Research Article

Approximate Analytic and Numerical Solutions to Lane-Emden Equation via Fuzzy Modeling Method

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A novel algorithm, called variable weight fuzzy marginal linearization (VWFML) method, is proposed. This method can supply approximate analytic and numerical solutions to Lane-Emden equations. And it is easy to be implemented and extended for solving other nonlinear differential equations. Numerical examples are included to demonstrate the validity and applicability of the developed technique.

1. Introduction

Lane-Emden equations are used to describe singular initial value problems (IVPs) relating to second-order ordinary differential equations which have been used to model several phenomena in mathematical physics, thermodynamics, fluid mechanics, and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres, and theory of thermionic currents.

Lane-Emden equations, first introduced by Jonathan Homer Lane in 1870 and further explored in detail by Emden, have the following form:

$$y'' + \frac{2}{x}y' + f(y) = 0, \quad 0 < x \le 1, \tag{1.1}$$

subject to the conditions

$$y(0) = A, \qquad y'(0) = B, \tag{1.2}$$

where A and B are constants and $f(y)$ is a continuous real valued function.

Since Lane-Emden type equations have significant applications in many scientific fields, various forms of $f(y)$ have been investigated in many research works. Among them, many attentions have been carried on the generalized Lane-Emden type equations,

$$y'' + \frac{k}{x}y' + f(x, y) = g(x), \quad 0 < x \leq 1, \ k \geq 0, \tag{1.3}$$

subject to condition (1.2), where $f(x, y)$ is a continuous real valued function, and $g \in C[0, 1]$.

Many different methods have been used to obtain solutions for the generalized Lane-Emden equations. Wazwaz [1, 2] got approximate solutions by using the Adomian decomposition method (ADM) and obtained the analytic solutions of some equations. But it may be an intricate problem to calculate the so-called Adomian polynomials involved in ADM sometimes. He [3, 4] developed a more convenient analytical technique, called the homotopy perturbation method (HPM). Chowdhury and Hashim [5] and Yildirim and Öziş [6] gave the solutions for a class of singular second-order IVPs of Lane-Emden type by using HPM. Sajid et al. [7] pointed out that HPM is a special case of the homotopy analysis method (HAM) and that it is valid only for weakly nonlinear problems. Liao [8] and Van Gorder and Vajravelu [9] used HAM to increase the radius of convergence of series solutions for Lane-Emden equations. Recently, Yiğider et al. studied a numerical method for solving Lane-Emden type equations by Padé approximation in [10]. Generally, when all the above cited analytical approaches are used to solve Lane-Emden equation, a truncated power series solution of the true solution is obtained. By the methods such as HPM and HAM, a series of newly achievements on the analytical solving for some nonlinear differential equations have been proposed recently. By HAM, Ziabakhsh et al. [11, 12] studied the natural convection of a non-Newtonian fluid between two infinite parallel vertical flat plates and the effects of the non-Newtonian nature of fluid on the heat transfer. Jalaal et al. [13, 14] investigated the settling behavior of solid particles using HPM, which show the capability and effectiveness of the method and exhibit new application of it further.

Besides, many soft computing technologies are developed to deal with all types of models for dynamic systems [15–18]. It is important to note that fuzzy modeling technology can transfer data information into a mathematical model which can approximate the original system with high accuracy [19–24]. Li et al. [25] used fuzzy modeling method to approximate the solutions of a class of autonomous differential equation. In order to obtain the analytical solution of the fuzzy system, Li et al. [26] introduced fuzzy marginal linearization method. Further, Li et al. [27] proposed fuzzy inference modeling (FIM) method to approximate the time-variant system. Wang et al. [28] proposed a dynamic fuzzy inference modeling (DFIM) method and proved that fuzzy system generalized by this method was universal approximators to the solutions of some nonautonomous differential equations.

However, above fuzzy modeling technology could not be used to solve the differential equation (1.3). And when $f(x, y)$ and $g(x)$ in (1.3) are unknown and only input-output data of them are obtained, how to obtain the corresponding solution is an interesting question. Motivated by this fact, the aim of this paper is to propose a novel fuzzy modeling method to

solve the Lane-Emden equation. This paper is organized as follows. In Section 2, we introduce some preliminary knowledge. In Section 3, we propose a novel fuzzy modeling technology and use it to obtain the approximate analytical solutions of the Lane-Emden equation. Some examples are used to illustrate the validity of the proposed method in Section 4. Finally, conclusions are presented in Section 5.

2. Preliminaries and Basic Ideas of FIM

Firstly, we introduce some basic concepts which will be used in sequel.

Definition 2.1. A fuzzy set A of X is a function from the reference set X to the unit interval $[0,1]$, that is, $A : X \rightarrow [0,1]$, $x \mapsto A(x)$.

Definition 2.2. Let A_i $(i = 1, \ldots, n)$ be a group of normal fuzzy sets of X, where x_i is the peak point of A_i, that is, $A_i(x_i) = 1$. If A_i $(i = 1, \ldots, n)$ satisfies the condition: $\sum_{i=1}^{n} A_i(x) = 1$ (for all $x \in X$) and for all i, j $(i \neq j \Rightarrow x_i \neq x_j)$, then $\mathcal{A} \triangleq \{A_i\}_{1 \leq i \leq n}$ is called a fuzzy partition of X.

Definition 2.3. The mappings w_r $(r = 1, 2)$ from $[a, b]$ to $[0, 1]$ are variable weights if the following conditions hold

    (a) for any $x \in [a, b]$, $w_1(x) + w_2(x) = 1$;

    (b) $w_1(a) = 1$, $w_2(b) = 1$.

Example 2.4. Let $X_k = [x_k, x_{k+1}]$. The following mappings are variable weights on X_k: $w_{1k}(x) = (x_{k+1} - x)/(x_{k+1} - x_k)$, $w_{2k}(x) = (x - x_k)/(x_{k+1} - x_k)$.

In the following, we will introduce FIM method. Consider a second order ordinary differential equation

$$y'' = \phi(x, y, y'), \tag{2.1}$$

where x is the independent variable, y is the unknown function, X, Y, Y', and Y'' are the universes of x, y, y', and y'', respectively, and ϕ is a real-valued continuous function, which explicitly contains the independent variable x. Obviously (1.3) is the special case of (2.1). In this paper, we assume that X, Y, Y', and Y'' are real number intervals, that is, $X = [a_0, b_0]$, $Y = [a_1, b_1]$, $Y' = [a_2, b_2]$, and $Y'' = [a_3, b_3]$.

Let $(x_i, y_j, y'_k, y''_{ijk}) \in X \times Y \times Y' \times Y''$ $(i = 1, \ldots, n;\ j = 1, \ldots, m;\ k = 1, \ldots, l)$ be a group of known data from (2.1), which satisfy the following two conditions:

    (1) $a_0 = x_1 < \cdots < x_n = b_0$, $a_1 = y_1 < \cdots < y_m = b_1$, $a_2 = y'_1 < \cdots < y'_l = b_2$;

    (2) $y''_{ijk} = \phi(x_i, y_j, y'_k)$, $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m;\ k = 1, 2, \ldots, l)$.

Then, we use above data information to construct fuzzy rule base. For each index i $(i = 1, \ldots, n)$, we, respectively, take y_j, y'_k, and y''_{kij} as the peak points of fuzzy sets A_j, B_k, and C_{ijk}, such that $\{A_j\}_{1 \leq j \leq m}$ is a fuzzy partition of Y, $\{B_k\}_{1 \leq k \leq l}$ is a fuzzy partition of Y', and $\{C_{ijk}\}_{1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq l}$ is a fuzzy partition of Y''. Similarly, we take y''_{ijk} as peak point of fuzzy

set D_{ijk}, such that $\{D_{ijk}\}_{1\leq i\leq n,1\leq j\leq m,1\leq k\leq l}$ is a fuzzy partition of Y''. In this way, fuzzy rules based on data information are represented as follows:

$$\text{If } y \text{ is } B_j \text{ and } y' \text{ is } C_k \text{ then } y'' \text{ is } D_{ijk},$$
$$i = 1,\ldots,n; \ j = 1,\ldots,m; \ k = 1,\ldots,l. \tag{2.2}$$

For fuzzy rules (2.2), fuzzy system generalized by FIM method can be expressed by

$$y'' = \sum_{i=1}^{n-1}\sum_{j=1}^{m}\sum_{k=1}^{l}(A_j(y)\cdot B_k(y'))\cdot y''_{ijk}\cdot\chi_i(x), \tag{2.3}$$

where the characteristic function $\chi_i(x)$ is defined as

$$\chi_i(x) \triangleq \begin{cases} 1, & x\in[x_i,x_{i+1}], \\ 0, & x\notin[x_i,x_{i+1}]. \end{cases} \tag{2.4}$$

Remark 2.5. It can be seen that fuzzy system (2.3) is a nonlinear differential equation with variable coefficients. When $A_j(y)$ and $B_k(y')$ are, respectively, chosen as triangular membership functions, in each local region $[x_i,x_{i+1}]\times[y_j,y_{j+1}]\times[y'_k,y'_{k+1}]$, (2.3) is changed into an autonomous differential equation with constant coefficients. This fact means that (2.3) is a two-order differential equation with piecewise constant coefficients. In [27], it is proved that solutions of (2.3) can approximate the numerical solutions of some nonautonomous differential equations with high accuracy.

3. Approximate Analytic Solutions of the Lane-Emden Equation Based on Variable Weight Fuzzy Marginal Linearization Method

From (2.3), we find that in each local region it is still a nonlinear differential equation. Hence, it is difficult for us to get the corresponding analytical solution for it. In this section, we propose a novel fuzzy modeling method, called variable weight fuzzy marginal linearization (VMFML) method, and utilize this technology to obtain the approximate analytical solution for Lane-Emden equation. For simplicity, we introduce some denotations.

Let $\{[x_i,x_{i+1})\times[y_j,y_{j+1})\times[y'_k,y'_{k+1})\}$ ($i=1,\ldots,n$; $j=1,\ldots,m$; $k=1,\ldots,l$) be a partition of $X\times Y\times Y'$, where $[x_n,x_{n+1})\triangleq\{x_n\}$, $[y_m,y_{m+1})\triangleq\{y_m\}$ and $[y'_l,y'_{l+1})\triangleq\{y'_l\}$. Similarly, $\{(x_{i-1},x_i]\times[y_j,y_{j+1})\times[y'_k,y'_{k+1})\}$ ($i=1,\ldots,n$; $j=1,\ldots,m$; $k=1,\ldots,l$) is another partition of $X\times Y\times Y'$, where $(x_0,x_1]\triangleq\{x_1\}$. Further, we, respectively, divide $[x_i,x_{i+1})\times[y_j,y_{j+1})\times[y'_k,y'_{k+1})$ and $(x_{i-1},x_i]\times[y_j,y_{j+1})\times[y'_k,y'_{k+1})$ into 4 pieces and let

$$(i,j,k)_{r_1r_2} \triangleq [x_i,x_{i+1})\times[y_{j+r_1/2},y_{j+(r_1+1)/2})\times\left[y'_{k+r_2/2},y'_{k+(r_2+1)/2}\right),$$
$$\overline{(i,j,k)}_{r_1r_2} \triangleq (x_{i-1},x_i]\times[y_{j+r_1/2},y_{j+(r_1+1)/2})\times\left[y'_{k+r_2/2},y'_{k+(r_2+1)/2}\right), \tag{3.1}$$

where $r_1 = 1, 2; r_2 = 1, 2; y_{j+1/2} \triangleq (y_j + y_{j+1})/2, y'_{k+1/2} \triangleq (y'_k + y'_{k+1})/2, j = 1, \ldots, m - 1; k = 1, \ldots, l - 1$. The characteristic functions on $(i, j, k)_{r_1 r_2}$ and $\overline{(i, j, k)}_{r_1 r_2}$ are, respectively, denoted as $\chi_{(i,j,k)_{r_1 r_2}}$ and $\chi_{\overline{(i,j,k)}_{r_1 r_2}}$, that is,

$$
\chi_{(i,j,k)_{r_1 r_2}}(x, y, y') \triangleq
\begin{cases}
1, & (x, y, y') \in [x_i, x_{i+1}) \times [y_{j+r_1/2}, y_{j+(r_1+1)/2}) \times \left[y'_{k+r_2/2}, y'_{k+(r_2+1)/2}\right), \\
0, & (x, y, y') \notin [x_i, x_{i+1}) \times [y_{j+r_1/2}, y_{j+(r_1+1)/2}) \times \left[y'_{k+r_2/2}, y'_{k+(r_2+1)/2}\right).
\end{cases}
$$

$$
\chi_{\overline{(i,j,k)}_{r_1 r_2}}(x, y, y') \triangleq
\begin{cases}
1, & (x, y, y') \in (x_{i-1}, x_i] \times [y_{j+r_1/2}, y_{j+(r_1+1)/2}) \times \left[y'_{k+r_2/2}, y'_{k+(r_2+1)/2}\right), \\
0, & (x, y, y') \notin (x_{i-1}, x_i] \times [y_{j+r_1/2}, y_{j+(r_1+1)/2}) \times \left[y'_{k+r_2/2}, y'_{k+(r_2+1)/2}\right).
\end{cases}
$$

$$(3.2)$$

In the following, we will introduce the basic idea of VMFML method.

For any $(x, y, y') \in X \times Y \times Y'$, without loss of generality, we assume that $(x, y, y') \in (i, j, k)_{r_1 r_2}$. For any $j \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, l\}$, by fuzzy marginal linearization technology, we take $A_j(y)$ as triangular membership function and $B_k(y')$ as rectangle-shaped membership function, then fuzzy system (2.3) can be changed into

$$
y'' = \frac{y_{j+1} - y}{y_{j+1} - y_j} \cdot y''_{ij(k+r_2)} + \frac{y - y_j}{y_{j+1} - y_j} \cdot y''_{ij(k+r_2)}.
$$

$$(3.3)$$

Similarly, when $A_j(y)$ is chosen as rectangle-shaped membership function and $B_k(y')$ is chosen as triangular membership function, fuzzy system (2.3) can be changed into

$$
y'' = \frac{y'_{k+1} - y'}{y'_{k+1} - y'_k} \cdot y''_{ij(k+r_2)} + \frac{y' - y'_k}{y'_{k+1} - y'_k} \cdot y''_{ij(k+r_2)}.
$$

$$(3.4)$$

Furthermore, we take sum of the right side of expressions (3.3) and (3.4) and subtract a constant, the corresponding fuzzy system in the local region $(i, j, k)_{r_1 r_2}$ is represented as

$$
y'' = d_{ijk}^{(1)} + d_{ijk}^{(2)} y + d_{ijk}^{(3)} y',
$$

$$(3.5)$$

where

$$
d_{ijk}^{(1)} = \frac{y_{j+1} y''_{ij(k+r_2)} - y_j y''_{i(j+1)(k+r_2)}}{y_{j+1} - y_j} + \frac{y'_{(k+1)} y''_{i(j+r_1)k} - y'_k y''_{i(j+r_1)(k+1)}}{y'_{k+1} - y'_k} - y''_{i(j+r_1)(k+r_2)},
$$

$$(3.6)$$

$$
d_{ijk}^{(2)} = \frac{y''_{i(j+1)(k+r_2)} - y''_{ij(k+r_2)}}{y_{j+1} - y_j},
$$

$$(3.7)$$

$$
d_{ijk}^{(3)} = \frac{y''_{i(j+r_1)(k+1)} - y''_{i(j+r_1)k}}{y'_{k+1} - y'_k}.
$$

$$(3.8)$$

By characteristic function, we can obtain fuzzy system on $X \times Y \times Y'$ as follows:

$$y'' \triangleq f(x, y, y') \triangleq \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} \left(d_{ijk}^{(1)} + d_{ijk}^{(2)} y + d_{ijk}^{(3)} y' \right) \cdot \chi_{(i,j,k)_{r_1 r_2}}(x, y, y'). \tag{3.9}$$

Remark 3.1. It is easy to see that (3.9) is a piecewise linear differential equation. In each local region $(i, j, k)_{r_1 r_2}$, (3.9) transfers into a linear differential equation and the corresponding coefficients can be computed by input-output data of the original system. By expressions (3.5)–(3.8), it is easy to prove that the right-hand side $f(x, y, y')$ is an interpolation function of $\phi(x, y, y')$, that is, $f(x_i, y_j, y'_k) = y''_{ijk} = \phi(x_i, y_j, y'_k)$, $i = 1, \ldots, n$; $j = 1, \ldots, m$; $k = 1, \ldots, l$. By numerical analysis theory, we know that when we obtain enough data information on (2.1), $f(x, y, y')$ can approximate $\phi(x, y, y')$ with the specified accuracy. This fact means that (3.6) can be used to describe the nonlinear differential equation (2.1).

Then, we will solve the initial value problem of (3.9).

Given an initial value problem as

$$y'' = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} \left(d_{ijk}^{(1)} + d_{ijk}^{(2)} y + d_{ijk}^{(3)} y' \right) \cdot \chi_{(i,j,k)_{r_1 r_2}}(x, y, y'), \tag{3.10}$$

subject to the conditions $y(0) = y_0$ and $y'(0) = y'_0$.

First, we can determine the local region which the initial vector $(0, y_0, y'_0)$ locates in. Suppose that $(0, y_0, y'_0) \in [x_1, x_2) \times [y_j, y_{j+1/2}) \times [y'_k, y'_{k+1/2})$. By (3.9) the corresponding piecewise equation can be written as

$$y'' = d_{1jk}^{(1)} + d_{1jk}^{(2)} y + d_{1jk}^{(3)} y', \tag{3.11}$$

and the corresponding coefficients can be computed by expressions (3.6)–(3.8). Since it is a linear ode, we can get the corresponding analytical solution and denote it as y_1. Then, we take $(x_2, y_1(x_2), y'_1(x_2))$ as the initial vector and search the next region which it moves in. And we can solve the corresponding piecewise linear equation with the initial value $(y_1(x_2), y'_1(x_2))$. In this way, by transferring initial value piece by piece and solving the corresponding piecewise linear equation, we can obtain analytical solution of (3.9).

On the other hand, if we take $\overline{(i, j, k)}_{r_1 r_2}$ ($r_1 = 1, 2$; $r_2 = 1, 2$; $j = 1, \ldots, m - 1$; $k = 1, \ldots, l - 1$) as the local region of universe $X \times Y \times Y'$, using fuzzy marginal linearization technology, we can obtain another fuzzy system as follows:

$$y'' = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} \left(\overline{d}_{ijk}^{(1)} + \overline{d}_{ijk}^{(2)} y + \overline{d}_{ijk}^{(3)} y' \right) \cdot \chi_{\overline{(i,j,k)}_{r_1 r_2}}(x, y, y'), \tag{3.12}$$

where

$$\overline{d}_{ijk}^{(1)} = \frac{y_{j+1}y''_{(i+1)j(k+r_2)} - y_j y''_{(i+1)(j+1)(k+r_2)}}{y_{j+1} - y_j} + \frac{y'_{(k+1)}y''_{(i+1)(j+r_1)k} - y'_k y''_{(i+1)(j+r_1)(k+1)}}{y'_{k+1} - y'_k}$$

$$- y''_{(i+1)(j+r_1)(k+r_2)},$$

$$\overline{d}_{ijk}^{(2)} = \frac{y''_{(i+1)(j+1)(k+r_2)} - y''_{(i+1)j(k+r_2)}}{y_{j+1} - y_j},$$

$$\overline{d}_{ijk}^{(3)} = \frac{y''_{(i+1)(j+r_1)(k+1)} - y''_{(i+1)(j+r_1)k}}{y'_{k+1} - y'_k}.$$

(3.13)

Obviously, both (3.9) and (3.12) can describe (2.1) and the analytical solutions of them can be obtained. The differences between them are that the coefficients of (3.9) in the local region $[x_i, x_{i+1}] \times [y_{j+r_1/2}, y_{j+(r_1+1)/2}) \times [y'_{k+r_2/2}, y'_{k+(r_2+1)/2})$ are computed by data $(x_i, y_j, y'_k, y''_{ijk})$, $(x_i, y_j, y'_{k+1}, y''_{ij(k+1)})$, $(x_i, y_{j+1}, y'_k, y''_{i(j+1)k})$, $(x_i, y_{j+1}, y'_{k+1}, y''_{i(j+1)(k+1)})$.

On the other hand, the corresponding coefficients of (3.12) in the local region $[x_i, x_{i+1}] \times [y_{j+r_1/2}, y_{j+(r_1+1)/2}) \times [y'_{k+r_2/2}, y'_{k+(r_2+1)/2})$ are deduced by data $(x_{i+1}, y_j, y'_k, y''_{(i+1)jk})$, $(x_{i+1}, y_j, y'_{k+1}, y''_{(i+1)j(k+1)})$, $(x_{i+1}, y_{j+1}, y'_k, y''_{(i+1)(j+1)k})$, and $(x_{i+1}, y_{j+1}, y'_{k+1}, y''_{(i+1)(j+1)(k+1)})$. For a given initial value $y(0) = y_0$ and $y'(0) = y'_0$, by transferring initial value technology, we can solve the corresponding analytical solutions of (3.9) and (3.12) denote them as $\varphi_1(x)$ and $\varphi_2(x)$, respectively. In order to describe (2.1) better, we take variable weighted sum of $\varphi_1(x)$ and $\varphi_2(x)$, and denote

$$\varphi(x) = \sum_{k=1}^{n-1} (\omega_{1k}(x) \cdot \varphi_1(x) + \omega_{2k}(x) \cdot \varphi_2(x)) \cdot \chi_k(x),$$

(3.14)

as the approximation analytical solution of (2.1), where $w_{1k}(x) = (x_{k+1} - x)/(x_{k+1} - x_k)$ and $w_{2k}(x) = (x - x_k)/(x_{k+1} - x_k)$.

In this way, we use variable weight fuzzy marginal linearization (VWFML) technology to obtain the approximation analytical solution for (2.1).

Next, we also take (2.1) as example to summarize basic processes of VWFLM method.

Step 1. Determine the universes of x, y, and y' and denote them as X, Y, and Y'.

Step 2. Divide the universes and let x_i, y_j, y'_k be the partition points of X, Y, and Y', where $i = 1, \ldots, n$; $j = 1, \ldots, m$; $k = 1, \ldots, l$.

Step 3. By (2.1), compute the corresponding output data $y''_{ijk} = \phi(x_i, y_j, y'_k)$.

Step 4. Using expressions (3.6)–(3.8) and data information $(x_i, y_j, y'_k, y''_{ijk})$ $(i = 1, \ldots, n-1;$ $j = 1, \ldots, m;$ $k = 1, \ldots, l)$, deduce the coefficients of piecewise equation and construct the corresponding fuzzy system (3.9).

Step 5. For given initial value, by transferring initial value technology, solve above fuzzy system and denote the solution as $\varphi_1(x)$.

Step 6. By expression (3.13) and data information $(x_{i+1}, y_j, y'_k, y''_{(i+1)jk})$ $(i = 2, \ldots, n;\ j = 1, \ldots, m;\ k = 1, \ldots, l)$, we can also deduce the coefficients of piecewise equation and construct another fuzzy system (3.12).

Step 7. Similarly, by transferring initial value technology, we can solve this fuzzy system (3.12) and denote the solution as $\varphi_2(x)$.

Step 8. We take weighted sum of $\varphi_1(x)$ and $\varphi_2(x)$, that is,

$$\varphi(x) = \sum_{k=1}^{n-1} (\omega_{1k}(x) \cdot \varphi_1(x) + \omega_{2k}(x) \cdot \varphi_2(x)) \cdot \chi_k(x), \tag{3.15}$$

where $w_{1k}(x) = (x_{k+1} - x)/(x_{k+1} - x_k)$ and $w_{2k}(x) = (x - x_k)/(x_{k+1} - x_k)$.

Remark 3.2. In many physical problems when differential equation models are unknown and only some data information of the investigated systems are known, by VWFML method we can still set up dynamic models and obtain corresponding approximation analytical solutions for the problems.

4. Numerical Results

In this section, we will use VWFML method to solve four Lane-Emden equations. The former two equations are of original type Lane-Emden equations and the latter two are of generalized type.

Example 4.1. Consider the linear singular initial value problem:

$$y'' + \frac{2}{x}y' + y = 6 + 12x + x^2 + x^3, \tag{4.1}$$

subject to the initial conditions $y(0) = 0$ and $y'(0) = 0$. The exact solution for this equation is $y(x) = x^2 + x^3$.

The solution, which is generated from this Lane-Emden equation of classical astrophysics, has a proven physical foundation and can be used to calculate the radius of electron in an electromagnetic mass model.

Let $n = 5$, $m = 5$, and $l = 5$. Figure 1, respectively, shows curves of exact solution and approximate solutions obtained by VWFML method, DFIM method, and FIM method. Table 1 shows the coefficients of piecewise (3.9) and (3.12). In Figure 2, the absolute error curves among them are given, where "crossed line" denotes the absolute error curve between the approximate solution obtained by VWFML method and the exact solution, "$\cdots$" denotes the absolute error curve between the approximate solution obtained by FIM method and the exact solution, and "–∗–" denotes the absolute error curve between the approximate solution obtained by DFIM method and the exact solution. Table 2 presents the errors among the exact solution and the approximate solutions obtained by them.

In this simulation, the approximation results are satisfying. In fact, the number of divided local regions is not large, which means that we can get all the coefficients of (3.9) and (3.12) without a great deal of computation. Accordingly, the approximate analytic solution
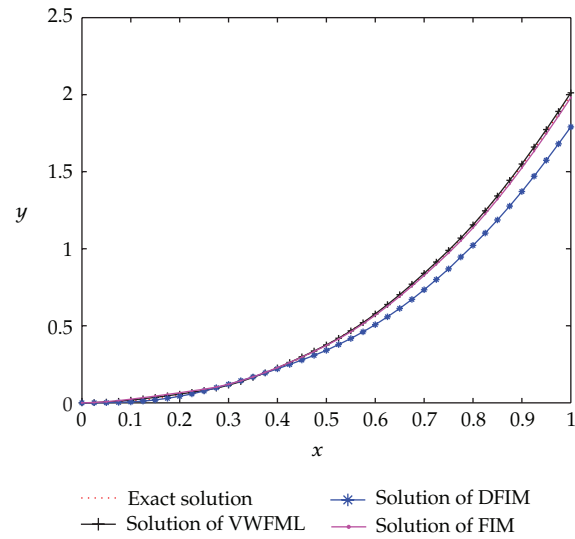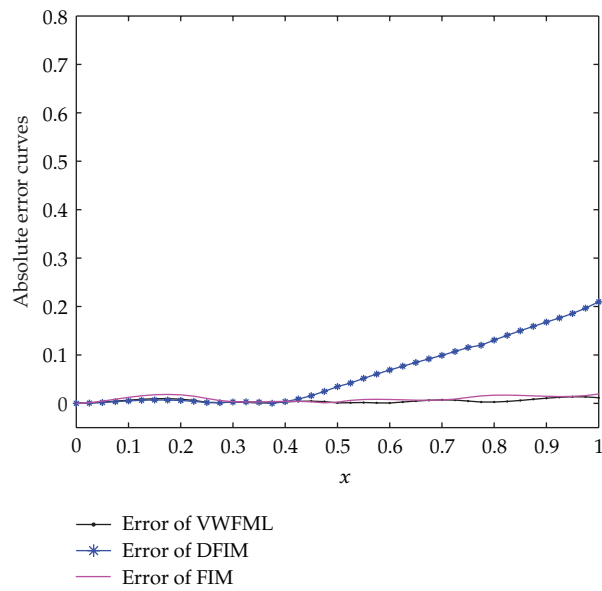
Figure 1: Solution curves of Example 4.1.

Figure 2: Error curves of Example 4.1.

can be found for Example 4.1. From Figure 2 and Table 2, it is clear that for solving this equation VWFML technology is more effective than DFIM and FIM, and the corresponding amount of error data is smaller.

Example 4.2. Consider isothermal gas spheres equation:

$$y'' + \frac{2}{x}y' + e^y = 0, \tag{4.2}$$

subject to the initial conditions $y(0) = 0$, $y'(0) = 0$.

Table 1: Coefficients of (3.9) and (3.12).

		Example 4.1 ($n = 5$, $m = 5$, $l = 5$)					
k	x	$d_{ijk}^{(1)}$	$d_{ijk}^{(2)}$	$d_{ijk}^{(3)}$	$\overline{d}_{ijk}^{(1)}$	$\overline{d}_{ijk}^{(2)}$	$\overline{d}_{ijk}^{(3)}$
1	(0, 0.25]	0	0	0	−1	−8	9.0781
2	(0.25, 0.5]	−1.0000	−8.0000	9.0781	−1.0000	−4.0000	12.3750
3	(0.5, 0.75]	−1.0000	−4.0000	12.3750	−1.0000	−2.6667	15.9844
4	(0.75, 1]	−1.0000	−2.6667	15.9844	−1.0000	−2.0000	20.0000

Table 2: Errors of different approximate solutions for Example 4.1.

	Example 4.1 ($n = 5$, $m = 5$, $l = 5$)		
x	Error of VWFML	Error of DFIM	Error of FIM
0.1	0.0067	0.0052	0.0125
0.2	0.0087	0.0059	0.0176
0.3	−0.0027	0.0028	0.0035
0.4	0.0029	−0.0035	0.0041
0.5	0.0010	−0.0345	−0.0027
0.6	0.0008	−0.0689	−0.0081
0.7	0.0072	−0.0991	−0.0070
0.8	0.0027	−0.1306	−0.0169
0.9	0.0107	−0.1678	−0.0146
1	0.0117	−0.2096	−0.0193

This type of equation has been used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules. Isothermal gaseous sphere, in which the temperature remains constant, subjects to the classical laws of thermodynamics when one seeks to determine the density and electric force of an electron gas in the neighborhood of a hot body in thermal equilibrium. It is worthy to notice that this equation is nonlinear and has no analytic solution.

From [1, 5], we know that the 6-term series solutions of this equation obtained by ADM and HPM are the same. Obviously, in Figure 3 we find that solutions obtained by VWFML method are very close to solutions of ADM method, where "$\cdots$" denotes the solution obtained by ADM method and "crossed line" denotes the solution obtained by VWFML method.

For the consistency of the comparison and without loss of generality, chosen the same initial value and the same independent variable values, Table 3 shows the approximations of solutions for Example 4.2, respectively, obtained by ADM method and VWFML method. Further, in Figure 4, the curves obtained by the above two methods are given. Since Lane-Emden equations are singular initial value problems, the accuracy of solutions near zero point is important, whatever the solution is gotten from approximate analytical method or from numerical method. From simulation results, we can see that if the number of local regions increased in VWFML method, the accuracy of the approximation solution can be improved evidently, especially when the independent variable is near the initial value 0. Besides, the solutions of Lane-Emden equations converge rapidly in a very small region ($0 < x < 1$), and
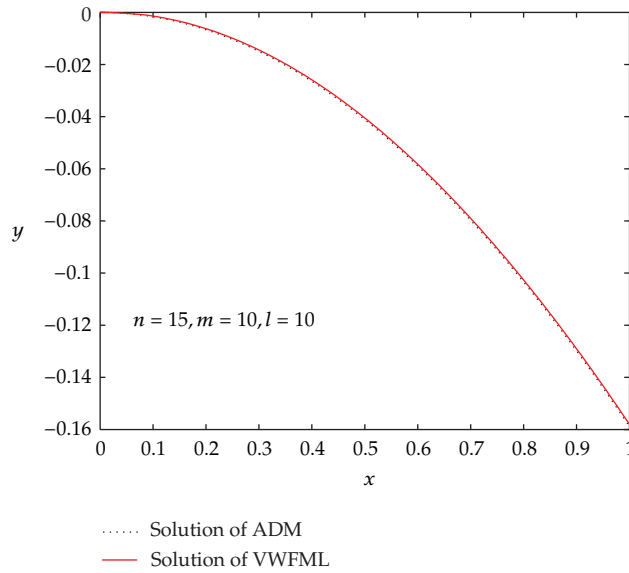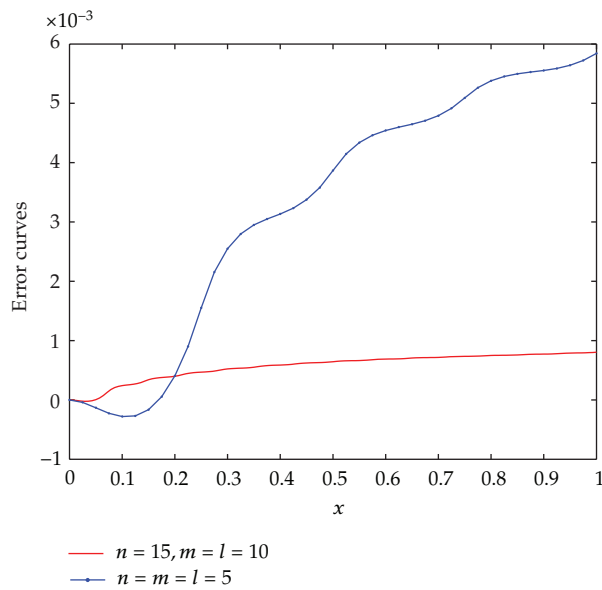
Figure 3: Solution curves of Example 4.2.

Figure 4: Error curves of Example 4.2.

in Figure 3 the approximate curve follows the trend appropriately. This implies that VWFML method can describe the isothermal gas spheres equation with high accuracy degree.
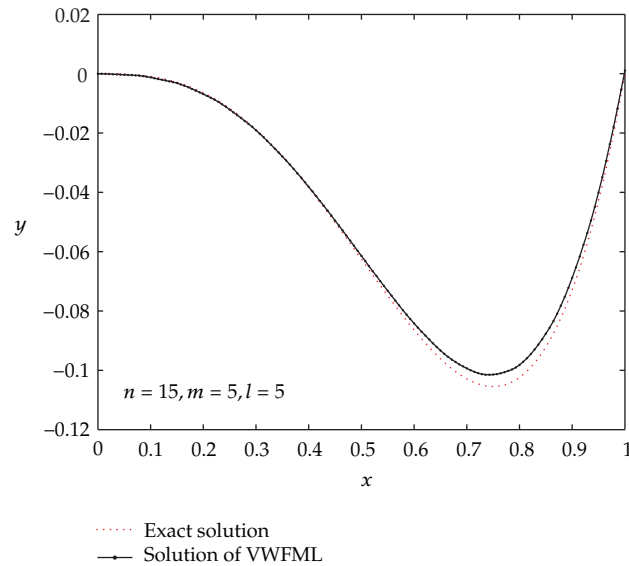
Example 4.3. Consider the linear initial value problem:

$$y'' + \frac{8}{x}y' + xy = x^5 - x^4 + 44x^2 - 30x, \tag{4.3}$$

subject to the initial conditions $y(0) = 0$, $y'(0) = 0$. The exact solution is $y(x) = x^4 - x^3$.

Table 3: Solutions of Example 4.2.

x	6-term ADM solution	VWFML solution ($n = m = l = 5$)	VWFML solution ($n = 15,\ m = l = 10$)
0.1	−0.0016658	0.0019471	0.0014253
0.2	−0.0066534	−0.0062487	−0.0062547
0.3	−0.014933	−0.012383	−0.014411
0.4	0.026456	−0.023321	−0.025868
0.5	−0.041154	−0.037287	−0.04051
0.6	−0.058945	−0.054402	−0.058256
0.7	−0.079728	−0.074936	−0.079011
0.8	−0.10339	−0.098012	−0.10264
0.9	−0.12981	−0.12426	−0.12904
1	−0.15886	−0.15302	−0.15806

Figure 5: Solution curves of Example 4.3.

Let $n = 15$, $m = 5$, and $l = 5$. The curves of exact solution of Example 4.3 and the corresponding approximate solution obtained by VWFML method are shown in Figure 5, where "$\cdots$" denotes the exact solution and "crossed line" denotes the approximate solution obtained by VWFML method.

Example 4.4. We consider the nonlinear initial value problem:

$$y'' + \frac{6}{x}y' + 14y = -4y \ln y, \tag{4.4}$$

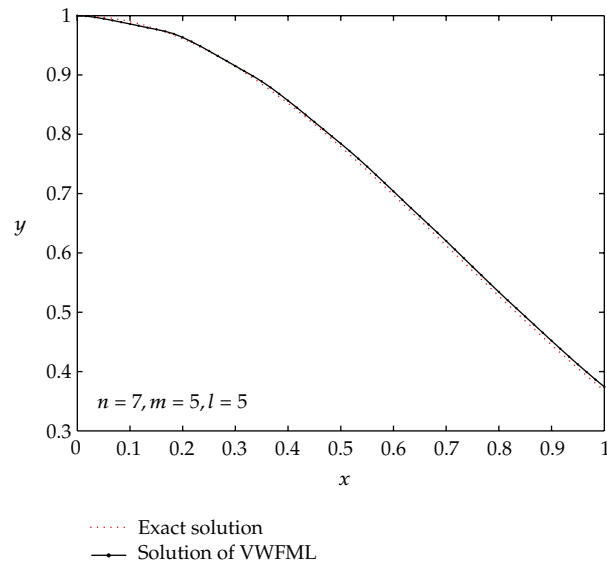subject to the initial conditions $y(0) = 1$, $y'(0) = 0$. The exact solution is $y(x) = e^{-x^2}$.

$n = 7, m = 5, l = 5$

· · · · · Exact solution
—•— Solution of VWFML

Figure 6: Solution curves of Example 4.4.

$n = 15, m = 10, l = 10$
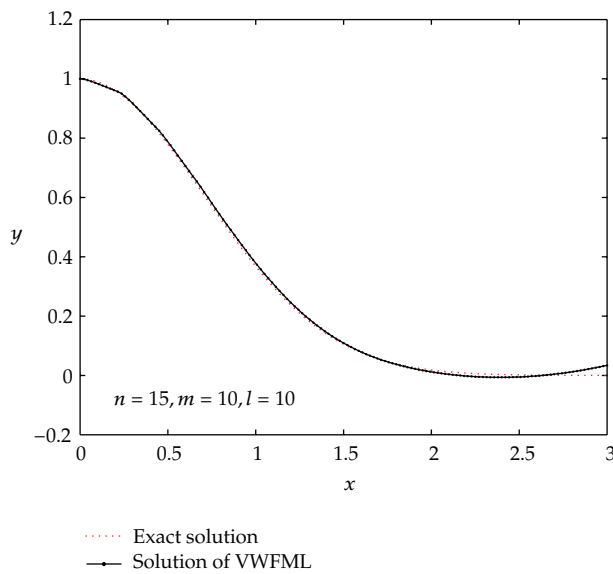
· · · · · Exact solution
—•— Solution of VWFML

Figure 7: Solution curves of Example 4.4 ($x \in [0, 3]$).

Let $n = 7$, $m = 5$, and $l = 5$. The interval of independent variable is chosen as $[0, 1]$. The curves of exact solution of Example 4.4 and the corresponding approximate solution obtained by VWFML method are shown in Figure 6, where "$\cdots$" denotes the exact solution and "crossed line" denotes the approximate solution obtained by VWFML method. Furthermore, the interval of independent variable is extended as $[0, 3]$ and let $n = 15$, $m = 10$, and $l = 10$. The simulation result is shown in Figure 7.

Remark 4.5. From Examples 4.3 and 4.4, it can be seen that VWFML method is effective for solving generalized Lane-Emden type differential equations. Furthermore, if the solving interval is extended, VWFML method can also provide approximate solutions in the larger domains with high accuracy degree.

Remark 4.6. Comparison with some analytical solutions, solutions of VWFML methods are totally dependent on the dividing points of independent variable interval, which means that the proposed technique can be presented in a general way. In particular, when the objective equation is unknown and only some data information can be obtained, equation determined by VWFML method can be solved analytically and the corresponding solution can approximate the solution of objective equation with high accuracy.

5. Conclusion

In this paper, we apply VWFML method to obtain the approximate analytical and numerical solution for Lane-Emden type differential equation. Some numerical examples show that by relatively minor data information, solutions obtained by VWFML method can approximate the corresponding solutions of Lane-Emden type equations with high accuracy. This means that VWFML method can be utilized to solve and analyze complex nonlinear differential equations in practical application.

Acknowledgments

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