Application of the LISS Lyapunov-Krasovskii small-gain theorem to autonomously controlled production networks with time-delays

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Abstract— In this paper we consider general autonomously controlled production networks. A production network consists of geographically distributed plants, which are connected by transport routes such that transportation times (time-delays) have to be taken into account. In autonomous controlled production networks logistic objects (e.g., parts, orders) route themselves through a network based on local information. In this paper these kinds of logistic networks are investigated in view of stability to avoid negative outcomes such as high inventory costs or loss of customers. We use the local input-to-state stability (LISS) property and the tool of an LISS Lyapunov-Krasovskii functional for the stability investigation. By the application of the LISS Lyapunov-Krasovskii small-gain theorem we derive conditions, which guarantee stability of the production network.

I. INTRODUCTION

Production, supply networks and other logistic structures are typical examples of complex systems with a nonlinear and sometimes chaotic behavior.

Their dynamics is subject to many different perturbations due to changes on market, changes in customers behavior, information and transport congestions, unreliable elements of the network etc. One of the approaches to handle such complex systems is to shift from centralized to decentralized or autonomous control.

The term production network is used to describe company or cross-company owned networks with geographically dispersed plants.

The main idea of autonomous cooperating logistic processes is to enable intelligent logistic objects to route themselves through a logistic network according to their own objectives and to make and execute decisions, based on local information [22], [23]. In this context intelligent logistic objects may be physical or material objects, e.g., parts or machines, as well as immaterial objects (e.g., production orders, information). For further investigations on autonomous control methods see for example [15], [16].

However a system emerging in this way may become unstable and hence be not effective.

Typical examples of unstable behavior are unbounded growth of unsatisfied orders or unbounded growth of the queue of the workload to be processed by a machine which cause high inventory costs or loss of customers. To avoid

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instability of a network it is worth to investigate its behavior in advance. In particular mathematical modelling and analysis provide helpful tools for design, optimization and control of such networks and for deeper understanding of their dynamical properties.

Roughly speaking, for production networks stability means that the state of the network remains bounded over time in spite of disturbances.

The state is the set of parameters, which we are interested in, for example the queue length of the workload to be processed by a machine, the work in progress (WIP) or the number of unsatisfied orders. In this contribution we identify the state as the number of unprocessed parts, which is the sum of the queue length and the WIP. Thus stable behavior of the network is decisive for the performance and vitality of a network. To design stable logistic networks we are going to apply tools from mathematical systems theory.

In this paper a production network is described as an interconnection of many dynamical subsystems that are logistic locations, which are described by retarded functional differential equations. These equations take into account time-delays, which are interpreted here as transportation times. Based on these equations we apply Lyapunov tools to investigate the subsystems in view of stability, where we use the stability property input-to-state stability (ISS), introduced in [19].

For time-delay systems there are two types of Lyapunov functions: ISS Lyapunov-Razumikhin functions ([21]) and ISS Lyapunov-Krasovskii functionals ([12]). Taking a network of many subsystems into account, it was proved in [6], that the whole network is stable provided that each subsystem has an ISS Lyapunov-Razumikhin function or ISS Lyapunov-Krasovskii functional and that the property of the small-gain condition in matrix form (see [7]) is satisfied. The used matrix in this condition describes the interconnection structure of the network.

In practical reality it may happen that the subsystems or the network are not ISS, but locally ISS, which means that the ISS property only holds for certain initial values and external inputs. This property was introduced in [20] for single systems and investigated for networks in [8].

For manufacturing systems stable parameters can be found by using fluid models [2], re-entrant lines [3] or manufacturing systems with different job types [4]. An approach with flows of multiple fluids was used to analyse the stability region of an autonomously controlled shop floor scenario [17]. Scholz-Reiter et al. [18] presented a fluid model of a production network and obtained a stability region for

a scenario with two locations and three types of products. A special case of such networks, where a certain type of topology was used, and no transportation times (time-delays) were considered, has been already investigated in [5].

In this paper we apply ISS or LISS Lyapunov-Krasovskii functionals to autonomously controlled production networks with transportation times to perform a stability analysis for such networks and derive parameter conditions under which the whole network is stable.

The structure of the paper is as follows. In Section II we give some basic notions used in the whole paper. The modelling method, retarded functional differential equations, the stability notions and the stability tool are described in Section II-A. In Section III autonomously controlled production networks are modeled and the stability analysis is performed. We derive conditions which guarantee stability of the network and illustrate the results in a certain scenario of a production network. Finally Section IV concludes the paper.

II. PRELIMINARIES

By x^T we denote the transposition of a vector $x \in \mathbb{R}^N$, $N \in \mathbb{N}$, furthermore $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{R}_+^N$ denotes the positive orthant $\{x \in \mathbb{R}^N : x \geq 0\}$ where we use the partial order for $x, y \in \mathbb{R}^N$ given by

$$x \geq y \Leftrightarrow x_i \geq y_i, \ i = 1, \ldots, N \text{ and } x \ngeq y \Leftrightarrow \exists i : x_i < y_i,$$
$$x > y \Leftrightarrow x_i > y_i, \ i = 1, \ldots, N \text{ and } x \ngtr y \Leftrightarrow \exists i : x_i \leq y_i.$$

We denote the Euclidean norm in $\mathbb{R}^N$ by $|\cdot|$. For $x = (x_1, \ldots, x_N)^T$ defined on an interval $I \subset \mathbb{R}$, we define $\|x\|_I := \max_i \max_{t \in I} \{|x_i(t)|\}$. $L_\infty(\mathbb{R}_+, \mathbb{R}^M)$ denotes the set of essentially bounded measurable functions from $\mathbb{R}_+$ into $\mathbb{R}^M$, $M \in \mathbb{N}$.

Let $\theta \in \mathbb{R}_+$ be the maximum involved delay. The function $x_t : [-\theta, 0] \to \mathbb{R}^N$ is given by $x_t(\tau) := x(t + \tau)$, $\tau \in [-\theta, 0]$ and we define $\|x_t\| := \max_{t-\theta \leq s \leq t} |x(s)|$. For $a, b \in \mathbb{R}$, $a < b$, let $C([a, b]; \mathbb{R}^N)$ denote the Banach space of continuous functions defined on $[a, b]$ equipped with the norm $\|\cdot\|_{[a,b]}$ and taking values in $\mathbb{R}^N$.

In general production networks are modeled by nonlinear functional differential equations. We call a production location within a production network subsystem and the production network whole system. Consider $n \in \mathbb{N}$ interconnected subsystems with time-delays, where the ith subsystem can be written as

$$\dot{x}_i(t) = f_i(x_1^t, \ldots, x_n^t, u_i(t)), \tag{1}$$

where $t \in \mathbb{R}_+$ is the (continuous) time, $x_i^t(\tau) := x_i(t+\tau) \in C([-\theta, 0]; \mathbb{R}^{N_i})$ and $u_i \in L_\infty(\mathbb{R}_+, \mathbb{R}^{M_i})$ are external inputs. θ denotes the maximal involved delay and x_i^t can be interpreted as the internal inputs of a subsystem. The functionals $f_i : C([-\theta, 0]; \mathbb{R}^{N_1}) \times \ldots \times C([-\theta, 0]; \mathbb{R}^{N_n}) \times \mathbb{R}^{M_i} \to \mathbb{R}^{N_i}$ are locally Lipschitz continuous on any bounded set to guarantee that the ith subsystem admits a unique locally absolutely continuous solution $x_i(t)$ on a maximal interval $[-\theta, b), 0 < b \leq +\infty$, satisfying the initial condition $x_i^0 = \xi_i$ for any $\xi_i \in C([-\theta, 0], \mathbb{R}^{N_i})$, $i = 1, \ldots, n$ (see [9], [10]).

If we define $N := \sum N_i$, $M := \sum M_i$, $x := (x_1^T, \ldots, x_n^T)^T$, $u := (u_1^T, \ldots, u_n^T)^T$ and $f := (f_1^T, \ldots, f_n^T)^T$, then the network (1) becomes the system of the form

$$\dot{x}(t) = f(x_t, u), \tag{2}$$

which we call the whole system.

The upper right-hand side derivative D^+V of a locally Lipschitz continuous functional $V : C([-\theta, 0]; \mathbb{R}^N) \to \mathbb{R}_+$ is given by (see [1], Definition 4.2.4, pp. 258)

$$D^+V(\phi, u) = \limsup_{h \to 0^+} \frac{1}{h} (V(\phi_h^\star) - V(\phi))$$

where $\phi_h^\star \in C([-\theta, 0]; \mathbb{R}^N)$ is given by

$$\phi_h^\star(s) = \begin{cases} \phi(s+h), & s \in [-\theta, -h], \\ \phi(0) + f(\phi, u)(h+s), & s \in [-h, 0]. \end{cases}$$

With the symbol $\|\cdot\|_a$ we indicate any norm in $C([-\theta, 0]; \mathbb{R}^N)$ such that for some positive reals b, c the following inequalities hold

$$b|\phi(0)| \leq \|\phi\|_a \leq c\|\phi\|_{[-\theta, 0]}, \ \forall \phi \in C([-\theta, 0]; \mathbb{R}^N).$$

See also [12].

Definition 2.1: Classes of comparison functions are:

$$\begin{aligned} \mathcal{K} \quad &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous}, \gamma(0) = 0 \\ &\qquad \text{and strictly increasing}\}, \\ \mathcal{K}_\infty \quad &:= \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\}, \\ \mathcal{L} \quad &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and decreasing} \\ &\qquad \text{with } \lim_{t \to \infty} \gamma(t) = 0\}, \\ \mathcal{KL} \quad &:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous}, \\ &\qquad \beta(\cdot, t) \in \mathcal{K}, \ \beta(r, \cdot) \in \mathcal{L}, \ \forall t, r \geq 0\}. \end{aligned}$$

Note that for $\gamma \in \mathcal{K}_\infty$ the inverse function γ^{-1} always exists and $\gamma^{-1} \in \mathcal{K}_\infty$.

A. Stability analysis

In this section we present the stability notions and tools to verify stability.

We define the stability notion for the system of the form (1) used in this paper as follows:

Definition 2.2: The ith subsystem of (1) is called locally input-to-state stable (LISS) if there exist constants ρ_i, ρ_{ij}, $\rho_i^u > 0$ and functions $\beta_i \in \mathcal{KL}$ and $\gamma_{ij}^d, \gamma_i^u \in \mathcal{K}$, $j = 1, \ldots, n$, $i \neq j$ such that for all initial functions $\|\xi_i\|_{[-\theta, 0]} \leq \rho_i$, for all inputs $\|x_j\|_{[-\theta, \infty)} \leq \rho_{ij}$, $j \neq i$, $\|u_i\|_{[0, \infty)} \leq \rho_i^u$ and for all $t \in \mathbb{R}_+$ it holds

$$\|x_i(t)\| \leq \max\{\beta_i(\|\xi_i\|_{[-\theta, \infty)}, t), \gamma_i^u(\|u_i\|_{[0, \infty)}), \\ \max_{j, j \neq i} \gamma_{ij}^d(\|x_j\|_{[-\theta, \infty)})\}. \tag{3}$$

Note that if $\rho_i = \rho_{ij} = \rho_i^u = \infty$, then the ith subsystem of (1) is called (globally) input-to-state stable (ISS).

The notion of LISS for systems of the form (2) is the following:

Definition 2.3: System (2) is called *LISS* if there exist constants $\rho, \rho_u > 0$ and functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such

that for every initial condition $\|\xi\|_{[-\theta,0]} \leq \rho$, every external input $\|u\|_{[0,\infty)} \leq \rho_u$ and for all $t \in \mathbb{R}_+$ it holds

$$\|x(t)\| \leq \max\{\beta(\|\xi\|_{[-\theta,0]}, t), \gamma(\|u\|_{[0,\infty)})\}.$$

Note that if $\rho = \rho_u = \infty$, then the whole system is called (globally) ISS.

We investigate under which conditions the whole system has the LISS property. A useful tool to verify LISS of a system is a Lyapunov function or functional. For time-delay systems of the form (2) one can use Lyapunov-Razumikhin functions (see [21]) or Lyapunov-Krasovskii functionals (see [12]). It was shown in [21], [12] that the existence of a (L)ISS Lyapunov-Razumikhin function or (L)ISS Lyapunov-Krasovskii functional implies the (L)ISS property.

In this paper we utilize Lyapunov-Krasovskii functionals, which are defined as follows:

Definition 2.4: A locally Lipschitz continuous functional $V_i : C([-\theta, 0]; \mathbb{R}^{N_i}) \rightarrow \mathbb{R}_+$ is called an *LISS Lyapunov-Krasovskii functional of the ith subsystem* of (1) if there exist a given functional V_j, which is continuous, proper and positive definite and locally Lipschitz continuous on $C([-\theta, 0]; \mathbb{R}^{N_j}) \backslash \{0\}$, $j = 1, \ldots, n$ and there exist constants ρ_i, ρ_{ij}, $\rho_i^u > 0$ and functions $\chi_{ij} \in \mathcal{K} \cup \{0\}$ and $\chi_i, \mu_i \in \mathcal{K}$, $j = 1, \ldots, n$, $i \neq j$ such that

$$
V_i(\phi_i) \geq \max\{\max_{j, j \neq i} \chi_{ij}(V_j(\phi_j)), \chi_i(|u_i|)\}
$$
$$
\Rightarrow D^+ V_i(\phi_i, u_i) \leq -\mu_i(V_i(\phi_i)),
\tag{4}
$$

$\forall \phi_i \in C\left([-\theta, 0], \mathbb{R}^{N_i}\right)$, for all initial functions $\|\xi_i\|_{[-\theta,0]} \leq \rho_i$ and all inputs $\|\phi_j\|_{[-\theta,\infty)} \leq \rho_{ij}$, $\|u_i\|_{[0,\infty)} \leq \rho_i^u$, $\chi_{ii} \equiv 0$. Functions χ_{ij}, χ_i are called (nonlinear) Lyapunov-Krasovskii gains. The gain-matrix is defined by $\Gamma := (\chi_{ij})_{i,j=1}^n$ and the map $\Gamma : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ by

$$
\Gamma(s) := \left(\max_j \chi_{1j}(s_j), \ldots, \max_j \chi_{nj}(s_j)\right)^T, \; s \in \mathbb{R}_+^n.
\tag{5}
$$

The Lyapunov-Krasovskii functionals are chosen to be locally Lipschitz continuous according to the results in [11], [13].

We say that the gain-matrix Γ satisfies the local small-gain condition (LSGC) if $\exists w^* \in \mathbb{R}_+^n$, such that it holds

$$
\Gamma(w^*) < w^* \text{ and } \Gamma(s) \not\geq s, \; \forall s \in [0, w^*], \; s \neq 0.
\tag{6}
$$

Lemma 2.5: Let Γ be a gain matrix as in (5) and let $w^* \in \mathbb{R}_+^n$ satisfy $\Gamma(w^*) < w^*$. Consider the trajectory $\{w(k)\}$ of the discrete monotone system $w(k+1) = \Gamma(w(k))$, $k = 0, 1, 2, \ldots$ with $w(0) = w^*$. Then $w(k) \rightarrow 0$ for $k \rightarrow \infty$ if and only if Γ satisfies the LSGC (6).

The proof and more information about the LSGC can be found in [8].

Note that if $\rho_i = \rho_{ij} = \rho_i^u = \infty$, $\forall i, j = 1, \ldots, n$ in the Definition 2.4, then V_i is called an ISS Lyapunov-Krasovskii functional of the ith subsystem. The (global) small-gain condition (SGC) is denoted as

$$
\Gamma(s) \not\geq s, \; \forall s \in \mathbb{R}_+^n, \; s \neq 0.
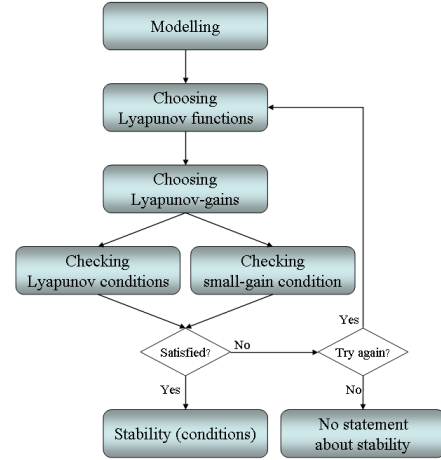\tag{7}
$$

Fig. 1. Scheme of the stability analysis procedure

More details about the global SGC can be found in [7], [14]. Note that (7) is equivalent to the cycle condition (see [14], Lemma 2.3.14): for all $(k_1, \ldots, k_p) \in \{1, \ldots, n\}^p$, where $k_1 = k_p$, it holds

$$
\chi_{k_1 k_2} \circ \chi_{k_2 k_3} \circ \ldots \circ \chi_{k_{p-1} k_p} < \text{Id}.
\tag{8}
$$

Now we state the main theorem we are going to apply to production networks in this paper.

Theorem 2.6: (LISS Lyapunov-Krasovskii theorem for general networks with time-delays)
Consider the interconnected system (1). Assume that each subsystem has an LISS Lyapunov-Krasovskii functional V_i, which satisfies the conditions in Definition 2.4, $i = 1, \ldots, n$. If the corresponding gain-matrix Γ, given by (5) satisfies the LSGC (6), then the whole system of the form (2) is LISS from u to x.
The proof can be found in [6] with corresponding changes according to the LISS property.

Note that if the functionals V_i in Theorem 2.6 are ISS Lyapunov-Krasovskii functionals and Γ satisfies the SGC (7), then the whole network is ISS (see [6]).

With these considerations we can perform a stability analysis for production networks, which will be done in the following section. We apply the tools and the Theorem 2.6 presented in this section. The procedure of the stability analysis is summarized in the Figure 1.

III. STABILITY ANALYSIS OF PRODUCTION NETWORKS

In this section we perform the stability analysis of production networks and apply the tools presented in the previous section.

We consider a general production network, consisting of n market entities: raw material suppliers (e.g. extracting or agricultural companies), producers, logistic companies, consumers etc. and denote each entity as a subsystem of the whole network. The state of each subsystem x_i is the number of unprocessed material within the i-th subsystem. A

subsystem can get material from an external source, which is denoted by u_i, and from subsystems of the network (internal inputs). We assume, that there is only one type of material.

An entity handles the material with a certain rate, called the process or production rate and transports the processed material to another subsystem or external customers according to the topology of the network with a rate, called distribution rate. This distribution rate can be constant or it changes, depending on the behavior of the network. We interpret the constant distribution rates as central planning and on the other hand changeable distribution rates can be used for some autonomous control method.

We model production networks as an interconnection of many dynamical subsystems that describe the dynamics of market entities, where we use functional differential equations as in (1). The state of the i-th entity or subsystem, respectively, is denoted by $x_i(t) \in \mathbb{R}_+$, where $t \in \mathbb{R}_+$ is the continuous time. The production rate of the i-th subsystem is referred to as $p_i(x_i(t)) \geq 0$ and depends on the state. By $c_{ij}(t) \geq 0$, $i \neq j$ we denote the distribution rate of material from subsystem j to i. Note that we can also write $c_{ij}(x_1(t), \ldots, x_n(t))$ and use $c_{ij}(t)$ for short, which means that the distribution rate depends on the state of the whole network.

The internal input of subsystem i from subsystem j, $j \neq i$ is denoted by $c_{ij}(t)p_j(x_j(t - T_{ij}))$, where $T_{ij} \geq 0$ is the time-delay and can be interpreted as the time needed for the transportation of the material from subsystem j to i. The external input from external suppliers (not considered in the network) is denoted by $u_i \in L_\infty(\mathbb{R}_+, \mathbb{R})$. The i-th entity processes the material with rate $\tilde{c}_{ii}(t)p_i(x_i(t))$ and delivers the ready-made material further, where $\tilde{c}_{ii} > 0$. Then the dynamics of the subsystems can be described by differential equations as follows:

$$\dot{x}_i(t) = \sum_{j=1}^n c_{ij}(t - T_{ij})p_j(x_j(t - T_{ij})) + u_i(t), \quad (9)$$

$i = 1, \ldots, n$, where we denote $c_{ii} := -\tilde{c}_{ii}$ with $T_{ii} = 0$.

For the stability analysis we apply the framework shown in Figure 1. At first we choose a Lyapunov-Krasovskii functional for each subsystem described in (9) and the corresponding Lyapunov-Krasovskii gains.

Consider the case where all functions $p_i \in \mathcal{K}_\infty$, in particular p_i, $i = 1, \ldots, n$ are unbounded. Later we show how with minimal modifications the same method can be applied for bounded $p_i \in \mathcal{K} \setminus \mathcal{K}_\infty$.

Note, that the conditions $p_i \in \mathcal{K}_\infty$, $c_{ii}(t) < 0$ and $c_{ij}(t) \geq 0$, $i \neq j$ imply, that if $x_i^0 = \xi_i \geq 0$ for any $\xi_i \in C([-\theta, 0], \mathbb{R}_+), \forall i = 1, \ldots, n)$, then $x_i^t \geq 0$ for all $t > 0$, where $\theta := \max_{ij}\{T_{ij}\}$.

We choose $V_i(x_i^t) = x_i(t)$ as the ISS Lyapunov-Krasovskii functional candidate for the i-th entity. Obviously, $V_i(x_i^t)$ is continuous, proper and positive definite and locally Lipschitz continuous on $C([-\theta, 0]; \mathbb{R}_+)\backslash\{0\}$. To prove, that

the condition (4) is satisfied, we choose the gains

$$\chi_{ij}(V_j(x_j^t)) := p_i^{-1}\left(\frac{a_i}{a_j}\frac{1}{1+\varepsilon_i}p_j(\|V_j(x_j^t)\|_{[t-T_{ij},t]})\right),$$

$$\chi_i(|u_i|) := p_i^{-1}\left(\frac{1}{r_i}|u_i|\right),$$

where a_j, $j = 1, \ldots, n$ and ε_i, r_i are some positive real numbers (see the certain scenario below for an explicit choice). From the assumption $V_i(x_i^t) \geq \max\{\max \chi_{ij}(V_j(x_j^t)), \chi_i(|u_i(t)|)\}$ it follows

$$x_i \geq \chi_{ij}(V_j(x_j^t)) \Rightarrow p_j(\|x_j\|_{[t-T_{ij},t]}) \leq \frac{a_j}{a_i}(1+\varepsilon_i)p_i(x_i),$$

$$x_i \geq \chi_i(|u_i|) \Rightarrow |u_i| \leq r_i p_i(x_i).$$

Using these inequalities and assuming that the condition

$$\sum_{j=1}^n c_{ij}(t - T_{ij})\frac{a_j}{a_i}(1+\varepsilon_i) + r_i \leq -\rho_i \qquad (10)$$

holds true, where $\rho_i > 0$, we obtain

$$D^+V_i\left(x_i^t, u_i\right) = \sum_{j=1}^n c_{ij}(t - T_{ij})p_j(x_j(t - T_{ij})) + u_i(t)$$

$$\leq \left(\sum_{j=1}^n c_{ij}(t - T_{ij})\frac{a_j}{a_i}(1+\varepsilon_i) + r_i\right)p_i(x_i(t))$$

$$\leq -\mu_i(V_i(x_i^t)),$$

where $\mu_i(r) := \rho_i p_i(r)$, $r > 0$ and we conclude that the condition (4) is satisfied for the ith subsystem. Thus, under condition (10), $V_i(x_i^t) = x_i(t)$ is the ISS Lyapunov-Krasovskii functional for the i-th entity. These calculations hold true for all $i = 1, \ldots, n$.

Now we check if the SGC (7) is satisfied, where we use the cycle condition (8). Consider a composition $\chi_{k_1 k_2} \circ \chi_{k_2 k_3}$ and $s > 0$:

$$\chi_{k_1 k_2} \circ \chi_{k_2 k_3}(s)$$
$$= p_{k_1}^{-1}\left(\frac{a_{k_1}}{a_{k_2}}\frac{1}{1+\varepsilon_{k_3}}p_{k_2}\left(p_{k_2}^{-1}\left(\frac{a_{k_2}}{a_{k_3}}\frac{1}{1+\varepsilon_{k_3}}p_{k_3}(s)\right)\right)\right)$$
$$= p_{k_1}^{-1}\left(\frac{a_{k_1}}{a_{k_3}}\frac{1}{(1+\varepsilon_{k_3})(1+\varepsilon_{k_2})}p_{k_3}(s)\right).$$

In the same way we obtain for $k_1 = k_p$

$$\chi_{k_1 k_2} \circ \chi_{k_2 k_3} \circ \cdots \circ \chi_{k_{p-1} k_p}(s)$$
$$= p_{k_1}^{-1}\left(\frac{1}{\prod_{i=2}^p(1+\varepsilon_{k_i})}p_{k_1}(s)\right) < s$$

Thus the cycle condition (8) and therefor the SGC (7) is satisfied for all $\varepsilon_i > 0$ and by Theorem 2.6 for the ISS property the whole network is ISS. We summarize these considerations in

Proposition 3.1: Consider a network as in (9) with $p_i \in \mathcal{K}_\infty$ and assume that the conditions (10) holds $\forall t > 0$ and all $i = 1, \ldots, n$, then the whole network is ISS.

Remark 3.2: The inequality (10) can be simplified, if $\exists M > 0 : c_{ij}(t - T_{ij}) \leq M$ for all $i,j = 1,\ldots,n$, $i \neq j$. Really, $\forall w_i > 0$ $\exists \varepsilon_j, j = 1,\ldots,n$ such that

$$\sum_{j=1,j\neq i}^{n} c_{ij}(t - T_{ij})\frac{a_j}{a_i}\varepsilon_j \leq M\left(\sum_{j=1,j\neq i}^{n} \frac{a_j}{a_i}\varepsilon_j\right) < w_i.$$

Using these estimates, we can rewrite (10) as

$$\sum_{j=1}^{n} c_{ij}(t - T_{ij})a_j \leq \tilde{\varepsilon}_i,$$

where $\tilde{\varepsilon}_i = -a_i(r_i + \rho_i + w_i)$. In matrix notation, with $a = (a_1,\ldots,a_n)^T$, $\tilde{\varepsilon} = (\tilde{\varepsilon}_1,\ldots,\tilde{\varepsilon}_n)^T$ and $C(t) := (c_{ij}^t)_{n\times n}$, where $c_{ij}^t := c_{ij}(t - T_{ij})$, it takes the form

$$C(t)a < \tilde{\varepsilon} \tag{11}$$

and we get the following proposition.

Proposition 3.3: Consider a network as in (9) with $p_i \in \mathcal{K}_\infty$ and assume that $\exists M > 0 : c_{ij}(t) \leq M$ for all $i,j = 1,\ldots,n$, $i \neq j$. If $\exists a \in \mathbb{R}^n$, $\tilde{\varepsilon} \in \mathbb{R}^n$, $a_i > 0$, $\tilde{\varepsilon}_i < 0$, $i = 1,\ldots,n$ such that the condition $C(t)a < \tilde{\varepsilon}$ holds $\forall t > 0$, then the whole network is ISS.

Remark 3.4: If the matrix C does not depend on t, then condition $Ca < \tilde{\varepsilon}$ can be replaced by $Ca < 0$. In the case, when $C = C(t)$, it cannot be done.

Now we consider $p_i \in \mathcal{K} \setminus \mathcal{K}_\infty$, i.e., function p_i is monotone increasing, but only up to a certain limit $\alpha_i := \sup_{x_i}\{p_i(x_i)\}$. For such p_i the ISS property cannot be achieved, but we can establish the LISS property. We choose again $V_i(x_i^t) = x_i(t)$ as the LISS Lyapunov-Krasovskii functional candidate for the i-th subsystem and the corresponding gains as follows

$$\chi_{ij}(V_j(x_j^t)) := p_i^{-1}\left(\frac{\alpha_i}{\alpha_j}\frac{1}{1+\varepsilon_i}p_j(\|x_j\|_{[t-T_{ij},t]})\right),$$

$$\chi_i(|u_i|) := p_i^{-1}\left(\frac{\alpha_i}{\|u_i\|_\infty r_i}|u_i|\right).$$

Note, that in contrast to the previous case, where the coefficients a_i involved in the gain functions were chosen arbitrarily, the α_j are taken from the boundedness assumptions of the functions p_i. The reason is to obtain a range of a function $\frac{\alpha_i}{\alpha_j}p_j(s)$ equal to the region where p_i^{-1} is defined.

Applying the same methods as for $p_i \in \mathcal{K}_\infty$, where we use the LSGC (6) and Lemma 2.5, we can prove that V_i, $i = 1,\ldots,n$ are the LISS Lyapunov-Krasovskii functionals of the subsystems.

Proposition 3.5: Consider a network as in (9) with $p_i \in \mathcal{K} \setminus \mathcal{K}_\infty$. Define $\|u\|_\infty := (\|u_1\|_\infty,\ldots,\|u_n\|_\infty)^T$ and $\alpha_j := \sup_{x_i \in \mathbb{R}}\{p_i(x_i)\}$, $i = 1,\ldots n$, $\alpha := (\alpha_1,\ldots,\alpha_n)^T$. If $\exists \tilde{\varepsilon} \in \mathbb{R}^n$, $\tilde{\varepsilon}_i < 0$, $i = 1,\ldots,n$ such that

$$C(t)\alpha + \|u\|_\infty < \tilde{\varepsilon}, \tag{12}$$

holds, then the whole network is LISS.

Remark 3.6: The stability analysis for functions $p_i \in \mathcal{K}$ is skipped here, because some more technical details are necessary, which will blast the size of the paper. The result is similar to the previous cases.
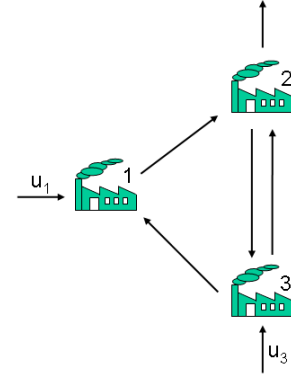

Fig. 2. A certain scenario of a production network

The conditions (10), (11) or (12) in the Propositions 3.1 and 3.3 and Remark 3.4 are based on the parameters of the network: the distribution rates c_{ij} and inputs u_i to guarantee ISS or LISS. This means that for a general production network one has to check if one of the conditions (10), (11) or (12) for the special kind of production rates p_i is satisfied to guarantee stability of the network.

A. Stability analysis of a certain scenario of a production network

In this section we provide a certain scenario of a production network to demonstrate the application of the stability analysis.

We consider a production network, consisting of three production locations as shown in Figure 2. Subsystems one and three get some raw material from external sources, denoted by u_1 and $u_3 \in \mathbb{R}_+$. $x_i(t) \in \mathbb{R}_+$, $i = 1,2,3$, denotes the amount of unprocessed parts within subsystem i. 50% of the production of subsystem three will be sent to subsystem one and two in each case ($c_{13} = c_{23} = 0.5$). There the parts enter the subsystems with the time-delay T_{13} and T_{23}, which denotes the transportation time from subsystem three to one and two, respectively.

Subsystem one processes the parts with the rate $p_1(x_1(t))$ and sends the processed parts to subsystem two ($c_{21} = 1$), where they arrive with the time-delay T_{21} and will be processed with the rate $p_2(x_2(t))$. 50% of the processed parts of subsystem two will be sent to subsystem three ($c_{32} = 0.5$) with T_{32} and 50% will leave the system. This can be interpreted as customer supply. Note that $c_{ii} = -1$, $i = 1,2,3$.

The production rates are given by $p_i(x_i) := x_i^2$ and the subsystems are modeled by

$$\dot{x}_1(t) = u_1(t) + \tfrac{1}{2}p_3(x_3(t - T_{13})) - p_1(x_1(t)),$$
$$\dot{x}_2(t) = p_1(x_1(t - T_{21})) + \tfrac{1}{2}p_3(x_3(t - T_{23})) - p_2(x_2(t)),$$
$$\dot{x}_3(t) = u_3(t) + \tfrac{1}{2}p_2(x_2(t - T_{32})) - p_3(x_3(t)).$$

By $a_1 = 0.6$, $a_2 = 1.2$, $a_3 = 1$ we have

$$Ca = \begin{pmatrix} -1 & 0 & \frac{1}{2} \\ 1 & -1 & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} 0.6 \\ 1.2 \\ 1 \end{pmatrix} < 0,$$

such that the condition (11) above (or in Remark 3.4) to guarantee stability of the network is satisfied. To illustrate this we do the following calculations:

We choose $V_i(x_i^t) = x_i(t)$, $i = 1, 2, 3$ as the ISS Lyapunov-Krasovskii functional candidates. By

$$\chi_1(|u_1(t)|) := \sqrt{\frac{|u_1(t)|}{r_1}}, \chi_{13}(V_3(x_3^t)) := \sqrt{\frac{a_1(\|x_3\|_{[t-T_{13},t]})^2}{a_3(1+\varepsilon_1)}},$$

where we choose $r_1 = 0.1$, $\varepsilon_1 = 0.01$ and the assumption $V_1(x_1^t) \geq \max\{\chi_{13}(V_3(x_3^t)), \chi_1(|u_1(t)|)\}$ we get

$$\mathrm{D}^+ V_1(x_1^t) = u_1(t) + \tfrac{1}{2}(x_3(t - T_{31}))^2 - (x_1(t))^2$$
$$\leq -\alpha_1(V_1(x_1^t))$$

for subsystem one, where $\alpha_1(r) := \frac{5}{100} r^2 \in \mathcal{K}$, $r \geq 0$. By

$$\chi_{ij}(V_j(x_j^t)) := \sqrt{\frac{a_i(\|x_j\|_{[t-T_{ij},t]})^2}{a_j(1+\varepsilon_i)}},$$
$$\chi_3(|u_3(t)|) := \sqrt{\frac{|u_3(t)|}{r_3}},$$

with $(i,j) \in \{(2,1), (2,3), (3,2)\}$, where we choose $\varepsilon_2 = \varepsilon_3 = 0.01$, $r_3 = 0.3$ and similar calculations for the other subsystems as for the first subsystem, we conclude that $V_i(x_i^t) = x_i(t)$, $i = 1, 2, 3$ are the ISS Lyapunov-Krasovskii functionals for the subsystems.

The small-gain condition is satisfied, because it holds

$$\chi_{13} \circ \chi_{32} \circ \chi_{21}(s) = \sqrt{\frac{1}{(1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3)}} s < s,$$
$$\chi_{23} \circ \chi_{32}(s) = \sqrt{\frac{1}{(1+\varepsilon_2)(1+\varepsilon_3)}} s < s,$$

and by application of Theorem 2.6 for the ISS property the whole network is ISS.

IV. CONCLUSIONS

We applied stability notions and LISS Lyapunov-Krasovskii functionals to check if a general autonomously controlled production network, modeled by retarded functional differential equations is stable. We derive conditions, which guarantee the stability of the whole production network. A certain scenario of a production network was investigated to demonstrate the derived conditions.

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