UNIVERSITY OF AGDER

# Venture Capital: Risk and Return

*Optimal Asset Allocation in a Venture Capital Portfolio*

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*Contact: orjan.aukland@hotmail.com. This Master's Thesis is carried out as a part of the education at the University of Agder and is therefore approved as a part of this education. However, this does not imply that the University answers for the methods that are used or the conclusions that are drawn. I thank my supervisors Dennis Frestad, Associate Professor at University of Agder, and Terje Berg-Utby, Analyst at Skagerak Venture Capital, for excellent guidance and helpful comments. All errors are my own.

# Abstract

*This thesis examines the impact of including higher moments than the mean and variance when optimizing an investment portfolio. As prior research on venture capital portfolio strategy has focused on diversification across industries and the optimal number of investments, this thesis adds insight to portfolio prioritization by focusing on the effect of portfolio diversification across different risk levels. More specifically, this study uses Monte Carlo to simulate returns from different risk levels and then determines how a "higher moments" optimal allocation change if the returns come from a non-normal as opposed to a normal distribution with everything else being equal. Although this study may provide insight for asset allocation in general, the relevance for the venture capital setting is recognized as high because extreme outcomes are more often observed in these portfolios compared to portfolios of ordinary noted stocks. I review some of the current literature on venture capital returns and find that the individual returns seem very well explained by a lognormal probability distribution. In my analysis, I find that constructing an optimal venture capital portfolio based on the skewness and kurtosis of the distribution, in addition to the mean and variance, should indicate minimal degrees of diversification between different risk profiles. This result does not align with the fact that many venture capital practitioners use stage diversification as a risk reduction strategy. The result can probably also explain some of the differences in performance between U.S. and European venture capital funds.*

# Contents

# 1 Introduction

In modern portfolio theory, an important assumption is that a rational investor's preference to invest in a risky asset compared to a risk-free asset is determined by the expected return of the two assets and the investor's tolerance for the underlying risk of the risky asset. Risk can be defined in various ways, but is commonly associated with *uncertainty* and *the tails of a distribution (Diebold, Doherty et al. 2010).* Variance and standard deviation are well-known tools for measuring risk but using these measures as the only measure of risk is intuitively very limited. First, it does not consider the direction of a deviation from the mean, and second, it is not a reliable measure for the likelihood of extreme outcomes on either side of the mean (Bodie, Kane et al. 2009). These are both moments of a probability distribution that could affect an investor's asset allocation decision.

The expected return and variance criteria (M-V) for optimal portfolio allocation was introduced by Markowitz and implies the assumption that asset returns are normally distributed or that the investor has a quadratic utility function[2] (Markowitz 1952). Therefore, when dealing with returns that are close to normally distributed, studies have shown that an optimal allocation decision based on the expected return and portfolio variance has little deviation from the direct optimal asset allocation (Levy and Markowitz 1979; Simaan 1993; Jondeau and Rockinger 2006). A well-known problem in modern finance is that returns in the financial markets seem to be non-normal, and many authors argue that extreme returns occur too often for an assumption of normality to hold (Fama 1963; Mandelbrot 1963; Taleb 2007). When returns are non-normally distributed, the M-V criteria has been shown to give a poor approximation of the direct optimal allocation (Jondeau and Rockinger 2006).

To argue that returns are non-normally distributed is easier in a venture capital (VC) setting. Research shows that a return on ten times the investment or a bankruptcy are not uncommon scenarios in a VC portfolio, although most returns are centered around a low normal rate of return (Sahlman 1990). Some of the most impressive returns from VC-backed companies come from Apple, Lotus, and Compaq, and these companies gave a return at public offering of 235, 63, and 38 times their initial VC financing, respectively (Bygrave and Timmons 1992). In general, this research implies that VC has a higher probability of extreme outcomes than a normal distribution would assume. Therefore, it seems likely that the venture capitalists (VCs) should consider the higher moments of the return distribution when putting together their portfolios.

This thesis examine the characteristics of VC returns and asses the similarity to a probability distribution. With Monte Carlo simulation and Crystal Ball™, I put together the optimal portfolio given different degrees of risk aversion and assess how this optimal asset allocation change when

[2] A quadratic function can only be derived twice.

returns are assumed to come from two different probability distributions. I contend that this will offer new input on risk reduction strategies used in the VC industry. The rest of the thesis is organized as follows. In section two, I introduce the VC market. Next, I explain the traditional portfolio selection theory and argue why one should account for higher moments when returns are non-normal distributed. Section four is a thorough review of VC returns and an assessment of a probability distribution that fit this setting. Section five introduces higher moments in a portfolio selection, and section six presents the Monte Carlo simulation and the optimization results. In the last section I discuss the impact of these results.

## 2 The Venture Capital Market

VC has grown to be a very important industry for promoting economic growth and innovation. The VCs serve as important financial intermediaries that provide capital to firms that otherwise would have difficulty to acquiring necessary financing. The firms that receive this financing are usually newly founded and have few tangible assets on their balance sheet (Gompers and Lerner 2001). The VCs then analyzes human capital, expansion potential, and other aspect of the business, and finances those that seem most promising.

As the companies that receive financing often have a great amount of uncertainty regarding their future, there is also a significant potential upside to the investment. History shows that while many of these companies go bankrupt, some VC investments do hit the jackpot. Some examples of this are Apple, Lotus, and Compaq as previously mentioned, but other examples include Yahoo, EBay, Cisco Systems, Starbucks, and Intel as they have all produced enormous return to their investors (Sahlman 1990; Gompers and Lerner 2001; Swensen 2005). The VC market is very large, and its range of service extends from providing capital in the earliest phase of a company's life to financing expansion phases through to public offering (Berg-Utby 2010). The level of uncertainty and individual risk is reduced as the VC firm invests in more mature companies, but this reduction in uncertainty also restricts the possibility of investing in a firm that can produce an extremely high return. The market is often divided into a seed, early growth and expansion phase, corresponding to where in its life-cycle the investment is (Robinson 1987).

Endowments, foundations, and pension funds are the largest investor group in the VC market and they mainly invest in these young companies through managed VC funds. The VCs normally take an annual fee of 2 to 3 percent of the total invested capital in their funds to select and actively manage the investments (Sahlman 1990; Cochrane 2005; Cumming 2010). VC is, today, a multibillion dollar industry with large amounts invested in funds worldwide. The U.S. market is much larger than the European market, and research also show that the U.S. VC funds give on average a higher return than the European funds (Hege, Palomino et al. 2003). In 2009, €3.2 billion was invested in European VC-backed companies and €14.9 billion in U.S. VC-backed companies from corresponding 1,234 and 2,489 deals, respectively (Ernst&Young 2009).

## 2.1.The Venture Capital Investment Cycle

VCs have three core activities: to buy, own, and sell companies. It is a cyclical process that starts over and over again, as illustrated in figure 1 (Berg-Utby 2010).

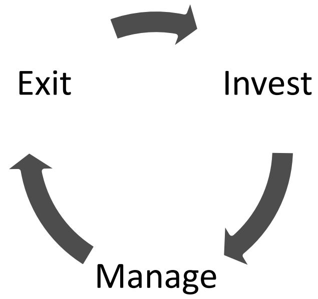


**Figure 1: The Venture Capital Investment cycle.**

The first core activity is the investment/buy decision. This task is a four-stage process where the VC firm decides to reject or go further with a possible investment based on how they interpret information that comes to their attention at each stage (Cumming 2010). Research has shown that this decision is based on a set of established criteria. The entrepreneur's capabilities, the attractiveness of his product, the market conditions, and the potential returns if the venture is successful (Robinson 1987; Hall and Hofer 1993; Cumming 2010). All criteria are present in every stage, but the level of importance may vary from stage to stage.

The first stage of this process is the **Sourcing** or **Deal Flow**. This stage is all available investment opportunities that the VC firm gets knowledge of when searching for the best investment opportunity. The Deal Flow can come to a VC firm in many ways, but it is often direct contact from entrepreneurs, referrals from trusted sources, or from participating in relevant forums such as entrepreneur networking events that gives the firm knowledge of which potential investments are available. The VC firm tries to interpret gathered information as effectively as possible before deciding which companies it will further explore. The next stage is the **Screening** process. The main focus in this process is in on the market conditions rather than on the entrepreneur himself (Hall and Hofer 1993). Thus, the firm analyzes and determines the project's feasibility and the magnitude of its market potential. Stage three is the **Due Diligence** stage, which is the most time-consuming phase. This stage is a more thorough evaluation of the firm's capabilities and potential. The focus in this stage is mainly on the entrepreneur and his team, and the objective is to determine whether they have the capabilities to realize potential returns. If the analyzed firm receives a favorable rating in this stage, then it moves to the final phase, the **Negotiation** stage. In this final stage, the VC firm and its potential investment discuss the specific terms of this particular case. This include terms of the contract, the staging of fund infusion, the valuation, and the board structure (Sahlman 1990; Cumming 2010). Providing all goes well in this final phase, the VC firm will invest in the venture; however, if any one of the areas under discussion is not satisfactorily resolved, the investment can still be rejected.

Through board representation, the VCs actively manage/own their investments and guide them to reach their potential. Their role as an active investor is to **monitor** and **advise** the entrepreneur (Sahlman 1990). While monitoring relates to controlling the entrepreneur’s actions so as to minimize wasteful expenditures (Jensen and Meckling 1976; Sahlman 1990), advising is a more supporting and value-adding role. This role includes supporting key decisions and providing advice with respect to strategic orientation, efficiency processes, and resource allocation (Cumming 2010). Research indicates that VCs have, on average, 40% of the board seats (Kaplan and Strömberg 2003), but their representation may often exceed this in situations where the entrepreneur is struggling to reach preset goals (Lerner 1995). These findings are consistent with agency theory and suggests that the VC firm will invest more resources for monitoring and advising when there is a greater risk that the entrepreneur will engage in non-pecuniary activities (Jensen and Meckling 1976).

The third and last core activity is the exit/sell process. VC is an illiquid market; thus, to realize profit for the initial fund investors, the VC firm must turn their illiquid investments into a realized return (Gompers and Lerner 2001). To do this, the VC firm has two options for exit. The investments can either **go public** or be **privately sold**. Each of these two possibilities has its own set of advantages and disadvantages; hence, the best method is dependent upon the individual situation of the investment and the VC’s preferences. Some of the benefits of an initial public offering (IPO) are that it increases the investment’s liquidity and that the information about the firm becomes more accessible. A drawback to an IPO is the risk that the market value of the investment will be lower than what the VC firm could obtain from a trade sale. Another disadvantage is that there are regulations that prohibit the VC firm from not selling all of its shares within a given period when the investors want cash return rather than shares. With a trade sale, however, the initial investors receive immediate cash returns, and the investment can have a strategic value for a buyer that potentially can result in a very high exit price. One of the disadvantages of this exit strategy is determining when to sell because the value of the investment may vary significantly over time. A second disadvantage with respect to a trade sale is the issue of trust that may arise when there are two private actors involved in the deal an no neutral party to endorse the information (Berg-Utby 2010).

## 2.2.Corporate Finance Challenges in Venture Capital

While there are many aspects in the VC setting that may differ from mainstream corporate finance, these can be categorized into one of four main concepts. They are *Information asymmetry*, *liquidity*, *ownership structure*, and *risk and return*. These concepts symbolize the distinctiveness of VC, but they also entail a discrepancy when considering the underlying assumptions in modern finance for describing the VC market. While these differences are a challenge to all VCs, it is wrong to think of them as only a threat because they can also create great opportunity.

In VC, **information asymmetry** is much greater between a buyer and a seller than it would be on a stock exchange because it is very difficult for the buyer (the VC firm) to actually obtain all of the information about the firm from the seller and it is in the seller's best interest to withhold bad information about the prosperity of the company. This withholding of information is not a major problem on the stock market because there regulations require companies to make information public. The stock market is also so liquid that modern finance theory assumes that stock prices will self-regulate when new information is made public. This is not the case for the VC market where the buyer must conduct a thorough screening and evaluation of the company before making an investment decision. This problem is often referred to as the adverse selection problem within agency theory (Milgrom and Roberts 1992).

Before the VC firm even begins the screening and evaluation process, an information problem arises that is primarily due to the **liquidity** of the investments. In VC, the investor must create the Deal Flow. This situation is unfamiliar to investors on the stock exchange where all shares are freely available to all investors. Research has shown that inexperienced VCs have a more difficult time obtaining attractive investment opportunities and have less bargaining power with their potential investments than their larger and more experienced colleagues (Gompers, Kovner et al. 2009).

After investing, another **information asymmetry** problem arises. The entrepreneur can attempt to maximize his own utility at the expense of the company. The reason why the entrepreneur can and will do this is because the VC firm cannot cost-efficiently monitor his behavior all of the time, and the entrepreneur will get a positive benefit-cost tradeoff when engaging in non-pecuniary expenditures[3] (Jensen and Meckling 1976). In the VC setting, this moral hazard problem is greater than for an investor investing in public stocks because there are fewer investors providing

[3] If an entrepreneur now owns 60% of the company, then he will pay only 60% for a private jet, better office etc., of which he has the full use.

capital on each investment and they have more capital at stake. The VC industry is trying to overcome this problem by making incentive contracts and performing **active ownership**.

The active management role has yet another risk reduction function. In modern portfolio theory, one assume that a passive investor can eliminate all firm-specific risk by diversifying his portfolio and invest in a large number of stocks that respond different to economic events. In a VC setting it is more difficult to diversify a portfolio in large numbers because the investment possibilities are not as liquid, there is little information about their past, and there is much uncertainty around their future. By active managing their investments, the VC firm can attempt to eliminate some of the firm-specific risk. To do this most effectively, the majority of VC partners only invest in industries of which they have knowledge (Gompers, Kovner et al. 2009). New research shows that a VC firm actually follows either a strategy to specialize in a specific industry or to spread investments across industries. The same study reveals that a firm's overall strategy is not a critical success factor for the company, but if the individual partners in the VC firm are specialized in a specific industry, this will yield a better return (Gompers, Kovner et al. 2009). This is consistent with the assumption that industry specialization helps to reduce firm-specific risk more effectively. There are probably two reasons for this finding. First, the specialized partners are better able to find the best investment opportunities, and second, they are better suited to add value and reduce risk for their investments. This does not mean that a VC firm should allocate all of their capital in one company in that industry. Weiding and Mathonet (2004) argue that because of the individual investment's risk profile, a VC fund should diversify in numbers to some degree. In another study, Kanniainen and Keuschnigg (2003) show that because of a VC's active management role, the optimal portfolio will be a trade-off between the number of investments held and the intensity of advice. If the marginal return from higher advising effort is diminishing this call for a larger portfolio, but if the entrepreneur receives less support he wants a higher profit share which can make a portfolio expansion unprofitable (Kanniainen and Keuschnigg 2003). Normally, a VC portfolio consists of ten to twenty companies (Weidig and Mathonet 2004).

The last challenge for the VC industry is that the research shows that the investment's **risk and return** profile differs from that of public stocks (Chiampou and Kallett 1989; Sahlman 1990; Ruhnka and Young 1991; Weidig and Mathonet 2004; Cochrane 2005). This asset class is naturally more risky than a public stock on, for example, the S&P 500. The reason why it is more risky is because VCs deals with young firms that are in a startup or expansion phase and the uncertainty about their future is great. In a VC portfolio returns of ten times the investment and bankruptcy are expected as normal scenarios. When investing in small companies on a stock exchange, some of the same characteristics as high expected return and high volatility are expected. In VC, however, some outcomes are more extreme. Robinson (1987) found in a survey of 53 VCs that another strategy for dealing with the high uncertainty and risk in VC is to include companies in all three phases of the market in the portfolio. He found that, on average, VCs diversify across the different stages of development by nearly an equal amount with approximately one-third in seed, early growth, and the expansion phase (Robinson 1987).

Despite the challenges presented above, a VC fund manager has the same objective as any other portfolio manager, i.e., to construct the optimal portfolio with the highest amount of expected return and the lowest risk. As previously mentioned, most VCs are focusing on eliminating firm-specific risk by having more than one investment in the portfolio, by industry specialization on the partner level, and by diversifying the portfolio across the different company phases. These strategies all seem sensible, and the empirical data support their use in practice (Robinson 1987; Sahlman 1990; Weidig and Mathonet 2004; Gompers, Kovner et al. 2009).

# 3 Traditional Portfolio Selection

The optimal portfolio of risky investments can be defined as the best way to allocate capital in a given number of risk bearing securities. For a rational investor, this portfolio is the one that gives the highest expected return for a given level of risk or the lowest level of risk for a given expected return (Markowitz 1952).

## 3.1. A Portfolio's Risk and Return

Suppose that $x_1, \ldots, x_n$ are the portfolio weights of $n$ securities in a given portfolio. If these investments have expected returns $E[R_1], \ldots, E[R_n]$, the expected return for the entire portfolio is the weighted average of the expected returns on these assets in the portfolio (Berk and DeMarzo 2007):

$$E[R_p] = x_1 E[R_1] + x_2 E[R_2] + \cdots + x_n E[R_n] = \sum_i x_i E\,[R_i] \qquad (1)$$

The uncertainty or risk regarding the expected return on a given security can be divided into two broad sources. First, there are the general economic conditions, such as inflation, interest rates, exchange rates, and market conditions. All of these economic factors can affect the price or dividend flow on a given security and, thus, the return to its investors. This type of uncertainty is often referred to as market risk, systematic risk or nondiversifiable risk. The other source of uncertainty is the firm-specific risk, nonsystematic risk or diversifiable risk. This uncertainty is influenced by resource and development processes, personnel changes, and other conditions that only affect a specific firm (Bodie, Kane et al. 2009).

The total level of risk in a portfolio will depend on whether the individual securities have risk that is systematic, firm-specific or both. This is because an impact on firm-specific risk to one single security, for example, valuable human capital leaves the company, will only affect that firm and no other security in the portfolio. Theoretically, this type of risk can be reduced to a negligible level through diversification, also investing in a large number of securities. Markowitz was the first to analytically show the role of diversification in forming an optimal portfolio through his mean-variance criteria, using mean as a measure of expected return and variance as a measure of risk (Markowitz 1952). As illustrated in figure 2, if a portfolio only has securities with firm-specific risk, it is theoretically possible to eliminate all risk through diversification.

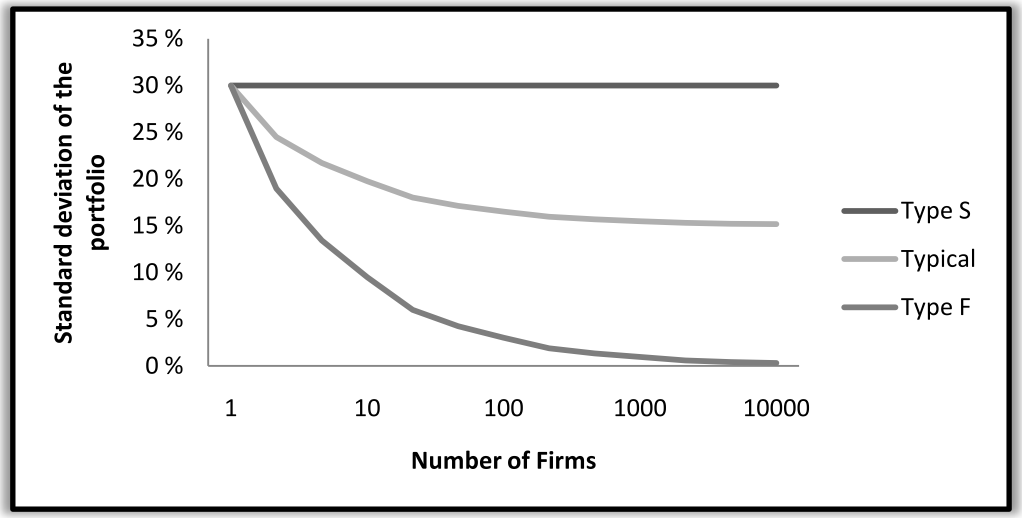


**Figure 2: The Standard deviation of a Portfolio, consisting of only systematic risk firm (Type S), only firm-spesific risk (Type F) or both types of risk (Typical).**

Using the covariance matrix, the total variance of a portfolio of $n$ securities is obtained as the following formula (Markowitz 1952):

$$var(R_p) = \sum_i x_i Cov(R_i, R_p) = \sum_i \sum_j x_i x_j Cov(R_i, R_j) \qquad (2)$$

If we divide both sides of the first presentation of the equation by the standard deviation of the portfolio, it gives the following expression (equation 3) for a portfolio's volatility. This expression illustrates that when combining stocks into a portfolio that places positive weights on each stock, unless all of the stocks have a perfect correlation with the portfolio, the risk of the portfolio is lower than the weighted average volatility of the individual stocks (Berk and DeMarzo 2007).

$$SD(R_p) = \sum_i x_i\, SD(x_i) Corr(R_i, R_p) \leq \sum_i x_i\, SD(R_i) \qquad (3)$$

From equations 1 and 2, it is obvious that an investor can obtain different expected portfolio returns and variance by changing portfolio weights for the $n$ securities available. Then, to obtain the optimal portfolio of these risky assets, one must find the optimal allocation of capital invested in the portfolio. This allocation is the unique allocation that yields the highest expected utility to the investor and can, therefore, be thought of as a tradeoff between the expected return and the risk the investor is willing to hold.

## 3.2.The Investor's Utility Function

Based on the expected utility theory framework, I consider an investor who wants to allocate initial wealth to maximize expected utility from end-of-period wealth, $E[U(W)]$. $W$ is the investor's uncertain end-of-period wealth, which is expressed as $W = 1 + R_p$, the gross return from a portfolio[4]. Because a VC manager has no real opportunity to invest in a risk-free asset, I assume that the portfolio only consists of risky assets and short selling is not allowed (Simaan 1993; Jondeau and Rockinger 2006). The investor's optimal asset allocation can be obtained by finding the portfolio weights, $x_i$, that maximize the expected utility (Zakamouline and Koekebakker 2009):

$$E[U^*(W)] = \max_{x_i} E[U(1 + R_p)] \qquad (4)$$

I assume that the utility function is increasing concave and that a solution to this optimal allocation problem exists and is unique (Jondeau and Rockinger 2006; Zakamouline and Koekebakker 2009). The first-order-condition of this optimization problem is

$$\frac{\partial E[U(1 + R_p)]}{\partial x_i} = 0$$

As my objective is to examine the impact of including higher moments of a distribution when selecting a portfolio, I now approximate the expected utility by a Taylor's series expansion around the expected end-of-period wealth[5]. Suppose that $U$ is a continuously differentiable function and the expected end-of-period wealth is $E\left[1 + R_p\right] = \bar{W}$. Then a utility function can be expressed as:

[4] The initial wealth $W_o$ is assumed to be 1, and all wealth is invested in the portfolio.

[5] Taylor's theorem states that any continuous differentiable function can locally be approximated by polynomials (the reminder term represent the difference between the approximation and the function):

$$f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_{n+1}$$

$$U(W) = U(\bar{W}) + \sum_{n=1}^{N} \frac{U^{(n)}(\bar{W})}{n!}(W - \bar{W})^n + R_{N+1}$$

Here, $(W - \bar{W})^n$ is the $n$th moment of the portfolio's return distribution (Zakamouline and Koekebakker 2009). The expected utility from the uncertain end-of-period wealth can be written as:

$$E[U(W)] = U(\bar{W}) + \sum_{n=1}^{N} \frac{U^{(n)}(\bar{W})}{n!} E[(W - \bar{W})^n] + R_{N+1} \qquad (5)$$

This Taylor's series expansion is not suitable for numerical implementation. A solution to approximate the expected utility is to truncate this function at a given value $n$, assuming that all moments of the distribution over value $n$ will be negligible (Jondeau and Rockinger 2006).

## 3.3.Expected Utility with Two Moments

At $n = 2$, equation 5 is the equivalent to the mean-variance criteria proposed by Markowitz (1952) that the investor will choose the portfolio that yields highest expected return for a given level of variance or the lowest variance for a given expected return. This can be shown analytically as

$$E[U(W)] \approx U(\bar{W}) + U'(\bar{W})E[W - \bar{W}] + \frac{1}{2}U''(\bar{W})E[W - \bar{W}]^2$$

But since $E[W - \bar{W}] = E[W] - \bar{W} = \bar{W} - \bar{W} = 0$ then,

$$E[U(W)] \approx U(\bar{W}) + \frac{1}{2}U''(\bar{W})E[W - \bar{W}]^2 \qquad (6)$$

I define the portfolio's variance as[6]

$$\sigma_p^2 = E[W - \bar{W}]^2$$

Equation 6 can then be rewritten as:

$$E[U(W)] \approx U\big(E[1 + R_p]\big) + \frac{1}{2}U''(\bar{W})\sigma_p^2 \qquad (7)$$

Given that the investor wants to maximize the expected utility, this expression shows that an investor will choose the portfolio that yields the highest grand total from the first two moments of a probability distribution, i.e., the expected return and variance. Note that because it is assumed that a risk-averse person has an increasing marginal utility with a decreasing rise (the utility function is increasing concave) (Bernoulli 1954), this formula correspond with the mean-variance rule: that the

[6] Calculating variance using gross return $(1 + R)$, will give the same results as using the return $(R)$.

rational investor considers expected returns as a desirable occurrence and variance of return as an undesirable occurrence (Markowitz 1952).

The first assumption underlying this approximation of expected utility is that the uncertain end-of-period wealth, also the gross portfolio return, is a normally distributed variable. The variable is then explained by the two first moments of the probability distribution; its mean and variance. The second assumption is that the investor utility function is quadratic, implying that any derivative over the second will be zero. This means that the investor would not appreciate higher moments of the distribution and only give a utility value to the first two. This last assumption is not realistic. Figure 3 illustrates a normally distributed random variable. The probability distribution is then symmetrically shaped as a bell. This means that there is the same probability for a random variable, in this case a return, to be over or under the mean. It also means that the uncertain return is most likely to be the mean of the distribution and that the return is more likely to be close to the mean than far away. A normally distributed random variable has an approximately 68% chance of being within one standard deviation, 95% chance of being within two, and better than 99.7% chance of being within three standard deviations from its mean (Studenmund 2006). Thus, unless the variance of a return distribution is very high, assuming a normal distribution will exclude the probability of any extremely high or low outcomes.

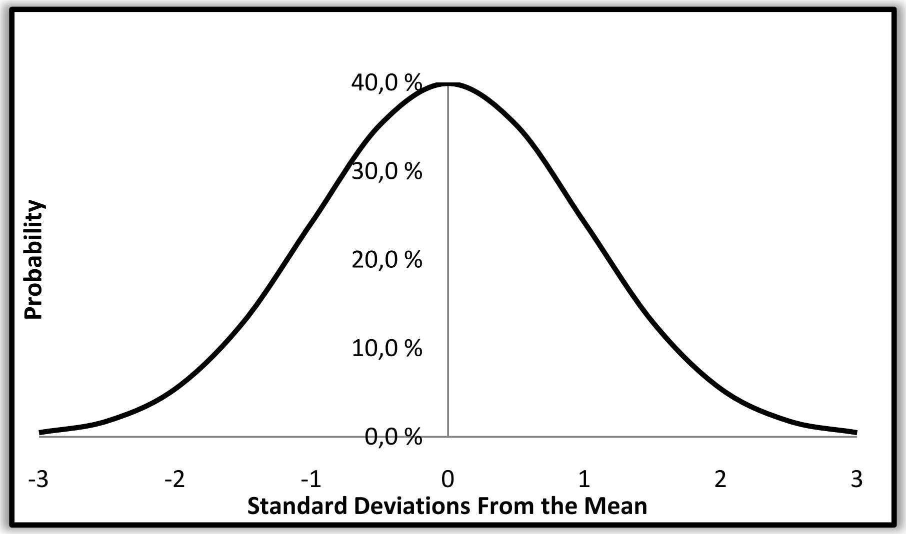


**Figure 3: Normal distribution with mean 0 and SD 1. The probability density function is $\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-(x-\mu)^2}{2\sigma^2}}$**

Studies have shown that the mean-variance utility function provides a good approximation to the direct optimization when returns are close to normally distributed (Levy and Markowitz 1979; Simaan 1993; Jondeau and Rockinger 2006). However, one intuitive problem that emerges with the mean-variance criteria is what if one are dealing with a variable that is not normally distributed, for example, the distribution is asymmetric or perhaps has a higher probability of extreme outcome than a normal distribution implies (Bodie, Kane et al. 2009).

If the probability distribution is non-normal, then it may yield a different asset allocation than a mean-variance optimal would imply. Given two distributions that have the same variance and expected return but where one is positively skewed and the other negatively skewed, the investor should choose the positively skewed distribution because it yields a higher probability of positive returns (Bodie, Kane et al. 2009). If the investor makes the decision based on only expected return and variance, then she would as likely choose the investment option with a negatively skewed distribution even if this is not the optimal decision. The kurtosis, i.e., the probability of extreme outcomes, should also affect the investor’s optimal decision. If an investor makes decisions based on the mean and variance, then she can underestimate the probability of extreme outcomes because she assumes that the returns are normally distributed. Figure 4 shows a normal and a fat-tailed distribution with the same mean and variance and a skewed and normal distribution with the same mean and variance (Bodie, Kane et al. 2009).

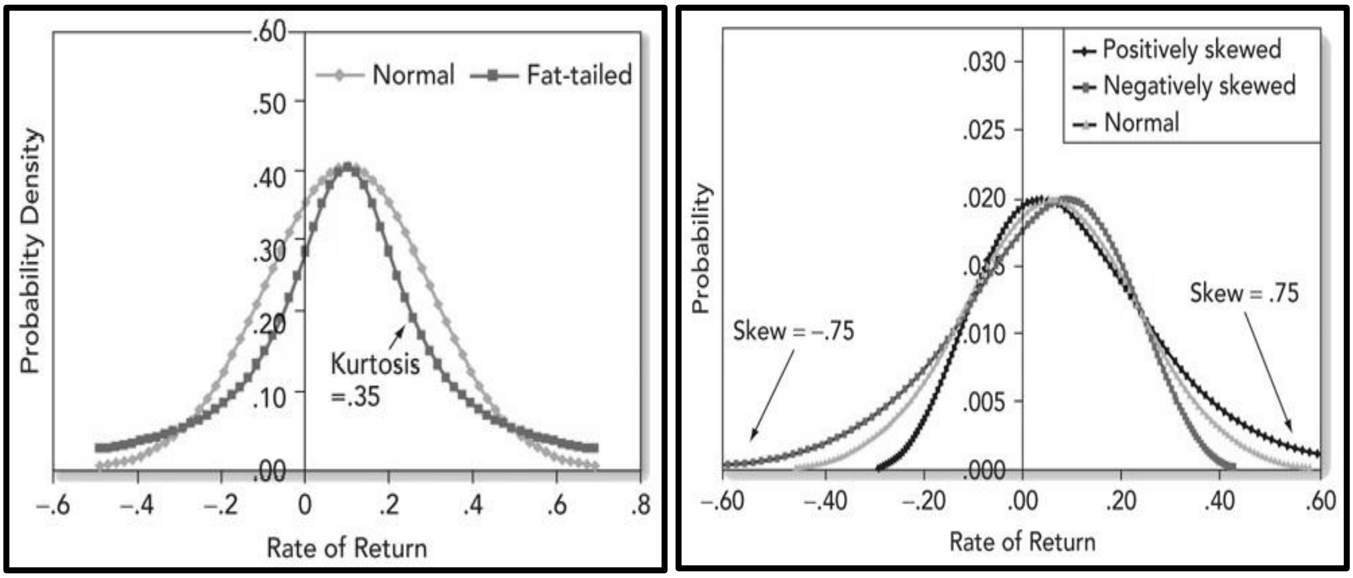


**Figure 4: The distributions to the left have both a mean of 10% and SD of 20%. The distributions to the right have a mean of 6% and a SD of 17% (Bodie, Kane et al. 2009).**

These two examples illustrate that the third moment, the skewness, and the fourth moment, the kurtosis, of a probability distribution could impact an investor’s optimal asset allocation if returns are non-normal distributed[7]. Thus, if VC returns are non-normal distributed, the portfolio allocation decision should also include the skewness and kurtosis of the distribution.

[7] The skewness is zero and the kurtosis is three if returns are normally distributed. Figure 4 shows these moments measured as statistically defined. That is $\frac{(X-\bar{X})^3}{\sigma^3}$ for skewness and $\frac{(X-\bar{X})^4}{\sigma^4}$ for kurtosis. Kurtosis is denoted as excess kurtosis, which is the kurtosis minus three.

# 4 Venture Capital Returns

Before assessing which probability distribution best fits the VC setting, I review some of the literature on VC returns. The most common measures used in the literature to present the returns are the interim internal rate of return (IRR) or the total value to paid-in capital (TVPI) (Kaplan and Schoar 2005; Mathonet and Meyer 2007; Ludovic 2009). The interim IRR is the discount rate that establishes the net present value of cash flows plus the latest net asset valuation (NAV) from an individual investment or fund, equal to zero. The net asset valuation is the VC's share of the investment's market capitalization at the measurement time, usually net of the management fee and carried interest (Kaplan, Sensoy et al. 2002; Ludovic 2009). The interim IRR is often called the dollar-weighted return because it considers both the size and the timing of cash flows. It can be presented as a discount rate at time $t$ (Weidig and Mathonet 2004; Bodie, Kane et al. 2009)

$$\sum_{i=0}^{t} \frac{CF_i}{(1+IRR)^i} + \frac{NAV_t}{(1+IRR)^t} = 0 \tag{8}$$

The TVPI is the multiple obtained by dividing the sum of cash outflows $(d)$ from the investment plus the latest NAV, on total cash inflow/invested capital $(CI_i)$ (Mathonet and Meyer 2007). Therefore, the timing of financing and cash returns will not affect this measure (Weidig and Mathonet 2004). The TVPI at time $t$ can be expressed as

$$TVPI_t = \frac{\sum_{i=0}^{t} d_i + NAV_t}{\sum_{i=0}^{t} CI_i} \tag{9}$$

Because financial returns are normally presented as annual returns, the TVPI multiple is often annualized using the following formula ($k$ is one year) (Campbell, Lo et al. 1997; Das, Jagannathan et al. 2002):

$$Annualized\ [TVPI_t] = [TVPI_t]^{\frac{1}{k}} - 1 \tag{10}$$

In the appendix, I provide an example on the timing difference between the IRR and the annualized TVPI.

## 4.1. Present Literature on Venture Capital Returns

VC authors normally use the VentureOne or the Venture Economics database when collecting industry data. Both databases provide self-reported valuations (NAV's) on individual and fund levels. These self-reported valuations made by the VC firm or the investment itself, can give a biased return estimate. In a survey of 143 VC financings, Kaplan, Sensoy and Strömberg (2002) find that the valuations are relatively unbiased compared to the actual situation, but that the databases are missing the valuations on a significant amount of the financing rounds that they report. Most valuations are from large financing rounds and on investments that eventually go public (Kaplan, Sensoy et al. 2002). Cochrane (2005) argues that because one are more likely to observe valuation data for a firm that is successful, that is, receives new financing, goes public, or is acquired, this will give an upward biased result if only those valuations are used to estimate an average return. He corrects for this bias using a maximum-likelihood estimate to measure the probability for a company to be successful, go bankrupt, or remain private but not receive new financing at the end of sample. In this study, Cochrane finds that the VC returns have similar characteristics as the smallest NASDAQ stocks (Cochrane 2005). Among industry participants and researchers, there has long been an understanding that the VC returns are very different from the normal stock market, but even if this study shows that VC returns are not as special as previously thought, one thing is certain: the VC market is a very uncertain and volatile market that has the ability to produce extreme outcomes.

An earlier survey by Chiampou and Kallett (1989) estimated the risk and return profile for 35 privately held VC *funds* that are six years or older. The reason why they choose these mature funds is because, normally, all committed capital is then invested and the fund begins to get known returns on the portfolio companies (Chiampou and Kallett 1989). The sample gives an average annual geometric return of 24.4%, which is measured by annualizing the product of each period's *gross realized return*[8] and not the last period's TVPI (Berk and DeMarzo 2007). The geometric return is often called the time-weighted return because it values the timing of returns in the same way as the IRR. This measure considers only returns for one single unit invested, e.g., one stock or one dollar, and the gain or loss on that unit. This measuring method eliminates the effect of the size on cash flows from an individual investment, which also is valued by the IRR, but it serves as a better measure of return than the annualized TVPI for comparison with other asset types, such as public stocks (Bodie, Kane et al. 2009). The annual standard deviation for the sample is 51.2%. In the same period, the S&P 500 had an annual geometric average return of 15.9% and a SD of 12.3%. Chiampou and Kallett also find the standard deviation as an inappropriate measure for risk. One reason for this is due to the illiquidity of the market and the characteristics of a typical successful VC fund (Chiampou and Kallett 1989). For a typical successful VC fund, the internal rate of return can be illustrated by a J-curve (figure 5) because write-offs and management fees are acknowledged immediately in the portfolio, but the value of the firms in the portfolio (NAV's) is valued at cost until a new round of financing (Berg-Utby 2010).

[8] The gross realized return at time $t$ is the value of the investment at time $t$ (NAV) plus cash outflow at time $t$, divided on previous period's valuation $t-1$, $1+R_t = \frac{d_t + NAV_t}{NAV_{t-1}}$.

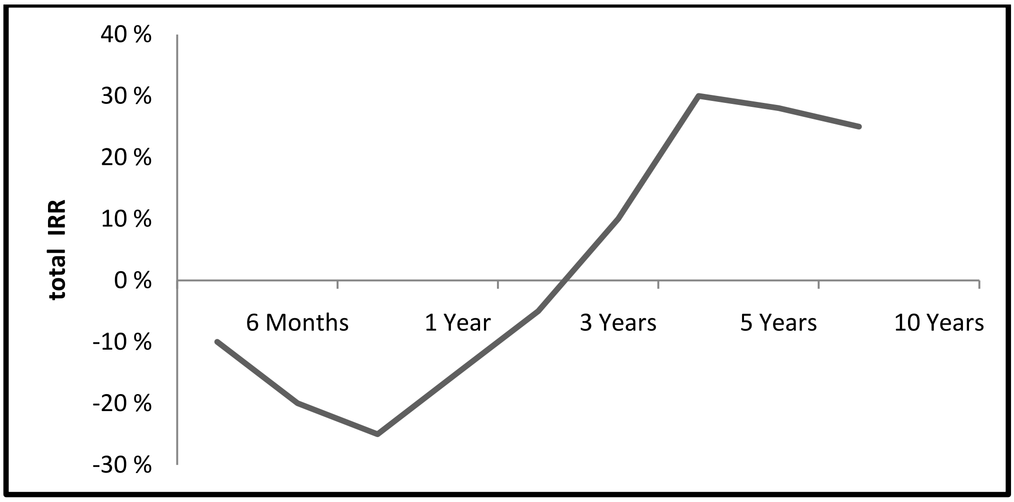


**Figure 5: An illustration of the typical J-curve for a successful VC fund.**

Venture Economics presented in the end of 2008 the average pooled IRR for the top quarter funds in their database. These returns are divided into stages based on the company phase the fund focus. The seed funds have an IRR of 12.9%, early growth funds of 18.8%, and expansion funds of 12.4%[9]. The survey also presents the IRR for generalist funds, i.e., funds that widely spread their investments across different stages[10]. The generalist fund has an IRR of 22.2% (Berg-Utby 2010).

Sahlman (1990) reports another Venture Economics survey of 383 *individual* investments made by 13 VCs between 1969 and 1985. These returns are measured as the TVPI (Kaplan and Schoar 2005). More than one-third of these investments results in a loss and the majority of the sample gives a small return, but a small portion of the investments (6.8% of the total amount invested) have a TVPI multiple of 11 or more. This finding is consistent with a usual assumption in the industry that one in ten investments in the fund will produce a “home-run”, and give a sufficient return for the entire portfolio. From the reported returns, it seems this return distribution is non-normal, and it appears positively skewed with the highest probability for low returns. This article does not say mention an annual return, but I might presume from the information provided that a preliminary annualized TVPI as an average of all investments is 9.5%[11]. Figure 6 shows the distribution of returns (Sahlman 1990).

[9] Venture Economics name the stages differently from me. They term the second stage “development” and the third stage “balanced”.

[10] This survey is of the whole private equity market, thus the generalist fund also includes the buyout stage.

[11] $TVPI = \frac{total\ value\ 1.049\ billion}{total\ investment\ 245\ million} = 4.3,\ Annualized\ TVPI = 4.3^{\frac{1}{16}} - 1 = 9.5\%$

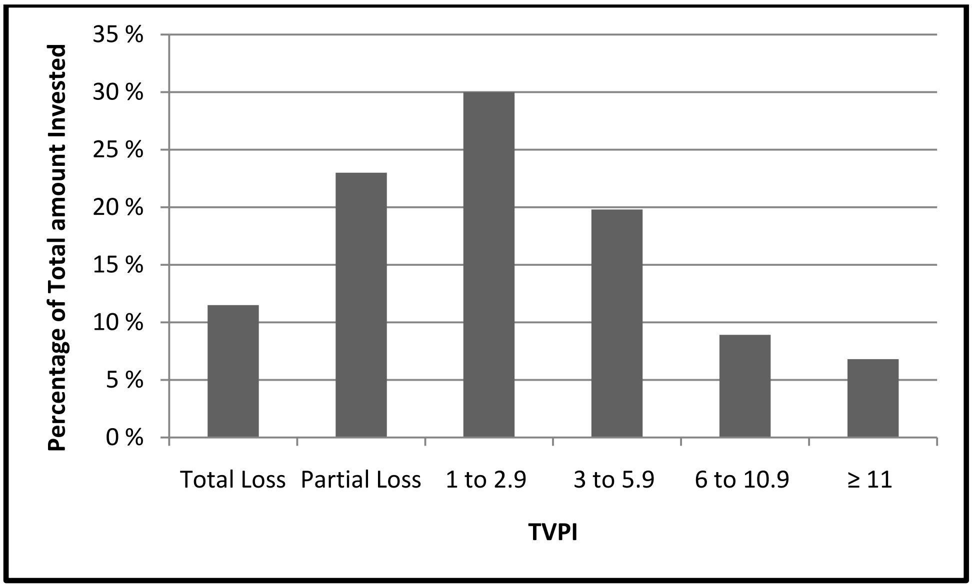


**Figure 6: The distribution of returns presented by Sahlman (1980)**

As mentioned previous, Cochrane (2005) presents one of the most comprehensive studies on VC risk and return using data provided from the VentureOne database, ranging from 1987 to June 2000. Cochrane argues that the *individual* return distribution is highly skewed and well described by a lognormal distribution. After correcting for the selection bias previously described, he finds an average annual geometric log (continuously compounded) return of 15% and a standard deviation of 89%. The gross return he uses is defined as the valuation of the investment at time $t$, divided by the valuation at time $t-1$, without subtracting management fee. A VC investment will obtain a new valuation at every financing round and when it goes public or is acquired. Because of the illiquidity of the VC market, Cochrane argues that the current value of an investment at one financing round cannot be regarded as realized return for a VC firm. He, therefore, calculates a return for the investment from each VC financing round to eventual IPO, acquisition or failure (Cochrane 2005). This calculation will likely yield a more correct realized return in this illiquid market, but Cochrane does not mention how he handles intermediate cash distributions. Because previous cash distributions affect a company's valuation at IPO (it will be worth less), but is not included in Cochrane's calculations of return, this will probably give a downward biased return estimate.

A study conducted by Weiding and Mathonet (2004) reveals a remarkable distribution of returns from both *individual* investments and *funds.* Their findings support Cochrane's assumption with respect to a lognormal distributed VC market. They report an average TVPI of 6.2 and a very high standard deviation of 53.8 for a direct investment. For a fund, the average TVPI is 1.7 and the SD is 1.9. It is important to note that these return and risk numbers are for the total investment period and cannot directly be compared with measures from time series, as I have presented earlier. The return distribution is illustrated in figure 7 (Weidig and Mathonet 2004).

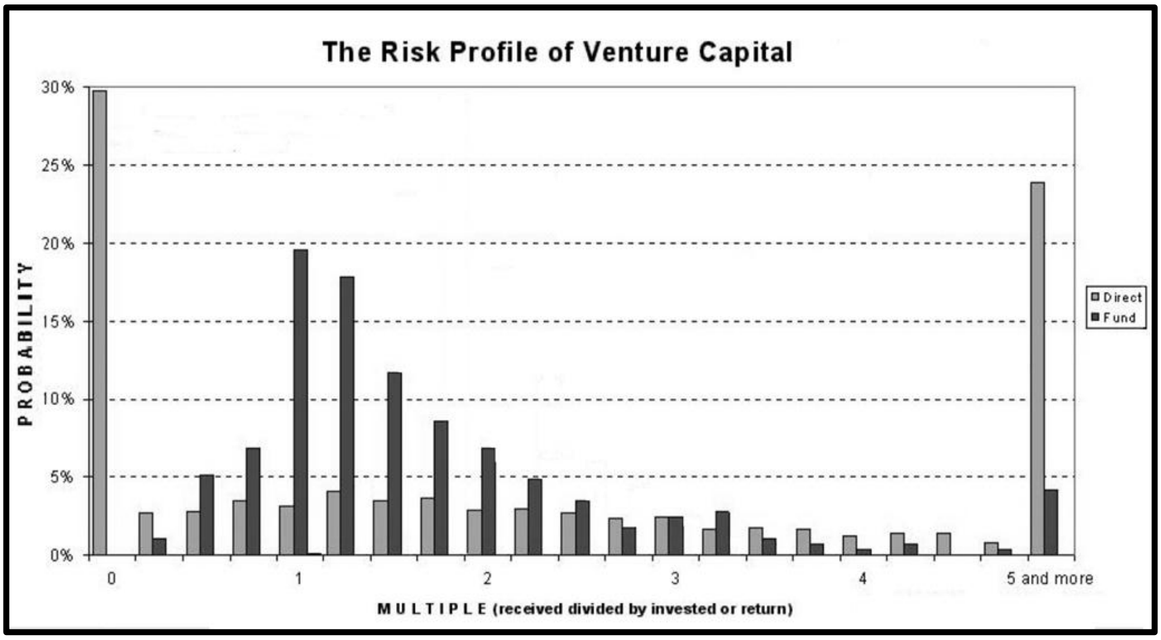


**Figure 7: The Risk Profile of a Venture Capital fund and the direct investment (Weidig and Mathonet 2004)**

## 4.2. A Probability Distribution That Fits the VC Setting

Therefore, what probability distribution best fits the VC industry? VCs are investing in inventions and entrepreneurs, an asset class that is very uncertain and has very low predictability. From the literature presented above, I can first draw the conclusion that the VC return distribution has thick tails, or high kurtosis. A distribution that well describes this situation is the t-student distribution (Campbell, Lo et al. 1997). A problem with this distribution that does not seem to fit the VC setting is that it is symmetrical. It assumes the same probability for a random variable to be over or under the mean. In VC, it appears there is a high probability for low returns, but there is also a higher probability for extreme high returns than extreme low returns. This higher probability of extreme high returns exists because the VC firm can only lose its initial investment but can gain nearly unlimited returns. Like Cochrane (2005), I find that the probability distribution that best fits the VC setting is the lognormal distribution.

A random variable, for example, a VC gross return $(1 + R)$, is lognormal distributed if the logarithm to this variable, the return continuously compounded, is normally distributed (Campbell, Lo et al. 1997). The lognormal distribution is a positively skewed distribution with most values near a lower limit of zero (Limpert, Werner et al. 2001). The uncertain variable cannot fall below this lower limit, e.g., the price of a stock cannot fall below zero. This distribution seems to have the properties to describe the return characteristics of VC. This is because the VC firm can, at the most, lose its investments, i.e., obtain a gross return 0, but there is also a possibility for unlimited gains. As presented earlier, the majority of returns are low normal rates, which also is consistent with a lognormal distribution. Figure 8 illustrates a random variable that is lognormal distributed.

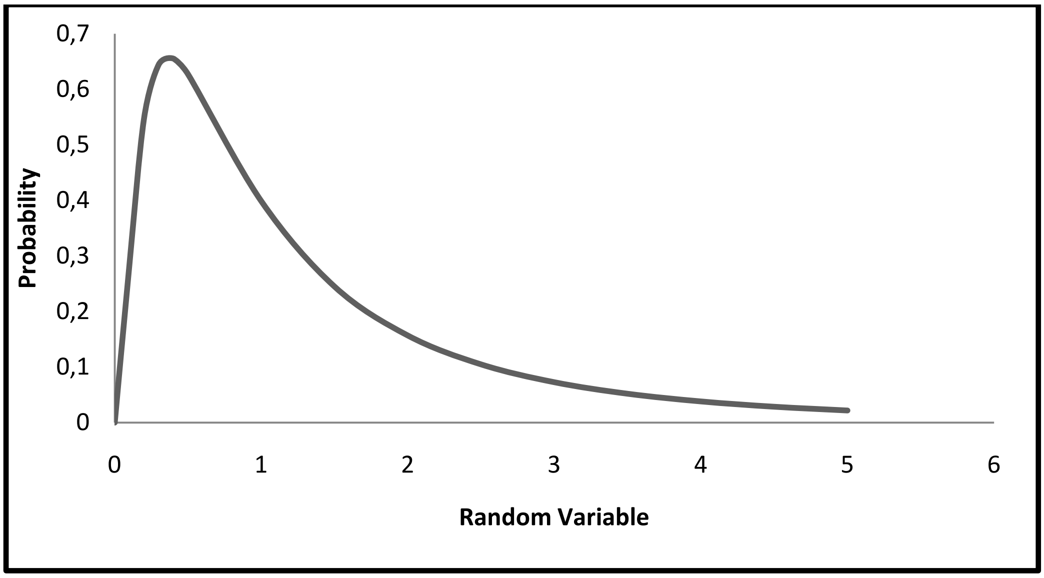


**Figure 8: A lognormal distributed random variable with standard deviation 1 and Mean 0. The probability density function is** $\frac{1}{x*\sigma\sqrt{2\pi}}e^{(-\frac{1}{2\sigma^2}(ln(x)-\mu)^2)}$

# 5 Portfolio Selection with Higher Moments

Considering the VC characteristics previously described and the underlying assumptions for using the mean-variance criteria when putting together a portfolio, it appears that a mismatch exists. When returns are non-normal distributed, the higher moments of a distribution will impact the optimal allocation. As VC returns appears positively skewed and good described by a lognormal distribution, a VC fund manager could probably yield a more optimal portfolio by accounting for higher moments of the distribution.

## 5.1.Expected Utility with Four Moments

I use the expected utility equation that I previously found (equation 5) and now truncate it at four moments, $n = 4$. The Investor's expected utility from the uncertain end-of-period wealth can then be approximated using the same approach as Jondeau and Rockinger (2006).

$$E[U(W)] \approx U(\overline{W}) + \frac{1}{2}U''(\overline{W})E[W-\overline{W}]^2 + \frac{1}{3!}U^3(\overline{W})E[W-\overline{W}]^3 + \frac{1}{4!}U^4(\overline{W})E[W-\overline{W}]^4$$

I define the portfolio's skewness and kurtosis of the end-of-period return as [12]

$$Skewness = s_p^3 = E[(W - \bar{W})^3]$$

$$Kurtosis = k_p^4 = E[(W - \bar{W})^4]$$

The expected utility can then be written as the following function:

$$E[U(W)] \approx U\big(E[1 + R_p]\big) + \frac{1}{2}U''(\bar{W})\sigma_p^2 + \frac{1}{3!}U^3(\bar{W})s_p^3 + \frac{1}{4!}U^4(\bar{W})k_p^4 \quad (11)$$

This expected utility function shows that a risk-averse investor responds positively to the expected return and skewness but negatively to variance and kurtosis if the following inequalities hold for the derivatives of the person's utility function (Scott and Horvath 1980).

$U^{(n)}(W) > 0 \quad \forall W$ If $n$ is odd and

$U^{(n)}(W) < 0 \quad \forall W$ If $n$ is even.

These preferences for the moments of a probability distribution are often used and are generally accepted (Dittmar 2002).

Let's now consider the case of a risk-free investment opportunity where the investor receives a certain gross return. Although I earlier stated that there is no real opportunity for a VC manger to invest in a risk-free asset, an assumed risk-free investment opportunity can be used to show what the manger should minimally expect in return from the risky portfolio to be satisfied with the risky investment. Let $W^{ce}$ be a certain end-of-period wealth, assumed to be very close to the same expected end-of-period wealth as earlier $\bar{W}$. Then, the utility function for this certain amount can be approximated around the expected end-of-period wealth as (Pratt 1964; Milgrom and Roberts 1992).

$$U(W^{ce}) \approx U(\bar{W}) + U'(\bar{W})(W^{ce} - \bar{W}) \quad (12)$$

Because the expected return and the certain amount are assumed to be very close, the terms over the first derivative are regarded as zero and ignored (Milgrom and Roberts 1992). A risk-averse investor will prefer the risky investment if its expected utility has a higher value than the utility of a certain investment $U(W^{ce}) < \mathrm{E}[U(W)]$. For the risky investment to give the same expected utility as the utility of a risk-free investment, I combine equation 11 and 12.

$$U(\bar{W}) + U'(\bar{W})(W^{ce} - \bar{W}) = U(\bar{W}) + \tfrac{1}{2}U''(\bar{W})\sigma_p^2 + \tfrac{1}{3!}U^{(3)}(\bar{W})s_p^3 + \tfrac{1}{4!}U^{(4)}(\bar{W})k_p^4$$

[12] These definitions differ from the statistical definitions, which I have presented in a footnote earlier.

This equation can then be rearranged and expressed as the Investor's *certainty equivalent*

$$W^{ce} = E[1 + R_p] - \frac{1}{2}A_2\,\sigma_p^2 - \frac{1}{3!}A_3 s_p^3 - \frac{1}{4!}A_4 k_p^4 \qquad (13)$$

$A_n = -\frac{U^{(n)}}{U'}$, is a measure of the investor's risk aversion, i.e., her aversion towards the different moments in a probability distribution (Pratt 1964; Arrow 1970). Equation 13 shows that a rational investor needs a higher return from a risky investment to obtain the same utility as one would receive from a certain investment. The *risk premium*, which is what the investor obtains in expected excess return for taking the risky investment, is expressed as $E[1 + R_p] - W^{ce}$ (Bodie, Kane et al. 2009).

Now I have developed the tools that can help analytically solve the allocation problem for a VC manager. However, before going further, I first must assume a specific utility function that can be general for all rational VC managers and, thus, contains certain properties because the risk aversion is dependent on the utility function itself.

## 5.2. The Optimal Asset Allocation of a Risky Portfolio using CARA Utility

The *negative exponential utility function,* also called the CARA utility function can be used as a general expression for a VC mangers utility (Jondeau and Rockinger 2006; Pennacchi 2008)

$$U(X) \approx -e^{-bw}\ , b > 0 \qquad (14)$$

This function has some characteristics that are sound for this setting. First, it assumes a marginal increasing utility $U' > 0$ , and second, it assumes a reduction in the change in marginal utility $U'' < 0$. The third feature is that this expression assumes that the risk aversion is constant and does not change with different levels of wealth (Pennacchi 2008)[13]. This assumption may not be good for an individual personal investor because one can assume a desire to be less risk-averse and take riskier bets as personal wealth increases. However, for a VC manager this is as a good assumption because one is not investing personal money and should, therefore, have the same tolerance towards risk regardless of the amount of wealth they manage. The exponential utility function implies that the investor's aversion towards the different moments of a distribution is a constant number $b$ raised to the power of $n - 1$ regardless of wealth level and is illustrated as

$$U' = be^{-bw}$$

$$U'' = -b^2e^{-bw}$$

[13] In other words, the investor has constant absolute risk aversion (CARA).

Given the formula for risk aversion described earlier $A_n$, the investor's aversion towards variance and skewness regardless of the wealth level $w$, is

$$A_2 = \frac{b^2 e^{-bw}}{be^{-bw}} = b$$

$$A_3 = -\frac{b^3 e^{-bw}}{be^{-bw}} = -(b^2)$$

The investor's *certainty equivalent* can then be expressed as

$$E\left[1 + R_p\right] - \frac{1}{2} b\, \sigma_p^2 + \frac{1}{6} b^2 s_p^3 - \frac{1}{24} b^3 k_p^4$$

From this expression, I can assume, *ad hoc,* that the VC manager gives a *utility score* to competing portfolios of risky investments based on the expected gross return and risk of the portfolio (Bodie, Kane et al. 2009). The risk is then defined as the sum of the variance, skewness, and kurtosis of the portfolio. The utility score can be expressed as

$$U_s = E\left[1 + R_p\right] - \frac{1}{2} b\, \sigma_p^2 + \frac{1}{6} b^2 s_p^3 - \frac{1}{24} b^3 k_p^4 \qquad (15)$$

Portfolios are given higher utility scores for higher expected returns, but lower scores for higher levels of risk. From available portfolios, the investor will always choose the portfolio that yields the highest utility score. The score of a risk-free investment will be its expected gross return; thus, for an investor to choose the optimal risky portfolio, its utility score must be higher than the risk-free return. If the investor is risk neutral $b = 0$, and does not emphasize risk, then the portfolio that yields the highest score will always be the one with the highest expected return. This means that for every value of risk aversion, the investor will put together the portfolio from available risky assets that give the highest utility score, and this will be the optimal portfolio. Actually, the VC fund-structure makes the VC manager quite risk seeking. As mentioned, the firm will always receive a management fee for managing the investments in the fund but obtains only a carried interest in cases where the fund generates a very high return (Berg-Utby 2010). This gives the manager incentives to take high-risk investments. One moderating factor is that the funds have a limited lifetime, and the VC firm is depending on obtaining new funds to survive in the long run (Berg-Utby 2010). A fund often lasts between 5 to 10 years (Gompers and Lerner 2001).

# 6 Analysis

In this analysis, I put together a “four moments” optimal portfolio when assuming returns are normal and lognormal distributed, everything else being equal. I assume different distributions to determine if the difference in optimal allocation between the distributions implies different risk reduction strategies. In the VC setting, it is difficult to eliminate firm-specific risk in a portfolio through diversification in large numbers because of its illiquid nature. One risk reduction strategy can then be to diversify invested capital across investments with different risk profiles rather than invest all capital in one investment type. I now test if that is a good strategy for obtaining the optimal portfolio when assuming a normal and lognormal returns distribution. To simplify the analysis I assume that a VC manager can chose between three investment opportunities with different risk profiles. My objective is to determine how much percentage capital the VC manager will allocate in each investment to obtain the optimal portfolio. This allocation is where the VC manager gets the highest utility score. To obtain the solution, I insert the utility score formula (15) in Excel™ and use Crystal Ball™ to find the optimal allocation for these three investment opportunities. As mentioned, I find an optimal solution when the returns come from a normal and a lognormal distribution, everything else being equal.

## 6.1. Monte Carlo Simulation and Optimization

I first use Monte Carlo simulation to simulate a thousand returns for each investment opportunity individually. This simulation is based on assumptions about the probability distribution, mean, and standard deviation. Because these investments are individual firms in the very early stages of development, I do not assume a correlation between the returns. This will actually provide a high incentive to diversify between the investments[14]. The expected returns used for the simulation is also assumed to be arithmetic and not geometric. The reason for this is as follows. If measured historic returns give a good representation of the underlying probability distribution of an asset, then the arithmetic average return will provide a good forecast for the investment’s expected return because the arithmetic return assumes that every return in a time series is equally likely to happen again, irrespective of when it occurs. In contrast, the geometric return is better to calculate the realized return of an asset because, as previously mentioned, this is a time-weighted measure and considers when the return occurred (Bodie, Kane et al. 2009).

These simulated trials are used to calculate the skewness and kurtosis for each investment opportunity when returns come from a normal and lognormal distribution. I use the formulas for

[14] Equation 3, as earlier presented, shows that the lower the correlation is the higher the risk reduction is from portfolio diversification.

skewness and kurtosis as presented earlier. Because I calculate from a simulated sample, I also have to correct for degrees of freedom and do not simply divide the total sample by the numbers of trials. Degrees of freedom can be defined as the number of observations/trials minus the number of assumptions needed to calculate the statistical measure (Zikmund, Babin et al. 2010)[15]. The skewness of the normal distribution is zero because the distribution is symmetrical. The kurtosis, as statistically defined, is three and, therefore, it is usual to calculate the excess kurtosis of a distribution, defining the excess kurtosis for a normal distribution as zero. Because I use the formulas presented earlier, I also calculate values for kurtosis when returns come from a normal distribution. To calculate the total skewness and kurtosis for a portfolio, I use Jondeau and Rockingers (2009) definitions for co-skewness and co-kurtosis:

$$Coskew = S_{ijk} = E[(R_i - \bar{R}_i)(R_j - \bar{R}_j)(R_k - \bar{\bar{R}}_k)] \quad i,j,k = 1,\dots,n \qquad (16)$$

$$Cokurt = K_{ijkl} = E[(R_i - \bar{R}_i)\big(R_j - \bar{R}_j\big)(R_k - \bar{R}_k)(R_l - \bar{R}_l)])] \quad i,j,k,l = 1,\dots,n \qquad (17)$$

By following the same procedure as the covariance matrix for calculating portfolio variance, one can develop the matrix one step further to apply for skewness and kurtosis. For a portfolio with portfolio weights $x_i$ and return $R_i$, the co-skewness matrix for $i = 1,2,3$ is

| | $i=1$ | | | | $i=2$ | | | | $i=3$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $j=1$ | $j=2$ | $j=3$ | | $j=1$ | $j=2$ | $j=3$ | | $j=1$ | $j=2$ | $j=3$ |
| $k=1$ | $S_1^3$<br>$x_1^3$ | $S_{121}$<br>$x_1^2x_2$ | $S_{131}$<br>$x_1^2x_3$ | $k=1$ | $S_{211}$<br>$x_1^2x_2$ | $S_{221}$<br>$x_2^2x_1$ | $S_{231}$<br>$x_2x_3x_1$ | $k=1$ | $S_{311}$<br>$x_3^2x_1$ | $S_{321}$<br>$x_3x_2x_1$ | $S_{331}$<br>$x_3^2x_1$ |
| $k=2$ | $S_{112}$<br>$x_1^2x_2$ | $S_{122}$<br>$x_2^2x_1$ | $S_{132}$<br>$x_1x_3x_2$ | $k=2$ | $S_{212}$<br>$x_2^2x_1$ | $S_2^3$<br>$x_2^3$ | $S_{232}$<br>$x_2^2x_3$ | $k=2$ | $S_{312}$<br>$x_3x_1x_2$ | $S_{322}$<br>$x_2^2x_3$ | $S_{332}$<br>$x_3^2x_2$ |
| $k=3$ | $S_{113}$<br>$x_1^2x_3$ | $S_{123}$<br>$x_1x_2x_3$ | $S_{133}$<br>$x_3^2x_1$ | $k=3$ | $S_{213}$<br>$x_2x_1x_3$ | $S_{223}$<br>$x_2^2x_3$ | $S_{233}$<br>$x_3^2x_2$ | $k=3$ | $S_{313}$<br>$x_3^2x_1$ | $S_{323}$<br>$x_3^2x_2$ | $S_2^3$<br>$x_3^3$ |

When $i = j = k$, $S_{ijk}$ gives the skewness of the asset $i$ and is, therefore, denoted as $S_i^3$ for $i = 1,2,3$. The co-skewness between the three assets can also be collected into seven entities[16]. This yields the

[15] I multiply the total sum of variance with $\frac{1}{n-1}$, the total sum of skewness with $\frac{n}{(n-1)(n-2)}$, and the total sum of kurtosis with $\frac{n(n+1)}{(n-1)(n-2)(n-3)}$ to correct for degrees of freedom.

[16] Like these examples: $S_{112} = S_{121} = S_{211}$ and $S_{113} = S_{131} = S_{311}$ and $S_{123} = S_{132} = S_{213} = S_{231} = S_{312} = S_{321}$

following expression for the skewness of a portfolio with three assets (the calculation of kurtosis uses the same procedure; however, it will not be presented in this thesis):

$$S_p^3 = x_1^3 S_1^3 + x_2^3 S_2^3 + x_3^3 S_3^3 + 3x_1^2 x_2 S_{112} + 3x_1^2 x_3 S_{113} + 3x_2^2 x_1 S_{221} + 3x_2^2 x_3 S_{223} + 3x_3^2 x_1 S_{331} + 3x_3^2 x_2 S_{332} + 6x_1 x_2 x_3 S_{123}$$

This yields the following general formula for a portfolio's skewness and kurtosis.

$$Skew(R_p) = \sum_i \sum_j \sum_k x_i x_j x_k Coskew(R_i, R_j, R_k) \tag{18}$$

$$Kurt(R_p) = \sum_i \sum_j \sum_k \sum_l x_i x_j x_k x_l Cokurt(R_i, R_j, R_k, R_l) \tag{19}$$

After the calculated skewness and kurtosis are added to the model in Excel™, I use the optimization function in Crystal Ball™ to find the allocation between the three investments that yields the highest utility score. This is a deterministic optimization[17] that is based on the same expected return and standard deviation as the simulation. I find the optimal allocation by using a thousand iterations. The table below provides an overview of some of the relevant measures used in the optimization (and the simulation):

| | | | **Normal distribution** | | **Lognormal distribution** | |
|---|---|---|---|---|---|---|
| Investment: | E(R) | SD | Skewness | Kurtosis | Skewness | Kurtosis |
| **High risk = $X_1$** | 0,25 | 1,10 | - | 4,100 | 43,021 | 1301,574 |
| **Medium risk = $X_2$** | 0,15 | 0,55 | - | 0,256 | 5,138 | 75,309 |
| **Low risk = $X_3$** | 0,10 | 0,275 | - | 0,016 | 0,536 | 3,572 |

**Table 1: The expected return and volatility is based on the literature review as presented earlier.**

The three investment opportunities represent common risk profiles expected in the VC industry. Assuming that a risk-free investment give a return on 5%, the *risk premium* from these three investments grows proportionally with the growth in volatility (Bodie, Kane et al. 2009). As previously mentioned, the investor should then choose the optimal risky portfolio if it yields a utility score over 1.05. Because I am only interested in finding the allocating that yields the highest utility score from available risky investments, that the value of the utility score is over or under a risk-free investment, will not affect my results.

[17] This is an optimization that finds the optimal forecast value based on the assumptions (including the calculated skewness and kurtosis) without simulating returns.

## 6.2.Results

I obtain the optimal portfolio using values for the risk aversion parameter $b$ ranging between 0 and 20. This range should cover most of the values used in the literature on portfolio optimization (Jondeau and Rockinger 2006). For explanatory reasons, I assume that a constant parameter value between 0.1 and 0.7 describes a person with low risk aversion. A constant value between 0.9 and 1.5 is defined as medium risk aversion, and, finally, all values over 1.5 are regarded as high risk aversion. The constant value $b$ raised to the power of $n-1$ indicates the investor's aversion toward the $n$th moment of a distribution. I also include $b=0$, a risk neutral person, to show the allocation if the person does not care about risk.

When returns come from a *normal distribution*, the asset allocation that yields the highest utility score is

| | **Asset Allocation** | | |
|---|---|---|---|
| Risk aversion (*b-value)* | $X_1$ | $X_2$ | $X_3$ |
| **0** | 1 | - | - |
| **0,1** | 0,86 | 0,14 | - |
| **0,3** | 0,40 | 0,54 | 0,06 |
| **0,5** | 0,25 | 0,39 | 0,36 |
| **0,7** | 0,19 | 0,33 | 0,49 |
| **0,9** | 0,15 | 0,29 | 0,57 |
| **1,1** | 0,12 | 0,26 | 0,62 |
| **1,3** | 0,10 | 0,24 | 0,66 |
| **1,5** | 0,08 | 0,22 | 0,70 |
| **2** | 0,05 | 0,19 | 0,76 |
| **5** | - | 0,03 | 0,97 |
| **10** | - | - | 1 |
| **20** | - | - | 1 |

From these results, one can see that a strategy to diversify investments between the three investment opportunities will be the optimal solution for most values of risk aversion. For the extreme low value of risk aversion, the optimal solution will be to allocate the most capital in the high risk investment. This will be consistent with a person who does not emphasize risk. For extreme high values of risk aversion, the optimal solution is all capital in the low risk/low return investment. This is also consistent with a person who is strongly inclined to avoid risk.

When returns are assumed *Lognormal distributed,* the asset allocation that yields the highest utility score is

| | Asset Allocation | | |
|---|---|---|---|
| Risk aversion (*b-value)* | $X_1$ | $X_2$ | $X_3$ |
| **0** | 1 | - | - |
| **0,1** | 0,96 | - | 0,04 |
| **0,3** | 0,43 | - | 0,57 |
| **0,5** | 0,24 | - | 0,76 |
| **0,7** | - | 0,02 | 0,98 |
| **0,9** | - | - | 1 |
| **1,1** | - | - | 1 |
| **1,3** | - | - | 1 |
| **1,5** | - | - | 1 |
| **2** | - | - | 1 |
| **5** | - | - | 1 |
| **10** | - | - | 1 |
| **20** | - | - | 1 |

These results indicate that in a VC setting where returns are better described by a lognormal distribution, the optimal allocation will include little or no diversification between the different investment types for most values of risk aversion. For medium and high risk aversion, the VC manager will actually prefer to invest 100% in the low risk investment. One drawback from this four moment optimization with a non-symmetric distribution is that the kurtosis is assumed to have a negative impact on the investor's expected utility. When the distribution is symmetrical and has the same possibility for positive or negative returns, it is logical that a risk-averse investor will prefer that most returns are close to the mean. When returns are lognormal distributed this assumption may not be as intuitive. The returns are now positively skewed; thus, there is a higher probability for extreme positive returns with no threat of extreme low returns. It seems likely that an investor would appreciate a high possibility for extreme returns when they are positive, as is the case in my simulation. For this reason, I also find the optimal portfolio when leaving the kurtosis out of the utility score

These are the result for a *lognormal distribution* when using a *three moment portfolio selection.*

| | Asset Allocation | | |
|---|---|---|---|
| Risk aversion (*b-value)* | $X_1$ | $X_2$ | $X_3$ |
| **0** | 1 | - | - |
| **0,1** | 1 | - | - |
| **0,3** | 1 | - | - |
| **0,5** | 1 | - | - |
| **0,7** | 1 | - | - |
| **0,9** | 1 | - | - |
| **1,1** | 1 | - | - |
| **1,3** | 1 | - | - |
| **1,5** | 1 | - | - |
| **2** | 1 | - | - |
| **5** | 1 | - | - |
| **10** | 1 | - | - |
| **20** | 1 | - | - |

This yields the same result with respect to diversification between the three investments as previously found. The difference is that now the investor wants to allocate all capital in the high risk opportunity for every value of risk aversion. This difference can be explained by the fact that the VC manager wants to exploit the probability of obtaining extremely high positive returns when there is no real risk of obtaining extremely low returns. It is important to stress that these results do not mean that a VC firm should allocate all capital in a single company. I still believe Weiding and Mathonet (2004) make a good point when arguing that a VC fund should contain several investments. From my analysis, however, one can conclude that the optimal portfolio strategy will be to allocate all capital in one type of investment and not diversify between different risk profiles. A sample of the optimization reports produced by Crystal Ball™ is presented in the appendix.

# 7 Conclusion

In this thesis, I have studied the impact on asset allocation when including the higher moments of a return distribution for optimizing a VC portfolio. I first present arguments for why an optimal allocation decision should include the higher moments of a return distribution and then show how this optimal allocation can be derived analytically. I do this by using Taylor's series expansion to approximate an investor's expected utility of end-of-period wealth as a function of higher moments. I find that the optimal asset allocation, when returns come from a normal distribution, is very different from the optimal solution when returns are lognormal distributed. When returns are normally distributed, the optimal allocation involves diversification between investment opportunities for nearly every value of risk aversion. When the returns are lognormal distributed, an optimal allocation will include no or minimal degrees of diversification for every value of risk aversion. These results imply that diversification between investments with different risk profiles may not be a good risk reduction strategy in the VC industry. Because the perception of a negative preference for kurtosis when returns are lognormal distributed is questionable, I also find the optimal solution for three moments. The result from this test shows that a good portfolio strategy in the VC setting is to invest all capital in high risk profile investments.

In general, a parallel can be drawn between the three investment opportunities used in the simulation and the different stages in the VC Industry. As mentioned earlier, the level of uncertainty and risk is reducing as the company becomes more mature. The high, medium, and low risk investment, therefore, can be thought of as an investment in the seed, early growth, and expansion phase. My result is very different from Robinson (1987) who states that VCs are diversifying their investments at a fairly equal amount across the three stages. If returns were normally distributed, then I would conclude that this would be a good risk-reduction strategy. However, based on my results I have determined that a strategy to invest all capital in only one stage would be the optimal solution for VC. More specifically, to invest all capital in seed stage companies would probably give the best results. These findings should entice the VC practitioners to evaluate the asset allocation strategy that they are currently using.

The findings herein may also explain some of the differences in performance between European and U.S. VC funds. On average, the U.S. funds show a significantly higher performance than their European colleagues. That the U.S. VC market is more mature than the European market is one possible explanation for this (Hege, Palomino et al. 2003), but business cultural differences between these two markets may also account for some the difference. My results further indicate that if one market is including more seed stage companies in their portfolios, this should result in higher performance, on average. If, for example, the European VC funds are more concerned with the company's reputation, then they are reluctant to include in their portfolios too many seed stage companies that have a higher probability of going bankrupt. If their U.S. counterparts, however, are not as risk averse, and are more willing to invest in seed stage companies, then this could explain higher returns from the U.S. funds.

There are also some limitations with the analysis that is worth noticing. First, the model does not consider the time perspective of a VC fund. It will be impossible to invest in a very early stage investment as the fund approaches closing because there will not be sufficient time to create value. Usually an investment in a company in the very early stage of development is preferred in the first year of the VC fund because the time to exit in these cases is very long. An investment in later stages of development will be preferred when the fund is approaching closing (Berg-Utby 2010). Second, the expected returns that is used for the different risk profiles, is not consistent with the fund IRR measured by Venture Economics. They presents that the early growth funds have a higher average return than the seed phase. The risk profiles used in my simulation are traditional, thus, higher risk gives higher expected return. The same study also shows that the generalist fund has a higher return than the funds focusing on one single phase. There can be many reasons for these findings. Inconsistency to allocate enough capital to investments in the stage of development the VCs focus on can be one of them. Another reason can be that the study is of the whole private equity market and, therefore, also includes the buyout phase.

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# Appendix

## A. Example on the Timing Difference between the IRR and the Annualized TVPI

Assume that a VC firm invests 100 in a promising company at year 0. The next year, there is a second financing on 50, and the investment pays out 25 in year 3. In year 4, the investment goes public and produces a return on 500. The cash flow from this investment is

| **Year** | **0** | **1** | **2** | **3** | **4** |
|---|---|---|---|---|---|
| **Cash Flow** | -100 | -50 | 25 | - | 500 |
| **IRR** | 42% | | | | |
| **Annualized TVPI** | 36.8% | | | | |

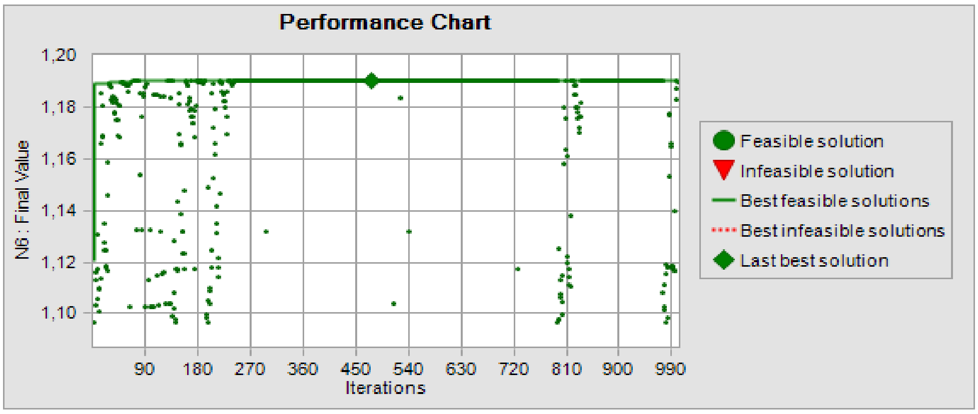
Because not all of the financing is paid in year 0, and not all cash outflows are received in year 4, the IRR correctly gives a higher annual rate of return for the investment. The TVPI multiple is nevertheless, often used in the literature because it is simple to calculate and understand. It expresses how much money is received from the capital invested. If there only is one financing round and the investment only pay a return on exit, then the IRR and the TVPI will give the same annualized return.

## B. Optimization Reports from Crystal Ball™

On the following pages, I present some of the reports produced by Crystal Ball™ after each optimization. The reports for a risk aversion on 0.1, 1.1, and 20 are included. For more reports, the author can be contacted.

## OptQuest Results

**Primary workbook: Final Test Normal 2.xlsx**




| | | |
|---|---|---|
| **Objectives** | Best Solution: | |
| Maximize the Final Value of N6 | 1,19 | |
| **Constraints** | Left Side: | Right Side: |
| 1 Ark1'!M2 = 1 | 1,00 | 1,00 |
| **Decision variables** | Best Solution: | |
| L2 | 0,86 | |
| L3 | 0,14 | |
| L4 | 0,00 | |

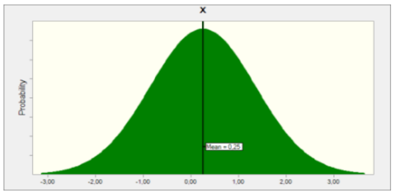
End of OptQuest Results

## Assumptions

### Assumption: X

Normal distribution with parameters:

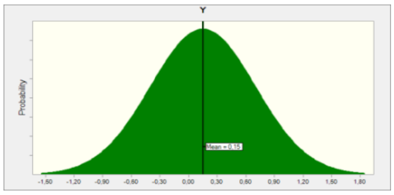
| | |
|---|---|
| Mean | 0,25 |
| Std. Dev. | 1,10 |




### Assumption: Y

Normal distribution with parameters:

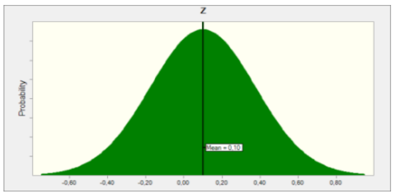
| | |
|---|---|
| Mean | 0,15 |
| Std. Dev. | 0,55 |




### Assumption: Z

Normal distribution with parameters:

| | |
|---|---|
| Mean | 0,10 |
| Std. Dev. | 0,28 |




End of Assumptions

## Decision Variables

**Decision Variable: L2**

Variable bounds:

| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

**Decision Variable: L3**

Variable bounds:

| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

**Decision Variable: L4**

Variable bounds:

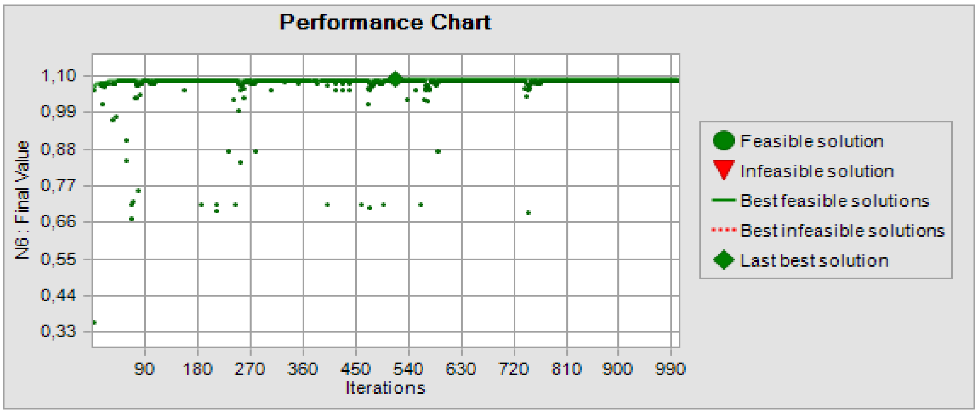
| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

End of Decision Variables

**OptQuest Results**

**Primary workbook: Final Test Normal 2.xlsx**




| | | |
|---|---|---|
| **Objectives** | Best Solution: | |
| Maximize the Final Value of N6 | 1,09 | |
| | | |
| **Constraints** | Left Side: | Right Side: |
| 1 Ark1'!M2 = 1 | 1,00 | 1,00 |
| | | |
| **Decision variables** | Best Solution: | |
| L2 | 0,12 | |
| L3 | 0,26 | |
| L4 | 0,62 | |

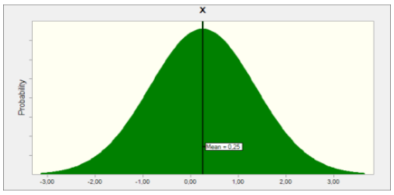
End of OptQuest Results

## Assumptions

### Assumption: X

Normal distribution with parameters:

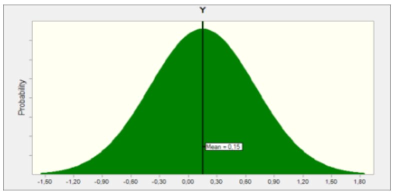
| | |
|---|---|
| Mean | 0,25 |
| Std. Dev. | 1,10 |




### Assumption: Y

Normal distribution with parameters:

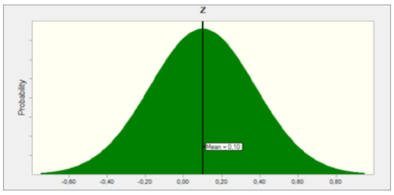
| | |
|---|---|
| Mean | 0,15 |
| Std. Dev. | 0,55 |




### Assumption: Z

Normal distribution with parameters:

| | |
|---|---|
| Mean | 0,10 |
| Std. Dev. | 0,28 |




End of Assumptions

## Decision Variables

**Decision Variable: L2**

Variable bounds:

| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

**Decision Variable: L3**

Variable bounds:

| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

**Decision Variable: L4**

Variable bounds:

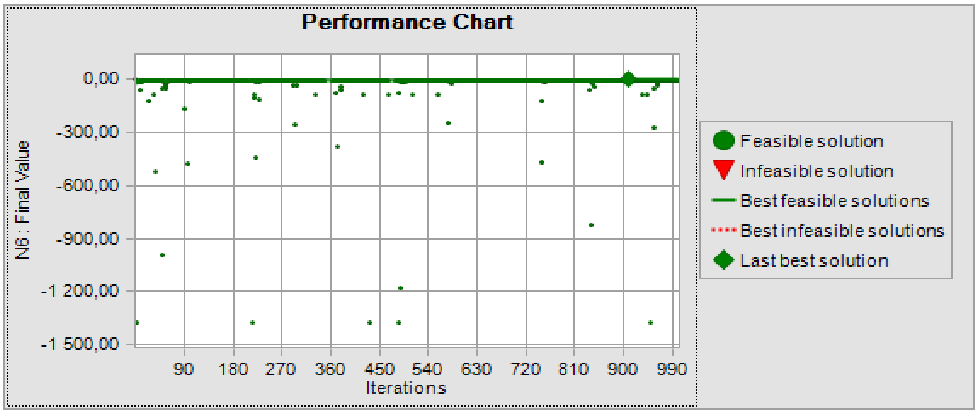
| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

End of Decision Variables

**OptQuest Results**

**Primary workbook: Final Test Normal 2.xlsx**




| | | |
|---|---|---|
| **Objectives** | Best Solution: | |
| Maximize the Final Value of N6 | -4,99 | |
| | | |
| **Constraints** | Left Side: | Right Side: |
| 1 Ark1'!M2 = 1 | 1,00 | 1,00 |
| | | |
| **Decision variables** | Best Solution: | |
| L2 | 0,00 | |
| L3 | 0,00 | |
| L4 | 1,00 | |

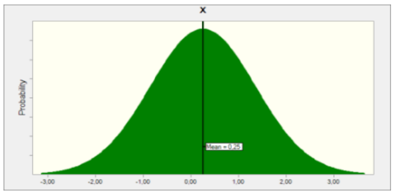
End of OptQuest Results

## Assumptions

### Assumption: X

Normal distribution with parameters:

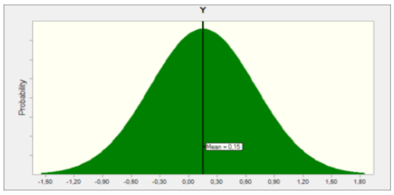
| | |
|---|---|
| Mean | 0,25 |
| Std. Dev. | 1,10 |




### Assumption: Y

Normal distribution with parameters:

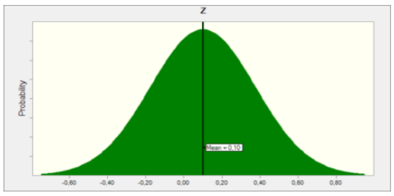
| | |
|---|---|
| Mean | 0,15 |
| Std. Dev. | 0,55 |




### Assumption: Z

Normal distribution with parameters:

| | |
|---|---|
| Mean | 0,10 |
| Std. Dev. | 0,28 |




End of Assumptions

## Decision Variables

**Decision Variable: L2**

Variable bounds:

| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

**Decision Variable: L3**

Variable bounds:

| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

**Decision Variable: L4**

Variable bounds:

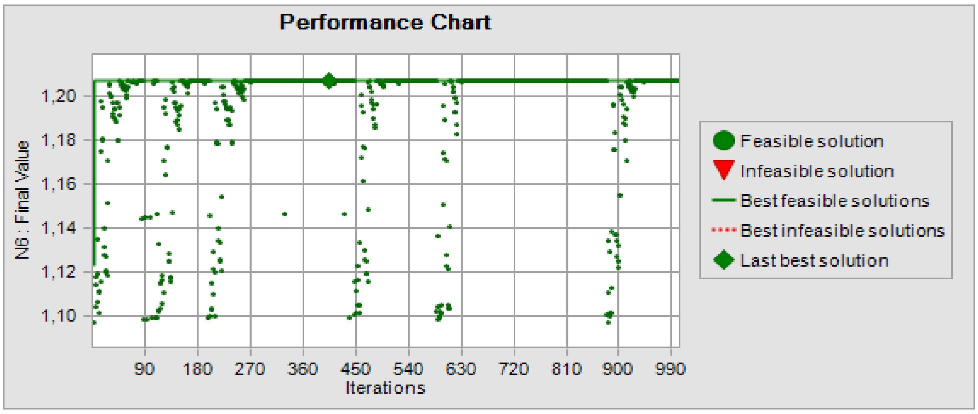
| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

End of Decision Variables

## OptQuest Results

**Primary workbook: Final Test Lognormal 2.xlsx**




| | | |
|---|---|---|
| **Objectives** | Best Solution: | |
| Maximize the Final Value of N6 | 1,21 | |
| **Constraints** | Left Side: | Right Side: |
| 1 Ark1'!N2 = 1 | 1,00 | 1,00 |
| **Decision variables** | Best Solution: | |
| M2 | 0,96 | |
| M3 | 0,00 | |
| M4 | 0,04 | |

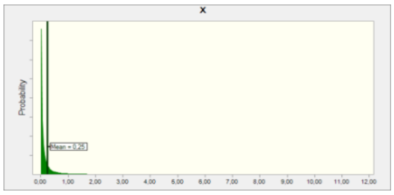
End of OptQuest Results

## Assumptions

### Assumption: X

Lognormal distribution with parameters:

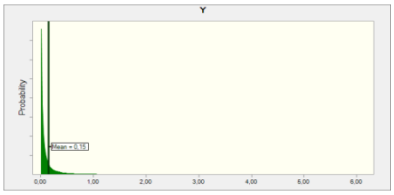
| | |
|---|---|
| Location | 0,00 |
| Mean | 0,25 |
| Std. Dev. | 1,10 |




### Assumption: Y

Lognormal distribution with parameters:

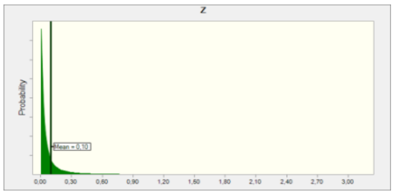
| | |
|---|---|
| Location | 0,00 |
| Mean | 0,15 |
| Std. Dev. | 0,55 |




### Assumption: Z

Lognormal distribution with parameters:

| | |
|---|---|
| Location | 0,00 |
| Mean | 0,10 |
| Std. Dev. | 0,28 |




End of Assumptions

## Decision Variables

**Decision Variable: M2**

Variable bounds:

| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

**Decision Variable: M3**

Variable bounds:

| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

**Decision Variable: M4**

Variable bounds:

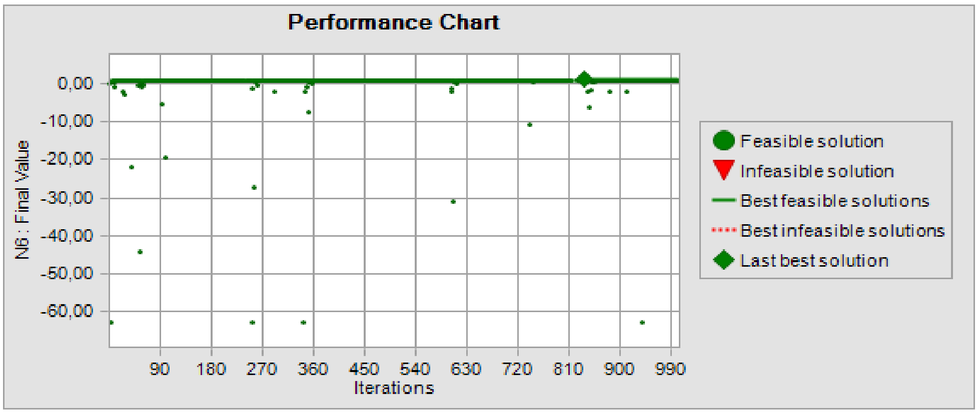
| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

End of Decision Variables

**OptQuest Results**

**Primary workbook: Final Test Lognormal 2.xlsx**




| | | |
|---|---|---|
| **Objectives** | Best Solution: | |
| Maximize the Final Value of N6 | 0,97 | |
| **Constraints** | Left Side: | Right Side: |
| 1 Ark1'!N2 = 1 | 1,00 | 1,00 |
| **Decision variables** | Best Solution: | |
| M2 | 0,00 | |
| M3 | 0,00 | |
| M4 | 1,00 | |

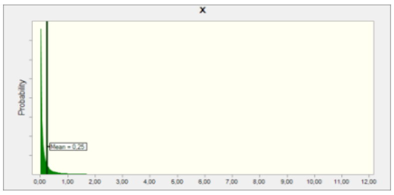
End of OptQuest Results

## Assumptions

### Assumption: X

Lognormal distribution with parameters:

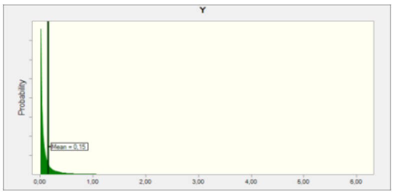
| | |
|---|---|
| Location | 0,00 |
| Mean | 0,25 |
| Std. Dev. | 1,10 |




### Assumption: Y

Lognormal distribution with parameters:

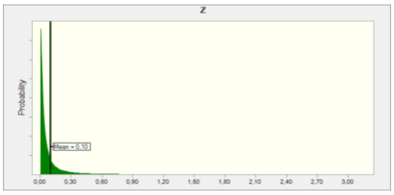
| | |
|---|---|
| Location | 0,00 |
| Mean | 0,15 |
| Std. Dev. | 0,55 |




### Assumption: Z

Lognormal distribution with parameters:

| | |
|---|---|
| Location | 0,00 |
| Mean | 0,10 |
| Std. Dev. | 0,28 |




End of Assumptions

## Decision Variables

**Decision Variable: M2**

Variable bounds:

| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

**Decision Variable: M3**

Variable bounds:

| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

**Decision Variable: M4**

Variable bounds:

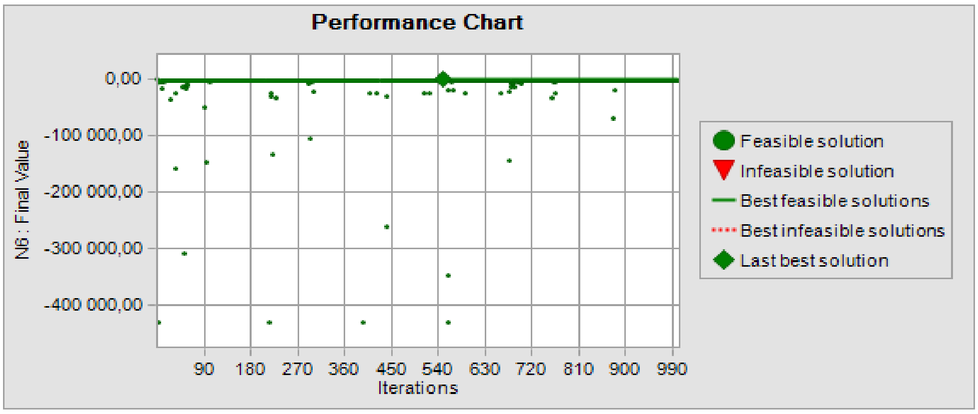
| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

End of Decision Variables

## OptQuest Results

**Primary workbook: Final Test Lognormal 2.xlsx**




| | | |
|---|---|---|
| **Objectives** | Best Solution: | |
| Maximize the Final Value of N6 | -1 154,47 | |
| **Constraints** | Left Side: | Right Side: |
| 1 Ark1'!N2 = 1 | 1,00 | 1,00 |
| **Decision variables** | Best Solution: | |
| M2 | 0,00 | |
| M3 | 0,00 | |
| M4 | 1,00 | |

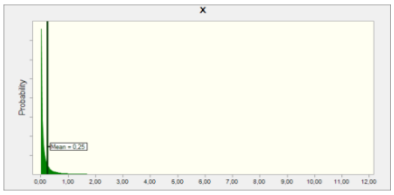
End of OptQuest Results

## Assumptions

### Assumption: X

Lognormal distribution with parameters:

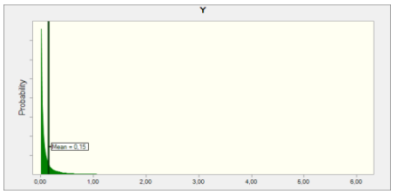
| | |
|---|---|
| Location | 0,00 |
| Mean | 0,25 |
| Std. Dev. | 1,10 |




### Assumption: Y

Lognormal distribution with parameters:

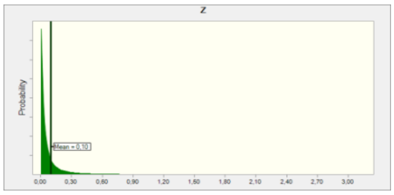
| | |
|---|---|
| Location | 0,00 |
| Mean | 0,15 |
| Std. Dev. | 0,55 |




### Assumption: Z

Lognormal distribution with parameters:

| | |
|---|---|
| Location | 0,00 |
| Mean | 0,10 |
| Std. Dev. | 0,28 |




End of Assumptions

## Decision Variables

**Decision Variable: M2**

| | |
|---|---|
| Variable bounds: | |
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

**Decision Variable: M3**

| | |
|---|---|
| Variable bounds: | |
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

**Decision Variable: M4**

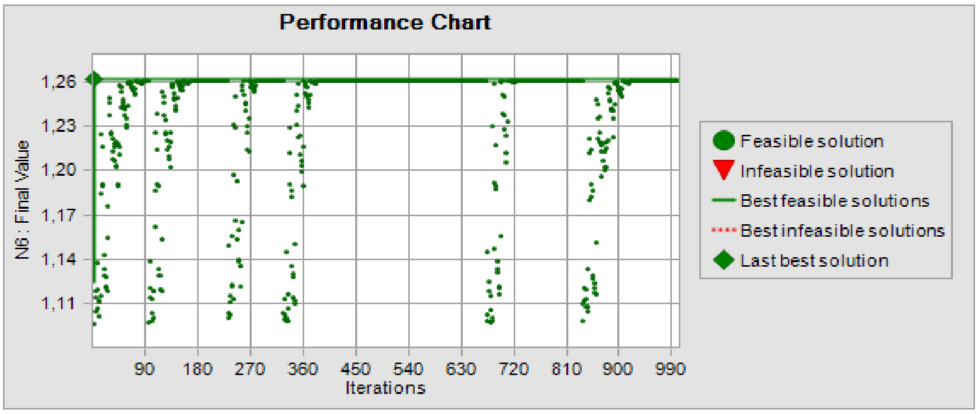
| | |
|---|---|
| Variable bounds: | |
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

End of Decision Variables

**OptQuest Results**

**Primary workbook: Final Test Lognormal 2.xlsx**




| | | |
|---|---|---|
| **Objectives** | Best Solution: | |
| Maximize the Final Value of N6 | 1,26 | |
| **Constraints** | Left Side: | Right Side: |
| 1 Ark1'!N2 = 1 | 1,00 | 1,00 |
| **Decision variables** | Best Solution: | |
| M2 | 1,00 | |
| M3 | 0,00 | |
| M4 | 0,00 | |

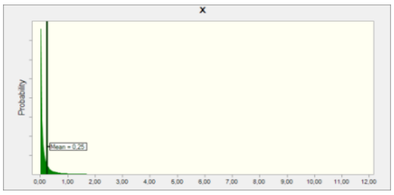
End of OptQuest Results

## Assumptions

### Assumption: X

Lognormal distribution with parameters:

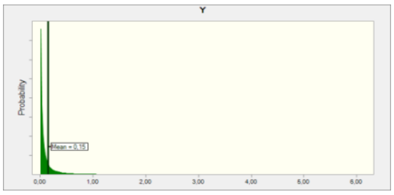
| | |
|---|---|
| Location | 0,00 |
| Mean | 0,25 |
| Std. Dev. | 1,10 |




### Assumption: Y

Lognormal distribution with parameters:

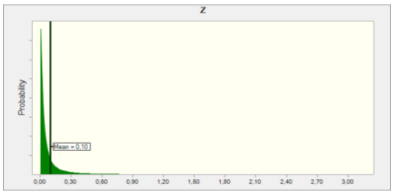
| | |
|---|---|
| Location | 0,00 |
| Mean | 0,15 |
| Std. Dev. | 0,55 |




### Assumption: Z

Lognormal distribution with parameters:

| | |
|---|---|
| Location | 0,00 |
| Mean | 0,10 |
| Std. Dev. | 0,28 |




End of Assumptions

## Decision Variables

**Decision Variable: M2**

Variable bounds:

| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

**Decision Variable: M3**

Variable bounds:

| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

**Decision Variable: M4**

Variable bounds:

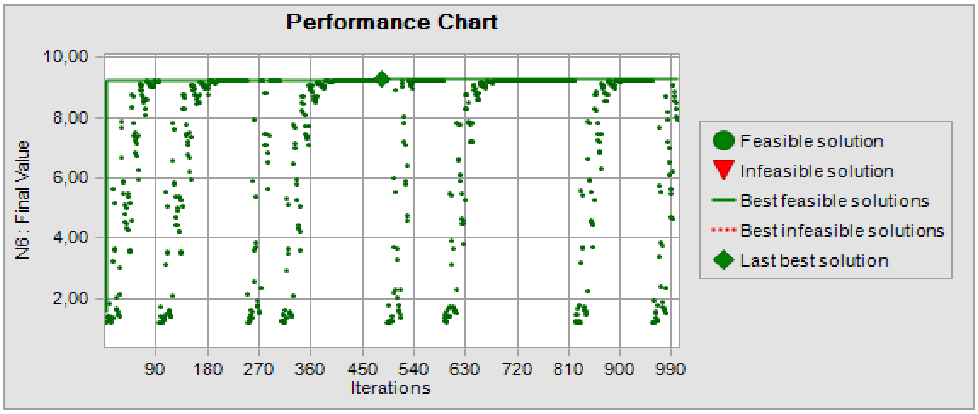
| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

End of Decision Variables

## OptQuest Results

**Primary workbook: Final Test Lognormal 2.xlsx**




| | | |
|---|---|---|
| **Objectives** | Best Solution: | |
| Maximize the Final Value of N6 | 9,26 | |
| **Constraints** | Left Side: | Right Side: |
| 1 Ark1'!N2 = 1 | 1,00 | 1,00 |
| **Decision variables** | Best Solution: | |
| M2 | 1,00 | |
| M3 | 0,00 | |
| M4 | 0,00 | |

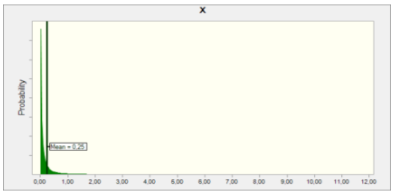
End of OptQuest Results

## Assumptions

### Assumption: X

Lognormal distribution with parameters:

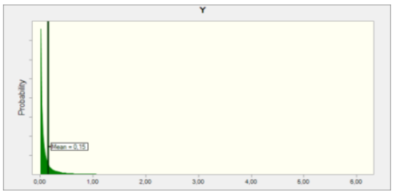
| | |
|---|---|
| Location | 0,00 |
| Mean | 0,25 |
| Std. Dev. | 1,10 |




### Assumption: Y

Lognormal distribution with parameters:

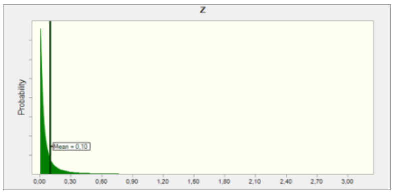
| | |
|---|---|
| Location | 0,00 |
| Mean | 0,15 |
| Std. Dev. | 0,55 |




### Assumption: Z

Lognormal distribution with parameters:

| | |
|---|---|
| Location | 0,00 |
| Mean | 0,10 |
| Std. Dev. | 0,28 |




End of Assumptions

## Decision Variables

**Decision Variable: M2**

Variable bounds:

| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

**Decision Variable: M3**

Variable bounds:

| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

**Decision Variable: M4**

Variable bounds:

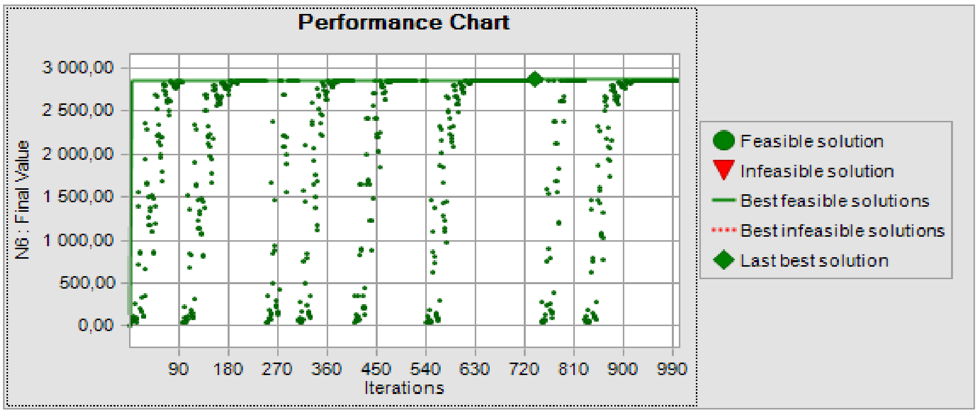
| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

End of Decision Variables

## OptQuest Results

**Primary workbook: Final Test Lognormal 2.xlsx**




| | | |
|---|---|---|
| **Objectives** | Best Solution: | |
| Maximize the Final Value of N6 | 2 857,29 | |
| **Constraints** | Left Side: | Right Side: |
| 1 Ark1'!N2 = 1 | 1,00 | 1,00 |
| **Decision variables** | Best Solution: | |
| M2 | 1,00 | |
| M3 | 0,00 | |
| M4 | 0,00 | |

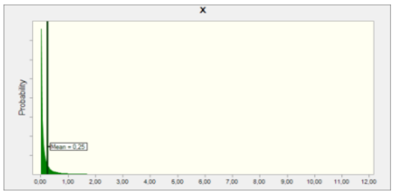
End of OptQuest Results

## Assumptions

### Assumption: X

Lognormal distribution with parameters:

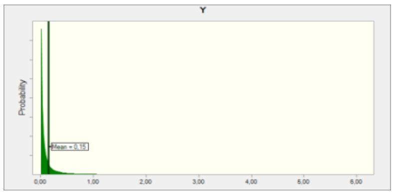
| | |
|---|---|
| Location | 0,00 |
| Mean | 0,25 |
| Std. Dev. | 1,10 |




### Assumption: Y

Lognormal distribution with parameters:

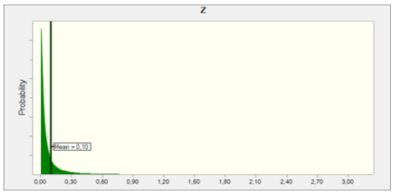
| | |
|---|---|
| Location | 0,00 |
| Mean | 0,15 |
| Std. Dev. | 0,55 |




### Assumption: Z

Lognormal distribution with parameters:

| | |
|---|---|
| Location | 0,00 |
| Mean | 0,10 |
| Std. Dev. | 0,28 |




End of Assumptions

## Decision Variables

**Decision Variable: M2**

Variable bounds:

| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

**Decision Variable: M3**

Variable bounds:

| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

**Decision Variable: M4**

Variable bounds:

| | |
|---|---|
| Lower | 0,00 |
| Upper | 1,00 |

Variable type: Continuous

End of Decision Variables