purpose to know pupils and their mathematics level, and then to decide which class to follow. So, the teacher chose the class for me, which follows advanced mathematics. Another reason for choosing this class concerns its timetable of mathematics lessons distribution, because I and the teacher wanted to have one activity for each school day. The timetable of mathematics lessons of this class was more convenient even though I conducted interviews about written test and my first teaching lesson in one day.

In this class there are thirty pupils, seventeen girls and thirteen boys, and they are around sixteen years old. Furthermore the pupils have five mathematics lessons per week, three times basic mathematics and two times advanced mathematics, they have two different books, one for basic and one for advanced mathematics.

I will introduce a setting chart to show the environment in which participants of my study work every day and also during my class observations:

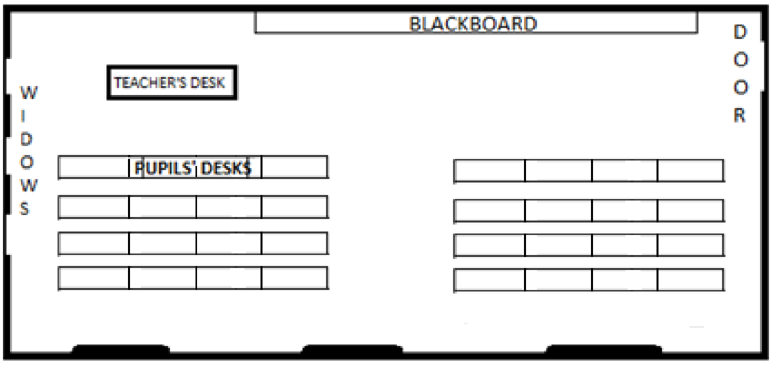


Figure 4.2. The class setting chart

The class had this seating during my data collection. This way of organizing desks is static because the way of teaching is almost the same. So the form of organizing teaching is: the teacher lectures new lesson and pupils listen and take notes (if they want), pupils work on assignments the teacher gives to them (individually or on the blackboard helping by teacher), and pupils explain homework (from the last lesson) mostly on blackboard while the teacher check them. Often explaining and checking homework takes part as the first activity. The teacher tries to include pupils when she was a lecturer by asking questions along the way, and pupils also initiates contact, when they ask about things they do not understand. The teacher mostly grades pupils based on the given tests (almost always in the end of each chapter), and sometimes based on the pupils’ answers while they are on the blackboard.

As I have mentioned above the test has been developed for all pupils of the class, while interviews have been conducted between me and the four pupils (individually) that I selected after checking written tests. This selection of four pupils is an important decision for my study since they will be interviewed and I will analyze their performance with purpose to look for any misunderstanding or misconception and any improvement concerning brackets' use. Firstly I chose five pupils, and I consulted with the teacher, asking her to give me a hand to save only four. She asked me to exclude one of the five pupils because he/she had 'family problems' in that period of time. I was totally agreed and, I had four seeking pupils. I checked thirty tests for a half-day and I was looking for pupils who:

- Have misconceptions about brackets
- Have misconceptions about mathematical objects
- Are unsure about what they know
- Use the rules in the simplest tasks
- Have no creativity
- Show no practice

4.6 Data collection

During my data collection I spent two weeks in this class, observing, doing test, interviewing and teaching, from 18th of October to 28th of October, 2011. Here is an overview of the way I organized my data collection:

<table>
<tr><th>Date</th><th></th><th>18/10/11</th><th>19/10/11</th><th>20/10/11</th><th>21/10/11</th></tr>
<tr><td>First week</td><td></td><td>Observation</td><td>Observation</td><td>Observation</td><td>Observation</td></tr>
<tr><td>Date</td><td>24/10/11</td><td>25/10/11</td><td>26/10/11</td><td>27/10/11</td><td>28/10/11</td></tr>
<tr><td rowspan="2">Second week</td><td colspan="2">Pre-test</td><td rowspan="2">Second lesson (participant observer)</td><td rowspan="2">Third lesson (participant observer)</td><td rowspan="2">Post-test (task-based-interview)</td></tr>
<tr><td>Test</td><td>Interviews and first lesson (participant observer)</td></tr>
</table>

Table 4.3. Overview of the organization around data collection

There were only a couple of weeks because the lessons which were relevant for my study in the chapter "Expressions with variables" in Albanian mathematics textbook (Babamusta and Lulja, 2009), that I was interested in, lasted one week. My impression was that if I would extend it, it would be boring for pupils. During the first week, for four lessons, I observed the class with intention of getting known with pupils and the way teaching is organized in this class. There is no change compared to the organization of classes when I was in this age. During forty-five minutes, pupils engaged with tasks from previews lesson (individually or in the blackboard for all the pupils) and in the second part of the lesson the teacher presented and explained the new lesson and the new tasks.

During the first week I was a complete observer, but in the second one I was active as a teacher and a researcher, becoming participant-as-observer. During this second week, I did the pre-test, my own teaching, and the post-test, furthermore I have used video recorder in each activity. This was like a pack, in which is involved: evaluation of actual situation; the attempts for improvement; and the evaluation of efficiency of the previous step.

4.6.1 Observation

Only the first day of the observation (18/10/11) was somewhat different from three next days, because I presented myself and pupils agreed to take part in test and interviews writing their names in fieldwork permission (see Appendix E). Pupils did not show any rejection to participate in my project. After that, for the whole rest of the week, I was a complete observer because I did not participate in any of the teaching lessons. The purpose of my observation was to get know the way of organizing teaching lessons and the pupils. I cannot get an overview of pupils' tackling brackets because the four lessons I observed were not related to the use of brackets. During this period of time I stayed passive and took notes.

4.6.2 Pre-test (test + interview)

In the first day of the second week of my data collection (24/10/2011), I conducted the test for all pupils of the selected class. After I had checked the tests and selected four pupils, I interviewed these pupils about their written tests. The purpose of this test was to classify the pupils' work and to pick four students for the interviews and the further activities. So, the first stage of my data collection was the test and the interviews about this test and both together I have called the pre-test.

The test lasted 45 minutes, which involved 30 pupils and took place in the special-class (mostly used during exams). Pupils were not informed before about the topic and the content of the test because I wanted to check their background related to tackling brackets since it was the first day in the new chapter. In the beginning of the test I explained exercises and the distributive of points among exercises, and also I was disposable to provide any explanation for pupils through this activity.

To be clear, "the distributive of points" means to associate an amount of points to each task in relation to its difficulty and importance for the test and the test's aim. So, I will introduce a table to show points for each exercise's correct answer.

Order of exercises	I	II	III	IV	V a)	V b)
Points for one task	2	4	3	6	4	5
Total points for exercise	18	16	15	30	16	5

Table 4.4. Test's grading points

I need grading points of the test because the teacher wanted the pupils' results about this test, but I did not use it for selecting pupils for interviews.

The first interview is about written test and it involves only four pupils. I conducted it right after the test and before my first teaching activity. On 25/10/11 my participants had mathematics class, in accordance to the timetable, in the fifth school hour. So, I conducted the four interviews during the second and third school hour. The interview for each pupil lasted twenty minutes. I required pupils to leave other teaching lessons (during the second and third school hour) because I did not have other possibility since I had only five days available to conduct my data collection. On the other part I did not have objections by other teachers.
I conduct interviews in a small meeting-room, which has three glass walls. It has its advantage and disadvantage since the interview was not private and, others that passed near these walls drew attention to my interviewer. During the interview in this room there were only me and one pupil. Firstly I made camera and everything else ready for interviews in the meeting-room and pupils came one after another because they knew the timetable of interviews, which I compiled before with the teacher. The camera has been fixed on the tripod and it was focused on the sheet of the test paper and not on the face of pupil. I was introduced before and also I knew pupils from classroom observation, so we got started immediately with discussions

about tasks. I asked the pupil about reasons, explanations and justifications of his/her answers given in test. And depending on his/her answers I developed further the interview. There was a standard question for the four pupils, about what they thought about the test and its level of difficulty. This was because I wanted to know if the test has been affordable for them.

4.6.3 Teaching lessons

My own teaching is composed by three teaching activities and it got started on 25/10/11. I developed my teaching activities in the same class that I observed and I fixed camera on the tripod most of the time because I had to organized and to lead the activities and teaching. And, I involved in these activities the thirty pupils of the class. I developed the following three teaching activities: “Evaluating mathematical statements”, “Interpreting multiple representations”, and “Creating problems”.

The first lesson that contains “Evaluating mathematical statements” was conducted on 26/10/11. I distributed a sheet of paper with algebraic statements, which pupils had to decide whether the statements are always, sometimes or never true. Also in this sheet of paper there is a table where pupils should put the number of the statement in the column that express statement’s validity. During this activity pupils worked in pairs for twenty minutes and then all the pupils had a discussion in which I also participated. I leaded the discussion emphasizing evaluating of several statements, for which different pairs gave different evaluation. Throughout this process, I encouraged pupils to think more deeply, to try further examples and to provide more convincing reasoning and justifications through examples or counterexamples.

During the next mathematics lesson, on 27/10/11, the activity “Interpreting multiple representations” was developed for forty-five minutes, and pupils worked in pairs. I distributed for each pair four sets of cards with representations, by: words, symbols, tables, and area diagrams. Also, each pair had a poster where they have to paste cards thus attempted to group cards which have an equivalent meaning. My role in this activity was to explain firstly what they have to do and the aim of this activity, suggesting not to rush through the tasks but to take time and evaluate each mathematic statement in those cards. I asked to explain their matching and reasoning, and to discuss with his/her mate about agreements and disagreements. I often challenged them discussing about cards that were not the same but for somebody it looks like, as for example, $3x^2$ and $(3x)^2$.

The third activity “Creating problems” was developed by all pupils in the same time. I wrote on the blackboard what pupils suggested to me about the steps of creating equations. I picked up pupils to answer, sometimes pupils that were quite or those who answered me leading by their uncertainty. Pupils did not communicate only with me but also with each other consulting and justifying for their answers. Regardless of whether responses were correct or not I insisted on asking pupils for arguments and justifications.

4.6.4 Task-based-interview

Task-based-interviews were conducted on 28/10/11 for the four pupils, and each interview lasted for around twenty minutes. Except the tasks, everything else was similar with the first interviews. For more details see section 4.7.2.2.

My data analysis is introduced in the three next chapters, such as: Analysis of tests, in which I am focused on analyzing pupils’ tests; Portrayal of Alba, in which I have analyzed even her performance during interviews and my teaching lessons in order to introduce her development during my data collection; and Portrayal of Dea (the same organization as the previous one).

5 Analysis of Tests

In this chapter I am focused on tests' analysis investigating for answers' correctness. This is because tests' answers are written answers without justifications and argumentations. I will give a general overview of mistakes that pupils did in their test performance concerning brackets' tackling. From tests' analysis I expect to get only some types of pupils' mistakes concerning the use of brackets and I cannot call them as misunderstandings or misconceptions that pupils have with brackets since these data are only written answers of several "ordinary" tasks.

From tests' analysis, I attempt to have an answer for my first research question, which is: *What kind of mistakes pupils in grade 10 (in the Albanian school) do?*
The answer of this question will be an overview of what do pupils involved in this study, related to the process of bracketing and to brackets itself. To achieve an overview of pupils' misconceptions and mistakes I have been interested on wrong answers and their density, and tasks in which these mistakes are occurred. This is because for each task I had my expectations related to my theoretical approach.

Even based on theoretical approach and my expectations of designing tests, I have generated three questions that would help me to answer the first research question, categorizing the analysis of tests.

- How are tackled mathematical objects by the pupils?
- Which is more present: *instrumental* or *relational* understanding?
- Are there any *Slips* or *Bugs*?

This is only to organize my analysis of tests and to lead my tests' analysis, because I cannot achieve definitive answers about them since written tests do not give me more access than this. It is necessary to emphasize that these answers will remain open.

This chapter contains two sections, such as: Types of Mistakes and Participants' Overview. In the first section I have involved charts, concerning correctness of participants, and types of mistakes for each exercise included in the test. While in the second section, I have involved some conclusions considering pupils' tackling of brackets and trying to have answers for three helping questions with purpose to achieve an overview of participants' written performance.

5.1 Types of mistakes

In this part, I am focused more on quantitative analysis introducing elements such as, charts concerning answers' correctness, and tables concerning density of different types of pupils' mistakes for each exercise.

5.1.1 Exercise 1

As it is described in Test section of Methodology chapter, the first exercise of test contains nine tasks and each of them has multiple correct answers. This property has been new for participants and it is reflected in their answers since no one has given all possible correct answers. On the other hand, since these assignments require basic knowledge to be solved and are situations previously countered by pupils, twenty-five (21+4) pupils have given at least one correct answer for each task and only four of them have solved all nine tasks and have given only one correct answer for each task. I think that these four pupils have not read carefully the demand of this exercise, which is: "Choose the right alternative (there are more than one for each task)". Mistakes are countered in only four pupils' answers, and one pupil has selected no alternative for one task even though he/she has written it near the respective task. All these results are presented in Chart 5.1.

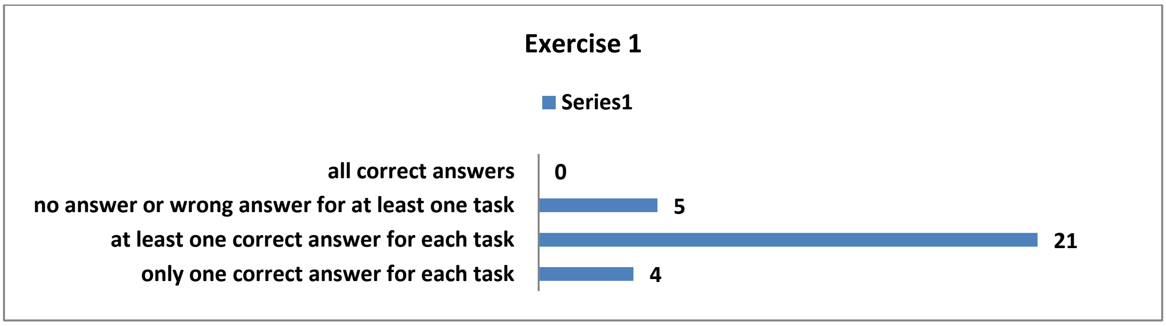


Chart 5.1. Results for Exercise 1

Since I am interesting on pupils' mistakes I will present types of mistakes that participants of my study did in the first exercise of test concerning tackling of brackets. In the following tables, I use "density" to indicate the number of pupils that have given the particular wrong answer.

CORDINATES OF TASK	TASK	ANSWERS	DENSITY
E. 1-T. 3	$(2x-y)+y=$	$2xy-y^2$	2
E. 1-T. 4	$(x+4)(x-5)=$	x^2-20	1
E. 1-T. 6	$x^2y-xy^2=$	$x^2(y-y^2)$	3

Table 5.1. Wrong answers for the first exercise

All these wrong answers are found in four pupils' performances. There are two pupils that have considered addition of one bracket with one monomial as multiplication. This is a wrong meaning attached to brackets that is mentioned even by Kaur (1990, p. 36), who classifies as misconception that "brackets indicate multiplication".

Concerning the fourth task, there is one pupil that has multiplied two brackets in a wrong way, since he/she has multiplied the two first terms of each bracket with each-other and the same with two second terms of brackets. I can consider this as an evidence of brackets' incorrect-tackles. This is because he/she has not considered the two polynomials 'x+4' and 'x+5' as mathematical objects and each monomial (element) of them should be multiplied with the rest.

There are three pupils that have given wrong answers for the sixth task showing that they have not found the common factor of two monomials.

5.1.2 Exercise 2

I included the second exercise in test with purpose to require from pupils an active use of brackets. Concerning the results from pupils' answers I can say that, this type of exercise has been as an obstacle for them. Even though this exercise requires for solving basic knowledge such as distributive law, it has been the new presentations of tackling brackets that I think it has affected pupils' answers. So, there are sixteen pupils that have not solved or correctly solved any task of the second exercise. Among these pupils, there are seven pupils that have made no notes in the four tasks of the second exercise; there are six pupils that have given no correct answer to any of tasks (these are pupils that have incorrect answer for at least one and no answer for the rest of tasks); and there are three pupils that have not understood the demand of the exercise since they have made some notes that are not related to the demand, such as to tackle algebraic statements as equations and to look for their solutions. Less than the half of pupils have worked correctly with these tasks but only one has solved this exercise correctly and completely. There are thirteen pupils that have one, two or three correct answers among their answers, and the rest of these pupils' answers might be wrong or there is no answer. All these data are presented in Chart 5.2.

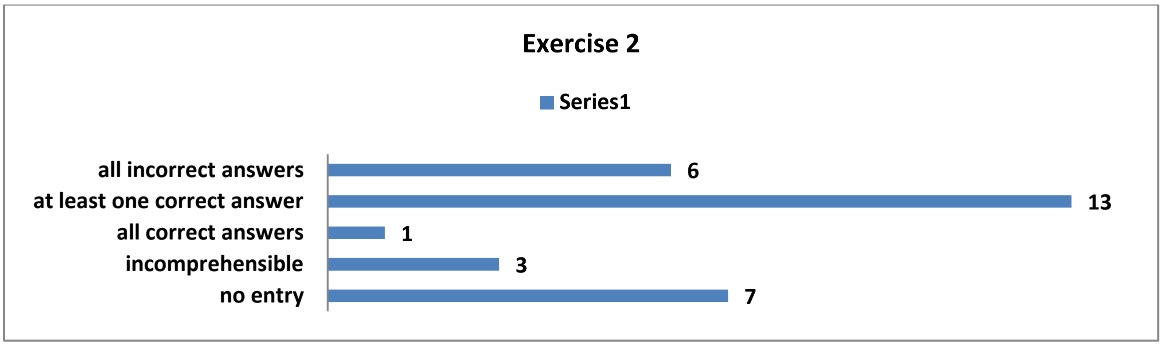


Chart 5.2. Results for Exercise 2

Among answers of nineteen (6+13) pupils, concerning the second exercise, there are eight types of mistakes introduced by nine pupils. These are organized in Table 5.2.

CORDINATES OF TASK	TASK	ANSWERS	DENSITY
E. 2-T. a)	$x+x^2+3\cdot x+4= x^2+4x+4$	$[x+x^2+(3\cdot x)+4]= x^2+4x+4$	1
		$(x+x^2+3\cdot x)+4= x^2+4x+4$	1
		$x+x^2+(3\cdot x+4)= x^2+4x+4$	1
		$x+(x^2+3\cdot x+4)= x^2+4x+4$	1
		$x+x^2+3\cdot(x+4)= x^2+4x+4$	2
E. 2-T. b)	$x+x^2+3\cdot x+4= x^2+4x+12$	$(x+x^2)+3\cdot(x+4)= x^2+4x+12$	1
		$(x+x^2)+(3\cdot x+4)= x^2+4x+12$	1
		$(x+x^2+3\cdot x)+4= x^2+4x+12$	1

Table 5.2. Wrong answers for the second exercise

The first, the second, the third, and the fourth placement of brackets are evidences of a non-well tackling of brackets, even though algebraic expressions in two sides of algebraic statements are equivalent. In the first answer, introduced in Table 5.2, the square brackets are not necessary because they are not preceded or followed by another algebraic operation. Also, in the second, third and fourth answers, brackets are not necessary since they are followed or preceded by addition.

In these four cases, I think that four pupils have not been able to *identify* and to *discriminate* (Sierpinska, 1994) these cases (in which are used brackets) and their analogues, in which are not used brackets. From the fifth answer, which is countered in two pupils' answers, I can say that these pupils have forgotten to multiply the '3' with the '4', showing a non-tackling of algebraic expressions as *objects* (products) (Gray and Tall, 1991, 1992).

From the task b) I could show three answers, in which brackets are not necessary and not in the right place. In the first case, the first brackets are not necessary and the second ones are correctly placed. While in the second case, algebraic expressions in two sides of the algebraic statement are not equivalent and the two pairs of brackets are not necessary. It is the same explanation also for the third case. I think that these pupils could not achieve a *generalization* for using brackets in addition and multiplication since they cannot *discriminate* (Sierpinska, 1994) them.

5.1.3 Exercise 3

Findings from the third exercise have been entirely different from the previous exercise. There is no pupil that has done the all tasks (5 tasks) wrongly or has not tried to solve them. Ten pupils have solved this exercise correctly giving right answers for each task, and twenty pupils have solved one, two, three or four tasks and have given wrong answer or no answer for the rest of tasks.

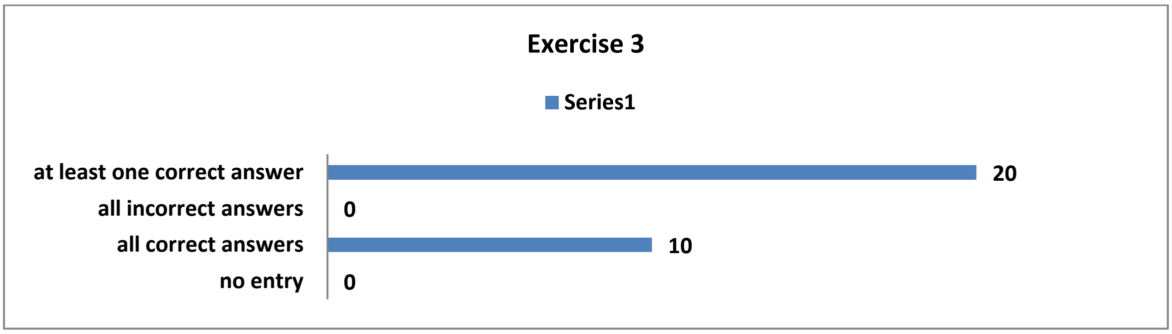


Chart 5.3. Results for Exercise 3

The most of mistakes done in this exercise have been related to calculations and operations with signed numbers, for example: ‘-2x-5x = -8x’. But, related to the use of brackets I have isolated two following mistakes, which could have been avoided by pupils if they would have a well-understanding of brackets placed there. This is because these two pupils could notice their own mistakes by substituting question marks with values they found and by operating with brackets. These mistakes are introduced in Table 5.3.

I think that the mistake in task c) comes from non-considering the polynomial ‘x+3x’as an *object* because he/she has obtained ‘9x’ only by multiplying the ‘3’ with ‘3x’ and has not considered the monomial obtained by multiplications of the ‘2x’ with ‘x’, and others following. Otherwise this pupil could notice that he/she is not obtaining the same monomials in two sides of the equal sign. It seems that in the next mistake, task e), the pupil has a well tackled concerning brackets and operating with brackets but the error here is that he/she has considered the question mark as variable. This is because this pupil has looked for only one value. It is interesting how this pupil have accepted the value ‘3’ while he/she has evaluated it in algebraic statement.

TASK	DENSITY
c) $(2x+3)(x+?)= 2x^2+9x+?$ 3x 3	1
e) $(x+?)(x+?)= x^2+6x+8$ 2 4 $(x+?)^2 = x^2+6x+8$ $x^2+6x+8 = x^2+2\cdot 3x+8$ $=(x+3)^2$?=3 $x(x+?)+?(x+?) = x^2+x?+x?+??$ $= x^2+2x?+?^2$ $2x? = 6x \Rightarrow \frac{6x}{2x} = \frac{2x?}{2x} \Rightarrow 3 = ?$	1

Table 5.3. Wrong answers for the third exercise

5.1.4 Exercise 4

Findings for the fourth exercise are an evidence of rote learning and passive use of brackets. This is because all pupils have worked with tasks of this exercise, which consist in expanding brackets and making calculations. Twenty-five pupils have solved one, two, three or four tasks and have given wrong answer or no answer for the rest of tasks; only one pupil has given correct answers for all tasks; and four pupils have achieved wrong results for all tasks.

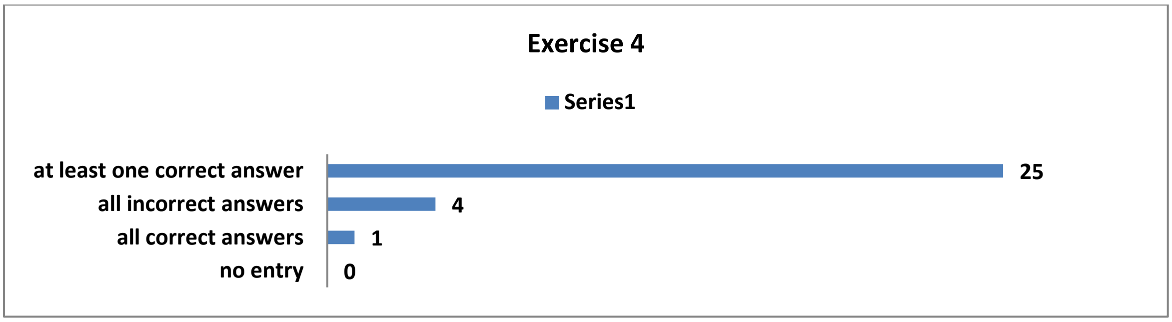


Chart 5.4. Results for Exercise 4

Even though this type of exercise was not new for participants, there are several mistakes in their solutions for each task, which are introduced in Table 5.4.

CORDINATES OF TASK	TASK	ANSWERS	DENSITY
E. 4-T. a	... $(6x+3x)^2$=	... $36x^2+9x^2$	1
		... $6x^2+2\bullet 6x\bullet 3x+3x^2$= ... $6x^2+36x^2+3x^2$	3
E. 4-T. b	$(-4x+3)(-1+2x)$ + $(-4x+3)$-$(-1+2x)$=	$4x-6x-3+6x-4x+3+1-2x$	1
		$4x-8x^2-3+6x+(-4x)(1-2x)+3(1-2x)$	1
E. 4-T. c	... $-3x(2x+1)(2x-1)$=	$-3x[(2x)^2-(1)^2]= -3x\bullet 4x^2-1= -12x^3-1$	2
E. 4-T. d	$(\sqrt{2x}-2\sqrt{y})^2$ ...=	$(\sqrt{2x})^2-(2\sqrt{y})^2$	4
		$(\sqrt{2x})^2-2(\sqrt{2x})(2\sqrt{y})+(2\sqrt{y})^2$= $2x-2\sqrt{2x}\cdot 2\sqrt{y}+4y^2$	1
		$2x^2-2\sqrt{2x}\cdot 2\sqrt{y}+4y$	2
		$2x-2\sqrt{2x}\cdot 2\sqrt{y}+2y$	2
E. 4-T. e	... $[5y-(3x-6y):3]$=	$5y-3x+6y:3=5y-1x+2y$	1
		$5y-3x+6y:3=11y-3x:3$	1

Table 5.4. Wrong answers for the fourth exercise

I think that all these mistakes concerning tackling of brackets are related to non-considering algebraic expressions as objects.

For task a) there are two types of mistakes (related to bracketing) that both are based on non-considering of algebraic expressions as *objects* (products) (Gray and Tall, 1991, 1992). In the first case, it seems that, the pupil did not know the formulae ($(a+b)^2 = a^2+2ab+b^2$) and he/she has operated as distributive law works. On the other hand, he/she has made correctly the square of ‘6x’ and ‘3x’ tackling them as objects. Three pupils have tried to use the formulae with purpose to expand brackets but they have not considered monomials as objects, and this is showed in the second case. In addition, it is interesting that no one has added firstly ‘6x’ with ‘3x’ since it is possible, and then to square the whole ‘9x’. And I would like to relate it with rote learning and rote applying of their knowledge without assessing the situation in advance.

Concerning the second task (b), one pupil has added monomials instead of multiplying them during the brackets’ multiplication. There are four multiplications that should be executed during multiplication of two brackets and concerning this case, the ‘-6x’ is obtained by adding ‘4x’ and ‘2x’, and by attaching the minus sign of ‘-4x’, while the three other monomials are obtained by multiplication. I can mention here Ayres (2000), who concludes that pupils make more errors during the second operation while they are multiplying one monomial with one bracket. The next mistake concerns in replacing subtraction of two brackets with multiplication. So, not only addition indicates multiplication (Kaur, 1990) but also subtraction indicates multiplication. In addition, in this case, the good part is that multiplication of two brackets is executed well and it is followed by considering algebraic expressions as objects.

In two pupils’ answers of the third task are found mistakes during operating with brackets concerning algebraic expression in the third task. Even though pupils have used firstly two types of brackets even in correct way, they have “put all brackets off” in the same time followed by non-considering polynomials as objects. I think that this type of mistake could be classified as *slip* since in the first step of solution everything is correct. But, in the next step he/she has forgotten to put brackets and this pupil is leaded to wrong result.

Even in task a) and in task d), it is repeated the same type of mistake since there are four pupils that agreed with $(a-b)^2 = a^2-b^2$. In the second, the third and the fourth cases of mistakes in task d), the non-well tackling of monomials as objects is introduced by five pupils.

It is interesting the way of task e) solution’s writing of one pupil without using brackets while he/she has calculated considering brackets. I cannot classify this as a mistake with careless nature. This is because, in front of that I can put Kieran’s conclusion (Kieran, 1979) concerning pupils that even though do not put brackets to refer the first operation that should be done, they execute that first. The second type of mistake concerning the last task e), is related firstly to the order of operations and then to non-tackling of ‘3x-6y’ as a product (object) that firstly should be divided by ‘3’.

5.1.5 Exercise 5

Also findings from the fifth exercise show a great number of pupils, twenty-nine (24+1+4) that have worked with this exercise. Twenty-four pupils have solved one, two, three or four tasks and have given wrong answer or no answer for the rest of tasks; four pupils have solved correctly all tasks; and only one pupil has mistaken in his/her solutions.

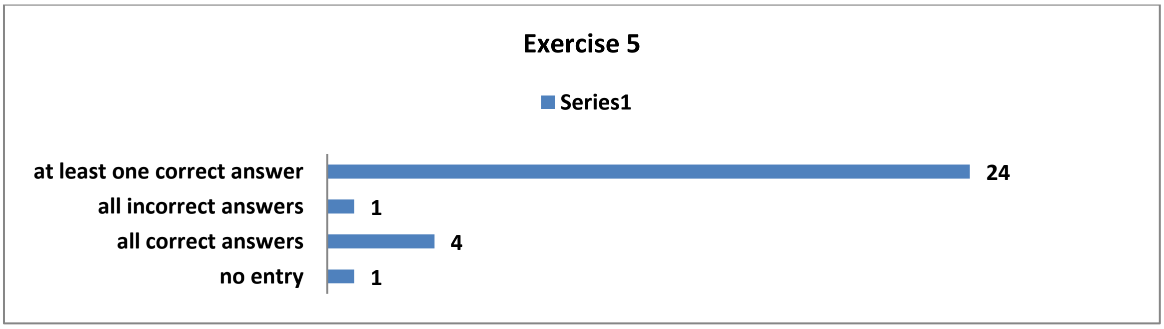


Chart 5.5. Results for Exercise 5

Among twenty-eight pupils' answers I have isolated some types of mistakes concerning using of brackets and tackling of mathematical objects. These are introduced in Table 5.5:

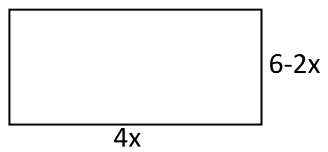
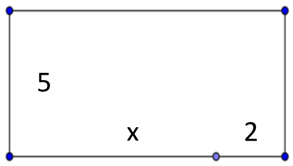
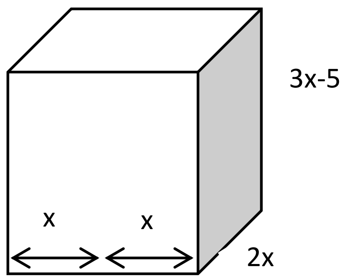
CORDINATES OF TASK	TASK	ANSWERS	DENSITY
E. 5- T. a). 1		$A=4x\bullet 6-2x$	1
E. 5- T. a). 2		$A=2x\bullet 5$	2
E. 5- T. b)	 V=	$V=(x\bullet x)\bullet 2x\bullet(3x-5)$	1
		$V= x^2[2x\bullet(3x-5)]$	2
		$V=(3x-5)(2x)(x+x)$ {with brackets} $V=3x-5\bullet 2x\bullet 2x = 3x-5\bullet 4x^2$ {without brackets}	2
		$V= (2x\bullet 2x)\bullet 3x-5= 4x^2\bullet 3x-5$ $=12x^3-20x^2$ {with brackets} $V= 2x\bullet 2x\bullet 3x-5$ {without brackets}	1

Table 5.5. Wrong answers for the fifth exercise

The first and the second mistake, respectively in task 1 and 2, which are countered in three pupils' answers, and are related to non-considering algebraic expressions as *products* (Gray and Tall, 1991, 1992) that is analogue with considering sides of rectangle as mathematical objects. Similar evidences are found in Kaur's study (Kaur, 1990), in which pupils did not use brackets for the sides of rectangle.

The first two cases of task's b) mistakes involve mistakes such as, incorrect determination of the cuboid's side since these three pupils have written 'x^2' or 'x·x' instead of '2x'; and the unnecessary-use of brackets because they have involved factors such as 'x·x' and '2x·(3x-5)' into brackets. And, this way of using brackets is showed even in two next cases.

The second demand in this task has been to write the volume of cuboid without brackets, which means to expand brackets and achieve an algebraic expression with no brackets. And there are three pupils that have written the volume of cuboid as multiplication of three sides of cuboid but without using brackets. I think these pupils have considered the two algebraic expressions for the cuboid's volume, with and without brackets, as equivalent. And similar considerations take place also in Booth's study (Booth, 1984).

In the fourth case it is written the volume of cuboid in a wrong way putting brackets not in the right place but he/she has gotten a correct result achieving an algebraic expression without brackets, which express the volume of cuboid. Once more, I have evidences of Kieran's (1979) results concerning the non-use of brackets and operating as if brackets are present.

5.2 Participants' overview

There are very few pupils that have not tried to work with any of the exercises involved in the test. And there is no participant that have not worked with all exercises getting zero points from the total of one-hundred points, but the lowest total amount of points taken by one of them is forty-five points.

As it is showed in previous section 'Types of Mistakes' there are several types of mistakes in pupils' answers for each exercise, as might be expected by me since these types of mistakes concerning the use of brackets are emphasized by other authors (included in 'Review of Literature') and are encountered by me during my experience as a teacher. In addition, concerning the analysis of tests with written answers I can state that participants of my study make some mistakes related to tackling of brackets, such as: non-use of brackets, bad-use of brackets, and wrong-meaning of brackets (that means, wrong meaning attached to brackets). In the first and second class of mistakes, non-use and bad-use of brackets, I will classify mostly of mistakes done in exercises 2, 4, and 5, in which is presented also the active use of brackets. While mostly of mistakes done in exercise 1 and 3, I will classify in the third class of mistakes, wrong-meaning of brackets, since in these exercises is present only the passive use of brackets. But I cannot exclude the possibility of evidences of different types of tackling brackets in particular tasks, such as in tasks of the fourth exercise. I think that wrong results in this exercises come from a wrong-meaning of brackets and no-use of brackets concerning evidences in Table 5.4 in previous section.

Regardless of exercise's type, algebraic or geometric, and in this latest diagrams are equipped, and regardless of exercises are known in advance or not, I have countered these types of mistakes:

- Brackets indicate multiplication
- Wrong-apply of distributive law
- Use of brackets where it is not necessary
- Use the distributive law's principle in situations where it is not involved
- Non-use of brackets for mathematical objects, such as:
    - Algebraic expressions
    - Sides of diagrams
- Non-use of brackets and in the same time operating as brackets are present
- Consider algebraic expressions with and without brackets as equivalent
- Factorization
- Order of operations.

Concerning these types of mistakes I could say that these participants (who have done these mistakes) have a wrong-tackle for algebraic expressions. They do not consider algebraic expressions also as *objects* (Gray and Tall, 1991, 1992), because otherwise would be avoided mostly of mistakes. Considering the number of pupils who have given all correct answers or at least one correct answer for each task, I could say that mostly of my study's participants tackle algebraic expressions as mathematical objects and they involve them in brackets with purpose to get their product firstly or to discern this product from others. On the other hand, algebraic expressions involved into brackets mean for them that, they should manipulate with that algebraic expressions as a whole object or product, which is obtained from the process presented by algebraic expressions. Regardless of the way of presenting mathematical objects, such as algebraic expressions or sides of diagrams, mostly of participants have tackled them as *objects* (Gray and Tall, 1991, 1992).

Since my tests' analysis is based on only written answers there are limitations concerning the absence of justifications, argumentations, and reasoning of pupils, I cannot have an answer for the second helping question. I have compiled a table to introduce how and how much *relational* and *instrumental understanding* (Skemp, 1976) is present in participants' mathematical understanding and learning. In order to have some "conclusions" for this issue, I referred to the exercises, which were not seen before by them and require an active use of brackets, since I have not concrete and convincing evidences.

Relational	Instrumental & Relational	Instrumental
It seems that they know what particular knowledge and why to use it, even in new tasks, justifying and explaining each step; synthesizing the adequate algorithm.	It seems that they have some knowledge and they use it also in new tasks getting the correct answers, but it is a superficial usage. So, relational understanding is not widely present.	It seems that they have some basic knowledge (maybe the wrongly one) and they use it in the simplest way.
3 pupils (10%)	19 pupils (63,3%)	8 pupils (26,7%)

Table 5.6. Participants' mathematical understanding

In terms of the third helping questions, I have noticed from pupils' answers some slips concerning operating with signed numbers ($-2x-5x=-8x$), with exponents ($2\cdot 6x\cdot 3x=36x^2$), with numerical data ($4\cdot 8=24$). In addition, I have classified some mistakes concerning brackets' use as slips and I have introduced them in previous section. On the other hand, I have not classified any mistake as bugs since I have no access to do that while I am analyzing written answers.

In this chapter I introduced the performance of participants (thirty pupils) concerning mistakes related to tackling the algebraic expressions and the use of brackets. As it is showed above different types of mistakes are not missing and some of them have similarities with what are introduced by previous researches. So, in this chapter I have compiled an overview concerning mistakes that Albanian pupils in tenth grade involved in my study did. Following in the next two chapters, I will introduce the portrayal of two pupils selected among four interviewed pupils in order to analyze in depth their performances. It was difficult to analyze and introduce in this dissertation the analysis of four completely performances (eight interviews) and this is a master thesis with sixty ECT, so I have selected the performances of two girls for deeper analysis. Concerning my second research question, in which is determined that there are improvements in pupils' performances during the post-test, I excluded one pupil's performances because his/her abilities in mathematics were in a low level and he/she could not have evidences for improvement during the post-test. I excluded another pupil because he/she was correct in his/her answers but his/her justifications were always the same. So, I have chosen Alba and Dea (pseudonyms) to analyze their performances since I noticed improvements in their answers during the post-test.

6 Portrayal of Alba

I have named this pupil Alba; she was a very interesting girl and she voluntarily agreed to be part of my study. I am so glad for working with her, and I appreciate her cooperation because she surprised me with her answers and, from her work I have interesting data to analyze.
I will describe all the activities she took part and her respective preparation, trying to highlight her difficulties, and improvements related to her conception for the process of bracketing and other conceptions closed to brackets. The portrayal of this pupil, Alba, will be divided in three sections, according to her preparation in: pre-test, teaching activities, and post-test.

The first section 'Results from the pre-test' is organized in different subsections in accordance to some criteria, such as: misconception that brackets indicate multiplication, superficial relational understanding, slips that are corrected and justified by her using instrumental understanding and followed by uncertainty, no-active use of brackets and relational understanding, instrumental proofs, difficulty to tackle algebraic expressions as mathematical objects and herself correction, no generalization concerning use of brackets related to mathematical objects.

These criteria are generated from keywords (misconception, understanding, mathematical objects, generalization, and syntheses, semantic, syntactic) used in my theoretical approach and in analysis of pupil's answers during my data collection. For each criterion I have chosen exercises or tasks, of which pupil's answers are relevant for that criterion. Also, I have emphasized for each task introduced how interview helps me to understand test's answers.

I agreed with this organization, which was discussed by me and my supervisor, because it is a way that makes explicit the link between my theoretical framework and data analysis. It is needed to emphasize that for each criterion I have chosen tasks in which dominates it but in the same time others criteria' properties might be involved. This is because elements in my theoretical approach are related to each other. On one side, relational understanding, tackling of algebraic expressions as product, discrimination, generalization and synthesis, semantic aspect, and correct-active use of brackets are related to each-other; and on the other side, instrumental understanding, tackling of algebraic expressions as one process, operation of identification, syntactic aspect, passive use of brackets are also related to each-other. It is needed to emphasize that in this section, especially in figures with Alba's test's answers *all writing in red is my own writing.*

While the second section, 'Her Activity during Teaching Activities' involves Alba's activities during my teaching lessons and any dialogue between me and her, concerning difficulties or misunderstanding during teaching activities.

Also the third section, 'Results from the post-test', is organized as the first one is. In this section are involved some criteria, and Alba's answers in dialogue with me from the task-based-interview that correspond to those criteria.

6.1 Results from the pre-test

This section is organized considering some criteria to classify Alba's answers to some tasks during the test and the interview about this test. In this section, the following criteria are involved: superficial relational understanding, misconception that brackets indicate multiplication, slips that are justified by instrumental understanding and followed by uncertainty, no-active use of brackets, difficulty to tackle algebraic expressions as mathematical objects and herself correction, instrumental proofs, no generalization concerning use of brackets related to mathematical objects (algebraic and arithmetic expressions). All these criteria emerged as a combination of both, reading research literature and analyzing data, and they could be considered as properties of Alba's preparations. More details for these criteria are given below.

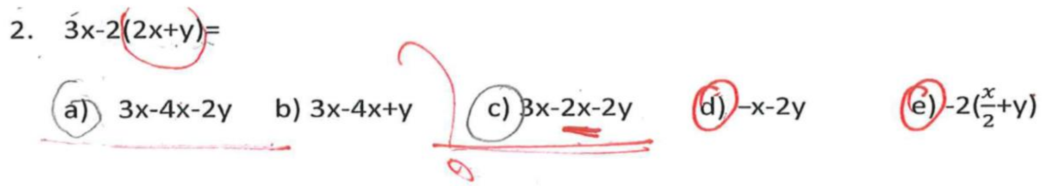
6.1.1 Alba's "superficial relational understanding"

This criterion is composed by one element taken from my theoretical approach, such as *relational understanding* (Skemp, 1976), and by another element taken by data analysis, such as superficially arguing and justifying. This criterion consists in justifying and arguing the task's results but Alba's understanding is superficial since she might be strongly based on applying distributive law and, I and Alba were discussing about situations that she has countered before. This criterion includes Alba's answers for the second task of the first exercise.

6.1.1.1 Exercise 1, Task 2

Alba's answer from the test

Since she has selected as a correct answer alternative c), which does not involve the multiplication of the first term of brackets with the factor before the brackets, and even the correct alternative a), I cannot have a prove for my doubt that is, if she knows how to use distributive law, and furthermore if she knows to tackle algebraic expressions as objects.



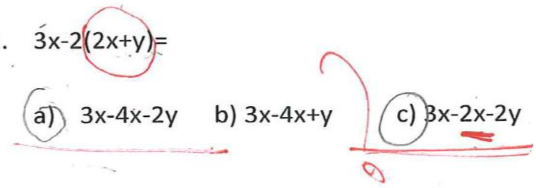
This is followed by non-marking alternatives d) and e) that are correct answers. My understanding is that, after she has marked a) and c) she thought that it is enough and she did not evaluate d) and e). In this task she did not achieve to decide about the equivalence of algebraic expressions that have the same product even though they introduce different processes. At this point, I can say that Alba has considered only the *syntactic aspect* (Berg, 2009; Demby, 1997; Drijvers et al., 2011) of algebraic expressions introduced as alternatives for this task. With *syntactic aspect* of algebraic expressions I am referring to transforming them using several rules. It is explained in detail in section 3.2.

According to these answers, I cannot even mention *relational* learning or *instrumental learning* (Skemp, 1976) of commutative property because I cannot see a correct way of *how* this property could be used. On the other hand, it seems like a *slip* (error of careless nature) because she has marked correct answers in the first task and one correct answer in the second task.

Alba's answer from the interview

Only from the written test I cannot decide if she knows how to do a multiplication of a number with an algebraic expression involved into brackets. I do not notice if she considers '2x+y' as a *mathematical object* and that whole one should be multiplied by '-2'.

During the interview I tried to clarify this issue, and I will present now this part from our dialogue:

Dialogue 1		
Name	Dialogue	Notes
D	At the second task. Tell me what you did.	Exercise 1/ Task 2 
A	I let ‘3x’ as it is, and I multiply ‘-2’ with ‘2x’ and ‘+y’.	
D	You have selected alternative a) and c), why have you?	
A	I let ‘3x’ as it is, and multiply the factor ‘-2’ with two other factors inside brackets, ‘2x’ and ‘+y’, and I got ‘3x-4x-2y’.	
D	Why did you choose c)?	
A	No, it is not correct.	
D	Why?	
A	Because ‘-2’ times ‘+2x’ is equal to ‘-4x’.	
D	What do you consider the whole group of terms ‘2x+y’ related to the multiplication with ‘-2’.	
A	As just *a number*.	
D	We can say as a *mathematical object*. Ok?	
A	Yes.	

From this dialogue I understand that Alba tends to tackle the algebraic expression included into brackets, ‘2x+y’, as a factor, which has two terms and ‘-2’ should multiply both of them; and with this reason she justifies her error done in test. It seems that she does not know to use the adequate words for that factor, which is a *mathematical object* (Gray and Tall, 1991, 1992).

She *identifies* the multiplication of a term with a bracket and *discriminates* (Siepinska. 1994) it from the wrong way of multiplying a term with a bracket. I think that this dialogue is an evidence of *“a superficial relational understanding”* (Skemp, 1976); it means that Alba has the idea why she is making two multiplications one after another, even though she is strongly based on distributive law. This is the reason why I classified these answers as “superficial relational understanding”.

This dialogue helps me to give answer questions that emerged from her written answer and proves my doubts.

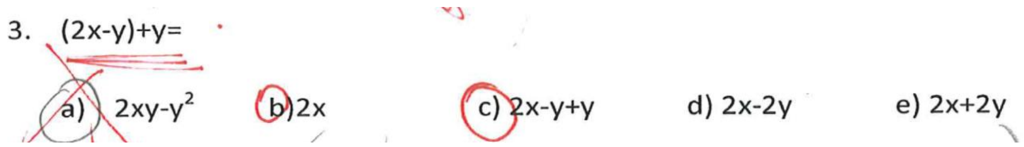
6.1.2 Alba’s “misconception that brackets indicate multiplication”

This criterion comes from analysis of Alba’s answers and this type of misconception is mentioned also in chapter 2 (Review of Literature). It consists in operating with algebraic expressions as they were two factors (at least one of them is involved into brackets) that should be multiplied regardless the operation between them. I have involved in this criterion Alba’s answers of two tasks, task 3 and task 5 of first exercise.

6.1.2.1 Exercise 1, Task 3

Alba's answer from the test

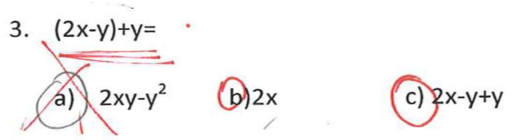
According to Kaur (1990) there is a misconception that brackets means multiplication, but not always brackets should be tackled as multiplication, as Alba did in this example (task 3):



If the operation between bracket and 'y' would be multiplication, then alternative a) will be a correct choice by Alba showing also that algebraic expression involved into brackets should consider as one object (Gray and Tall, 1991, 1992). Since she has selected only the alternative a) it seems that Alba has considered addition operation as multiplication but from the interview I will show if this misconception stands for Alba or not.

Alba's answer from the interview

Alba has shown in her test the tendency to make multiplication where brackets are involved even though there is the addition operation between the single term and brackets. Also, during the explanation of her solution for the third task in the first exercise, while she was being interviewed, Alba was convinced that she has to make multiplication. If there would be multiplication and not addition, the result will be correct because she did it very well; and showing that this time she noticed correctly the two factors. Only when I required showing me the operation she noticed that it was addition and, not multiplication. Furthermore Alba was precise during the process of bracketing for this task.

Dialogue 2		
Name	Dialogue	Notes
D	At the third task you have selected a) $2xy\text{-}y^2$.	Exercise 1/ Task 3 
A	Yes, I have.	
D	What do you get from: (2x-y)+y?	
A	I have to multiply 'y' with '2x' and '-y'. So '2x-y' *is multiplied* with a factor that is 'y'.	
D	You have multiplied it, what do you see there?	
A	There is an expression in brackets and *a factor*.	
D	What is the operation between the expression in brackets and 'y'?	
A	Addition.	
D	Ok, what should you do?	
A	Just to put away brackets. And the result is '2x'.	

As I understand the dialogue, Alba does not identify the addend included in brackets because her misconception is that *brackets indicate multiplication* but in the same dialogue she shows that she knows both the meaning and the use of distributive law in algebra. On one hand the statement: " '2x+y' *is multiplied* with a factor that is 'y' " is evidence of her misconception,

on the other hand it is evidence for identification of distributive law (even though she never names it) because it means to multiply a term (monomial) with a sum (polynomial).
This dialogue convinces me concerning my doubt for Alba's misconception that brackets indicate multiplication.

6.1.2.2 Exercise 1, Task 5

Alba's answer from the test

In the fifth task Alba shows a correct tackling of mathematical objects, as it is 'xy'.

5. $x^2y-(xy)^2=$

a) $2x^2y$ b) $x^2y-x^2y^2$ c) 0 d) x^2y-xy^2 e) $x^2y(1-y)$

My understanding, for this part, is that *relational understanding* (Skemp, 1976) has no evidence in her work because I cannot see any answer for *why* she has used particular rules (as distribution of an exponent over a multiply). Since she has selected only one correct answer instead of two, I think that she was focused only on syntactic aspect (Berg, 2009; Demby, 1997; Drijvers et al., 2011) and she has not considered the possibility of containing the same factor by two monomials, such as 'x^2y' and 'x^2y^2'. But, does Alba know what she is using and, *how* and *why*?

Alba's answer from the interview

In the fifth task she has selected during the test one of the correct alternatives, and I asked also about correct solutions to show her mathematical understanding. During the interview she did not justify her choice because it seems that Alba memorized the rule for multiplication and she wants to use it everywhere, and obviously because her misconception is that brackets indicate multiplication.

Dialogue 3		
Name	Dialogue	Notes
D	How did you operate in the fifth task?	Exercise 1/ Task 5 5. $x^2y-(xy)^2=$ a) $2x^2y$ b) $x^2y-x^2y^2$
A	Here, I have multiplied eee, 'y' with the whole expression 'xy squared'.	
D	Which is the first operation that you should do? Is it multiplication?	
A	No, I should expand brackets.	
D	Ok, then what?	
A	(She writes: = x^2y-xy^2.)	
D	What is squared?	
A	Even 'x' and 'y'.	
D	But what did you write?	
A	Ah, 'x^2y^2'.	
D	Ok. How do you notice that both 'x' and 'y' should be squared?	
A	Because *there are into brackets* and the entire bracket is squared.	
D	How do you consider 'xy'?	
A	As an only number.	
D	Ok, let say as a mathematical object.	

I classified this dialogue as part of criterion that brackets indicate multiplication since regardless her written answer she started to multiply because there were brackets. I made this conversation with purpose to know more about how she has achieved that answer, alternative b), but firstly I got one more time another evidence of her misconception. After that, we clarified if she should multiply or subtract, she gave me some answers concerning tackling of mathematical objects. My interpretation about how Alba tackles mathematical objects is that she has understood that everything involved into brackets is a whole. From this dialogue I understand that she has also errors by careless nature because she gives me correct answers during the interview after I emphasize what is wrong.

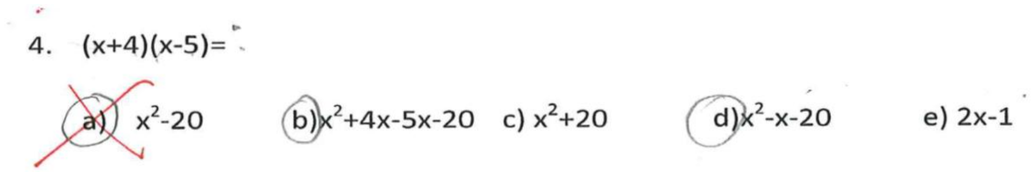
6.1.3 Alba's "slips that are corrected and justified by her using instrumental understanding and followed by uncertainty"

This criteria has emerged from data analysis referring to mistakes that Alba has done during the test and I considered them as mistakes of careless nature because she corrected and justified them based on arguing, but on another hand all this was followed by uncertainty since I intervened some times. This criterion is divided into two parts containing Alba's answers from first and third exercise.

6.1.3.1 Exercise 1, Task 4

Alba's answer from the test

Alba has selected three answers for this task, b) and d) that are correct answers, and a) which is not.

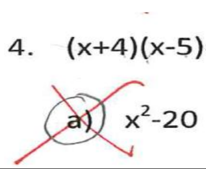

4. $(x+4)(x-5)=$

a) x^2-20 b) $x^2+4x-5x-20$ c) x^2+20 d) x^2-x-20 e) $2x-1$

Even though she marked two correct answers showing a correct use of distributive property, which is linked with tackling algebraic expressions into brackets as mathematical objects (Gray and Tall, 1991, 1992), Alba did her choice of alternatives b) and d) uncertain with marking alternative a). From this answer I can say that Alba has had two different thoughts concerning using of distributive law and furthermore, concerning tackling of algebraic expressions as objects.

Alba's answer from the interview

My intention of this dialogue was to discuss with Alba about her double choice (wrong and right) and to justify each of them.

Dialogue 4		
Name	Dialogue	Notes
D A D	What should you do in the fourth task? I have to multiply the first element of the first bracketwith the first one of the second brackets; And that first with the second element of the second bracket. And it is the same about '4'. Ok, let's start from the beginning. You should	Exercise 1/ Task 4  4. $(x+4)(x-5)=$ a) x^2-20

	multiply the first bracket with the second one. What is involved into the first bracket?	
A	The ‘x’.	
D	Is anything else?	
A	Also ‘4’.	
D	Ok. So you can consider ‘x+4’ as a mathematical object. And, only ‘x’, or only ‘4’, or both of them should multiply the second bracket?	
A	Both of them (and she writes ‘= x^2+4x-5x-20’).	
D	Can you write it in a longer way?	
A	Yes (she writes ‘= x(x-5)+4(x-5)’).	
D	Why did you select a)?	
A	Mmm, I have thought that ‘x’ multiplies the first term of the second brackets, and ‘4’ the second one.	

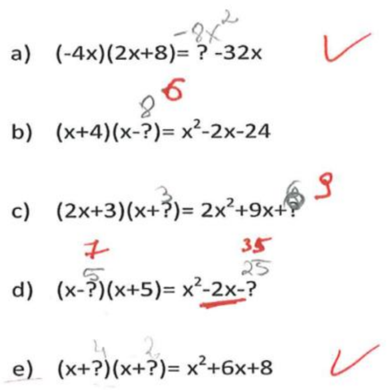
I think that Alba is entering into the logic of the multiplication *procedure* since she identifies multiplication of factors included into brackets, and she distributes firstly ‘x’ and then ‘4’ rejecting the alternative a). According to this part of our discussion, I think that Alba managed to identify two mathematical objects as factors, even though I interfere sometimes; and it is an evidence of *instrumental understanding* (Skemp, 1976) since she wants just to get the right result, without justifying steps that she passes to achieve that result. From this dialogue she might have achieved *discrimination* (Sierpinska, 1994) between multiplications of: one bracket with a term, and two brackets.

6.1.3.2 Exercise 3, Task b) and e)

Alba’s answer from the test

I think that the third exercise is easier than the second one, which requires no-active use of brackets, because the structure of algebraic statements is given; also brackets are placed and are missing only some terms.

For solving this exercise, the knowing of distributive property is sufficient, which could be presented as multiplication of one term with bracket or as multiplication of two brackets. In the first task Alba tackled very well mathematical objects ‘-4x’ and ‘2x’, multiplying numerical parts together and variables together. It is difficult to understand why the missing term in the second task would be ‘8’ as Alba wrote, because ‘8’ times ‘4’ means ‘32’ and not ‘24’. Even it would be a slip, Alba would be able to assess the soundness of her solutions doing other operations as ‘4x-8x=-4x’ that is different from term in the right side ‘2x’.


a) $(-4x)(2x+8)= ?\ -32x$ — $-8x^2$

b) $(x+4)(x-?)= x^2-2x-24$ — 8, 6

c) $(2x+3)(x+?)= 2x^2+9x+?$ — 3, 6, 9

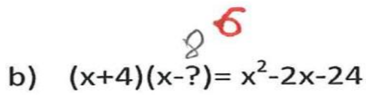
d) $(x-?)(x+5)= x^2-2x-?$ — 7, 5, 35, 25

e) $(x+?)(x+?)= x^2+6x+8$ — 4, 2

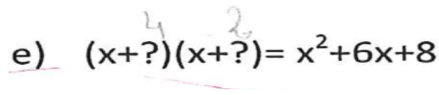
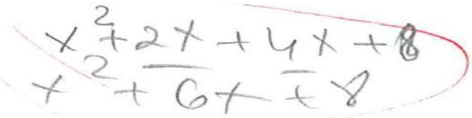
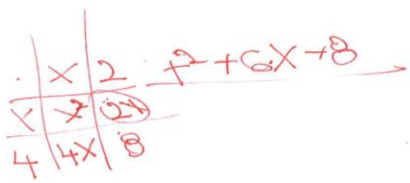
Even in d) task is the same situation as in b). Whereas c) is solved relatively well, because she decided firstly for ‘3’ and ‘9’ (which is written under the ‘6’) instead of question marks but during making calculations again I think she add ‘3’ and ‘3’ instead of multiply them. But Alba operated very well in the last task tackling correctly algebraic expressions into brackets as mathematical objects (Gray and Tall, 1991, 1992). And, only from the interview, I could know if these mistakes have really careless nature *(slips).*

Alba’s answer from the interview

As I supposed in the previous section, errors in the third exercise have careless nature (slips) because Alba corrects herself and was sure while she was giving below explanations:

Dialogue 5		
Name	Dialogue	Notes
D	Why did you write 8?	Exercise 3/ Task b 
A	Eee, it is wrong.	
D	Why do you think so?	
A	Because from ‘x’ times ‘x’ I get ‘x^2’ and from ‘4’ times ‘-6’ I get ‘-24’. Also, ‘4x’ and ‘-6x’ is equal to ‘-2x’.	

I think that this is an *instrumental proof* (Weber, 2002) for algebraic statement ‘$(x+4)(x-6)=x^2-2x-24$’ because Alba uses only rote process of multiplication of two brackets (formal definition) without referring steps and their justifications (relational conceptual understanding) in which this multiplication passes through. According to this task’s solution and to the next one, she did not achieve to have a *synthesis* (Sierpinska, 1994) for understanding of multiplying two brackets (distributive law). This consists on: since two polynomials, included into brackets, are multiplying to each other, each monomials of one polynomial should multiplied each monomials of other polynomial. And in this claim is involved understanding of non-routine and procedural multiplication of two brackets.

Dialogue 6		
Name	Dialogue	Notes
D	Explain me, why did you get ‘2’ and ‘4’?	Exercise 3/ Task e 
A	Because ‘4’ times ‘2’ equals ‘8’, and ‘x’ times ‘x’ equals ‘x^2’.	
D	And ‘6x’, where does it come from?	
A	It is a *formula*.	
D	It comes from a formula or from random multiplication.	
A	Yes, multiplication.	
D	Ok, make the multiplication of these two brackets.	
A	(She writes, see photo.)	
D	Ok. We can multiply these brackets also using the table (and I explained it). Have you ever used this way? What do you think, which is more practicable?	
A	The one with table.	

In this dialogue, since Alba does not have *identification* (Siepinska, 1994) of a random multiplication of two brackets because she thinks it is “a formula”, she cannot have the *synthesis* (Siepinska, 1994) announced above, even though her written results are correct. And, one more time, as in task 4 in exercise 1, for Alba it is enough to multiply the first and the second monomial of the first bracket with respectively the first and the second monomial of the second bracket.

By showing a method using the table, I introduce a structure in the process of multiplying two brackets. It seems that Alba recognizes this advantage since she agreed that using the table was more practicable. But even though I got this consideration by Alba she did not use it to multiply two brackets during the post-test.

6.1.4 Alba’s “no-active use of brackets and relational understanding”

I think that *relational understanding* (Skemp, 1976) and active use of brackets are related to each other since without *relational understanding* of the use of brackets one cannot have an active and correct use of them regardless if it is a new situation or not . This criterion is a mix of elements from theoretical approach and elements emerged from data analysis. Here are involved Alba’s answers of second exercise.

6.1.4.1 Exercise 2

Alba’s answer from the test

Alba seems to be unsure and not correct in putting brackets in their adequate place, with purpose to save the equal sign. Putting brackets in the same expression but in different places, and to get different results is the purpose of exercise 2, in which Alba grouped elements using brackets but with no difference in the result.

In the first case a), even though algebraic expressions in both sides have the same product, as it is required from the task’s demand, and I have put a tick sign. It seems like an accidentally placement of brackets, because I cannot find any reason of grouping elements as it is in the following example:

a) $x+(x^2+3\cdot x+4) = x^2+4x+4$ ✓

Also in the second and third tasks Alba has put brackets not in a reasonable way (red brackets are put during the interview):

b) $x+(x^2+3((x)+4)) = x^2+4x+12$?

c) $(x+x^2)+3\cdot x+4 = x^3+4x+4$

Based on these examples, I do not have any evidence to discuss about relational understanding (Skemp, 1976) of the usage of brackets because Alba seems to be so strict in the way of using her knowledge, and she does not attempt to make links between new situations confronted, as this above exercise, and her knowledge. The second exercise of the test is a very interesting one because for its solution is required basic knowledge in algebra and a good ability for using them.

Precisely it is required to know the distributive property: ‘a(b+c)=ab+ac’ and to emphasize (to have clear) the pass from right side to the left one. It means that ‘b’ and ‘c’ are both multiplied by ‘a’ and in the left side they should be included into brackets composing an algebraic expression that is a factor in multiplication with ‘a’.
Alba has tried to consider this type of understanding of distributive property, in the following case but she did not achieve success:

$$d)\ (x+x^2)+3\cdot(x+4)= x^3+5x^2+7x+12$$

I think that the first bracket is not necessary since next to it is the addition operation, whereas the second bracket is place to get ‘12’ but it is multiplied by ‘3’, which is the wrong factor for this case; it should be ‘$x+x^2+3$’.

She does not achieve to use in the right way the rules that she knows even in new situations. It seems that she is not able to create links between what she knows and situations not seen before, which would be evidence of relational understanding (Skemp, 1976). For example, she uses the rule of multiplication of two brackets: (a+b)(c+d)=ac+bc+ad+bd, but she did not know to pass from the right side to the left one as it looks in the second exercise. So, I can say that this is an evidence of having no flexibility in her knowledge.

Alba’s answer from the interview

Putting brackets in adequate place in algebraic statement was difficult for Alba. It seems that Alba does not have clear the intended use of brackets. From the analysis until now, I can see that she captures ideas from our discussions and memorizes them, such as the table for brackets’ multiplication in the previous section. But since this memorization has no basis it could not be helpful in different and new situations. As I showed Alba’s wrong answers about finding right place for brackets, in previously section, I will introduce some culminating points of our dialogue related to the second exercise, and I will try to determinate if it was a *relational* or *instrumental proof* (Weber, 2002).

During the interview, Alba said that she has not understood the task, and firstly I solved by myself the first task, then I helped her to solve the second, and her solution for the third task is introduced in the next dialogue:

Dialogue 7		
Name	Dialogue	Notes
A	There is ‘x^2’ in the left side and ‘x^3’ in the right one.	Exercise 2/ Task c $c)\ (x+x^2)+3\cdot x+4= x^3+4x+4$?
D	How can you obtain ‘x^3’ considering ‘x^2’?	
A	I can multiply ‘x^2’ with ‘x’.	
D	Ok, but which is that ‘x’?	
A	(She puts brackets involving ‘$x+x^2$’, see photo.)	
D	One moment. Next to this bracket is the operation of addition. It means that with or without brackets it is the same.	
A	Ah yes, brackets should involve ‘x+4’.	
D	But you will get ‘12’, and you are looking for ‘x^3’.	
A	... (She does not give another answer.)	

I cannot pretend to have a proof for these statements by Alba, even less to look for *instrumental* or *relational understanding* (Skemp, 1976). Furthermore, this dialogue proves my assumptions concerning her answers in test that she has no flexibility in her knowledge.

6.1.5 Alba's "difficulty to tackle algebraic expressions as mathematical objects and herself correction"

This criterion is related to difficulties caused by the concept of *procept* (Gray and Tall, 1991, 1992) in Alba's understanding. This criterion contains Alba's answers concerning the eighth task of first exercise.

6.1.5.1 Exercise 1, Task 8

Alba's answer from the test

Difficulties with tackling of mathematical objects are occurred also in this task's solution, where '$(2x)^2$' is considered as '$2x^2$', and I think this is the reason why Alba marked alternative a) in the following task:

8. $\frac{1}{4}(x^2)^2(2x)^2=$

a) $\frac{1}{2}x^6$ b) $\frac{1}{2}x^5$ c) x^6 d) $\frac{1}{4}\cdot 4\cdot x^4x^2$ e) $\frac{x^2}{4}$

This is the reason why I classified this task's answer as evidence of difficulty to tackle algebraic expressions as mathematical objects (Gray and Tall, 1991, 1992).

Alba's answer from the interview

My interpretation about how Alba tackles mathematical objects is that she has understood that everything involved into brackets is a whole; also she identifies it in the next dialogue:

Dialogue 8		
Name	Dialogue	Notes
D	In the eighth task you have selected alternative a), why?	Exercise 1/ Task 8 8. $\frac{1}{4}(x^2)^2(2x)^2=$ a) $\frac{1}{2}x^6$ $\frac{1}{4}x^4\cdot 4x^2 = \frac{\cancel{1}}{\cancel{4}}\cdot\cancel{4}x^4x^2 = x^6$
A	Eee, I have multiplied 'x squared, all squared', eee, this is 'x in fourth'.	
D	Ok.	
A	'2x squared' is ... (she writes, like in photo).	
D	Did you finish?	
A	Yes.	
D	Why '$4x^2$'?	
A	Because even '2' and 'x' are squared.	
D	Ok. Keep going.	
A	I can simplify '4 with 4', and get 'x^6'.	

This is evidence about tackling of algebraic expressions as *products* (objects) but answers during the interview are different from ones in test concerning this task. From this dialogue I can see Alba while she corrects her wrong answer; she has understood that everything involved into brackets is a whole.

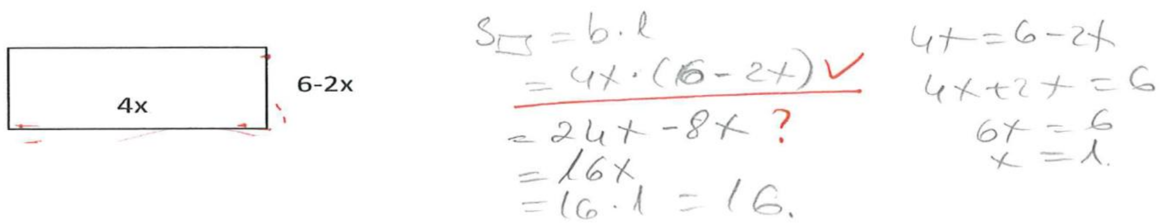
6.1.6 Alba's "instrumental proofs"

This criterion is directly emerged from my theoretical approach and it involves proving a statement or solving of a task based on formal definitions. Alba's answers for task 1, 2 and 5 of the fifth exercise are involved in this section.

6.1.6.1 Exercise 5, Task 1, 2, and 5

Alba's answer from the test

I cannot achieve an estimate for tackling of mathematical objects in area/volume diagrams, because she has made other calculations not related to the demand of the task. I will introduce only the first case:

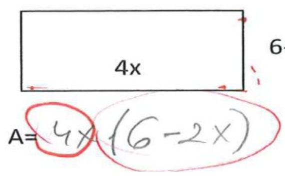


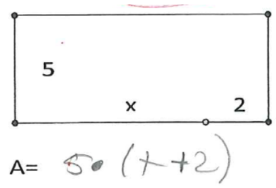
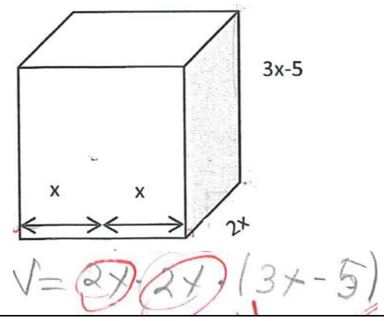
I think that Alba has equated two algebraic expressions that submit two sides of rectangle to find a value for 'x' because she cannot accept the area of rectangle expressed by algebraic expression. I cannot see here the consideration of an algebraic expression as a *product* without having values for its variables (Gray and Tall, 1991, 1992) because Alba makes an attempt to evaluate the value of 'x' by elaborating an equation '4x=6-2x' that is transforming the rectangle into a square. After solving the equation Alba was able to find a numerical value for the area of the figure; therefore it seems that Alba has not the concept of *procept* (Gray and Tall, 1991, 1992).

In addition, even though she presented '6-2x' as one object including it into brackets she did not manipulated with it as well as to manipulate with an mathematical object because from multiplication of '4x' with '2x' she should get '$8x^2$'. So, for evaluating her answers for this exercise I should base only on analysis of the interview.

Alba's answer from the interview

In this part of the interview I am introducing three dialogues concerning task 1, 2 and 5 of the last exercise, which are relevant for this criterion.

Dialogue 9		
Name	Dialogue	Notes
D	Can you write the area of the rectangle, which has sides '6-2x' and '4x'. We know that the area of a rectangle is the product of two of its sides.	Exercise 5/ Task 1 
A	(She writes, '4x(6-2x)', and she hesitates to put brackets.)	
D	Why did you hesitate to put brackets?	
A	… (pause)	
D	Ok, why did you use brackets?	
A	In order to express this side (points '6-2x') as a whole object.	
D	From this expression ('4x·(6-2x)') you show that	

	the '4x' is …, what it is?	
A	One side of the rectangle.	
D	Ok, and the '6-2x' is, what?	
A	The other side.	
	….	….
D	Can you write the area of the next rectangle?	Exercise 5, Task 2
A	Yes. (She writes '5·(x+2)')	
D	Why it is 'x+2' and not '2x' or 'x-2'?	
A	We should add these since they are next to each other.	
D	Why do you use brackets?	
A	Because it ('x+2') shows only one side of the rectangle.	
	….	….
D	Write down the volume.	Exercise 5/ Task 5
A	V=2x•2x•(3x-5).	
D	Why did you use brackets?	
A	Because '3x-5' is one side.	
D	If you will not use brackets, would you refer that side in the diagram?	
A	No.	
D	As what do you consider this side?	
A	As only *one object*.	

For the first, the second, and the last diagram Alba has given three correct algebraic expressions that express diagrams' areas. Sometimes the result shows little in comparison with thoughts of task solver at the time of solving the task, so Alba hesitated to put brackets that should include algebraic expressions which correspond to sides of rectangles. I will justify this hesitation by mentioning the fact that it was the first answer concerning this type of exercise. She caught the idea and by rote usage achieved correct results. I am classifying these as *instrumental proofs* (Weber, 2002) since with appearance of new situations she would operate wrongly such as in examples of the next criterion.

6.1.7 Alba has "no generalization concerning use of brackets related to mathematical objects (algebraic and arithmetic expressions)"

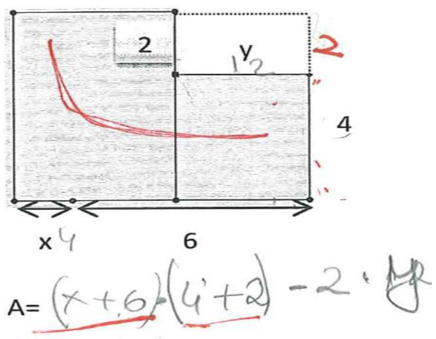
Since generalization of the use of brackets is identification of one situation of the use of brackets as a particular case of another situation and Alba does not achieve it because she did not identify the involvement of arithmetic expressions into brackets as particular case of the involvement of algebraic expressions into brackets. This criterion is emerged from analysis and is based on my theoretical framework. And I have classified Alba's answers for the third task of last exercise as relevant to this criterion.

6.1.7.1 Exercise 5, Task 3

Alba's answer from the test

As I emphasized in the previous section Alba's answers for the fifth exercise have been not understandable since she has equalized two sides of rectangle and is tried to find the variable ‘x’ or ‘y’. So, I can analyze only her answers for this task during the interview, and these are introduced in the following dialogue.

Alba’s answer from the interview

Dialogue 10		
Name	Dialogue	Notes
D	Write down the area of the big rectangle.	Exercise 5/ Task 3
A	(She writes) A=(x+6)·6	
D	Ok, but can you write the side ‘6’ as ‘4+2’?	
A	(She writes) A=(x+6)·4+2.	
D	Now subtract the area of the small rectangle.	
A	(She writes) A=(x+6)•4+2-2y.	
D	Here you have multiplied brackets with ‘4’. From the diagram you said that you should multiply this side with ‘6’.	
A	Yes.	
D	What is happening? How can you express that you are multiplying brackets with ‘6’?	
A	Eee.	
D	Ok, which is one side of the big rectangle?	
A	‘x+6’ (she points it in the expression).	
D	Ok. Which is the other side?	
A	‘4+2’ (she shows it in the diagram).	
D	What about expression?	
A	‘4+2’ (and she puts brackets).	

She does not have the same justification for sides that could be expressed by a numerical sum, as she had for sides that are expressed by algebraic expressions. She does not arrive to *generalize* the use of brackets (Sierpinska, 1994) in case when we refer to the side of one diagram although it is expressed by numerical expression. This would be considered as a reason for determining that the use of brackets for diagrams’ sides that are expressed by algebraic expressions was based on *instrumental understanding* (Skemp, 1976), with which are linked and rote learning and imitative justification.

First of all, it is needed to emphasize that participants of my study did not repeat their knowledge linked to the usage of brackets because it was *the first lesson of the chapter*. Therefore it is difficult to decide about the origin of errors, so I will comment only what is written in the paper-test. Even though the interview about the test is conducted only one day after the test and there was no mathematics lesson between them, I can see how Alba had a self-correction attitude during the interview of the written test, even though she was still followed by her insurance and my interventions were frequent.

Even from the way of organizing the analysis of Alba’s pre-test, using several criteria, I have introduced some properties of Alba’s answers. I think that, Alba did not know to manage very well the challenge offered from having multi correct answers since the first exercise required not only to find the correct answer but also to find different types of presenting the same answer. For some tasks she has selected only one correct answer or one wrong answer, for some others she selected one wrong answer and one or two correct ones. On the other hand, the answers are given and this gives the solver the possibility to operate firstly with answers and then to achieve the task's algebraic expression. According to the written tests, this feature is not exploited by her, she considered it only during the interview while we were discussing. In mostly of tasks (of the first exercise) she focused only on syntactic aspect (Berg, 2009; Demby, 1997; Drijvers et al., 2011) of given answers since she is not trying to capture the meaning of each algebraic expressions presented by alternatives, and to make links between them. This means to manipulate even with semantic aspect (Berg, 2009; Demby, 1997; Drijvers et al., 2011), which was not evident in Alba's answers.

In terms of misconceptions concerning the use of brackets that Alba has during the pre-test, I can mention two of them: brackets indicate multiplication, and non-use of brackets for arithmetic expressions. Related to the second misconception, I think that Alba could not achieve a generalization (Sierpinska, 1994) of use of brackets related to algebraic expressions and arithmetic expressions, which are a particular case of the first ones. Related to this non-achievement of active use of brackets I can introduce the difficulty that Alba has in tackling of algebraic expressions as *mathematical objects* (Gray and Tall, 1991, 1992).

I think that Alba has no flexibility in her knowledge, which comes from *instrumental understanding* (Skemp, 1976) and rote learning. This is because she does not have a good performance in emerging with new tasks which are different from what she is used to solve before. I will relate this with the absence of *relational understanding* (Skemp, 1976) of the concept of brackets, and *procept* (Gray and Tall, 1991, 1992), and tackling algebraic expressions concerning their *semantic aspect* (Berg, 2009; Demby, 1997; Drijvers et al., 2011).

I have classified mostly of mistakes that Alba has done in test as *slips*, because during the interview she has corrected and justified them. This justification has been mostly based on *instrumental understanding* and superficial *relational understanding* (Skemp, 1976).
This pupil seems to be characterized by fragility in her knowledge since the answers to some tasks are correct while other answers to the same type of task are wrong. In her test's tasks' solutions and their interpretation during the interview, there are some errors that are repeated many times, or there are some errors that appear in one task's solution but in the solution of another task similar to the above one there is no error; my interpretation is that she could not notice that some tasks are similar in principle and she has lack of self-confidence.

6.2 Alba's activity during my 'teaching lessons'

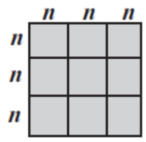
During my teaching activities Alba has been present, but she has not been very active. When I and all pupils have discussed about, for example: evaluating algebraic statement or creating equations, she was passive and rarely participated in discussion, but she was attentive and followed activities and discussion in her tranquility. I paid attention to Alba, more than others, as one of the four pupils I was following during the "Interpreting multiple representations" activity, but I have not had too much space to go and to discuss with pupils individually. During some minutes when I and Alba discussed together helping her to understand what she has to do and what she did, I noticed some difficulties that she had.

During "Interpreting Multiple Representations" activity, pupils are asked to put in a set cards that have the same meaning but with different ways of representation. They have to put closer card(s) with words, card(s) with algebraic expression, card(s) with area diagram, and card(s) with table values, which express the same idea that means they have the same *semantic aspect* (Berg, 2009; Demby, 1997; Drijvers et al., 2011).

Returning to Alba's performance, after she selected one card with words, to pass from this card to the adequate one with algebraic expression was a little difficult, and I can mention later some of these confusions. In general, matching the area diagram card was difficult, also in interpreting it she belonged difficulties. In the end she looked for the tabular card. While she was reading the card: "Multiply *n* by 3 then square your answer", she was not convinced which algebraic expression correspond to it: $(3n)^2$ or $3n^2$. We had this conversation:

Name	Dialogue
D	What do you get from multiplication of 'n' by '3'?
A	Three 'n' (3n).
D	Now, can you square it?
A	Three 'n' square (she pointes $(3n)^2$). The whole '3n'.
D	And, at three n square (I point $3n^2$), what is squared?
A	Only 'n'.

She found by herself the area diagram, and she explained that the big square has the length side 3n, so its area should be $(3n)^2$.



I could notice this as evidence of developing relational understanding about the expression '$(3n)^2$', while she is noticing the semantic aspect of this expression. This means that Alba has understood the "square" as the expression of area and not as multiplication of '3n' with itself.

There was another card that had these words: "Square *n* then multiply your answer by 9", and we agreed to put it in the same set with the previous one because they were different representations of the same idea. But Alba did not know how to explain why that area diagram could correspond with the new card. And one way of presenting that is the area diagram of '$(3n)^2$', and we had this dialogue.

Name	Dialogue
D	Can you write the area of this diagram?
A	$A=(3n)^2=9n^2$.
D	From the diagram, how many small squares there are?
A	There are nine squares with side n.
D	Which is the expression of the area of one of these squares?
A	'n' square (n^2).
D	What about nine squares?
A	$9n^2$.

I would link Alba's difficulties to her consideration for *semantic aspect* (Berg, 2009; Demby, 1997; Drijvers et al., 2011), which is not too evident in her answers and it is strongly associated with *relational understanding* (Skemp, 1976). And from analysis of the pre-test the best evidence of this type of understanding is as superficial since her justifications and argumentations might be based on rote learning.

Considering Alba's activity during my teaching lesson I want to emphasize the correctness in her answers while we discuss and these answers are followed by argumentations and not rote explanations. In addition, I emphasize Alba's *relational understanding* which seems to be a superficial one because even though she operates and decides about cards she has to explain the reason why and to refer to *semantic aspect* of representations.

6.3 Results from the post-test

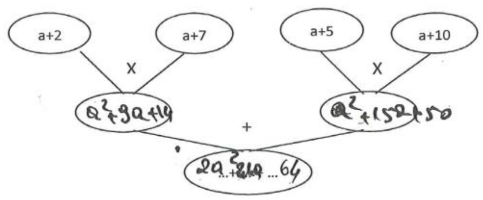
This section is organized in different subsections according to some criteria, which are linked to those presented in analysis of pre-test.

6.3.1 Alba's "active use of brackets and relational understanding

This criterion is related to criterion in 6.1.4 section, which emphasized the link between active use of brackets and *instrumental understanding* (Skemp, 1976). I have involved in this criterion Alba's answers from the first and third exercise.

6.3.1.1 Exercise 1, Task 1

During the interview Alba should write tasks' solutions and to comment on them in the same time. After she wrote '(a+2)·(a+7)' for the first operation, and we had the following dialogue.

Dialogue 11		
Names	Dialogue	Notes
D A D A D A	Why are you using brackets? Because I got the first one as a single number, also the second; and I should multiply these two. And what would happen if you don't use brackets? *Only '2' will be multiplied with 'a'.* ... What does 'a(a+7)+2(a+7)' mean? It means that each elements of the first bracket multiply the second bracket.	Exercise 1/ Task 1  $(a+2)\cdot(a+7)=$ $\cancel{a^2+7}$ $a\cdot(a+7)+2\cdot(a+7)$ $a^2+7a+2a+14$ $a^2+9a+14$

In this first task of the post-test, Alba goes further, in comparison with her tackling of algebraic expressions, since she uses brackets to multiply two algebraic expressions. She seems to be convinced by what she said and she is tackling very well algebraic expressions, as: 'a+2' and 'a+7', as products, *objects* (Gray and Tall, 1991, 1992).

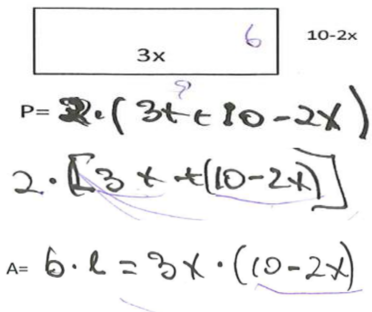
In this dialogue I can see a creative justification by Alba for the use of brackets while she aims to tackle two algebraic expressions as *objects* (Gray and Tall, 1991, 1992). I would like to categorize this as *relational proof* (Weber, 2002) because, to calculate the product of two algebraic expressions Alba considers her understandings for mathematical objects and distributive law, and passes to formal definitions that is the process of multiplication.

From the last dialogue, it seems that Alba could *identify* (Sierpinska, 1994) one case of using brackets while she is multiplying two algebraic expressions, and she *discriminates* (Sierpinska, 1994) it with the case of multiplying these two algebraic expressions using no brackets. She claims that only two terms adjacent to the sign of multiplication will be multiplied in the second case, and she shows also for differences that cause tackling and non-tackling of polynomials as whole objects.

Considering Alba's two answers, during the pre-test and post-test, concerning these two criteria, I can see differences in her way of tackling algebraic expressions, which is in the way she used brackets.

6.3.1.2 Exercise 3, Task 1

The third exercise requires the pupils to be creative and to decide by themself where brackets are necessary in order to present geometrical elements (sides) distinct from each other.
But how did Alba tackle mathematical objects presented in several diagrams?
Alba starts being correct and convinced on her writing and on justifying that each side of the rectangle could be considered as a given product (object) (Gray and Tall, 1991, 1992).

Dialogue 12		
Names	Dialogue	Notes
D	Why did you use brackets?	Exercise 3/ Task 1 
A	Because, ee, *it is just the high*, it is just a number. And, the perimeter is ‘2(3x+10-2x)’.	
D	If I will ask to write this expression using two different types of brackets, how could you do that?	
A	Ok (see the second algebraic expression).	

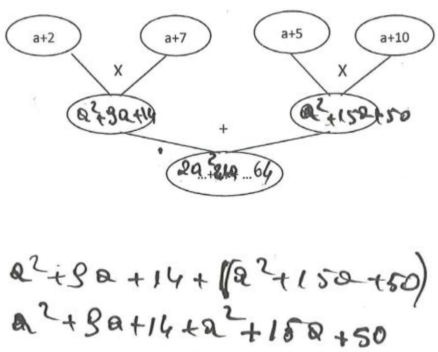
Furthermore, she is right while she uses brackets and square brackets to write algebraic expression for perimeter of given rectangle. I think that she is introducing two different kind of brackets since she is considering ’10-2x’, ‘3x’ and ‘3x+(10-2x)’ as different *mathematical objects* (Gray and Tall, 1991, 1992).

6.3.2 Alba’s uncertainty in discriminating cases of using brackets

To achieve a synthesis of the use of brackets, according to Sierpinska (1994) it is necessary to discriminate cases of the use of brackets referring different conditions. Concerning this criterion I have selected three task’s answers and they introduce non-use or bad-use of brackets from Alba during task-based-interview. I have involved in this criterion Alba’s answers of three different tasks from task-based-interview.

6.3.2.1 Exercise 1, Task 1

Concerning the second level, Alba should add two polynomials ‘$a^2+9a+14$’ and ‘$a^2+15a+50$’.

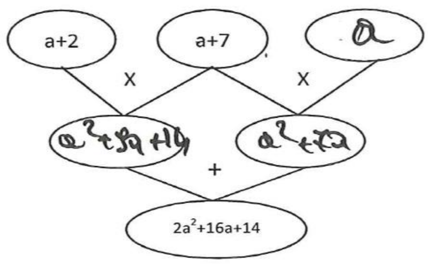
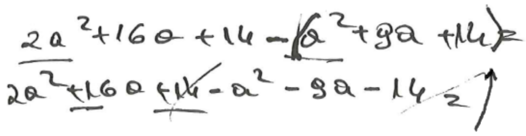
Dialogue 13		
Names	Dialogue	Notes
D	In this step, we have to add ‘$a^2+9a+14$’ and ‘$a^2+15a+50$’, *is it necessary (essential) to use brackets? Is it possible the non-usage of them?*	Exercise 1/ Task 1 
A	Eee, yes, eee, no.	
D	Which is your definitive answer?	
A	It is necessary because without brackets I will add only one ‘number’ (term).	
D	Ok, without brackets you got ‘$2a^2+24a+64$’. Make this addition using brackets, as you said.	
A	Ok (and she wrote as in photo). Oh, it’s the same.	

During our dialogue I made a question with tendency related to the essentiality of brackets usage, getting started from two facts: during *addition* of two polynomials (algebraic expressions) we get the same result even using brackets even not; and Alba included into brackets only the second polynomial, ‘a^2+15a+50’, even though ‘a^2+9a+14’ is also a polynomial. And her answer is not completely correct since she does not determine that without brackets she will add ‘14’ and ‘a^2’, and not ‘a^2’ with ‘a^2+9a+14’. Her *proof* is *instrumental* (Weber, 2002) because she does not care about presenting two algebraic expressions as *objects* (Gray and Tall, 1991, 1992), using brackets for both (relational conceptual understanding), and then to go through formal procedure of clearing brackets, grouping similar monomials and adding them. Her ability to capture the reason *why* brackets are used in particular cases, as the previous ones, which is related to *relational understanding* (Skemp, 1976), seems to be “unstable”, as I saw it in this dialogue.

6.3.2.2 Exercise 1, Task 2

At this point Alba has to find the second addend in the middle level, by subtracting ‘a^2+9a+14’ from ‘$2a^2$+16a+14’.

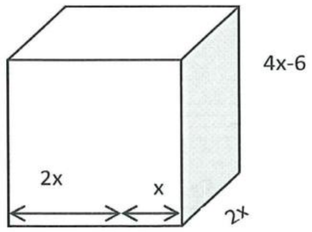
<table>
<tr><td colspan="3">Dialogue 14</td></tr>
<tr><td>Name</td><td>Dialogue</td><td>Notes</td></tr>
<tr><td>D
A
D
A
D

A
D
A
D
A</td><td>Here is the sum and one addend, how can you find another addend?
I have to subtract.
Which one?
(She writes) ‘$2a^2$+16a+14-a^2+9a+14=’
Wait. You said that you should subtract the whole ‘a^2+9a+14’. And in your writing, what is subtracted?
Eee, *I have subtracted only ‘a squared’*.
So, what is missing here?
Brackets.
And where is their place?
Here (see photo).</td><td>Exercise 1/ Task 2


For the first time she wrote it without brackets</td></tr>
</table>

I can categorize the first operation that Alba did as an *instrumental proof* (Weber, 2002), even though it is not correct, because I think she is confined only with putting the minus sign (formal definitions) between two polynomials without considering them as whole objects. So, she does not achieve to tackle polynomials as one whole *object* (Gray and Tall, 1991, 1992) considering the subtraction of two algebraic expressions. So, she does not *identify* (Sierpinska, 1994) another case of using brackets with purpose to group monomials with the same “rights”. After my intervention she argues correctly what she did before and identifies another case of putting brackets with purpose to subtract the whole polynomial and not only one monomial. Also this was a case that Alba could make two *discriminations* (Sierpinska, 1994): one between two cases of subtraction, with and without brackets; and between two cases of using brackets, multiplication and subtraction of two algebraic expressions (polynomials).

6.3.2.3 Exercise 3, Task 5

In this task Alba is asked to write the volume of the cuboid and we had this conversation:

Dialogue 15		
Name	Dialogue	Notes
D	Here, you have to write down the expression for the volume. The volume is the product of three sides.	Exercise 3/ Task 5  
A	(She writes) $V=[(2x+x)\bullet 2x\bullet 4x-6$.	
D	Where do you close the square bracket?	
A	Ah, $V=[(2x+x)\bullet 2x]\bullet 4x-6$	
D	Did you finish?	
A	Yes.	
D	We know that, $V=a\bullet b\bullet c$. Which is ‘a’?	
A	‘2x+x’.	
D	What about ‘b’?	
A	‘2x’.	
D	And ‘c’?	
A	‘4x-6’.	
D	You should multiply these three sides. What have you done in your written expression?	
A	I have multiplied just two sides: a and b.	
D	To multiply the three sides, what should you do?	
A	Eee, I put in square brackets everything. $V=[(2x+x)\bullet 2x\bullet 4x-6]$ (see the above photo).	
D	Usually, when we use brackets, we have an operation related to them.	
A	Yes.	
D	Do you have any operation related to this square bracket?	
A	No.	
D	Does this fail?	
A	Yes.	
D	Ok, in this expression, $V=(2x+x)\bullet 2x\bullet 4x-6$, which is the third side? Where is the error?	
A	To put away the brackets.	
D	Ok, and we have, $V=2x+x\bullet 2x\bullet 4x-6$. Please, tell me one more time, three sides.	
A	‘2x+x’, ‘2x’, ‘4x-6’.	
D	Are you multiplying these three sides, referring to the last expression?	
A	No, I have to use brackets.	
D	Where?	
A	For ‘2x+x’ and ‘4x-6’. $V=(2x+x)\bullet 2x\bullet(4x-6)$.	
D	Ok. Now expand the brackets.	
A	$V=2x+x\bullet 8x^2-12x$.	
D	Tell me what you did.	
A	I expand brackets for the first side.	
D	Did you execute operations related to the first brackets?	
A	No, it should be brackets ‘$V=(2x+x)\bullet 8x^2-12x$’.	
D	Is something else missing here?	
A	Mmm, I don’t know.	

During this dialogue Alba seems to be confused and she is showing bad-use and non-use of brackets while she writes down the volume of the cuboid. However she is able to identify all three sides and their expressions, but I think that this is out of my expectations considering the background of pupils in tenth grade. From the beginning of her answering she is not correct since she decides to use two types of brackets instead of using twice brackets. In addition, I would like to relate this to non-considering neither *semantic aspect* nor *syntactic one* (Berg, 2009; Demby, 1997; Drijvers et al., 2011) of algebraic expressions '[(2x+x) 2x•4x-6' and '[(2x+x)•2x•4x-6]' since respectively she does not close the square bracket and involves the whole algebraic expression into brackets without any action that will be executed for the whole.

She does not consider as similar the two sides which are expressed by algebraic expressions, such as '2x+x' and '4x-6', and as such they should be involved into brackets. So, she is not identifying the object for which she should use brackets. From Alba's answers I can see an uncertainty about tackling algebraic expressions as *objects* (Gray and Tall, 1991, 1992) even after some steps of our discussion when she wrote correctly the algebraic expression for cuboid's volume. She can interpret correctly what she stares, as for example the multiplication of the two first sides, but not to make the opposite action, putting brackets in correct place by herself. This is an evidence of Alba's no-active use of brackets.

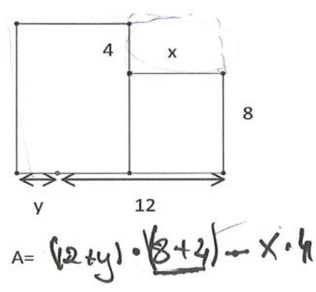
From these three presentations, it is evident that Alba has not achieved a *synthesis* for usage of brackets because she has not *generalized* the use of brackets in each case (Sierpinska, 1994).

6.3.3 Alba has "no generalization concerning use of brackets related to mathematical objects (algebraic and arithmetic expressions)"

This criterion has its analogue in the analysis of the pre-test and it is evident in Alba's answers of the same task as in pre-test but with different data. In this criterion I have involved some of Alba's answers for task 3 of exercise 3.

6.3.3.1 Exercise 3, Task 3

This task requires the algebraic expression which shows the diagram's area (presented by the figure) as the difference of area of the big rectangle and that of the small one.

Dialogue 16		
Name	Dialogue	Notes
A	Area of the big rectangle is, 'A=(12+y)•8'.	Exercise 3/ Task 3
D	But '8' is only one part of its side, and the other part is '4', so the length of this side is '8+4=12'.	
A	Yes, '12' (and she writes 'A=(12+y)•8+4').	
D	Which is the second side in this expression?	
A	'8+4'.	
D	The whole '8+4'?	
A	Yes.	
D	To calculate the area, you should multiply these two sides.	
A	Yes.	
D	Have you multiplied them in what you wrote?	
A	Ah (she puts brackets), A=(12+y)•(8+4).	
D	Now, you should subtract the area of the small rectangle, with sides: 'x' and '4'.	
A	Ok, now I have to subtract '4x' (see photo).	

She falls back into the error, even though it is a case countered before in pre-test, but except of this I think the reason is that during the whole post-test, this is a new way of presenting the side of rectangle, as a numerical expression by the sum ‘8+4’, and deviating from the usual and routine presentation. This is evidence that Alba cannot avoid her rote learning and cannot *discriminate* (Sierpinska, 1994) two ways of presenting the cuboid’s side as an object by using algebraic and arithmetic expressions. And from this she does not *generalize* (Sierpinska, 1994) the use of brackets related to mathematical objects. Comparing with the same task (but with different data) in pre-test Alba has the same answer but I can make a distinction regarded to how fast she corrects her mistake.

6.3.4 Alba’s “instrumental proofs”

This criterion is analogue to that in section 6.1.6. Referring to this criterion I have chosen Alba’s answers for task 1 of exercise 2, because of her formal definitions and rote justifications concerning tackling of mathematical objects.

6.3.4.1 Exercise 2, Task 1

The dialogue below is extracted from task-based-interview related to the second exercise, which presents several steps of creating equations starting from a particular value of the variable. I am concentrating on saving mathematics identity that consists in conducting the same operation for left and right side of algebraic statement, and sometimes operating with the whole side means to include it into brackets. In other words, it is essential considering algebraic expressions of each side of algebraic statement, as a *procept* (Gray and Tall, 1991, 1992). It seems that Alba has captured the idea of saving mathematics identity and determines if it is saved or not, by checking if the two sides are involved in the same operation.
What has interest for me is that she argues the reason of the use of brackets since she notices that what is involved into brackets should be tackled as an *object* (Gray and Tall, 1991, 1992); even though in the previous step this algebraic expression has shown for her a *process*.

Dialogue 17		
Name	Dialogue	Notes
D	What is multiplied by 6?	Exercise 2/ task 1
A	‘x+2’.	x=5
D	How do you notice that?	+2 x+2=7
A	There are brackets for ‘x+2’.	·6 (x+2)•6=42
	…	-20 6(x+2)-20=22
D	Which is the next operation?	:2 [6(x+2)-20]:2=11
A	Divided by ‘2’.	-1 {[6(x+2)-20]:2} -1=10
D	Are both sides divided by ‘2’ and how do you notice it?	
A	Yes, because ‘6(x+2)-20’ is into square brackets.	
	…	
D	What do you think, do braces ‘{}’ are necessary?	
A	Yes, they do.	
D	Can I write without braces (see photo)?	$\frac{6(x+2)-20}{2}-1=10$
A	Yes.	
D	So, are they necessary?	
A	No, but eee, it depends from the way of writing.	

It looks that Alba is doing a rote process for proving these statements since her justifications are imitation and so they cannot take place in each step. The last case is an evidence of another *instrumental proof* (Weber, 2002) based on formal ideas since braces ‘{}’ are not necessary because even without them '-1' will be subtracted from the product of algebraic expression '[6(x+2)-20]:2'.

From the very last Alba's sentence, I understand that according to her the use of brackets depends from the way of presenting division operation in algebraic statement, using (/) sign or (÷) sign. On the other hand, the usage of braces for the first time during the post-test, makes her unsure, even though there is the same principle as for brackets or square brackets, but since she has no flexibility in her knowledge she cannot secede from her routine.

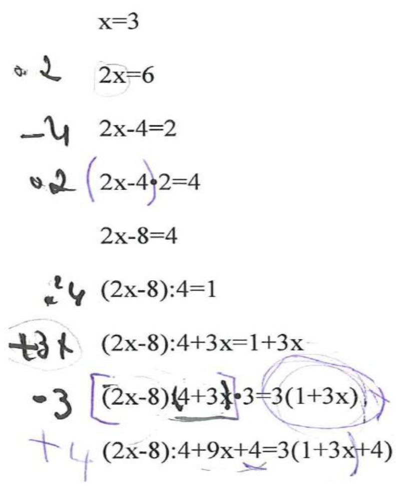
6.3.5 Alba's "difficulty to tackle algebraic expressions as mathematical objects"

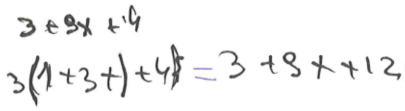
This criterion has its analog in section 6.1.5 and involves Alba's answers for the second task of the second exercise.

6.3.5.1 Exercise 2, Task 2

From the whole second task solution, I have involved in this criterion only Alba's answers concerning the last operation that is done in the process of creating this equation.

Uncertainty continues to follow Alba because she reaches conclusions previously opposed by her and I think the cause of this might be instrumental way of using brackets and formal definition about considering an algebraic expression as *procept* (Gray and Tall, 1991, 1992). I think that she has not achieved to capture the idea tackled in this task through several steps. It seems that she does not care about considering shifting from step A to the next one, step B, as an isolation of a *product* (Gray and Tall, 1991, 1992) given by the algebraic expression in step A, and attaching a certain operation related to a particular *object* (product). And in most of cases is necessary the use of brackets to give the right to find the product firstly.

Dialogue 18		
Name	Dialogue	Notes
D	Ok, go to the next step.	Exercise 2/ task 2
A	It is added '4'.	
D	Is this shift equivalent?	
A	No.	
D	Why?	
A	*It is not saved the algebraic identity neither in left side nor in the right side.*	
D	Why?	
A	In the left side '*4' is added only to '9x'.*	
D	Is the use of brackets necessary in addition?	
A	No.	
D	Is '4' added in other side?	
A	*No.*	
D	Why?	
A	Eee.	
D	Can you write the expression '3(1+3x)' from the previous step and then add '4'?	
A	Ok. '3+3x+4'.	
D	Why?	
A	No, '3+9x+4'.	
D	And if you should not expand brackets.	
A	*Ok, '3(1+3x+4)'.*	

D	Where does the last bracket end?	
A	After number ‘4’.	
D	Ok, do the calculations.	
A	It is equal to ‘3+9x+12’.	
D	See above: ‘3+9x+4’ and ‘3+9x+12’. Are these equal.	
A	No.	
D	Where is the error?	
A	Eee.	
D	Ok, where should you close the bracket?	
A	Eee, after ‘3x’.	

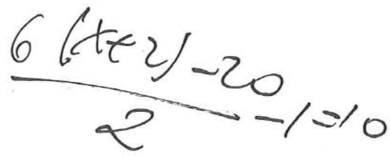
Considering the first Alba’s answers: “In the left side ‘4’ is added only to ‘9x’”, I can say that Alba is in repetition comparing with the same answer in dialogue13 (section 6.3.2.1). This means that this answer has not careless nature but it is rooted in her misunderstanding of the use of brackets.

6.3.6 Alba’s “superficial relational understanding”

In this criterion, which is analogue with section 6.1.1, I have involved two dialogues concerning two different tasks.

6.3.6.1 Exercise 2, Task 1

Even though there are some evidences that Alba has achieved to understand the consideration of an algebraic expression as mathematical object, but she does not explain it completely to the others. Taking encouragement from the dialogue 17 of the criterion in section 6.3.4, in which Alba answered correctly concerning tackling of mathematical objects, I attempted to know how Alba would tackle algebraic expressions in a new situation referring to the opposite way of creating equations. So, I asked her to order operations, from the first to the last one, which are executed concerning the variable ‘x’.

Dialogue 19		
Names	Dialogue	Notes
D	Which operations are executed starting from the ‘x’, in this algebraic statement (see photo).	Exercise 2/ Task 1 
A	Addition with ‘2’.	
D	Why are you saying this?	
A	Because there is, ‘x’ plus ‘2’. Eee.	
D	Does the reason is the presence of brackets?	
A	Yes.	
D	Ok, what is next?	
A	After, ‘20’ are subtracted.	
D	‘20’ are subtracted from (x+2).	
A	No.	
D	What?	
A	Firstly it is multiplied by ‘6’.	
D	Ok.	
A	And then ‘20’ are subtracted.	
D	What comes after?	
A	Division by ‘2’, and then it is subtracted ‘1’.	

In this new way of tackling the creation of an algebraic expression I can see that Alba ordered correctly operations showing that she was looking for products that are manipulated step by step until the creation of the algebraic expression.

6.3.6.2 Exercise 2, Task 2

In the second task of the second exercise, several steps of creating one equation are introduced but some shifts do not save the algebraic identity since brackets are missing or are placing in a wrong way. And, it is important to mention that operations are executed considering algebraic expression in the previous step even though it was not an equivalent shift. One more time, tackling of algebraic expressions as products is emphasized, and it is connected with involving algebraic expressions into brackets with purpose that the expansion of brackets will be the first operation which should be performed. I think this is an example of the *semantic aspect* (Berg, 2009; Demby, 1997; Drijvers et al., 2011) of understanding the use of brackets in these cases and it would be *discriminated* (Sierpinska, 1994) with non-usage of brackets even though algebraic expressions are tackled correctly.

Dialogue 20		
Names	Dialogue	Notes
A	The third operation is: multiplying by '2'.	Exercise 2/ Task 2
D	Is this shift equivalent to the previous one? Is algebraic identity saved?	$x=3$
A	Yes.	$\cdot 2$ $2x=6$
D	Why?	-4 $2x-4=2$
A	No, it is not.	
D	Why?	$\cdot 2$ $(2x-4)\cdot 2=4$
A	Eee, brackets should be present.	$2x-8=4$
D	What does it mean?	
A	To multiply the whole, eee, ...	$:4$ $(2x-8):4=1$
D	What?	
A	*Even '2x' and '-4'.*	$+3x$ $(2x-8):4+3x=1+3x$
D	Is '2x-4'the left side?	$\cdot 3$ $[(2x-8):4+3x]\cdot 3=3(1+3x)$
A	Yes.	$+4$ $(2x-8):4+9x+4=3(1+3x+4)$

During this task Alba has achieved an *instrumental proof* (Weber, 2002) because she is based on formal definitions and rote justifications to explain what is going on through these steps. Furthermore, she can *discriminate* (Sierpinska, 1994) two ways of multiplying two algebraic expressions, with and without brackets, but for each case she cannot shift from multiplication with any brackets to the one with brackets in order to achieve a proper result.

In the first part of this dialogue, Alba makes a *discrimination* of two ways of multiplying, like: multiplying the whole expression '2x-4' by '2', and multiplying only '4' by '2'; furthermore she puts brackets in correct place including algebraic expression in the left side of previous step and she justifies her action saying: "To multiply … even '2x' and '-4'".

6.3.7 Alba's "slips that are corrected and justified by her using instrumental understanding and followed by uncertainty"

This criterion is mentioned also in section 6.1.3 and involves Alba's answers for task 2 of exercise 2.

6.3.7.1 Exercise 2, Task 2

In this part I have involved Alba's answers concerning the sixth operation in the second task.

Dialogue 21		
Name	Dialogue	Notes
A	The next operation is multiplying with '3'.	Exercise 2/ task 2
D	Is this shift equivalent?	$x=3$
A	No, *because is multiplied only '3x'*.	$\cdot 2$ $2x=6$
D	What should you do to multiply the whole left side? Is something that you have to use?	-4 $2x-4=2$ $\cdot 2$ $(2x-4)2=4$
A	Yes, *brackets*.	$2x-8=4$
D	And, what should be included into brackets?	$:4$ $(2x-8):4=1$
A	Eee, '4+3x'.	$+3x$ $(2x-8):4+3x=1+3x$
D	Ok, the left side should be into brackets, where does it start?	$\cdot 3$ $[(2x-8):4+3x]\cdot 3=3(1+3x)$ $+4$ $(2x-8):4+9x+4=3(1+3x+4)$
A	From '2x-8'.	
D	Where is the right place of brackets?	
M	Starting to '2x-8'.	
D	Can we use square brackets?	
M	Yes, and '3x' is the last term included into these brackets.	
D	What about right side? Is it multiplied by '3'?	$[(2x-8):4+3x]\cdot 3=3(1+3x)$
M	Yes.	
D	Why are you saying this?	
M	*Because brackets are present.*	

So, she *discriminates* (Sierpinska, 1994) two "ways of multiplying with '3'" the algebraic expression in the left side of the seventh step, and she explains what is presented in the eighth step, considering that as the wrong one. Following, I required putting brackets in order to save algebraic identity and, her argumentations and justifications look to be formal, imitation or routine. So, in this shift Alba does not have a *relational understanding* (Skemp, 1976) for the aim of using brackets. It seems that she is not thinking about square brackets, in which brackets that are already used will be involved. In addition, in this step she cannot achieve a *generalization* (Sierpinska, 1994) for the case of involving brackets into square brackets with purpose to order the consideration of the algebraic expression included into the latest as object (product) (Gray and Tall, 1991, 1992), such as the first operation that should happen before multiplying by '3'. It might be a rote process of putting brackets as it was in the fourth step (in which '2x-4' is included into brackets) by taking in consideration only terms that are not involved into existing brackets. Furthermore, I think that on this *instrumental understanding* (Skemp, 1976) is based also her justification about multiplication of the right side by '3'.

Since I have organized my task-based-interview's analysis considering several criteria, which are analogue to those in pre-test's analysis, the comparison between two performances is more explicit. So, concerning Alba's performance during the post-test I can notice two evidences of her improvement. The first evidence is that in some tasks she is able to use brackets by herself and to express a *relational understanding* (Skemp, 1976) while she argues the use and non-use of brackets. The second improvement is that there is no evidence of misconception that brackets indicate multiplication.

Even during the post-test Alba is characterized by the fragility of her knowledge. This is because she has uncertainty in *discriminating* (Sierpinska, 1994) cases of using brackets for mathematical objects in different cases. So, she cannot achieve having *generalization* (Sierpinska, 1994) concerning the use of brackets related to mathematical objects. Parallel with this is also tackling of *mathematical objects* (Gray and Tall, 1991, 1992) that Alba expresses difficulties in some cases or she is leaded by instrumental learning or superficial *relational learning* (Skemp, 1976) since she uses formal definitions and rote justifications while she is arguing. Even if I will refer to criteria, almost all of them are the same as in the pre-test's analysis.

I will emphasize more the comparison of Alba's answers for the last exercise of the post-test with her answers for the fifth exercise in the test since they are the same type. For the first task's answers (referring exercises with area diagrams) there is an improvement from 'instrumental proof' criterion in pre-test to 'active use of brackets and relational understanding' criterion in post-test. There is a similarity in her answers during the pre-test and post-test for the third task and they are involved in the same criterion 'no generalization concerning use of brackets related to mathematical objects (algebraic and arithmetic expressions)'. I do not have evidence for improvement in Alba's answers for the fifth task comparing the pre-test with the post-test, but it is the opposite. Her answers for this task pass from 'instrumental proofs' criterion in the pre-test's analysis to 'uncertainty in discriminating cases of using brackets' criterion in the post-test's analysis.

It is difficult to emphasize the role or the effect of three teaching activities from Swan's *collaborative orientation* (Swan, 2005, 2006) of doing mathematics in second Alba's performance, since she still has confusions and uncertainties in her answers. On the other side I can consider the activity "Interpreting multiple representations" (because during this activity she was more active) and especially matching the relevant card with area diagram to other representations as an inspiration for Alba, since her answers in the post-test are more related to relational understanding (Skemp, 1976).

6.4 Summary of Alba's portrayal

In Alba's performance during the test I have noticed a lot of mistakes concerning the use of brackets and tackling of algebraic expressions, by which I mean that I have not seen Alba considering neither the syntactic aspect of using brackets, nor the semantic one. Since these are written answers I could not classify them as slips or bugs. I tried to clarify her answers during the test and to justify them during the interview about the test, in which Alba had a better performance even though it was conducted one day after the test. During this interview, in Alba's answers I noticed her self-correction, even though it was followed by uncertainty, no flexibility in her knowledge and lack of self-confidence. Based even on her oral answers during the interview I thought to classify mostly of mistakes observed in the written test as slips, since during the interview she has corrected them and have justified her answers.

Considering her performance during the pre-test I can mention two misconceptions that Alba had concerning the use of brackets, such as: brackets indicate multiplication, and non-use of brackets for arithmetic expressions.

About Alba's performance during my teaching lessons, even though my contacts with her have not been frequent, I concluded that she had difficulties to her consideration for *semantic aspect* of different representations, which were presented by cards of Swan (2006). Considering these difficulties related to *semantic aspect* (Berg, 2009; Demby, 1997; Drijvers et al., 2011) of the content of cards, even though Alba gave me correct answers and associated them by arguments, I think that she had a superficial *relational understanding* (Skemp, 1976) concerning algebraic expressions.

Even during the post-test Alba is characterized by no-flexibility in her knowledge. She did not achieve generalization concerning the use of brackets related to *mathematical objects*. Also she had difficulties for tackling of mathematical objects, which is related to her *instrumental understanding* or sometimes to her superficial *relational understanding* (Skemp, 1976) concerning the concept of procept (Gray and Tall, 1991, 1992). Also, it is important to emphasize that Alba has no improvement concerning her misconception related to tackling of arithmetic expressions as *mathematical objects*. I have seen the same way of tackling of arithmetic expressions by Alba.

Considering the criteria, based on which I have organized my analysis, there are mostly the same criteria involved in the analysis of pre-test and post-test. I have noticed in the post-test's analysis two evidences of improvement of Alba's mathematical thinking concerning the use of brackets. As the first, I have considered *no evidences of her misconception that brackets indicate multiplication*. The next one is concerning *her ability to use brackets mostly then during the pre-test*. Even related to the criteria, I have used *"active use of brackets"* in the post-test analysis, instead of *"no-active use of brackets"*, which is used in the pre-test analysis.

Considering Alba's improvement and her performance during my teaching lessons, in which I tried to lead Alba to the meaning of algebraic expressions, area diagrams and representations by words, and the links between them. However, I am not convinced and I cannot have the responsibility of deciding for the cause of Alba's improvement concerning her mathematical thinking related to tackling of brackets. This is because, I could not see that Alba captured the idea and the aim of activities involved in my teaching lessons and then she used it to develop mathematical thinking related to the use of brackets and tackling of algebraic expressions. This could be shown by improving her performance during the post-test basing on understanding development during my teaching lessons.

7 Portrayal of Dea

"Dea" is the pseudonym of the second pupil that I will involve in my analysis. She voluntary agreed to be part of my study and we had a very good cooperation. Dea passed two selections; the first one is related to being interviewed and the second is related to analyzing of her performances. I selected her to be interviewed because she did not have good performance related to tasks with multi correct answers and I did not notice any evidence of relational understanding among her test's answers. She has not done a lot of mistakes during the test but the blank answers that were numerous in her test made me curious and I decided to go further in her understanding concerning use of brackets. Through our discussions during interviews I convinced that Dea has more than instrumental understanding concerning use of brackets and tackling of algebraic expressions. Referring to what I am looking for my second research question and Dea's answers during interviews I decided to analyze her performances. In addition, her portrayal will be organized in three sections, in accordance to her preparation in: pre-test, teaching activities, and post-test.

7.1 Results from the pre-test

The analysis of the pre-test is organized in accordance to some criteria concerning the use of brackets and tackling of algebraic expressions involved into brackets. These criteria are the same as those used in analyzing the performance of Alba during the pre-test. It is important to mention that I did not organize analysis of Dea's performance in accordance to criteria used in Alba's portrayal. Firstly I analyzed Dea's answers and then I noticed that criteria emergimg from this analysis were the same as those used in analyzing the performance of Alba during the pre-test. This part of analysis is divided into several sections that are named from the criterion and they involve Dea's answers of different tasks during the test and during the interview about the test. The way how criteria are emerged is the same as in analysis of Alba's performance, referring to keywords from research of literature and my theoretical approach. Also, *it is needed to emphasize that all writing in red is my own comments.*

7.1.1 Dea's "slips that are corrected and justified using instrumental understanding"

In this criterion are involved Dea's answers concerning task 6 of the first exercise and of task d) of the fourth exercise.

7.1.1.1 Exercise 1, Task 6

Dea's answer from the test

Since Dea has also written by herself the answer of this task, the same as alternative a), and it seems that she has thought more about it than just to mark an alternative.

6. $x^2y-xy^2=$ $x^2(y-y^2)$

a) $x^2(y-y^2)$ b) $y(x^2-xy)$ c) $xy(x-y)$ d) $xy(y-y^2)$ e) $x-y$

Considering only her notices I think that Dea is not referred to even *syntactic aspect* (Berg, 2009; Demby, 1997; Drijvers et al., 2011) because these two algebraic expressions are not equivalent and this can be proved by using only distributive law. The mistake stands in the exponent of the term factorized 'x^2'.

Dea's answer from the interview

Only from the written test I cannot decide if this mistake has its roots in Dea's understanding concerning factorizing and its opposite operation expanding brackets. During the interview we had this conversation.

Dialogue 1		
Name	Dialogue	Notes
D	Why have you selected the alternative a)?	Exercise 1/ Task 6
Dea	I have factorized 'x^2' but I see that it is not involved in the second element 'xy^2'. It should be 'x' instead of 'x^2'.	6. $x^2y-xy^2= x^2(y-y^2)$
D	Is there any other change that you have to do?	~~a)~~ $x^2(y-y^2)$
Dea	'x' times 'y' gives 'xy', yes I substitute the 'y' with 'xy' (she writes).	$x(xy-y^2)$
D	Which alternative does comply with this expression?	
Dea	None, if I will factorize even the 'y' it will be c).	c) $xy(x-y)$

I think that Dea is considering *syntactic aspect* (Berg, 2009; Demby, 1997; Drijvers et al., 2011) of using brackets while she is checking for relevant common factor. Since her justifications are based on the way of executing distributive law I will consider these answers as evidence of her *instrumental learning* (Skemp, 1976). But it is needed to emphasize that Dea identified two factorizations, one after another, showing that she can execute the opposite operation of expanding brackets and she can use correctly brackets while she is passing from the subtraction of two monomials to the multiplication of a monomial and a polynomial. This is a passing from one way of presenting a *mathematical object* (Gray and Tall, 1991, 1992), such as the value (product) of 'x^2y-xy^2', to another one, in which are involved brackets; and I have classified it as evidence of *instrumental learning* (Skemp, 1976) since it is a situation seen before by Dea. This is because, factorizations are introduced and pupils have worked with, even in previous grades.

7.1.1.2 Exercise 4, Task d

Dea's answer from the test

In the fourth exercise's Dea's answers I notice a mistake among her calculations.

d) $(\sqrt{2x}-2\sqrt{y})^2+(x+x^2+2)(5-x)=$

$(\sqrt{2x})^2-(2\cdot\sqrt{2x}\cdot 2\sqrt{y})+(2\sqrt{y})^2+5x-x^2+5x^2-x^3+10-2x=$

$2x-4\sqrt{2xy}+4\sqrt{y}+3x+4x^2-x^3=$

$5x-4\sqrt{2xy}+4\sqrt{y}+4x^2-x^3$

Dea has squared only the '2' without considering the square root of 'y' as part of brackets that are squared. This is evidence of non-well tackling of algebraic expressions but I think that this would be a slip since she has correctly squared other monomials in this task.

Dea's answer from the interview

I asked Dea why she has not manipulated with '$2\sqrt{y}$)' considering it as on object, as it is in real. And during our discussion she just corrects the written answer without giving explanation for what she has written.

Dialogue 2		
Name	Dialogue	Notes
D Dea D Dea D Dea	Why have you calculated '$(2\sqrt{y})^2$' as '$4\sqrt{y}$ '. It should be '4y'. Why? Because '$(\sqrt{y})^2$' is equal to 'y'. Why the square root of 'y' should be squared? Because there are brackets.	Exercise 4/ Task d) $(2\sqrt{y})^2$ $4\sqrt{y}$

During this dialogue Dea justify squaring of the square root of 'y' and she *identifies* (Sierpinska, 1994) the use of brackets and the consequences of brackets' presence.

7.1.2 Dea's "instrumental proof"

In this section are involved two subsections concerning Dea's answers for task 5 of the first exercise, and the second exercise.

7.1.2.1 Exercise 1, Task 5

Dea's answer from the test

In the fifth task Dea has written her answer tackling the monomial as one mathematical object (Gray and Tall, 1991, 1992) and then she has selected one correct answer b). I asked her during the interview why the alternative e) is correct with purpose to go further in her mathematical understanding for factorization process.

5. $x^2y-(xy)^2=$ $x^2y - x^2y^2$

a) $2x^2y$ b) $x^2y-x^2y^2$ c) 0 d) x^2y-xy^2 e) $x^2y(1-y)$

Dea's answer from the interview

Dialogue 3		
Name	Dialogue	Notes
D Dea	Why is the alternative e) correct? Here, eee, the 'x^2y' is factorized and 'x^2y' times '1' is 'x^2y', and 'x^2y' times 'y' is 'x^2y^2'.	Exercise 1/ Task 5 5. $x^2y-(xy)^2=$ $x^2y - x^2y^2$ e) $x^2y(1-y)$

In this dialogue, I can see how Dea is based on manipulations powered by distributive law without referring to her intuitive conceptual understanding of factorization. I think she has captured the principle which stands in the roots of this process but she did not explain me *why* it is allowed to disjoint the 'y'. This is the reason why I consider this answer as an *instrumental proof* (Weber, 2002). Also, this is evidence of consideration of two equivalent algebraic expressions' *syntactic aspect* (Berg, 2009; Demby, 1997; Drijvers et al., 2011).

7.1.2.2 Exercise 2

Dea's answer from the test

In Dea's test paper has no note concerning the second exercise. In the following photo, notes with pencil are Dea's ones and red notes are mine. During the test Dea has used a blue pen.

a) $x+x^2+3\cdot x+4= x^2+4x+4$

b) $x+x^2+3(x+4)= x^2+4x+12$

c) $x+x^2+3\cdot x+4= x^3+4x+4$

d) $(x+x^2+3)(x+4)= x^3+5x^2+7x+12$

Dea's answer from the interview

During the interview, Dea said that she has not understood this type of exercise. So, I explained its demand and I solved by myself the first task, then I asked her to solve the second one. She put brackets around the 'x+4' and she argued it like in the following dialogue.

Dialogue 4		
Name	Dialogue	Notes
D	Why have you put brackets around 'x+4'?	Exercise 2/ Task b) b) $x+x^2+3(x+4)= x^2+4x+12$
Dea	Because '3' times 'x' is '3x' and '3' times '4' is '12', then '3x' plus 'x' is equal to '4x'. Also 'x squared' is in two sides and algebraic expressions in two sides are equal.	

Dea puts brackets correctly, involving 'x+4' and she justified it correctly since two algebraic expressions are equivalent. But since her justification is based on calculations using distributive law I have considered as an *instrumental proof* (Weber, 2002). Dea has stopped in *syntactic aspect* (Berg, 2009; Demby, 1997; Drijvers et al., 2011) of transforming algebraic expressions, and since she has executed two operations it seems that she has tackled 'x+4' as an *object* (Gray and Tall, 1991, 1992).

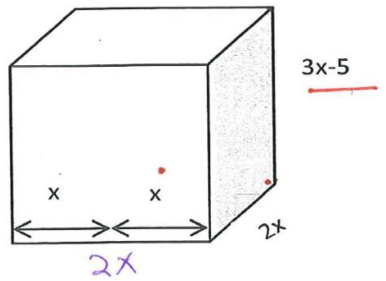
7.1.3 Dea's "misconception of considering algebraic expressions with and without brackets as equivalent"

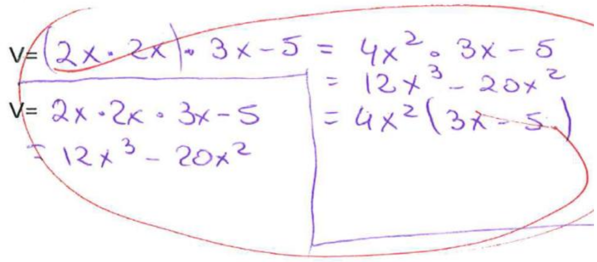
In this criterion are involved Dea's answers for task b) in fifth exercice.

7.1.3.1 Exercise 5, Task b

Dea's answer from the test

The demand of this task is: "Write down the algebraic expression of the cuboid using brackets and them without brackets". Considering her written answer, I think, Dea has started by writing the algebraic expression for the volume using brackets, but not using them in the right place. And she has put brackets away multiplying '2x' and '2x', then she has multiplied '$4x^2$' with '3x-5' in the same way as brackets are present. And finally she has factorized the '$4x^2$' achieving the algebraic expression for this cuboid's volume. It is interesting how she has started from an algebraic expression that is not adequate for this demand and has achieved the right one. This trend is introduced even in writing algebraic expression without brackets for cuboid's volume, which means to expand brackets starting from the algebraic expression with brackets. This time Dea has written the algebraic expression even without including into brackets '2x·2x', and she has multiplied three mathematical objects as if the brackets are present.



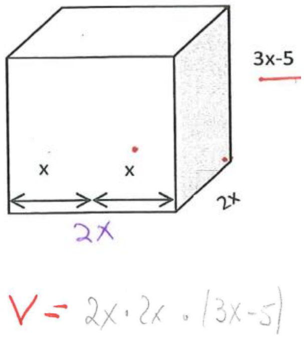


Also, I cannot have evidence of considering *syntactic aspect* (Berg, 2009; Demby, 1997; Drijvers et al., 2011) of multiplying a monomial with a polynomial included into brackets, which means that the monomial multiplies even the first and the second term involved into brackets. But in cases when there are not brackets the monomial should multiply the term next to the multiplication sign.

This answer is evidence of the misconception of considering algebraic expressions with and without brackets as equivalent. This misconception is introduced also by Booth (1984, pp. 53-55), which concludes that a large proportion of pupils appear to regard brackets as irrelevant in algebraic expressions and they consider these expressions with and without brackets to be equivalent.

Dea's answer from the interview

During the interview I wanted to have an explanation for the strange calculations Dea had done during the test.

Dialogue 5		
Name	Dialogue	Notes
D Dea D Dea D Dea	Explain me, what have you written for this task? Oh, it is a mess. Write down the volume of the cuboid. (She writes, see fig.) Why are you using brackets? Because I should refer to the whole side '3x-5'.	Exercise 5/ Task b) 

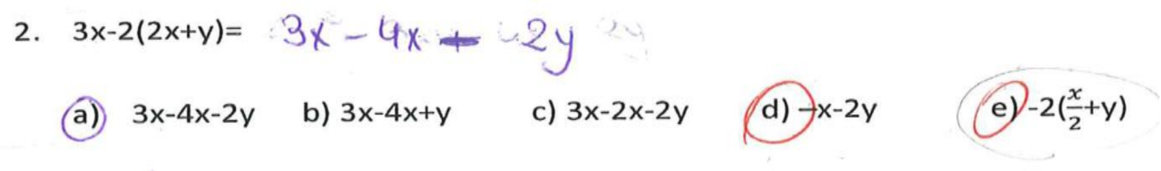
I will emphasize her answer: "Because I should refer to the whole side '3x-5'", which expresses a correct justification based on considering both the side of the cuboid as an object and the algebraic expression that present that side. Since Dea is interesting just on the length of this side, which is analogue with the *value* of '3x-5' (Gray and Tall, 1991, 1992), she is using brackets.

7.1.4 Dea's "superficial relational understanding"

7.1.4.1 Exercise 1, Task 2

Dea's answer from the test

Mostly of test's tasks have are not solved by Dea, also in the first exercise there are only three tasks for which she is answered correctly and completely. The second task is one of tasks that have only one answer given by Dea.


2. 3x-2(2x+y)= 3x − 4x + −2y

a) 3x-4x-2y b) 3x-4x+y c) 3x-2x-2y d) -x-2y e) $-2(\frac{x}{2}+y)$

Dea's answer from the interview

I leaded our conversation toward evaluating algebraic expressions of other alternatives.

Dialogue 6		
Name	Dialogue	Notes
D Dea D Dea D Dea	In the second task, why d) and e) are correct? Ok, (she writes, as it is in the picture) '3x-4x-2y', I can factorize the 'x' and I have '-x-2y'. What about alternative e)? I can write the '-x' as a fraction. And eee. You can start from the algebraic expression in alternative e) and you can make calculations. Ok (and she writes).	Exercise 1/ Task 2 $3x-4x-2y$ $x(3-4)-2y$ $x\cdot-1-2y$ $-x\cdot-2y$ $-2(\frac{x}{2}+y)$ $-\frac{2x}{2}-2y$ $-x-2y$

I can see that Dea explains and justifies the correctness of these two alternatives of the second task. It is needed to emphasize the transformation of algebraic expression in alternative d), step by step, showing the reason *why* '3x-4x' is equal to '-x'. The factorization of 'x' is evidence of Dea's learning with meaning. In addition, "I can write the '-x' as a fraction" is a statement which express Dea's intuitive conceptual understanding concerning mathematical objects. This is because she is aware concerning the *semantic aspect* of algebraic expressions and this understanding is helped by *syntactic aspect* (Berg, 2009; Demby, 1997; Drijvers et al., 2011), which enables presenting of 'x' as '$2\cdot\frac{x}{2}$'. This is the reason why I selected these answers for the criterion related to Dea's *relational understanding* (Skemp, 1976). But it is a brackets' absence for '-1' and this absence have not effect Dea's answer. She has considered both the '1' and the minus sign as one object. So, one more time is showed Dea's misconception of considering algebraic expressions with and without brackets as equivalent.

7.1.5 Dea's "no-active use of brackets and relational understanding"

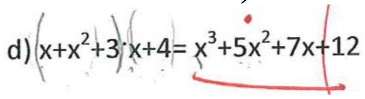
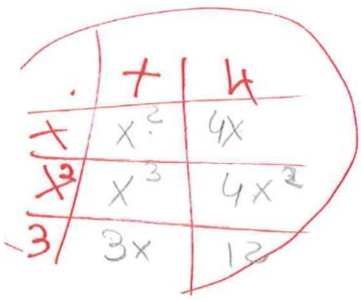
7.1.5.1Exercise 2, Task d

Dea's answer from the test

There is no answer in Dea's test concerning this task.

Dea's answer from the interview

During the interview I asked Dea to solve task d) of the second exercise, which means to put brackets in correct place in the left algebraic expression with condition that this latest should be equal to algebraic expression in the right place. And, firstly she put brackets around the 'x+4'.

Dialogue 7		
Name	Dialogue	Notes
Dea	(She puts brackets around 'x+4') It is necessary to multiply 'x^2' with 'x'.	Exercise 2/ Task d) 
D	Ok and how can you achieve this? Be careful, while you are multiplying two brackets, you should multiply each element of the first bracket with all elements of the second bracket (she says it in one voice with me).	
Dea	Ah yes, so only 'x squared' should be involved into brackets.	
D	But this is not going to change the expression, you should involve into brackets more than one element, two or three.	
Dea	Ok. (She put brackets around 'x^2+3'.)	
D	Ok, make the operations in the left side.	
Dea	'x^2' times 'x' is 'x^3', 'x^2' times '4' is '$4x^2$', but, eee, we have '$5x^2$' in the other side.	
D	So, we need another one 'x squared'. Is there any other element that you should involve it into brackets?	
Dea	Even the 'x'. (She put the bracket as in the fig.)	
D	Make operations in the left side using multiplying with table. (And I explained its principle.)	
Dea	(She writes as in fig.)	
D	And which is the result?	
Dea	'$x^3+3x^2+7x+12$'.	
D	Have you ever seen before this way of multiplying polynomials?	
Dea	No.	
D	Is it understandable?	
Dea	Yes, it is, and it is more practicable than the usual one.	

I tried to help her recognizing the multiplication of two brackets, even though she knew it. But I was surprised by her answer to involve into brackets only 'x^2'; it is a case of unnecessary use of brackets. It seems that Dea has not achieved a *generalization* (Sierpinska, 1994) of the use of brackets. It seems to be difficult for Dea the active use of brackets, which is followed by *relational understanding* (Skemp, 1976) of brackets' use.

By showing a method using the table, I introduce a structure in the process of multiplying two brackets. It seems that Dea recognizes this advantage since she agreed that using the table was more practicable and she will use it for multiplying two brackets during the post-test (see Dialogue 8).

From Dea's performance during the pre-test, I can see that Dea was under the pressure of being tested and under the time restriction during the test since in her test I found a few mistakes and some blank spaces in tasks' solution. Also, at the begging of the interview she stated that: “when I became aware that I should select more than one answer for all tasks, it was too late”. But, during the interview Dea justified correctly the correctness of other alternatives since she argued the equality of algebraic expressions in alternatives, by syntactic aspect.

In Dea’s test are noticed not only blank spaces for tasks’ solutions but also mistakes, which I have classified as *slips*. This is because she has corrected them by herself during the interview. Among these slips I would like to emphasize mistakes related to the process of factorizing, even though this process is introduced in earlier grades. But during the interview Dea corrected herself showing that she can execute the opposite operation of expanding brackets and she can use brackets correctly while she is passing from the subtraction of two monomials to the multiplication of a monomial and a polynomial.In terms of Dia’s performance during the pre-test she has a misconception about considering algebraic expressions with and without brackets as equivalent. This is followed by uncertainty in active use of brackets.

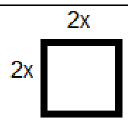
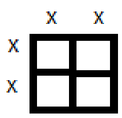
Summarizing what is said above, I think that Dea has mathematical understanding for the use of brackets and tackling algebraic expressions involved into brackets, more *instrumental* than *relational* (Skemp, 1976). This is because she is not correct in using brackets in active way and she is based on *syntactic aspect* (Berg, 2009; Demby, 1997; Drijvers et al., 2011) of using brackets and transforming algebraic expression.

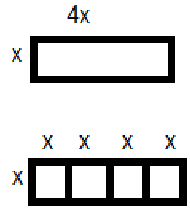
7.2 Dea’s activity during my teaching lessons

Dea has been present during my three teaching lessons and also has been active during our discussions. I tried to pay attention to Dea as one of four selected pupils with intention to look closer her work and to discuss with her.

During the “Evaluating algebraic statement” activity, I distributed each pair the sheet of paper with ten algebraic statements that should be evaluated. I required the four selected pupils to write down in that sheet of paper their name and their solutions and justifications. From Dia’s notes, I notice that she has considered only the syntactic aspect of algebraic statements, which means she has tackled algebraic statements as equations and tried to solve them. If the equation has one or two solutions she has evaluated it as sometimes true. If the equation has no solution then she has evaluated it, as never true. And if algebraic expressions in two sides of equal sign are equal she considered it as always true.

So, in Dea’s answers dominates instrumental understanding of tackling algebraic expressions. I could have only one short discussion with Dea during this activity since I noticed her consideration for syntactic aspect of algebraic expressions. I focused on the second algebraic statement $(2x)^2=4x^2$. Dea wrote, $4x^2=4x^2$, and this algebraic identity is always true for each value of ‘x’. Following, we had this conversation:

Name	Dialogue	Notes
D	Can you design a diagram that has the area ‘$(2x)^2$’?	
Dea	Yes, it is a square with sides ‘2x’.	
D	Ok. And I am thinking to divide this square into four small squares with side ‘x’ by dividing the side of big square in two equal parts. Do you understand?	
Dea	Yes, I do.	
D	Can you divide the square into four small ones?	
Dea	Ok.	

D	Now, can you design a diagram with area ‘$4x^2$’?	
Dea	It is the same one, since there are four squares with area ‘x^2’.	
D	I am thinking about expressing ‘$4x^2$’ as ‘4x’ times ‘x’, which is the area of a rectangle.	
Dea	Yes, but if I will divide its side in four parts, there are four squares with side ‘x’.	
D	Ehe, but the difference is this diagram is a square and another one is a rectangle.	

In this part I tried to introduce another way of considering algebraic expressions referring to their meaning and other ways of representing them.

A conversation similar to that previous we had during the “Interpreting multiple representations” activity, in which she had organized all cards in different sets (each set contains cards with different representation but with the same meaning) and tried to make links between sets. So, Dea was saying that $9n^2$ and $(3n)^2$ has the same area diagram, but I showed one more time two different types of area diagrams (one rectangle with sides ‘9n’ and ‘n’, and one square with side ‘3n’) introducing different meanings of two algebraic expressions that are equivalent in accordance to *syntactic aspect* (Berg, 2009; Demby, 1997; Drijvers et al., 2011).

What I noticed during my teaching lessons concerning the performance of Dea is that she had correct answers basing on *syntactic aspect* of tackling algebraic expressions, and she knew the reason of using brackets. This latest is showed by Dea’s writing, ‘(n+6)·2’, in a blank card which correspond to the same set with card: “Add ‘6’ to ‘n’ then multiply by ‘2’”. So, she used brackets to insulate the product of the addition that should be multiplied by ‘2’, and this is also evidence of active use of brackets.

7.3 Results from the post-test

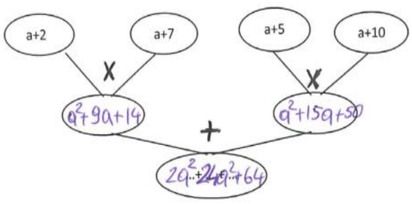
This section is organized in different subsections in accordance to some criteria which are linked those used in pre-test’s analysis, in order to introduce Dea’s performance during the post-test and to compare it to her performance during the pre-test.

7.3.1 Dea’s “superficial relational understanding”

In this section I have involved Dea’s answers concerning three tasks, such as: task 1 and 2 of the first exercise, and task 2 of the second exercise.

7.3.1.1 Exercise 1, Task 1

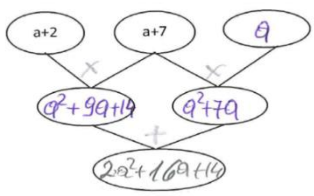
This task demands simple calculations but the “difficulty” stands in involving polynomials into brackets.

Dialogue 8		
Name	Dialogue	Notes
D	In this exercise you are asked to make calculations.	Exercise 1/ Task 1 
Dea	Would I multiply?	
D	There is a multiplication, another one multiplication and then an addition.	
Dea	Ok, ‘a+2’ times, eee.	
D	Sorry, if you remember, you can use the table for multiplication.	

Dea	Firstly, I am writing the multiplication (see photo).	
D	Why are you using brackets?	
Dea	I am using brackets because this is an *addition* [sum] (she points ‘a+2’) and this is another one (she points ‘a+7’). And if I do not use brackets I am going to multiply only the ‘2’ with the ‘a’, and the others will be as they are.	$(a+2)(a+7)$
D	Are you considering ‘a+2’ as a single element?	
Dea	Yes, both of them, even ‘a+2’ and ‘a+7’.	
D	Ok, go on with this table. (see photo) (she makes both two multiplications) Ok, now you should add these two polynomials. Does the use of brackets is necessary in this case?	$\begin{array}{c\|c\|c} & a & 2 \\ \hline a & a^2 & 2a \\ \hline 7 & 7a & 14 \end{array}$
Dea	No, I do not think so, since there is an addition. It would be necessary if there is subtraction.	

In this first task of the post-test Dea has an algebraic expressions’ tackle as *objects* and she used brackets while she was multiplying two polynomials. Since, this task is a new situation for her, she has an active use of brackets, and she justified this use of brackets considering polynomials as single elements, I can classify this dialogue as evidence of *relational learning* (Skemp, 1976) of use of brackets. Since she *discriminates* (Sierpinska, 1994) two cases of multiplying two polynomials, with and without brackets, it seems that Dea has achieved a *generalization* (Sierpinska, 1994) concerning the use of brackets related to multiplying two polynomials. It is necessary to emphasize Dea’s discrimination of necessity of the use of brackets in addition and subtraction of two polynomials. I have called this criterion “superficial relational understanding” since I notice an embarrassment of terms for ‘a+2’ and ‘a+7’. She called them ‘addition’ and I think that she focused more on the processes of addition than on their product. The adequate term would be ‘sum’, which I put it in square brackets, that is more close to considering polynomials as *procept* (Gray and Tall, 1991, 1992). It is important that Dea used the table to make the multiplication of two polynomials since for her it was a new structure in the process of multiplying two brackets. It seems that Dea captures new ideas and assimilates them as hers.

7.3.1.2 Exercise 1, task 2

Dialogue 9		
Name	Dialogue	Notes
D	Here is the sum and one addend, how can you find the other addend?	Exercise 1/ Task 2 
Dea	I will subtract this given addend from the sum. And I have to use brackets.	
D	Why?	
Dea	There is subtraction and if I do not use brackets only ‘a squared’ would be subtracted.	$2a^2+16a+14-(a^2+9a+14)$ $2a^2+16a+14-a^2-9a-14$ a^2+7a

From this dialogue I can understand that Dea has a *relational understanding* (Skemp, 1976) for using brackets while she is subtracting two polynomials. This is because she *discriminates* (Sierpinska, 1994) two cases of subtracting these two polynomials, with and without brackets, associating to this tackling of algebraic expression as *object* (Gray and Tall, 1991, 1992) and involving it into brackets. I can notice that Dea has captured the idea of grouping elements concerning to the use of brackets. I classify this answer "if I do not use brackets only the 'a^2' would be subtracted" as evidence that Dea considers *syntactic aspect* of using brackets. If Dea would consider both *semantic* and *syntactic aspect* (Berg, 2009; Demby, 1997; Drijvers et al., 2011), she would give me another explanation, such as: "the minus sign before the bracket means change for the signs of monomials involved into brackets, and then I can add terms of the first polynomial with opposite terms of the second polynomial".

7.3.1.3 Exercise 2, Task 2

Dialogue 10		
Name	Dialogue	Notes
Dea	Also in the third step is saved algebraic identity, since eee, there are no brackets so only the '4' is multiplied by '2'; and the right side is multiplied by '2'.	Exercise 2/ Task 2 $x=3$ $\cdot 2$ $2x=6$ -4 $2x-4=2$ $\cdot 2$ $(2x-4)\cdot 2=4$ $2x-8=4$ $:4$ $(2x-8):4=1$ $+3x$ $(2x-8):4+3x=1+3x$ $\cdot 3$ $[(2x-8):4+3x]\cdot 3=3(1+3x)$ $(2x-8):4+9x+4=3(1+3x)+4)$
D	Ok, we know that for saving identity it is necessary to operate with both sides. Is the left side multiplied by '2'?	
Dea	No, only the '4' is multiplied.	
D	Is this shift equivalent?	
Dea	No.	
D	What is going wrong?	
Dea	It is needed to use brackets.	
D	Where?	
Dea	(She puts into brackets '2x-4'.)	
D	And what does it mean?	
Dea	It means that the whole '2x-4' is multiplied by '2'.	
D	Ok, go on.	
Dea	The next operation is divided by '4', it is saved the identity since both sides are divided by '4'. I can notice this from brackets that involve the '2x-8'.	

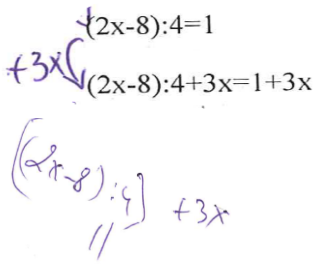
I can notice from this dialogue Dea's *discrimination* (Sierpinska, 1994) of two cases of using brackets in multiplication of '2x-4' by '2' while brackets are used involving '2x-4', or not. Considering this discrimination, I can say that Dea has considered two types of mathematical objects, the '4' and the '2x-4'. And I like to emphasize that she has considered even '2x-4' as an object and she is interested on its *product* (Gray and Tall, 1991, 1992). So, she does not know only where the brackets are necessary but also she understands what the use of brackets means. This is expressed in the last part of our dialogue since she captures the idea of involving '2x-8' into brackets, and this is that its product is divided by '4'. I think, Dea is considering both *syntactic aspect* and *semantic aspect* (Berg, 2009; Demby, 1997; Drijvers et al., 2011) of using brackets achieving a *relational understanding* (Skemp, 1976) of brackets' use. Considering that this dialogue is related to the second task of the second exercise and justifications and argumentations become routine, which means that they might be just repetition, I have classified this dialogue in criterion 'superficial relational understanding'.

7.3.2 Dea's "uncertainty in discriminating cases of using brackets"

In this criterion are involved Dea's answers for, task 2 of the second exercise and task 5 of the third exercise.

7.3.2.1 Exercise 2, Task 2

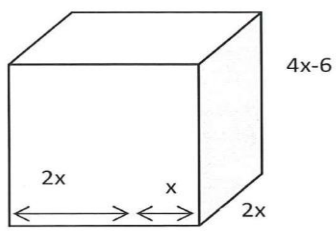
While Dea is going correctly with the task solution I was vigilant all the time looking for spaces and possibilities for interventions with purpose to provoke her mathematical thinking concerning the use of brackets. In the following dialogue is introduced one of my interventions related to the use of square brackets.

Dialogue 11		
Name	Dialogue	Notes
Dea	The '3x' is added to both sides, so it is saved the identity.	Exercise 2 / Task 2 
D	While you are adding (as in this case), is it necessary the use of brackets? And what about square brackets, are they necessary?	
Dea	It is necessary.	
D	Ok, you can put them and do calculations.	
Dea	So, first I have to put the square brackets away, and, eee ah, it is the same. They are not essential.	

Sometimes it is enough just to pick out mathematical object emphasizing operations involved in the whole algebraic expression even though it is not involved into bracket. So, in this case the '(2x-8):4' is a whole object, of which product is obtained by multiplying the value of 'x' with '2', subtracting '8', and finally divided by '4'. Dea does not tackle this algebraic expression, which is not involved into square brackets, as an object.

In addition, in this part is dominating her tackling of algebraic expressions considering the *syntactic aspect* and not the *semantic* one (Breg, 2009; Demby, 1997; Drijvers et al., 2011). I think that her answer "it is necessary" is evidence of her uncertainty and she was stuck from my provocation. But I will emphasize that she ordered correctly which brackets should be cancelled, square brackets.

7.3.2.2 Exercise 3, Task 5

Dialogue12		
Name	Dialogue	Notes
D	Write down the algebraic expression for the volume of the cuboid.	Exercise 3/ Task 5 
Dea	(She writes, see photo.)	
D	Why are you using brackets?	
Dea	To show that '2x+x' is for the first side, and '4x-6' for the third one.	
D	You mean that you should multiply these three algebraic expressions to find the volume of the cuboid. But why are you using brackets?	
Dea	To multiply even the '4x' and the '-6'.	
D	So, your brackets are related to one operation, the multiplication.	
Dea	Yes.	

D Dea	But what about square brackets? Why are you using them? With which operation are related them? Ah, yes. They are not necessary.	$V=[(2x+x)2x(4x-6)]$ $V=[2x\cdot(3x)(4x-6)]$ $=6x^2\cdot(4x-6)$ $=24x^3-36x^2$

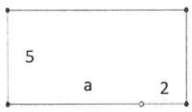
From this dialogue I can see Dea's uncertain in active use of brackets. She is right with using brackets for two sides of cuboid that are expressed by algebraic expressions, and in this line she is tackling sides of cuboid as mathematical objects by considering algebraic expressions, with which two sides are expressed, as *products* (Gray and Tall, 1991, 1992). But I think that Dea has not achieved a *synthesis* of the necessity of brackets' use since she did not *discriminate* (Sierpinska, 1994) the use of brackets concerning two cases that one of them was related to an operation and the other was not. I think that the use of brackets in algebraic expressions means not only to group elements but to group elements, which will undergo a change based on a law. I will relate Dea's uncertainty to non-considering even *syntactic aspect* (Berg, 2009; Demby, 1997; Drijvers et al., 2011) of brackets' use in algebraic expressions concerning the use of square brackets, since there is no operation that act on these square brackets.

7.3.3 Dea's "active use of brackets"

In this criterion I have involved Dea's answers for task 2 of the third exercise.

7.3.3.1 Exercise 3, Task 2

Dea wrote algebraic expressions for the perimeter and area of the rectangle in the second task of third exercise, and we had this conversation:

Dialogue 13		
Name	Dialogue	Notes
D Dea D Dea	Why are you using brackets and square ones? I have involved the side '2+a' into brackets, while the '5+(2+a)' is involved into square brackets. I needed to use twice brackets and, if I use once brackets then I should use square brackets. Ok. Which operation is related to square brackets? Multiplication by '2'.	Exercise 3/ Task 2  $P=2\cdot[5+(2+a)]=2(5+2+a)$ $A=5\cdot(a+2)$

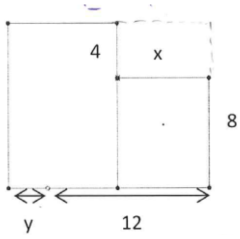
This is evidence of active use of brackets by Dea since she has used brackets for two different algebraic expressions and has discriminated cases of using brackets and square brackets. I want to emphasize that Dea has used brackets in relation to one algebraic expression and to one operation, and the same with square brackets. Based on *syntactic aspect* (Berg, 2009; Demby, 1997; Drijvers et al., 2011) of using brackets she has passed correctly from the first to the second algebraic expression of the perimeter.

7.3.4 Dea's "generalization concerning use of bracket related to mathematical objects (algebraic and arithmetic expressions)"

In this criterion are involved Dea's answers for task 3 of the third exercise.

7.3.4.1 Exercise 3, Task 3

In the third task of the last exercise during task-based-interview, Dea wrote the algebraic expression for area of the diagram. Referring to the figure (diagram) this area is the difference between the area of the big rectangle with sides, ‘y+12’ and ‘8+4’, and the area of the small one with sides, ‘x’ and ‘4’. But Dea has chosen to find the area of the diagram by adding area of rectangle with sides, half of ‘y+12’ and ‘8+4’, and the area of another one with sides, half of ‘y+12’ and ‘8’. I think she has considered these two rectangles as if they have one side equal. Regardless of her interpretation, I am interesting on the way she has tackled arithmetic expression, such as ‘8+4’. After the following dialogue we generated the correct algebraic expression for the diagram’s area.

Dialogue 14		
Name	Dialogue	Notes
D Dea D Dea D Dea	Why did you use brackets? To show sides of rectangles. Is it necessary the use of brackets for the half of ‘12+y’? No, it is not because even without brackets I should add the value of ‘y’ with ‘12’ and then to divide it by ‘2’. Is it necessary the use of brackets for ‘4+8’? Yes, otherwise only four will multiply the other side.	Exercise 3/ Task 3  $A=\left(\frac{12+y}{2}\right)\cdot(4+8)+\left(\frac{12+y}{2}\right)\cdot 8$ $=\left(\frac{12+y}{2}\right)\cdot 12+\left(\frac{12+y}{2}\right)\cdot 8$

Dea has used brackets for algebraic expressions which even without brackets they have only one product thanks to the way of representation. For example, ‘$\frac{12+y}{2}$’ does not need brackets to be involved into, but if I will present this algebraic expression using ‘:’ instead of ‘/’ I have to write, ‘(12+y):2’. So, I will consider this use of brackets as case “to show sides of rectangles”. It is necessary to emphasize that Dea *has generalized* (Sierpinska, 1994) the use of brackets for algebraic and arithmetic expressions with purpose to tackle them as a whole *object* (Gray and Tall, 1991, 1992).

Considering the four criteria involved in the analysis of Dea’s performance during the post-test, I could see a correct tackle of algebraic expressions as mathematical objects, since she involves polynomials into brackets while she multiplies or subtract them. On the other side, she explains which the result in absence of brackets is, discriminating cases of using and non-using brackets. She justifies tackling of algebraic expressions as objects by using brackets for them, but she uses the term ‘addition’ to refer to the process and to the object (sum). This makes me think about *semantic aspect* of using brackets which is also related to her uncertainty in discriminating cases of using brackets and furthermore she uses brackets even where they are not necessary. In this point I will mention Dea’s misconception in the pre-test’s answers, related to considering algebraic expressions with and without brackets as equivalent, as the opposite of using brackets even if they are not necessary. It is needed to emphasize that she has an active use of brackets concerning tackling of algebraic expressions and arithmetic expressions as objects, generalizing the use of brackets, of which base stands relational understanding of the use of brackets.

Comparing to the pre-test’s performance, I can see improvements concerning Dea’s use of brackets and tackling of algebraic expressions. In post-test’s answers she does not have no-active use of brackets, contrary she uses brackets even if they are not necessary or if they are not related to algebraic operations. But also this is not a good way of using brackets because it shows uncertainty and non-capturing the principle of using brackets. In her post-test's answers there is no evidence of her misconception in considering algebraic expressions with and without brackets as equal; and instrumental proofs. This latest is because she tried to explain and to justify her tasks' solutions basing on her intuitive conceptual understanding. She had a very good performance during the post-test, on the other side our discussions during my teaching lessons were short but meaningfully and with attention to lead Dea toward *semantic aspect* (Berg, 2009; Demby, 1997; Drijvers et al., 2011) of algebraic expressions.

7.4 Summary of Dea's portrayal

Dea did not have a good performance during the test. There were unsolved tasks and not accurately solved tasks. In addition, I had my doubts even for her correct answers. But, during the interview I could get even correct answers from Dea but also argumentations and justifications concerning the use of brackets and tackling of algebraic expressions. This is the reason why I have considered her mistakes as *slips*. In Dea's performance during the pre-test I have noticed that she has a misconception about considering algebraic expressions with and without brackets as equivalent. This is followed by uncertainty in active use of brackets. I think that Dea has mathematical understanding for use of brackets and tackling algebraic expressions involved into brackets, more *instrumental* than *relational* (Skemp, 1976). This is because she is not correct in using brackets in active way and she is based on *syntactic aspect* (Berg, 2009; Demby, 1997; Drijvers et al., 2011) of using brackets and transforming algebraic expression.

Also during my teaching lessons, I could notice that Dea had considered only the syntactic aspect of algebraic expressions. This leads me to conclude that in Dea's answers dominates *instrumental understanding* (Skemp, 1976) of tackling algebraic expressions. I tried to lead Dea to considering also *semantic aspect* of algebraic expressions (Berg, 2009; Demby, 1997; Drijvers et al., 2011). All these are related to my first activity "Evaluating algebraic statements", while during the second activity "Interpreting multiple representations" I could emphasize her active use of brackets and also her consideration for *syntactic aspect* of tackling algebraic expressions (Berg, 2009; Demby, 1997; Drijvers et al., 2011).

Considering Dea's performance during the post-test, she achieved to *discriminate* (Sierpinska, 1994) cases of using and non-using brackets, which is linked to tackling of algebraic expressions as *objects* (Gray & Tall, 1991, 1992). It is needed to emphasize that she has an active use of brackets concerning tackling of algebraic expressions and arithmetic expressions as *objects* (Gray & Tall, 1991, 1992), *generalizing* the use of brackets (Sierpinska, 1994), in whose base stands relational understanding of the use of brackets. But the misconception that I noticed in Dea's performance is that she used brackets even if they were not necessary.

Comparing the pre-test's analysis with the post-test one, I can conclude that in the latest there was no evidence of *considering algebraic expressions with and without brackets as equivalent* (Booth, 1984) (this was the misconception from her performance during the pre-test); and no evidence of *instrumental proofs* (Weber, 2002), since she tried to explain and to justify her answers basing on her intuitive conceptual understanding. Furthermore she achieved to *generalize* the use of brackets (Sierpinska, 1994) concerning tackling of algebraic expressions as *procept* (Gray & Tall, 1991, 1992), and I think she has considered even the *semantic aspect* of use of brackets and of algebraic expressions (Berg, 2009; Demby, 1997; Drijvers et al., 2011). This was missing in her performance during my teaching lessons while we were discussing. However, I am not considering our conversation and my work during teaching lessons that I conducted, as the cause of improvement of Dea's performance. This is because I have not concrete evidences (in terms) of argumentations and justifications used by Dea referring to what she understood during my teaching lessons. I mean, during task-based-interview Dea did not refer to examples or argumentations done during activities of my teaching lessons.

Following, I introduce briefly my findings, from tests and interviews, and discuss about them considering my theoretical approach and results from previous research studies, in order to have answers for my research questions. Also, in the next chapter I indicate suggestions for further research and some improvement concerning my education.

8 Discussion and conclusions

In three previous chapters I presented the results of analysis of all tests and the whole performance of two pupils, concerning participants' mistakes and their understanding related to the use of brackets. In addition, I achieved my data analysis basing on my theoretical approach proposed in chapter 3 and comparing to previous evidences presented in chapter 2.

In this chapter, I am having a flashback to recall my research questions, and results from findings with purpose to discuss these results and to achieve conclusions for my dissertation. In addition I will have some reflections in order to indicate pedagogical implications and direction for further research.

8.1 The main aspects of my study

The topic of my research study is the use of brackets and one of my aims is to point out mistakes and misconceptions concerning the understanding of brackets' use in algebraic expressions. My second aim is to follow four pupils during pre-test and post-test and to compare the answers. Therefore I formulated the following two research questions:

1. What kind of mistakes pupils in grade 10, involved in my study, do?

Form data analysis, it seems that some pupils show evidence of improvement in their performance during the post-test comparing to the pre-test. Thereby I want to address the following research question, which is composed by two parts:

2. a) What kind of improvement is showed in performance during the post-test?
2. b) What kind of causes can be identified concerning the improvement?

To answer these research questions I have implemented methods such as, test, semi-structured interviews, task-based-interview and my teaching lessons. I have involved thirty pupils in test and in my teaching lessons, and only four pupils are interviewed twice. Because the time constrains I have described the portrayal of only two pupils concerning their performances during my data collection. I decided for Alba and Dea, because their tests contained mistakes concerning the use of brackets and then during their one after another interviews I noticed improvements in their performances. In addition, by presenting an analysis of two pupils and not of the four pupils, I had the opportunity to analyze in much more detail their answers.

Through the first research question, I achieved to have an overview of the types of mistakes that pupils in tenth grade in upper secondary level involved in my study do. While I was looking for mistakes in thirty written tests I also was referring to types of mistakes evidenced by other authors and by me during my experience as a teacher, and these are introduced in chapter 2. In addition I concluded some results concerning my theoretical approach, and I elaborated it on the basis of several authors' theoretical approaches, such as Skemp (1976), Sierpinska (1994), Gray and Tall (1991, 1992), Swan (2005, 2006), Berg (2009), Demby (1997), Drijvers et al. (2011), Weber (2002), which is introduced in the third chapter. It is necessary to emphasize that these results are open because I took in consideration only written answers of thirty pupils during the test.

Again by virtue in my theoretical framework and especially in some key words such as, relational and instrumental understanding and learning (Skemp, 1976), instrumental and relational proofs (Weber, 2002), procept (Gray and Tall, 1991, 1992), identification, discrimination, generalization and synthesis (Sierpinska, 1994), syntactic and semantic aspect (Berg, 2009; Demby, 1997; Drijverset al., 2011), I have compiled portrayals of two pupils.

This is because my theoretical approach enables me to have access for analyzing the development of participants' mathematical thinking concerning the use of brackets during my data collection, which has three stages: pre-test, my teaching lessons, and post-test. And I can address this through the second research question's answer, which consists in analyzing improvement's evidences concerning the two participants' use and understanding of brackets. Also, in the second research question's answer is involved the cause of these improvements, and I chose to be careful since it seems that I cannot find evidence of links between my own teaching and Alba's and Dea's performances in post-test. This is because my teaching activities were new activities during pupils' mathematics lessons and as something new might cause changes. Also my teaching lessons lasted three school hours, which means it was a short period of time during which pupils could become familiar with these methods and could capture their meaning and intentions, and then to be able to show what they have understood, have changed, and have learned.

8.2 The main findings and discussion about them

As I have presented also in previous section and previous chapters, I have generated two research questions and I tried to bring them in my mind all the time while I was analyzing in order to achieve answers for them. I will introduce in this section the main findings from my data analysis in order to discuss them with other authors' results introduced in chapter 2, and my theoretical approach introduced in chapter 3, with purpose to achieve answers for my research questions.

8.2.1 Main findings from tests

Considering only test's answers, I will list the types of mistakes according to the frequency that they are displayed. This implies that type of mistake Nr. 1 is observed with highest density while type of mistake Nr. 9 with lowest density.

Nr	Type of mistake		Example	Table
1	Non-use of brackets for mathematical objects	Algebraic expression	$-3x[(2x)^2-(1)^2] = -3x•4x^2-1$	5.4
		Sides of diagrams	A = 4x•6-2x	5.5
2	Use of brackets were it is not necessary		$(x+x^2+3\cdot x)+4 = x^2+4x+4$	5.2
3	Wrong-apply of distributive law		$(x+4)(x-5) = x^2-20$	5.1
4	Use the distributive law's principle in situations where it is not involved		$(\sqrt{2x}-2\sqrt{y})^2 = (\sqrt{2x})^2-(2\sqrt{y})^2$	5.4
5	Considering as equivalent the algebraic expressions with and without brackets		V=(3x-5)(2x)(x+x) {with brackets} V=3x-5•2x•2x {without brackets}	5.5
6	Brackets indicate multiplication		$(2x-y)+y = 2xy-y^2$	5.1
7	Incorrect factorization		$x^2y-xy^2 = x^2(y-y^2)$	5.1
8	Non-use of brackets and in the same time operating as if they are present		$V=(2x•2x)•3x-5=4x^2•3x-5=12x^3-20x^2$	5.5
9	Order of operations		5y-(3x-6y):3=5y-3x+6y:3=11y-3x:3	5.4

Table 8.1. Types of mistakes ordered in density ordered

Based on five tables introduced in section 5.1, I have compiled the Table 8.1 for types of mistakes that pupils did during the test. In the very last column is introduced the table from section 5.1 in which the particular incorrect answer is announced during tests' analysis.

Almost all these types of mistakes I have countered even in other authors' studies, and they are introduced in chapter 2. The most wide spread mistake in pupils' test is no-active use of brackets involving algebraic expressions, even if they present sides of diagrams. I think this is related to not tackling algebraic expressions as *procept* (Gray and Tall, 1991, 1992). Related to this type of mistake I can mention studies of authors, such as: Kaur (1990), who has concluded that several pupils are ignorant of the use of brackets or they ignore the use of brackets since they consider them unnecessary; Booth (1984), who states that there are pupils that regard brackets as irrelevant in algebraic expressions; and Kieran (1979), in whose study there are pupils that has no-use of brackets for algebraic expressions.

Except of pupils that do not use brackets in order to involve algebraic expressions with purpose to tackle them as objects, there are pupils that use brackets even where they are not necessary, involving algebraic expressions (monomials, polynomials) followed by addition or no operation. I think that using brackets if they are not necessary means to have not consideration for syntactic aspect (Berg, 2009; Demby, 1997; Drijvers et al., 2011) since brackets are used to group several elements that are related to the same operation. And this type of mistakes is so wide spread in tests of my participants and I have not found it in previous research studies. I have mentioned in "Review of Literature" the study of Ayres (2000) since it was a study based on expanding brackets using the distributive law.

Also, I have evidences from my participants for this type of mistake, and they do not achieve a correct multiplication of two brackets, or one monomial with a bracket. And mostly of mistakes have happened during the second multiplication than the first one. I will relate this type of mistakes to non-considering at least syntactic aspect (Berg, 2009; Demby, 1997; Drijvers et al., 2011) of use of brackets.

Since pupils do not consider semantic aspect (Berg, 2009; Demby, 1997; Drijvers et al., 2011) of an algebraic expression and try to use the distributive law in each situation, they achieve incorrect results. So, squaring a polynomial means multiplying it by itself, and not squaring the first monomial and then the second one. This type of mistake is *not* mentioned before in chapter 2 (Review of Literature).

Considering as equivalent the algebraic expressions with and without brackets is another type of mistakes encountered in my participants' tests, but also this is a conclusion in Booth's study (Booth, 1984). I think that this misunderstanding is related to non-considering neither syntactic nor semantic aspect (Berg, 2009; Demby, 1997; Drijvers et al., 2011) of the use of brackets and algebraic expressions; non-tackling of algebraic expressions as procept (Gray and Tall, 1991, 1992); no discrimination for cases of using and non-using of brackets; and finally I cannot classify both as instrumental or relational proofs (Weber, 2002).

Kaur (1990) identified in her study one misconception for several pupils that brackets indicate multiplication, and the same one is showed in my several participants' answers.

Incorrect factorization, as the inverse operation of expanding brackets, is another type of mistake for which I have evidences from test's answers. I will relate non-finding of the correct common factor with non-considering semantic aspect of algebraic expressions and with non-proving its correctness by considering syntactic aspect of the use of brackets (Berg, 2009; Demby, 1997; Drijvers et al., 2011).

I can link the last two types of mistakes encountered in test's answers, such as several pupils do not use brackets and in the same time they operate as if brackets are present, and the order of operations that means brackets first, with Kieran's conclusions (Kieran, 1979). In both cases, if pupils would consider at least syntactic aspect of using brackets (Berg, 2009; Demby, 1997, Drijvers et al., 2011) will not achieve incorrect results.

8.2.2 Main findings from pupils' portrayals

Since it was difficult to achieve conclusions based only on pupils' written tests, I selected four pupils to be interviewed and to follow their performances during my all data collection. It is explained also in previous chapters, even the reason and the criteria of choosing two pupils to be introduced in my dissertation.

Since Alba and Dea are selected to be interviewed and their performances to be analyzed, I have seen a lot of mistakes and uncertainty in their test concerning the use of brackets and tackling of algebraic expressions. In addition, I could not classify them as slips or bugs, and I could not decide about which aspect is taken in consideration, syntactic or semantic aspect. Both of two girls have had a better performance during the interview about their test while I was present and asking them in order to achieve an overview concerning their algebraic thinking related to the use of brackets. It is necessary to mention that this interview is conducted one day after the test and there was no mathematics lesson between. And, their improvement has started to bear from this step. This is because they had self-correction followed by justifications and argumentations, and this is the reason why mostly of their mistakes I have classified as slips.

On the other side, considering even their answers during the first interview, I was convinced that Alba and Dea have misconceptions concerning the use of brackets and tackling of algebraic expressions. I can mention misconceptions, such as: brackets indicate multiplication, considering algebraic expressions with and without brackets as equivalent, no-active and uncertainty use of brackets. These are mentioned even from other authors but also I will relate them to my theoretical framework. Brackets indicate multiplication, is the misconception introduced by Kaur (1990), also the third misconception is emphasized in Kaur's (1990) study. In addition, Booth (1984) concludes that, several pupils consider algebraic expressions with or without brackets as equivalent. The existence of these misconceptions has emerged instrumental understanding (Skemp, 1976) that pupils have concerning their mathematical thinking for the use of brackets (Berg, 2009; Demby, 1997; Drijvers et al., 2011), and their no-consideration for at least syntactic aspect of the use of brackets and algebraic expressions because they do not consider algebraic expressions after some transformations based on several rules (syntax) of algebraic expressions with brackets and to compare it to algebraic expressions without brackets.

While during my teaching lessons I have noticed no consideration about semantic aspect of algebraic expressions (Berg, 2009; Demby, 1997; Drijvers et al., 2011). This is the reason why I think that they had instrumental understanding or mostly a superficial relational understanding (Skemp, 1976) concerning tackling of algebraic expressions as procept (Gray and Tall, 1991, 1992). My intention, during my teaching lessons, was to lead them towards understanding also meaning of algebraic expressions' representations. I had the opportunity to achieve this through my teaching activities, since these latest are based on confronting misconceptions and discussing about them with purpose to learn from them.

In both pupils' performance during the post-test I could see improvements concerning the use of brackets and tackling algebraic expressions as mathematical objects. In this step I will focus shortly in each pupil's performance during the task-based-interview.

Alba's improvements concerning her performance during the post-test and comparing with it during the pre-test, has evidences in these points: having no misconception that brackets indicate multiplication; the criterion "no-active use of brackets" is substituted by criterion "active use of brackets".

Dea's improvements consist in: having no misconception that algebraic expressions with and without brackets are equivalent; active use of brackets for algebraic and arithmetic expressions; discriminating cases of using and non-using brackets for algebraic expressions, which is related to considering semantic aspect of using brackets and algebraic expressions; no evidences of instrumental proofs.

8.2.3 Looking back to the first research question

Referring to my first research question: *what kind of mistakes pupils in grade 10, involved in my study, do?*

As I have mentioned and explained before, my first research question addressed the types of mistakes concerning the use of brackets, the way of considering brackets, and tackling of algebraic expressions. Considering previous research studies and my experience as a teacher, I had my expectations concerning errors that pupils involved in my study could do while they operate with brackets in algebraic expressions.

In addition, I generated this research question with purpose to design an overview of pupils' mistakes concerning the use of brackets, the process of bracketing and factorizing, and tackling of algebraic expressions as mathematical objects. To have the answer for my first research question I am based on analyzing of test's answers of thirty pupils.

Since I was confronting only with written answers, which may be wrong or correct, and without justifications and explanations, it was a little difficult to determine pupils' misconceptions and misunderstandings concerning the use of brackets. This is because *"a misconception is not wrong thinking but is rather a concept in embryo or a local generalization"* (Sierpinska, 1994, p. 82), which, I think, could not be identified by only the correctness of several tasks' answers. This is the reason why I am referring to mistakes pupils did during their test.

Considering my findings from analysis of tests, presented even in previous section 5.2, my participants' mistakes, which are emphasized even by previous researchers, are such as: non-use of brackets for mathematical objects, such as algebraic and arithmetic expressions; wrongly-apply of distributive law; considering as equivalent the algebraic expressions with and without brackets; brackets indicate multiplication; incorrect factorization; non-use of brackets and in the same time they operate as brackets are present; and confusion about the order of operations that brackets are the first.

Except of these kind of mistakes concerning the use of brackets, I have noticed two other types of mistakes, such as: use the distributive law's principle in situations that it is not involved; and the use of brackets if they are not necessary, even if they are not related to any operation.

So, the overview of test's pupils' answers contains several types of mistakes concerning tackling of brackets, which involves the use of brackets, and operating with them; and tackling of algebraic expressions, which involves considering them as mathematical objects, or referring to them as they are composed by several operations (processes).

8.2.4 Looking back to the second research question

Referring to my second research question that is composed by two parts: *what kind of improvement is showed in performance during the post-test,* and *what kind of causes can be identified concerning the improvement?*

I formulated this research question as it is after I analyzed my data and I noticed improvements from one to the next pupils' performance during my data collection. In order to have answer for my second research question, I leaded analysis of pupils' performances during the pre-test and the post-test toward the process of understanding the use of brackets in algebraic expressions. I noticed mistakes in pupils' test that I decided to interview four of them with purpose to investigate deeper their use of brackets and tackling of algebraic expressions. Since I noticed improvements in two pupils' performance during the post-test comparing to the pre-test, I introduced only the portrayal of these two participants in order to go in depth in analyzing their answers.

Considering my data analysis and portrayals of two participants, which are organized based on several criteria, I have noticed improvements in their mathematical thinking concerning the use of brackets and in tackling algebraic expressions as mathematical objects. Evidences of improvement are such as: having no misconception that brackets indicate multiplication; the criterion "no-active use of brackets" is substituted by criterion "active use of brackets"; having no misconception that algebraic expressions with and without brackets are equivalent; active use of brackets for algebraic and arithmetic expressions; discriminating cases of using and non-using brackets for algebraic expressions, which is related to considering semantic aspect of using brackets and algebraic expressions; no evidences of instrumental proofs.

These improvements I have classified as improvements in the level of understanding and mathematical thinking concerning the use of brackets. This is because there are changes in: the way of perception and tackling the algebraic expressions, as objects or as processes (Gray and Tall, 1991, 1992); considering semantic aspect instead of syntactic aspect (Berg, 2009; Demby, 1997; Drijvers et al., 2011) of the use of brackets; understanding the use of brackets in relational way (Skemp, 1976) instead of instrumental understanding; and in confronting with cases of the use brackets discriminating them in order to generalize (Sierpinska, 1994) each case and finally to synthesize the use of brackets in algebraic expressions.

As I have emphasized several times during writing of my dissertation that the test is conducted in the beginning of the chapter "Expressions with variables" (Babamusta and Lulja, 2009) and the interview about the test is conducted in the next day. In addition the post-test is conducted after several mathematics lessons concerning operating with algebraic expressions, which contain brackets. This means that, regardless the way of organizing mathematics lessons and teaching, pupils have operated with brackets in algebraic expressions and they are used to work with brackets. Three activities involved in my teaching lessons were introduced for the first time to my participants during my data collection. As new things cause changes, these three activities might cause changes in pupils' performance during the post-test.

On the other side, according to Swan (2005, p. 4), the main aims of conducting activities of 'collaborative orientation' of doing mathematics are: challenging learners to become more active participants, to engage them in discussion, to share their thoughts, ideas, misconceptions and results with purpose to become more confident and effective learners.

However, as I mentioned before, I chose to be careful and, based on data analysis, I cannot see clear evidence explaining the cause of improvement of two participants' mathematical thinking concerning the use of brackets in algebraic expressions.

8.3 Results' credibility, suggestions for improvements, and limitations

During designing the frame of my research and the tasks involved in my research I was faced with the difficulty of non-existing another study similar to mine or which tackles the use of brackets as I do. At least, I have not read any of these studies even if they exist. Another challenge of my study was that I elaborated by myself my theoretical approach concerning mathematical thinking for the use of brackets, based on several authors' theoretical basis. In addition, I designed tasks and I analyzed pupils' answers basing on my theoretical approach. I do not think that everyone will agree with me in my choice of methods and tasks, but since I have indicated in details how and why I have chosen and designed them (in chapter 4) I show to the readers my reasons and then they can assess the relevance. In addition, it is important to have relevance between my research questions and results from my research. I tried to keep in mind my research questions all the time during designing tasks, and analyzing data.

As I have explained several times that the purpose of my study is the highlighting of the ways that pupils have concerning the use and understanding of brackets in algebraic expressions. I expected to see two different ways of tackling brackets by pupils, which during the pre-test had done mistakes in their answers. So, I decided to implement, after the test, a semi-structured interview about the test with pupils that did not tackled correctly brackets and algebraic expressions included into brackets, in order to point out their difficulties related to brackets' concept. I tried to cause some changes in my participants' mathematics lessons, introducing some new activities from the collaborative orientation of doing mathematics (Swan, 2005, 2006), with purpose to make more visible and clearer the use of brackets in algebraic expressions and other branches of mathematics. In the end, I decided to conduct task-based-interview to look for the changes that I thought would be caused by my three teaching lessons. Since task-based-interview is a semi-structured interview I thought that it would give me the opportunity to seep toward participants' mathematical thinking concerning the understanding and the use of brackets. According to my interpretation I think that this combination of methods has been valuable to collect the data I needed to achieve the purpose of this inquiry and to have evidences to answer my research questions. But on the other side, I have some suggestions for improvements concerning the content and implementations of methods I used during my data collection.

Pre-Test
Pupils were informed about the test planning, which will be conducted on Monday (24/10/2011), but not about its content. This was because I attempted to check pupils' knowledge about the use of brackets, emphasizing what they kept remembered from previous grades. Considering this criterion and the fact that in test are involved several new types of exercises, I would develop the test for 60 minutes and not 45 minutes. Several pupils have expressed their discontent concerning the time that was in disposal for the test, and I have noticed blank spaces in several pupils' tests. However there are pupils that are answered correctly for all tasks or mostly of them.

My teaching lessons
Since my teaching lessons were new activities for participants and I planned only three lessons to involve them, I think that the number of lessons was not enough for pupils to capture the idea and the aim of activities. In terms of organization of my teaching activities, I think, it would be more efficient for my data if I would have more time and space to discuss with my selected pupils (four pupils) and maybe to have an assistant, who could lead activities for the rest of the class.

Post-test
The task-based-interview contains three types of exercises, the first two are new ones for participants and different from the pre-test. But these exercises contain two tasks each, and I was thinking that if they will have one task, then each pupil would not repeat the same explanations and justifications, which lead them to rote learning. And instead of them I could involve more exercises similar to exercises in the pre-test with purpose to make more visible and concrete the comparison of two performances during the pre-test and the post-test.

According to Goldin (2000), in a task-based interview as a semi-structured interview, the interview is preplanned, but it also gives space to the interviewer to act in the moment. So, during the interview about the test and the task-based-interview I should be vigilant concerning pupil's answers in order to act with the next question. And while I reflect over pupils' responses and my questions I think that I should insist more on getting the answer that I was looking for. For example, I asked "why have you written this…?", and I got this response "oh, it is not correct, it should be like this…".

Since I am discussing about interviews in my study, it is necessary to emphasize that I transcribed almost all interviews and I translated into English several relevant dialogues. And I had *"benefits in terms of bringing me closer to the data"* (Bryman, 2008, p. 456).

In addition I would like to mention limitations that I think they have affected in my study:

- My first time as a researcher
    a. Finding relevant research literature
    b. Transcription and translation in English
    c. Time restriction for analysis
- The effect of the test to participants
- The effect of camera to participants
- The effect of being interviewee
- The environment, out from the class, during interviewing
- The age of participants

In accordance to Hart (1998) in Bell (2010, p. 103), review of literature is important in order to acquire *"an understanding of your topic, of what has already been done on it, how it has been researched, and what the key issues are"*. So, I wanted to have a picture from previous research studies concerning the topic and the aim of my study. But this was difficult because, as I have expressed in chapter 2, there are not so many studies concerning the use of brackets in algebraic expressions and the tackling of algebraic expressions involved into brackets as objects. I spent a lot of time reading previous literature related to algebra and algebraic errors with purpose to capture relevant ideas for my aim. In addition, I also spent a lot of time transcribing parts from interviews that I thought are relevant for data analysis with purpose to achieve answers for my research questions. On the other side, the transcription is very time-consuming but also translating in English has its difficulties because it was needed to make adaption.

So, the time spent for finding review of literature, collecting data in Albania, and transcribing interviews effect on the analyzing time. On the other hand, according to Bryman (2008, p. 451), "qualitative researchers are frequently interested not just in what people say but also in the way that they say it" (p. 451), was a very big work to analyze in details eight interviews of four pupils. In addition, I decided to compile the portrayal of only two pupils and the criteria for choosing these two pupils are explained earlier (section 5.2), and this choice allow me to present in depth analysis of Alba and Dea.

I think these are some difficulties that I caused myself during my study by being for the first time in position of a researcher. By the side of participants in my study, I could list some causes that might have influenced in their performances. So, some of the limitations of the study can be associated with the test instrument and the way the students performed the tests. Since mostly of exercises involved in the test had something new, such as the demand (multy correct answers), or it is completely a new type of exercise, pupils might be more stressfully and do not concentrate on tasks. While, during the interviews I have used video recorder and this might affect in pupils' answers making them to feel uncomfortable or in front of a responsibility. I can consider the same even the effect of being interviewed. Also I would like to emphasize the possibility that taking pupils out of the usual context (classroom) in order to interview them could affect pupils' performances.

In addition, it is necessary to emphasize the behavior's properties of teenagers and especially their non-serious way of considering lessons and teaching activities. Focusing on my participants' willingness and referring to how much I could understand it since I was alone in front of thirty pupils, I state that mostly of them were engaged during my data collection. However, I have my doubts concerning their involvement with seriousness in test and activities during my teaching lessons. This is because during the test, the work was individual and I could not interfere; and during my teaching lessons I was too tasked and it was impossible to check thirty pupils in the same time.

8.4 Pedagogical implications and further research

I got the idea to tackle an algebraic topic for my master thesis from my experience as a teacher. And I chose tackling of brackets in algebraic expressions since brackets are an important element in algebra language and on the other part, algebra is inserted and useful in other branches of mathematics. Skemp (1976), Weber (2002), Sierpinska (1994), Gray and Tall (1991, 1992), Berg (2009), Demby (1997) and Drijvers et al. (2011) offered me a theoretical basis within which are involved ways of tackling the use of brackets in algebraic expressions. I have called these ways, Way A and Way B, and they are introduced in Table 8.2, followed by respective properties.

Way A	Way B
Instrumental understanding Instrumental proof Syntactic aspect Process	Relational understanding Relational proof Semantic aspect Concept (Object) Mental operations

Table 8.2. Ways of tackling use of brackets

And in general, based on my data analysis, it is more preferable and more usable by pupils the Way A of tackling the use of brackets. But, thanks to this way pupils fail in new situations. I appreciate the way B, in which brackets are used significantly and meaningfully. I chose my teaching activities from Swan's (2005, 2006) collaborative orientation of doing mathematics, within which are important the discussion of pupils' ideas and the confrontation with their mistakes and misconceptions, and considering mathematics as an interconnected body of ideas, which means to attempt to show a mathematical concept in front of different branches of mathematics. In accordance to Swan (2005, p. 4), the main aims of conducting activities of 'collaborative orientation' of doing mathematics are: challenging learners to become more active participants, to engage them in discussion, to share their thoughts, ideas, misconceptions and results with purpose to become more confident and effective learners.

This is because I consider Swan's model as a bridge to pass from the way A to the Way B. If pupils discuss their ideas about the use of brackets they would show to each other new ideas and justifications, their misunderstandings and their bad-use or lost-use of brackets, by comparing to their mathematical understanding and to understand their ideas. Facing pupils with the use of brackets in different situations from different mathematics' branches lead them toward the principle of the brackets' use.

In this point, I need to address the transition between different representations (by words, algebraic expressions, tabular data, and area diagrams) in "Interpreting multiple representations" activity; and evaluating the equality of algebraic expressions referring even counterexamples or examples by different representations, such as area diagrams, in "Evaluating algebraic statements" activity, in order to help learners to make connections between their different mathematics branches' ideas. I think this is the manner of having the way B of tackling brackets even though I do not have compelling evidences for improvements caused by my teaching lessons.

On the other side, since arithmetic is introduced before algebra in mathematics textbooks and elementary school curricula, Kieran (1979), and Linchevski (cited in Welder, 2012, p. 257) suggest to introduce and to expand the concept of brackets during the study of arithmetic prior to algebra. So, the need for tackling the use of brackets in earlier studies is emphasized and by other researchers, and I think that understanding of the concept of brackets in arithmetic would be helpful for pupils to understand the concept of brackets in algebra because concretization (as arithmetic is for algebra) helps in understanding concepts.

In terms of generalizing my results, I think it is difficult since this is a case study, which involves only thirty pupils. In order to have a generalization for the way Albanian pupils tackle brackets and algebraic expressions involved into brackets, it is needed an extensive research.

Also, another reason is that this school is a private one, in which pupils are selected by an acceptance test and my participants follow the course of advanced mathematics. This means that these pupils are motivated to work with mathematics, which does not mostly happen.

Since this is a case study, I think that if I would have chosen other participants, the results might have been different. Still my findings are supported by previous studies and are emerged by analyzing my data considering my theoretical approach in each step; and, new difficulties concerning the use of brackets and not reported in previous literature are highlighted among my other findings.

In my study are involved several variables, and referring to some of them I could have suggestions for further research studies. It could be possible to involve younger participants since pupils in tenth grade are involved in my study. And in lower grades, eight or nine grade, operating with brackets in algebraic expressions is more widespread and more emphasized, so I could consider this as a stimulus for further research.

Also I could suggest the composition of two groups of participants, such as the control group and experimental group, in order to compare their results with purpose to assess the effect of teaching activities taken from Swan's "collaborative orientation" (Swan, 2005, 2006) of doing mathematics.

8.5 My development as a teacher and as a researcher

Since this is my first study, in which I had the role of the researcher, and since I have not a long experience in teaching, I think this was an opportunity to improve my being as a researcher and a teacher in the same time.

Firstly I had the idea for tackling the use of brackets not only as part of the order of operations, but I was also confused concerning the way of tackling brackets. In this step, I knew what I did not want but I did not really know to explain what I want. I explained my ideas to my supervisor more easily using examples from my experience as a teacher, because I did not have theoretical part, in which I should base on for the way to tackle brackets. And so from my supervision meetings, discussions with my supervisor, and reading previous literature I started to concretize the theoretical basis and my theoretical framework concerning tackling of brackets as part of algebraic language. So, the elaboration of my theoretical approach is followed by the evolution of my own understanding and algebraic thinking concerning the use of brackets in algebraic expressions.

Also I would like to emphasize another effect of reading a lot of other research studies, that I used to decide about what was relevant for my topic even though each of them was interesting; also I used to consider advantages and disadvantages of methods, which I could involve in my research.

The process of analyzing my data had influenced my development as a researcher since from one supervision meeting to the next one I achieved to link my theoretical approach with my comments for pupils' answers. On the other side, the insights (results) emerging from the analysis of my data would be useful for my improvement as a teacher in order to be able to create an effective learning environment for the pupils. This is because I developed my own mathematical teaching concerning the use of brackets, basing on confronting my theoretical approach and concrete data.

It is important to emphasize my agreement with Swan's "collaborative orientation" (Swan, 2005, 2006) of doing mathematics concerning the way of teaching and creating effective learning environment for pupils.

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10 Appendices

10.1 Appendix A: Test

PRE-TEST ------------------------------------

I. Choose the right alternative (there are more than one for each task).

1. -3(2x-5)=

a) -6x+15 b) -6x-15 c) -6x-5 d) 15-6x e) $-2x^3-5$

2. 3x-2(2x+y)=

a) 3x-4x-2y b) 3x-4x+y c) 3x-2x-2y d) –x-2y e) $-2(\frac{x}{2}+y)$

3. (2x-y)+y=

a) $2xy-y^2$ b)2x c) 2x-y+y d) 2x-2y e) 2x+2y

4. (x+4)(x-5)=

a) x^2-20 b)$x^2+4x-5x-20$ c) x^2+20 d)x^2-x-20 e) 2x-1

5. $x^2y-(xy)^2$=

a) $2x^2y$ b) $x^2y-x^2y^2$ c) 0 d) x^2y-xy^2 e) $x^2y(1-y)$

6. x^2y-xy^2=

a) $x^2(y-y^2)$ b) $y(x^2-xy)$ c) xy(x-y) d) $xy(y-y^2)$ e) x-y

7. $15x-24x^2$=

a) 3x(5+24x) b) 3x(5x-8x) c) 3x(5-8x) d) x(15-24x) e) -(9x)

8. $\frac{1}{4}(x^2)^2(2x)^2$=

a) $\frac{1}{2}x^6$ b) $\frac{1}{2}x^5$ c) x^6 d) $\frac{1}{4}\cdot 4\cdot x^4x^2$ e) $\frac{x^2}{4}$

9. $4x-2[5y-x+3(2x-y)]$=

a) -x-8y b) 14x-4y c) -6x-4y d) -2(3x+2y) e) 14x-16y

II. Place parentheses in these expressions to the left of the equal sign to create equality.

c) $x+x^2+3\cdot x+4= x^2+4x+4$

c) $x+x^2+3\cdot x+4= x^3+4x+4$

d) $x+x^2+3\cdot x+4= x^2+4x+12$

d) $x+x^2+3\cdot x+4= x^3+5x^2+7x+12$

III. Write out the following expansions, filling in the missing terms:

a) (-4x)(2x+8)= ? -32x

b) $(x+4)(x-?)= x^2-2x-24$

c) $(2x+3)(x+?)= 2x^2+9x+?$

d) $(x-?)(x+5)= x^2-2x-?$

e) $(x+?)(x+?)= x^2+6x+8$

IV. Expend the brackets

a) $(-2x)\cdot 8x+(6x+3x)^2=$

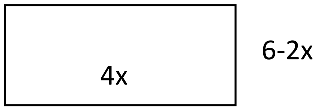
b) (-4x+3)(-1+2x)+(-4x+3)-(-1+2x)=

c) $(12x^3-x^2)-3x(2x+1)(2x-1)=$

d) $(\sqrt{2x}-2\sqrt{y})^2+(x+x^2+2)(5-x)=$

e) -{5x-(11y-3x)-[5y-(3x-6y):3]}=

V. a) Write down the area of the figures:



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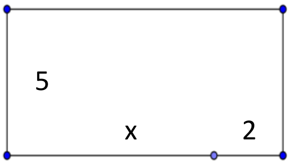

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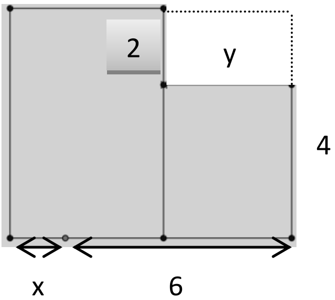

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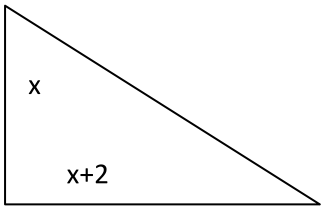
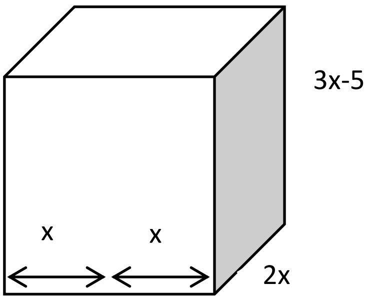


A=

b) Write down the expression for the volume of the object:
1)containing brackets;
2) with no brackets



V=

V=

10.2 Appendix B: Evaluating Algebraic Statements

1. $2(x+5)=2x+10$

2. $(2x)^2=4x^2$

3. $(2x)^2 =2x^2$

4. $(x+5)(x-2)=x^2-10$

5. $\frac{4x+6}{2x+3} = 1/2$

6. $3(x-6)=3x-6$

7. $5x+25x^2=x(5+25x)$

8. $(2x+5)/5=2x$

9. $(x^2+3x-5)-(4x-5)=x^2+3x-5-4x-5$

10. $3(2x-5)^2=(6x-15)^2$

ALWAYS TRUE	SOMETIMES TRUE	NEVER TRUE

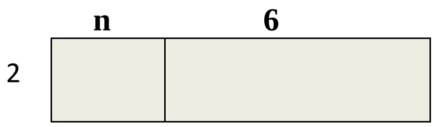
10.3 Appendix C: Interpreting Multiple Representations

Algebraic Expressions

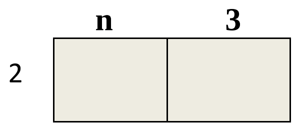
SH1 $\frac{n+6}{2}$	SH2 $3n^2$
SH3 $2n + 12$	SH4 $2n + 6$
SH5 $2(n + 3)$	SH6 $\frac{n}{2} + 6$
SH7 $(3n)^2$	SH8 $(n + 6)^2$
SH9 $n^2 + 12n + 36$	SH10 $\frac{n}{2} + 3$
SH11 $n^2 + 6$	SH12 $n^2 + 6^2$
SH13	SH14

Area Diagrams

S1



S2



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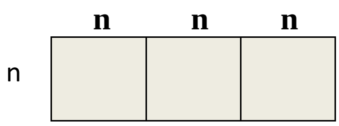

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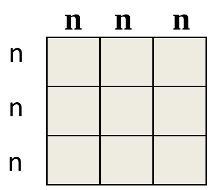

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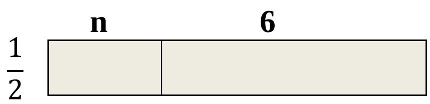

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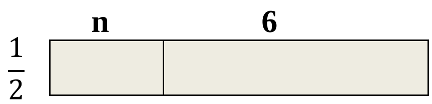

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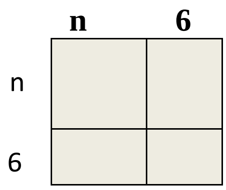

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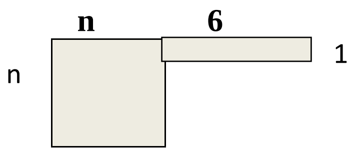


S9

Tables

T1

n	1	2	3	4
The product	14	16	18	20

T2

n	1	2	3	4
The product			81	144

T3

n	1	2	3	4
The product		10	15	22

T4

n	1	2	3	4
The product	3		27	48

T5

n	1	2	3	4
The product			81	100

T6

n	1	2	3	4
The product		10	12	14

T7

n	1	2	3	4
The product		4		5

T8

n	1	2	3	4
The product	6.5	7	7.5	8

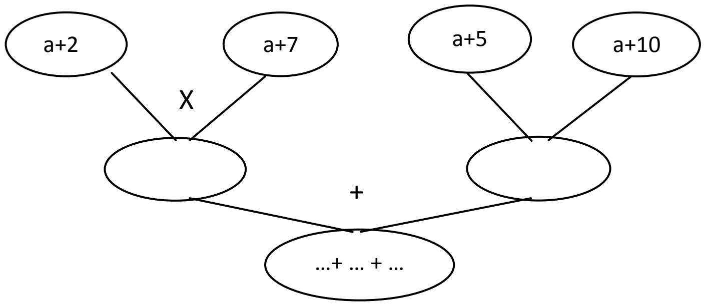
T9

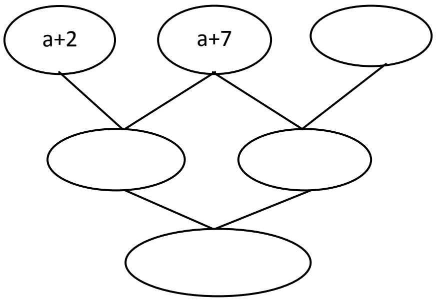
n	1	2	3	4
The product				

Word Statements

F1 Multiply n by two, then add six	F2 Multiply n by three, then square the answer
F3 Add six to n, then multiply by two	F4 Add six to n, then divide by two
F5 Add three to n, then multiply by two	F6 Add six to n, then square the answer
F7 Multiply n by two, then add twelve	F8 Divide by two then add six
F9 Square n, then add six	F10 Square n, then multiply by nine
F11	F12
F13	F14

10.4 Appendix D: Post-Test





x=5

x+2=7

(x+2)•6=42

6(x+2)-20=22

[6(x+2)-20]:2=11

{[6(x+2)-20]:2} -1=10

x=3

2x=6

2x-4=2

2x-4•2=4

2x-8=4

(2x-8):4=1

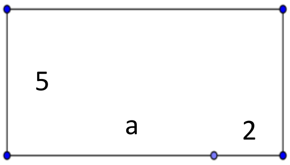
(2x-8):4+3x=1+3x

(2x-8):4+3x•3=3(1+3x)

(2x-8):4+9x+4=3(1+3x+4)

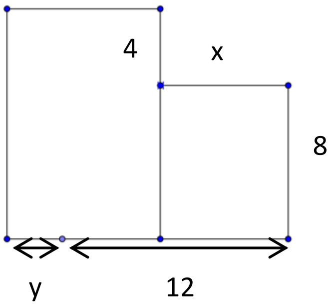
P=

A=



P=

A=



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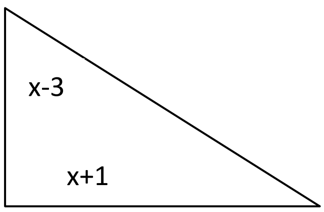

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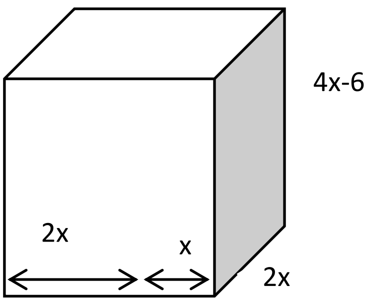

V=

V=

10.5 Appendix E: Fieldwork permission

Dear Pupils!

My name is Dorela Kraja and I am a master student in Agder University in Kristiansand, Norway. I study mathematics education and my supervisor is Claire V. Berg.

During being a pupil, studying mathematics and practicing as student-teacher, I have noticed some of difficulties and misconceptions of pupils, related to algebra and especially to the usage of brackets.

I would like to achieve a general description about the algebraic mistakes that Albanian pupils, in the 10^{th} grade, do.

I want to classify the errors of the pupils and to see the causes of those errors.

I need your help to develop my project (completing tests, answering the questions of interviews and participating in my teaching lessons). And, according to them I am going to write a master thesis and article, but your names will be anonymous.

I can assure you that:

The data collected from your tests, interviews and lessons will be used only for the intentions that I mentioned above.

Your names will be coded in other names and in my papers I will use only coded names (the paper with names and respective codes will be secret and can be used only by me and my supervisor).

In the end of my project, June 2012, I will destroy the whole tests, interviews, videos and the paper with coded names.

If you feel uncomfortable during the process, you are free to back out at any given time and withdraw your consent, and everything yours (the test, the interviews and the video) will be destroyed in your presence.

I agree to make the test ……………………………….

I agree to be interviewed ……………………..

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