UNIVERSITETET I AGDER

Albanian Upper Secondary students' ways

of working with equations

A case study based on task-based interviews

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Preface

It has started as a challenge to encompass mathematics with didactics and it has turned into a reality, which would not be possible without the support of a number of people to whom my gratitude goes.

Warm thanks to my supervisors, Cyril Julie and Hans Erik Borgersen. Their ideas, advices, and support have encouraged me to overpass difficulties, and to achieve my goals. I feel privileged to have had the possibility to work with you.

I was not left alone on my way, along with me has been Maria Luiza Cestari, whom I would like to thank for the guiledlines given through the project of MERG; all professors here in Kristiansand, whom were always available with useful advices, and the community of Albanian teachers and pupils that kindly offered their attention and made possible the data collection. This is not just an end but a first step towards a future with much more colleagues: these are all my friends of Agder University, whom made me feel home in a new country, and my highest appreciation goes for the state of Norway which laid the foundation of my professional career.

My work goes as dedication to my parents for every effort they made to build more then just a professional: a worthy human being.

I deeply wish that you will be proud!

Kristiansand, June 2010

Besara Kadija

Summary

Equations are an important part in algebra in the Albanian school mathematics curriculum. This case study focuses on the way six Albanian students approach and solve equations. The students that took part were in the first year of upper secondary school (10th school year), and they have chosen to have more mathematics than the knowledge required. Students are motivated to work with mathematics, and their mathematics background is approximately the same. They came from different lower secondary schools but the programs that they have followed are the same.

The research question that has directed this research is:

How do the Albanian students in the study approach and solve equations?

I had in focus a class with 34 students who firstly did a test dealing with equations. According to their performance on the test, I divided the students into three groups: high performing, middle performing, and low performing. I have picked two students from each group: one girl and one boy, and I have interviewed them. The method that I have used to interview the students is the task-based interview where the questions are based on the five requests of Newman's technique. The students were presented with four cards, where they had to choose two of them: the most difficult and the easiest, and to solve one of them. The cards contain two equations of these kinds: linear, quadratic, rational, and irrational. On the cards was also a word problem which was the same for all the cards.

During the interview, I have made questions that are related with their reasoning in the solution process, and many times I have also tried to give hints to them. The interviews have lasted for approximately 45 minutes, where some of the time was used to ask some additional tasks, but I have analysed only the part where the students have solved tasks on the cards.

The students in general showed that they knew how to solve equations, even if they lack some parts of the complete solution. They have shown that the word problems (simple ones) are not a problem for them, and the translating from word problem to equation is not a difficult for them. They have demonstrated to have a general knowledge on equations, even if sometimes this knowledge seems to be more of an instrumental understanding than a relational understanding.

The students have shown difficulties in dealing with domain and checking part, because for some of them these are not seen as part of the solution of an equation. They have also shown some difficulties with the quadratic equations, and the main difficulty that the students have shown is the solution of an irrational equation. Most of the students have said that irrational equations are more difficult than the other kinds of equation or they have shown lack of understanding during their solutions.

Sammendrag

Ligninger er en viktig del av læreplanen for matematikk i Albania. Denne casestudien fokuserer på hvordan seks albanske elever nærmer seg og løser ligninger. Elevene som deltok gikk første året på videregående skole (deres tiende skoleår), og de har valgt å ha mer matematikk enn hva som kreves. Elevene er motivert for å jobbe med matematikk og har ganske lik matematisk bakgrunn. De kommer fra forskjellige ungdomsskoler, men har fulgt den samme læreplanen.

Forskningsspørsmålet som har ledet denne forskningen er som følger:

Hvordan tilnærmer og løser de albanske studentene i studien likninger?

Jeg har studert en klasse med 34 elever som først gjennomførte en test med ligninger. Avhengig av deres prestasjoner delte jeg elevene i tre grupper: høyt presterende, middels presterende og lavt presterende. Jeg plukket ut og intervjuet to elever fra hver gruppe: en jente og en gutt. Metoden jeg har brukt for å intervjue elevene er et oppgavebasert intervju hvor spørsmålene er basert på de fem spørsmålene fra Newmans fremgangsmåte. Elevene ble presentert for fire kort, hvor de måtte velge to av dem: de som etter deres oppfatning var det vanskeligste og det enkleste. Videre ble de spurt om å løse ett av dem. Hvert av kortene inneholdt to ligninger innenfor de følgende kategoriene: lineær, kvadratisk, rasjonal og irrasjonal. Kortene hadde også et tekstproblem som var likt på alle kortene.

Underveis i intervjuene stilte jeg spørsmål relatert til elevenes resonnement i løsningsprosessen, og jeg gav dem hint hvis de sto fast. Intervjuene varte i omtrent 45 minutter. Noe av tiden ble brukt til tilleggsoppgaver, men jeg har kun analysert den delen hvor elevene løste oppgaver fra kortene.

Generelt viste elevene at de visste hvordan de skulle løse ligninger, selv når de manglet deler av den fullstendige løsningen. De har vist at de behersker enkle tekstproblemer, og at de ikke har vanskeligheter med å omdanne tekstproblemer til ligninger. De har demonstrert å ha generell kunnskap om ligninger, selv om kunnskapen av og til er mer av instrumentell forståelse enn av relasjonell forståelse.

Elevene har vist vanskeligheter med å bestemme mulighetsområdet for den er ukjente og å sjekke svaret. Dette viser seg hos noen elever å være fordi de ikke ser på dette som en del av det å løse ligninger. De har også vist at de har noe vanskeligheter med kvadratiske ligninger. Det største problemet er å løse irrasjonale ligninger. De fleste av elevene har sagt at irrasjonale ligninger er vanskeligere enn andre ligningstyper og de har vist manglende forståelse i løsningsprosessen, av slike likninger.

Content

1 Introduction

This thesis is a study on how Albanian students approach and work with equations. The research is a case study and the purpose is to understand the mathematical thinking of six Albanian students during their solution of equations.

I have always liked mathematics, especially algebra. This is why my focus was the algebra part. I decided to choose equations since it is a very important part of algebra, and of the Albanian curriculum. The concept of equation starts in the third year of primary school and it develops through the years. The development of this notion through the Albanian curriculum is given in chapter 2.

I have completed a full mathematics teacher education programme of four years at Luigj Gurakuqi University in Albania. I got the opportunity to continue my studies, so I accepted the chance to develop my knowledge in mathematics didactics at the University of Agder, Kristiansand, Norway. During my education through the years, the school system has changed in Albania. There was the old system which I have followed for 10 years, and the new system which is now in operation. In the middle of these systems there was an experimental system which I have followed in the two last years of upper secondary school. In the experimental system the students in the third year of upper secondary school could either follow a science direction or a social science direction. This division I have also found operative in Norway in the first year of upper secondary school, as in Albania in the new school system.

I have always wondered why students in general do not like mathematics or consider it very difficult. I would like to know what pushes students to have these considerations about mathematics. Equations are an important part of algebra. Students often treat the solution of an equation as a mechanical procedure; the only purpose is the finding of the unknown. This is the reason why I have chosen to conduct this study solving equations focusing on students' perspective. I would like to get to know their difficulties, and find the reasons with what they are connected.

The research question that I have posed is:

How do the Albanian students in the study approach and solve equations?

My goal is to see the variety of their approaches and solutions. I am interested in their way of reasoning. There are studies done about the solution of equations by students and the problems that they show during their solutions (Kieran, 2007), and some of the will be presented in chapter 3. My study is a qualitative one based on task-based interviews with the purpose to analyse how students approach the solution of equations and engage with related word problems.

The framework of my study is Kieran's framework (op. cit.), which is based in three activities of school algebra. Since I am interested in the Albanian students' ways of working with equations, Kieran's framework was extended in four students' skills (based in the Albanian curriculum) as the following:

1- The skill to build an equation for a given word problem.

2- The skill to transform the equation into simpler one but saving the equality.

3- The skill to solve the equation completely.

4- The skill to justify (checking) the solution.

The data collection was done in a first year class of an upper secondary school in north of Albania. Based on the new curriculum the students of this class have chosen mathematics as their favourite subject, so they have more hours of mathematics than the general classes. Data were collected from 9th November to 21th December 2010. The class that I observed had 34 students, 11 boys and 23 girls.

The methods that are used in this thesis are: *Classroom observations*, *Test*, and *Task-based interviews*.

Classroom observations lasted for approximately one month and the reasons for these observations were: to get an overview on the students' understanding of equations, to get to know better the students, and them to know me.

Test was done on 4/12/2009 and all the students took part in it. The test has five equations and one word problem. Equations picked for this test are: linear, rational, quadratic and irrational. The purpose of the test was to choose students for the task-based interviews based on their performance.

Tasks-based interview. Based on the definition that Davis (1984) has given for this method, I asked the students to solve equations and in the same time I was posing questions to them. Firstly, the students were given four cards and requested to select two; the most difficult and the easiest card. These cards contain two equations and one word problem. Secondly, I had prepared some additional tasks similar to the test tasks. The structure of the interviews uses Newman's technique (Vaiyavutjamai and Clements, 2006). The questions that I have used during the interviews are:

1- Read the task.

2- How do you think to solve it?

3- What do you need to solve it?

4- Why do you think that these steps are equivalent or which is the reasoning behind these passages?

5- How can you check that the answer is the correct one?

The task-based interviews were all audio-taped and video-taped, and I have analysed only the part where the students have worked with the solution of tasks in cards. The analysis of my data is divided in three parts as follows:

Students' choice of cards, *Students' works with the first and second task of the chosen card*, and *Students' work with the word problem in the chosen card*. The students' reasoning for the choice

of the two cards is presented in the first part. They have also picked one of these cards, and the second part contains the students' work with the first task in the chosen card. The third part focused on students' solution of the word problem, which was the same for all the cards.

I do not intend to make a generalization of my study but only to present six Albanian students' work with equations.

This thesis is separated in eight chapters as the following: *introduction, development of the notion of an equation through the Albanian curriculum, review of the literature*, *methods*, *findings*, *discussion and conclusion*, *pedagogical implications and further research,* and *references*.

Development of the equation notion through the Albanian curriculum gives a short description of the way the equation concept is treated in the first nine years of school. This chapter is divided into eight subchapters, which contains a short summary of how the notion of equation is treated through the years.

Review of the literature contains a description of the main literature that I have used in this research. It is divided into three parts. The first part is about equations. Some studies are presented here and relate directly to the notion of equation and word problem. The second part is about the theoretical framework of this thesis. A general overview of the Kieran's framework is given and then the extended version that I have used in my thesis is presented. The third part is about the task-based interview that I have used to collect my data. I present a general overview of this method and then the way I have used it.

The method chapter contains a description of the school and the class where I have done the data collection. I describe the students, the way they were chosen, the way I have gathered the data, and the way I have analysed them.

The findings chapter is the longest chapter and contains the data analysis and results that I have achieved in this research. It is divided into three main parts as introduced above, following the way the students worked during the task-based interviews, which became my main data.

The discussion and conclusion contain a short discussion of my thesis in relation to other research results. At the end of this part all the results that I have found from my data will be summarized.

The pedagogical implications and further research is the last chapter of my research and contains a short description of what I think about the students' difficulties that have emerge during the analysis, how they can be dealt with, and further research needed.

There is also an appendix chapter which is divided into four parts that are: *the headmaster permission for the data collection*, *the list of the symbols for the transcriptions*, *transcription of six interviews*, and s*tudents' work papers.*

2 Development of the notion of an equation through the Albanian curriculum

In this chapter I will make a short presentation of the development of the notion of equation through the Albanian curriculum. Firstly, I will give a short overview of the Albanian school system, and secondly I will present how the concept develops from the first year of elementary school, until the ninth year of lower secondary school.

The Albanian school system has changed in 1999 - 2000. The old system for many years was:

> 4 years of Elementary school
>
> 4 years of Lower Secondary school
>
> 4 years of Upper Secondary school

In 2000 the system changed into this structure,

> 5 years of Elementary school
>
> 4 years of Lower Secondary school
>
> 3 years of Upper Secondary school

With the change in the structure of the school system, the Albanian mathematics curriculum has also changed, especially the curriculum of upper secondary school. The curriculum of upper secondary school for many years has been the same for all the students, and then around 1999 an experimental curriculum was introduced. It was applied only in some upper secondary schools of the country. However, everything has been abolished, because in 2000, the Minister of Education decided to introduce another system that starts from the first year of the Elementary school. This new mathematics curriculum includes four to five mathematics books from which one has to be chosen for a school. These books are chosen by the teachers of the school. So some upper secondary schools have different mathematics books. Also the students have to choose if they want to have extra mathematics or not. In the classes where the students have extra mathematics, the knowledge that they get is divided in two parts: it is the main book that is for all the students of the school and the advanced book that is for some of the students, who have chosen to do more mathematics.

The books that I have used to write this part of my thesis are one each grade for grade 1 to grade 9. Since there are many textbooks available I have chosen to present one of them for each grade.

2.1 The first and the second year

The books that I have chosen for the first and the second year are: Matematika 1 (Dedej, Koçi, Spahiu and Konçi, 2008) and Matematika 2 (Dedej, Spahiu and Konçi, 2009).

The notion of equation starts in the first year of elementary school. But the notion is not yet the exact definition of equation; it is closely related to the notion of numerical equivalence (barazim numerik).

During the first year the students mostly do arithmetic. So they work more with numbers and calculations with these numbers. They deal with natural numbers not bigger than 100, but the concept of the natural number is not yet settled. They start to get the concept of adding and subtracting two given numbers and they start to deal with zero.

During the second year the students work with natural numbers not bigger than 1000. They learn how to multiply and divide two given numbers. They even start to deal with fractions but still the concept of natural number is not given.

I will describe parts of how the students in the first and second year of elementary school work with equations.

Task: Find the missing number(s):

These tasks require from the students to find the number in the missing places to establish a numerical equivalence. The first tasks have the same logic. They are simple tasks where the students have to find the sum between two numbers $1 + 2 = \underline{\quad}$ (Dedej, Koçi, Spahiu and Konçi, 2008, p. 20), $4 + 0 = \underline{\quad}$ (op. cit., p. 25). There are tasks where to find the missing addend in the numerical equivalence $1 + \square = 4$ (op. cit., p. 21). Another task is related with the third one but with a distinction, because the sum is in the left of the equal sign $5 = 4 + \underline{\quad}$ (op. cit., p. 25), so the students have to understand the symmetry property of the equal sign. The next tasks differ from the ones above because the numerical equivalence is given by two unknowns that in this case are the addends of the equivalence. Both of the examples are of the same type but the unknowns are signed with different figures. $5 = \underline{\quad} + \underline{\quad}$ (op. cit., p. 25) and $6 = \triangle + \triangledown$ (Dedej, Spahiu and Koçi, 2009, p. 13), and the question is "Perform the actions". So the students have to find the numbers in the missing places, but in this case there will be a lot of solutions, for example for the first: 0+5, 1+4, 2+3, and for the second: 0+6, 1+5, 2+4, 3+3. So, these tasks request to find the unknowns where we can have multiple solutions for both of the tasks. The example below is a numerical equivalence which helps the students to make the difference between the addition and subtraction. The students have to find the two elements that need to complete this equivalence: $6 - \triangle = \triangledown$ (op. cit., p. 13), so the solutions for this task can be: 0 and 6, 1 and 5, 2 and 4, 3 and 3.

The three equalities above have multiple choices. The task below is multiple choices too, but it has three addends. In this way the students learn how a number can be divided into three numbers: $7 = 4 + \underline{\quad} + \underline{\quad}$ (op. cit., p. 13) and $3 + 4 + 0 = \underline{\quad}$ (op. cit., p. 13). These tasks involve also the sum with the zero. They have to know that the sum of a number with zero, remains the same, as the example below (op. cit.):

$$6 = 6 + \_$$
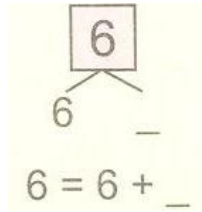
The example above gives a very easy way to understand that a number, which in this case is 6, can be divided with itself and with the zero.

The next example treats how a number can be divided into three addends (op. cit.):

$$7 = 2 + \_ + 1$$
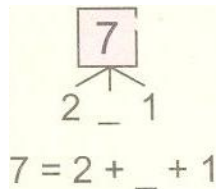
The students, in this task have to find the unknown by finding how 7 can be divided into three numbers, so 1, 2, and 4. They have to calculate that the sum of the two numbers in the right is 1+2=3, and then they have to find that 7-3=4, which is the answer of this equation.

The last task explains the use of number line. The first addend of the equality is two, which is represented with the first doted arrow in the figure. Now the students have to find the other addend and the sum of these two numbers.
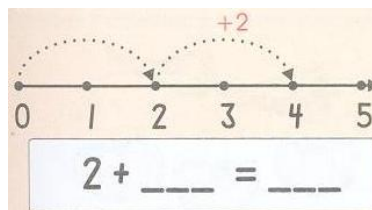
(Dedej,Konçi, Spahiu and Koçi, 2008, p. 25)The students have to find that the first addend is 2, and the second one is also 2, and their sum gives the number 4.

2.2 The third year

In the third year the students start the notion of equation and it is explained by comparing the numerical equivalence as $52 - 24 = 28$ (Starja and Shkoza, 2009, p. 137) with the equation $18 + \square = 29$ (op. cit.). So in this year the notion of equation is given as numerical equivalence with one unknown; and the students have to find it. The difference between these two notions is with two directions,

1- We have the unknown which is denoted by a square or a triangle.

2- The numerical equivalence is always true, but for the equation you have to find the value of the unknown to make it true. So, with this logic we can say that equation is a general form of the numerical equivalence.

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In the third year there are no more than two sections on equations, and it is asked only to find the unknown or if it is given a set of numbers and the students have to find which of the numbers fit in the equation by trying all of the numbers, as it is given below (op. cit.):

$$B = \{6, 13, 25, 32\}$$

$$35 - \square = 10$$

$$\boxed{}$$

This task consists in finding the missing number from the B set. We can name B the domain (mjedisi), because it is asked to find the unknown in this set of numbers. In this way the students have to try each of the numbers that are in B set and to see which fits and write it in the big rectangle.

The students also have to solve few word problems that are related to equations. An example about a word problem can be as it follows (op. cit., p. 137):

Problem 7. Blerta has a book with 230 pages, and she wants to read it in one week. She reads 30 pages per day. Can she finish reading the book in one week? How many pages she will have left in the end?

This word problem has two operations: multiplication which is related with the first question, and subtraction which is related with the second question.

2.3 The fourth year

During the fourth year the students get the definition of the natural numbers as below: "The set of consecutive numbers which starts from 1, it is named with $\mathbb{N}$."(Starja and Shkoza, 2009, p. 14) So the set of the natural numbers is $\mathbb{N} = \{1,2,3,4,5,6,\ldots\}$ and they deal with numbers not bigger than 10000. They also start to learn about the set of integers, which they name with $\mathbb{Z}$, and it is given as a union between the natural set, the set of negative numbers (numbers that are the opposite of the natural numbers) and zero.

In the fourth year the notion of equation is given as in the third year by making a comparison with the equivalence notion, but the only difference is the unknown which the students have to find. Here we have an emphasis between the differences when we have addition, subtraction, multiplication and division and all parts of the equation are given names. For example for the sum and its relations with addition and subtraction it is given the scheme below (Starja and Shkoza, 2009, p. 143):
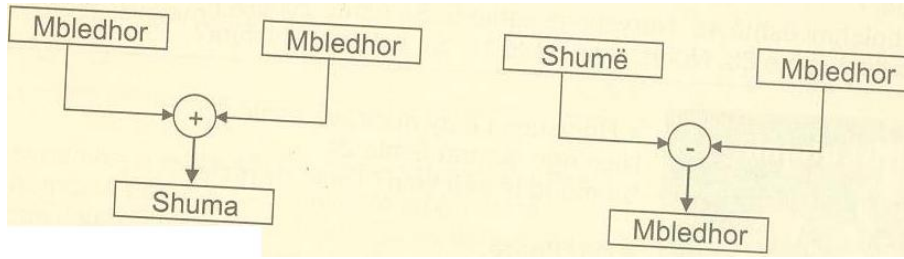
where sum=shuma; addend= mbledhor

2.4 The fifth year

During this year the concepts of the natural numbers and integers are given more complete. The goal is to solve equations in a given set of natural numbers. The definition of equation in the textbook is given as follows: "*Equation is called the numerical equation that contains unknown.*" (Starja and Shkoza, 2009, p. 99) And the definition for the solution of an equation is "The value of the unknown that turns the equation into a real numerical equivalence is called solution of the equation." (op. cit.)

The fifth year comprises more detailed the way of working with the equations because here the students learn even to manipulate in the opposite side and to justify the result. This way of working is given by schemes as below (op. cit., p. 100):
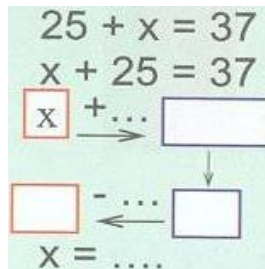
The task, which is given by this scheme, is a good one for the students to understand the opposite action of equality. The students have to follow the arrows; the first arrow means the sum of the unknown with the addend 25 so; they have to write 25+x, the second arrow means the sum which is equal with 37 and we write it in the big box. The small box has the number 37, and the last arrow means that from the sum that is 37, we take away the number 25, we get x which in this case is 12. The scheme of how the students have to fulfil is given below:

So x=12.

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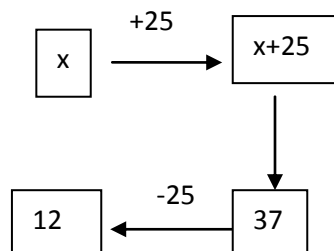
The students have to follow the arrows that are in the figure above. This tells to the students that the expression x+25 is equal with the number 37, and the expression 37-25 it gives the value of x, which in this case is 12. All these passages have to be with a reason behind, which the students have to write about. Also the students have to check it by this scheme (op. cit.):

$$25 + \ldots = 37$$
$$\ldots = \ldots$$

to see if the equation is solved correctly or not. The students have to consider this as part of the solution of the equation. This check consists of putting the number found above, which in this case is 12, at the dotted line and then show that 25+12 is equal with 37 in the last row or:

25+12=37

37=37

This means that the number x=12, is the solution of the equation: x+25=37. They have to know that; if they get in the end to equal numbers it means that they have found the right solution, otherwise their solution is wrong.

Here the definition of equation stays the same with that of the years before but now the authors has stressed it more.

2.5 The sixth year

There is nothing valuable about set of numbers to be mentioned. The students work with those concepts that they already know from the early years.

During this year the difference between three main types of equivalences is stressed (Perdhiku, 2007, p. 122):

We have two expressions A(x) and B(x), where x is the unknown in these expressions.

1- *If the equivalence A(x)=B(x) between these two expressions is true for some values of x (unknown), it is called an equation (ekuacion).*

   Examples: 5x-2=x or 3x-5(x-5)=6

2- *If the equivalence A(x)=B(x) between these two expressions is true for all the values of x (unknown), it is called identity (identitet).*

   Examples: $4x\text{-}6=\dfrac{1}{2}(8x\text{-}12)$ *or* $x\text{-}3=\dfrac{1}{3}(3x\text{-}9)$

3- *The equivalence A(x)=B(x) between these two expressions is called a numerical equivalence (barazim numerik).*

   Examples: 5+6=11-5+2+3 or $\dfrac{2}{3}+\dfrac{4}{3}=2$

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The last definition is connected with the equality between the expressions of the both sides of the equal sign. This is closely related with what the students have done in the earlier years, but now they notice this better. Here the students understand, with this sharing, that the numerical equivalence is always true (see Third Year).

The sixth year starts with the exact definition of equation. We have the differences between an equation, an identity, and numerical equivalence. The students start to work with two expressions on both sides of the equal sign. They learn the explicit rules of manipulating an equation, for example how the sign changes when you pass an addend to the other side of an equation.
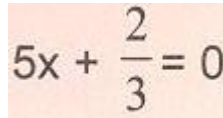
Now it comes out the simplest kind of equation: ax+b=0 (Perdhiku, 2007, p. 124), where all the equation in the form:

Expression= Expression are turned into this form. Some of the tasks that the students have to solve are referring to the general form ax+b=0 (op. cit., p. 125):

1- Write the equation when you have given a and b:

a=-2 and b=-3

2- There is given the equation below, find a and b:

$$5x + \frac{2}{3} = 0$$

At the end of these two sessions the students have to solve many equations of this form and few word problems.

2.6 The seventh year

During this year the students work with the set of the numbers that they already know from the earlier years.

In the seventh year the concept of equation is given in a more theoretical way. Here the rules are stressed and the equations treated in these sessions are the linear ones and the quadratic equations of the form: $x^2+b=0$. It is discussed how we can make equivalent passages, emphasizing them with the appropriate rules, and the problem when two equations are logically equivalent.

Equivalent passages are the reductions (transformations) that an equation has to be submitted with the purpose to make it easier and to save its equivalence during these steps.

Making equivalent passages in both of the sides of an equation, we get a new equation, equivalent with the given one. (Kopliku, 2007, p. 190)

Here are many different equations if we compare with the years before. The amount of the tasks is not many but the way of treating them is quite different because all the passages are justified by the author during a solution of an equation as below (op. cit., p. 192):

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Zgjidhje 3(2-x)-5=7-4(1+0,75x)

Heqim kllapat duke pasur parasysh ligjin e përdasisë. Kujdes duhet të kemi kur para kllapës kemi shenjën – (duhet të ndryshojmë shenjat)

⇔ 6-3x-5=7-4-3x

Tani kalojmë kufizat me ndryshore në njërën anë dhe numrat në anën tjetër

⇔3x-3x=7-4-6+5

Bëjmë reduktimin e kufizave të ngjashme

⇔ 0x=2

Ekuacion i fundit 0x=2 nuk ka zgjidhje pasi nuk mund të ketë ndonjë numër që duke u shumëzuar me 0 të dalë një numër i ndryshëm nga zero. Kështu që edhe ekuacioni i dhënë nuk ka zgjidhje.

Translation: "Solution of equation: 3(2-x)-5=7-4(1+0.75x). We take away the brackets using the rules. We have to be attentive when we have the minus before the brackets (we have to change the signs)

⇔ 6-3x-5=7-4-3x. Now we pass the terms with the unknown in one side and the numbers in the other side

⇔3x-3x=7-4-6+5. We make the reduction of the similar terms and we have

⇔0x=2

The last equation 0x=2 has no solution since we cannot find any number that we can multiply with 0 and get a number different from zero. So the given equation has no solution." The students have to explain this solution steps in the same way.

 Some of the tasks that are given in the textbook are as below (Kopliku, 2007, p. 194):

1- How many solutions do the equations below have:

a) 2(0,5x-3)-3(x-1)=7-(1-x)·2 b) 3(2-x)-5=7-4(1+0,75x) c) 12-3(x-6)=2(1-1,5x)+28

2- Solve the equations and justify the results:

$$\frac{2-3a}{5}=4 \qquad 2-\frac{m-3}{9}=\frac{2-3m}{3} \qquad \frac{7-x}{2}+\frac{x+5}{3}-\frac{1}{3}x=\frac{31-5x}{6}-\frac{1}{3}x$$

2.7 The eighth year

The textbook in the eighth year treats a review of the definition of natural numbers and integers but also gives the definition of rational numbers as a union between integers set and fractions and it is named ℚ (Polovina, Gjoka, Kovaçi, 2007, p. 29).

In the eighth year the equations are developed in three chapters which are (op. cit.):

1- Equations and inequalities with one unknown (p.83-93). (These are linear equations and inequalities)

2- Systems of equations of the first degree with two unknowns. (p. 94-105)

3- Quadratic equations with one unknown. (p. 106-113).

I will treat the first and the third chapter because they are related with what I am studying:

The first chapter is very close to the seventh year. Linear equations of the form ax+b=0 are treated (Polovina, Gjoka, Kovaçi, 2007, p. 83). Equations, which are built on the equality of two expressions A(x) =B(x) or A(x) B(x) =0 or $\frac{A(x)}{B(x)}$=b or $\frac{A(x)}{B(x)}$=C(x) such that B(x) $\neq$ 0 and, where A(x), B(x) and C(x) are expressions related to the unknown x, can be turned into the form ax+b=0. This chapter contains many types of equations, and many word problems are treated.

The third chapter is more advanced because there equations of the form ax^2+bx+c=0, ax^2+bx=0 and ax^2+c=0 are treated, and many equations that are turned into these forms.

The second kinds of equations are solved by the discriminator or from Vieta's formulas. This is the first time that the students start to name equations and define the difference between the kinds of equations. The tasks that are most common in this chapter ask to find the solution of the quadratic equations and some word problems of the type:

"Find the two consecutive even numbers, product of which is 8."

2.8 The ninth year

This is the last year of lower secondary school, so the level of the information that the students get is higher compared with the other years. In this year the students do a repetition of the sets of numbers that they have learned and also start with the set of irrational numbers and real numbers. But firstly they deal with the precise definition of the rational numbers which is: "All the numbers of the form $\frac{m}{n}$ where m$\in \mathbb{Z}$ and n$\in \mathbb{N}$, are named rational numbers" (Lulja and Babamusta, 2008, p. 9). These numbers can be represented as finite or infinite periodic decimal numbers.

Irrational numbers are presented with a simple problem: If we have a square with the side 1 and we want to find its diagonal, the result that we have is the square root of 2, ($\sqrt{2}$). This number is an infinite no periodic number, so numbers like it are called irrational numbers and are named I (op. cit., p. 11).

The set of the real numbers is given as a union between the rational numbers and irrational numbers and it is named $\mathbb{R}$ (op. cit., p. 13).

The students learn some of the passages that are not logical equivalent. There are given some examples which are:

1- Equation x(x-2) =x is not equivalent with $\frac{x(x-2)}{x} = \frac{x}{x}$ (Lulja and Babamusta, 2008, p. 87), because the first one has as a root the number zero, and the second one has not. This

is related with the domain of these equations. For the first equation the domain is all the real numbers, and for the second one are all the real numbers except zero, which is the solution of the first one.

2- Equation (x-2) =1 it is not equivalent with $(x-2)^2=1^2$ (Lulja and Babamusta, 2008, p. 88), because the first equation has as a solution the number 3, and the second one has two solutions, number 1 and 3. Based on what we have called equivalent equations, "The equations that have the same set of solutions" (Kopliku, 2007, p. 190), these equations are not equivalent.

These are some special cases of the passages, that look as equivalent passages but they are not. Equivalent passages (see Seventh Year) are the one that transform one equation into and equivalent one, so they have the same set of solutions.

The students learn that there are some conditions which are called domain (bashkesi përcaktimi) of a function that is a set of numbers where the equation has meaning. They treat again the quadratic equation as in the eighth year, but something more: the students start to work with equations of the form: f(x) g(x)=0 (Lulja and Babamusta, 2008, p. 95). They have to know to solve these equations and even to turn some equations into this form. The students in this year also treat irrational equations, $(x^2-1)\sqrt{x}=0$, and $(2x^2-5x+3)\sqrt{x-5}=0$ (op. cit., p. 96) but they do not name them yet.

This chapter will serve as a background for the students' knowledge about equations. I will use the chapter during my analysis and especially in chapter 6, where I will compare what the students have learned during the years with what they know now.

3 Review of literature

This chapter deals with the theory which sustains my thesis, and is divided into three parts: equations, theoretical framework, and task-based interviews. The first part has an overview of research on equations and word problems. The second part, which is the theoretical framework, contains an overview of the framework that I have used in my thesis. It is presented a short review, and then I explain the theoretical framework in my perspective. The third part is related to the method chapter and I have presented my method of data collection, which is the task-based interview.

3.1 Equations

This section deals with the main theoretical parts of my paper. It is divided into three parts. Firstly, a short introduction about the history of equations through the years is given. Secondly, some research on equations, and thirdly a short part about word problems and their translations to equations are presented.

3.1.1 History of research on equations

Equations are an important part of algebra. They have been developed through the years together with algebra. Carolyn Kieran (2006) analysed all the PME researchers done on the topic from 1977 until 2006. She claims that in the beginning the researches dealt more with the transition of arithmetic in algebra, the misconceptions and the difficulties that the students have shown. They were also focused more in algebraic concepts, procedures, and word problems. During this time the researchers developed many theoretical frameworks to analyse their data in learning / teaching algebra. These changes also resulted due to the development of technology. According to Kieran (2006) the use of technology started from mid-1980s to 2006. The focus was mostly on the multiple representations and the use of the new technology to develop understanding. Then, mid-1990s to 2006 the studies were based on the algebraic thinking of the students, the ways how algebra were taught in the classes especially in the elementary school, and the students' understanding in dynamic algebra environments. Table 1 gives the major themes developed through the years in the PME conference proceedings (Kieran, 2006, p. 12). I present a short overview of each of the periods and research findings that are relevant for my study.

During the first period the following topics were treated (op. cit.):

- *Interpreting algebraic signs, unknowns, and variables.*

As pointed out by Kieran (op. cit), the concepts used in arithmetic are the same as the ones in algebra, with the difference that in algebra these concepts need many conceptual adjustments to make this shift.

> *Early research on the ways in which students interpret algebraic symbols tended to focus on cognitive levels (Kücheman, 1981), prior mathematics experience and methods of thinking (Booth, 1984), and difficulty with notation such as the equal sign with its multiple meanings (Kieran, 1981;Vergnaud, 1984) and the use of brackets (Kieran, 1979). (op. cit, p. 13)*

Time Period	Theme-Groups that emerged
1977 to 2006	Transition from arithmetic to algebra, variables and unknowns, equations and equation solving, and algebra word problems.
Mid-1980s to 2006	Use of technological tools and a focus on multiple representations and generalization.
Mid-1990s to 2006	Algebraic thinking among elementary school students, a focus on the algebra teacher/teaching, and dynamic modelling of physical situations and other dynamic algebra environments.

Table 1. Major themes that have emerged over the 30-year history of PME algebra research from 1977 to 2006.

There are also some studies done about the symmetric and transitive character of equality (Vergnaud, 1988 from Kieran, 2006) and the use of these patterns in the solution of equation (Herskovics and Kieran, 1980 from Kieran, 2006). Another difficulty which is seen in the students' transition from arithmetic to algebra is related to the unknown. As claimed by Kieran (2006), this concept has to develop from label to unknowns and variables, and later as parameters. Some studies focused on older students and it was found that only a small number of students could adequately described the differences between these concepts (Furinghetti and Paola, 1994, from Kieran, 2006). From this we can see that even if students have taken the knowledge of the basis of algebra they still have problems in understanding the notions of unknown, variables and parameters, and make the difference between them.

- *Working with expressions, equations, and equations solving.*

According to Kieran (2006), the early PME research found that students have difficulty in interpreting expressions such as $a+b$, both as process and name/object. It was suggested by Herscovics and Kieran (1980, from Kieran, 2006) that students could easier construct meaning for the equations than for expressions. The same result was also achieved by Wagner, Rachlin and Jensen (1984, from Kieran, 2006) who noted that students tried to add $"=0"$ to any expression that they were asked to simplify. These difficulties that the students have shown are also related to the considerations that the students have towards the equations and expressions, this is one the results that was found by Kieran (1989, from Kieran, 2006).

- *Solving algebra word problems.*

Another difficulty which is found by the researchers is the solution of the word problems with the help of the equations. This part will be also treated in a separate section, but here I will present a short overview in the historical perspective. Mostly the ways of solving these word problems are done in an intuitive way (Kieran, 2006). This is related to the fact that arithmetic is more seen as a procedure, and when the students pass from arithmetic to algebra, they continue with that way of reasoning. According to Kieran (2006), the students are mostly focused on the operations of the equation that they have built to solve the word problem than to the relations that are given from the problem. This transition does not need only a different way of thinking of the students towards the problem but also a way to solve the equation saving the equivalence between the passages. Filloy and Rojano (1989, from Kieran, 2006) found that a didactical cut occurs between

the linear equations of the type: $ax+b=c$ and the linear equations of the type: $ax+b=cx+d$. The first type of equations can be solved by arithmetic methods and the second type by formal algebraic methods. They have also tried to use concrete materials for the solving the equations, but they arrived at the result that it does not increase significantly the students' abilities to operate formally with equations of the second type. Other researches (see Theoretical Framework section) have shown that students mostly like to solve the word problems in the arithmetic way than in the algebraic way.

The topics discussed in the second time period are related with the use of technological tools in algebra learning and focused on the multiple representations and generalization. The use of the technology started in the mid-1980s. The topics discussed in this part related to my research are the followings (Kieran, 2006):

- *Algebra as generalization activity*

- *Word problems and multiple representations*

The topics treated in this part are related with the way how the introduction of the technological tools influences the students' understanding of algebra. Since I do not focus on technological tools I will not go deeper in these sections. The development of the technological tools related to algebra, made possible the development of other new theoretical perspectives to observe better the students' understanding. I will focus on the results that are related with the expressions, equal sign, equations, and word problems even if in this part the equation notion is closely related with the concept of the function and its graph.

Researches on the word problems in connection to the technological tools were also conducted. These researches focused on the solving of word problems by using multiple representations. According to Huntley, Rasmussen, Villarubi, Sangtong and Fey (2000, from Kieran, 2006) for the students which were exposed to word problems and could use context clues and used the graphing calculators they performed better than the students which were required to formulate and interpret the situation. As claimed by Yerushalmy (2000, from Kieran, 2006) the representation of a given problem situation evolved (p. 23): "… *from numbers as the only means of modeling, to intensive work with graphs and tables, to the use of more symbolic representations.*"

Researches were also done about the technological tools and the students' understanding of the equivalence. An example for such study is the one made by Ball, Pierce and Stacey (2003, from Kieran, 2006) which concluded that the recognition of the equivalence is an important difficulty for students. According to them,

> *The ability to recognize equivalent algebraic expressions quickly and confidently is important for doing mathematics in an intelligent partnership with computer algebra; it is also a key aspect of algebraic expectation, the algebraic skill that parallels numeric expectation. (p. 23)*

As we can see from what is presented above, the main focus of this period has been the technological tools and the relation of the students' understanding of algebra.

The topics discussed in the third time period are related with the algebraic thinking of the students in elementary school which are focused in the teacher/teaching, and dynamic modelling of physical situations and other dynamic algebra environments. The topics, related to my research, discussed in this part are the following (Kieran, 2006):

- *Algebraic thinking among elementary school students*

- *Dynamic modelling of physical situations and other dynamic algebra environments*

In this third period, many of the research done previously were further developed, and also a new theoretical perspective evolved. This theoretical perspective take also into account the gestures, bodily movement, and language (op. cit.).

The first topic of this part is related with the research done for elementary school in algebra. From PME studies, Kieran (2006) claim that the students in beginning high school algebra are more focused on calculating rather than in relational aspects of the operations. This is related with the concept that they have about the equal sign. According to Kieran they consider the equal sign as a signal to compute an answer. This is related with the gap which is form from the transition between arithmetic and algebra. The studies done by the PME community include (op. cit., p.26): *"Relational thinking about numeric equalities, symbolizing relationships among quantities, working with equations, developing functional thinking, and fostering an understanding of mathematical properties."*

There are many studies that are done in relation to these topics but I will mention only the one that is related to working with equations. The findings by Schliemann, Carraher, Brizuela, Earnest, Goodrow, Lara-Roth and Peled (2003, from Kieran, 2006) showed that the students of 9 and 10 years old are: *"able to develop an enlarged sense of the equality sign, represent unknown quantities with a letter, represent relations with variables, work with unknowns, write equations, and even solve letter-symbolic linear equations."* (p. 26).

But other studies such as the one from Warren (2003, from Kieran, 2006) arrived at the result that the students 8 and 9 year-olds have difficulty in handling problems with unknowns.

3.1.2 Research on equations

The literature used in this research is related to equations in general. It has been very difficult to find a survey which is similar to mine. According to Kieran (2007) there are many researchers that deal with linear equations but not with quadratic equations. Most of the researches that I have found are related with the understanding of the equal sign. They claim that students' understanding of the equal sign is inadequate especially when the students are at an early age. The equal sign is interpreted by the students as an operation, not a relational symbol (McNeil, Grandau, Knuth, Alibali, Stephens, Hattikudur and Krill, 2006). But I could find some researches that have equations as an object.

I could find two articles that are related to each other. There are two surveys from Vaiyavutjamai, Ellerton and Clements (2005) and the second one is from Vaiyavutjamai and Clements (2006) where the first research is done for quadratic equations and the second one is done for linear equations and inequalities.

The goal for the first study dealt with students having to solve two elementary quadratic equations. This is a quantitative study, in which 231 students from six year 9 classes in two secondary schools in Thailand, 205 year 10 students attending the secondary school in Brunei Darussalam, and 29 second-year students attending a mid-Western university in the United States participated. All of these groups have had classes on quadratic equations. The first group have had 11 lessons of quadratic equations, the second group 10 lessons of quadratic equations and the third group has studied quadratic equations on the early years.

There are few researches that are done on the quadratic equations as claimed by Vaiyavutjamai et al. (2005), and students' thinking during the solution of these type of equations is focused more on the procedure, which many times do not guarantee that the relational understanding between the steps of the procedure is achieved. According to Lim (2000, from Vaiyavutjamai et al. 2005), this is also related with what teachers believe is more important, the algebraic manipulation skills. This is the reason that many "students acquire procedural skills without comprehending what they are doing" (Vaiyavutjamai, 2004a from Vaiyavutjamai et al.2005) or as used by Skemp (1976, from Vaiyavutjamai et al., 2005) they acquire "instrumental understanding".

All the 465 students were posed to similar equations of the type:

$x^2 = K(K \succ 0)$ and $(x-a)(x-b)=0$ where a and b can be real numbers or more specifically they were posed to solve these two equations:

$x^2 = 9$ and $(x-3)(x-5)=0$ on a paper-and-pencil test and they were asked to check any solution that they have obtained for the second equation. Some of the students from Thailand and Brunei were also asked in a post-interview teaching to explain the way how they solved the equation $(x-3)(x-5)=0$ and the US students, after they have solved the equations, they were also posed a series of true/false questions to ascertain the way how they have approached to the solution of second equation.

The difficulty that the students have shown during the solution of the first task $x^2 = 9$ is related with the fact that the students did not know that this equation has two solutions. The majority of the Thai and Bruneian students have committed this error, and 12 of 29 US students. According to Vaiyavutjamai et al. (2005), this error is also related with the misconception that students have about the square root of a number.

The results that were taken from the solution of the second equation: $(x-3)(x-5)=0$ showed that Thai students were better in solving this equation than the Bruneian students. The Bruneian students expanded the brackets of this equation. The same tendency was also seen in some of the Thai students. 11 of the 29 university students in US followed the same procedure as mentioned above.

These were the results that the researchers got from the paper-and-pencil test of the students. But the some of the students were also interviewed. There were 18 Thai students which were interviewed, three students (a high-performer, a medium-performer, and a low-performer) for each of the six classes. The researchers found that students have serious misconceptions about the

solution of quadratic equations. During the solution of $(x-3)(x-5)=0$ the students, even if they have solved it correct, they said the unknown in this equation have different values. This means that the value of x in the first brackets has a different value from the x in the second bracket. This was also sustained when they did the checking of the solution. They substituted $x=3$ into $(x-3)$ and $x=5$ into $(x-5)$ and achieved that $0 \cdot 0 = 0$, so their solution was correct. Some of them seem not to have any idea of how to solve this equation, or what the instruction "solve the equation" actually meant. The problem was because they did not realise that the equation have two solutions.

The Bruneian students were more confused about the quadratic equation $(x-3)(x-5)=0$ than the Thai students. Also the US students showed confusion during the solution of this task. They were posed to a True/False statement interview, and many of them were confused about the concept of the unknown in the given equation context.

Vaiyavutjamai et al. (2005), according to this study concluded that it is needed other researches on quadratic equations since they are an important part of algebra, and an important part of school curriculum in all the countries. There is need to increase the understanding of the students regarding this topic.

The second study was done by Vaiyavutjamai and Clements (2006) about the students' understanding of linear equations and inequalities. There were 231 students from two upper secondary schools in Thailand who took part in this survey. The students were taken from 6 classes of 9 grades (three classes for each school). They were given a pre- and post-test, retention, and 18 of these students were chosen to interviews.

> The aims of the study were: (a) to quantify the extent to which traditional lessons in linear equations and linear inequalities generated lasting improvement in student performance on standard tasks involving linear equations and inequalities and (b) to explore how the lessons affected student knowledge, skills, concepts and understanding with respect to linear equations and inequalities. (Vaiyavutjamai and Clements, 2006, p. 118)

The authors believed that by comparing students' test performance, and what students answered in the interview one by one, they could understand better the students' level of understanding of equations and inequalities. The students were posed to two tests: Test 1 (Language of equations and inequalities test) and Test 2 (Linear equations and inequalities test). The results of these tests were evaluated by Cronbach alpha reliabilities, which are respectively 0.80 and 0.92. Each of the tests contains 27 tasks. For the pre and post-test, and retention the evaluation of the papers of the second test was done by grading the way 231 students answered the tasks. This research is done for both equations and inequalities. The results that were achieved for the equations are given from the figure below (Vaiyavujamai and Clements, 2006, p. 120):

Solve the equation:

- Q1: $2x = -6$
- Q2: $x/3 = 3/2$
- Q3: $2/x = 3$
- Q4: $2x + 4 = 20$
- Q5: $6 - 3x = 12$
- Q6: $x/3 - 1/3 = 4$
- Q7: $3 - x/2 = 8 + 5x$
- Q8: $5x - (3x - 1) = 23$
- Q9: $4(2x + 1) = 17 - 2x$
- Q10: $2(x + 1) = 3 - (1 - 2x)$
- Q11: $3 - 2x = 2 - (1 + 2x)$
- Q19: $x - x - x - x = 4$
- Q20: $1/3(x + 2) = 1/2$
- Q21: $(1 + x)/3 = 3/2$
- Q22: $(2x - 1)/2 = (3x + 1)/6$
- Q23: $(2x - 3)/4 + x/2 = (2x + 1)/4$

Legend: ■ Pre-Teaching ▨ Post-Teaching □ Retention
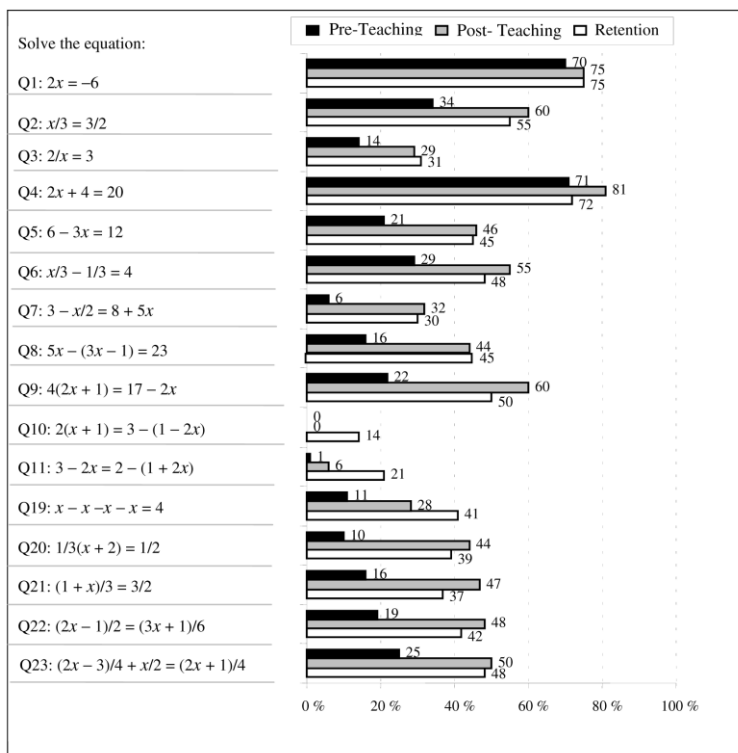
Figure 1. Pre- and postteaching, and retention percentages correct for the linear equation tasks on the Linear Equations and Inequalities Test.

As we can see from the figure above, the students have increased their way of working with equations.

I cannot use Cronbach alpha reliabilities in my study because firstly I will have only 30 students for the test, and secondly, I do not want to compare the progression of the students from pre-teaching to post-teaching but to analyse the way that Albanian students approach and solve equations in a qualitative study.

From this article I have taken the idea of interview; the Newman's technique of interviewing (Vaiyavutjamai and Clements, 2006). They use five steps that are included in that way of interviewing, that consists in five key requests for the interviewees that will be presented in the task-based interview section.

The main sources for the data for this study are 36 transcripts of interviews that were conducted with 18 student interviewees, two interviews for each student. These students were chosen based on their performance. The students were selected from each class, based on their high, middle and low scores in both of the pre-teaching tests (Test 1 and Test 2). The questions of the Newman's technique are related closely to error classification, which is related with: reading, comprehension, transformation, process skills and encoding.
Vaiyavutjamai and Clements (2006) have measured the understanding of students during the interviews by using the Battista's levels that link knowledge, skills and concepts. The rating of

the understanding of the students during the interview is done by using 5 points scores as given below (Vaiyavutjamai and Clements, 2006, p.127):

> *0: the student cannot proceed to apply procedures because he/she does not comprehend the meanings of the symbols, both individually and collectively, that appear in the statement of the equation.*
> *1: the student has some idea of the meanings of the symbols used, and is able to transform the task by choosing appropriate mathematical procedures, but is not able to apply those procedures correctly, and cannot explain the meaning of any answer obtained in relation to the original task.*
> *2: the student not only identifies an appropriate procedure for tackling the task but also applies the procedure accurately. He/she is able to encode his/her answer in an appropriate way, and has some idea of the meaning of the answer obtained in relation to the original task. However, he/she is not aware of the mathematical principles that underlie the procedures used when responding to the task and not able to check whether his/her answer is correct.*
> *3: the student can apply an appropriate procedure, or set of procedures, has some of the awareness of the mathematical principles that underlie the algorithms, and can explain how the answer relates to the original task. However, he/she is either not able to check whether his/her answer is correct.*
> *4: the student can apply an appropriate procedure accurately, has some awareness of the mathematical principles that underlie the approach adopted, and knows how the answer obtained relates to the original task. He/she can check the answer and can link the various representations of the answer(s)-written, verbal and symbolic-to each other and to the original task.*

Vaiyavutjamai and Clements (2006) concluded that students have increased the number of correct answers in the post-test but from the interview they saw that the increase of correct answers is due to an instrumental understanding. The high-performing interviewees were the one that tended to improve more their understanding. Another difficulty which emerged was also the misconception that the students have about inequalities. They treat inequalities as equations. The researchers pose the demand to find forms of teaching that could help not only the high-performing students to increase their understanding but also the students of middle and low-performing.

3.1.3 Translating word problems into equations

In this section I will present a short overview of studies that are done in relation to the translation of word problems into equations.

Firstly I would like to present a study which is done by MacGregor and Stacey (1996) in which 90 students aged 14-16 participated, and lasted for 10 months. The students were exposed to tests three times during this period and they were given six problems of which three were presented as follows (MacGregor and Stacey, 1996, p. 290):

> **Problem 1**- *A group of scouts did a 3-day walk in a long weekend. On Sunday they walked 7 km farther than they had walked on Saturday. On Monday they walked 13 km farther than they had walked on Saturday. The total journey was 80 km. how far did they walk on Saturday?*
>
> **Problem 2**- *Jeff washes three cars. The second car takes 7 minutes longer than the first one. The third car takes 11 minutes longer than the first one. Jeff works for 87 minutes altogether. How many minutes does he take to wash the first car?*
>
> **Problem 3**- *The three sides of a triangle are different lengths. The second side is 3 cm longer than the first side, and the third side is twice as long as the first side. The side lengths add up to 63 cm altogether. How long is the first side?*

The problems which were posed to the students in these tests were chosen by the researchers in such a way that they could be simple to understand by them, and to avoid linguistic difficulties.

According to MacGregor and Stacey (1996), they found that approximately 70% of the students have found the correct answers for all the problems, but most of the solutions were obtained by non-algebraic method. The result of this research showed that only 35 students (39 %) in the final test attempted to use algebraic notations. The progression of the students for using algebra in their solution has not increased very much. The researchers concluded that (MacGregor and Stacey, 1996, p. 292):

> *53 students continued to make no attempt to use algebra, 11 moved from "No algebra" to "Partial use" and only two students moved from "No algebra" to writing a correct equation. ... eight students moved from being partial users to writing an equation, and nine did not progress. There were 20 students at the time of the final test who used some algebraic notation but had not yet progressed to a stage where they could write an equation for these relatively straightforward problems. Two students used some algebra in early tests but no algebra in the final test.*

This regression that the researchers noticed is explained by them as a teaching effect because during this time the students were trained to use the trial-and-error approach to the problem-solving.

The difficulties shown in this study are shortly described as follows (MacGregor and Stacey, 1996):

1- *Naming the unknown quantities referred to in the problem.*

    This difficulty is related with the problem that the students have to name the unknowns given by the task, even if they can understand the relationships between the unknowns. The number of the students that have shown this difficulty (identifying and naming the unknowns) in this study is low.

2- *Expressing the relationships between the parts.*

    Many students could express the relationships between the unknowns in a correct way. But there are still some students that have done some errors related with this. These errors are: a) reversal, b) concatenation for addition and c) exponential notation for the product.

3- *Writing a useful equation that integrates the problem information.*

    Many students could express very well the relations between the quantities of the given problem but they could not express them into an equation. This is related also because some of the students did not know to write equations in a standard form. But it is seems that with the right instruction they could quickly learn how to write equations in the standard way.

4- *Equations as descriptions of procedures used for calculating.*

    Some of the students in this study could solve the problems using arithmetic reasoning but have written their calculations as "equation". These pseudo-equations can be used to solve simple problems, but when they were posed to harder problems these students did not success by using this method.

The difficulty that the students have shown in the study is related with passing from a word problem to an equation. According to MacGregor and Stacey (1996) these

difficulties were due to misuse of algebraic notation, including the difficulty of writing an equation.

3.2 Theoretical framework

Carolyn Kieran has given an extensive contribution to research on learning and teaching. I have chosen her framework as the theoretical framework of my study. I got to know this frame in one of the courses done earlier during my studies in Norway.

The framework was developed in 1996 (Kieran, 2007) by treating algebra as an activity. This consideration about algebra is given by Lee (1997, from Kieran, 2007). She developed a model that divided the activities of school algebra into three parts: *Generational*, *Transformational* and *Global/ meta-level*. The scheme of this model is given below (Kieran, 2007, p. 713) and it is called GTG model:
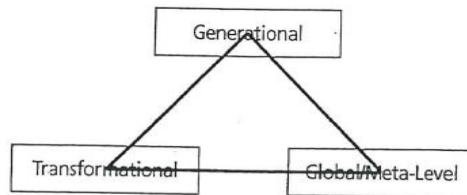
Figure 2. GTG model for conceptualizing algebraic activity.

According to Kieran (2007, p. 713-714) the definitions of these three types of algebraic activities are:

- *Generational activities involve the forming of the expressions and equations that are the objects of algebra.*

- *Transformational activities include, for instance, collecting like terms, factoring, expanding, substituting one expression for another, adding and multiplying polynomials, solving equations and inequalities, simplifying expressions, working with equivalent expressions and equations, and so on.*

- *Global/ meta-level activities include problem solving, modelling, working with generalizable patterns, justifying and proving, making predictions and conjectures, studying change in functional situations, looking for relationships or structure, and so on -activities that could indeed be engaged in without using any letter-symbolic algebra at all.*

I would like to explain these mathematical activities in more detailed based on what Kieran (2007) has claimed about them.

The Generational activities help in building many of the algebraic meanings that we use in algebra as for example: work with the variables, unknowns, and equality. This activity is also closely related with the notion of the solution of an equation.

The Transformational activities contain several types of activities. These types are also related with the changing of the symbolic form of a mathematical expression or an equation in order to maintain the equivalence between these two expressions. This is what, in my study is called in this study "equivalent passages" of an equation (see chapter 2). These kinds of activities are not

only based on the skills but also on conceptual/theoretical elements, especially in the beginning when they are presented.

The Global/ meta-level activities are more general mathematical processes and activities. As we can see from these activities, they use algebra as a tool but they are not exclusive to algebra. This is the reason why these activities represent a general view. As pointed out by Kieran (2007, p. 714) "*the global/meta-level activity of algebra is both broader than and at the same time not quite as broad as algebra*".

The research from which I got the first idea about the Kieran's frame is the article by Tabach and Friedlander (2008), which is about the understanding of the equivalence of symbolic expression in algebra. The environments used in this research is spreadsheet-based (excel spreadsheet) and paper-and-pencil. The goal was to make a comparison between the understandings of the equality by the students in these two environments.

The research (op. cit) was done when the students started to work on algebraic notions. Data for this research were: 41 Excel files of 74 pupils that worked mostly in pairs on the spreadsheets, audio-taped work of four randomly selected pairs of students on spreadsheet activity, and the individual work of 136 pupils on paper-and-pencil activity. They were asked to express equivalence of the following expressions: 2A+2B and 10A+10B for the first and second activity, and 2(A+B) and -2(A-B) for the third activity.

They arrived at the conclusion that both activities: paper-and-pencil activity and spreadsheet activity help students in working in generational perspective (numerical level that is promoted by spreadsheet) and transformational perspective (symbolic level that is promoted by paper-and-pencil). Further details for this study are not relevant for my study but I am interested more on the way how they used the Kieran's framework in their research. The way that this frame is used in this study is seen by which activities of this frame the students develop mostly when they work on the spreadsheet and on paper-and-pencil. They concluded that working with "*the integrating learning sequence described in this research, allows students to consider the same tasks as generational, transformational, and global/meta-level activities*" (op. cit., p. 45). So this way of working gives students to a better understanding on the equivalent passages (transformations), and equivalence.

My focus is equations and for this reason I have chosen research that is connected with topic as: equivalent expressions, equations, and solving a word problem in Kieran (2007).

The generational activity that is related with the notions of variables, expressions, and equations are treated in the following studies (op. cit., p. 716):

> *Interpretation of algebraic symbols that tend to focus on the cognitive levels (Kücheman, 1981), prior arithmetic experience and method of thinking (Booth, 1984), and difficulty with notation such as brackets (Kieran, 1979) and the equal sign (Behr, Erlwanger and Nichols, 1976; Kieran, 1981), more recent work suggests additional factors impinging upon students' interpretation of algebraic notation: (a) what one is able to perceive and prepared to notice (Sfard and Linchevski, 1994), (b) difficulties with operation signs (Cooper et al., 1997), (c) the nature of the question asked and the medium in which they are asked (Waren, 1999), (d) the presence of multiple referents and shifts in the meaning of unknowns (Stacey and MacGregor, 1997) and (e) the nature of the instructional activity (Wilson, Ainley, and Bills, 2003).*

There are many studies on the students' understanding of the equal sign, but there are few on the difficulties that the students have with the operational minus and the negative numbers (op. cit.). According to Kieran (2007) in a study by Vlassis (2004) various uses of minus used in the algebraic expressions and in equations are counterintuitive, which makes pupils (eighth graders) to have difficulties in giving meaning to algebraic symbols and processes.

According to Kieran (2007), the solution of a word problem consists in two phases that are: a) The setting-up of an equation to represent the relationships inherent in the word problem and b) Solving the problem.

The transition from a word problem to an equation is a difficult area for the students. From some studies it is shown that students rather prefer arithmetic reasoning than to solve word problems with the help of equations (Bednarz and Janvier, 1996; Cortes, 1998; Swafford and Langall, 2000, from Kieran, 2007).

As pointed out by Kieran (2007), a study done from Stacey and MacGregor (1999) shows those students, during the solution of word-problem, passed from an algebraic approach to an arithmetic reasoning. Another study was done from Malara (1999, from Kieran, 2007) that arrived at the conclusion that students do numerical substitutions to achieve the solution. A study conducted by Nathan and Koedinger (2000a, from Kieran, 2007), shows that word problems which are given in verbal form are easier for the students to solve than when they are given in other formats as equations or "word-equations". Another conclusion that these researchers (Nathan and Koedinger, 2004, from Kieran, 2007) found is that simple algebra stories problems were easier for the students to solve than the mathematically equivalent equations. According to Nathan and Koedinger (2004, from Kieran, 2007) this is related not only with the situated world knowledge but mostly with the difficulties that the students have to comprehend the formal symbolic representation of quantitative relations.

The transformational activity, based on what is defined with this activity contains terms like factoring, expanding, substituting an expression with another, adding and multiplying polynomial expressions, exponentiation with polynomials, solving equations and inequalities, simplifying expressions, substituting numerical values into expressions, working with equivalent expressions and equations, and so on (Kieran, 2007). This activity is related with the manipulative processes that students have to do, but in later research it is seen that the algebraic transformations are not only procedural but also theoretical. One of the important points here is the saving of the equivalence. This is one of the bases for the solution of an equation. The saving of the equivalence, which in the Albanian curriculum is denoted by equivalent passages (see chapter 2), is a difficult area for the students (op. cit).

According to Kieran (2007), a study by Filloy and Rojano (1989) concluded that students found it difficult operating in both sides of an equation. These studies were made for students in lower secondary school. Let's focus now in some studies which are made in upper secondary school and college level in relation to the transformational activity in equations.

It is a study done by Vaiyavutjamai, Ellerton, and Clements (2005) where students of the tenth year in Brunei Darassalam, and 2nd-year university in the US, were posed to solve the same quadratic equations as the followings (Kieran, 2007, p. 732) all of the form:

" $x^2 = K(K \succ 0)$ and $(x-a)(x-b) = 0$ where a and b are any real numbers. "

This study is presented in the section above (Research on equations). The researchers found that the responses related with the second type of equation of the students showed serious gaps in the theoretical thinking even if they have worked in these kinds of equations for a long time. When the students were posed to the equation $(x-3)(x-5) = 0$, during interview, some of them found the correct answers that the solutions of this equation are $x = 3$ and $x = 5$. The justification that they gave was $0 \cdot 0 = 0$. So they have treated the unknown in the first bracket as different from the value with the unknown in the second bracket. This shows the difficulty that students have with quadratic equations, even if they are an important part of mathematical curriculum through the years.

As mentioned above, the global/meta-level activity includes activities that need higher knowledge and more complicated activities. As pointed out by Kieran (2007), these activities are also related with the generalizing activity, for example the generalization of a solution of a word problem in problem solving. A study by Lee (1987, from Kieran, 2007) and Lee and Wheeler (1987, from Kieran, 2007) concluded that students do not like to use algebra to justify a general statement. These justifications are also related with proof and proving. The difficulty that some 14-year-old students have shown dealing with the sum of two even numbers, shown that these are related not only the difficulty that the students have with the algebra competences but also with the mathematical knowledge (Miyakawa, 2002 from Kieran, 2007), so they have difficulty in proving this statement.

Kieran (2007) concluded that the students' understanding about the algebraic objects increase from their multiple representations. But still there is a gap between the transformational activities in algebra and the generational one. Difficulties are also shown about word problems, were the students try to use arithmetic methods instead of algebraic ones. According to Kieran (2007), based on the researches that are mentioned by her, students need more time to deal with the algebraic concepts; this is why it is always difficult for the students to make the passage from arithmetic to algebra.

Until now I have explained Kieran's framework, and described some researches done in relation with the activities of this frame. To better fit with my research, these activities were turned into students' skills. Based on Kieran's three types of algebraic activities (GTG model) I have built the following skills that students have to demonstrate during the task-based interviews:

1. The skill to build an equation for a given word problem.

2. The skill to transform the equation into simpler one but always saving the equality.

3. The skill to solve the equation and to justify the solution.

I saw that the third level included too much information for the student's and I splitted it in two parts. So the framework that I will use is extended to:

1. The skill to build an equation for a given word problem.

2. The skill to transform the equation into simpler one but always saving the equality.

3. The skill to solve the equation

4. The skill to justify the solution.

Based on the first skill that I am looking for I have given a word problem to the students, both in the test and in the task-based interviews. The word problem is a geometric problem that can be solved by turning it into an equation. The second skill is related with the ability of the students to transform the given equations into simpler equivalent equations (making equivalent passages). The third skill is the ability of the students to solve the equation, so each of the transformations has to be an equivalent passage, and the fourth skill is related with the justification (checking) of the result that they get when they solve an equation. This means that students have to know how to check, so they can see if the result that they have found is the right one or not.

3.3 Task-based interview

Task-based interview is the method used in this research to observe the mathematical thinking of the students during their solution of equations. I got to know this method in a course on the research methods (MA-404 at the University of Agder) or know as the Mathematics Education Research Group (MERG) project. It is a pilot project which helps the master students in mathematics education to collect and analyse classroom data, and prepare them to write a master thesis. I had this course in spring 2009, and I used task-based interviews in my MERG project. The method gave me the opportunity to ask questions that go deeper and deeper in the students reasoning and I can also act in the moment. For these reasons I decided to use task-based interviews in my master thesis as well.

The definition of this method is given by Goldin (2000). This is a chapter which focuses completely on the task-based interviews. It gives a scientific perspective of this method.

The definition of task-based interviews is given as:

> *Structured, task-based interviews for the study of mathematical behaviour involve minimally a subject (the problem solver) and an interviewer (the clinician), interacting in relation to one or more tasks (questions, problems, or activities) introduced to the subject by the clinician in a preplanned way. (Goldin, 2000, p. 519).*

Based on this definition, we can see that the persons that take part in this interview are the interviewer and the interviewee. Another element in this interview is the task(s) or the problem(s) that the interviewer poses to the interviewee. The interview is based on these tasks (problem or questions), and for this reason the method has its name. The subject (the student in my research) interacts not only with the interviewer, but also with the task that has been prepared for him/her to solve. These questions, problems, and tasks are preplanned by the interviewer.

3.3.1 A short history of task-based interviews.

In 5 decades, researchers were developed clinician interviews. Earlier data collection in mathematics education was mostly based on standardized multiple-choice achievement tests and so-called 'curriculum-specific' tests (Davis, 1984). The clinical interview method was started firstly by Piaget as an instrument for psychological research (Ginsburg, 2010). Piaget's goal was to understand the students' thinking; he wanted to explore the richness of the students' thinking.

A similar method, about thinking aloud protocol has been developed by Schoenfeld (1987) in his research. His goal was to make students able to understand what they were working on or in other words to develop the metacognition of the students. Metacognition is the way a person thinks about his/her own thinking (Schoenfeld, 1987)

> The metacognition is focused in three related but distinct categories of intellectual behaviour:
>
> 1. *Your knowledge about your own thought processes. How accurate are you in describing your own thinking?*
>
> 2. *Control, or self-regulation. How well do you keep track of what you're doing when (for example) you are solving problems, and how well (if at all) do you use the input from those observations to guide your problem solving actions?*
>
> 3. *Beliefs and intuitions. What ideas about mathematics do you bring to your work in mathematics, and how does that shape the way that you do mathematics? (op. cit., p. 190).*

If we shortly analyse each of these categories we can see that:

The first category is related with the ability that the students have to express themselves, how can they describe their thoughts. For some students this is very difficult.

The second category is related to the control and self-regulation that the students have when they solve a problem (in this case). How they can recall the knowledge that they have to make a short and clear solution of the problem. According to Schoenfeld (1987) students are more focused to make conjecture and then prove it. They do not go back and analyse thoroughly the problem, but they stay in that conjecture. Schoenfeld has done a study where he compares the solutions of a geometric problem of a student and a mathematician that has not worked for a long time with geometric problems. The figures below show their ways of approaching the problem, how different activities are activated in the problem solving process (Schoenfeld, 1992, p. 356):
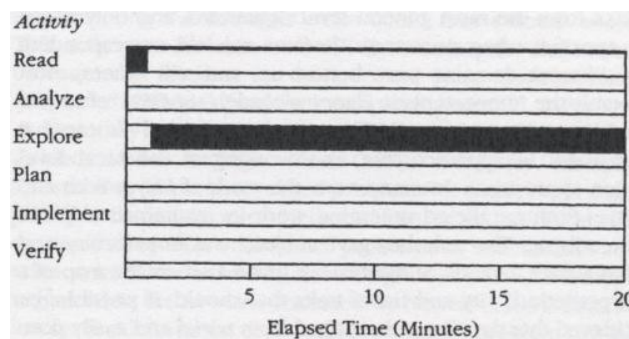
Figure 3. The time-line graph of a typical student attempting to solve a non-standard problem.

From this graph we can see that the student, when posed to a non-standard problem, spends some time in reading and all the other time in exploring the solution.
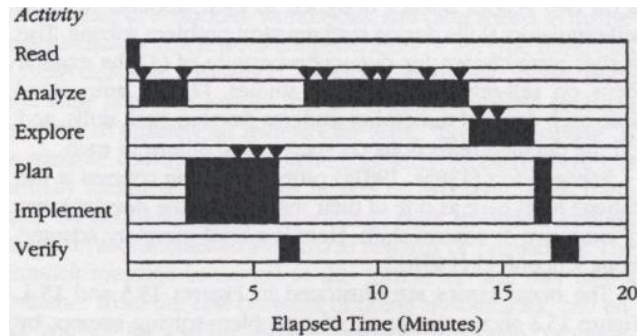
Figure 4. Time-line graph of a mathematician working a difficult problem.

From this graph we can see that the mathematician read the problem and passes through all the activities (Read, Analyze, Explore, Plan, Implement, Verify). The problem he was solving had two parts; this is shown from the bipartite nature of the graph above.

After having finished the problem solving course, Schoenfeld exposed the student(s) to a similar situation, and a graph different from the first one appeared:
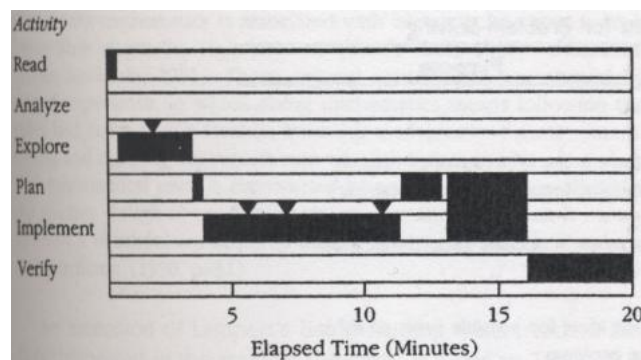
Figure 5. Time-line of a solution attempt after explicit training in monitoring and control.

During this course Schoenfeld (1987) encouraged the students to discuss the problem by talking aloud of their ideas of the solutions. His role was to direct questions as (op. cit, p. 206):

1. *What (exactly) are you doing? (Can you describe it precisely?)*

2. *Why are you doing it? (How does it fit into the solution?)*

3. *How does it help you? (What will you do with the outcome when you obtain it?)*

The third category, the way of making the students talking aloud helped them to control or self-regulate their way of solving a problem.

Beliefs and intuitions, is related to how the students interpret or conceptualize mathematics. Teachers try to teach concepts and procedures but the problem is how the students interpret these. In Schoenfeld (1987, p. 196) two examples of misconceptions, are presented. I would like to present one of them, which shows the difficulty that the students have to relate the real word with mathematical tasks. The example is the following: "*An army bus holds 36 solders. If 1128*

soldiers are being bused to their training site. How many buses are needed?" 70% of the students could do the correct division, but when they wanted to answer to the question how many buses are needed, the result is the following: 295 said that the buses needed are "31 remainder 12", 18% said that the number of buses has to be "31", and 23% said that the number of buses has to be "32", which is the correct answer.

During the years the researchers of mathematics education felt the necessity to reveal the students' thought processes. According to Goldin (2000), this was firstly started by Newell and H. A. Simon in 1972 as an analysis of the subjects' verbal problem solving protocols and then it was developed into a method task-based interview.

3.3.2 The method of task-based interview

A simple definition about this method is given by Davis (1984), as follows

> *The basic idea is very simple. A student is seated at a table, paper and pencil are provided, and the student is asked to solve some specific mathematics problem. One or more adults are present collecting data. (Davis, 1984, p. 87).*

According to Davis (1984) the adults that collect the data (researchers) participate by posing to the students some questions related to the tasks that the students are working with. But this interference of the researchers can vary considerably. Sometimes the interference of the researchers can be minimal, and other times the students can be left with no interference at all. They are let free to express themselves.

> *These interferences are used from the researchers in order to pose further questions, in order to provide hints, or in order to correct an error or misunderstanding. They may also intend to provide more motivation or perhaps some encouragement. (Davis, 1987, p. 88).*

The researchers can be the students' teacher or a complete stranger and the materials that the students can use during these interviews can also be rulers, compasses, textbooks, hand-held calculators, graph paper, computer terminals or manipulatable materials (op. cit.).

But this method has also some difficulties which are mentioned by Davis (1987, p. 89-90) [I have slightly shortened them] as given below:

1- *The difficulty to remain a neutral observer. Usually the interviewers are the teachers or the former teachers, and they are oriented to help the teacher to succeed. Their goal makes them to have difficulty to remain neutral in their way of posing the questions, and giving hints.*

2- *Questions have to be phrased in that way that they do not give extraneous cues or clues, and the interviewer's inflections, gestures give no indication of desired responses.*

3- *It is needed that the observer to establish a rapport with the student. This can help the student to feel more comfortable during the interview, so he/she can express better his/her thoughts.*

4- *It is important to establish who is leading whom. Sometimes the interview can take another direction, it is the duty of the interviewer to lead and to direct the conversation in the focus that he has. So the interviewer has to be persistent in his questions, but this persistence is also a problem. If the observer is too persistent can make the student into a mirror of his thinking and not the student's thinking.*

5- *The last, there are the ethical questions. Students must not be exploited. The observer has to be attentive when he gives answers because the students cannot have any signal as "right" or "wrong".*

In the beginning I presented a definition the task-based interviews given by Goldin (2000) and a short explanation of that. He has done research on the structured, task-based interviews, and has collaborated with many researchers. He claims that the interviewer is free to give hints and heuristic questions related to the problems in sequence, retrospective questions, or other interventions depending on the focus. The interviews are transcribed and analysed. Usually this method is used to refine or elaborate a conjecture. But the structured based-interviews are built depending on the research purposes (op. cit).

> *These may be exploratory investigation; refinement of observation, description, inference, or analysis techniques; development of constructs and conjectures; investigation or testing of advance hypothesis; and/or inquiry into the applicability of a model of teaching, learning, or problem solving. (Goldin, 2000, p. 519).*

Goldin in collaboration with other researchers as Zang (1994), DeBellis (1996), and Passantino (1997), work on a series of five task-based interviews. According to Goldin (1998, p. 52):

> *The thesis of Zang (1994) examines the development of strategic thinking in four of the children, comparing Interview 1 and Interview 4; the thesis of DeBellis (1996) studies affect in four of the children, using Interviews 1, 3, and 5; and the thesis of Passantino (1997) looks at the development of fraction representations for all the children, comparing Interviews 2, and 5.*

They were designed to explore the individual children's mathematical development longitudinally, through case studies focusing on the growth of internal representational capabilities and their interplay with external representations constructed by the children. The questions, which were posed by the clinician during these task-based interviews, were relatively "neutral", so they did not influence on the students thinking. The questions were as: "Why do you think so?" or "Can you show me what you mean?". The analysis of each of the questions proceeds in four stages (Goldin, 2000, p.523 and Goldin, 1998, p. 45) that are:

- *Posing the question (free problem solving), with sufficient time for the child to respond and only nondirective follow-up questions, such as, "Can you tell me more about that?"*

- *Minimal heuristic suggestion, if the response is not spontaneous, such as, "Can you show me using some of these materials?"*

- *The guided use of heuristic suggestions, again only when the requested description or anticipated behaviour does not occur spontaneously, such as, "Do you see a pattern in the cards?"*

- *Exploratory, metacognitive questions, such as, "Do you think you could explain how you thought about the problem?"*

Part of the data were also the imagistic representation (especially visual and kinaesthetic) and/or numerical representation, verbal discussion, and affective responses. The interviews with the children were videotaped, transcribed, and analysed.

According to Goldin (2000) a task-based interview, the researcher is the one who control or partially control the tasks, the interview questions and any hints or suggestions offered, the interview setting, the choice of the subject, the physical materials available to the subject, the

time allotted for the problem solving, and related task and situational variables. The control of the task is related to their mathematical content and structure, complexity, and linguistic and semantic structure. All these choices that have to be done are not accidental; they are done focused on the purpose of the research.

An example of these structured, task-based interviews that are used from Goldin et al. is given below:

It is the first task-based interview of five task-based interviews (Goldin, 1998). The interview started by posing to the students three cards as the ones given below:
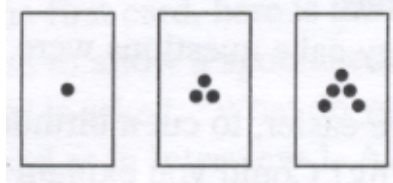
Figure 6. The first three cards presented in the Task-Based Interview 1

These cards were taken out from an envelope. The purpose was that the child could understand the pattern of these cards, and think that there are also other cards with more dots. Then he/she was asked: "*What do you think would be the next card*". The questions are (Goldin, 1998, p. 45):

- *What card do you think would follow that one?*

- *Do you think that this pattern keeps going?*

- *How would you figure out what 10th card would look like?*

- *Here's a card [showing 17 dots in the chevron, or inverted V, pattern]. Can you make the card that comes before it?*

- *How many dots would be on 50th card?*

The script of these answers to each main question is explored in the four stages cited above. The goal of the researcher for this interview was to get a complete and coherent verbal reasoning of the child, and an external representation of his thinking. This is what I wanted for my research.

3.3.3 The method of my study

I have also used this method but I have tried to fit it to my goal. My goal was to understand the way six Albanian students are thinking when they solve equations. The students have to choose among four cards (presented in the method part) the most difficult and the easiest. These cards contain two equations and a word problem. I was the interviewer. Some of the questions were preplanned but the hints and the clues were given to the students in the moment. The questions that I have used are based in the Newman's diagnostic interview technique which has five key requests that are given below (Vaiyavutjamai and Clements, 2006, p. 121-122):

1- *Please read the question to me.*

2- *Tell me, what does the question mean?*

33

3- *What will you need to do to answer this question?*

4- *Now answer it, and tell me what you are thinking as you do it.*

5- *Now, write down you final answer.*

According to Newman (op. cit) these requests are related with the error classification called Reading, Comprehension, Transformation, Process skills, and Encoding which is not used in my study.

These were the five requests that I had in mind when I asked the students but I would like to divide them into two parts: the choice of cards, and solution of the chosen card

During the choice of the cards the main questions that were posed to the students are: which cards do you consider the easiest and which the most difficult? Why do you think that this is the easiest and this is the most difficult?

These have been the main questions that I have posed to the students during their choice of cards. My main focus during this part was to find out what students consider easy and what difficult.

For the solution of the chosen card the questions that were used to understand the student's way of approaching are: What is the strategy that you want to follow to solve this task? What do you need to solve this equation? What did you think when you pass from this step to this step? How can you check that this is the right solution for the equation? (see chapter 4)

I have tried to be neutral in my questions and in my hints but many times it has been difficult (Davis, 1984). The materials that the students have worked with in my research are paper and pencil. The interviews have been audio-taped and video-taped. The focus of the camera was on the work paper of the students. So, my data are the audio and video-recorded interviews, and the students' papers work.

Since I was a complete foreign person to the students, I used the classroom observations to get to know each other. In this way they could be more open with me during the interviews. I tried to make them feel comfortable during the interview. I agree with the difficulties that Davis (1984) has mentioned because I have had those kinds of difficulties. It has been very difficult to be a neutral observer since I wanted to get the best from the students, and sometimes the hints have not been neutral. I used a lot of time to be accepted by the students. I wanted them to trust on me and to understand that I was not there to evaluate them but for my own project. I did not find any resistance from them and they were comfortable with the interview.

4 Methods

This chapter is divided into four parts that are: research context, participants, data collection, and data analysis. In the first part is a description of the context where the study is done. Here I have described a short overview on the changing of the school system, and especially difference in the books. I have done a short description on the class and on environment where the study is done. The second part is a description of the participants, and it follows with the data collection section, where I have described the way how I have collected the data. Here are described all the stages of my data collection, and the reasons to pick the students for the task-based interview, which is also part of this section. The last part is the data analysis, where I have described how I will analyse the data in my thesis.

4.1 Research context

The data collection was done in the first year of an upper secondary school in Albania. I got contact with the teacher on December 2008, and explain that I needed to make a project in one of his classes. The teacher was informed with the project on summer of 2009. The school is one of the biggest upper secondary schools in the centre of the city, in the north of Albania. The language that is used in this school is Albanian.

During the last years the school system of Albania has changed as cited above (see chapter 2). Now, the upper secondary school lasts three years instead of four years as was the case before. Textbooks that students work with have also changed because they have to fit to the new curricula.

Before the new school system started around 2000, the textbooks of the upper secondary school were divided in this way:

1- If the upper secondary school was a normal one, the students work with two text books; that are divided into an algebra book and a geometry book.

2- If the upper secondary school was an experimental one, the students in the third year had to choose between the social and science orientation. So the first two years were similar for all the experimental schools, but different from other normal schools. After the students were divided into two orientations, then they started to work within different textbooks. The students in the social orientation work only with one mathematics textbook, which includes the basics of mathematics, and the students in the science orientation worked with two textbooks that were divided in geometry and algebra. During the fourth year they also treated some part of mathematical analysis.

The new system also requests a division which starts at the first year of upper secondary school. The students decide if they want to study more mathematics or not. The textbooks have also changed. There are many textbooks, four or five textbooks, which are approved by the Ministry of Education in Albania, for the first year of upper secondary school and the school has to choose one of them. In this way, two schools may not work with the same textbooks. These books are written by different authors and they are called basic books. All the students of a school have to work with it. This choice between textbooks has to be done by the mathematics teachers of that

school. The students that have chosen to work more with mathematics also use another textbook which contains advanced knowledge.

The first contact that I have to this school was with the teacher who helps me to gather the data. He was the first that got to know my project, and presented me with the headmaster of the school. I needed the permission of the headmaster to start my data collection. When I met her, I explained my project, and what I was going to do during that time to gather my data. I explained to her that I needed an hour to make the test, and after that according to the students' performances in the test I pick 6 students for the task-based interview. I explained that the interviews will be audio and video taped, and I will use an audio recorder and a camera. I assured her that I will keep the anonymous of the students. She was very willing to help me, so she also provided to me a room to make my interviews. For her acceptance, she has also signed a letter (see Appendix 1).

There are nine classes in the first year in this upper secondary school, and three of these nine classes have chosen to study more mathematics, so they work with two textbooks. My data collection is focused on one class, in which the students have chosen to do more mathematics. I observed only the lessons, where the students worked with the basic book since the topics treated in the advanced book were different from my focus. My data collection started November, 9 and finished December, 21 2009 where I got the last test that the teacher did in the beginning of the chapter: "Equations, inequalities and systems of equations".

It is necessary to see the environment in which the Albanian students work so I did a seating chart that shows how students sat during my class observations:
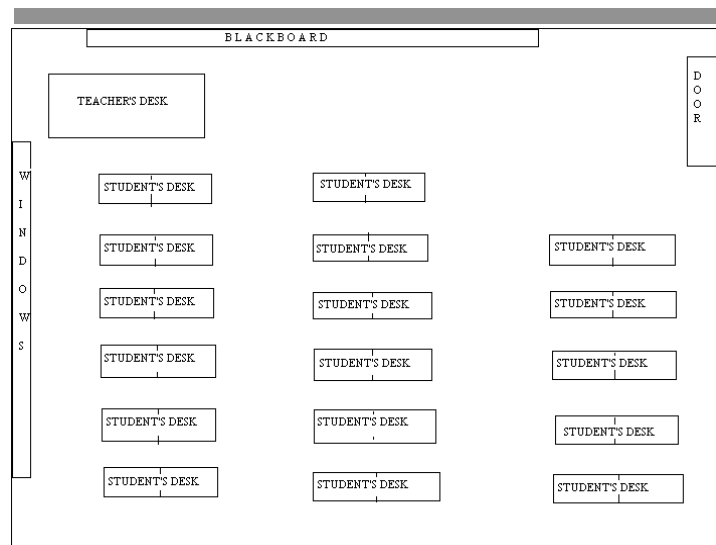
Figure 7. The class seating chart.

This class had this seating when I did my data collection. The student's desks are static because the way of teaching is always the same. So the teacher lectures and the students listen and take notes. This is the common way of the organising and teaching class in Albania. The teacher mostly grades the students based on the given tests, and sometimes based on the students' answers when they are at the blackboard.

4.2 Participants

The class that I have observed had 34 students where 11 of them are boys and 23 of the students are girls. These students have chosen mathematics as their favourite subject. So they work with the books, the general book and the advanced one.

Until now the students have worked with linear equations, quadratic equations and some irrational equations of the form $f(x) \cdot g(x)=0$.

The students are 16-17 years old, and had a choice if they wanted to participate in my project or not. Firstly, when I was presented to them, the teacher explained in general what my goal was, and then he asked me to give more details about my research. I explained to the students that my goal was to understand their way of reasoning, and their thinking. And also I told to them that I was not going to grade them, and their anonymous will be saved.

4.3 Data collection

My data collection started on November, 9 and lasted until December, 21. During this time I have done:

1- Classroom observations

2- One test for the whole class

3- Task-based interviews of six students.

4.3.1 Classroom observations

The first day in the school I presented myself to the headmaster, from who I asked the permission to gather the data. She accepted that. There after the teacher introduced me to the class and with few words my project.

The students did not show any rejection to participate in my project. So for more than one month I was part of their mathematics classes. I was a complete observer (Bryman, 2008) because I did not participate in any of the teaching hours. There were two purposes for my presence during the mathematics lessons to get to know the students, and them to know me and to have an overview of the students' work with equations

The first goal was achieved, but the second one was not fully achieved. Because when I started my data collection the teacher was in the beginning of the "Function" chapter. He tried to make a connection between function and equation, and my observations of students working with equations have been when the students were working on the domain of a function or when they needed to solve equations of the type $f(x)=0$.

Seeing that it would be too late to do the test and the task-based interviews during the teaching of the "Equation" chapter, I did the observations before the students started this chapter. So I needed to change some of my goals such that I can discuss the student's knowledge about equations mostly based on their background knowledge.

During the observations the teacher gave tests. The first two tests were done during the "Function" chapter, one in the middle and one at the end of this chapter. I have named them First Test and Second Test. The teacher did a last test at the end at my observation, on December, 21 that I have named Third Test.

4.3.2 Tests

The tests were done on 23/11/2009 and 2/12/2009 and the third one on 21/12/2009.

The two first tests were done based on the "Function" chapter and the third one in the beginning of the "Equations, inequalities and systems of equations". These were the teacher's tests.

I also gave the students a test. The purpose of this test was to classify the students' work and to pick six students for the task-based interviews. This test was planned to last for 45 minutes (real time) and there were 29 students out of 34 that took part in it. The other 5 students were absent that day. The students finished the test in less than 45 minutes. (The first paper was delivered after 20 minutes and the last paper was delivered after 35 minutes.) There are two tasks in the test: 1- Solve the equations and justify the solution, and the second is a word problem which is: 2- It is given a square. If we decrease one side of the square with 1m and the side next to it with 3m, we get a rectangle which area is $8m^2$. Find the side and the area of the given square.

The results of my test helped me to divide the students in three categories according to their grades: Students that got the lowest grades (4-5-6) made one category; the middle grades (7-8) made a second category, and the highest grades (9-10) made the third category.

After my test the teacher posed the students to his last test in the beginning of the chapter "Equations, inequalities and systems of equations". The teacher gave me all the students' grades of the tests.

Below I will show the table of the grades that the students have taken on the four tests, three from the teacher and one by me (Table 3). These grades serve as a background to understand the level of the students, but they are not part of my analysis. I have used only my test to pick the students for the interview.

Below it is given the test that I have posed to the students. The tasks of this test are got from the sources:

Task1-a (Equation website, a); Task1-b (Equation website, a); Task1-c (Equation website b); Task1-d (Equation website, a); and Task1-e (Mesi and Boriçi, 1997, p. 41). The word problem is done by me, based on the word problems of the students' textbook (Lulja and Babamusta, 2009). I have presented the English version of it.

Test

1- Solve the equations given below and check the answer:

   a) 5x-6=3x-8

   b) $\sqrt{2} \cdot x - \sqrt{3} = \sqrt{5}$

   c) $(x-5)^2 - 100 = 0$

   d) $\sqrt{x-8} = 0$

   e) $\dfrac{1-x}{1+x} - \dfrac{1+x}{1-x} = \dfrac{1}{x^2-1}$

2- It is given a square. If we decrease one side of the square with 1m and the side next to it with 3m, we get a rectangle which area is 8m². Find the side and the area of the given square.

Figure 8. The test given on 9/12/2009.

Tests grading

The purpose of the test for all the students was to divide them into three groups. In this way I could pick two students: one boy and one girl from each group, and interview them. I could grade the students. The correct answers in my test were graded as follows:

The task	The points
1a	1
1b	1,5
1c	1,5
1d	2
1e	2
2	2

Table 2. Tests' grading.

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I have tried to put these points to the tasks according to the following reasons, where the value of the checking is 0,5 points for all the tasks.

Task 1a) is an easy one so I gave 1 point; it is a linear equation and the students are used to solve them from the third year of elementary school.

Task 1b) for many students is difficult because they mix it with the irrational equation, so I give it 1,5 points.

Task 1c) has much work to do, but it is an easy one because there is no problem for the students to use the given formulas, so I give it 1,5 points.

Task 1d) is an irrational equation. Since the students had to find the domain to which I have given 0,5 points, and 0,5 points to the check. So the solution of equation is 1 point, so the complete task has 2 points.

Task 1e) has a lot work to do, a lot of calculations, and to find the domain so I give it 2 points.

Task 2) has three sections to it: expressing the problem by an equation, solving the equation, and making the check. So I give this problem, 2 points.

If I do a summary of my grading I defined that finding the domain and checking the results values of 0,5 points each and the solution of the equation the value of 1 point. An exception is the first task because it is very easy basing on the students' background.

Then the grading of the students is done by adding the points, and the total point is 10, which is the best mark in the Albanian grading system. So in my grading, the number of points fits to the grade that the students get. With 1-4 the student fails, 5 is the pass mark and 10 is the best mark.

The teacher grading is similar to mine but his test had 8 points in total and you need to get more than 2,5 points to pass. In this way to all the points of the students he add 2, for instance if one student has 5 points that means that he gets the grade 7 (5+2).

All the grades obtained in the four tests are presented. This table shows the number of the students that were present during this test. The missing students have a (-) in the box. The first column is divided in two parts the number of the student and the sex. The other four columns represent chronologically the tests. I have used in the third column (*) for the students that I have picked for the interview. The last five students are the ones that were absent on my test.

Based on the grades that the students got during the four tests we can see that they vary. From the table above we can see, especially from the teacher tests that the students are mostly divided in two groups: high performing students, low and middle performing students even if I have divided them into three groups. The two last columns are the ones which give the students' results for the tests that are done for the equation topic and we can see an improvement of the students' grade in the last teacher's test, which had the same form as mine, but the tasks were more difficult.

Here I will show the table of the student's grade:

Students' number and sex:		First test done from the teacher	Second test done from the teacher	My test	Third test done from the teacher
1	M	8,1	8,2	3,9	6,5
2	F	8	-	3,9	9,5
3	M	5,1	3,9	5,6	7,9
4	M	3,6	5	1,7	7,6
5	M	6,6	2	3,5*	-
6	F	4,9	8,3	6	7,8
7	F	-	-	6,3	7
8	M	3,8	2,6	2	3,4
9	F	4,2	-	4,6	8,8
10	F	-	5	1,7*	4,9
11	M	-	3,2	4	-
12	F	3,9	7,5	6,4	6,8
13	F	7,2	-	8	9
14	M	3,8	-	7,9	-
15	F	7	6	7	8,6
16	F	5,9	-	6,9	8,3
17	M	-	8	7,9	8,8
18	F	5,7	7,2	7,7	8,1
19	F	8,5	9,4	8,2	9,5
20	F	8,9	9	8*	9,6
21	M	8,6	8,9	8*	9,6
22	F	8,1	5,9	6,5	8,7
23	F	8	8,6	7,5	9,3
24	F	7,9	6,7	7,2	7,7
25	F	9,5	-	9,5	8,8
26	F	7,2	6,8	9	9,2
27	F	5,1	7,3	8,5	8,9
28	M	8,1	6,1	9*	9,1
29	F	8,9	7,4	9,6*	9,7
30	F	9,3	8,1	-	9,7
31	F	9,6	8,4	-	9,6
32	M	5,2	3,7	-	6,6
33	F	3,6	2,7	-	5,8
34	M	2,6	3,2	-	7,4

Table 3. Students' grades in four tests

The students with the * are the one that I have interviewed.

4.3.3 Task-based interviews

The task-based interviews lasted for two days and six students took part in these interviews. Two students of different gender were picked from each grade category: 4-5-6, 7-8, 9-10. Each of the task-based interviews lasted around 45 minutes, and it was structured based on the five points of

Newman's technique of interviewing (Vaiyavutjamai and Clements, 2006) but with some changes to fit better with my purposes. The interviews were audio and video taped. For this I have used an audio recorder and a camera. The camera was focused on the students' paper work, and not on their face.

Choice of the students

The choice of the students for the task-based interviews was done in this way:

When I had divided the class into three categories by writing their names in three different columns, I started to pick the students with the intention to choose two students, one girl and one boy from each category. Based on the Table 3, the student number 1-12 there are part of the group 4-5-6, the student number 13-24 are part of the group 7-8 and the student number 25-29 are from the group 9-10.

The choices of the students when I picked them from these groups are as below:

From the 9-10 group: when I picked the boy, I did not have another choice because he was the only boy in this group. The girl that I have chosen was placed at the first desk near to me. I noticed that she never raise her hand during my classroom observation, even if she was very well prepared as it is seen from her grades. So I wanted to hear her reasoning.

From the 7-8 group: I have no specific reasons for the two students. I did not know them. I just picked the names that I have listed and called them in the class. The only purpose was to pick one boy and one girl from this group.

From the 4-5-6 group: I tried to pick the student that got the lowest number of points and the one in the middle. The students with the lowest points are a girl, and the boy got 3,5 points, which is like in the middle of the points.

When I had chosen the students from the list, I just called them in the class where the teacher came with me. I did not have any rejection from the students to take part in this project, to the in contrary they liked to know more and help me in my project. The naming of the chosen students for the interview is done based on the order that they have been interviewed. In this way Student 1 is the first student that I have interviewed, Student 2 is the second, and so on.

The interviews

The interview consists of presenting four cards with the tasks to the students and they had to choose the most difficult and the easiest among them. Each of the cards contains two types of tasks, given equations and a word problem. The first task which had two equations and the students were requested to solve theme and to justify the solution that they found. The equations are of four types: linear, rational, quadratic and irrational. The word problem, which is the second task, is the same for all the cards and it requests to turn a simple mathematical word problem into an equation. This problem makes a connection between geometry and algebra, and is formulated by myself based on the students textbook (Lulja and Babamusta, 2009). The equations are chosen from mathematical sources, as follows:

Card1, Task1-a (Equation website, b); Card1, Task1-b (Mesi and Boriçi, 1997, p.40); Card2, Task1-a (Equation website, a); Card2, Task-b (Lulja and Babamusta, 2008, p. 96); Card3, Task1-a (Equation website, b); Card3, Task1-b (Equation website, a); Card4, Task1-a (Equations website, a); Card4, Task1-b (Equation website, c).

The cards which were given to the students are given below:

Card 1:

1- Solve the equations given below and justify the answer:

a) $x^2-3=2x$

b) $\dfrac{x-1}{x+2} - \dfrac{x+5}{2-x} = \dfrac{2(x^2+2)}{x^2-4}$

2- It is given a rectangle, where the base is twice bigger than the height. Find the dimensions of the rectangle if its perimeter is 120m.

Card 2:

1- Solve the equations given below and justify the answer:

a) $\dfrac{6x-7}{4} + \dfrac{3x-5}{7} = \dfrac{5x+78}{28}$

b) $x^2\sqrt{x} - \sqrt{x} = 0$

2- It is given a rectangle, where the base is twice bigger than the height. Find the dimensions of the rectangle if its perimeter is 120m

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Card 3:

1- Solve the equations given below and justify the answer:

a) $x^2 = -5x$

b) $\dfrac{1}{x+3} - \dfrac{1}{3-x} = \dfrac{10}{x^2-9}$

2- It is given a rectangle, where the base is twice bigger than the height. Find the dimensions of the rectangle if its perimeter is 120m.

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Card 4:

1- Solve the equations given below and justify the answer:

a) 2(3x-7)+4(3x+2)=6(5x+9)+3

b) $\sqrt{x+3} - \sqrt{2-x} = 1$

2- It is given a rectangle, where the base is twice bigger than the height. Find the dimensions of the rectangle if its perimeter is 120m.

Coding of the tasks:

The purpose of the use of the coding for the tasks has come as a necessity to denote them in an easier way. The coding is of the form [C_, P _]. This means that C_ represent the number of the card and P _ represents the number of the problem in that card. For example if we have [C2, P 1a], this means that the student is working with problem 1a of Card 2.

The students were asked to choose the most difficult card and the easiest one for them and to tell the reasons why they think in this way. After they had chosen the two cards, I asked them to solve one of them. The structure of the part of interview was:

1- Read the task.

2- How do you think to solve it?

3- What do you need to solve it?

4- Why do you think that these steps are equivalent or which is the reasoning behind these passages?

5- How can you check that the answer is the correct one?

These questions are based on the Newman's technique for interviewing (Vaiyavutjamai and Clements, 2006, p.121). The method used during these interviews is the task-based interview (see Section 3.3.2). But seeing that the students had finished my test in a shorter time than expected, I prepared some additional tasks. Some of the additional tasks are taken from the test, which are: the second, the third, and the seventh. I asked the students in the same way as with the cards, what they think about these tasks, which is the most difficult and which is the easiest one. Even in this case the structure of the questions has been the same as with the cards.

The additional tasks were given to all the students, but some were questioned more and some less around these.

Additional questions for the interview:

1. 5x-6=3x-8*

2. $\sqrt{2}x - \sqrt{3} = \sqrt{5}$ *

3. $(x-5)^2 - 100 = 0$ *

4. $\sqrt{6 + \sqrt{x-3}} = 3$ [1]

5. $\dfrac{1-2x}{x+1} + \dfrac{5}{x} = \dfrac{2x}{1-x}$ [2]

6. $\sqrt{x-6} + \sqrt{6-x} = 1$ [3]

7. $\sqrt{x-8} = 0$ *

4.4 Data analysis

The data analysis was done step by step, until I got the full picture of it. The ways to analyse were many and they have changed from time to time until I found the ones that fit to my data. The sequence of my data analysis has been as follows:

1- Transcription and translation of the interviews.

2- Choosing the episodes.

3- Listing of the data.

I will explain them in the order given above.

4.4.1 Transcription and translation of the interviews

I finished the data collection around mid-December and started my data analysis. Since I have interviewed six students, the interviews are my main data, and I started to transcribe them. The table that I have used for the transcription of the interviews has the form as below:

Nr.	Person	Time	Text	Comment

Number=Nr. denote the number of the utterance of the interview. Person means the person who talks and they are: student (St._) or interviewer (Int.). Time is defined by the time when the

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*These tasks are taken from test on page 39.

[1] Equation website, d.

[2] Equation website, a.

[3] Teacher example during the classroom observations.

utterance finishes. Text includes the utterances that are expressed during the interview. Comment is what I explain or comment about the utterance. I have also put student's works in this part.

An example for this table is given below and is taken from Student 3, when she was checking the solution of the word problem in the chosen card:

Nr.	Person	Time	Text	Comment
28.	Int.	07:41 07:45	How you can check that the perimeter is 120m?	
29.	St.3	08:09	The perimeter is 2 times a plus b. In this case a plus b was the base and the height each. The perimeter is 120m, two times 40m plus 20 and it comes 120 equal with 2 times 60, 120 equal with 120. This means that the sides of this rectangle are 40 and 20.	Then she decided to check it using the writing: $P = 2 \cdot (a+b)$ $120 m = 2 \cdot (40m + 20m)$ $120 = 2 \cdot 60$ $120 = 120$

From this example we can understand that these are the utterances 28, 29. Utterance 28 has lasted for 4 seconds, (07:41-07:45) and the interviewer is the person who asks the question: "How you can check that the perimeter is 120m?"

Student 3 is talking during this time which has finished in the minute 08:09 and said:

"The perimeter is 2 times a plus b. In this case a plus b was the base and the height each. The perimeter is 120m, two times 40m plus 20 and it comes 120 equal with 2 times 60, 120 equal with 120. This means that the sides of this rectangle are 40 and 20".

I have commented and shown her writing in the comment part. During the transcriptions I have used some signs that are listed in the appendix but below I will give a short example.

Nr.	Person	Time	Text	Comment
157.	St.2	27:06	Yes, I can do for example base 2 times h, it can be (3) I don't have [I want your strategy] yes, I would like more to work with the picture because it is more clear for me. So () where the base is twice as height, if this is the height so …	

Signs comments (see appendix 2):

(3) This sign represents a pause that is made by the person who talks and it has lasted 3 seconds. I denote only the time which is 3 seconds or longer.

[ ] This means that it is an overlapping of the discussion by another person (in this case the interviewer).

() This is used to denote that the level of hearing is not good, or it is impossible to hear the words.

The original transcriptions are in Albanian, since the data collection was done in Albania. During the interviews the students were posed even to some additional tasks. I have translated the part related to the students work with tasks in the chosen cards into English.

4.4.2 Choice of episodes.

After the transcription and the translation of the interviews, I started to read carefully through them and try to make a full picture of the students' solutions. In this way I could notice some difficulties that the students have during their solutions of the tasks and some of their abilities in relation to these tasks. They were of different types of difficulties. The choice of the episodes is done due to the research question which is: *How do Albanian students in the study approach and solve equations?* I have analysed all the students' solutions and picked the episodes that include the difficulty that the students have. I have showed parts of students' competences in solving the equations to have a full picture of their solutions. I have also presented complete solutions of the students. The presentation of the episodes is done based on the students' order of interviewing.

4.4.3 Organising the data analysis

It has been very difficult to organize the data analysis. We have tried different ways before choosing the actual one. The data analysis is divided into three parts: *Students' choice of the cards*, *Students' works with the first and second task in the chosen cards*, and *Students' work with the word problem in the chosen cards*.

Student's choice of the cards

The choice of the cards includes a general view of the students' choice among four cards. They have to define what they call the most difficult and the easiest card among the four cards presented, and they have to give their reasoning about the choice of the cards. I have chosen episodes where all the students have given their reasons for their choices. In the end of this section, I have done a summary of the analysis. This part shows which types of equations students considered easy and difficult.

Students' work with the first and second task

Students' works with the first problem of the card include the students' solutions for different kinds of equations, linear, quadratic, rational, and irrational. This part has more information because the students had to solve a diversity of equations, and have shown some difficulties. I tried also to show what students can solve without difficulties. I tried to make a general view about the situation.

This section is divided into eight parts where in the beginning I have presented a complete solution. I have considered a solution complete when the students have done correctly three steps of a solution which are: *finding the domain*, *solving the equation*, and *checking the result* which in Albanian curriculum is done in this way for the rational equations (Lulja and Babamusta, 2008, p. 94), and then I have categorized the difficulties that the students have shown. Firstly, I have presented the difficulty that the students have to find the domain of an equation. The other difficulties are as: squaring an irrational equation, solving and quadratic equations, calculation errors, and other errors that are related to the solution of an equation. The last mistakes are as: difficulty to define the type of equation, difficulties to find the common denominator of a fraction, and having difficulties to understand when an equation has solution or not. They have also shown difficulties in checking of the solutions. I have treated this part in section 5. Episodes are chosen to illustrate these types of difficulties. In each case students' works with these

difficulties are shown, both how the difficulty are handled and are not handled. The last section is a summary or a general view of the seven parts. Here it is given a general picture of the students' solutions of the first problem in the chosen cards.

Students' work with the word problem in the chosen cards

The solution of the word problem is the third and the last part of my data analysis. Here, I have treated how students manage to solve the word problem of the cards. The solution of the task was divided into three parts: *the strategy and the passage from the word problem to equation*, *the solution of equation,* and *the justification of the result*. In the beginning I have shown a complete solution and then three other students which have had some difficulties with their solutions. Mostly I have presented the episodes that contain the difficulty and not the parts that were solved correctly by them. In the end I have done a summary to this section as in the two other sections of my data analysis.

5 Data Analysis

At the start of each interview, I made a short presentation of my project and my goals. I also informed the six students that I will pose questions to them during the interview. In this chapter I present the analysis of my data. The chapter is divided into three parts where the first part is about the *students' choice of the cards*, the second part is about the *students' works with the first and second task in the chosen card* (Problem 1), and the third part is about the *students' works with the word problem in the chosen card* (Problem 2).

The first section contains the episodes where the students have expressed their reasons about the choice of the cards. I have presented them chronologically. The episodes in this section are longer than in other sections since the students' explanations include even the strategies that they wanted to solve the tasks. At the end I made a summary of the students' choices.

The second section is divided into eight parts where the in the first part I present a complete solution which is done by Student 4, and then in the six other sections are included the difficulties that the students have shown during their solution of Problem 1 (the two tasks) and also their ability to solve them. In the end I have made a summary to present the general view of the students' performance during the interviews.

In the third section the solutions of the students for the word problem are presented. It is divided into six parts. The first part is a general presentation of the students' strategies to solve this task. The other four consecutive parts contains students' working with this problem. I present a complete solution and then episodes where the students had difficulties. Finally, a general overview of the whole section is presented.

5.1 Students' choice of cards

I informed the students that they had to pick two cards out of the four: the one that they consider as the most difficult and the one that they consider the easiest. I asked them different questions during their selection of cards and their solutions of the tasks. The episodes include all the reasons of the students for picking both of the cards. The episodes presented in this section terminate when the students start to solve the task of the chosen card. They also contain parts of the students' strategies to solve these tasks. The order in which I have presented the strategies is according to students' order of interviewing (the same as the students' number). The two first students (Student 1 and Student 2) are low performing students, Student 3 and Student 4 are middle performing students, and Student 5 and Student 6 are high performing students.

5.1.1 Student 1 Unclear choice

Nr.	Person	Time	Text	Comment
8.	Int.	01:03 01:08	Have you decided it?	
9.	St.1	01:30	(11) Here, one is difficult, the other is easy, so I don't know (3) which to take as the most difficult because all here are easiest ones…	She is talking in general for the tasks of the cards.
10.	Int.	01:33	So you think all are easy?	

11.	St.1	01:42	No, no they are alternated, so some are <u>very easy</u>, for example I got the idea to take this but not (3) but since this is easy	She picks Card 2 as easy.
12.	Int.	01.44	How do you consider this?	
13.	St.1	01:53	This is simple, this is not that I can't solve it, but I thought to solve this since it is more interesting.	She thinks that the equation: $x^2\sqrt{x}-\sqrt{x}=0$ is interesting.
14.	Int.	02:00	Ok, but which are the two cards; one that you think is the most difficult and one that you think is the easiest. Why?	I insist to get two cards.
15.	St.1	02:06	The easiest for me? I don't know, a simplest for me is of this form, mostly they are simple.	She meant the form of the Card 1.
16.	Int.	02:17	So, you think that this is the easiest for you? Ok, then (3) which would you like to solve?	
17.	St.1	02:18	I will get these two,	She got Card 1 and 2.
18.	Int.	02:20	They are the easiest, why do you think so?	
19.	St.1	02:38	We take this as the easiest*, (3) eee I don't know **. I am getting this as the most difficult*** and this as the easiest one because even this doesn't have any difficulty, all are easy, <u>neither this has no difficulty</u>, (3) these two.	*is the Card 4 and she is very unsure which is showed in the ** part. *** The most difficult is the Card 2, but when she picked this card, she made quotation marks with her fingers. Now she picked Card 4 as the easiest. The underlined parts show that she is very unsure about her choice.
20.	Int.	02:39	Ok,	
21.	St.1	02:41	Well	
22.	Int.	02:50	So, as I can see these two types of equations are very easy for you, (3) who would you like to solve?	I meant the irrational equations.
23.	St.1	02:54	Eee, let's get this here and this one.	The only two irrational equations that are in the cards.
24.	Int.1	02:57	No, from the cards…	She hasn't understood my request in the beginning.
25.	St.1	02:59	E, from the cards,	
26.	Int.	03:05	From these that you have picked which would you like to solve, the entire card.	
27.	St.1	03:06	You mean as a whole one?	
28.	Int.	03:07	Ehe	
29.	St.1	03:09	I will take this.	Card 4.
30.	Int.	03:11	So this card you consider as the easiest one?	
31.	St.1	03:12	Yes.	

Student 1 is unclear; she does not know what to pick as the most difficult (9). She thinks that the tasks in the cards are alternated and she picked Card 2 as the easiest (11). She thinks that the irrational equation $x^2\sqrt{x}-\sqrt{x}=0$ is interesting (13). I ask her again to pick the cards (14) but she is unsure and chooses Card 1 as an easy one (15). Student 1 picks Card 1 and Card 2 (17) but when I ask her the reason (18), she is not very sure because none of them are difficult for her

(19). She defines Card 2 as the most difficult but she makes the quotation marks with her fingers. She does not think that any of them are difficult (19). She has no problems with the irrational equations (22) because she wants to solve the only two irrational equations that are in the four cards (23). I explain to her that I mean two cards not tasks (26) and she picks Card 4 as the easiest (29). For her the solution of the irrational equation is not a problem and she call them more interesting equations, and for the other kinds of equations she did not have any problem. We can see from this that she is much undecided to pick the two cards. In the end she picked Card 4 as the easiest and Card 2 as the most difficult. She is not very clear about the reasons of her choice.

5.1.2 Student 2 Quadratic equation

Nr.	Person	Time	Text	Comment
2.	St.2	00:26 01:57	Ok, yes (79) the most difficult? (3) In my thinking the most difficult is this.	
3.	Int.	01:58	One moment	
4.	St.2	02:05	Ok, (4) this one.	He picked Card 3.
5.	Int.	02:08	Why do you think so?	
6.	St.2	02:29	In my opinion, I will have an idea for the solution of the second equation, and according to a figure I will solve even the second problem, but I would think more about the solution of the other equation.	For him the second task: $\frac{1}{x+3} - \frac{1}{3-x} = \frac{10}{x^2-9}$ and the word problem weren't very difficult, but the problem is the equation $x^2 = -5x$.
7.	Int.	02:30	The first one?	
8.	St.2	02:31	Yeah the first.	
9.	Int.	02:32	Why?	
10.	St.2	03:01	Because I am unsure if I can solve it with the discriminant, for example I can take 5x and we can make x simply with, 1 equal with zero or simply it will be a common number that these inequalities will be equal. So, a simple number that these inequalities to be equal. So, I have an idea who this number will be, but I thought since I am in the middle of the discriminant and this idea, I thought to let it and to be more sure…	He is giving his reasons why he is picking the Card 3 as the most difficult for him.
11.	Int.	03:04	Ehe, so this is the most difficult for you? Ok.	
12.	St.2	03:09	Since I am an dilemma, between these two, this is the one that I can solve …	
13.	Int.	03:12	The easiest?	Card 1
14.	St.2	03:49	Yes, because the second task is similar for both, this means that I would have some ideas to solve it. The first task will be very easy to solve, so it will be solved with the discriminant if we pass 2x, and equalize it with 0 and this*, so the equation of the second grade with one unknown, 2x^2 so, since we can have x^2, we compare both of them and find the common denominator x^2-4, which can be 2-x and 2+x, and then it can	*he is explaining the task 1b) During this reasoning behind, he is giving even the strategy how to solve these tasks.

			be solved by multiplying with this.	

Student 2 says that Card 3 is the most difficult because of the equation $x^2 = -5x$ (6). The other tasks are not a problem for him. He is in a dilemma for this task (10). He can solve it with the discriminant or to find a value that fits to this equation. The second idea is very unclear for me (10). But he can solve quadratic equations of the form $ax^2 + c = bx$ as the one that is in Card 1 (14) which he considers as the easiest (12).

Student 2 has picked Card 3 as the most difficult and Card 1 as the easiest. He is very clear in his reasoning, and he does not have problems to solve the second task of the Card 3 and the word problem, but he has difficulties to solve the first task of this card which is $x^2 = -5x$. He considered the card with equation $x^2 - 3 = 2x$ as the easiest. The reason behind this picking is the difficulty that the student has in making a connection between a quadratic equation of the form $ax^2+bx+c=0$ and $ax^2=bx$. He had two ideas to solve the last equation: with the discriminant or finding a common number through substitution.

5.1.3 Student 3 Clear choice

Nr.	Person	Time	Text	Comment
1.	St.3	00:37	Since there is no unknown in the denominator, it is easier to solve it, so we find the common denominator and we can solve very easy even 1b) since it is equal with zero, we can factorize square root of x and in the same time we can eliminate it because one of the factors of this product has to be zero. So we can find the both solutions of this equation. In this other card* the second task, 1b), since we have a square root, we need to square both of the sides of the equation and maybe it needs more work to do.	She picked Card 2 as the easiest, and she gives the reasons why she thinks that. * the Card 4 is the most difficult for her, and the reason is the second equation which is an irrational equation: $\sqrt{x+3} - \sqrt{2-x} = 1$ It is not that she doesn't know how to solve it but she thinks that it has more work to do.

Student 3 is very clear in her reasoning, and she has picked Card 4 as the most difficult and Card 2 as the easiest. The reason for this choice is that Card 2 has an easy equation as the first task and the second one can be solved by factorization. But she considered Card 4 as the most difficult because of the second task, which is an irrational equation. It is not that she cannot solve it but she think that it has more work to do than the others. This does not mean that she knows to solve it correctly, but she knows the strategy which she has to follow.

5.1.4 Irrational equation

This section is focused on the difficulties that the students have with the irrational equations during the choice of the cards.

Student 4 Linear irrational equation

Nr.	Person	Time	Text	Comment
3.	St.4	01:57 02:00	I am thinking that this is more difficult than the others.	He thinks that Card 4 is the most difficult.
4.	Int.	02:03	So, you think that this is the most difficult, why?	
5.	St.4	02:05	Problem with square roots.	He has problems with irrational equations.
6.	Int.	02:09	So you think that eee,	
7.	St.4	02:12	If I compare with the others, it is more difficult.	The equation is: $\sqrt{x+3} - \sqrt{2-x} = 1$
8.	Int.	02:24	So you think that the irrational equations are the most difficult? But you have chosen this as the easiest, or no? Why is this the easiest because it includes also …	He has chosen as the easiest Card 2 which includes an irrational equation too which is: $x^2\sqrt{x} - \sqrt{x} = 0$
9.	St.4	02:38	But here the square root can be eliminated in a simple way, but in the other card, the denominator can be eliminated very easy, with one simple step, so I think that this is the easiest.	
10.	Int.	02:43	Ehe, which would you like to solve?	
11.	St.4	02:47	(3) I will solve the easiest.	Card 2

Student 4 chooses Card 4 as the most difficult (3). The reason is the difficulty that he has to manipulate with irrational equations of the form $\sqrt{ax+b} + \sqrt{cx+d} = 0$ (5) but he can manipulate with the irrational equation like the one in Card 2 (9) which he picks as the easiest (11).

He has problems with irrational equations of the form $\sqrt{x+3} - \sqrt{2-x} = 1$. For this he picked Card 4 as the most difficult and Card 2 as the easiest one. Even both of the cards have irrational equations, for the equation $x^2\sqrt{x} - \sqrt{x} = 0$ he knows how to solve it. His reason of choice is based on these two equations; he did not have problems with the other tasks of cards.

Student 5 Irrational equation as a product

Nr.	Person	Time	Text	Comment
1.	St.5	03:01	(180) this seems the easiest.	Card 3
2.	Int.	03:07	You think that this is the easiest? Ehe, why do you think so?	
3.	St.5	03:29	Because the first equation can be solved with the discriminant, and this is a very easy way, and in the second the denominators are common*, both of	* Denominators are similar because they have a relation between each other

			these have no difficulty in my opinion to solve and the problem is the same for all the cards.	
4.	Int.	03:35	Yes, yes same for all the cards. And the most difficult?	
5.	St.5	03:36	Just a moment,	
6.	Int.	03:37	No take your time.	
7.	St.5	04:09	(26) I think this is the most difficult, and I don't know if I can solve it or not. This second …	He is much undecided. He thinks that Card 2 is the most difficult. He is not sure if he can solve it or not. The equation is $x^2\sqrt{x}-\sqrt{x}=0$
8.	Int.	04:10	Why do you think so?	
9.	St.5	04:13	I don't, these with the square roots are not very …	
10.	Int.	04:21	I understand, you haven't work very much with irrational equations, but this one is …	I meant the second equation of Card 4 that is: $\sqrt{x+3}-\sqrt{2-x}=1$
11.	St.5	04:22	This?	
12.	Int.	04:23	Yes, why they are not similar?	These two irrational equations.
13.	St.5	04:44	But if I square both of the sides in this equation, we can take out from the square root and then it is easy to solve it. But I don't know in this (3) I don't know how to reason.	
14.	Int.	04:49	So you don't know how to turn it in a simple form?	
15.	St.5	04:50	No, I don't know,	
16.	Int.	04:52	So, which would you like to solve?	
17.	St.5	04:53	I can solve this.	Card 3

Student 5 picks Card 3 as the easiest (1) by explaining that the first equation can be solved with the discriminant and the second one by finding the common denominator (3). He says that the irrational equation of Card 2 is the most difficult, which makes him to define that Card 2 is the most difficult (7). He does not know how to manipulate with this kind of equation (9) but according to him it is possible to solve the irrational equation of Card 4 by squaring both of the sides of the equation (13).

He has given all the strategies for the solution of all the problems on the cards that he has picked. Firstly he has chosen Card 3 as the easiest, because the first equation is a quadratic one and can be solved by the discriminant and the second one can be solved by finding the common denominator, and multiplying it on both sides of the equation. The reason for his choice of Card 2 as the most difficult card has been the second task. He has problems with the equation of the form $x^2\sqrt{x}-\sqrt{x}=0$. He said that the equation $\sqrt{x+3}-\sqrt{2-x}=1$ is easy for him since he can square both of the sides of the equation, but for the first one he does not know how to manipulate with it.

5.1.5 Student 6 Determined choice

Nr.	Person	Time	Text	Comment
1.	St.6	00:02	The easiest,	Card 1
2.	Int.	00:22	One second… () we can start. You think that this is the easiest, why?	
3.	St.6	00:59	This is the easiest because the first equation we have a quadratic equation of the form ax^2+bx+c, which we can solve with the discriminant. So it has two solutions and we can find them very easily and in the second equation we have to multiply with the common denominator, that in this case is x minus, x-2, x+2 and we have to <u>do a change</u>. And * here the first equation is a simple one but the second has more work to do maybe then the others.	She explains the reasons why she picked Card 1 as the easiest and then she passed to the choice for the second card as the most difficult * which is Card 4. This happened because of the irrational equation: $\sqrt{x+3}-\sqrt{2-x}=1$ The underline means that she got the difference between the denominators in the task: $\frac{x-1}{x+2}-\frac{x+5}{2-x}=\frac{2(x^2+2)}{x^2-4}$.
4.	Int.	01:03	So, do you consider this as the most difficult with this equation? As an equation?	I wanted to know if it was only the equation that made the card difficult.
5.	St.6	01:05	What?	
6.	Int.	01:08	Do you consider this as the most difficult because of this equation? Why?	
7.	St.6	01:12	But if I compare with the others this seems to be the easiest.	
8.	Int.	01:15	But even this card has a similar equation.	Card 2 that has $x^2\sqrt{x}-\sqrt{x}=0$ another irrational equation.
9.	St.6	01:24	Yes, but there we can factorize x, the square root of x and then it is very easy to solve.	
10.	Int.	01:26	Very good, which do you like to solve?	
11.	St.6	01:28	I would like to solve this.	

Student 6 picked Card 1 as the easiest (1) and she explains the strategies to solve them (3). She picked Card 4 as the most difficult because of the irrational equation (3) because the other task is not a problem for her. She says that the equation $\sqrt{x+3}-\sqrt{2-x}=1$ is the most difficult if she compares with the others (7). The irrational equation of Card 2 is a simple one for her (9) and she tells the strategy to solve it.

She has chosen Card 1 as the easiest and Card 4 as the most difficult. Her reasoning is related to both of the tasks in the cards. She has chosen Card 1 as the easiest because it includes two easy tasks, as she called them the quadratic equation and the fraction equation. She thinks that the irrational equation $\sqrt{x+3}-\sqrt{2-x}=1$ is the most difficult. It is not that she does not know how to solve it, but the reason is that it will need more work.

5.1.6 Summary

The students have chosen all the four cards. Most of them have given clear reasons about their choices. The diversity of these choices is shown in the table below:

Student Card(s)	Low Performance Grade: 4, 5, 6		Middle Performance Grade: 7, 8		High Performance Grade: 9, 10	
	Student 1	Student 2	Student 3	Student 4	Student 5	Student 6
1		Easy				Easy
2	Difficult		Easy	Easy	Difficult	
3		Difficult			Easy	
4	Easy		Difficult	Difficult		Difficult

Table 4. The diversity of students' choices.

Based on the table we can see that the Card 2 and Card 4 are the most picked by the students. The next table shows the number of times that the students have considered a given card as difficult or easy.

Card	Easy	Difficult
1	2	0
2	2	2
3	1	1
4	1	3

Table 5. Number of choices for each card.

Card 1 is chosen as the easiest one two times. This has happened because it includes a quadratic equation and a fraction equation. One out of the six students has chosen a quadratic equation as difficult, and it is in Card 3. This student said that the quadratic equation $x^2 = -5x$ is a difficult one. He has ideas of how to solve it but he is not sure. This is shown in the transcription of Student 2 (p.10). Card 3 is similar with the Card 1, because it includes a quadratic equation and a fraction equation. It is chosen one time as the most difficult for the reasons mentioned above and one time as the easiest one.

There is also a similarity between Card 2 and Card 3. The students' choices for Card 2 have been two times as the most difficult and two times as the easiest. The reasons to pick it as the easiest has been:

1- Student 3 told that it includes only a linear equation, which denominator can be taken away very easy and the second equation $x^2\sqrt{x} - \sqrt{x} = 0$ we can solve just factorizing (p.120, 1). The same reason was also given by Student 4 (p. 122, 9).

And Card1 is chosen as the most difficult by the following reasons:

1- Student 1 said that the equation of the form $x^2\sqrt{x} - \sqrt{x} = 0$ is more interesting, and she categorized it as the difficult one. This is shown in the utterance 13 and 19 (p. 108).

2- Student 5 was not sure how he can manipulate the equation $x^2\sqrt{x} - \sqrt{x} = 0$. Since he did not have any idea about it, he considered Card 2 as the most difficult.

Card 4 is chosen as the most difficult card by half of the students that I have interviewed. Only Student 1 considered this an easy card. Her reason was that she was very fond of irrational equations. She was not very clear about her decision (19, 23, 29, p. 108). The students have chosen this Card 4 as the most difficult because of the equation 1b): $\sqrt{x+3} - \sqrt{2-x} = 1$. Student 3 (p. 120,1) and Student 6 (p. 130 , 3) said that is not very difficult but it will need much work to do in comparison with the others, and Student 4 accepted that he has problems with irrational equations (p. 122, 5).

5.2 Students' works with the first and second task of the chosen card.

The cards that are used in the interviews include three tasks: two equations that I have put under one request (Problem 1a and 1b) and one word problem (Problem 2). In this section I deal with the way how the students have solved the equations in Problem1. I will show a complete solution of one task and also some difficulties that the students have demonstrated during their work with the equations.

5.2.1 A complete solution:

I have considered the following solution a complete one, because the student has found the domain, has solved it, has found the set of the solutions and has checked of the result. This is the solution of Student 4 for the second task, the equation: $x^2\sqrt{x} - \sqrt{x} = 0$ [C2, P 1b]:

Nr.	Person	Time	Text	Comment
53.	Int.	16:53 17:48	… (21) How do you understand this task, or which is your strategy to solve this task?	I asked him about the strategy that he want to use.
54.	St.4	18:14	I will factorize the square root of x, to separate it, and the equation can be turned into a product of factors. So in this way it will be easier to solve it. It is simply equal with zero (). In the beginning I have to find the not allowed values.	The strategy is to factorize the square root of x and to express this equation as a product of factors.
55.	Int.	18:20	Very good. Why do you find these values?	
56.	St.4	18:43	Because for these values this equation doesn't have solution, they are not allowed values. So for negative x this equation doesn't have meaning, because the square root of a negative number doesn't exist.	He also found the domain of this equation; x has not to be negative. The student has written: $x < 0$
57.	Int.	19:05	(20) eee, this last passage ...	The passage is: $x^2-1=(x-1)(x+1)$
58.	St.4	19:10	Ee, it is a formula, a formula that we have done in	

59.	Int.	19:43	So you have just applied a formula? (27) So do you think that this equation has three solutions?	He has solved the equation and has found three values, 0,-1 and 1.
60.	St.4	19:51	No, since for negative x the equation has no meaning, so we have two solutions which are 0 and 1.	He gives the reason why he is taking away -1.
61.	Int.	19:56	Ehe, ok, but how can you check that these are the right solution?	
62.	St.4	20:20	I will, I will replace (18) zero equal with zero.	He did the checking of the result.
63.	Int.	20:50	Ehe, (25) so you mean that when zero is equal with zero we have that the solution is right?	
64.	St.4	20:52	Yes, yes	

The solution and the check of this equation are:

Figure 9. The solution of the task

Figure 10. The checking of the result

In the beginning he explains the strategy that he wanted to follow by factorizing out the square root of x and expressing the expression on the left side of the equal sign as a product of two expressions (54). So in this way one solution is x=0 (54) and the other solution he gets from manipulating the other factor which is x^2-1 that was divided into two factors x-1 and x+1 (57). Since their product is equal with zero this means that at least one of them has to be zero, so we get that x-1=0 or x+1=0. The values of x that the student found are: x=0, x=-1 and x=1, with the condition that x has not to be negative since the domain of this equation is x bigger or equal with zero. In this case the solutions of this equation are: x=0 and x=1 (60). The explanations that the students gave for this solution are very clear and correct. As we can see from the episode and from the student's work with this task that he knows how to solve it.

Other students have also given complete solutions. But not all the solutions are as complete as the one above. Only Student 1 and Student 6 had no complete solution. I will give a general table of the students' solutions of these tasks, to have a full picture of their answers. The table is divided in six parts, where each of them represents respectively, student's number and the card that he/she has chosen to solve, the problem that he/she is solving, and for each of the tasks the student's performance. The column of the problem (task) is divided in two parts. The first part has the number of the task and the second part has the task in itself. The next three columns

represent the demands (steps) that I have listed for a complete task. The signs that I have used in this table are given below:

Student	Problem		Domain	Solution	Checking	Complete solution
Student 1 Card 4	1a	$2(3x-7)+4(3x+2)=6(5x+9)+3$	——	Wrong	Wrong	No
	1b	$\sqrt{x-3}-\sqrt{2-x}=1$	Asking and helping	Wrong	Wrong	No
Student 2 Card 1	1a	$x^2-3=2x$	——	Right	Right	Yes
	1b	$\dfrac{x-1}{x+2}-\dfrac{x+5}{2-x}=\dfrac{2(x^2+2)}{x^2-4}$	Asking, helping	Wrong	Asking, right	No
Student 3 Card 2	1a	$\dfrac{6x-7}{4}+\dfrac{3x-5}{7}=\dfrac{5x+78}{28}$	——	Right	Asking, orally	Yes
	1b	$x^2\sqrt{x}-\sqrt{x}=0$	Asking, orally	Right	Asking, orally	Yes
Student 4 Card 2	1a	$\dfrac{6x-7}{4}+\dfrac{3x-5}{7}=\dfrac{5x+78}{28}$	——	Wrong	Asking, wrong	No
	1b	$x^2\sqrt{x}-\sqrt{x}=0$	Right	Right	Asking, right	Yes
Student 5 Card 3	1a	$x^2=-5x$	Asking, wrong	Right	Asking, right	No
	1b	$\dfrac{1}{x+3}-\dfrac{1}{3-x}=\dfrac{10}{x^2-9}$	Asking and helping, orally	Right	Right	No
Student 6 Card 1	1a	$x^2-3=2x$	——	Wrong	Asking, right	No
	1b	$\dfrac{x-1}{x+2}-\dfrac{x+5}{2-x}=\dfrac{2(x^2+2)}{x^2-4}$	Right	Right	Asking, right	Yes

Table 6. Students' solutions

Domain: the line (------) that I have used means that I took for granted the fact that students know to find the domain of these linear and quadratic equations, since it is all real numbers, $\mathbb{R}$. By *asking*, I mean that I have asked the students to find the domain. *Helping* to show when the students needed to get oriented and helped by me. But many have not achieved to find the domain, even with help, so I have signed it *wrong*. For students that found the domain, I used the

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word *right*. I have used the word *orally*, to show that the students have obtained the result by explaining it orally. I have written for example for Student 5, "Asking, orally and helping", which means that he needed to be asked to find the domain, and he has done find orally and I have helped him to find the correct result.

I have used two words to classify the students' solutions which are: wrong and right. This is based on their results, if they have got the right one or not.

After the solution of the tasks, many students needed to be asked to check their solutions. I have used "asking" to show this and "orally" when they have explained orally, and the words "wrong" and "right" to show if they have done errors during the checking or not.

I will give an example to explain these: Student 4, for the task [C2, P 1a] I have used *asking, wrong* which means that he needed to be asked to check the result and during his checking he has made some errors (calculation errors).

The last column gives the result if the solution of the task is complete or not. I have used "yes" and "no" to show this.

We can see from this table that Student 3 is the only student that has complete solutions of both tasks in the chosen card. We can see that the checking of the results is done orally and right. I accepted her solution as complete even if she has checked the result orally. Based on the word problem, she knows how to check the result. Student 1 and Student 5 have no complete solutions at all. They have made mistakes in each of the tasks of the first problem. These errors are of different kinds. Some are calculation errors, and some other understanding equations. The other students, Student 2 and Student 4 have one task complete and the other not. The errors showed in this table are related to the difficulties that students have to find the domain, for instance Student 5 has solved and checked the result very well of both of the tasks but when he wanted to find the domain of these equations, he found difficulties.

As we can see from this table, most of the students needed to be asked to find the domain and to check their results. They did not consider them as part of the solution, even if the request of the task was clear.

The students' difficulties have been organized in the following areas:

1- Finding the domain
2- Squaring an irrational equation
3- Solving quadratic equations
4- Calculation errors
5- Solving an equation
6- Control of the results

Now, I will present them in the above order.

5.2.2 Finding the domain.

During the solution of Problem 1a and 1b, some of the students have shown difficulties in finding the domain. Student 1 was one of them when solving the task [C4, P 1b]: $\sqrt{x+3} - \sqrt{2-x} = 1$.

One of the demands for its complete solution is finding its domain. The episode that I have chosen for this part is given below:

Nr.	Person	Time	Text	Comment
106.	Int.	24:41 24:49	So, according to your opinion this equation exists for all x from $\mathbb{R}$.	I asked her since she solved the equation and did not treat the domain.
107.	St.1	24:50	Yes.	
108.	Int.	25:06	If you have [()] if you have the square root, what do you need to do except finding the solution? It is even another step that you have to do. Eee because the square root is not always true for every x out of $\mathbb{R}$.	
109.	St.1	25:12	No, I have to find the domain in which I can solve this equation.	

Theoretically, she knows that something has to be bigger or equal with zero, but she is not sure that the expression under the square root has to be such. Her work is as the following:

Figure 11. Student 1's finding the domain

Her reasoning is not wrong mathematically, but the first line is redundant. So she find two sets, and now she has to find their intersection. She has shown difficulties because she was unsure in which direction the arrow has to go. Here is her writing; in the end she could find the right solution.
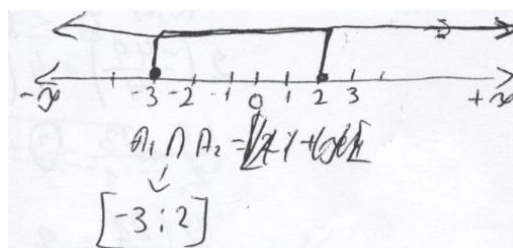
Figure 12. Expressing the inequalities in number line.

The following is an extract of her reasoning:

Nr.	Person	Time	Text	Comment
124.	Int.	27:41 27:49	How do you understand that x is bigger than -3 in the number line?	I tried to help her to express the inequality in the number line.

125.	St.1	28:10	x bigger than -3; we have the entire x in the left side because if they will be in the right side, the numbers will be bigger than -3. Eee, in the left side, we can say even as many apples that we have, as much in the left, right side of the number line, means bigger numbers, and in left of the number line we have smaller numbers.	She is trying to make it concrete, by using "apples" in the number line.
126.	Int.	28:13	What have you done in this case?	
127.	St.1	28:30	Eee, we have the numbers, I have given the numbers bigger than -3 and so -3 and -∞, will go in the left, and we always have bigger numbers.	She thinks that the arrow will go on the left.
128.	Int.	28:33	Are you sure?	
129.	St.1	28:48	(3) So for x bigger then -3, eee, ooo, infinite, 2, minus infinite, is infinite. It is infinite because we have bigger numbers.	
130.	Int.	28:52	So, is here minus or plus infinite?	I needed to be sure that she has found correctly the sets.
131.	St.1	28:53	Yes, plus infinite.	
132.	St.1	28:56	I took away the minus, let's do this plus?	
133.	Int.	28:59	And in the other case?	
134.	St.1	29:01	It is minus infinite.	
135.	Int.	29:08	So, how you can do the cutting of these sets, because…	
136.	St.1	29:17	We have the number line, cutting is also the solution and to say the truth this year we have started to work with them, with irrational equations.	The students have not worked very much on the irrational equations.
137.	Int.	29:19	Yes, I know it.	
138.	St.1	29:52	So I am writing here, plus infinite, to not forget (3) plus 3 is a stressed point, so even 2 takes part (3) I didn't say it since in the beginning 2 and +∞ () so the cutting is 2 plus infinite. A1 cut with A2…	
139.	Int.	29:57	I can read, [2, -∞[at least this is what you have written here.	
140.	St.1	30:00	It is the smallest	
141.	Int.	30:04	Don't erase, it has no importance, but simply how do you think about this?	
142.	St.1	30:08	It is in the other side and it is simply till 2.	
143.	Int.	30:09	Is it just in the other side?	
144.	St.1	30:15	I am putting it in this way so we won't get confused.	
145.	Int.	30:16	Ok.	
146.	St.1	30:31	(3)() even the cutting with 2 and minus 3, 2. They are both in use because both of them take part here.	
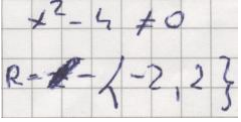
She is very unclear about finding the intersection between these two sets (125). In the beginning she says that x bigger than -3, are the entire x in left side because if they were in the right side they will be smaller than -3. Student 5 tries to explain this by making it more "concrete", something which she did not achieve. But in the end of this utterance she can fix it, because she says that "as much to the right of the number line we have bigger numbers and as much to the left
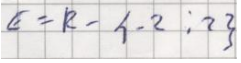
we have smaller numbers". She is right, now but what she has written, it was wrong. So she tries to correct the error. She has written [-3,-∞[and she thinks that they have to go to the left because we have bigger numbers (127). I try to make her think about the signs that have to have the infinites (130) because she mixes them (127, 129). She finds the right signs for the infinites (131, 134). She finds the right intervals, but the difficulty is to put them in the number line. She knows that the numbers -3 and 2 are part of the intervals, and she does well the arrow for the interval [-3, +∞[, but for the interval]-∞, 2] she draws the arrow in the opposite side. She gets the answer [2, +∞[(138). I stress the interval]-∞, 2] (139) so she can see better what she has written. She finds the right direction and even the right answer (146).

Student 2 has the same difficulty. During the solution of the [C1, P 1b]:

$$\frac{x-1}{x+2} - \frac{x+5}{2-x} = \frac{2(x^2+2)}{x^2-4}$$, he made two mistakes. He had difficulty to find the domain, and he was mixing domain with the values that are not allowed and with the solution set. The domain in Albanian mathematics is usually denoted by E. Here is the whole episode that shows Student 2's work with the domain:

Nr.	Person	Time	Text	Comment
44.	Int.	09:04 09:07	But can you multiply with zero?	I asked if he can multiply with zero both of the sides of an equation.
45.	St.2	09:08	No.	
46.	Int.	09:09	Then,	
47.	St.2	09:32	The expression doesn't have sense, so only except zero, with numbers bigger than zero, we can multiply the expression (6) here we can find the not allowed values for the equation, that is x^2-4 different from zero or bigger than zero	He thinks that only when the expression is bigger than zero.
48.	Int.	09:34	Why bigger than zero?	
49.	St.2	10:12	So, because it needs to be bigger than zero because if it is negative it has no sense. No, no it has sense for negative numbers; it has sense when it is not equal with zero. It doesn't matter so it can be negative or positive, it is enough just to be different from zero. So, x^2 (5) then ℝ minus -2 and 2, that is the domain of this equation, and these values can be taken when we equalize x^2-4 with zero. Then we start to solve the equation for values different from zero, having in mind that …	He got the problem and he wrote: $x^2 - 4 \neq 0$ $R = E - \{-2, 2\}$ where instead of the spot was E.
50.	Int.	10:13	What do you understand with ℝ ?	
51.	St.2	10:14	All the real numbers.	
52.	Int.	10:15	And E?	E is called the domain.
53.	St.2	10:18	It is the domain of the equation...	
54.	Int.	10:22	But why you have written ℝ equal with E minus -2, 2?	
55.	St.2	10:33	It is the set of the solutions, the set of all the numbers except -2, so E is the set of the not allowed values that in this case are -2, 2.	And then he said that E is the set of not allowed values.
56.	Int.	10:47	But if I tell you that E is equal with minus -2, 2 (34). You called E domain, [domain, yes], so you have done …	

| 57. | St.2 | 11:16 | The domain, the domain will be indeed *E* equal $\mathbb{R}$ minus -2, 2 () it will be according in my opinion, so if we take away *E* from here, it is the set of real numbers without, only without these two, and *E* will be $\mathbb{R}$ minus -2,2, will be the demand set. | I have cut this part of the utterance. He wrote: |

Student 2 makes an error saying that the domain of a rational equation is when its denominator is bigger than zero (47). This means, the student is not very clear about the finding of the domain, because he realises later that he could accept even the negative values of the denominator (49). Then he mixes the domain with the set of not allowed values and with the set of solution. We can see this if we compare the utterances 53 and 55. He knows that -2, and 2 are not allowed values for the equation (55). He achieves to find the exact solution (57).

Student 5 has shown difficulties finding the domain for the task [C3, P 1b], $x^2 = -5x$. There are no exceptional values in this case. So the domain is $\mathbb{R}$. I asked him to see if he can make the difference between the solution set and the domain. He made a mistake, since he mixed these two sets. The extract of how he explained it is given below:

Nr.	Person	Time	Text	Comment
38.	Int.	09:02 09:09	But, what do you think about the domain of this equation?	I am asking just to know.
39.	St.5	09:29	The domain is x (7) the domain is the set with eee … 0 and -5.	He mixed it with the set of solution that he has found.
40.	Int.	09:32	How do you understand the domain of an equation?	
41.	St.5	09:46	Domain is, so the values that (10) eee	
42.	Int.	10:01	Can make a difference between the domain and the set of solutions of this equation? (3) Which is the difference between them? Or which is the subset of the other.	I am trying to see if he can make the differences.
44.	St.5	10:39	The domain is the subset … eee wait, (13) the set of solutions is a subset of the domain and in another example the domain can be an interval, eee interval eee from zero to 5 and the set of solutions is	

Student 5 mixes the set of the solution with the domain (39). I needed to know if he can do the difference between the set of the solutions and the domain for this equation (42). He tries to find it and he is sure that the set of solution is a subset of the domain, but he cannot tell me the right answer (44). We can see that he is not very sure about the differences between these sets, and it was difficult for him to make the reasoning. We can see this from the big pauses that he has taken during the answer.

The second task is [C3, P 2b]: $\dfrac{1}{x+3} - \dfrac{1}{3-x} = \dfrac{10}{x^2-9}$. Student 5 has not written anything in his paper in relation to the domain, but gave an oral explanation as cited below:

Nr.	Person	Time	Text	Comment
66.	St.5	17:57	Yes, so multiplying both of the sides with the same	He is explaining

		18:03	expression, it will keep its equivalence.	how he can save the equivalence.
67.	Int.	18:20	Ok, but do you think that they are some conditions for this expression? (3) For example if this " $x^2 - 9$ " expression is zero, do we have the right to multiply with this?	I am trying to orient him to find the domain.
68.	St.5	18:23	So, zero, multiplied with a number gives always zero.	
69.	Int.	18:34	So, you don't have the right to multiply with this number. Basing on this, which is the first thing that you have to do in this case?	
70.	St.5	18:41	The denominator has to be different from zero ()	He knows it.
71.	Int.	18:43	What does that mean?	
72.	St.5	19:04	That, (4) the domain are all the numbers, the numbers eee all the whole numbers, eee whole, all the rational numbers except square root of, all the real numbers except zero.	He thinks that the domain is $\mathbb{R}$ -{0}
73.	Int.	19:39	Taking away zero, ehe ok, if I replace x with zero this means that the 1/3 -1/3 is equal with zero, equal with 10/-9, that means that this is not a solution but you can't say that it doesn't exist. It doesn't give a solution for the equation but it isn't a non allowed value for this equation. Because the non allowed values when we have fraction, when we have square root …	And then I tried to replace x with zero in the equation, so he can see the difference.
74.	St.5	19:52	X+3 is different from zero, I wanted to say so but then we have that x is different from zero. 3-x different from zero, this means that –x is different from -3, so x is different from 3.	He got that the expression has to be different from zero.
75.	Int.	19:59	So, what is the domain of this equation?	
76.	St.5	20:06	The set of the not allowed values for these equations are 3, and the domain is the real numbers except 3.	So his answer is: $\mathbb{R}$ -{3} the domain
77.	Int.	20:13	But if I say to you that (3) that I want to take -3 for example.	
78.	St.5	20:22	Then there are two solutions, -3 and +3 ()	
79.	Int.	20:25	So, how you can find the domain?	
80.	St.5	20:26	Now is over, you told me everything.	
81.	Int.	20:28	It doesn't matter.	
82.	St.5	20:30	The domain is the set of real numbers except 3 and -3.	

Student 5 thinks that since the expression that we have to multiply the equation has to be different from zero, this means that the not allowed value of this equation is zero (72). I try to explain by substituting the zero to the equation (73). He says that this was in his mind, and he finds that the not allowed values are 3 and -3, by the inequalities x+3 different from zero and 3-x different from zero (74). But he says after all that 3 is the not allowed value and the domain is the set of the real numbers except 3 (76). He understands that they are the two values not allowed for this equation (78) when I replaced -3 to the equation (77).

We can see from this episode the students' way of reasoning to find the domain. He has found the domain orally. I helped him to find the solution. In the beginning he said that the unknown has to be different from zero. I asked him to replace x with zero and to see what he will get, and then he

told that the expression (denominator) has to be different from zero. He showed that $\mathbb{R}$ -{3} is the domain of this equation, even if he has found that the solution of x^2-9 equal to zero has two solutions: -3 and 3. But in the end, he achieved the exact answer for the domain.

From the general table and from what I have showed above, we can see that half of the students that I have interviewed have difficulties with finding of the domain. The three others do not have difficulties, but still for example Student 3 needed to be asked about the domain. Student 4 and Student 6 did not need to be asked to find the domain. These three students have found the domain correctly. The others do not treat domain as part of the solution of the task. Finding the domain means for these students, to find firstly the not allowed value(s) of the equation, and then they find the domain as the real numbers except this/these number(s). The domain is defined by *E* and it is given as the set of real numbers $\mathbb{R}$ minus the not allowed values of the equation. Sometimes they keep in mind that the unknown has to be different from this/these value(s). So mostly they keep in mind the complementary set of *E*. Student 4 is a perfect example of such behaviour.

5.2.3 Squaring an irrational equation.

One of the difficulties that I have noticed in the interviews and even on the last teacher' test is squaring an irrational equation. Two out of six students in the interviews have shown this difficulty. Student 1 is part of the low performing group, and Student 5 is in the high performing group (the task that Student 5 solved is part of the additional tasks). The task is [C4, P 1b].

Nr.	Person	Time	Text	Comment
91.	St.1	22:54 23:04	Eee, good. The first thing that we do [the first thing], is to square the equation.	
92.	Int.	23:08	Is this the first thing? What kind of equation is this?	
93.	St.1	23:09	Irrational?	She knows the type of equation.
94.	Int.	23:10	Yes.	
95.	St.1	23:15	So, we square both of the sides	She knows what to do
96.	Int.	23:16	So, is this your strategy?	
97.	St.1	23:18	I guess,	
98.	Int.	23:20	Ok	
99.	St.1	23:52	To get away the brackets, eee sorry the square root (11) so in this case it can be reduced and we have (7) plus 2, it is zero.	The reason why she want to square.
100.	Int.	23:55	What is the logic behind this reasoning?	
101.	St.1	24:10	We know that the square root of a number is that number raised in the second power and to take away the square root of a number or of an expression that is inside of a number, we raised it in the second power so we have the square root reduced.	

In this episode we can see that Student 1 knows which type of equation she is dealing with (93), and she knows what to do to solve this equation (95), but she does not know how to do it. She was very clear that she will square both of the sides of the irrational equation (95). She wants to cancel the square root; this is the reason why she is squaring both of the sides of equation (99). I

need to know her logic (100), but she is very unclear. She says that the square root is a number raised in the second power (101). But she knows that to cancel this root she needs to raise the expressions in the second power (101). The way of squaring is wrong.

$$\sqrt{x+3}^{\,\rceil} - \sqrt{2-x}^{\,\rceil} = 1$$

$$\sqrt{x+3}^{\,2} - \sqrt{2-x}^{\,2} = 1$$

Figure 13. Student 1's approach to irrational equation.

The same error is done by Student 5, but the squaring is clearer in his writing. His work is given below:

$$\sqrt{x-6} + \sqrt{6-x} = 1$$

$$\left(\sqrt{x-6}\right)^{2} + \left(\sqrt{6-x}\right)^{2} = 1^{2}$$

Figure 14. Student 5's approach to irrational equation

It is clear that he has squared both of the sides of equation. He has squared each of the square roots of the left side separately, and not treated the left side as a sum.

We can see that in both cases the students; do not treat the left side of the equation as an expression which needs to be raised in second power. They divide it according to the terms that it is composed of, so they square separately, each of the square roots of the left side.

During the interviews, other students have worked with irrational equations. The cards that I have used to interview the students contain two irrational equations: [C2, P 1b] and [C4, P 1b], the first one is picked two times as the most difficult and the second one three times as the most difficult. They have been chosen as the most difficult for the reason of being irrational equations. But for some students as Student 3 and Student 6 the equation $\sqrt{x+3} - \sqrt{2-x} = 1$ is not impossible to solve, but it is more laboriously than other tasks. Some of the students knew how to square correctly as shown in the following cases:

Student 3 (this is a task from the additional part):

$$\sqrt{x-6} + \sqrt{6-x} = 1$$

$$\left(\sqrt{x-6} + \sqrt{6-x}\right)^{2} = (1)^{2}$$

$$\left(\sqrt{x-6}\right)^{2} + 2\cdot\sqrt{x-6}\cdot\sqrt{6-x} + \left(\sqrt{6-x}\right)^{2} = 1$$

Figure 15. Student 3' s approach to irrational equation

Or the same task but for Student 4, who has seen over this task and has solved it by using the domain:

Figure 16. Student 4's solution of irrational equation.

He could check that this equation did not have any solution. I would like to give a full picture of the students that have worked with the irrational equations, so I will present Student 6:

Student 6 (even this task is part of the additional tasks):

Figure 17. Student 6's approach to irrational equation.

This last example is a little different from the other two above because the left side of the equation is a term in itself but she could solve this irrational equation correctly.

The examples in this section tell that students know the strategy to reduce the square root; by squaring both of the sides of the equation, but some of them do it in a wrong way. They squared separately the square roots of the sides of equations. This error is related to algebraic misunderstanding, because in this case the students do not see the relation between the formula $(a+b)^2 = a^2 + 2ab + b^2$ and this squaring. All the students except Student 2, have worked with irrational equations. Student 1 and Student 5 have showed the problem of squaring during the solution of the task [C4, P 1b]. But the others have solved these equations correctly.

5.2.4 Solving quadratic equations.

Another difficulty showed during the interviews is connected to quadratic equations. Student 2 that I have interviewed has shown difficulties in solving the equation [C3, P 1a]: $x^2 = -5x$. He does not know how to manipulate it, he has two ideas but he is not sure and instead of this equation he picked the quadratic equation [C1, P 1a]: $x^2 - 3 = 2x$, as the easiest.

This difficulty is expressed in the following episode:

Nr.	Person	Time	Text	Comment
10.	St.2	02:32 03:01	Because I am unsure if I can solve it with the discriminant, for example I can take 5x and we can make x*, simple 1 equal with zero or simply it will be a common number that these inequalities will be equal. So, a simple number that these inequalities to be equal. So, I have an idea who this number will be, but I thought since I am in the middle of the discriminant and this idea, I thought to leave it and to be more sure…	*his idea is not very clear to me, but at least with what I understand from the interview, he is trying to find numbers, by guessing, that fulfil this equation.
11.	Int.	03:04	Ehe, so this is the most difficult for you? Ok.	
12.	St.2	03:09	Since I am an dilemma, between these two, this is the one that I can solve …	

Student 2 knows to solve a quadratic equation, especially if it is given in the formal form. He has problems with the task [C1, P 1a] because he does not know how to solve it. He has two ideas to solve this task; with the discriminant which is the common way when you have a quadratic equation and a second idea which is very unclear (10). He does not feel sure about the way which he wants to solve this task so he changes the card and pick another one with a quadratic equation of the formal form (12).

The first task of the card that he has chosen [C1, P 1a] as the easiest is also a quadratic equation $x^2 - 3 = 2x$. But when he wrote it, he turned it into the formal form ax^2+bx+c=0. He defined a=1, b=-2 and c=-3 and solved it correctly by using the formula and checked it as it is given below:

Figure 18. Student 2's solution of the first task.

Figure 19. Students' control of the results.

For the first when x=3 and for the second when x=-1, he found that these two values (results) are the solutions of this equation, so he knows to solve quadratic equations.

Student 6 has done another error which it is not connected with an understanding error but it is more a focus error (executive error according to Orton (1983a, 1983b)). When she has solved the task [C1, P 1a]: $x^2-3=2x$, which she has solved with the help of the discriminant, she writes very well the formulas and defines the coefficients but when she replace them she makes mistakes.

Figure 20. Student 6's approach to quadratic equation.

Her replacement was related to the number b. Firstly, she replaced it with 1 and got the results 1,5 and 2,5. She made the check which is shown in the following episode:

Nr.	Person	Time	Text	Comment
17.	St.6	03:32 03:57	So, 1,5 square, minus 3 is equal with 2 times 1,5. Then 1,5 square is 2,25 minus 3 is equal with 3. We have- 0,5 equal with 3. That is not true.	She made the check of the solutions and she got that one of them it is not true.
18.	Int.	04:04	What has happened? What do you think that....?	
19.	St.6	04:29	(4) So, (6) maybe it has solution for the other result. (6) no,	And she could check that even for the other it was not true.
20.	Int.	04:49	You have written in the formula $x_{1,2}$ is equal with –b plus, minus the square root of b square minus 4a times c, all this divided by 2a. And this passage you have taken it just replacing.	I helped her to see what was wrong.
21.	St.6	05:07	Yes, I replaced and 2 squared is 4, minus 4 times 1 times 3, is 12, this is turned into plus and it is plus 12. It is 16, and the square root of 16 is 4.	She is finding again the discriminant.
22.	Int.	05:16	Ehe, ok, but this passage, in the first,	
23.	St.6	05:47	U, it is -2, even here, (4) so minus () so minus 1, and we have that x_1 is minus 1, and minus 6 over 2, that means that x_2 is -3.	

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Student 6 checks her results and she finds that the solution is wrong (17). She is not finding the error so I try to help her by asking what she has replaced in the formula of the solution (20, 22). She understands the error and she replaces b with 2 (23), even if she says that b is -2.

Later in the interview, I asked her to make the check of the solution again, which were: -1 and -3. She got that one of the numbers is a solution and the other not. She said that "… the equation has only one solution, I don't know why." (27).

She had problems understanding when a quadratic equation has no solution, one solution or two solutions. This was the reason why she made the mistake because she was not basing her reasoning in the value that she has found for the discriminant. She has found correctly that the value of the discriminant is 4, so bigger than zero and the equation has to have two solutions. She did not take in consideration this, and said that the equation one solution.

Three of six students worked with quadratic equations. They were Student 2, Student 5 and Student 6. The solution of this kind of equation was not a problem for Student 5, and just a procedure for Student 6. She knows the formulas but showed problems with her reasoning during checking of the results. Student 2 had some difficulties with [C3, P 1a], which is connected to the formal form of the quadratic equation. He was not sure how to solve it, so he just gave up and picked another card to solve, an quadratic equation but given in the form $ax^2 + c = bx$ (with all the terms).

5.2.5 Calculation errors

The calculation errors are the most usual things that the students do during the solution of the tasks. These errors are not related to the understanding of an equation but they are related to attention and speed that the students have when they solve equations. During the interviews some of the calculation errors that the students have done are shown in the following episodes. I have placed them in the order they were interviewed. The first episode is part of [C4, P 1a] during its solution. Student 1 has to calculate: 6x+12x-30x= 54+3+14-8 and to pass to the next step she said:

Nr.	Person	Time	Text	Comment
65.	St.1	15:06 15:12	18x minus 30 equals -22.	
66.	Int.	15:15	How you can check it?	I wanted from her to explain the way that she makes that calculation
67.	St.1	15:22	Eee, (4), eee x to check it, how?	
68.	Int.	15:26	That it is -22.	
69.	St.1	15:34	18x minus 30 eee how, how I can check it, don't…	
70.	Int.	15:50	When you have a numerical equivalence, this one for example, 18 minus 30 you said that is equal with 22, how you can check it?	It is something related to the beginning of mathematics.

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| 71. | St.1 | 16:50 | So, 22 plus 18 is equal with 40, eee so it is 12 (3) () (4) we reduce with 3 and it comes 21, eee of fourth. ¾ cannot be reduced more. -21over 4. Eh, all this work. Can I continue the check? | |

She was wrong in her calculations, so I asked her to check that the result was exact. The only way that she could do it was by adding the numbers to get the result. And she succeeded. Her work is given below:

Figure 21.

It is very difficult to see the error here since she has written over several times. She achieved the idea that she was wrong when she was checking the result, which tells that the solution was not right. But even during the check she has done some mistakes:

Figure 22.

Here she has forgotten the minus in the second part of the left side of the equation.

Figure 23.

She has forgotten that the first 3 on the right side has to be divided by 2. And even in the last check she has made error:

Figure 24.

It is very difficult to understand the way that she has passed from the first to this second step, because the right sides of these two equations are not equivalent. When she has calculated 6 times minus 51 over 4 and the result is -153, she has forgotten that it is divided by 2.

Figure 25.

Now, in this case she has forgotten the minus, which has happened even in the first check,

Figure 26.

In general from her errors we can see that she mostly makes mistakes when she multiplies the fraction because she forget the remaining denominator and when she adds two numbers with different signs, she loose the final sign. She has done many calculation errors during the solution of this task.

Student 2, has made calculation errors during the solution of the second task, [C1, P 1b]:
$\frac{x-1}{x+2} - \frac{x+5}{2-x} = \frac{2(x^2+2)}{x^2-4}$.

The first error that he has done is when he made this passage:

Figure 27.

It is not seen in the picture, but he wrote minus instead of plus in the left side of the equation. It is the sign that joins the two product factors. This is shown in the following episode:

Nr.	Person	Time	Text	Comment
122.	Int.	20:27 20:36	() no it doesn't matter. So this passage here, you tried before to make it in a regular form to cancel and…	He has tried to change the sign.
123.	St.2	20:37	Yes	

124.	Int.	20:39	And you found that this has to be plus	I was showing what he has found.
125.	St.2	20:42	Plus, since we have multiplied with minus	
126.	Int.	20:47	Ehe, (3) why?	
127.	St.2	21:46	Yes, because that small problem can make me loose all the solution of the task…	*I have taken a part of the utterance

He has tried to change the sign of the second term of the equation, so he would cancel the denominator of this fraction with the common denominator that he has found. This is what I am alerting him to (122). He has found that the sign was plus, but he has written minus, and he give the reason why he found that (125). Then I ask him, why he has put a minus there (126). He understands where the error is (127).

This was just a replacement error. Another error is given below:

Figure 28.

He did not solve correct the other part of the equation, because when he has calculated he said that12 minus 8 is equal with -4. This can be seen from the passage above.

Even Student 4 has done some mistakes during the solution of the equation [C2, P 1a]:
$\dfrac{6x-7}{4}+\dfrac{3x-5}{7}=\dfrac{5x+78}{28}$ and the control of the result, which are:

In the beginning he got 28 instead of 78 in the nominator of the right side of the equation. He has done also some errors during calculating as:

Figure 29.

Where in the beginning he wrote that 42x +12x is equal with 24x. And during the control of the result he has had difficulties to multiply with common denominator. After he has replaced the result instead of the unknown, he found 59 as the common denominator for a part of the equation:

Figure 30.

He found the common denominator of the complete equation which is 59 times 28. But when he multiplied with this common denominator both of the parts of the equation, he forgot to multiply the right side of the equation respectively; the first term with 7 and the second term with 4, each of the parts so he got the last step in the figure below:

Figure 31.

There are many students that have made these kinds of errors during their solutions especially when they have solved equations that need a lot of calculations. The problems that are shown in this section are connected to their ways of working. Some try to work quickly, so many times they lose numbers and signs, or some of them make errors when they re-write the equation, or even have difficulties when they multiply an expression/number with a fraction. Usually they go back and forth to find the error, until they will get the right result, but this time I asked them to pass in the next task.

5.2.6 Solving an equation

Students do not tend to make a connection between equations and what they have learned in algebra. This pushes students to make mistakes, which I have categorized as algebraic errors. This section is closely related to the section 6.3, *Squaring an irrational equation.*

Student 2 has made errors during the solution of the second task, [C1, P 1b]: $\dfrac{x-1}{x+2}-\dfrac{x+5}{2-x}=\dfrac{2(x^2+2)}{x^2-4}$

Student 2, after he has found the domain of this equation, he wrote:

Figure 32. Error in transfering the task.

The first error that he made is the on the coping of the task. This has happened because he has got $2(x^2-4)$ (the right side of the equation) instead of $2(x^2-2)$ which is from the original equation. He knew to find the common denominator which is x^2-4, and he multiplied both of the sides of the equation with the common denominator. I tried to help him to manipulate with the second term on the left side (62), and he passed from the first to the second equation.

Student 2 cancelled the denominator of the first term in the left side with the common denominator that he found and said that "…Then if we get x^2-4, if we multiply with $x+2$, then the result will be x-2. This means that x-1 over x-2, …" (57). He could get the first term of the second equation. Then we have a talk about the sign of the second term in the second equation. He had some difficulties to manipulate with the sign (71). He was sure that the passage from the first to the second equation is equivalent (91) and gave his reasons (93). Even I have tried to show that the second equation was not equivalent with the first one he could not understand me. He is very unclear about the second equation, and the passage to the third one is very difficult to understand. He based on the second equation, to pass in the third one. As we can see from the figure above, it is right, but I cannot understand the logic behind (105). I gave an explanation to him by reminding him what happens when we multiply a fraction with a number (110 and 112). Then he got the third equation equivalent with the first one.

It is very difficult to understand some of his reasons. He knows theoretically what to do, but the way that he acts shows many misconceptions that he has when he solves an equation.

Student 5 chose to solve the equation [C3, P 1b]: $\dfrac{1}{x+3} - \dfrac{1}{3-x} = \dfrac{10}{x^2-9}$ but when he found the common denominator of the fraction, he just multiplied the denominators. He could not find the relation between x^2-9, $x+3$, and $-x+3$. The same problem has shown by Student 6 when she found the common denominator of the task [C1, P 1b]: $\dfrac{x-1}{x+2} - \dfrac{x+5}{2-x} = \dfrac{2(x^2+2)}{x^2-4}$. She just multiplied the equation with the product of the three denominators excluding the common $x+2$:

Nr.	Person	Time	Text	Comment
42.	Int.	09:23 09:27	So, are you finding the common denominator or no?	
44.	St.6	09:43	(7) I multiply here with x-2, x+2 and with 2-x, so in this case I can simplify that () in fraction.	

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The difficulty to find the smallest common denominator of this equation made her to get the same result, but what she found in the end are two values, x=2 and x=-2. If she could get as common denominator (x-2) times (x+2), she will get x=-2 in the end.

Students have shown even difficulties in defining the kind of equation. This is shown in the following:

Student 2, for the task [C1, P 1b] he said that "… this is an equation of the second grade with one unknown …" (32).

Student 3, for the task [C2, P 1b], said that "this is an equation of the second power with one unknown …" (17).

Another problem that Student 5 has shown is about defining when an equation has a solution or not. He was solving [C3, P 1b] very well, and found that the solution is x=5. He made the check of this result and he got that 5=5. Then he said to me that the equation has no solution. I cannot say that he is sure about that based on his solution of the first task of this card; he made a mix between the solution set and the domain. This is shown in this extract:

Nr.	Person	Time	Text	Comment
38.	Int.	09:02 09:09	But, what do you think about the domain of this equation?	I am asking just to know.
39.	St.5	09:29	The domain, is x (7) the domain is the set with eee, … 0 and -5.	He mixed it with the set of solution that he has found.
40.	Int.	09:32	How do you understand the domain of an equation?	
41.	St.5	09:46	Domain is, so the values that (10) eee	
42.	Int.	10:01	Can make a difference between the domain and the set of solutions of this equation? (3) Which is the difference between them? Or which is the subset of ()	I am trying to see if he can make the differences.
44.	St.5	10:39	The domain is the subset … eee wait, (13) the set of solutions is a subset of the domain and in another example the domain can be an interval, eee interval eee from zero to 5 and the set of solutions is ()	

This part is presented even before[4], but I thought that it is also connected to this section.

There are students that finding of the common denominator is not a problem for them, for example Student 2 who has the same task as Student 6, and he has written that:

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[4] Section 5.2.2, Student 5.

He could find that the common denominator is x^2-4.

This section includes difficulties that are directly related to the notion of equation. Here we have seen that students have to deal with different things when they solve an equation, as for example to save the equivalence between passages[5]. The equivalent passages are the manipulations that the students have to do to both of the sides of the equation, to get a simpler equation. These manipulations that save the equivalence between the equations sometimes form a difficulty for the students. For instance they have difficulties to find the smallest common denominator (the appropriate one), but they have also difficulties to define the kind of equation, and to decide if the given equation has solutions or not. The defining of the type of equations is related with the irrational equations. Students have difficulty to determine the irrational equation, which is probably related with the short time that they are working with it.

5.2.7 Checking the results

Some of the students that I have interviewed have been very resistant towards the checking of the results of their solutions of equations. Many times I needed to ask them to do the checking of the result, because many of them did not treat it as part of the solution of an equation. Below I will give some of the ways that I asked the students to check the result:

"How can you check that this is the right solution?" (Question to Student 2, p.118, 132)

"This is the right solution, how can you check it?" (Question to Student 2, p.120, 172)

"How can check that this is the right one?" (Question to Student 4, p.123, 34)

"Yes, but do you think that these are the right solutions of this equation. How can you check this?"(Question to Student 5, p.126, 34).

"You think that this is the right solution of this equation, how can you check it?" (Question to Student 6, p.131, 61).

The most resistant to this problem was the third student who did not want to write down the check. This is the episode that I have extracted:

Nr.	Person	Time	Text	Comment
15.	St.3	03:26 03:46	Ok, to check, to check that this is the solution of the equation, we replace x in equation and we get a, a numerical equivalence. In this case, if numerical equivalence is true that means that this is the right solution.	She knows theoretically how to make the check.

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[5] Something that they have learned very detailed in lower secondary school (see chapter 2).

She did not make the check for the first solution, and not for the second but for the third one she did. And her checking was clear and correct. This means that she knows how to check the result but it seems like she does not want to do it, one may wonder why. Maybe if I have asked her to write it down, this will not be a problem for her.

The other students have done the checking of the solution not just orally but even writing them on the paper. Some have needed to be asked to make the check for the solution and the others have done it as part of the solution of the tasks.

Student 1 considered checking as part of the solution for the first task, as "…find the value of x and then we do the check to be sure that this is the right answer." (37). She has not done the check for the other tasks because she mixed the solution of the task with the checking in itself. She did this error which is shown below:

Figure 33. Student 1's solution of irrational equation.

As it can been seen from the students' writing that she got that zero is equal with zero, and she said that this is the solution of the equation. This is the extract that I have taken from this part:

Nr.	Person	Time	Text	Comment
104.	Int.	24:21	=in this case? Sorry, which is the solution?	
105.	St.1	24:41	The solution is 0 because 0 is eee equal with zero. If in the end we take that the zero is equal to another number, we say that the equation has no solution. in this case 0 is the solution because eee zero has sense because zero is equal with zero ()	

She made a connection between checking the result and solution of equation. Based on the way how the students work with equation over the years, this is a big mistake because they have to know that when they have zero equal with zero, the equation has solution for all the values of the unknown that are part of the domain. This is a misconception of the student. She does not have clear what means to solve an equation or when an equation has a solution, a multiple solution or no solution.

Student 2 has been very clear in making his checking for the results that he got after the solution of equation. He said that: "We replace the found values for x and can I do the check?" (26)[6] For

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[6] Appendix 3, interview with Student 2.

the other I asked him to check if it is the right solution. Base on how he has performed during the interview of his solutions, he knows to check solutions for a given equation.

I needed to ask Student 4 to make the check of his solutions. For both of the tasks I needed to ask him to check his result, but in both cases he was able to do it.

I needed to ask Student 5 twice for the check of his solutions. He did not think that the check was part of the solution of a given equation. And for the second task he did the check as part of the solution. But when he check and got that 5=5 he said that the equation has no solution. It is strange because for the first task he has answered well.

I have asked Student 6 twice to make the check of her solutions. I have asked her:

"How can you check that these solutions are right?" (14 and 61)[7].

Checking the result of the solution is not very difficult for the students. The errors that are showed in this part are related to the *calculation errors* (Section 5.2.5). Even Student 1, which has made a mistake in the task [C4, P 1b], knows how to check because she did it for the first task. But what is important to remember from this section is the resistance that the students have shown towards checking their answers. Almost all the students needed to be asked by me to make the checking. From the general table we can see that only Student 1 has done the checking without being asked, but the others, for example Student 2 and Student 5, needed to be asked at least in one of the tasks. Student 5 is a special case because she did not write the checking but she did them by talking aloud. In general students do not consider the checking as part of the solution of a task.

5.2.8 Summary of the section

This is a very important part of the data analysis, which includes the difficulties that the students have and what they have not. This is the second part of my data analysis, and as mentioned in the beginning it deals with the way students solved the equations. The equations are: linear, quadratic, rational and irrational. I have tried to include all the types of equations that students solve in the first year of upper secondary school.

During the solution of these equations, the students have shown that they know to manipulate an equation but sometimes they have some difficulties. This manipulation consists in working with both the sides of the equation with the purpose to get an equivalent and simpler equation after each step. Some of the students have solved the tasks without any mistake. There are 5 tasks out of 12 completely solved, but the others have errors. A complete task is defined in this study when the student can correctly found the domain, the solution, and done correctly the checking of the solution. This is based on the Albanian curriculum (Lulja and Babamusta, 2008, p. 94). Only Student 3 has complete solution in both of the tasks of Problem 1. Student 2, Student 4 and Student 6 have solved correctly only one of the tasks, and Student 1 and Student 5 have no complete solutions.

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[7] Appendix 3, interview with Student 6.

In this section we can see that students are able, sometimes with some help, to find the domain of an equation. All of them could get the right domain in the end except Student 5 for the task [C3, P 1a], because he was stuck even if I tried to help him. Some of them have needed to be asked to find the domain, because they do not see it as part of the solution of an equation. Students have also shown some difficulties as: expressing an interval in the number line, or even mix it with the set of not allowed values of an equation or the set of solutions. The same situation has also happened for the checking of the results. Many of them seem to be resistant towards it. Student 1 is the only student that did not need to be asked to make the check. The others needed to be asked at least in one of the tasks. During the checking, the difficulties that the students have shown are the calculation errors and mixing it with the solution as Student 1 did.

During the solution of the given equations students have also shown some difficulties. The kinds of mistakes that are done by the students are different. Some are related to their previous knowledge and others to their way of working. These last difficulties are related to the focus of the students and how careful they are when they solve a task. But the first ones, they are connected to the understanding in the earlier years.

Students know that they have to save the equivalence when they manipulate an equation but during this procedure they meet many difficulties, from calculation errors to finding common denominator, and replacement errors. But the difficulties that the students have shown related to the notion of equation are: to name the kind of equation they have worked with, to determine when the equation has solution or not, and to solve a quadratic equation when it is not in the formal form.

The squaring of an irrational equation is directly related to the concept of equation and it is closely related to saving the equivalence between the passages. Two out of 5 students that have worked with irrational equations have shown this problem. They square separately the square root of the equation, and eliminate the square roots not in a correct way. This difficulty is shown by Student 1 and Student 5 for the task [C4, P 1b]. Another problem which is related to the concept of the equation is to name the irrational equation. The cause of this difficulty is probably related with the short time that the students have worked with irrational equations.

The determination when an equation has a solution or not is shown firstly from Student 1, who said that if we get in the end of a solution of an equation 0=0 [C4, P 1b], its solution is 0. During her solution of the quadratic equation [C1, P 1a] Student 6 got that the discriminant of that equation was 4, bigger than zero, but after a replacement error, during the checking she said that one of the results is a solution and the other no. So according to her this equation had only one solution. Another problem which is related to the concept of equation is difficulty in solving quadratic equations when it does not have the term bx. This is the case of Student 2 who has shown difficulties in solving the task [C3, P 1a]. But he solved correctly another quadratic equation given in the form $ax^2 + c = bx$.

In general students have shown that they do not have problems to solve equations and they had no resistance towards working with these equations.

5.3 Students' work with the word problem in the chosen card

This section deals with the students' solutions of the word problem. The analysis of the data related to the students' solutions of the word problem gave three categories as following:

1- Passage from word problem to equation.

2- Solution of the built equation

3- Justification and discussion of the result.

But like in section 5.2, it was difficult to make these divisions because most of them have talked and have written at the same time. I first present a general view of the students' strategies to solve this task, and then a complete solution with all the parts. I have also shown episodes where students have had difficulties.

5.3.1 Students' strategies

All the students without any exception used a figure as a helping tool to approach the word problem. Some of their responses were:
"… I like very much to draw a helping figure because in the case that you have cleared the data and also the figure, it is much easier to solve the problem." Student 1 (p. 113, 148)
"…, I would like to work more with the picture because it is clearer for me." Student 2 (p. 119, 157)
"I use it as an orientation tool." Student 3 (p. 122, 25)
"In the beginning I think it is better if I draw a scheme as a base" Student 4 (p. 124, 66)
"In this way I won't get lost when I will start to solve," Student 5 (p. 129, 106)

Student 6

Nr.	Person	Time	Text	Comment
71.	Int.	20:04 20:14	Ok, very good, the first thing that you did was to read the problem and draw a scheme. Why did you do that?	I asked for the reason about her scheme.
72.	St.6	20:19	I use that to help me to solve the problem, to understand it better.	The scheme is a helping tool.
73.	Int.	20:22	So, do you think that the scheme helps you?	
74.	St.6	20:35	Yeah, it shows in a simple way what you are looking for and helps to find the height in this case.	

They say that making a figure helps them to understand the problem better. The words used by them are: easier, clear, orientation, not to get lost, and understand. All these words mean that the figure is an object that makes the problem much easier for the students to solve, and it clears their thinking during the solution.
This way of working is part of students' strategy to solve this task. They did not show many difficulties in solving this task. They built the equation by using the formula for the perimeter of the rectangle.

The ways of approaching this task has been similar for all the students: but I will show four episodes; one student that has solved the task very well, one that has done calculation errors during the checking of the result, one that has made a mistake in reading the problem, and one which mix the formula of the area with the one of the perimeter.

5.3.2 Complete solution

It was very difficult to choose between three of the students that have solved the task very well, but I picked Student 3 since the division between the strategy part and the solution part is clearer than the others.
The strategy of Student 3's solution is:

Nr.	Person	Time	Text	Comment
22.	Int.	05:36 05:41	(4) Which is your strategy to solve this problem?	
23.	St.3	05:58	To solve this problem I express it with the help of, I mark one of the sides with x and I express the other next side in relation with x, so in this way I can build an equation and find the unknown.	She uses one unknown to define the other.

We can see that the student's strategy to solve this task is by marking one of the sides of the rectangle with x (23). The x is used as an unknown to denote the height. The base is expressed by 2x, based on the relationship that the adjacent sides have. Student 3's idea is to build the equation which will give her the solution for the problem (23).
Her solution is:

Nr.	Person	Time	Text	Comment
24.	Int.	05:58 06:01	Now, I would like to make you a question, why are you drawing a figure?	
25.	St.3	06:57	I use it as orientation tool. It says that the base is twice of the height. This means that if I mark the height with x, the base will be 2x. We need to find the base and the height when its perimeter is 120m. We know that the perimeter is 2 times 2x plus x, that means twice of the sum of two of the sides. The perimeter is 120m and here we have 4x plus 2x. 120 is equal with 6x, here we find the x is 20m. So we found the height that is 20m, and the base that is twice as the height, this means 2 times 20m, 40m.	

She has started to draw the figure for the clarification of the data which is given in the task. She is reading the problem carefully and is starting to make the connection of the relation that is given in the problem by turning the word problem into an equation. She got the equation and solved it. Here is her solution:

Figure 34. Students' solution of the task.

Then I asked her about the checking of the result. She has been very resistant to write down the checking of the other tasks but in this task she managed to explain it very well. This is the episode in which she explained the checking of her solution.

Nr.	Person	Time	Text	Comment
26.	Int.	06:57 07:04	If I ask you to check that this is the right solution of this problem, how can you do it?	
27.	St.3	07:41	Yes, eee we see 20m, ee we see that the base is 40m, that is twice of the height, plus these both are positive number that fits that they are length of sides of the rectangle. If they were negative, they will be wrong since we have side length. So, since the base is twice of the height, then the perimeter of this rectangle 120m.	She has explained orally the check in this part.
28.	Int.	07:45	How can you check that the perimeter is 120m?	
29.	St.3	08:09	The perimeter is 2 times a plus b. In this case a plus b was the base and the height each. The perimeter is 120m, two times 40m plus 20 and it comes 120 equal with 2 times 60, 120 equal with 120. This means that the sides of this rectangle are 40 and 20.	Then she decided to check it using the writing:

She explains orally the checking for the solution (27). Her explanation is correct. I ask for the way, how she can check it (28). She gives the correct way of checking it (29) by using the formula of the perimeter but in this case having as unknown the perimeter and getting as known the base and the height of the rectangle, and she gets that her solution is right (29).

Student 3 has done a clear solution of the third problem. In the beginning she gave her strategy (23) and then solved it. Her strategy was by marking one of the sides on the rectangle with the unknown x and then the other next side, in relation with x. This relation is given by the problem. Then she has drawn a picture as a tool to solve this problem and started to solve it. Her solution was correct by marking the height with x and the base 2x, based on the problem. She got that the solution was 20m. I have also shown her work. She needed to be asked to check her solution. In the beginning she checked it orally but after my insistence she could write it. I considered it as a complete solution since she was very clear in her solution and very sure on each step that she has made.

5.3.3 Calculation error

I would like to show an interesting part of the checking of another student, the first one who has solved the task very well, but when it comes to the checking of the result, she just gets lost as is shown in the following:

Nr.	Person	Time	Text	Comment
155.	Int.	33:43 33:48	Very good. But if I ask you to check this, how can you do? How can you check that this solution is right?	
156.	St.1	33:52	Eee, we can replace x or we can replace different.	She is not very clear in her explanation.
157.	Int.	33:54	What do you know?	
158.	St.1	33:59	The height or height and base.	
159.	Int.	34:01	In the beginning of the problem?	I tried to make clear.
160.	St.1	34:28	E, only the perimeter. We can replace x in the formula and we have to see if it is equal with the perimeter. Two times 20 plus 20 plus 20 plus two times 20 is 120. So, 40 plus 20, I am writing it again (4) 40 plus 20, 80. 80 plus 80	She got the idea of what to do to check this and she starts the calculation. She just got lost during the calculations.
161.	Int.	34:42	Don't hurry. Eee look better at the last step. (5) The left side.	
162.	St.1	34:50	(4) 40 plus 20, plus 40 plus 20.	
163.	Int.	34:51	So?	
164.	St.1	35:03	60, plus 40 plus 20, 80 I was perplexed. So, it is 120 equal to 120 and it is true.	
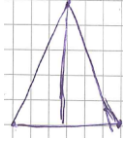
She knows that she has to replace x, but then she get lost (156). I try to help her, by reminding what she had done in the beginning (159). She gets the idea, how to check it and she builds the numerical equivalence, but then she makes a calculating error (160). She finds the mistake and concludes that the solution is right (164).

Student 1 has solved correctly the problem but when I asked her to check the solution she was perplexed. In the beginning she was not clear (156), but after my suggestions she could manage to make the check. During the calculations she made an error in adding and she did not get that her answer was right. She made this error because she was working quickly. I advised her to slow down and focused her attention to the error (161). She could find it and fix (162 and 164).

Now I will show two students who have done something different from the others.

5.3.4 Difficulty in focusing

Student 2 made a mistake during the reading of the problem, instead of rectangle he reads triangle and started to make the figure for the problem. But then I asked him to read it better. Here is an extract of the interview:

Nr.	Person	Time	Text	Comment
154.	Int.	26:19 26:30	... So can you start in the second problem (3) I would like you to read it and tell me your strategy to solve it.	
155.	St.2	26:43	Yes, it is given the triangle where the base is twice as its height. So, as a beginning we draw a picture. (3)	He reads triangle and start to draw its picture.
156.	Int.	26:45	So the first thing that you want to do is a picture?	
157.	St.2	27:06	Yes, I can do for example base 2 times h, it can be (3) I don't have [I want your strategy] yes, I would like more to work with the picture because it is clearer for me. So () where the base is twice as height, if this is the height so …	
158.	Int.	27:08	Read it again, please.	
159.	St.2	27:11	It is given a triangle where the base is twice as its height.	He insists in the triangle word.
160.	Int.	27:13	It is given a …	Then I put my finger at the written word
161.	St.2	27:41	Eee (11) rectangle where its base is twice as its height. So the base, if we take this as the base and this the height, so if this is x this will be 2x in this equation.	He makes a mistake in the relation between the height and the base by putting x to base and 2x to the height.
162.	Int.	27:42	Read it again.	
163.	St.2	27:48	It is given a rectangle where its base is twice of its height.	He re-reads it again.
164.	Int.	27:54	It is twice of its height, so the base is two times the height.	I try to help him.
165.	St.2	28:03	So, yes 2x it is right, find the base and the height of this rectangle where its perimeter is 120m, yeah.	He can fix it.

He starts to read the problem but instead of rectangle he reads triangle, and tries to draw a figure (155). I make clear his strategy (156). Student 2 can solve it without a figure, but he likes more to work with a figure because it makes the task clearer and easier (157). I see that he is drawing a triangle and I ask him to read the problem again (158), but once more he reads the word triangle instead of rectangle (159). I start to read the problem aloud and stop in front of the word that he was mixing (160). He notices the word and starts to make the figure of a rectangle (161). The difficulty that he has is to express the relationship between the adjacent sides of the rectangle (161). I ask him to read again because he makes a mistake in marking the height and the base because he exchanged their places (162) and he reads it again (163). I try to help him, to make it clearer (164) and he gets the difference and find the correct relation between the sides (163).

Student 2 has the strategy as the ones above but in the beginning he could not get that the figure was a rectangle. Instead of it he has drawn a triangle. I needed to tell to him by showing. He has also had difficulties to express the sides in relation to the unknown. He added to the base 2 in front of x and deleted it from the height, which is shown in the picture below.
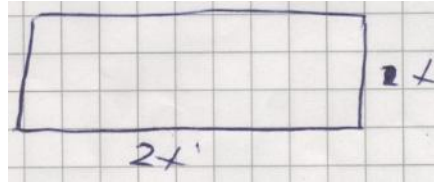
Figure 35. Student 2's approaching to the word problem.

This relation has not been understood very well by the student in the beginning, but he got it very soon. After he had built the equation, and solved it correctly he could also check it correctly.

5.3.5 Mixing area with the perimeter

Student 5 used the area formula instead of the perimeter formula of the rectangle. The solution of the equation that he built was correct. He could understand the mistake when I asked him about the difference between the perimeter and the area of a rectangle (p. 127, 117)

I would like to present his work starting from the solution part since the strategy part is like the ones that I have presented before.

Student 5 was very clear in the strategy that he wanted to follow the solution of the task. He started by drawing a picture and marking the sides of the rectangle with the unknown x correctly. The way of solving is shown in the following episode:

Nr.	Person	Time	Text	Comment
108.	St.5	26:34 27:31	(10)* The base and the height. The perimeter can be found by adding**, multiplying base times height, and it is equal with 120 and we have: 2x times x is equal with 120. So we have $2x^2 = 120$, $x^2 = 60$, x is equal with the square root of 60. Square root of 60 is (4) so,	*He makes some calculations. ** He told in the beginning adding but then he changed it.

He is not sure of the formula of the perimeter because he says in the beginning to add and then to multiply the adjacent sides of the rectangle (108). He gets an equation which is the square root of 60 (108).

In this episode the focus is on building the equation and finding the value of the unknown. But the equation was not correct, so the result was not the right one for this problem. The equation built by him is correctly solved. I asked him about the finding of the square root of a number, which has to get two results, one negative and one positive. He has not mentioned the negative value because he knew that the length of sides can be only positive.

I asked him to explain the difference between the concept of area and perimeter of a rectangle:

Nr.	Person	Time	Text	Comment
115.	Int.	28:30 29:27	… (11) If you can read again the problem, it says the perimeter is 120m. Which is the difference between the perimeter and the area of a rectangle?* If we get a rectangle, we have different elements as the base, the height, the perimeter, the area. How can you give the	*I was orienting him to explain the difference between the area and the perimeter

			formula of the perimeter and of the area using the base and the height of the given rectangle?	of the rectangle.
116.	St.5	29:43	Ehe, this is base time height and the area of the rectangle, eee (7).	
117.	Int.	29:45	What do you call perimeter? How do you define it?	
118.	St.5	29:50	The perimeter is the sum of all the sides.	
119.	Int.	29:53	And what have you written in the equation?	
120.	St.5	29:56	I have found the area.	

I tried to help him to see that the formula that he has used to solve this task is wrong. I use all the elements of a rectangle: base, height, perimeter, area to orient him (115). He is unsure (116). I ask him to define the perimeter (117). He knows that the perimeter is expressed by the sum of the sides (118). He understands the mistake (120).

Then Student 5 turned the problem into a linear equation, and it was very simple for him to solve it and to check the result was the correct one.

He was not sure about the difference between the area and the perimeter of a rectangle. This is why he did a mistake and found a wrong answer. In the beginning he said adding the sides (108) but then he changed it and wrote the area formula. The solution of the equation that he got using the area formula is solved correctly by him. I needed to ask him for the difference between these two concepts. After he managed to write the correct equation he could solve it without any difficulty, and to make the check orally.

5.3.6 Summary

This section is composed by the students' solutions of the word problem, which was the same for all the four cards. Three students out of six have solved this task without any difficulty. Students' strategies are similar. They use a figure to solve it. This problem relates geometry and algebra, because based on the geometric notions and their relations, the students build the equations (algebraic notion) by introducing a variable.

The students' main strategy is to draw a figure. They consider it a helpful tool that makes the problem clearer and easier to solve. Having a figure they introduce the unknown x to denote one of the sides (in this case the height of the rectangle) and 2x the other side based on the relation that the two sides have. This relation between the adjacent sides is given in the problem. They built the equation and solved it correctly. The solution of a linear equation was not a problem for them and they could also make the checking of the result. They needed to be asked to make the checking of their results.

I have chosen Student 3 to present a complete solution, since her solution was very clear. She solved it correctly and without difficulty. In the beginning she was resistant towards the writing of the checking, but in the end she could complete it.

Student 1 also achieved to solve the problem correctly. She was clear in her strategy and her solution. But during the checking of her solution, she got perplexes and made a calculation error. She could manage to fix it with some help by me.

Student 2 misread the word rectangle into triangle, but with my orientation he could manage to see it. I needed to insist so he could understand what he was missing in the words of the problem. He made also a mistake when he was denoting the unknown in the figure. But he could fix it with my advice. When he built the equation he could solve it, and check the result correctly.

Student 5 mixed the concept of perimeter with area, so instead of the perimeter formula he used the area formula which made him to find a wrong result. He needed to be asked about the difference between area and perimeter notion. He could make the difference and understand the error. He succeeded to complete the task without any other intervention by me.

I see that the students do not have difficulties to pass from this word problem to an equation, to solve this kind of equation (linear one), and check the result. The three mistakes that were shown in this section are not connected to the notion of equation.

The first and second errors are related to the students' focus. They are the calculation errors and the misreading of the word.

The third one is related to the mixing of area and perimeter (geometric error).

From the data presented in this section we can see that the students are capable to solve word problems of this type.

6 Discussions and Conclusion

This chapter contains an overview of the findings and conclusions of my research. Firstly, I will present my research question as a recall of the focus that I had working with this project. Secondly, I will give some explanations about the choice of my theoretical framework, and method. Thirdly, I will present the results and I explain them related with the theory. In this part I will also make the conclusions for my research.

The project is a qualitative case study that took place in an Albanian school. It focused on the ways six students of a class approach and solve linear, quadratic, rational, and irrational equations. The research question that leads the research is:

How do the Albanian students in the study approach and solve equations?

This is the question that I will answer in this section but firstly I would like to make a short overview of the main points that have lead this project and these are: theoretical framework and task-based interview.

Kieran's framework is the theoretical framework which is used in this study, which is also called the GTG model for conceptualizing algebraic activity. This theoretical framework is based on three types of algebraic activities which are: generational activities, transformational activities, and global meta-level activities (Kieran, 2007). I have chosen this frame because it is an important frame for algebra, and according to Kieran (2007) these kinds of activities include solving equations, equality, equivalent transformations, and work with word problems, which are key words for my research.

The method used in this study is task-based interview. I used this method because in my opinion it helps the students to express better their reasoning, based on its own simple explanation of this method *"The basic idea is very simple. A student is seated at a table, paper and pencil are provided, and the student is asked to solve some specific mathematics problem. One or more adults are present collecting data."* (Davis, 1984, p. 87).

But a more complete definition of this method is given by Goldin (2000, p. 519):

> *Structured, task-based interviews for the study of mathematical behaviour involve minimally a subject (the problem solver) and an interviewer (the clinician), interacting in relation to one or more tasks (questions, problems, or activities) introduced to the subject by the clinician in a preplanned way.*

According to Goldin (2000) the interview is preplanned, but it also gives space to the interviewer to act in the moment, and to give hints and neutral orientations to the students (Davis, 1984). The main reason of this method, which has its beginning with Piaget (Ginsburg, 2010) is to let the students express their reasoning on a given task.

For the interview I have used, as a preplanned way of interviewing, the five requests of Newman's technique (Vaiyavutjamai and Clements, 2006) which is presented in the literature chapter. The requests of this technique are closely related with what is called an open-ended questioning (Weber, 2002). According to Weber (2002) these kinds of questions are designed to probe the students' understanding.

6.1 Discussion of the findings

In this section I will discuss the main findings of my paper. I will discuss the findings based on the skills that I have posed to the students in the extended theoretical framework that I have developed for this study (see chapter 3):

1. The skill to build an equation for a given word problem.

2. The skill to transform the equation into simpler one but always saving the equality.

3. The skill to solve completely the equation

4. The skill to justify the solution.

The tasks that the students were posed to solve during the task-based interview were two equations which was part of the first task, and the word problem, which is the second task on each of the cards. Below I present the results that I have taken from the data analysis.

6.1.1 Solution of the word problem

During task-based interviews the students were given the four cards where they have to choose two of them: one as the most difficult and the other as the easiest. On each of the cards was the same word problem. I have tried to give an easy geometric problem so they will not have difficulties to understand what the problem is (MacGregor and Stacey, 1996). In this problem the students had to relate geometry with algebra. The purpose of this problem was to see the ability of the students to translate this kind of word problem into equations. They have shown that they do not have any difficulties to turn this word problem into equation. What is more interesting is the way of approach towards this problem. They use a figure (a drawing) to solve this problem. The reasons they gave to use this drawing are because they think that it is a helpful tool, a tool that makes the understanding of the problem very clear. The words that the students have used to denote the use of the figure (or the draw) are the following: it makes the problem clearer, and easier to solve; it is an orientation tool; it serves as a base; it helps to not get lost; and the figure shows in a simple way what you are looking for and helps to find it. All the students could complete this problem, even if some of them needed few hints. The main purpose of posing this problem to the students was to see their ability to translate a word problem into equation, and the performance of them on this task has been very good. I cannot generalise this result for all the word problems that can be posed to them because this is a special problem, which mix algebra with geometry. According to MacGregor and Stacey (1996) and Pawley, Ayres, Cooper and Sweller (2005) students have difficulties to translate the word problems into equations. This is also related to the cognitive load of the problem (Pawley et al., 2005) but in this case I can say that the students that I have picked have translated the word problem into equation very well, even if two of them have had some difficulties.

The difficulties that the students have shown during the translation of problem into equation are related with the focus. The first error was a misreading of the problem, because the student read triangle instead rectangle. The other error that another student did is by mixing the formula of the perimeter with the one of the area.

Both of these errors are not related with the difficulty of the translating of the problem. They are related to the focus of the student, and to the geometrical concept. The students did not show any difficulty to solve the built equation (linear equation), and to check that it was the correct solution of the problem with one exception of Student 1. In the end when I asked her to check the result for the task she got perplexed, and she made wrong calculations, but very soon she could understand the error. So based on the extended Kieran's framework (Kieran, 2007) the students have the skill to translate this kind of word problem into an equation. It has to be stressed that this is a special word problem, and since this is a qualitative case study (Bryman, 2008) in a class with motivated students in mathematics, I cannot generalize this skill to all the Albanian students.

6.1.2 Students' abilities related to their mathematical background

In this section, I would like to make a relation between the students' abilities and the curriculum that they have worked through the 9 first years of school. Firstly I would like to make a short overview of what kinds of abilities students are expected to have when they finish the lower secondary school (the 9th year).

According to the section *Development of the notion of an equation through the Albanian curriculum*, the students from the first year of elementary school start to work on the notion of the unknown which is usually given in a symbolic way as a line, a triangle or a square. But the notion of equation is introduced in the third year of elementary school. During these years the students learn to solve equations, the rules of the equivalent passages, especially linear equations, and in the eighth year they start to deal with quadratic equations.

The ninth year, which in the old system has been the first year of upper secondary school; importance is given to the equations' notion, especially to equivalent passages. They start to know about the domain, and its role. They start to understand that an equation can be taken from another one, and the passage can be seen equivalent but it is not. Two equations can look similar, but they are not because they can have different domains, or different set of solutions (Lulja and Babamusta, 2008). There are listed some passages that look to be equivalent passages (save the equivalence of the equation) but indeed they are not. The students here start also to deal with irrational equations, even if they do not name them yet. The kinds of equations that they deal with this year are the ones that can be factorized, and turned into a product of two factors as the following example: $(x^2-1)\sqrt{x}=0$ (op. cit., p. 96). This is the same example I have given to the students in one of the cards. Even if they have to know this equation to solve, since it is done before, some of the students have shown difficulties during the solution of it, but some others have considered it easy. This is related with the difficulties that the students have to remember because their understanding is more an instrumental than a relational one (Skemp, 1976 from Vaiyavutjamai et al. 2005). According to the data analysis, it is seen that the students know theoretically what they have to do to solve the tasks of the chosen card, but sometimes they have some difficulties during the solution. I will treat these difficulties of the students in the next section.

6.1.3 Students' difficulties during the solution of chosen card

In this section I present the difficulties that the students have shown during the task-based interviews. Firstly, I will present a short overview of what they think is difficult, and then what is easy.

The considerations about the cards of the students are shown in the first section of data analysis where the students select what they think it is most difficult, and the easiest, and give their reasons according to my questions. From Table 4 we can see that with one exception students think that solving linear equations, rational equations, and quadratic equations is not difficult. We can see this from their choices of the cards. One out of 6 students thinks that a quadratic equation is difficult even if he has picked as easy the other card with quadratic equation too. His difficulty is related with the form in which the equation is given. But what the students think that the irrational equations are more difficult. They know theoretically that they have to square both of the sides of the equation, but when they do it, they lose terms. The irrational equations seemed to be the type of equations that the students mostly struggle with. This is also related to the fact that the students have not worked for a long time with this type of equation.

Based on the performance of the students I can say that in general students are capable to solve equations, even if during the solutions they made some mistakes. The difficulties that they have shown are of different kinds such as calculation errors, difficulties in finding the domain, difficulties with quadratic equations, and checking the result.

I explain each of these difficulties that the students have shown during their solution of the first task in the chosen card.

Calculation errors are the most common errors that the students show during a solution of a task. The group of the errors that I have named with this name is big, because there errors that are connected with multiplying or dividing wrong an expression or a number, losing a minus sign. These errors are also named as executive errors (Donaldson, 1963 from Orton, 1983a, 1983b). The meaning that Orton gives to these errors in relation to equations and expressions is the same that I have used in this paper. An example that I want to show from him about what he considers as an executive errors is given below: in his study Orton saw that when the students were solving a task some of them have written the following (Orton, 1983b, p. 237):

"$3x^2 - 6x = 0$ *so we have that* $3x(x-6) = 0$"

This is related with the factoring of the same expression in two terms. The errors that the students have made during this study; Orton (1983) has denoted them as: *executive errors*, *structural errors*, and *arbitrary errors*. The meanings of these errors are given by Donaldson (Orton, 1983a, p. 4) gives to these errors are:

- *Executive errors: are those which involved failure to carry out manipulations, though the principles involved may have been understood*

- *Structural errors: are those who arose from one failure to appreciate the relationships involved in the problem or to grasp some principle essential to solution.*

-   Arbitrary errors: are those in which the subject behaved arbitrarily and failed to take account of the constraints laid down in what was given.

This is the reason why I related the calculations mistakes with the executive errors of Orton (1983 a). I will also try to relate the other difficulties of the students with the Orton's' errors presented above, if it will be possible.

Another difficulty that the students have shown during their work with the first task in the chosen card is finding the domain. Even if the students have worked in finding the domain in the early years (Lulja and Babamusta, 2008) they do not consider finding the domain as part of the solution of equation. I needed to ask 4 of 6 students to find the domain during the interview. They could manage to find it correctly, with the exception of Student 5 who, could not manage to find the domain of one task, even if I gave some hints to him. For most of the students I needed to give hints or orient them towards the right solution. Other difficulties that the students have shown during the finding of the domain are different. I will try to give a general overview of them, and classify these errors as structural errors, because they are related with the students' lack of understanding. One difficulty that I can mention is the error that the students do when they have an inequality and they have to express (draw) it in a number line, and they are undecided in which direction the arrow should go. This is related with the lack of understanding that the students have with inequalities. But they can also mix the set of the domain with the set of the not allowed values, and with the set of solutions, or also lose one of the not allowed values of the equation. Each of these difficulties above is related with structural errors that students make during the solution of an equation, even if these are stressed by the curriculum and the teachers many times, especially in the eighth and ninth grade of their study.

Another difficulty that the students have shown during the solution of the first task in the chosen card is the solution of a quadratic equation. The difficulties shown in this part are three but two of them are structural errors and one of them is executive error. The structural errors founded here are: related with the form of the equation, and with the number of solutions of a quadratic equation. In the first error the student was uncertain how to solve the equation of the form: $ax^2 = bx$ but he could solve the quadratic equation of the form: $ax^2 + c = bx$. Probably the problem has been the missing of one of the terms in the equation. Many times it is difficult to believe that the students at this age have difficulties in solving quadratic equation, but it is true (Vaiyavutjamai, Ellerton, Clements, 2005). This is related to their instrumental understanding that they got during the lessons (Skemp, 1976, from op. cit.). The second structural error that appears in this study is the lack of explaining the solutions that they get in the end (Vaiyavutjamai, Ellerton, Clements, 2005). The student could not make the relation between the value of the discriminant and the number of solutions. According to the textbook for grade 8 and 9 (Polovina, Gjoka and Kovaçi, 2007 and Lulja and Babamusta, 2008), when the discriminant of a quadratic equation is bigger than zero then the equation always has two solutions.

The executive error done by Student 6 was the substitution of a number in the formula. This error caused her lack of understanding of a solution of quadratic equation (structural error) which is mentioned above.

Checking the result causes the same problems as finding the domain. Many students lack this part, and they do not treat it as part of solving an equation. The students only needed to get asked

to make the checking of their results with an exception of Student 1, who in her first task did not need to be asked. But in the second equation, she mixed the solution of the equation with the checking of the equation. This shows instrumental understanding of the students towards the equations; this is the reason why I call this a structural error too.

Except for these errors and misconceptions that the students have shown during the solution of the first task, there are also some other misunderstandings of the students which I have denoted as directly related with the solution of the equation, and I have categorized these as algebraic errors. These errors are the following:

The first one is a *transfer error* or a *slip* (executive error according to Orton's categorization) because the student copied the equation differently from the original one. I can call this a transfer error since the error is done during the transfer from one step of equation into another one.

The second one is difficulty in finding the smallest common denominator, even if this is repeated many times in the Albanian curriculum. Two students find a common denominator, not the smallest one. This does not make them to make any error in finding the solution of that equation, but just give them extra calculations. This has happened to two of the students. It is difficult to categorize this kind of error in relation with the Orton's categorization, but I think it is mostly a structural error.

The third difficulty that has emerged during the data analysis is the difficulty that the students have to define the type of equation. This is related again with the superficial knowledge that the students get during their studies. Students tend to learn the procedures of the solution, without reflecting on what they are solving.

A fourth difficulty that pops out in this study is the student's difficulty to decide if the equation has solution or not. This is another structural error that was showed in this study, because after the student made the checking of the result and got that the left side is equal with the right side, he found it difficult to decide whether the equation has solution or not. This shows also the procedure of checking has a superficial understanding in this case, since the purpose for which it is done is not achieved from the students.

These were all the difficulties that the students have shown during the solution of the first task. There are many difficulties, where most of them are structural errors. This means that the students are more focused on the procedure than to the understanding of solving equations in general.

6.2 Conclusions

The conclusions that emerge from this case study show different things in different types of equations. We can see that linear equations are not a problem for them. They have not considered them difficult, and they have not made mistakes during their solutions. There was only one student that picked the card with the linear equation, considering it as the easiest card. We can also see it when they solve the word problem because the equation that is obtained is a linear one. All the students could solve it without any difficulty, and check the result. But we can see another

picture about the irrational equations, where most of the students consider them difficult, or find difficulties (squaring error) in solving them.

Rational equations are not difficult for the students. The parts that students have shown difficulties in relation to rational equations are: finding the domain, which many times they do not consider as part of the equation, and finding the smallest common denominator, which does not influence the students' result but just give to them more work to do.

During the solution of quadratic equations students' difficulties have been of two types according to Orton's categorizations: executive and structural. The executive one is related to substitution of a number in the formula, and the structural errors are related to its domain and the number of solutions depending on the value of the discriminant.

Other difficulties presented by the students during the solutions of the equations in general are finding the domain of the equation and checking the result. Some of the students know how to find them, but most of them do not consider these two steps as part of the solution of the equation. The domain part was more difficult for the students because some of the students needed many hints to find it.

So I conclude that the students that I have interviewed know how to solve equations in general, but some students have superficial knowledge of parts of the solving process and of quadratic and irrational equation in special. This is shown from the difficulties that they have during the solution of the equations, and which are mentioned above.

7 Pedagogical implications and further research

Equations are a difficult part of the curriculum for many students. My purpose with this study was to identify the ways students approach and reason when they solve different kinds of equations. The students in this study are highly motivated because they have chosen to have more mathematics during their classes than the normal basic school mathematics course.

In general the results demonstrate that most of the students know what to do to solve a given equation. They lack some of the elements, especially when they solve irrational equations, but this is also related with the fact that the irrational equations are new for them. In my opinion they need some time to get used with the irrational equations. Their way of approaching word problems is mostly the same for all, by drawing a picture (scheme), and the solution of this task has been very simple for them. Two of the students showed some difficulties in the beginning of this task, but their difficulties were more related to their focus on the task and geometrical formulas.

The results indicate that there is a need to emphasize the finding of the domain and the checking of the solution of an equation. Some of the students see these parts as not related with the solution of equation, especially the checking part.

Another part that I would like to select for this section is the solution of quadratic equations. The students' performance on the quadratic equations is an important data, since they are a main part of the Albanian curriculum. The students start to work with quadratic equations in the seventh year (see chapter 2). But from the results that I have found they have still problems with this kind of equation. Their understanding is more a procedural understanding, than a deep understanding. In my opinion more work with the students' understanding on solving quadratic equations is needed.

I cannot generalize my results because it was a small group of students that took part in my research. If I wanted to give a general overview of the way Albanian students approach and solve equations I will need a more extensive study. This research has a lot of borders. The students in this study are motivated to work with mathematics, and in general not all students are motivated. If I have chosen another class where the students do not take extra mathematics, the results might have been different.

It is thus recommended that further research, particularly in grade 9 where the students are not yet divided into specific classes, be conducted. For such research, all the students of one class should be subjected to a test and task-based interviews. And use should be made of the qualitative interview data and the quantitative test data.

8 References

Bryman, A. (2008). *Social research methods*. Oxford: Oxford University Press.

Davis, R. B (1984). *Learning mathematics: The cognitive science approach to mathematics education*. London, Great Britain: Routledge.

Dedej, K., Koci, E., Spahiu, E., and Konçi, Z. (2008). In *Matematika 1*. Tirane, Albania: Shtepia Botuese e Librit Shkollor e Re.

Dedej, K., Spahiu, E., and Konçi, Z. (2009). In *Matematika 2*. Tirane, Albania: Print 2000.

Equations websites. These are websites used to select the students' tasks for the test and task-based interviews.

a) http://www.sosmath.com/algebra/solve/solve0/solve0.htm

b) http://www.purplemath.com/modules/solvquad.htm

c) http://www.math10.com/en/math-problems/equalities.html?q=Irrational-Equations

d) http://www.priklady.eu/en/Mathematics/Irrational-Equations.alej

Goldin, G. A. (1998). Observing mathematical problem solving through task-based interviews. In Teppo, A. R (Ed.), *Qualitative Research methods in Mathematics Education* (pp. 40-62). Reston, Virginia: The national council of teachers of mathematics.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In Kelly, A. E and Lesh, R. A (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 517-547). Mahwah, New Jersey: Lawrence Erlbaum.

Ginsburg, H. (2010). The clinical interview in psychological research on mathematical thinking. Aims, rationales, techniques. In A. J. Bishop (Ed.), *Mathematics education: Major themes in education* (Vol. 4, p. 183-197). Oxon, UK: Routledge.

Kieran, C. (2006). Research on the learning and teaching of algebra. In Gutiérrez, A. and Boero, P. (Eds.), *Handbook of Research on the Psychology of Mathematics Education Past, Present and Future*. Rotterdam, Netherlands: Sense publishers.

Kieran, C. (2007). Learning and teaching of algebra at the middle school through college levels: building meaning for symbols and their manipulation. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 707-762). United States of America: National council of teachers of mathematics.

Kopliku, F. (2007). Zgjidhja e ekuacioneve dhe inekuacioneve. In *Matematika 7* (pp. 190-196). Tirane, Albania: D&U.

Lulja, E. and Babamusta, N. (2008). Ekuacione dhe sisteme. In *Matematika 9* (pp. 86-100). Tirane, Albania: pegi.

Lulja, E. and Babamusta, N. (2009). Matematika 10: për klasën e 10 të arsimit të përgjthshëm. Tiranë, Albania: Pegi.

MacGregor, M. and Stacey, K. (1996). Learning to formulate equations for problems. In Puig, L and Gutiérrez (Eds.), *Proceedings of 20th Conference of the International Group for the Psychology for the Psychology of Mathematics Education* (Vol. 3, p. 289-296). Valencia, Spain.

McNeil, N. M., Grandau, L., Knuth, E. J., Alibali, M. W., Stephens, A. C., Hattikudur, Sh. And Krill, D. E. (2006). Middle-school students' understanding of the equal sign: the books they read can't help. *Cognition and instruction, 24(3)*, 357-385.

Mesi, M. and Boriçi, A. (1997). *Të mësojmë matematikë.* Shkoder, Albania: Camaj-Pipa.

Orton, A. (1983a). Students' understanding of integration. *Educational studies in mathematics, 14(1),* 1-18.

Orton, A. (1983b). Students' understanding of differentiation. *Educational studies in mathematics, 14(3),* 235-250.

Pawley, D., Ayres, P., Cooper, M. and Sweller, J. (2005). Translating words into equations: a cognitive load theory approach. *Educational Psychology, 25*, pp. 75-97.

Perdhiku, N. (2007). Algjebra e funksioni. In *Matematika 6* (pp. 114-142). Prishtine, Kosove: Albas.

Polovina, A., Gjoka, L. and Kovaçi, S. (2007). Ekuacioni i fuqise se dyte me nje ndryshor. In *Matematika 8* (pp. 83-92). Tirane, Albania: Lilo.

Polovina, A., Gjoka, L. and Kovaçi, S. (2007). Ekuacioni dhe inekuacioni me nje ndryshor. In *Matematika 8* (pp. 106-113). Tirane, Albania: Lilo.

Schoenfeld, A. H. (1987). What's all the fuss about metacognition? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189-217). Hillsdale, New Jersey: Lawrence Erlbaum.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). Reston, Virginia: National Council of Teachers of Mathematics.

Starja, D. and Shkoza, Z. (2009). Kutiza si vendmbajtese e numrit natyror. In *Matematika 3* pp. 134-142). Tirane, Albania: albPAPER.

Starja, D. and Shkoza, Z. (2009). Ekuacionet dhe inekuacionet. In *Matematika 4, per klasen e katert te shkolles 9-vjeçare* (pp. 142-145). Tirane, Albania: albPAPER.

Starja, D. and Shkoza, Z. (2009). Shprehjet shkronjore, ekuacione dhe inekuacione. In *Matematika 5* (pp. 93-102). Tirane, Albania: albPAPER.

Tabach, M. and Friedlander, A. (2008). Understanding equivalence of symbolic expressions in a spreadsheet-based environment. *Int J Comput Math Learning, 13,* 27-46.

Vaiyavutjamai, P., Ellerton, N. F., and Clements, M. A. (2005). Students' attempts to solve two quadratic equations: A study in three nations. In P. Clarkson et al. (Eds.), *Building connection: Research, theory and practice* (Proceedings of MERGA 28, Vol. 2, p. 735-742). Melbourne, Australia: MERGA.

Vaiyavutjamai, P. and Clements, M.A (2006). Effects of classroom instruction on student performance on, and understanding of, linear equations and linear inequalities. *Mathematical thinking and learning, 8(2),* 113-147.

Weber, K. (2002). *Students' understanding of exponential and logarithmic functions*. (Report No. EFF-088). U.S: EDRS (ERIC Document Reproduction Service No. ED 477 690).

Appendix

Appendix 1 The Headmaster permission for the data collection.

The Albanian version.

REPUBLIKA E SHQIPERISE   *Shkoder, 09.11.2009*

SHKOLLA E MESME "28 NENTORI"

DEKLARATE

Une e nenshkruara Ganimet Kastrati, drejtoreshe e Shkolles se Mesme "28 Nentori" ne Shkoder, deklaroj nen pergjegjesine time te plote se lejoj studenten e masterit Besara Kadija qe te beje çdo veprim qe i sherben mbledhjes se te dhenave studimore ne vitin e pare te shkolles sone, te dhena qe do t'i sherbejne ne punimin e temes se saj te masterit "Menyrat e Zgjidhjes se Ekuacioneve te Nxenesve Shqiptar te Shkolles se Mesme" per te cilen po ndjek studimet ne Universitetin "Agder" ne Kristiansand, Norvegji.

Drejtoresha e Shkolles se Mesme "28 Nentori"

Ganimet Kastrati

The Head Master permission for the data collection. The English version.

DECLARATION

I the undersigned Ganimet Kastrati, headmaster of High School "28 Nentori" in Shkoder, declare in all my responsibility that I allow the master student Besara Kadija to do all she needs to take her data collection in the first year of our school. These data will serve to her for the Master Thesis "Albanian Upper Secondary mathematics students' ways of working with equations", for which she is following the studies in Agder University, Kristiansand, Norway.

Headmaster of High School "28 Nentori"

Ganimet Kastrati

Appendix 2 The list of the symbols for the transcriptions

(3)   pause between the words and expressions (reaction time)

=   continuation of a statement after an interruption

[ ]   overlapping in the discussion

(( ))   non verbal activity

!   statements with strong emphasis

.   the end of the statement

,   change of intonation

( )   something that it is not heard

?   question mark

eee   hesitation statement

Appendix 3 *Transcription of six interviews*.

The original transcription is in Albanian but I have translated only the part that I have used in the analysis part.

Student 1

Nr.	Person	Time	Text	Comment
1.	St.1	00:02	( )	
2.	Int.	00:06	( ) read, read the cards	
3.	St.1	00:32	(26) So can I start to solve them?	
4.	Int.	00:35	No, firstly can you tell me your thinking during the selection of the cards?	
5.	St.1	00:39	Let's get this	
6.	Int.	01:01	Take your time, don't hurry (), (10) I am not taking your picture.	
7.	St.1	01:03	So we can take...	
8.	Int.	01:08	Have you decide it?	
9.	St.1	01:30	(11) Here, one is difficult, the other is easy, so I don't know (3) which to take as the most difficult because all here are easiest ones…	
10.	Int.	01:33	So you think all are easy?	
11.	St.1	01:42	No, no they are alternated, so some are very easy, for example I got the idea to take this but not (3) but since this is easy	
12.	Int.	01.44	How you can consider this?	
13.	St.1	01:53	This is simple, this is not that I can't solve it, but I thought to solve this since it is more interesting.	
14.	Int.	02:00	Ok, but which are the two cards; one that you think is the most difficult and one that you think is the easiest?	
15.	St.1	02:06	The easiest for me? I don't know, a simplest for me is of this form, mostly they are simple.	
16.	Int.	02:17	So, you think that this is the easiest for you? Ok, then (3) which would you like to solve?	
17.	St.1	02:18	I will get these two,	
18.	Int.	02:20	They are the easiest, why do you think so??	
19.	St.1	02:38	We take this as the easiest, (3) eee I don't know. I am getting this as the most difficulties and this as the easiest one because even this doesn't have any difficulty, all are easy, neither this has no difficulty, (3) these two.	
20.	Int.	02:39	Ok,	
21.	St.1	02:41	Well.	
22.	Int.	02:50	So, as I can see these two types of equations are very easy for you, (3) who would you like to solve?	
23.	St.1	02:54	Eee, this here and this here	
24.	Int.1	02:57	No, from the cards…	
25.	St.1	02:59	E, from the cards,	
26.	Int.	03:05	From these that you have picked which would you like to solve, the entire card.	
27.	St.1	03:06	You mean as a whole one?	
28.	Int.	03:07	Ehe	
29.	St.1	03:09	I will get this.	
30.	Int.	03:11	So this card you consider as the easiest one?	

31.	St.1	03:12	Yes?	
32.	Int.	03:25	Ok. (10) What do you understand to solve this equation?	
33.	St.1	03:32	Now, find the values of the unknown for which the expression has meaning	
34.	Int.	03:34	Which is your strategy that you want to follow now?	
35.	St.1	04:33	So, in the beginning I multiply the number with the elements that are in brackets, inside the brackets. During the multiplication we have to pay attention to the sign, usually the change of the sign when we have minus in front of the brackets, so the sign inside the brackets will change. If the expression contains round brackets, square brackets and {brackets we will do the multiplications and divisions. Firstly the multiplications, and then the divisions and then the additions and subtractions and reducing the brackets, from round brackets to square brackets till {brackets. We eliminate the brackets, as we call it in another way.(14)	
36.	Int.	04:38	So, what do you need to solve this equation?	
37.	St.1	05:14	Simply we can multiply the numbers with the bracket, take in one side the eee unknown and in the other side the numbers, we do the divisions and find the value of x and then we do the checking to be sure that this is the right answer. (20)	
38.	Int.	05:23	Why do you think that these two, or what did you think when you pass from this step to this step?	
39.	St.1	05:24	From here to here?	
40.	Int.	05:28	No, from the second to the third.	
41.	St.1	06:22	Yes, I have simply passed from one side to the other side the value of eee, the unknown x together with the value that is beside it, and in the other side we have taken the numbers. Even if we have put 14 before and -8, the value doesn't change. (20) So, eee 43. If () (5) eee when, the reduction, if we write it here, in this case …	
42.	Int.	06:23	From this step, to this step?	
43.	St.1	07:09	So, we have finished eee () we have finished the addition, if ee one expression, expressions that have the same letter part they can be added and subtracted, the same as the numbers which don't have. There is a problem if we can add or we can subtract. We do the calculations and we take apart the value of the unknown x that in the case where it is possible to make some reductions is easier for us. So we can replace, in that way we can have some reductions. The problem () and we cannot forget about the signs because they are very important and then we can make the checking and find the value of the expression. I have to see if I can reduce eee (4), can I write here?	
44.	Int.	07:12	Yes, you can write.	
45.	St.1	07:21	(8) so, eee,	
46.	Int.	07:24	Can you write the checking here, in this part of the paper? =	
47.	St.1	07:26	Yes but…	
48.	Int.	07:28	= you don't need () they reduce [reductions?] () yes.	
49.	St.1	07:33	Eee, how can I write this eee? ()	
50.	Int.	07:37	It doesn't matter, I can understand and they are video-taped.	

51.	St.1	08:06	So, now, they can be reduced with 3, because I don't have, eee 23 and 12 (6).I guess that I don't have anything else to reduce.	Student writes some calculations.
52.	Int.	08:09	There is no number, because 21 is a negative number and 12 is an …	
53.	St.1	08:12	Odd number, they don't reduce, so let's do the checking.	
54.	Int.	08:20	O, sorry, I lost 21 with …	
55.	St.1	08:22	3	
56.	Int.	08:24	3	
57.	St.1	09:26	So, (4) 7/4 () minus 7/4. (3) The checking (30) () (13) I am working slowly not to make an error.	
58.	Int.	09:30	Slowly because you have 45 minutes in your disposition.	
59.	St.1	11:05	(92) ()	
60.	Int.	11:06	From this step to this one?	
61.	St.1	14:04	(3) so, I multiply the number with the fraction, we take out for example a fraction, we multiply with a number with () and take out a fraction eee and do the multiplication, the addition of two, and we have again a number and we do the reduction to pass in the other step.(7) so (25) eee, here I have a mistake (9) so (32) oau I have done a mistake.(24) is the possibility to do the mistake here? (23) It is -8 and I have written it +8 (12) this is a mess.	
62.	Int.	14:09	And the left side of this equation, can you see it	
63.	St.1	14:58	(4) so, 6x+12x, 18x minus 30, I have done -22 minus 30, I have put the minus, and it is -22 (4) it is -22 minus 30 it is -4, 58 (4) 18, (163)()	
64.	Int.	15:06	Don't delete it, you can continue down, (3) are you sure that it is 22?	
65.	St.1	15:12	18x minus 30 equals -22.	
66.	Int.	15:15	How you can check it?	
67.	St.1	15:22	Eee, (4), eee x to check it, how?	
68.	Int.	15:26	That it is -22.	
69.	St.1	15:34	18x minus 30 eee how, how I can check it, don't…	
70.	Int.	15:50	When you have a numerical equivalence, this one for example, 18 minus 30 you said that is equal with -22, how you can check it?	
71.	St.1	16:50	So, 22 plus 18 is equal with 40, eee so it is 12 (3) () (4) we reduce with 3 and it comes 21, eee of fourth. ¾ cannot be reduced more. -21over 4. Eh, all this work. Can I continue the check?	
72.	Int.	17:00	Yes, (5) it is -3/4, but you haven't forgot anything?	
73.	St.1	17:01	-21.	
74.	Int.	17:03	Yes	
75.	St.1	20:25	(32) It seems like 4. (107) we write it like this (30) we can add to this, no, () (25) I don't believe.	
76.	Int.	20:27	Which was your reasoning to pass from this step to this one?	
77.	St.1	20:29	Here to here?	
78.	Int.	20:32	Yes, complete as two equations.	
79.	St.1	20:36	Like expressions, to reduce.	
80.	Int.	20:41	No, from this step to this one, just this passage.	
81.	St.1	20:44	So, only this part here?	
82.	Int.	20:46	No, both of these.	

83.	St.1	21:42	(3) eee, () I forgot, I thought that I have reduced (46) I think that I have worked how I wanted.	
84.	Int.	21:59	It doesn't matter, simply would like to know more about the way how you group these, so how you manipulate when you solve an equation. The exact solution is not very important to me. You can pass to the other...	
85.	St.1	22:03		This part I have taken away since
86.	Int.	22:10		
87.	St.1	22:13		
88.	Int.	22:15		student started to
89.	St.1	22:51		talk out of the
90.	Int.	22:54		focus.
91.	St.1	23:04	Eee, good. The first thing that we do [the first thing], is to square the equation.	
92.	Int.	23:08	Is this the first thing? What kind of equation is this?	
93.	St.1	23:09	Irrational?	
94.	Int.	23:10	Yes.	
95.	St.1	23:15	So, we square both of the sides	
96.	Int.	23:16	So, is this your strategy?	
97.	St.1	23:18	I guess,	
98.	Int.	23:20	Ok	
99.	St.1	23:52	To get away the bracket, eee sorry the square root (11) so in this case it can be reduced and we have (7) plus 2, it is zero.	
100.	Int.	23:55	Which is the logic behind this reasoning?	
101.	St.1	24:10	We know that the square root of a number is that number raised in the second power and to take away the square root of a number or of an expression that is inside of a number, we raised it in the second power so we have the square root reduced.,	
102.	Int.	24:15	Now, which is the next thing to do, the checking? =	
103.	St.1	24:16	Eee	
104.	Int.	24:21	=in this case? Sorry, which is the solution?	
105.	St.1	24:41	The solution is 0 because 0 is eee equal with zero. If in the end we take that the zero is equal to another number, we say that the equation has no solution. In this case 0 is the solution because eee zero has sense because zero is equal with zero ()	
106.	Int.	24:49	So, according to your opinion this equation exist for all x from $\mathbb{R}$.	
107.	St.1	24:50	Yes.	
108.	Int.	25:06	If you have [()] if you have the square root, what do you need to do except finding the solution? It is even another step that you have to do. Eee because the square root is not always true for every x out of $\mathbb{R}$.	
109.	St.1	25:12	No, I have to find the domain in which I can solve this equation.	
110.	Int.	25:14	Something that you haven't done.	
111.	St.1	25:25	For x bigger or equal with zero, because we don't have … so we find the domain, is that important to write "we find the domain"?	

112.	Int.	25:26	No, it's not a problem.	She has gone
113.	St.1	26:02		again
114.	Int.	26:04		out of the focus
115.	St.1	26:17		So I haven't
116.	Int.	26:18		Translated this part.
117.	St.1	26:37	(9) We multiply with, divide with minus to take away the minus and in this case when we divide with a negative number the inequality changes its direction.	
118.	Int.	26:46	Ok, to find the domain, so you have found two domains…	
119.	St.1	27:10	Yes, A1, A1 here is with -3 and A2, eee it is not right. We have -3, minus infinite, eee here I am not sure. Minus infinite or it is minus infinite, -3?	
120.	Int.	27:15	It, doesn't matter, this means that x is bigger or …	
121.	St.1	27:23	Equal 3. -3 takes part since it is bigger or equal with three eee meanwhile ()	
122.	Int.	27:30	So this passage from here to here ()	
123.	St.1	27:41	Eee, it is connected with the numerical equivalence; if it would be bigger than 3 we will have the bracket open.	
124.	Int.	27:49	How do you understand that x is bigger than -3 in the number line?	
125.	St.1	28:10	x bigger than -3; we have the entire x in the left side because if they will be in the right side, the numbers will be bigger than -3. Eee, in the left side, we can say even as many apples that we have, as much in the left, right side of the number line, means bigger numbers, and in the left of the number line we have smaller numbers.	
126.	Int.	28:13	What have you done in this case?	
127.	St.1	28:30	Eee, we have the numbers, I have given the numbers bigger than -3 and so 3 and $-\infty$, will go in the left, and we always have big numbers.	
128.	Int.	28:33	Are you sure?	
129.	St.1	28:48	(3) So for x bigger then -3, eee, ooo, infinite, 2, minus infinite, plus infinite. It is infinite because we have bigger numbers.	
130.	Int.	28:52	So, is here minus or plus infinite?	
131.	St.1	28:53	Yes, plus infinite.	
132.	St.1	28:56	I took away the minus, let's do this plus?	
133.	Int.	28:59	And in the other case?	
134.	St.1	29:01	It is minus infinite.	
135.	Int.	29:08	So, how you can do the cutting of these sets, because…	
136.	St.1	29:17	We have the number line, cutting is also the solution and to say the truth this year we have started to work with them, with irrational equations.	
137.	Int.	29:19	Yes, I know it.	
138.	St.1	29:52	So I am writing here, plus infinite, to not forget (3) plus 3 is a stressed point, so even 2 takes part (3) I didn't say it since in the beginning 2 and $+\infty$ because () so the cutting is 2 plus infinite. A1 cut with A2…	
139.	Int.	29:57	I can read, $\left[2,-\infty\right[$ at least this is what you have written here.	

140.	St.1	30:00	It is the smallest	
141.	Int.	30:04	Don't erase, it has no importance, but simply how do you think about this?	
142.	St.1	30:08	It is in the other side and it is simply till 2.	
143.	Int.	30:09	Is it just in the other side?	
144.	St.1	30:15	I am putting it in this way so we won't get confused.	
145.	Int.	30:16	Ok.	
146.	St.1	30:31	(3)() even the cutting with 2 and minus 3, 2. They are both in use because both of them take part here.	
147.	Int.	30:45	Ok. The second task, read it aloud, and please you tell me what do you understand with it and what do you think about it?	
148.	St.1	32:28	Firstly, after I read I always get out the data and I like very much do draw a helping figure because in the case that you have clear the data and also a figure, it is much easier to solve the problem. So, the base is twice of the height, let's do the rectangle where the base is x, 2x and its perimeter is 120m. (4) The perimeter is 2a+2bvso we can write as…so we find the base and the height (18) 4x plus 2x is 6x (16) 20m.	
149.	Int.	32:33	Which was your reasoning to solve this problem?	
150.	St.1	32:44	It is given a rectangle and firstly we write the data and put them in the figure. Eee, I didn't determine a and b, can I do it?	
151.	Int.	32:45	No, it is not necessary.	
152.	St.1	33:20	It requests to find the base and the height of this rectangle when its perimeter and the base is twice as the height. If we call the height with x, so the base will be twice of it so 2x. If there were 2 bigger it means 2+x. in this case the perimeter is given and we have different ways to solve it. 2x plus 2 times 2x is the easiest one, can I write it or …=	
153.	Int.	33:22	No it doesn't matter.	
154.	St.1	33:43	= and we replace with its perimeter, with 120m which is given. In this way we can finds x that is 20m and 2x will be twice of it.	
155.	Int.	33:48	Very good. But if I ask to check this, how can you do? How can you check that this solution is right?	
156.	St.1	33:52	Eee, we can replace x or we can replace different.	
157.	Int.	33:54	What do you know?	
158.	St.1	33:59	The height or height and base.	
159.	Int.	34:01	In the beginning of the problem?	
160.	St.1	34:28	E, only the perimeter. We can replace x in the formula and we have to see if it is equal with the perimeter. Two times 20 plus 20 plus 20 plus two times 20 is 120. So, 40 plus 20, I am writing it again (4) 40 plus 20, 80. 80 plus 80	
161.	Int.	34:42	Don't hurry. Eee look better the last step. (5) The left side.	
162.	St.1	34:50	(4) 40 plus 20, plus 40 plus 20.	
163.	Int.	34:51	So?	
164.	St.1	35:03	60, plus 40 plus 20, 80 I was perplexed. So, it is 120 equal to 120 and it is true.	

Student 2

Nr.	Person	Time	Text	Comment
1.	Int.	00:26	(24) The problem is the same.	
2.	St.2	01:57	Ok, yes (79) the most difficult?? (3) In my thinking the most difficult is this.	
3.	Int.	01:58	One moment	
4.	St.2	02:05	Ok, (4) this.	
5.	Int.	02:08	Why do you think so?	
6.	St.2	02:29	In my opinion, I will have an idea for the solution of the second equation, and according to the figure I will solve even the second problem, but I would think more about the solution of the other equation.	
7.	Int.	02:30	The first one?	
8.	St.2	02:31	Yeah the first.	
9.	Int.	02:32	Why?	
10.	St.2	03:01	Because I am unsure if I can solve it with the discriminant, for example I can take 5x and we can make x simply with 1 equal with zero or simply it will be a common number that these inequalities will be equal. So, a simple number that these inequalities to be equal. So, I have an idea who this number will be, but I thought since I am in the middle of the discriminant and this idea, I thought to let it and to be more sure…	
11.	Int.	03:04	Ehe, so this is the most difficult for you? Ok.	
12.	St.2	03:09	Since I am an dilemma, between these two, this is the one that I can solve …	
13.	Int.	03:12	The easiest?	
14.	St.2	03:49	Yes, because the second task is similar for both, this means that I would have some ideas to solve it. The first task will be very easy to solve, so it will be solved with the discriminant if we pass 2x, and equalize it with 0 and this*, so the equation of the second grade with one unknown, $2x^2$ so, since we have x^2, we compare both of them and find the common denominator x^2-4, which can be 2-x and 2+x ()	*he is explaining the task 1b)
15.	Int.	03:50	So would you like to solve this?	
16.	St.2	03:51	Yes,	
17.	Int.	03:52	Ok, you can continue	
18.	St.2	03:54	Can I start to solve them?	
19.	Int.	04:05	(9) Which is the strategy for the first one?	
20.	St.2	04:46	So, the strategy for the first one is with the discriminant because we can do x^2-3-2x (25).	
21.	Int.	04:50	How do you understand this equation?	
22.	St.2	04:57	With equation I understand a numerical equivalence where the left side is equal with the right side. =	
23.	Int.	04:58	Or the expression	
24.	St.2	05:07	=expression, so, left side has to be equal with the right side. In this case, the expression x^2+2x-3 to be equal with zero.	
25.	Int.	05:24	Ok (12) so to solve this equation you are using the formula of the discriminant?	
26.	St.2	06:23	Yes (46) so, since we have two solutions we can do the checking for each of them. We replace the found value for x	

			and () can I do the checking?	
27.	Int.	07:15	Yes (53) that means…	
28.	St.2	07:16	For this case …	
29.	Int.	07:17	What does that means?	
30.	St.2	07:23	It means that two of the values found for x, make this equation a real numerical equivalence.	
31.	Int.	07:39	So they are solutions for this equation, (10) what do you understand with that equation that you have started to solve now?	
32.	St.2	08:12	So, with this equation, that is an equation of the second grade with one unknown, since we want to find x, in this case is used the formula a-b, a+b since we have x^2-4, but since both of the divisors, both of the divisors that are x+2 and 2-x, that are equal with x^2-4, and then we find the common denominator that is x^2-4 and we multiply the equation. So, their sum gives this production and…	
33.	Int.	08:14	Their sum?	
34.	St.2	08:29	Eee, (10) their sum? No the sum, the product of them gives x^2-4.	
35.	Int.	08:34	So your strategy is simply to find the common denominator and [()]	
36.	St.2	08:36	Then we reduce and perform the calculations.	
37.	Int.	08:43	Ok, but do you think that is needed to do something else before to find the solution?	
38.	St.2	08:46	We can even, we can multiply with what is in antecedent …=	
39.	Int.	08:47	Only …	
40.	St.2	08:52	= to see if I can disassemble x^2-4.	
41.	Int.	08:55	When can you multiply with a number?	
42.	St.2	09:04	When I can multiply with a number? I have that right when I have found the common denominator ()	
44.	Int.	09:07	But can you multiply with zero?	
45.	St.2	09:08	No.	
46.	Int.	09:09	Then,	
47.	St.2	09:32	The expression doesn't have sense, so only except zero, with numbers bigger than zero, we can multiply the expression (6) here we can find the not allowed values for the equation, that is x^2-4 different from zero or bigger than zero	
48.	Int.	09:34	Why bigger than zero?	
49.	St.2	10:12	So, because it needs to be bigger than zero because if it is negative it has no sense. No, no it has sense for negative numbers; it has sense when it is equal with zero. It doesn't matter so it can be negative or positive, it is enough just to be different from zero. So, x^2 (5) then $\mathbb{R}$ minus -2 and 2, that is the domain of this equation, and these values can be taken when we equalize x^2-4 with zero. Then we start to solve equation for values different from zero, having in mind that …	
50.	Int.	10:13	What do you understand with $\mathbb{R}$?	
51.	St.2	10:14	All the real numbers.	
52.	Int.	10:15	And *E*?	*E* is called the domain.

53.	St.2	10:18	It is the domain of the equation...	
54.	Int.	10:22	But why you have written $\mathbb{R}$ equal with *E* minus -2, 2?	
55.	St.2	10:33	It is the set of the solutions, the set of all the numbers except -2, so *E* is the set of the not allowed values that in this case are -2, 2.	
56.	Int.	10:47	But if I tell you that *E* is equal with $\mathbb{R}$ minus -2, 2 (34). You called *E* domain, [domain, yes], so you have done …	
57.	St.2	12:40	The domain, the domain will be indeed *E* equal $\mathbb{R}$ minus -2, 2 () it will be according in my opinion, so if we take away *E* from here, it is the set of real numbers without, only without these two, and *E* will be $\mathbb{R}$ minus -2,2, will be the demand set. (30) Then if we get x^2-4, if we multiply with x+2 then the result will be x-2. This means that x-1 over x-2, minus 2-x. 2-x it will be (4) then 2-x if we divide by x-4, x^2-4 with 2-x, this means that we will get again 2-x. no, because 2, -2 …	
58.	Int.	12:43	How you can divide x-4, sorry x^2-4?	
59.	St.2	12:55	The division is x+2 multiplied with x-2. But then I will have a mistake into the solution because I won't have 2-x [()]	
60.	Int.	12:58	What can you do in this case?	
61.	St.2	13:08	I can divide x^2-4 in 2, in 2+x or I can do the multiplications here or simply find the common denominator.	
62.	Int.	13:19	Yes, how you can manipulate this to make it in your favour? (3) 2-x how can become x-2?	
63.	St.2	13:25	x-2, if we multiply with 1, -1 sorry =	
64.	Int.	13:26	So if you have...	
65.	St.2	13:34	= it can accompanied; I can accompanied this with -1 so it will have sense that x^2-4 to divide with it.	
66.	Int.	13:41	But if you multiply with -1 what you need to have in mind in this case? [()]	
67.	St.2	13:45	It will change even the sign, it is normal.	
68.	Int.	13:46	So in this case?	
69.	St.2	14:08	So this will become plus and we will have x-2, x+ () x-1, so the denominator will be -x-5. Except this, since we multiply both of the cases ...	
70.	Int.	14:14	Yes, but you have there the minus that you put in front. Did you finished with it?	
71.	St.2	14:28	Yes, it is, x+5, -5, since we have, we have minus and minus it becomes, since we have multiplication and with minus, we have that the side change and we have -5 or we take plus minus 5.	
72.	Int.	14:42	(4) This is complete as a brackets, this means that it multiplies the bracket [()] when you have operated.	
73.	St.2	14:45	All of the three will have the different sign.	
74.	Int.	14:46	Ehe,	
75.	St.2	15:12	So, both of the fractions are equal, 4 (8) and then 5 here is the same thing so they will be…[() sorry continue] I think that in one side we will have 2, x^2+4, and from the other side we will we will have x^2 …	
76.	Int.	15:13	One minute	
77.	St.2	15:14	Yes, sorry.	
78.	Int.	15:21	How you have passed from this step to this one? So, I understand that you have multiplied so you have…	

79.	St.2	15:22	Why I haven't done the calculations?	
80.	Int.	15:25	Do you think that these equations are equivalent?	
81.	St.2	15:34	(3) these two equations are equivalent because we will have () we will have 1 and it will be equivalent.	
82.	Int.	15:46	Yes, but when you solve an equation, it needs that all the steps to be equivalent, so in this case you have to try to make these two equivalent.	
83.	St.2	15:47	Yes,	
84.	Int.	15:49	Where is the mistake=	
85.	St.2	15:52	x^2…	
86.	Int.	15:54	= that you think that they are not equivalent?	
87.	St.2	15:58	That is not equivalent!	
88.	Int.	16:02	Or do you think that they are equivalent.	
89.	St.2	16:16	(4) maybe it needs to manipulate before the operations up that are up with 2x+4 so I have to see another possible solution to	
90.	Int.	16:28	Even your strategy, the one of finding the same common denominator and multiplying is right (3) do you think that these …	
91.	St.2	16:36	Are these steps equivalent? Yes, I guess yes.	
92.	Int.	16:37	Why do you think so?	
93.	St.2	16:54	Because since in both of the cases I have multiplied this equation, we have that $2x^2+4$ divided by x^2-4, since we have multiplied even the other side, I think that these two sides are equivalent; they are equivalent with each other.	
94.	Int.	16:57	Ok, when you have multiplied, you multiplied this with this.	
95.	St.2	16:58	Yes	
96.	Int.	17:01	This simply want to [I didn't] eliminate.	
97.	St.2	17:05	No I multiply, normal I will multiply with x^2-4	
98.	Int.	17:18	() No, what I mean is that you have done a passage here, do you understand? Here you have put this up but you have not used it at all. This means that you have done =	
99.	St.2	17:19	A mistake	
100.	Int.	17:24	= a mistake (3) how you can solve or how you can continue?	
101.	St.2	17:39	(3) so if, these two are equivalent with each other then I can disassemble even this x^2-4 in x-2, x+2	
102.	Int.	17:44	I would advise you to work more with the left side of the equations then with the right side.	
103.	St.2	17:45	This side?	
104.	Int.	17:46	Ehe.	
105.	St.2	18:39	So we will have that x^2, we will take the common denominator of both of these that we said that is x^2-4. So, x^2-4, that means x-2 multiplied, we said it before, and get this result x-2 multiplied with x+2, plus and then we will find the common denominator of these, so we will have the possibility to multiply. So the common denominator of this is x-2, oh sorry () x-2 that multiplies x+2 (4) so in my opinion the denominator, common antecedent of both of these is the multiplication of both of the sides, x-2 times x-2.	
106.	Int.	18:49	But you have here x-2 and x-2, why do you need to find another common denominator; these have the same	

			common denominator, the first expression and the second one.	
107.	St.2	19:12	Yes so I have to divide with, with x-2 that, so we can have only x-1, so we will get only the denominator that is 5 (), not to have in one side fraction and the other yes. And we will have in the end, we won't have, in my opinion it will be more difficult for me to make a mistake and ()	
108.	Int.	19:15	Another question, here you have reduced=	
109.	St.2	19:16	Yes,	
110.	Int.	19:23	= [()], in my opinion even these will not be reduced, but this will be reduced and these will remain as they are.	
111.	St.2	19:27	Yes, and in this case it is x-2.	
112.	Int.	19:37	But then, if this is, is down, so is the denominator, what remains? Will it be as an antecedent?	
113.	St.2	20:03	No, it multiplies with the denominator (4) so it will be x-4 that multiplies x-1, minus so, x-2 that multiplies x-5. (6) Sorry, can I write down here?	
114.	Int.	20:08	Since it is an equation, with what has to be equal …?	
115.	St.2	20:10	$2x^2+4$.	
116.	Int.	20:12	You can continue in the other page.	
117.	St.2	20:13	This one?	
118.	Int.	20:18	Yes, as you like.	
119.	Nx.2	20:20	()	
120.	Int.	20:21	Ok.	
121.	Nx.2	20:27	So, x^2-1, we can apply the formula of the multiplication between two expressions, sorry	
122.	Int.	20:36	() no it doesn't matter. So this passage here, you tried before to make it in a regular form to cancel and …	
123.	St.2	20:37	Yes	
124.	Int.	20:39	And you found that this has to be plus	
125.	St.2	20:42	Plus, since we have multiplied with minus	
126.	Int.	20:47	Ehe, (3) why?	
127.	St.2	21:46	Yes, because that small problem can make me loose all the solution of the task. () it will be x^2, x^2-x, no, but x^2-x-2x+2, plus x^2 (3) x^2+5x+2x+10, I operated in this way because this is the way since we don't have $(a+b)^2$ because it will be $a^2+2ab+b^2$, so we will have $2x^2+4$. I didn't do this in the beginning since I wanted to take away the brackets.	
128.	Int.	21:48	()	
129.	St.2	22:42	Yes, x^2 so we get x^2-3x+2+x^2+7x+10 (3) equal with (3) so we have, we have x^2 and since that they are with plus will get plus 2, plus x^2+7x+10. Then we will have to do the multiplication which in this case is $2x^2+8$. And then $2x^2$-3x+7x, no. yes, $2x^2$ since x^2 and x^2, so $2x^2$ …	
130.	Int.	22:46	So the strategy of your work now is just operating [from the left side]?	
131.	St.2	23:41	Yes, and then if we have something, we can equalize with zero, and putting in the right side. 2x -2 (4) no why, no we can do it even with one step because I can take it and equalize with zero. x^2 minus, - 3x+2 (4) 10-$2x^2$-8=0 so () with x^2, $2x^2$=0, 3x+7x=4x, 4x-4=0, so we will have 4x=4. x equal with, 4 divided by 4, equal with one.	
132.	Int.	23:43	How can you check that this is the right solution?	

133.	St.2	23:44	Solution of this equation	
134.	Int.	23:45	How you can do it?	
135.	St.2	24:04	So, the solution x=1 we can replace it in () (12) I am writing again the equation.	
136.	Int.	24:05	No it doesn't matter?	
137.	St.2	25:08	(6) so it will be 1-1 divided with 1+2 minus, minus eee 1-5, 1+5 divided with 2+1 equal with 2^2+4, 1^2-4. So we will have 0 over, zero over 3 minus 6 over 6 over 1. Here it is 1 over () with 1. The result will be 2+8, so it will be 2 times 1 it will be () 2 plus 8	
138.	Int.	25:10	And [that will be 2 times 5].	
139.	St.2	25:30	Yes, we have multiplication, so 1-3, minus 3. So, since we have 3 and -3 we will get (3) in my opinion it will be 9 in this case or ()	
140.	Int.	25:32	Zero over 3 how much is?	
141.	St.2	25:34	Zero over 3 gives zero.	
142.	Int.	25:36	That means [()] you can eliminate.	
143.	St.2	25:49	Yeah minus 6 equal with minus 2 plus 8, 10 over minus 3 we will have something as minus 3,33333()	
144.	Int.	25:50	So,	
145.	St.2	25:52	Or it will be the square root of 5.	
146.	Int.	25:56	It doesn't matter you can leave it -10/3...	
147.	St.2	25:57	Yes,	
148.	Int.	25:58	So,	
149.	St.2	26:03	This means that x is not a solution for this equation, so x=1 is not a solution.	
150.	Int.	26:04	Why?	
151.	St.2	26:17	Because or I have done a mistake or I have forgotten to multiply here with -1 () I don't know. I have done a mistake, so do I have to find	
152.	Int.	26:18	No, it doesn't matter,	
153.	St.2	26:19	()	
154.	Int.	26:30	I just want your way of reasoning during the checking. So can you start in the second problem (3) I would like you to read it and tell me your strategy to solve it.	
155.	St.2	26:43	Yes, let is given the triangle where the base is twice as its height. So, as a beginning we draw a picture. (3)	
156.	Int.	26:45	So the first thing that you want to do is a picture?	
157.	St.2	27:06	Yes, I can do for example base 2 times h, it can be (3) I don't have [I want your strategy] yes, I would like more to work with the picture because it is clearer for me. So () where the base is twice as height, if this is the height so …	
158.	Int.	27:08	Read it again, please.	
159.	St.2	27:11	Let is given a triangle where the base is twice as its height.	
160.	Int.	27:13	Let is given a …	
161.	St.2	27:41	Eee (11) rectangle where its base is twice as its height. So the base, if we take this as the base and this the height, so if this is x this will be 2x in this equation.	
162.	Int.	27:42	Read it again.	
163.	St.2	27:48	Let is given a rectangle where its base is twice of its height.	
164.	Int.	27:54	It is twice of its height, so the base is two times the height.	
165.	St.2	28:03	So, yes 2x it is right, find the base and the height of this	

			rectangle where its perimeter is 120m, yeah.	
166.	Int.	28:05	So, which is your strategy?	
167.	St.2	28:37	Since the perimeter is 2 times 2x so we will have 120m, and since the sides of a rectangle are two and two equal, so we will have 2 times 2x plus 2 times x, 2x. And we will have 4x plus 2x, 120m this means that 6x equal with 120m. x is equal with eeee 20.	
168.	Int.	28:38	Very good.	
169.	St.2	28:56	So we will see that, (3) 2x base means 2 times 20 so it will be 40, and the height of this rectangle will be 20cm, 20m.	
170.	Int.	29:00	But if we want to check that this is the solution, how you can check it?	
171.	St.2	29:03	If I want to check it?	
172.	Int.	29:06	This is the right solution, how you can check it?	
173.	St.2	29:12	We can replace x with 20 and we do the operations (3) if we want to check it.	
174.	Int.	29:14	Ehe, what do you have?	
175.	St.2	30:00	I have the eee the perimeter which is 120m, but not given exactly with numbers, and I have that the base of this rectangle is twice as its height. Since I have the solution of the equation, I suppose that it is the right solution and I replace x, the solution that we have taken so 20 and I put it to the smallest that is its height, and then basing on the smallest that is the height we multiply with 2 and we will have, () 2ab, 2aplus 2b and then we get the checking.	
176.	Int.	30:04	So you simply make the sum of these and multiply with 2.	
177.	St.2	30:19	No, we can put instead of, after I have solved, instead of x we can put 20 (). There will be 40, multiplied with 2 it will be 80. If I compare with 20 both of this then we will get 120 so ()	

Student 3

Nr.	Person	Time	Text	Comment
1.	St.3	00:37	Since there is no unknown in the denominator, it is easier to solve it, so we find the common denominator and we can solve very easy even 1b) since it is equal with zero, we can factorize square root of x and in the same time we can eliminate it because one of the factors of this product has to be zero. So we can find the both solutions of this equation. In this other card the second task, 1b), since we have a square root, we need to square both of the sides of the equation and maybe it needs more work to do.	
2.	Int.	00:48	It doesn't seem very difficult () [()]. Ok, which would you like to solve? Which you like more to solve?	
3.	St.3	00:49	I would like to solve this.	
4.	Int.	01:23	Ok (30) so the strategy for the first is [finding the common denominator] finding the common denominator.	
5.	St.3	01:27	And take away the fractions.	
6.	Int.	01:51	(19) Now which is your reasoning? ()	
7.	St.3	02:53	If we remove the denominator, we multiply the antecedents with the quotient of the common denominator and we turn it	

			in the form without fraction. And then we operate, so we find the productions with the brackets and then we reduce the similar terms as in this case 42x and 12x. Then we put in one side of the equation the terms with the unknown and in the other side the numbers. (10) And later we find x, which is the quotient between terms without an unknown with the one with the unknown.	
8.	Int.	02:54	That is?	
9.	St.3	03:05	157 over 49 that is… can I start with b)	
10.	Int.	03:10	Yes but before I would like to make you a question: if you want to check=	
11.	St.3	03:12	That is the solution of the equation	
12.	Int.	03:18	= what do you do? The result is 3, just to simplify it.	
13.	St.3	03:19	Have I done a mistake?	
14.	Int.	03:26	No, no it is right, no if you reduce it you will get 3.	
15.	St.3	03:46	Ok, to check, to check that this is the solution of the equation, we replace x in equation and we get a, a numerical equivalence. In this case, if numerical equivalence is true that means that this is the right solution.	
16.	Int.	03:59	(12) This equation is…	
17.	St.3	05:06	This is an equation of the second power with one unknown and to solve it we can factorize the square root of x and we express it as a product of two terms. Since the production of these two factors is zero this means that at least one of them is equal with zero. So the square root of x is equal with zero, so x is zero. We have that $x^2-1=0$. X^2-1 can be expressed and a product of x+1 times x-1. In this case since their product is zero, this means that one of the factors is zero so we have: x+1=0 or x-1=0. So we have x=-1 and x=1. This means that the set of the solutions of this equation is zero, -1, 1. To see that they are solutions of the equations we replace each of the solutions in the equation. And if the numerical equivalence is true, this means 0=0, these are solutions of the equation.	
18.	Int.	05:11	You have given the square root, what do you need to find before?	
19.	St.3	05:25	We have to find before, eee we have to find that x has not to be negative. So in this case we have to exclude from the set of solutions the -1, so the set of the solutions for this equation will be 0 and 1.	
20.	Int.	05:31	(3) Ok, and what's about the problem?	
21.	St.3	05:36	(3) so,	
22.	Int.	05:41	(4) Which is your strategy to solve this problem?	
23.	St.3	05:58	To solve this problem I express it with the help of, I mark one of the sides with x and I express the other next side in relation with x, so in this way I can build an equation and find the unknown.	
24.	Int.	06:01	Now, I would like to make you a question, why are you drawing a figure?	

25.	St.3	06:57	I use it as orientation tool. It says that the base is twice of the height. This means that if I mark the height with x, the base will be 2x. We need to find the base and the height when its perimeter is 120m. We know that the perimeter is 2 times 2x plus x, that means twice of the sum of two of the sides. The perimeter is 120m and here we have 4x plus 2x. 120 is equal with 6x, here we find the x is 20m. So we found the height that is 20m, and the base that is twice as the height, this means 2 times 20m, 40m.	
26.	Int.	07:04	If I ask you to check me that this is the right solution of this problem, how can you do it?	
27.	St.3	07:41	Yes, eee we see 20m, ee we see that the base is 40m, that is twice of the height, plus these both are positive number that fits that they are length of sides of the rectangle. If they were negative, they will be wrong since we have side length. So, since the base is twice of the height, then even the perimeter of this rectangle that has these as sides is 120m.	
28.	Int.	07:45	How you can check that the perimeter is 120m?	
29.	St.3	08:09	The perimeter is 2 times a plus b. In this case a plus b was the base and the height each. The perimeter is 120m, two times 40m plus 20 and it comes 120 equal with 2 times 60, 120 equal with 120. This means that the sides of this rectangle are 40 and 20.	

Student 4

Nr.	Person	Time	Text	Comment
1.	St.4	01:56	(114) I think …	
2.	Int.	01:57	Have you decided?	
3.	St.4	02:00	I am thinking that this is more difficult than the others.	
4.	Int.	02:03	So, you think that this is the most difficult, why?	
5.	St.4	02:05	Problem with square roots.	
6.	Int.	02:09	So you think that eee,	
7.	St.4	02:12	If I compare with the others, it is more difficult.	
8.	Int.	02:24	So you think that the irrational equations are the most difficult? But you have chosen this as the easiest, or no? Why is this the easiest?	
9.	St.4	02:38	But here the square root can be eliminated in a simple way, but in the other card, the denominator can be eliminated very easy, with one simple step, so I think that this is the easiest.	
10.	Int.	02:43	Ehe, which would you like to solve?	
11.	St.4	02:47	(3) I will solve the easiest.	
12.	Int.	03:11	Ok, it doesn't matter. (19) Which is you strategy for the first one, would you like to solve 1a)?	
13.	St.4	03:15	I can eliminate the common denominator, eliminate the denominator.	
14.	Int.	03:17	And how do you operate?	
15.	St.4	03:21	I multiply with the common denominator and cancel.	

16.	Int.	03:55	Ok, (17) how do you understand this equation? (4) What do you understand with this equation? Not a solution of equation, but an equation? It is a numerical equivalence …	
17.	St.4	04:04	A numerical equivalence with one unknown and the function of it is to find the unknown.	
18.	Int.	04:07	We always find a solution for the equation?	
19.	St.4	05:17	It can be with many solutions, infinite solutions, and one solution or without solution. (57) so the unknowns in one side	
20.	Int.	05:22	Do you think that these equations are equivalent?	
21.	St.4	05:24	These two equations?	
22.	Int.	05:25	Ehe,	
23.	St.4	05:26	Yes,	
24.	Int.	05:28	So, you think that this passage that you have performed is equivalent, why do you think so?	
25.	St.4	05:34	I have simply multiplied, I have cancelled the brackets and I will multiply.	
26.	Int.	05:56	(20) How did you get this last step?	
27.	St.4	06:12	So, passing in the other side of the equation but with the opposite sign, the elements that I want to separate, and then I just calculate.	
28.	Int.	06:15	But in the left side, what?	
29.	St.4	06:25	I took 5x in the other side of the equal sign with the minus, and it is -5x. And I do the calculations.	
30.	Int.	06:48	(18) Why do you think that these two equations are equivalent?	
31.	St.4	07:30	Ee, I have done the operations, I have done the addition between the first and the last term since it is easier to get added in this way and () (24) we can cancel.	
32.	Int.	07:36	So you get a solution, do you think that this is the right solution?	
33.	St.4	07:38	Yes, I think that this is the right solution.	
34.	Int.	07:39	How can you check that this is the right one?	
35.	St.4	07:42	Replacing x, replacing.	
36.	Int.	07:55	If you read again to the request of the task, you can see that it means solve the equations and check the result, so normally I would like from you to check that this is the right solution.	
37.	St.4	07:57	I will replace,	
38.	Int.	10:49	(84) If you want you can start from the beginning (81) what are you doing in this case, you are just …?	
39.	St.4	10:54	Just operating with numbers, so in the end to have an identity.	
40.	Int.	11:40	(45) How have you passed from this step to this?	
41.	St.4	12:03	I have put them in the common denominator, eee I divided multiplied with 4, eee I divided with 4 or I multiplied with ¼, in this way even the second and the third.	
42.	Int.	12:08	So, now what can you do, or which is you way of reasoning in this case?	
44.	St.4	12:14	59 have, I will put them in the common denominator ...	
45.	Int.	13:19	So, again you are finding the common denominator [common denominator] (21) why do you think that this step is equivalent? So you have and equivalent passage.	

46.	St.4	13:37	I haven't add, I haven't add terms I have got it equal with this. Since I have solved the equation and I have got an identity in the end.	
47.	Int.	15:33	(57) You can pass in another page to write if you need. (51) I am seeing that you are controlling your solution of equation, why?	
48.	St.4	15:35	Because in the end I didn't get an identity,	
49.	Int.	15:39	So you think that you have done any mistake during the solution of the equation?	
50.	St.4	15:40	Yes, I guess.	
51.	Int.	16:07	(24) But how did you find that there is a mistake there?	
52.	St.4	16:53	I checked the operations, (3) since I didn't get an identity in the end () numbers and this is equal, so, I notice that we have here a product. Since 147 with 5 (), so even the other side has to be similar, the sum product since we have an identity. Even in the other side 59 is multiplied with a number with 78 so I have done a mistake, I am sure about it.	
53.	Int.	17:48	(13) Ok, you can pass in the second task, it doesn't matter (21) How do you understand this task, or which is your strategy to solve this task?	
54.	St.4	18:14	I will factorize the square root of x, to separate it, and the equation can be turned into a production of factors. So in this way it will be easier to solve it. It is simply equal with zero (). In the beginning I have to find the not allowed values.	
55.	Int.	18:20	Very good. Why do you find these values?	
56.	St.4	18:43	Because for these values this equation doesn't have solution, they are not allowed values. So for negative x this equation doesn't have meaning, because the square root of a negative number doesn't exist.	
57.	Int.	19:05	(20) eee, this last passage ...	
58.	St.4	19:10	Ee, it is a formula, a formula that we have done in the school.	
59.	Int.	19:43	So you have just applied a formula? (27) So do you think that this equation has three solutions?	
60.	St.4	19:51	No, since for negative x the equation has no meaning, so we have two solutions which are 0 and 1.	
61.	Int.	19:56	Ehe, ok, but how you can check that these are the right solution?	
62.	St.4	20:20	I will, I will replace (18) zero equal with zero.	
63.	Int.	20:50	Ehe, (25) so you mean that when zero is equal with zero we have that the solution is right?	
64.	St.4	20:52	Yes, yes	
65.	Int.	21:17	So, for the problem, can you read it and tell me what do you understand with it (6) or how do you understand this problem? What do you need to solve this, which is your strategy?	
66.	St.4	23:03	(3) In the beginning I think it is better if I draw a scheme as a base. (26) We have that the base is twice of the height, so if I call x the height the base will be 2x, where x is the unknown. Ee, I have to find a relation between the two	

Nr.	Person	Time	Text	Comment
			bases and the height of this rectangle. The perimeter, 120m, so the perimeter is equal with the product of twice of the base with the twice of the height, the sum of the fourth sides. Since we have called the base 2x, we calculate with, and we get eee the unknown. (10) The perimeter is given so, (6) the height is 20m, and the base that is twice of the height has to be 40m.	
67.	Int.	23:29	So you just used the perimeter formula, so the definition of the perimeter. But if I ask you to check this solution, how can you do it? (3) That you have found the right solution? (7) What is given in the problem?	
68.	St.4	23:36	It is given the perimeter. So, if with this solution I get the perimeter of the rectangle…	
69.	Int.	23:38	What does that mean?	
70.	St.4	23:44	That I have to add all the sides again and see.	

Student 5

Nr.	Person	Time	Text	Comment
1.	St.5	03:01	(180) this seems the easiest.	
2.	Int.	03:07	You think that this is the easiest? Ehe why do you think so?	
3.	St.5	03:29	Because the first equation can be solved with the discriminant, and this is a very easy way, and in the second the denominators are common*, both of these have no difficulty in my opinion and the problem is the same for all the cards.	* Denominators are similar because they have a relation between each other
4.	Int.	03:35	Yes, yes same for all the cards. And the most difficult?	
5.	St.5	03:36	Just a moment,	
6.	Int.	03:37	No take your time.	
7.	St.5	04:09	(26) I think this is the most difficult, and I don't know if I can solve it or not. This second …	
8.	Int.	04:10	Why do you think so?	
9.	St.5	04:13	I don't, these with the square root are not very …	
10.	Int.	04:21	I understand, you haven't work very much with irrational equations, but this one is …	
11.	St.5	04:22	This?	
12.	Int.	04:23	Yes, why they are not similar?	
13.	St.5	04:44	But if I square both of the sides in this equation, we can take out from the square root and then it is easy to solve it. But I don't know in this (3) I don't know how to reason.	
14.	Int.	04:49	So you don't know how to turn it in a simple form?	
15.	St.5	04:50	No, I don't know,	
16.	Int.	04:52	So, which would you like to solve?	
17.	St.5	04:53	I can solve this.	
18.	Int.	04:57	Ok, you can solve it, the one that makes you feel more comfortable.	
19.	St.5	04:58	So do I have to solve this?	
20.	Int.	05:02	Yes, but I will make to you some questions during your solution.	
21.	St.5	05:04	Can I write here?	
22.	Int.	05:35	Where do you like, it doesn't matter (23) which is your	

			strategy or the way that you want () because of…	
23.	St.5	05:45	If we turn the terms, -5x in the same side of x^2 we have a quadratic equation with two unknowns and so we can solve it using the disriminant.	
24.	Int.	05:46	With two unknowns?	
25.	St.5	05:49	With one unknown and we solve it with the help of the discriminant.	
26.	Int.	06:00	So you are just manipulating, so the passage from the first step to the second one ()?	
27.	St.5	06:18	Is it equal with ()	
28.	Int.	06:20	What have you applied here?	
29.	St.5	06:52	Eee (3) the formula of the discriminant to find the solutions x1 and x2 and they are two values. If the discriminant is positive the equation has two solutions, *if the discriminant is negative it has one solution and if it zero the equation has no solution. () so the discriminant in this case is 25.	*he has done a mistake
30.	Int.	06:56	Ehe, so you got 25.	
31.	St.5	07:35	So, we say that x has two solutions, so one solution is (33) ()	
32.	Int.	07:36	Ehe,	
33.	St.5	08:23	And x2 (33) x has two solutions 0 and -5, that for this values the equation has a fix value	
34.	Int.	08:33	Yes, but do you think that these are the right solutions of this equation. How can you check this?	
35.	St.5	08:57	We can replace x with both of the values in the equation, and zero equal with zero, and then -5 times -5 (), 25 equal with 25.	
36.	Int.	08:59	So these are…	
37.	St.5	09:02	This means that these are the right solutions of the equation.	
38.	Int.	09:09	But, what do you think about the domain of this equation?	
39.	St.5	09:29	The domain, is x (7) the domain is the set with eee, … 0 and -5.	
40.	Int.	09:32	How do you understand the domain of an equation?	
41.	St.5	09:46	Domain is, so the values that (10) eee	
42.	Int.	10:01	Can make a difference between the domain and the set of solutions of this equation? (3) Which is the difference between them? Or which is the subset of ()	
44.	St.5	10:39	The domain is the subset … eee wait, (13) the set of solutions is a subset of the domain and in another example the domain can be an interval, eee interval eee from zero to 5 and the set of solutions is ()	
45.	Int.	10:56	Why does the second task have solution? (3) How do you understand this and which is your strategy to solve it? ()	
46.	St.5	11:13	(10) We can do a figure in the beginning and we call its height. ()[()]	
47.	Int.	11:24	Sorry the second equation (4) but if it is easier for you, you can solve the problem first.	
48.	St.5	11:25	No, no I just forgot it.	
49.	Int.	11:26	It is not a problem.	
50.	St.5	12:56	(84) now the common denominator of them is $-x^2+9$ and the common denominator of these is also $-x^2+9$, and we get $-(x^2-9)$ and we write it here.	

51.	Int.	13:10	It doesn't matter, because I understand. So you think that your strategy, so the first thing that you did is to find the common denominator, () why do you think that this is the best way?	
52.	St.5	14:35	I have to see the solutions of both of these and equalize in the equation, and then the denominators (4) one second, (39) can I solve this? ()[()] (23)	
53.	Int.	14:37	Why do you think that ()?	
54.	St.5	14:52	So, I can take the solution of this if I equalize this with this, and then we have that the common denominators of these two equations are the same to find x and to find the solution of the equation. () with zero.	
55.	Int.	15:00	How () [to find the common denominator] how did you pass from this step to this one? ()	
56.	St.5	15:43	So, this is the common denominator, it is multiplied with this and this and we have, $-x^2+9$. since x+3 is multiplied with 3-x, then even 1 multiplied with 3-x, and 3-x is –x+3 and then even 1 can be multiplied with –x+3. 3-x is multiplied by 1, -x-3 is multiplied by 1, -x-3 here we have a minus in front of and () *	Attention! *he is calculating with himself
57.	Int.	15:49	Why do you think so, or which is your reason to act in this way for x square?	
58.	St.5	16:35	So, 3-3=0, -x-x=-2x, $-x^2+9$. So, -2x divided with $-x^2-9$ ()	
59.	Int.	16:59	Why do you think that this is wrong? (17) So, which is the part that you fill this error or mistake, or where do you think that your reasoning is not right?	
60.	St.5	17:02	()	
61.	Int.	17:23	So, you have found this as the common denominator, and you have both of the sides with x^2-9, which is that part or which is that point that makes you to stop?	
62.	St.5	17:38	For the solution of the equation, I multiplied both of the sides with x^2-9 and cancelled x^2-9 and we have -2x divided -1, here we have 10, 10/1 is 10.	
63.	Int.	17:47	Yes, but do you have the right to multiply with x^2-9, and in which conditions you have the right to multiply?	
64.	St.5	17:56	So, the common denominator of both of these is x^2-9, and then we multiply both with the same number, both of the sides with the same number	
65.	Int.	17:57	Is this in this case?	
66.	St.5	18:03	Yes, so multiplying both of the sides with the same expression, it will keep its equivalence.	
67.	Int.	18:20	Ok, but do you think that they are some conditions for this expression? (3) For example if this expression is zero, do we have the right to multiply with this?	
68.	St.5	18:23	So, zero, multiplied with a number gives always zero.	
69.	Int.	18:34	So, you don't have the right to multiply with this number. Basing on this, which is the first thing that you have to do in this case?	
70.	St.5	18:41	The denominator has to be different from zero ()	
71.	Int.	18:43	What does that mean?	
72.	St.5	19:04	That, (4) the domain are all the numbers, the numbers eee all the whole numbers, eee whole, all the rational numbers except square root of, all the real numbers except zero.	
73.	Int.	19:39	Taking away zero, ehe ok, if I replace x with zero this means	

			that the 1/3 -1/3 is equal with zero, equal with 10/-9, that means that this is not a solution but you can't say that it doesn't exist. It doesn't give a solution for the equation but it isn't a non allowed value for this equation. Because the non allowed values when we have fraction, when we have square root …	
74.	St.5	19:52	X+3 is different from zero, I wanted to say so but then we have that x is different from zero. 3-x different from zero, this means that –x is different from -3, so x is different from 3.	
75.	Int.	19:59	So, which is the domain of this equation?	
76.	St.5	20:06	The set of the not allowed values for these equations are 3, and the domain is the real numbers except 3.	
77.	Int.	20:13	But if I say to you that (3) that I want to take -3 for example.	
78.	St.5	20:22	Then there are two solutions, -3 and +3 ()	
79.	Int.	20:25	So, how you can find the domain?	
80.	St.5	20:26	Now is over, you told me everything.	
81.	Int.	20:28	It doesn't matter.	
82.	St.5	20:30	The domain is the set of real numbers except 3 and -3.	
83.	Int.	20:58	But how can you find this domain with symbols not just with words? (19) you are just lost.	
84.	St.5	21:00	Yeah, I know but,	
85.	Int.	21:20	(5) Do you remember, you have had an example as the domain you have found very well? Which is the difference between these two; in the end the conditions are similar?	
86.	St.5	21:38	Yeah, yeah I don't (16) eee.	
87.	Int.	21:45	Ok you can continue; the solution doesn't matter. If you get two values you eliminate one of them.	
88.	St.5	22:15	X, () x^2- 9, (11) x^2- 9, we get that -2x divided by -1 is equal with 10. So	
89.	Int.	22:23	-2x, divided by -1 equal with that, how can you solve it?	
90.	St.5	22:42	We can find again the common denominator of both of this and we have that -1, and we multiply both of them with -1 and we have -2x=-10. –x=-10/2, -x=-5, so x is equal with 5.	
91.	Int.	22:45	Can you write it?	
92.	St.5	23:57	(21) So, ee we equalize x in the given equation to see, to make the checking, if the numerical equivalence is true. (26) the common denominator of both of these is 16, and …	He replaces the x in the given equation to check the solution.
93.	Int.	24:05	Ok, you have 10 over 16. Can you do anything to this fraction, a manipulation, or you can cancel them?	
94.	St.5	24:39	(6) It is 5/8. (3) and the common denom9inator is 8, and then since the denominator is 8, we multiply all with 8, and we have 1 minus, minus () can I write it because …	
95.	Int.	25:10	Yes, (26) why do you think that this passage is equivalent?	
96.	St.5	25:11	Which?	
97.	Int.	25:15	This one, from here to here?	
98.	St.5	25:17	This, this passage here?	
99.	Int.	25:18	Ehe,	
100.	St.5	25:45	Yes, the common denominator of all the functions is 8, so I multiplied all with 8, and cancelled 8 with 8, and we have here that -8 with -2, can be cancelled and we get 4, [()] minus and minus is plus, and here is 8 with 8 can be also	

			cancelled and we get 5, the equation is without solution.	
101.	Int.	25:47	Ok, the problem.	
102.	St.5	25:51	(3) Where can I write it, here or below?	
103.	Int.	26:05	It doesn't matter? (5) So, how do you understand this equation, what do you understand with it and which is the strategy to solve this?	
104.	St.5	26:10	So, we can draw a figure.	
105.	Int.	26:12	Why do you use the figure?	
106.	St.5	26:27	In this way I won't get lost when I will start to solve, and I call the base 2x and the height x since the base is twice of the height, this is x and this is 2x. Can I start? Can I do the figure or no?	
107.	Int.	26:34	Yes, yes. (3) You can solve it as simpler as you can.	
108.	St.5	27:31	(10)* The base and the height. The perimeter can be found adding, multiplying base times height, and it is equal with 120 and we have: 2x times x is equal with 120. So we have $2x^2=120$, $x^2=60$, x is equal with the square root of 60. Square root of 60 is (4) so,	He makes some calculations.
109.	Int.	27:53	So, when you find the square root of a number as in this case, x square is equal with 60. [x is equal with the square root of 60], x square root of 60. You haven't put any sign before the square root of 60, does exist any sign or is it like this? (4) Because according to the definition of the square root we will have plus, minus square root of 16, sorry 60. In this case, you don't think that you need to have sign	
110.	St.5	28:20	(11) Yes, but there is no chance that the side of a rectangle to be negative. The base and the height can't be negative so they will be always positive. So the square root of 60, eee it is… (3)	
111.	Int.	28:24	It doesn't matter, how can you check that this is the right solution of this problem?	
112.	St.5	28:26	Eee,	
113.	Int.	28:28	One question before, what number is the number 60?	
114.	St.5	28:30	Number eee irrational	
115.	Int.	29:27	How can you build, if we can think this rectangle as a field, with an irrational number? (11) If you can read again the problem, it says the perimeter is 120m. Which is the difference between the perimeter and the area of a rectangle? If we get a rectangle, we have different elements as the base, the height, the perimeter, the area. How can you give the formula of the perimeter and of the area using the base and the height of the given rectangle?	
116.	St.5	29:43	Ehe, this is base time height and the area of the rectangle, eee (7).	
117.	Int.	29:45	What do you call perimeter? How do you define it?	
118.	St.5	29:50	The perimeter is the sum of all the sides.	
119.	Int.	29:53	And what you have written in the equation?	
120.	St.5	29:56	I have found the area.	
121.	Int.	30:06	So, basing on this that you told me, you said that the sum of sides, how can you express the sum of the sides of a rectangle?	
122.	St.5	30:17	2x plus 2x plus x plus x it is 4x plus 2x, 6x equal with 120, x equal with, can I write it?	

123.	Int.	30:52	Ehe, (20) ehe. How, so in this case you found the solution. How can you check that this is the right solution for this rectangle, your solution is right?	
124.	St.5	31:02	Yes, x is 20 and I can do the checking, 2 times 20 plus 20 plus 2 times 20 plus 20 which is 120, so it is solution for the problem for …	
125.	Int.	31:04	And in this case it is 120.	
126.	St.5	31:07	Yes,	

Student 6

Nr.	Person	Time	Text	Comment
1.	St.6	00:02	The easiest,	
2.	Int.	00:22	One second… () we can start. You think that this is the easiest, why?	
3.	St.6	00:59	This is the easiest because the first equation we have a quadratic equation of the form ax^2+bx+c, which we can solve with the discriminant. So it has two solutions and we can find them very easily and in the second equation we have to multiply with the common denominator, that in this case is x minus, x-2, x+2 and we have to do a change. And here the first equation is a simple one but the second has more work to do maybe then the others.	
4.	Int.	01:03	So, do you consider this as the most difficult with this equation? As an equation?	
5.	St.6	01:05	What?	
6.	Int.	01:08	Do you consider this as the most difficult because of this equation? Why?	
7.	St.6	01:12	But if I compare with the others this seems to be the easiest.	
8.	Int.	01:15	But even that card has a similar equation.	
9.	St.6	01:24	Yes, but there we can factorize x, the square root of x and then it is very easy to solve.	
10.	Int.	01:26	Very good, which do you like to solve?	
11.	St.6	01:28	I would like to solve this.	
12.	Int.	01:29	Ok,	
13.	St.6	03:19	So, we have x^2, $x^2-3=2x$, and we turn it into the form: ax^2++2x, in this case it will be -2x minus 3 equal with zero. So, this is a quadratic equation and we have to find two solutions, and the formula to find the two solutions of the quadratic equation is: $x_{1,2}$ is equal with -b plus, minus the square root of b^2 minus 4 times a times c, divided by 2a. I define a that is 1, b that is -2 and c that is -3, and replace. So, we have that 1 minus 1, plus minus, minus 2^2, minus 4 times 1 times minus 3 divided by 2 times 1. $X_{1,2}$ is equal with -1, plus minus, I calculate and I have, 4 minus, it is 12, plus 16 it is, because we have minus and minus, plus 12 sorry, divided by 2. Then I find x_1 that is -1 plus 4 divided by 2 and it is 3/2 that is 1,5. And x_2 that is -1 minus 4 divided by 2 that is 5/2 that is, 2,5	
14.	Int.	03:25	How can you check that these solutions are right?	
15.	St.6	03:30	I have to replace in the given equation in the beginning.	

16.	Int.	03:32	Can you replace?	
17.	St.6	03:57	So, 1,5 square minus 3 is equal with 2 times 1,5. Then 1,5 square is 2,25 minus 3 is equal with 3. We have 0,5 equal with 3. That is not true.	
18.	Int.	04:04	What has happened? What do you think that....?	
19.	St.6	04:29	(4) So, (6) maybe it has solution for the other result. (6) no,	
20.	Int.	04:49	You have written in the formula $x_{1,2}$ is equal with –b plus, minus the square root of b square minus 4a times c, all this divided by 2a. And this passage you have taken it just replacing.	
21.	St.6	05:07	Yes, I replaced and 2 squared is 4, minus 4 times 1 times 3, is 12, this is turned into plus and it is plus 12. It is 16, and the square root of 16 is 4.	
22.	Int.	05:16	Ehe, ok, but this passage, in the first,	
23.	St.6	05:47	U, it is -2, even here, (4) so minus () so minus 1, and we have that x_1 is minus 1, and minus 6 over 2, that means that x_2 is -3.	
24.	Int.	05:51	Can you check that these are the right solutions?	
25.	St.6	06:22	So, minus 1 squared minus 3 is equal with 2 times minus 1, 1 minus 3 equal with minus 2 and it is -2 equal with -2. For -2 the numerical equivalence is true. For -2 it is true and for -3 we have, () 9 minus 3, 6 equal with - 6.	
26.	Int.	06:39	How is it possible, what has happened in this case? Because you have that one root is the solution of the equation and the other no. What do you propose that has happened in this case? The calculations seems to be good,	
27.	St.6	06:56	Yes, (7) the equation has only one root, I don't know why.	
28.	Int.	06:59	What do you understand to solve an equation?	
29.	St.6	07:06	To find the values of the unknown for which the numerical equivalence, numerical equivalence is true.	
30.	Int.	07:12	But, what do you understand with equation? How do you understand an equation?	
31.	St.6	07:22	Equation is an expression with an unknown; it is a numerical equivalence with one unknown, where the unknown is what we have to find.	
32.	Int.	07:37	(5) Ok, let's go back to our equation? You said here -2 and you write -2, why do you think that has to be -2?	*
33.	St.6	08:11	(5), it is +2, this means that has a solution () so we have 6 over 2 is 3, (3) and here is -2 over 2 is minus 1, that means that here we have 6 equal with 6.	She understands the error.
34.	Int.	08:21	Ok, let's pass in the second task. Which your strategy to solve this task, this means which is the first thing that comes in your mind when you see this equation?	
35.	St.6	08:30	In the beginning I find the values of x, for which the equation can be solved, the domain.	
36.	Int.	08:32	Yes, which in this case is…?	
37.	St.6	08:52	In this case, so (), x has to be different from -2 and x different from 2.	
38.	Int.	08:56	So, the domain is…?	
39.	St.6	09:01	The set of real numbers except -2 and 2.	
40.	Int.	09:06	Now, the first thing that you want to do to solve this equation is…?	
41.	St.6	09:23	I have x^2-4 that is x-2 and x+2, and I am trying to find the	

			common denominator with which I will multiply all the fractions.	
42.	Int.	09:27	So, are you finding the common denominator or no?	
44.	St.6	09:43	(7) I multiply here with x-2, x+2 and with 2-x, so in this case I can simplify that () in fraction.	
45.	Int.	10:40	(49) Which is the reasoning that makes you to pass from this step to this other? Which is the logic behind this passage?	
46.	St.6	10:48	So, () the denominator of each, so I can cancel as much as I can.	
47.	Int.	10:51	So, do you think that this is the common denominator?	
48.	St.6	10:54	Yes, and I can cancel all*	
49.	Int.	10:56	To cancel all the denominators?	
50.	St.6	10:57	Yes,	
51.	Int.	11:28	(29) How do you pass from here to here?	
52.	St^3.6	11:40	I did the cancellation and then I wrote what remains from the denominator, and multiplied with what remained from the denominator of the fraction. (3) because I *	
53.	Int.	11:43	So, do you think that these two equations are equivalent?	
54.	St.6	11:48	(4) yes,	
55.	Int.	11:55	(3) When you cancelled this with this what remains in this part?	
56.	St.6	12:50	We have x-2, 2-x. () –x^2-4, is ()	She is calculating.
57.	Int.	12:55	So in this case what are you doing?	
58.	St.6	15:20	Another term, this is given an identical way () so 2x^2-x^3-4x+2x^2-2x+x^2+4-2x-x^3-5x^2, we have plus since there is a minus before and it changes the sign, so +4x+20 equal with 4x^2+8-2x^3-4x. (4) so I take away the brackets and () -4x, I see if I can cancel, or I am underling the one that has the same <u>part of the unknown</u> (), no I am underling, and I have 2 time x^2 () no eee	
59.	Int.	15:22	Take your time	
60.	St.6	17:36	(10) x^2+x^2, 2x^2, plus 2x^2, 4x^2, plus x^2, 5x^2 minus 5x^2 we cancels all these. I leave this on this side. –x^3, -2x^3 and we can cancel with this and we have () -4x,-6x,-8x plus 4x (), then it is 4+20. This means that 4x+24 equal with 4x^2+8-2x^3-4x. I can cancel more, 24 equal with 4x^2 plus 8 minus 2x^3 minus 4x. I cancel this with this and these, and we get, 24=4x^2+8. 24-8=4x^2, <u>16=4x^2, x^2= 4</u>, x is equal with plus, minus 2.	
61.	Int.	17:40	You think that these are solutions of the equation, how can you check this?	
62.	St.6	17:42	I can replace,	
63.	Int.	17:43	Yes,	
64.	St.6	18:24	() 2 plus 2, (13) 1 over 4 minus 7 divided with 0 equal, divided with zero, (6). There has to be a mistake.	
65.	Int.	18:39	These values that you have found as solutions, if you compare them with the domain of this equation, what do you see?	
66.	St.6	18:46	Eee, this means that this equation doesn't have solution, because these values are not part of the domain.	
67.	Int.	18:49	This means?	

68.	St.6	18:55	This means that this equations doesn't have solution, for ee no solution.	
69.	Int.	19:10	So, the problem can you read it and tell me the way that you think to solve it.	
70.	St.6	20:04	It is given a rectangle, a rectangle where the base is twice of its height. I will call the height with x and the base with 2x in this case. Find the base and the height of this rectangle when the perimeter of it is 120m. The perimeter is 120m, so the perimeter is the sum of the length of the sides. Since in a rectangle the sides are two, and two equal we have that 2 times x+2x. The perimeter is 120 equal with 2x+4x. So we have 120 equal with 6x, x equal with20.	
71.	Int.	20:14	Ok, very good, the first thing that you did was to read the problem and draw a scheme. Why did you do that?	
72.	St.6	20:19	I use that to help me to solve the problem, to understand it better.	
73.	Int.	20:22	So, do you think that the scheme helps you?	
74.	St.6	20:35	Yeah, it shows in a simple way what you are looking about and helps to find the height in this case.	
75.	Int.	20:42	You find that x is equal with 20, and from the scheme I see that this is the height.	
76.	St.6	20:52	Yes, I found x, so I found the height that is 20, and the base that is twice of the height is 2x, so 2 times 20, that is 40.	
77.	Int.	21:02	Ok, but how can you check that this is the right solution of this problem	
78.	St.6	21:37	We have the perimeter that is 120, 120m and we have the formula of the perimeter, that is 2 times base plus height, that I am writing it as an equation, then I found the height that is 20m and I replace, the perimeter is 2 times base which is 40 plus the height which is 20. And the perimeter is equal with 2 times 60, equal with 120. This means that is the right solution.	

Appendix 4 Students' work papers

Student 1

$2(3x-7) + 4(3x+2) = 6(5x+9)+3$

$6x - 14 + 12x + 8 = 30x + 54 + 3$

$6x + 12x - 30x = 54 + 3 + 14 - 8$

$-12x = 63$

$x = \dfrac{63}{-12}$

$x = \dfrac{21}{-4}$

Prova

$2\left(3\cdot\left(\dfrac{-7}{4}\right)-7\right) + 4\left(3\cdot\left(\dfrac{-7}{4}\right)+2\right) = 6\left(5\left(\dfrac{-7}{4}\right)+9\right)+3$

$2\left(\dfrac{-21}{4}-7\right) + 4\left(\dfrac{-21}{4}+2\right) = 6\left(\dfrac{-35}{4}+9\right)+3$

$2\left(\dfrac{-21-28}{4}\right) + 4\left(\dfrac{-21+8}{4}\right) = 6\left(\dfrac{-35+36}{4}\right)+3$

$2\left(\dfrac{-49}{4}\right) + 4\left(\dfrac{13}{4}\right) = 6\cdot\dfrac{1}{4}+3$

$\dfrac{-49}{2} + \dfrac{13}{1} = 3+3$

$\dfrac{-49+26}{2} = 9$

$\dfrac{23}{2}$

$2\left(3\dfrac{(-21)}{4}-7\right) + 4\left(3\left(\dfrac{-21}{4}\right)+2\right) = 6\left(5\cdot\left(\dfrac{-21}{4}\right)+9\right)+3$

$2\left(\dfrac{-63}{4}-7\right) + 4\left(\dfrac{-63}{4}+2\right) = 6\left(\dfrac{-105}{4}+9\right)+3$

$2\left(\dfrac{-63-28}{4}\right) + 4\left(\dfrac{-63+8}{4}\right) = 6\left(\dfrac{-105+36}{4}\right)+3$

$2\left(\dfrac{-91}{4}\right) + 4\left(\dfrac{-55}{4}\right) = 6\left(\dfrac{-51}{4}\right)+3$

$\dfrac{-91}{2} + (-55) = -153 + 3$

$\dfrac{-91-55}{2} = -153 + 3$

$\dfrac{-91-110}{2} = -153 + 3$

$\dfrac{201}{2} = -153 + 3$

$\begin{array}{r} 105 \\ -\ 36 \\ \hline 51 \end{array}$

$\begin{array}{r} 51\,3 \\ \hline 153 \end{array}$

134

$$\sqrt{x+3} - \sqrt{2-x} = 1$$
$$\sqrt{x+3}^2 - \sqrt{2-x}^2 = 1$$
$$x+3-2-x = 1$$
$$0 = 1-3+2$$
$$0 = 0$$

Përcaktojmë nyjedisim z zgji dllyes svckwociou.?

$$\sqrt{x+3} \geq 0 \qquad \sqrt{2-x} \geq 0$$
$$x \geq -3 \qquad\qquad 2-x \geq 0$$
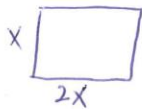$$A_1 = \{+3\} \qquad\qquad x \leq 2$$
$$[-3; +\infty[\qquad\qquad [2; +\infty[$$

$$A_1 \cap A_2 = \{-3\} \cup \{2\}$$

$$[-3; 2]$$

$$S = b \cdot e$$
$$P = 2a + 2b$$

$$b = ? \quad \ell = ?$$

$$2x + x + 2x + x = 170 \text{ m}$$
$$6x = 120$$
$$x = 20$$

$$\ell = 20 \text{ m}$$
$$b = 2 \cdot 20$$
$$b = 40 \text{ m}$$
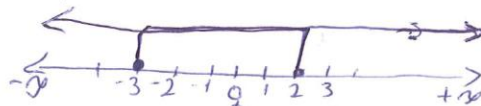
$$2 \cdot 20 + 20 + 2 \cdot 20 + 20 = 120$$
$$40 + 20 + 40 + 20 = 120$$
$$120 = 120 \quad\checkmark$$

$$x^2\sqrt{x} - \sqrt{x} = 0$$
$$\left(x^2\sqrt{x}\right)^2 - \left(\sqrt{x}^2\right)$$
$$x^4 x - x$$
$$x\left(x^4 - 1\right) = 0$$
$$x = 0 \qquad x^4 = 1$$
$$\qquad\qquad x = 1$$

Student 2

1) $x^2 + 2x - 3 = 0$ $\qquad\qquad a = 1 \qquad b = -2 \quad c = -3$

$D = b^2 - 4ac$

$\quad = (-2)^2 - 4 \cdot 1 \cdot (-3)$

$\quad = 4 + 12$

$\quad = 16$

$x_2 = \begin{cases} \dfrac{-b \pm \sqrt{D}}{2a} = \dfrac{-(-2) + 4}{2} = \dfrac{6}{2} = 3 \\[4mm] \dfrac{-b \pm \sqrt{D}}{2a} = \dfrac{-(-2) - 4}{2} = \dfrac{-2}{2} = -1 \end{cases}$

$x^2 - 2x - 3 = 0 \qquad\qquad\qquad x^2 - 2x - 3 = 0$

$\quad\quad \Downarrow \qquad\qquad\qquad\qquad (-1)^2 - 2 \cdot (-1) - 3 = 0$

$3^2 - 2 \cdot 3 - 3 = 0 \qquad\qquad\qquad 1 + 2 - 3 = 0$

$9 - 6 - 3 = 0 \qquad\qquad\qquad\qquad\qquad 0 = 0$

$\qquad 0 = 0$

b) $\dfrac{x-1}{x+2} - \dfrac{x+5}{2-x} = \dfrac{2(x^2+2)}{x^2-4}$

$\qquad x^2 - 4 \neq 0$

$\qquad R = \mathbb{R} - \{-2, 2\}$

$\qquad G = R - \{-2 ; 2\}$

$(x^2-4) \cdot \dfrac{x-1}{x+2} - \dfrac{x+5}{2-x}^{(x^2-4)} = \dfrac{2(x^2+4)}{x^2-4}(x^2-4)$

$\qquad\qquad\qquad\qquad \Downarrow$

$\dfrac{x-1}{x-2} + \dfrac{x+5}{x-2} = \dfrac{2(x^2+4)}{x^2-4}(x^2-4)$

$(x-2)(x-1) + (x+2)(x+5) = 2(x^2+4)$

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$(x^2-x-2x+2) + (x^2+5x+2x+10) = 2(x^2+4)$

$(x^2-3x+2) + (x^2+7x+10) = 2(x^2+4)$

$x^2-3x+2 + x^2+7x+10 = 2x^2+8$

$2x^2 \quad x^2-3x+2+x^2+7x+10-2x^2-8 = 0$

$4x-4 = 0$

$4x = 4$

$x = \frac{4}{4}$

$x = 1$

$\frac{x-1}{x+2} - \frac{x+5}{2-x} = \frac{2(x^2+4)}{x^2-4}$

$\frac{1-1}{1+2} = \frac{1+5}{2-1} = \frac{2(1^2+4)}{1^2-4}$

$\frac{0}{3} - \frac{6}{1} = \frac{2+8}{-3}$

$0 - 6 =$

2 →

(   )

$2x$
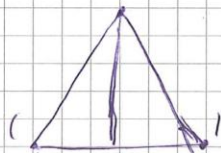
$120\,mm = 2 \cdot (2x) + 2x$

$120m = 4x + 2x$

$6x = 120\,mm$

$x = 20$

$Base = 2 \cdot 20$

$= 40\,mm$

$L\text{ortezue} = 20\,mm$

Student 3

$$\frac{6x-7}{4} + \frac{3x-5}{7} = \frac{5x+78}{28}$$

$$7 \cdot (6x-7) + 4 \cdot (3x-5) = 5x+78$$

$$42x - 49 + 12x - 20 = 5x+78$$

$$54x - 5x = 78 + 69$$

$$49x = 157$$

$$x = \frac{157}{49} = 3$$

$$x^2 \sqrt{x} - \sqrt{x} = 0$$

$$\sqrt{x} \cdot (x^2 - 1) = 0$$

$$\sqrt{x} = 0 \qquad x^2 - 1 = 0$$

$$x = 0 \qquad (x+1) \cdot (x-1) = 0$$

$$x+1 = 0 \qquad \vee \quad x-1 = 0$$

$$\boxed{x = -1} \qquad \qquad x = 1$$

$$A = \{0, -1, 1\}$$

$$A = \{0, 1\}$$
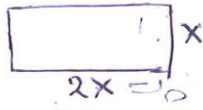
$$P = 100$$

$$P = 2 \cdot (2x + x)$$

$$120 = 4x + 2x$$

$$120 = 6x$$

$$x = 20\,m$$

$$l = 20\,m.$$

$$b = 2l = 2 \cdot 20\,m = 40\,m.$$

$$P = 2 \cdot (a+b)$$

$$120\,m = 2 \cdot (40\,m + 20\,m)$$

$$120 = 2 \cdot 60$$

$$120 = 120$$

138

$$\sqrt{x-6} + \sqrt{6-x} = 1$$
$$\left(\sqrt{x-6} + \sqrt{6-x}\right)^2 = (1)^2$$
$$\left(\sqrt{x-6}\right)^2 + 2 \cdot \sqrt{x-6} \cdot \sqrt{6-x} + \left(\sqrt{6-x}\right)^2 = 1$$
$$x-6 + 2 \cdot \sqrt{(x-6)\cdot(6-x)} + 6-x = 1$$
$$2 \cdot \sqrt{(x-6)\cdot(6-x)} = 1$$
$$\sqrt{(x-6)\cdot(6-x)} = \frac{1}{2}$$

$$(x-6)\cdot(6-x) = \frac{1}{4}$$
$$6x - x^2 - 36 + 6x = \frac{1}{4}$$
$$-x^2 + 12x - 36 = \frac{1}{4} = 0$$
$$x^2 - 12x + 36$$

$$\sqrt{x-6} + \sqrt{6-x} = 1$$

$$x - 6 \geq 0 \qquad\qquad 6 - x \geq 0$$
$$x \geq 6 \qquad\qquad x \leq 6$$
$$A_1 = [6, +\alpha[\qquad A_2 =]-\alpha; 6]$$

$$A = A_1 \cap A_2 = \boxed{6}$$

$$\sqrt{6-6} + \sqrt{6-6} = 1$$
$$\sqrt{0} + \sqrt{0} = 1$$
$$\underline{0 = 1}$$
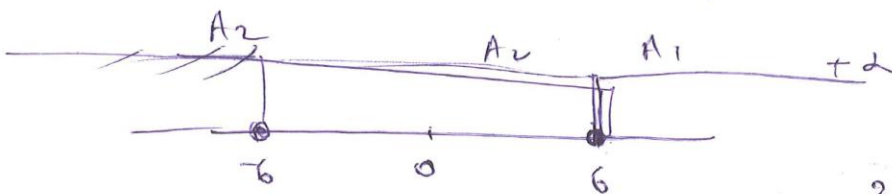
$$x^2 = -5x$$
$$x^2 + 5x = 0$$
$$x \cdot (x+5) = 0$$
$$x = 0 \qquad x+5 = 0$$
$$x = -5$$

Student 4

① a) $28 \cdot \dfrac{6x-7}{4} + \dfrac{3x-5}{7} \cdot 28 = 28 \cdot \dfrac{5x+10}{28}$

$7(6x-7) + 4(3x-5) = 5x+78$

$42x-49 + 12x-20 = 5x+78$

$42x+12x-5x = 78+49+20$

$54x-5x = 98+49$

$\dfrac{49x}{49} = \dfrac{147}{49}$

$x = \dfrac{147}{49}$

$$\dfrac{6 \cdot \frac{147}{59} - 7}{4} + \dfrac{3 \cdot \frac{147}{59} - 5}{7} = \dfrac{5 \cdot \frac{147}{59} + 78}{28}$$

$$\dfrac{6 \cdot 147 - 7 \cdot 59}{59} \cdot \dfrac{1}{4} + \dfrac{3 \cdot 147 - 5 \cdot 59}{59} \cdot \dfrac{1}{7} = \dfrac{5 \cdot 147 + 78 \cdot 59}{59} \cdot \dfrac{1}{28}$$

$$\dfrac{6 \cdot 147 - 7 \cdot 59 + 3 \cdot 147 - 5 \cdot 59}{59 \cdot 28} = \dfrac{5 \cdot 147 + 78 \cdot 59}{59 \cdot 28}$$

$$\dfrac{147(6+3) - 59(7+5)}{59 \cdot 28} = \dfrac{5 \cdot 147 + 78 \cdot 59}{59 \cdot 28}$$

$$x < 0$$

5) $x^2\sqrt{x} - \sqrt{x} = 0$

$\sqrt{x}(x^2 - 1) = 0$

$\sqrt{x}(x-1)(x+1) = 0$

$\begin{cases} \sqrt{x} = 0 \\ x-1 = 0 \\ x+1 = 0 \end{cases}$ $\begin{cases} x = 0 \\ x = 1 \\ x = -1 \end{cases}$

$x = 0$

$0^2 \cdot \sqrt{0} - \sqrt{0} = 0$

$0 - 0 = 0$

$0 = 0$

$x = 1$

$1^2 \cdot \sqrt{1} - \sqrt{1} = 0$

$1 - 1 = 0$

$0 = 0$

---

②

$P = 120 \, m$

$P = x + x + 2x + 2y = 6x$
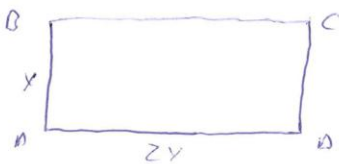
$6x = 120$

$x = 20 \, m$

$2y = 40 \, m$

$P = 20 + 40 + 20 + 40 = 120 \, m$

---

② $\sqrt{2} \, x - \sqrt{3} = \sqrt{5}$

$\sqrt{2} \, x = \sqrt{5} + \sqrt{3}$

$x = \dfrac{\sqrt{5} + \sqrt{3}}{\sqrt{2}}$

$x = \dfrac{(\sqrt{5} + \sqrt{3})\sqrt{2}}{2}$

$x = \dfrac{\sqrt{10} + \sqrt{6}}{2}$

---

6. $\sqrt{x-6} + \sqrt{6-x} = 1$

$x - 6 \geq 0 \qquad 6 - x \geq 0$

$x \geq 6 \qquad x \leq 6$

$\sqrt{6-6} + \sqrt{6-6} = 1$

$\sqrt{0} + \sqrt{0} = 1$

$0 \neq 1$

141

Student 5

Ushtrimi 1.

a) $x^2 = -5x$

$x^2 + 5x = 0$

$a=1 \quad b=5 \quad c=0$

$D = b^2 - 4ac$

$D = 5^2 - 4 \cdot 1 \cdot 0$

$D = 25$

$x_1 = \dfrac{-b + \sqrt{D}}{2a}$

$x_1 = \dfrac{-5 + \sqrt{25}}{2 \cdot 1}$

$x_1 = \dfrac{-5 + 5}{2}$

$x_1 = \dfrac{0}{2} = 0$

$x_2 = \dfrac{-b - \sqrt{D}}{2a}$

$x_2 = \dfrac{-5 - \sqrt{25}}{2 \cdot 1}$

$x_2 = \dfrac{-5 - 5}{2}$

$x_2 = \dfrac{-10}{2} = -5$

$0^2 = -5 \cdot 0 \qquad -5^2 = -5 \cdot (-5) \qquad A = \{0 ; -5\}$

$0 = 0 \qquad\qquad 25 = 25$

Ushtrimi 2.

$\dfrac{1}{x+3} - \dfrac{1}{3-x} = \dfrac{10}{x^2 - 9}$

$\dfrac{3-x-x-3}{-x^2+9} = \dfrac{10}{x^2-9}$

$\dfrac{-2x}{-x^2+9} = \dfrac{10}{x^2-9}$

$\dfrac{-2x}{-(x^2-9)} = \dfrac{10}{x^2-9}$

$x^2-9 \cdot \dfrac{-2x}{-(x^2-9)} = x^2-9 \cdot \dfrac{10}{x^2-9}$

$\dfrac{-2x}{-1} = 10$

$-1 \cdot \dfrac{-2x}{-1} = -1 \cdot 10$

$-2x = -10$

$x = \dfrac{-10}{-2} \qquad x = 5$

$\dfrac{1}{5+3} - \dfrac{1}{3-5} = \dfrac{10}{25-9}$

$\dfrac{1}{8} - \dfrac{1}{-2} = \dfrac{10}{16}$
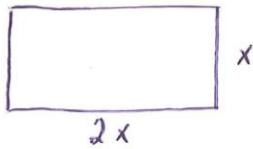
$\dfrac{1}{8} - \dfrac{1}{-2} = \dfrac{5}{8}$

$8 \dfrac{1}{8} - 8 \dfrac{1}{-2} = 8 \dfrac{5}{8}$

$1 + 4 = 5$

$5 = 5$

Usb. 2.

$$2x + x + 2x + x = 120$$

rectangle with sides x (height) and $2x$ (bottom), $P = 120$

$$6x = 120$$

$$2x \cdot x = 120$$

$$x = \frac{120}{6}$$

$$2x^2 = 120$$

$$x = 20$$

$$x^2 = \frac{120}{2}$$

$$x^2 = 60$$

$$x = \sqrt{60}$$

$$x - 3 \geq 0$$

$$x \geq 3$$

$$\sqrt{x-6} + \sqrt{6-x} = 1$$

$$\left(\sqrt{x-6}\right)^2 + \left(\sqrt{6-x}\right)^2 = 1^2$$

$$x - 6 \geq 0$$

$$x \geq 6$$

$$\mathbb{D}_1 = [6 ; +\infty[$$

$$\sqrt{6-6} + \sqrt{6-6} = 1$$

$$6 - x \geq 0$$

$$-x \geq -6$$

$$x \leq 6$$

$$\mathbb{D}_2 =]-\infty ; 6]$$

Student 6

$x^2 - 3 = 2x$

$a = 1$
$b = -2$
$c = -3.$

$x^2 - 2x - 3 = 0.$

$x_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x_2 = \dfrac{+2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1.}$

$x_2 = \dfrac{+2 \pm \sqrt{4 + 12}}{2}$

$x_2 = \dfrac{+2 \pm 4}{2}$

$x_1 = \dfrac{+2+4}{2} = \dfrac{2}{2} = 1.5 = -1$

$x_1 = 3$

$x_2 = \dfrac{+2-4}{2} = \dfrac{5}{2} = 2.5$

$x_2 = -1.$

$(1.5)^2 - 3 = 2 \cdot 1.5.$
$2.25 - 3 = 3$
$-0.5 = 3.$

$(-1)^2 - 3 = 2 \cdot (-1).$
$1 - 3 = -2$
$-2 = 2.$

$(-3)^2 - 3 = 2 \cdot (3)$
$6 = -6.$
$\underline{6 = 6.}$

144

$$\frac{x-1}{x+2} - \frac{x+5}{2-x} = \frac{2(x^2+2)}{x^2-4} \qquad x=-2$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad x=2$$

$$\frac{x-1}{x+2} - \frac{x+5}{2-x} = \frac{2(x^2+2)}{(x-2)(x+2)}$$

$$(x-2)(x+2)(2-x)\cdot\frac{x-1}{x+2} - (x-2)(x+2)(2-x)\cdot\frac{x+5}{2-x} =$$

$$(x-2)(x+2)(2-x)\cdot\frac{2(x^2+2)}{(x-2)(x+2)}$$

$$(x-2)(2-x)(x-1) - (x^2-4)\cdot(x+5) = (2-x)\cdot 2(x^2+2)$$

$$(2x-x^2-4+2x)(x-1) - (x^3+5x^2-4x-20) =$$

$$(2-x)(2x^2+4)$$

$$\left(2x^2-x^3-4x+2x^2-2x+x^2+4-2x\right)-x^3-5x^2+4x+20=$$

$$4x^2+8-2x^3-4x.$$

$$2x^2-x^3-4x+2x^2-2x+x^2+4-2x-x^3-5x^2+4x+20=4x^2+8-2x^3$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -4x$$
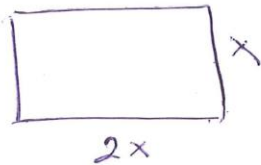
$$-2x^3-4x+24 = 4x^2+8-2x^3-4x.$$
$$24 = 4x^2+8$$
$$24-8 = 4x^2$$
$$16 = 4x^2$$
$$x^2=4 \qquad x=\pm 2.$$

145

$$\frac{2-1}{2+2} - \frac{2+5}{2-2} = \frac{2\cdot(2^2+2)}{2^2-4}$$

$$\frac{1}{4} - \frac{7}{0} = \frac{}{0}$$

$P = 120 \text{ m}$

$P = 2\cdot(x+2x)$

$120 = 2x + 4x$

$120 = 6x$

$x = 20$

$2x = 2\cdot 20 = 40$

$P = 2\cdot(b+l)$

$P = 2\cdot(40+20)$

$P = 2\cdot 60$

$P = 120$

$$\sqrt{6 + \sqrt{x-3}} = 3$$

$$\left(\sqrt{6+\sqrt{x-3}}\right)^2 = 3^2$$

$$6 + \sqrt{x-3} = 9$$

$$\sqrt{x-3} = 9 - 6$$

$$\left(\sqrt{x-3}\right)^2 = 3^2$$

$$x - 3 = 9$$

$$x = 6$$
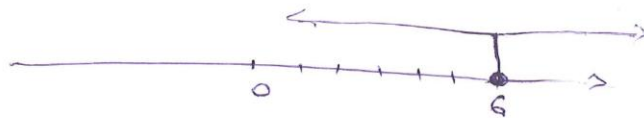
$x \geq 3$

146

$\sqrt{x-6} \geq 0$

$x - 6 \geq 0$

$x \geq 6$

$\sqrt{6-x} \geq 0$

$6 - x \geq 0$

$-x \geq -6$

$x \leq 6$

① $6 - 6$,