Modeling Alternatives to Exponential Discounting

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Abstract

One area that is often overlooked by economists and social scientists is discounting. Most economic models of intertemporal choice make use of Samuelson's (1937) DU model which leads to an exponential discount function. Divergences from what economic modelling predicts and empirical findings are on the most part attributed to factors other than the discount function employed. We review the literature on the DU model and identify its behavioral anomalies. We look into suggested quasi-hyperbolic and hyperbolic models that in part account for these anomalies. We analyze an infinite IPD game and demonstrate that under quasi-hyperbolic discounting, cooperation emerges as an SPE at a higher level of the discount factor. We further demonstrate that the unemployment equilibrium in the Shapiro & Stiglitz (1984) shirking model is not static under both hyperbolic and quasi-hyperbolic discounting.

Keywords: *intertemporal, exponential, quasi-hyperbolic, hyperbolic.*

1 Introduction

Economic decision making involves an analysis of expected payoffs over some time period. Hence, a critical issue that arises is what value an agent assigns to the present relative to the future and the tradeoffs she makes between near-term payoffs and payoffs that are far into the future. Decisions regarding the timing of a firm's entry into a new market, for example, are based on the tradeoffs between the value of waiting and the benefits from moving quickly. The higher the current dividends, residual resource value, and interest rate the greater the likelihood of entry now (Folta & Miller 2002, p. 662). To take another example, in an infinitely iterated prisoner's dilemma (IPD), the evolution of cooperation depends on the magnitude of the time discount factor. Cooperation is easier to sustain when players have a high time discount factor (Shy 1995, p. 33).

Discounting thus is central in intertemporal choice but an equally important consideration is what discounting model to employ, that is, one that adequately represents the preferences of an agent across time. Samuelson's (1937) Discounted Utility (DU) model is the standard discounting model in most economic analysis of intertemporal choice. The DU model represents an agent as selecting between choices based on a weighted sum of utilities: the weights being represented as discount factors. The main underlying assumption of the DU model is that the discount factor is constant over time. For example, if an agent believes that the utility derived from receiving a dollar falls by 5 percent from time t to $t + 1$, then its utility would be 5 percent less at $t + 2$ in relation to $t + 1$, 5 percent less at $t + 3$ in relation to $t + 2$ and so on. This assumption leads to an exponential discounting function which is commonly used in financial calculations of present value. Part of the success of the DU model as the enduring model of choice for economists and social scientists modeling social or economic behavior lies in its simplicity and mathematical tractability. Employing the model allows for an analytical solution even when dealing with an infinite series of payoffs since it has the convenient property of convergence over time. In addition, the model implies that agents have consistent time preferences or that their preferences remain the same over time. This is in line with rational choice theory and makes the model rather convenient for use.

However, a number of anomalies in the DU model have been identified and its assumptions and implications have been widely criticized. Empirical studies in experimental and behavioral economics and psychology e.g., (Thaler 1981, Benzion, Rapoport &

Yagil 1989, Slonim, Carlson & Bettinger 2007) suggest that animals and human beings in general appear to discount the future hyperbolically. This means that they are extremely impatient about payoffs that occur in the near future and more patient about payoffs that occur further into the future. Thus, they appear to discount the near-term more than exponentially and the long-term less. In addition, economic models of preference reversals and time inconsistency e.g., (Strotz 1956) challenge the DU model and call into question the empirical validity of conclusions drawn based on modeling agents as exponential discounters.

Accordingly, researchers in economics and the social sciences have taken note of the DU model's apparent lack of empirical validity sparking huge interest on alternative formulations. Among the most prominent include Phelps & Pollak (1968) and Laibson (1997) quasi-hyperbolic model, Herrnstein (1981) and Mazur (1987) hyperbolic model, and Loewenstein & Prelec (1992) hyperbolic model. A search on Science Direct, one of the world's largest online collections of published scientific research shows an over 800 percent increase in the number of published articles on hyperbolic discounting in the past decade alone, indicating that increased interest in the topic is a relatively new phenomenon.

In our analysis, we examine the problem of cooperation in an infinite IPD and demonstrate that under quasi-hyperbolic discounting, the subgame perfect equilibrium (SPE) is attained at a higher level of the discount factor. Further, we analyze the Shapiro & Stiglitz (1984) shirking model and demonstrate that wages set under the assumption of exponential discounting may not effectively deter shirking in the presence of hyperbolic or quasi-hyperbolic discounting employees. Incorporating Solow (1956), we demonstrate that the unemployment equilibrium in the Shapiro & Stiglitz model is not static under both hyperbolic and quasi-hyperbolic discounting.

The paper is organized as follows: In Section 2, we highlight anomalies in the DU model and look into suggested quasi-hyperbolic and hyperbolic alternatives. In Section 3, we model the interaction between two firms in the form of an infinite IPD and analyze how the equilibrium conditions necessary for sustaining cooperation change as we move from exponential to non-exponential discounting. Eventually, in Section 4 we analyze the Shapiro & Stiglitz (1984) shirking model under non exponential discounting. Incorporating Solow (1956), we investigate how the unemployment equilibrium changes if we shift from the standard exponential functional form. Our findings are summarized in Section 5.

2 Discounting Functions

2.0.1 Definition and Properties

The term *time discounting* can be broadly defined to include any reason for placing less concern on a future consequence including factors that reduce the expected utility generated by a future consequence, such as risk or uncertainty. In more specific terms, it can be defined as the fall in the value of a reward as a function of increasing delay or decreasing probability (for a review, see McKerchar, Green, Myerson, Pickford, Hill & Stout (2009); for a broader elaboration, see Fredrick, Loewenstein & O'Donoghue (2002)).

In most analytical economic models, the idea of discounting is compressed into a single construct, that is, the discounting function.[1] Thus at the onset, it is appropriate to define a general version of a discounting function before proceeding to specific forms. Most discounting functions of intertemporal choice can be expressed in the following way:[2]

$$D(\tau) = \prod_{t=0}^{\tau-1} \frac{1}{1+\rho_t} \tag{1}$$

where $D(\tau)$ is the discount function (which can take any form) and ρ_t is the discount rate at time t or the discount rate between t and $t+1$. Axtell & McRae (2006) present two "necessary conditions" which a function must satisfy inorder to be a discounting function. These are summarized below:

Axiom 1: $D(0) = 1$, that is, no discounting of the present;

Axiom 2: $D(t)$ must be strictly monotone decreasing, $D'(t) < 0$.

Both properties suggest that earlier nominal payoffs are preferable to later ones given that both are equal in magnitude. In particular, Axiom 2 implies that the value at two different future periods is always at least as large at the nearer time. One way of examining this is by way of a simple offer wherein you consider a choice between

[1]To cause no confusion, we do not differentiate between the terms discount function and discounting function.

[2]see Fredrick, Loewenstein & O'Donoghue (2002) for a broader review.

receiving a given amount of money today, say \$100 or the same nominal amount in a year's time. Naturally, you would choose the former given the opportunity which gives the properties some intuitive basis.[3] Thus, as is evident, the need for discounting arises from the notion that individuals in general have a bias towards present consumption over future consumption, ceteris paribus. There is a wide array of literature addressing this phenomenon and we briefly review the psychological motives underlying this form of time preference as well as other justifications for discounting.

2.0.2 Motives for Discounting

One of the most cited arguments for discounting is the tendency inherent in human beings to attach less importance to future payoffs even if there is no rational reason to do so. This fact of human psychology often referred to as 'pure time discounting' can trace its origins as far back as the emergence of intertemporal choice as a distinct topic.[4] Proponents of this view include Rae (1834, p.120) who contends that human beings have a passion for present consumption and face great discomfort from delaying utility gained from this consumption:

Such pleasures as may now be enjoyed generally awaken a passion strongly prompting to the partaking of them. The actual presence of the immediate object of desire in the mind by exciting the attention, seems to rouse all the faculties, as it were to fix their view on it, and leads them to a very lively conception of the enjoyments which it offers to their instant possession.[5]

Pigou (1932, Pt.1 ch.2 sec.3) refers to it as a 'brute fact of human psychology' and 'a psychological frailty' - a defect of the 'telescopic faculty' and maintains the innate nature of the phenomenon. Other proponents who have stressed this point in their works include philosophers Sidgwick (1907, Bk.4 ch.1) and Rawls (1972, sec.45). However, it should be noted that there is no uniform support for this argument among economists. The most notable criticism comes from Ramsey (1928, p.543) who terms it as a 'practice which is ethically indefensible and arises merely from the weakness

[3] Perhaps it would have been better to state that given the choice, a rational agent would choose the former to the latter but such an assumption is always implicit.

[4] It is widely noted that economist John Rae invented the topic of intertemporal choice following the publication of *The Sociological Theory of Capital* in 1834. This was a follow up on Adam Smith's *Wealth of Nations* and sought to determine why wealth differed among nations (Fredrick, Loewenstein & O'Donoghue 2002, pp.352-53).

[5] *The Sociological Theory of Capital,* cited in Loewenstein (2007, p.60)

of the imagination'.[6]

A second justification for discounting and one that is more appealing to rational minded individuals relates to uncertainty and risk. Unlike the previous which focuses on the psychological make up of human beings, this argument contends that there exists some exogenous factors which bring about the need for discounting. Absent from these factors, equal treatment of past and present or zero discounting would prevail. To take an example, a risk premium is included in the interest rate that a bank charges on a loan to a borrower. In this instance, the bank is discounting expected future receipts to account for the possibility of default or credit risk on the part of the borrower resulting from various sources such as an unexpected increase in the rate of inflation, adverse conditions caused by unanticipated changes in regulations, business climate, etc. One of the most common risk factors cited is death, 'for reproductive rewards, the organism may die before the reward is realized' (Sozou & Seymour 2003, p.1047).[7] Mill (1848, Bk1 Ch.11) identifies other factors related to risk and uncertainty and their effects on an individual's choices:

In weighing the future against the present, the uncertainty of all things future is a leading element; and that uncertainty is of very different degrees. "All circumstances" therefore, "increasing the probability of the provision we make for futurity being enjoyed by ourselves or others, tend" justly and reasonably "to give strength to the effective desire of accumulation. Thus a healthy climate or occupation, by increasing the probability of life, has a tendency to add to this desire. When engaged in safe occupations, and living in healthy countries, men are much more apt to be frugal, than in unhealthy or hazardous occupations, and in climates pernicious to human life. Sailors and soldiers are prodigals. In the West Indies, New Orleans, the East Indies, the expenditure of the inhabitants is profuse. The same people, coming to reside in the healthy parts of Europe, and not getting into the vortex of extravagant fashion, live economically. War and pestilence have always waste and luxury among the other evils that follow in their train.[8]

In this instance, Mill implies that an individual who is exposed to greater risk would discount the future more and this is reflected through maximizing present consump-

[6] references cited in (Goodin 1982, pp.54-55). We recommend that you review full article for a broader discussion of justifications for discounting.

[7] Ultimately, the fact that an individual is a mortal being and there is a positive relationship between the progression of time and the probability of death, it is reasonable to conclude that ceteris paribus, an earlier payoff is preferable to a later one.

[8] The Principles of Political Economy, cited in Fredrick, Loewenstein & O'Donoghue (2002, p.353)

tion.[9]

Another key justification for discounting relates to cost. Specifically, the argument contends that there is an *opportunity cost* to delaying present consumption and this can be viewed in terms of the next best alternative forgone. If we revert back to our earlier example wherein you had a choice between receiving $100 today or the same nominal amount in a year's time, an opportunity cost of accepting the latter can be the interest that you would have earned had you accepted to receive the money today and deposited it in an interest bearing account for the duration of a year. A major proponent of this argument is Baumol (1970, p.274)[10] , who claims that 'the correct discount rate is the opportunity cost in terms of the potential rate of return on alternative uses on the resources that would be utilized by the project' (See Goodin 1982, pp.58-60). The idea that one should use the prevailing risk-free rate of interest to discount a future payoff or reward primarily emanates from this argument. As Baumol argues, a firm that is evaluating a stream of future payoffs resulting from a project should discount such payoffs using a rate that is equal to the rate of return that the firm's resources would have earned had they not been employed in that particular project. In the same manner, an individual who has no ideas of what to do with $100 apart from immediate consumption can do no worse than investing the money at the risk free rate of interest. Thus, from an economic point of view at least, the opportunity cost argument can be viewed as the most orthodox justification for discounting.

A final major justification for discounting relates to the concept of diminishing marginal utility (DMU). In very simple terms, this is the idea that as an individual increases consumption of one good ceteris paribus, there is a decline in the marginal utility she derives from consuming additional units of the same good. If we allow for the progression of time, then we can extend the DMU argument over different generations of agents. In effect, given that utility is not invariant over time, there is a shift in the preferences of an individual or society from one period to the next leading to DMU as consumption increases. According to (Arrow 1976, p.122) this argument is especially relevant as it pertains to an emerging economy:

[9] Mill's conclusion is not extraordinary in any way. In the US for example where life expectancy is over 75, most employees maintain their 401(k) accounts until retirement although they may opt to liquidate part or all assets in the accounts before then.

[10] Baumol of course is more famous for his model of the transactions demand for money. For a further insight see *The Transactions Demand for Cash: An Inventory Theoretic Approach*, Quarterly Journal of Economics, vol. 66, pp.545-56)

In a growing economy, where people will be generally better off in the future, diminishing marginal utility implies that they would derive more satisfaction from any given unit of a good now (when they have less) than later (when they will have more anyway). Discounting is to be encouraged - and saving shunned - in order to avoid 'redistributing income from a present that is relatively poor to a future relatively rich.'

In the following sections, we review specific forms of discounting functions and identify their characteristics, advantages and drawbacks. We begin our review with the DU model and then progress to more recent quasi-hyperbolic and hyperbolic specifications.

2.1 The Discounted Utility Model

The DU framework developed by economist Paul Samuelson (1937) is the standard model for representing intertemporal preferences.[11] Since its introduction, the model has dominated economic analysis of intertemporal choice (Loewenstein & Prelec 1992, 573).[12] The model states that for a given consumption profile $(c_1, c_2....., c_N)$, where c_1 denotes consumption in period 1, c_2 in period 2, and so on, the time-separable intertemporal utility function denoted $U(c_1, c_2....., c_N)$ can be expressed simply as a sum of discounted cardinal instantaneous utility functions for each period, $\rho^{\tau-1}u(c_\tau)$ $(\tau \in \mathbb{Z}_+)$. Formally,

$$U(c_1, c_2, ..., c_N) = \sum_{\tau=1}^{N} \rho^{\tau-1} u(c_\tau). \tag{2}$$

This represents the functional form proposed by Samuelson wherein ρ is a time discount parameter.[13] The unique characteristic of the DU model is that the rate of discount of future utilities is a constant. The discount rate, denoted r, and the discount factor are expressed in the following way:

[11] Samuelson was suggesting an alternative multi-period model of intertemporal choice in an article titled *A Note on Measurement of Utility* (1937) and in the process making the point that representing tradeoffs at different points in time necessitated a cardinal measure of utility.

[12] Further credit for the popularization of the DU model should be extended to Koopmans (1960) for his axiomatic derivations of the model. For a review see *Stationary Ordinal Utility and Impatience*, Econometrica 18 (1960), p 207-309.

[13] In the case of continuous time, the DU model can be expressed in the following way: $U(c_1, c_2, ..., c_N) = \int_{\tau=1}^{N} e^{-r(\tau-1)} u(c_\tau)$

$$\rho = \frac{1}{1+r} \;\; and \;\; r = \frac{1}{\rho} - 1 \tag{3}$$

Thus, r is a parameter representing the agent's preferences or the rate at which she discounts the future.[14] Incorporating (3) into (2), we can restate (2) in the following way:

$$U(c_1, c_2, ..., c_N) = \sum_{\tau=1}^{N} \left(\frac{1}{1+r}\right)^{\tau-1} u(c_\tau). \tag{4}$$

Expanding (4), we have

$$U(c_1, c_2, ..., c_N) = u(c_1) + \frac{u(c_2)}{1+r} + ... + \frac{u(c_N)}{(1+r)^{N-1}}. \tag{5}$$

From (5), we have the familiar representation of the DU model as is presented in most literature. Taking the equation in isolation, it is evident that the DU model leads to an exponential discounting function where the per-period discount factor is constant and equal to $\frac{1}{1+r}$ or ρ. **Proof: See Appendix- Section 6.1 A2**

2.1.1 DU Model Assumptions

We can identify six main assumptions that the DU model makes. These are summarized below:

1. Agents have a positive rate of time preference, that is they prefer to receive a payoff sooner rather than later. In terms of the discount factor, $\rho \in (0, 1)$ implying the less the agent values the future, the closer is the discount factor to 0 and the future is never valued more than the present, $\rho < 1, \; \forall \, \tau \; (\tau \in (0, \infty))$.

2. The rate of discount of future utilities is a constant, that is the discount factor,

[14] Shy (1995) interprets the parameter ρ as the present value of one dollar to be received in the next period. If we assume a world with perfect capital markets, then r will adjust to equal the real interest rate (p.29)

ρ, is constant and applicable to each and every period.

3. Utility independence, that is no relation exists between utility in one period and every other period. The model explicitly assumes that total utility is simply a sum of discounted utility functions for each period. Hence, apart from that which is dictated by discounting, the distribution of utility across time makes no difference.

4. Independence of consumption, that is consumption in one period does not affect or is not affected by consumption in each and every other period. Specifically, the marginal rate of substitution between consumption in two given periods τ and $\tau^{'}$ is independent of consumption in period $\tau^{''}$ or putting it in another way, watching a particular movie on two consecutive days, say Monday and Tuesday should not affect your preference for watching the same movie over a choice of others on the third day, Wednesday or on any other day (Fredrick, Loewenstein & O'Donoghue 2002, p.357).

5. Stationary instantaneous utility, that is the utility function is constant across time implying that the preferences of an individual do not change over time.

6. Independence of discounting from consumption, that is the discount function or rate of time preference is constant across all consumption forms. Hence, preferences of an individual remain the same when comparing her choices across a diverse array of consumption forms or stated differently, discounting is same for food as it is for leisure, money, etc. (Fredrick, Loewenstein & O'Donoghue 2002, p.358)

Before identifying anomalies in the DU model, we first explore its mathematically desirable properties that make it convenient for use in analysis of intertemporal choice.

2.1.2 Desirable Properties of Exponential Discounting

One appealing property of exponential discounting is that it leads to *dynamic consistency*, that is, preferences that do not change over time. As an example, we consider an individual who given a choice prefers to add m units to her consumption at time t over adding n units ($n > m$) at a later time $t^{'}$. Given a constant initial consumption in every period which we denote c and utility which we denote u, we have:

$$u(c+m)\rho^t + u(c)\rho^{t'} > u(c)\rho^t + u(c+n)\rho^{t'}. \tag{6}$$

If we divide through by ρ^t, we have:

$$u(c+m) - u(c) > (u(c+n) - u(c))\rho^{t'-t}. \tag{7}$$

Comparing (6) and (7) above tells us that the preference between the 2 consumption adjustments is just simply a function of the separating absolute time interval, that is $(t' - t)$.

Taking another example, we consider a firm that identifies an investment project wherein it has to incur an initial cost C at time $t+2$ and earn a return R at time $t+3$. We further suppose that the company evaluates the project at time t and finds that it is worth pursuing since it generates a positive net present value or in formal terms:

$$-\rho C + \rho^2 R > 0. \tag{8}$$

If the company chooses not to pursue the project at time t and waits for the next period, i.e., $t+1$, the project will still be worth pursuing since:

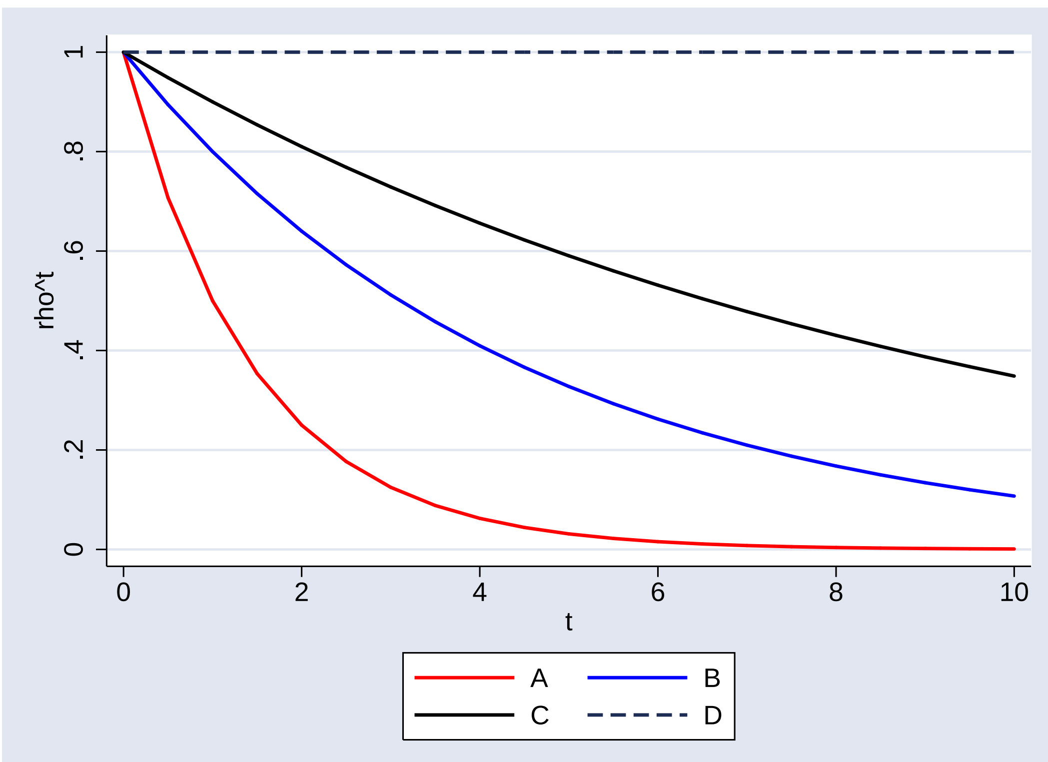
$$-C + \rho R > 0. \tag{9}$$

To prove that the above is true, we only have to note that (9) is just (8) multiplied by $\frac{1}{\rho}$. This stationarity property evident in both examples above is unique to the exponential discounting and thereby makes it extremely attractive especially in the field of rational choice theory.

A second appealing property of exponential discounting is its mathematical tractability. The function allows for an analytical solution when summing an infinite series of payoffs such as may be encountered in an infinite IPD since the series converges:

$$1 + \rho + \rho^2 + \rho^3 + ... = \sum_{t=0}^{\infty} \rho^t = \frac{1}{1-\rho} \quad \textbf{Proof: See Appendix - Section 6.1 A1} \tag{10}$$

These two key properties in particular have contributed to the popularity of exponential discounting as the standard mode of discounting among economists and social scientists. Figure 1 is a plot of an exponential discount function for different values of ρ. The dashed line represents the theoretical limit case wherein $\rho = 1$, implying zero discounting. We next explore anomalies inherent in this form of discounting.



$A : \rho = 0.5,\ B : \rho = 0.8,\ C : \rho = 0.9$

Figure 1: Exponential discount function (ρ^t *versus* t)
($u(c_1) = u(c_2) = ... = u(c_\infty) = 1$)

2.1.3 DU Model Anomalies

Samuelson (1937) when proposing the DU model recognized inadequacies inherent in the assumptions as they pertain to the real world. On more than one occasion, he stressed the 'arbitrariness' of these assumptions even to the extent of criticizing their implications, " it is completely arbitrary to assume that the individual behaves so as to maximize an integral of the form envisaged in (2)" (pp. 156, 159). To stress his point even further, he concluded the article by noting, "the idea that the results of such a statistical investigation could have any influence upon ethical judgments of policy is one which deserves the impatience of modern economists" (p. 161). However, despite Samuelson's misgivings and partly for reasons discussed earlier in this paper, economists and social scientists still persist with the model as the standard tool in intertemporal choice analysis. In the course of time, several anomalies have been identified and we review each sequentially citing specific studies along the way.

One of the most highlighted anomalies in the DU model concerns the empirical validity of the *constant rate of discount assumption*. Empirical evidence suggests that discount rates fall over time. This phenomenon also known as hyperbolic discounting contends that the discount rate, r, is not constant from one period to the next but declining in time.[15] In one study, Thaler (1981) asked respondents to state an amount over three distinct future time periods, that is, one month, one year and ten years that would be equivalent to receiving \$15 now. The median responses were \$20 for one month, \$50 for one year and \$100 for 10 years representing discount rates of 345%, 120% and 19% respectively (Fredrick, Loewenstein & O'Donoghue 2002, 360). In another experimental study, Slonim, Carlson & Bettinger (2007) asked subjects to choose between receiving a given amount of money by varying front-end delays (3 levels: 0 days, 2 days and y months). In each case, subjects got more patient the greater the front-end delay ($\rho_0 < \rho_2 < \rho_y$) at the mean and 25th, 50th and 75th percentiles implying falling discount rates over time. Other empirical studies have found patterns consistent with declining discount rates (Benzion, Rapoport & Yagil 1989, Redelmeier & Heller 1993, Chapman & Elstein 1995, Pender 1996).

A common finding in these studies is that there is a steep drop in the value of a reward in the very short term which is not captured by exponential discounting. Thus, individuals are extremely impatient about payoffs occuring in the very near future leading

[15]We do not go into specifics of hyperbolic discounting at this time since the topic is addressed broadly in later sections.

to relatively high discount rates over short horizons (Laibson 1997, p.445). On the other hand, payoffs far into the future are discounted more heavily under exponential discounting as the empirically implied rate falls below the constant discount rate. In addition, Fredrick, Loewenstein & O'Donoghue (2002) note that when mathematical functions are explicitly fit to measured data, a hyperbolic functional form or one that imposes declining discount rates is seen to fit the data better than the exponential functional form, that is, one that imposes constant discount rates (p. 360).

A second anomaly in the DU model relates to the *magnitude of payoff effect*. The DU model does not make any distinction between the size or magnitude of a payoff and thus implicitly assumes that an individual's rate of discount at any point in time is independent of this factor. However, empirical evidence suggests that individuals discount larger payoffs at a lower rate compared to smaller amounts. Benzion, Rapoport & Yagil (1989) examined the effect of varying the amount of money involved (4 levels: $40, $200, $1000, $5000) and the time period (4 levels: 0.5, 1, 2, and 4 years) on the discount rate. In all scenarios, they found that discount rates decrease as the size of cashflow increases. For example, for the two year time period, the mean discount rates were 0.228 for $40, 0.183 for $200, 0.16 for $1000, and 0.123 for $2000.

In another experimental study, Thaler (1981) asked respondents to state an amount that they would be willing to accept in a year's time in exchange for a given amount today. He varied the amounts (3 levels: $15, $250 and $3000). On average, the findings were $60, $350 and $4000 respectively implying rates of 139%, 34% and 29% (Fredrick, Loewenstein & O'Donoghue 2002, 363). Other empirical studies that have found the existence of a magnitude effect include (Holcomb & Nelson 1992, Green, Myerson & McFadden 1997, Kirby & Marakovic 1995).

A third anomaly that has been identified in the DU model relates to how individuals discount losses vis-*á*-vis gains or what is sometimes known as the *sign effect*. The DU model does not distinguish between positive payoffs and negative payoffs and thus implicitly assumes that if such payoffs occur at the same point in time, then they should be discounted using the same rate. Empirical evidence on the other hand suggests that individuals are quite anxious to receive a positive reward, especially a small one, but are less anxious to postpone a loss (Loewenstein & Thaler 1989, p.187). Therefore, they discount gains at a higher rate in relation to losses. In one study, Loewenstein (1988) found that on average, subjects were indifferent to receiving $100 now and $157 in a year's time whereas the same subjects were indifferent to losing $100 now and \$133 in a years time implying a rate of 0.45 for gains and 0.29 for losses.[16] The corresponding figures for \$10 were \$21 for gains and \$15 for losses implying rates of 0.74 and 0.41 respectively (Loewenstein & Prelec 1992, p.575).[17] In another study, Thaler (1981) asked subjects how much they would be willing to pay for a traffic ticket given that they had the option of either paying now or delaying (3 levels: 0.25, 1, 3 years). The findings resulted in lower discount rates being imputed in relation to those from comparable questions relating to positive payoffs (Fredrick, Loewenstein & O'Donoghue 2002, p.363). Other studies that have found similar patterns include (Redelmeier & Heller 1993, Benzion, Rapoport & Yagil 1989, Mischel, Grusec & Masters 1969).

A fourth anomaly in the DU model concerns the empirical validity of the stationarity property of exponential discounting or what is often referred to as the *common difference effect*.[18] Empirical evidence points to a preference reversal which is not accounted for by the DU model wherein an individual may choose A over B at time t only for the same individual to choose B over A at say $t + 50$ days. Strotz (1956) highlights this anomaly when he notes that "the optimal plan (as predicted by the DU model) of the present moment is generally one which will not be obeyed, or that the individual's future behavior will be inconsistent with his optimal plan" (p. 165). In one study for example, subjects preferred to receive \$100 now over receiving \$110 the next day and the same subjects when asked to state their preference after a specified time delay preferred \$110 in 31 days over \$100 in 30 days (Fredrick, Loewenstein & O'Donoghue 2002, p.361).

One implication of the common difference effect is that it results in dynamically inconsistent behavior. Empirical evidence shows this anomaly to hold in a synchronic sense, that is subjects are asked questions at a single point in time and show the preference reversal as evaluated from that particular point in time. Fredrick, Loewenstein & O'Donoghue (2002) possess an 'implicit belief' that preference reversals in a synchronic sense lead to preference reversals in a diachronic sense. Taking the study above as an example, the same subjects who prefer \$110 in 31 days over \$100 in 30 days if brought back to the laboratory in 30 days time will prefer \$100 at that time

[16]These discount rates are not explicitly given. However, we can impute them easily in the following way: $\$100 = \$157 * e^{-0.4511*1} = \$133 * e^{-0.2851*1}$. The general formula is of course $PV = FV * e^{-rt}$ and given that $t = 1$, it follows on that $r = -ln(\frac{PV}{FV})$.

[17]The studies cited were originally part of Loewenstein G.,"The Weighting of Waiting: Response Mode Effects in Intertemporal Choice," working paper, Center for Decision Research, University of Chicago, 1988.

[18]For a review of the property, refer to section 2.1.2

over \$110 one day later. They note that that this is contrary to hyperbolic discounting. However, to the extent that subjects anticipate such diachronic reversals and seek ways to avoid them such as through some commitment mechanism, the preference of commitment in itself on their part may be interpreted as evidence for hyperbolic discounting (p. 361). Other studies that have found the existence of preference reversals in human beings as well as animals include (Ainslie & Herrnstein 1981, Green, Fristoe & Myerson 1994, Kirby & Marakovic 1995, Green, Fischer, Perlow & Sherman 1981).

A fifth anomaly in the DU model first identified by Loewenstein (1988) relates the apparent existence of an asymmetric preference between delaying and speeding up consumption or what is commonly referred to as the *delay-speed up asymmetry*.[19]. In Loewenstein's study, subjects were willing to accept two to four times the amount of compensation for delaying a real reward by a given time interval t to $t + s$ than they were willing to give up in order to speed up consumption over the same interval, that is, $t + s$ to t (Loewenstein & Prelec 1992, p.578). In another study, subjects who were not expecting to receive a VCR for another year were on average willing to pay \$54 to receive it immediately whereas those who were expecting to receive it immediately on average demanded \$126 to delay receipt of the VCR by a year (Fredrick, Loewenstein & O'Donoghue 2002, 363). The findings constitute a framing effect that is inconsistent with the DU model. Other subsequent studies that have found similar patterns include (Shelley 1993, Benzion, Rapoport & Yagil 1989).

Finally, a sixth anomaly in the DU model that has been documented suggests that individuals show *preference over improving sequences* of outcomes. Principally, this anomaly contradicts the DU model's positive rate of time preference assumption.[20] [21] In a study by Loewenstein & Sicherman (1991), a majority of respondents expressed preference for an increasing payment sequence over a decreasing and flat sequence. In the study, it was noted that sequence took precedence over present value maximization. Thus, an increasing sequence was preferred to a decreasing sequence even though the latter yielded a higher net present value compared to the former. On average, respondents were willing to give up \$2351 to obtain their preferred payment

[19]Loewenstein's study was part of "Frames of Mind in Intertemporal Choice," Management Science, XXXIV (1988), 200-14

[20]Refer to section 2.1.1 for a review

[21]Fredrick, Loewenstein & O'Donoghue (2002) note that positive discounting is the norm in intertemporal choice studies e.g., choosing between X at τ and Y at τ' (p. 363). Thus, if $\tau' > \tau$, given a choice between receiving X at either period, positive discounting implies that one will always choose X at τ ($X > 0$).

option in stead of the present value-maximizing declining payment option. In another study by Varey & Kahneman (1992), it was found that subjects had preference for streams of decreasing discomfort to increasing discomfort eventhough taken over an interval, the overall sum of discomfort in both cases was equal (Fredrick, Loewenstein & O'Donoghue 2002, 363). Other studies that have replicated these findings include (Hsee, Abelson & Salovey 1991, Loewenstein & Prelec 1993, Chapman 2000).

Having highlighted behavioral anomalies in the DU model, we now explore alternative specifications that have been suggested in part to address these anomalies. We begin with the quasi-hyperbolic model.

2.2 The Quasi-Hyperbolic Model

The quasi-hyperbolic discount function was proposed by Phelps & Pollak (1968) in a model of intergenerational altruism. Their model took the form:

$$U = u(C_o) + \alpha\rho u(C_1) + \alpha\rho^2 u(C_2) + \alpha\rho^3 u(C_3) + ..., \quad 0 < \alpha < 1, \;\; 0 < \rho < 1 \tag{11}$$

where α reflects time preference or "myopia" and all other variables as previously defined under the DU model. (p.186)

2.2.1 Analysis of Phelps and Pollak's specification

An analysis of (11) reveals that it is identical to the DU model in (2) when $\alpha = 1$. To interpret the role of α in Phelps and Pollak's specification, we can rewrite (11) in the following way:

$$U = u(c_o) + \alpha[\rho u(c_1) + \rho^2 u(c_2) + \rho^3 u(c_3) + ...]. \tag{12}$$

From (12), we can interpret α as a weight factor applicable to all future periods. For example, taking the case where $\alpha < 1$, say $\alpha \simeq 0.33$ and $\rho \simeq 1$, we have:

$$\{1, \alpha\rho, \alpha\rho^2, \alpha\rho^3, ...\} = \{1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, ...\}$$

$$\Rightarrow U = u(c_o) + \frac{1}{3}[u(c_1) + u(c_2) + u(c_3)...] \tag{13}$$

From (13), we can derive some properties of a quasi-hyperbolic function. We note that:

- Relative to the present period, all periods in the future are worth less (weight $\frac{1}{3}$).
- Most of the discounting (in our case all) occurs between the present and the immediate future.
- Between future periods, there is little (in our case no) additional discounting.

Thus, the quasi-hyperbolic model partly addresses one of the suggested behavioral anomalies in the DU model, that is, hyperbolic discounting. By introducing the variable α, Phelps & Pollak were able to capture the extreme impatience individuals appear to have for payoffs that occur in the very near future. Since as we noted above, $\alpha \in (0, 1)$, it follows on that the smaller the value of α, the greater is the effective discount rate applicable to the immediate future.

$$\alpha \to 0 \Rightarrow \alpha\rho \to 0 \Rightarrow \alpha\rho u(c_1) \to 0 \tag{14}$$

From (14), we observe that if α is arbitrarily close to zero reflecting the case of *extreme myopia*, then the agent assigns a minute value to all payoffs that occur after the present reflecting very high discounting of the near future. The model still assumes a positive rate of time preference since the future is still never valued more than the present, that is, $\alpha < 1, \ \forall\, t$.

Another similarity to the DU model is that the change in discounting after the first period is constant and equal to ρ irrespective of the value assigned to α:

$$DU\ model;\ \frac{\rho^3}{\rho^2} = \rho. \quad quasi - hyperbolic\ model: \frac{\alpha\rho^3}{\alpha\rho^2} = \rho. \tag{15}$$

Hence, by virtue of its similarity to the DU model, the quasi-hyperbolic functional form retains the mathematical tractability of exponential discounting at the same time capturing many of the qualitative implications of hyperbolic discounting (Fredrick, Loewenstein & O'Donoghue 2002, 366)[22].

However, unlike the DU model, the quasi-hyperbolic function results in an agent having time-inconsistent preferences since ρ, the marginal rate of substitution between t and $t+1$ from the view point of any past period is replaced by $\alpha\rho$ at t. To illustrate this point, let us revert to our example above. Taking equation (13), we note that at a given time t, the agent will prefer to be patient between the subsequent future periods $t+1$ and $t+2$, i.e.,

$$U_t = u(c_t) + \frac{1}{3}[u_{c_t+1} + u_{c_t+2} + u_{c_t+3} + ...] \tag{16}$$

But at $t+1$, the agent is no longer patient between $t+1$ and $t+2$ but will have a 'present biased' time preference[23] implying:

$$U_{t+1} = u(c_{t+1}) + \frac{1}{3}[u_{c_t+2} + u_{c_t+3} + u_{c_t+4} + ...] \tag{17}$$

instead of,

$$U_{t+1} = \frac{1}{3}[u_{c_t+1} + u_{c_t+2} + u_{c_t+3} + ...]. \tag{18}$$

Strotz (1956) suggests two ways that an individual can overcome this namely through

[22]For a review of mathematically desirable properties of exponential discounting, refer to section 2.1.2

[23]the term is used by O'Donoghue & Rabin (1999) to emphasize the heavy discounting of the near future.

pre-commitment and consistent planning (p 165).[24]

2.2.2 Laibson's specification

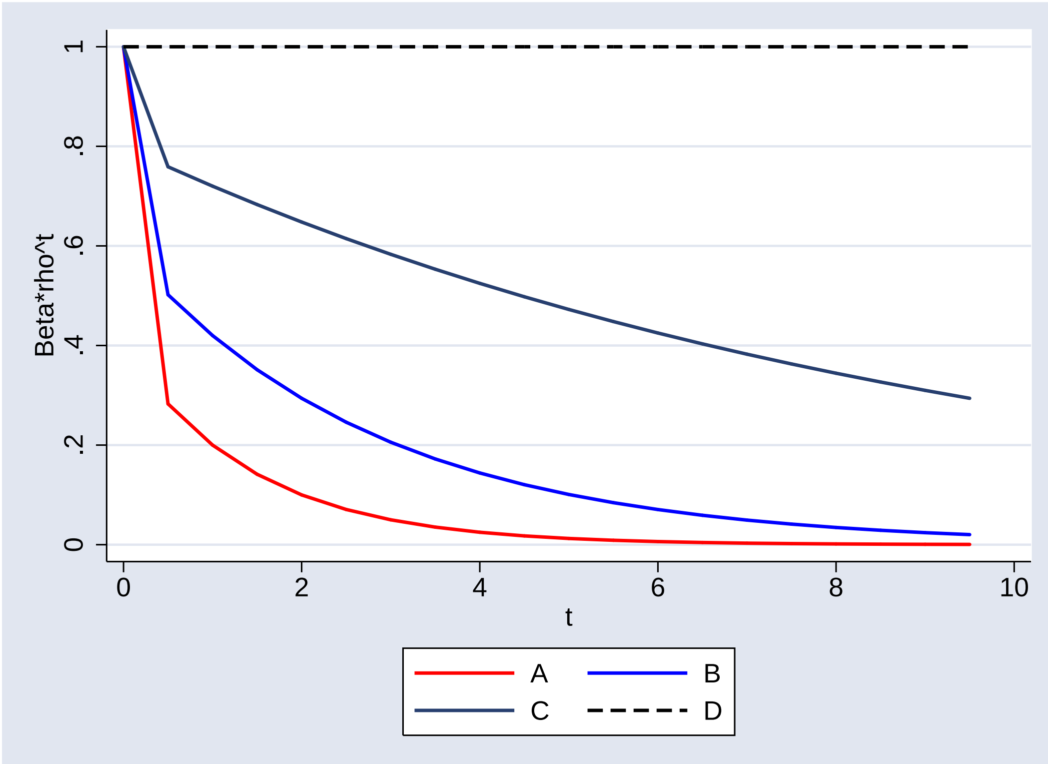
Further credit for the development and popularization of the quasi-hyperbolic functional form is due to Laibson (1997). In his paper, Laibson examines the role of illiquid assets, such as real estate, as an imperfect form of commitment emphasizing how an individual could restrict overconsumption by holding her wealth in these assets.

He expresses the model as a set of discrete values, $\{1, \beta\rho, \beta\rho^2, \beta\rho^3, ...\}$. Thus, the discount function is such that,

$$D(t) = \begin{cases} 1 & \text{if } t = 0\,, \\ \beta\rho^t & \text{if } t > 0\,. \end{cases} \tag{19}$$

Based on empirical evidence, Laibson further suggests that β should be calibrated in the interval $(0, \frac{2}{3})$ assuming that ρ is close to unity (p. 542). We plot a quasi-hyperbolic function in Figure 2 for different combinations of $\beta\rho$. Additionally, we include the theoretical limit case wherein $\beta = 1$ and $\rho = 1$ implying zero discounting. This is represented by the dashed line in the figure.

[24]Here, we should mention Strotz's (1956) paper which was the first to challenge the DU model. In his paper, he considers an individual who chooses a plan of consumption over a future time period that maximizes her present utility subject to a budget constraint. Given that the individual is free to reconsider her plan at a later date, Strotz asks whether she will carry through her initial plan at that future point in time and concludes that "the individual's future behavior will be inconsistent with her optimal plan" (p 165). Strotz also notes that any discount rate other than exponential would lead to time-inconsistent preferences.



(A: $\beta = 0.4, \rho = 0.5$, B: $\beta = 0.6, \rho = 0.7$, C: $\beta = 0.8, \rho = 0.9$).

Figure 2: Quasi-hyperbolic discount function ($\beta\rho^t$ *versus* t)
$(u(c_1) = u(c_2) = ... = u(c_\infty) = 1)$

2.3 Hyperbolic Models

Hyperbolic models incorporate two key behavioral aspects highlighted in the literature namely:

- extreme impatience for payoffs that occur in the immediate future,
- declining discount rates over time.

Chung & Herrnstein (1961) were first to propose that results from animal behavior experiments could be characterized by hyperbola-like functions. Their conclusions implicit in their “matching law” were later empirically shown to apply to human subjects as well (Laibson 1997, 449).[25]

Several hyperbolic functional forms have been proposed and we review three of the most prominent in the following sections starting with Ainslie (1975).

2.3.1 Ainslie's functional form

Ainslie (1975) suggested the following functional form:

$$D(t) = \frac{1}{t} \tag{20}$$

where $D(t)$ is the discount function and t is the length of the delay. Figure 3 exhibits a plot of this function.

An analysis of the hyperbolic discount function in (20) reveals that the value of the function falls as the time delay increases or $D(t)$ is strictly monotone decreasing.

$$D'(t) = -\frac{1}{t^2} < 0 \tag{21}$$

Taking the limit cases, we note:

$$\lim_{t \to 0} D(t) = \infty \tag{22}$$

$$\lim_{t \to \infty} D(t) = 0 \tag{23}$$

These properties thus account for two anomalies inherent in the DU model, that is, declining discount rates over time and the common difference effect. However, the other anomalies such as the magnitude effect, sign effect, preference for improving sequences and delay-speed up asymmetry are not accounted for by this specification.

[25]Richard Herrnstein's matching law is a quantitative relationship between relative rates of response and relative rates of reinforcement in two (or more) simultaneously available schedules of reinforcement. Herrnstein formulated the law following an experiment with pigeons. Formally, it can be expressed $P_1/(P_1 + P_2) = R_1/(R_1 + R_2)$ where P_1 andP_2 represent rates of responses on two schedules yielding rates of reinforcement R_1 and R_2 respectively.

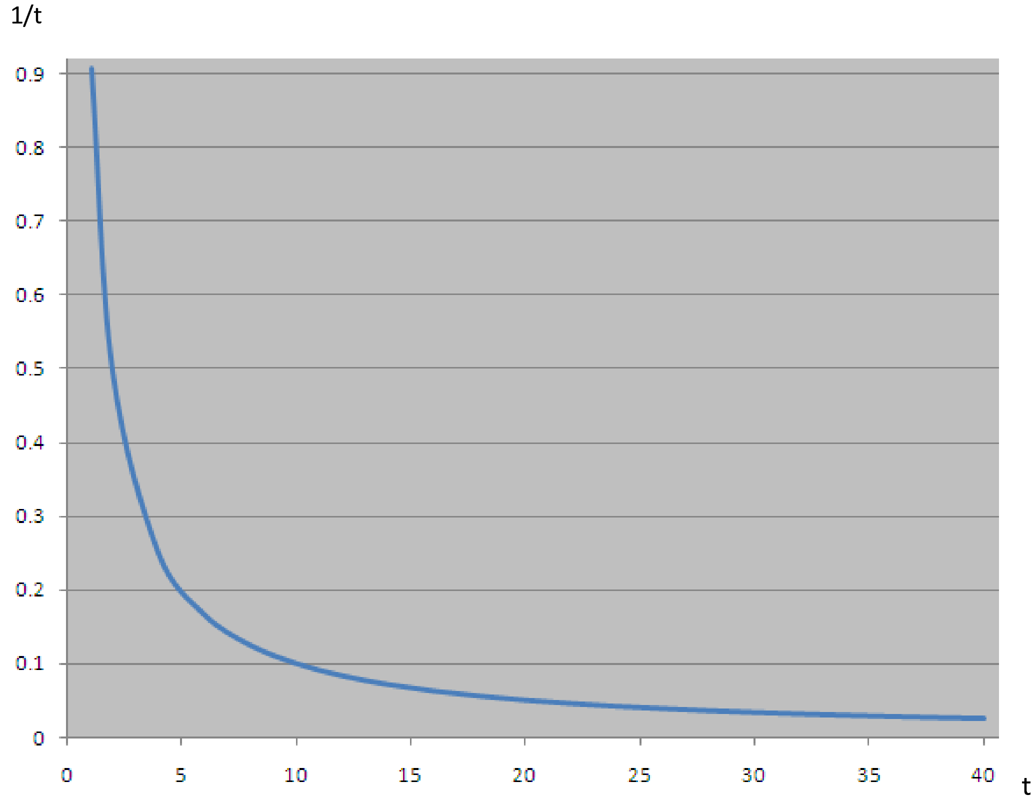


Figure 3: Hyperbolic discount function $D(t) = \frac{1}{t}$

2.3.2 Herrnstein and Mazur's functional form

Herrnstein (1981) and Mazur (1987) proposed the following functional form:

$$D(t) = \frac{1}{1+kt} \tag{24}$$

where k is the discount rate and the other variables as previously defined.

Unlike Ainslie's (1975) specification wherein the discount rate is not explicit, the above functional form sets out a clear relationship between the discount function and the discount rate. Analytically, this is easily established below:

$$\frac{d}{dk}\left(\frac{1}{1+kt}\right) = -\frac{t}{(1+kt)^2} < 0, \;\; t > 0, \;\; k > 0 \tag{25}$$

and similarly, with respect to a marginal time delay,

$$\frac{d}{dt}\Big(\frac{1}{1+kt}\Big) = -\frac{k}{(1+kt)^2} < 0, \;\; t > 0, \; k > 0 \tag{26}$$

It is evident from (25) and (26) that the specification implies an inverse relationship between the the discount function and both k and t thus accounting for the declining discount rates anomaly.

2.3.3 Loewenstein and Prelec's functional form

Loewenstein & Prelec (1992) suggested the following hyperbolic functional form:

$$D(t) = \frac{1}{(1+\alpha t)^{\gamma/\alpha}} \quad \alpha, \; \gamma \; > 0 \tag{27}$$

where α is the extent to which the function deviates from exponential discounting and γ is a time preference parameter and the other variables as previously defined.

Given the constraints placed on α and γ, that is, positive rate of time preference and a positive departure from exponential discounting, we still note an inverse relationship between the discount function and a time delay, t.

$$\frac{d}{dt}\Big(\frac{1}{(1+\alpha t)^{\gamma/\alpha}}\Big) = -\gamma(1+\alpha t)^{1-\frac{\gamma}{\alpha}} \; < \; 0 \tag{28}$$

since $\gamma \; > 0, \;\; (1+\alpha t)^{1-\frac{\gamma}{\alpha}} \; > 0$.

Streich & Levy (2007, p.209) note that this specification by Loewenstein & Prelec probably provides the best fit with experimental data but acknowledge its complex formulation citing that students of intertemporal choice prefer to use Herrnstein and Mazur's functional form.

2.3.4 Concluding Remarks

A limitation of hyperbolic discount functions including in all three specifications above is their non-convergence property as it relates to summing an infinite series of payoffs. Thus, economists and social scientists alike face great difficulty when trying to use these functions due to their non-tractability and more often than not choose the exponential form even though it appears not to be empirically valid. This property will be highlighted later in our analysis. Additionally, out of the six main DU model anomalies discussed in Section 2.1.3, only two are accounted for by the hyperbolic functional forms in Section 2.3. These are the declining discount rates anomaly and the common difference effect. The quasi-hyperbolic specification by Phelps & Pollak (1968) and Laibson (1997) only partly accounts for these two anomalies.

Nonetheless, even though hyperbolic models fail to account for a majority of the DU model's behavioral anomalies, there is still huge interest among researchers on the topic. A search on Science Direct, one of the world's largest online collections of published scientific research (key word: hyperbolic discounting) shows an over 800% increase in the number of published articles in the past decade alone indicating that developments in the subject area are still a work in progress. Figure 10 in the Appendix - Section 6.3 C1 shows the trend over the past century.

The challenge for economists and social scientists thus is to develop a discount function which adequately addresses each of the behavioral anomalies in the DU model. This is complicated by the fact that a number of the anomalies are psychological in nature. Table 1 relates each of the suggested non-exponential models of discounting and the DU model anomalies. We now consider an IPD model and examine the implications of changing the discount rate from the standard exponential form.

	Non-Exponential Model of Discounting	A	B	C	D
	DU Model Anomaly				
1.	Declining discount rates	*	X	X	X
2.	Magnitude of Payoff effect	-	-	-	-
3.	Sign effect	-	-	-	-
4.	Common difference effect	*	X	X	X
5.	Delay-Speed up asymmetry	-	-	-	-
6.	Preference over improving sequences of outcomes	-	-	-	-

A	Phelps & Pollak (1968) and Laibson (1997) quasi-hyperbolic model
B	Ainslie (1975) hyperbolic model
C	Herrnstein (1981) and Mazur (1987) hyperbolic model
D	Loewenstein & Prelec (1992) hyperbolic model
X	Accounts for anomaly
*	Partly accounts for anomaly
-	Does not account for anomaly

Table 1: Non-exponential models of discounting and DU model anomalies

3 An IPD Model under non-exponential discounting

3.1 Model description

We model the interaction between 2 firms in the form of an infinite IPD. Both firms produce a homogeneous non-durable and non-storable good which lasts for one period only. Demand for the good is of finite magnitude and both firms are aware of the aggregate industry demand curve. For simplicity, we assume the non-existence of production costs implying that each firm can satisfy the industry demand on its own. No alternative producers for the good exist and there are no substitutes. Hence, demand for the good is inelastic. We further assume that the firm's decision variable is the price and disregard quantity for purposes of this analysis. Each firm is owned by an independent entity whose key objective is to maximize the net present value of an infinite stream of future payoffs. The payoffs of one firm not only depend on the price it charges for the good but also the price charged by the other firm. Thus, the firm can choose between two actions namely cooperate or defect. We give a precise definition of these terms:

Definition $D-1$: *Cooperate - firm i, $i \in \{1,2\}$ is said to cooperate if it sells the good at time t for a given price which was pre-agreed at time $t-1$ by both, itself and the other firm.*

Definition $D-2$: *Defect - firm i is said to defect if it sells the good at time t for a lower price than that which was pre-agreed at time $t-1$ by both, itself and the other firm.*

We assume that consumers of this good and the firms are instrumentally rational (IR). This means that the consumers, for example, will buy the good from the firm charging a lower price and shun the firm charging a higher price. We represent this information in the form of a normal form game using standard notation below:

1. The set of players (firms) is given by

   $I \equiv \{1,2\}$.

2. The actions available to firm i, $i \in I$ denoted a_1^i and a_2^i:

a_1^i = cooperate, a_2^i =defect.

3. Firm i's action set is denoted by A^i : $A^i = \{a_1^i, a_2^i\}$.

4. The outcome of the game, denoted a; there are four possible outcomes:

$a^1 = \{cooperate, cooperate\}$,

$a^2 = \{cooperate, defect\}$,

$a^3 = \{defect, cooperate\}$,

$a^4 = \{defect, defect\}$.

Thus, having established preliminary notation for the model, we now introduce a notation for the payoffs of firm i under each possible outcome. We consider the first outcome wherein both firms cooperate. In this case, both firms will sell a specific quantity of the good at a pre-agreed price earning a cooperative profit which we denote C. In the case where firm i cooperates and the other firm defects, consumers will shun firm i and buy the good from the other firm charging a lower price. In this case, firm i nets a zero profit denoted Z and the other firm earns an abnormal profit denoted A. Finally, in the case where both firms defect, then both earn a subnormal profit which we denote S.[26] The following inequality holds with respect to the magnitudes of these payoffs:

$$A > C > S > Z^{27}.$$

Using standard notation, we represent the payoffs for firm 1 under the four possible outcomes previously listed:

$\pi^1(a^1) = C$,

$\pi^1(a^2) = Z$,

$\pi^1(a^3) = A$,

$\pi^1(a^4) = S$.

[26]For purposes of this model, we assume that deviation is to a specific price below that which was agreed by both firms. This means that if both firms deviate, they still end up charging the same price and not some other arbitrary lower price which may differ.

[27]Given this payoff structure, if we replace *cooperate* with *confess* and *defect* with *not confess* we obtain the famous Prisoners' Dilemma game originally framed by Merrill Flood and Melvin Dresher working at RAND in 1950 and later formalized by Albert W. Tucker.

Similarly, for firm 2 we have:

$$\pi^2(a^1) = C,$$

$$\pi^2(a^2) = A,$$

$$\pi^2(a^3) = Z,$$

$$\pi^2(a^4) = S.$$

We represent the normal form game in matrix form below:

Firm 2

Firm 1

	cooperate	defect
cooperate	C C	Z A
defect	A Z	S S

Table 2: 2-Firm PD game

The matrix above contains all the data necessary to define our 2-firm PD game. We suppose that the two firms interact an infinite number of times (i.e., $T = \infty$). Thus, as our 2 firms go to the market at the start of each period, they are confronted with the situation as is represented in Table 2 and this is repeated an infinite number of times.

Before proceeding to the analysis of the infinitely IPD model under different forms of discount functions, we first limit our analysis to one kind of strategy called *trigger strategies*. We use Shy's (1995) definition and adapt it to our model. Hence, in this class of trigger strategies, firm i cooperates in period t (playing a^i_τ = cooperate) as long as it and the other firm cooperated in period $\tau - 1$. However, if either firm i or the other firm played defect in period $\tau - 1$, then firm i "pulls the trigger" and plays defect forever or, a^i_τ = defect for every $t = \tau, \tau + 1, \tau + 2, ...$ Formally,

Definition $D - 3$: *Firm i is said to be playing a* ***trigger strategy*** *if for every period* $\tau, \tau = 1, 2, ...,$

$$a^i_\tau = \begin{cases} cooperate\, as\, long\, as\, a^i_\tau \ = \ a^j_\tau \ \ = \ cooperate\ \forall\ t = 1,\ ...,\ \tau - 1 \\ defect\ otherwise \end{cases} .$$

In other words, firm i plays *cooperate* as long as both, itself and the other firm, have not deviated from this outcome. However, in the event that a firm deviates even once, firm i punishes the deviator by playing $defect$ forever.

3.2 Analysis of equilibrium conditions in the 2-Firm IPD Model.

We now analyze the IPD model under three distinct discount functions namely exponential, quasi-hyperbolic and hyperbolic discounting. Specifically, we seek to establish conditions under which cooperation emerges as a subgame perfect equilibrium (SPE) under each of these specifications. It is worth noting that in a one shot PD game, discounting plays no role and the outcome where each firm $defects$ and charges a lower price for the good is a unique Nash equilibrium. By extension, if the game is repeated a finite number of times and the players are aware of this fact, then discounting plays no role either and the non-cooperative outcome still remains a unique SPE. One proof of this involves using backward induction (see Shy (1995, p.30) and Appendix - Section 6.2 B1).

The discount factor enters the frame if the IPD is played an infinite number of times. The main reason for such an analysis is the fact that under certain circumstances, outcomes which are not SPE's under a one shot game (namely *cooperate*) can emerge if the game is repeated an infinite number of times or if there is uncertainty with regards to the number of repetitions. Our analysis thus seeks to investigate specific conditions under which cooperation emerges but unlike most such analysis which implicitly use constant-rate discounting, we introduce two more specifications which appear to have some empirical basis as has been outlined in the literature. Thus, in our analysis the exponential case acts as the base.

Proposition 1. ***Under exponential discounting, the outcome where both players play their trigger strategy is SPE if*** $\rho \geq \frac{\boldsymbol{A - C}}{\boldsymbol{A - S}}$.

Proof. We first consider the case where firm 1 cooperates in all periods. In this scenario, it will earn a payoff of C in each period since $\pi^1(a^1) = C$.

PV of infinite cooperation:

$$C + \rho C + \rho^2 C + \rho^3 C + ... = C\Big(\frac{1}{1-\rho}\Big) \tag{29}$$

since

$$1 + \rho + \rho^2 + \rho^3 + ... = \sum_{t=0}^{\infty} \rho^t = \frac{1}{1-\rho}.$$

We consider the second possible case where firm 1 $defects$ in one period. Under the trigger strategy defined in $D-3$, firm 2 would deviate in all subsequent periods and play $defect$. Firm 1's payoff would thus be A in the initial period and S in each period thereafter since $\pi^1(a^3) = A$ and $\pi^1(a^4) = S$.

PV of deviation under trigger strategy:

$$A + \rho S + \rho^2 S + \rho^3 S + ... = A + S\Big(\frac{1}{1-\rho}\Big) - S \implies$$

$$= S\Big(\frac{A}{S} + \frac{1}{1-\rho} - 1\Big). \tag{30}$$

Comparing (29) and (30) yields the conclusion that deviation is not beneficial for firm 1 if:

$$\rho \geq \frac{A-C}{A-S}. \tag{31}$$

Proof: See Appendix - section 6.2 B2

Therefore, given the existence of exponential firms in our model, the discount factor must satisfy (31) for cooperation to emerge as an SPE. If the discount factor is below this level, both firms will find it beneficial to defect at all periods.

Proposition 2. ***Under quasi-hyperbolic discounting, trigger strategies con-***

stitute an SPE if $\rho \geq \frac{\boldsymbol{A - C}}{\beta(\boldsymbol{C - S}) + \boldsymbol{A - C}}$.

Proof. We use Phelps & Pollak (1968) specification of a quasi-hyperbolic discount function to analyze the model.[28] Taking the case where firm 1 cooperates in all periods, we have:

$$C + \beta\rho C + \beta\rho^2 C + \beta\rho^3 C + ... \implies$$

$$= \beta C\Big(\frac{1}{\beta} + \frac{1}{1-\rho} - 1\Big) \tag{32}$$

since

$$\beta + \beta\rho + \beta\rho^2 + \beta\rho^3 + ... = \beta\sum_{t=0}^{\infty}\rho^t = \beta\Big(\frac{1}{1-\rho}\Big).$$

If firm 1 defects in one period, we again note that under the trigger strategy defined in $D-3$, firm 2 would deviate and play $defect$ indefinitely or in other words "pull the trigger" and not cooperate forever. In this case, we have:

$$A + \beta\rho S + \beta\rho^2 S + \beta\rho^3 S + ... \implies$$

$$= A + \beta S\Big(\frac{1}{1-\rho} - 1\Big). \tag{33}$$

Comparing (32) and (33) yields the conclusion that deviation is not beneficial for firm 1 if:

$$\rho \geq \frac{A-C}{\beta(C-S) + A - C} \equiv \rho^*(\beta) \tag{34}$$

Proof: See Appendix - Section 6.2 B3

[28]For details on the specification, refer to Chapter 2, section 2.2

Differentiating (34) with respect to β, we note:

$$\frac{d}{d\beta}\,\rho^*(\beta) < 0 \qquad \frac{d^2}{d\beta^2}\,\rho^*(\beta) > 0 \tag{35}$$

Proof: See Appendix - Section 6.2 B4 & B5

Here, we observe that ρ^* is decreasing in β. We plot $\rho^*(\beta)$ versus β in Figure 4.

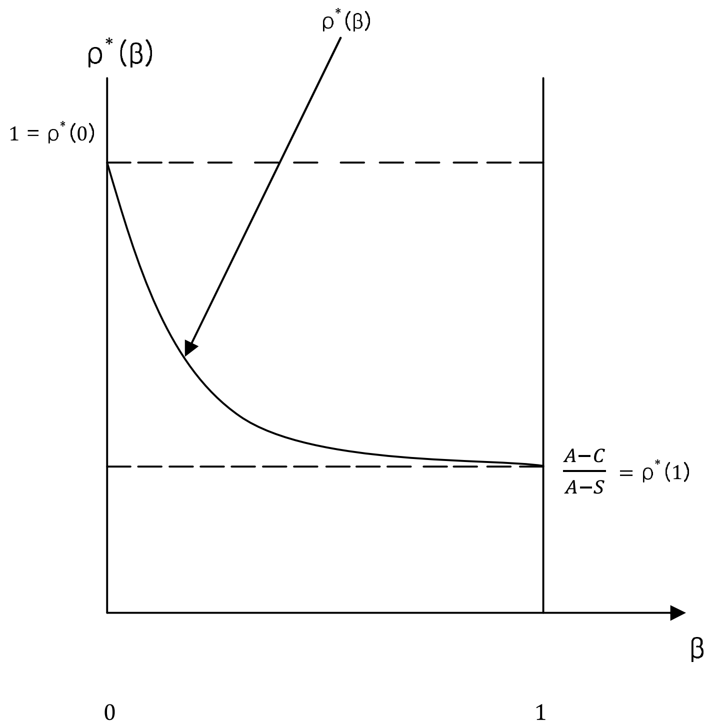


Figure 4: $\rho^*(\beta)\ versus\ \beta$

Taking the limits:

$$if\ \beta\ \rightarrow\ 1,\ \rho^*\ \rightarrow\ \frac{A-C}{A-S}\ \ and\ \ if\beta\ \rightarrow\ 0,\ \rho^*\ \rightarrow\ 1 \tag{36}$$

with

$$\frac{A-C}{A-S}\ <\ 1$$

Proof: See Appendix - Section 6.2 B6

From (36), we see that the discount factor under quasi-hyperbolic discounting necessary to sustain cooperation as an SPE equals that of exponential discounting if β is equal to 1. However, if $\beta < 1$, we require a higher ρ for cooperation to emerge. The implication therefore is that if preferences of the firms in our model are similar to what empirical evidence suggests, then the discount factor needs to be higher than that which exponential discounting implies for cooperation to emerge.

Proposition 3. ***Under hyperbolic discounting, there does not exist a level of ρ for which trigger strategies constitute SPE.***

Proof. Taking the case where firm 1 cooperates in all periods, under hyperbolic discounting, we have:

$$C + \beta_1 \rho C + \beta_2 \beta_1 \rho^2 C + \beta_3 \beta_2 \beta_1 \rho^3 C + ... + \beta_n \beta_{n-1} \beta_{n-2} ... \beta_2 \beta_1 \rho^n C + ... \tag{37}$$

$$\beta_n > \beta_{n-1} > \beta_{n-2} > \ ... \ (n \in \mathbb{N}).$$

Under the second strategy where firm 1 deviates in one period, firm 2 punishes firm 1 by playing $defect$ in all subsequent periods as a result of the trigger strategy yielding:

$$A + \beta_1 \rho S + \beta_2 \beta_1 \rho^2 S + \beta_3 \beta_2 \beta_1 \rho^3 S + ... + \beta_n \beta_{n-1} \beta_{n-2} ... \beta_2 \beta_1 \rho^n S + ... \tag{38}$$

Comparing (37) and (38), it is apparent that the series diverges and we are not able to specify a level for ρ at which deviation is not beneficial.[29] Thus, we note that an analytical solution is not possible due to infinitely declining discount rates. This property as was noted earlier in Section 2.3.4 is a limitation of a purely hyperbolic discount function as it relates to the analysis of an infinite series of payoffs.

[29] In our specification of a hyperbolic function, we drop the constraint $\beta_i \in (0, 1)$ imposed under quasi-hyperbolic discounting to account for falling discount rates into the future.

4 Applications to wage theory: Efficiency wages

Efficiency wages act as a mechanism to deter cheating among employees. Cheating may take various forms including officers of a police department accepting bribes, a captain of an oil tanker getting intoxicated while on the job, analysts in an investment bank recommending investment options without exercising diligence, salespeople in a marketing division entertaining family and friends instead of customers, etc. (Milgrom & Roberts 1981, 250).

4.1 The Shapiro-Stiglitz shirking model

Shapiro & Stiglitz (1984) efficiency wage model forms a basis in which an organization can determine at what level it will be beneficial to set an employee's wage in order to deter cheating. In their model, a worker who does not cheat or "shirk" is defined as one who performs at the customary effort level. She will receive a wage denoted w and will retain her job until exogenous factors result in separation. On the other hand, an employee who shirks and gets caught will lose her job and will receive unemployment benefits denoted $\bar{w}$ with effort e being equal to zero ($e = 0$). There is a q probability that the employee will be caught if she shirks. The probability that a worker will be separated from her job through some external factors such as relocation is denoted b and is assumed to be exogenous. Thus, the model states that the employee will opt not to shirk if the expected lifetime utility of an employed non-shirker denoted (V_E^N) is greater or equal to the expected lifetime utility of an employed shirker denoted (V_E^S). In formal terms:

$$rV_E^S = w + (b + q)(V_u - V_E^S) \tag{39}$$

$$rV_E^N = w - e + b(V_u - V_N^S) \tag{40}$$

Where V_u is the expected lifetime utility of an unemployed individual and r is the employee's pure rate of time preference. Thus, in terms of the lifetime utilities, (39) and (40) are rearranged to get:

$$V_E^S = \frac{w + (b + q)V_u}{r + b + q} \tag{41}$$

$$V_E^N = \frac{(w-e)+bV_u}{r+b} \tag{42}$$

and the non-shirking condition (NSC) being:

$$V_E^N \geq V_E^S \tag{43}$$

Finally, in terms of the wage rate, they conclude that:

$$w \geq rV_u + (r+b+q)\frac{e}{q} \equiv \hat{w} \tag{44}$$

where $\hat{w}$ is the efficiency wage.

Thus, the model states that the efficiency wage rate is a function of the interest rate, that is, the higher the interest rate, the greater should be the efficiency wage.

4.2 Shapiro-Stiglitz model under non-exponential discounting

One implication as it relates to the discount function is that a firm which unknowingly or otherwise believes that its employees are exponential discounters (when in fact they are quasi-hyperbolic or hyperbolic discounters) would consistently set the wage rate at a lower level than that which is required to effectively deter shirking. To illustrate this, let us assume that a firm is able to divide an employee's work contract into 4 distinct time periods which we denote τ^1, τ^2, τ^3 and τ^4, where τ^1 is the first period of the contract, τ^2 is the second period and so on. If we take the average duration of an employment contract to be 20 years, then each period lasts for 5 years representing the very near future, the near future, the far future and the very far future. We further assume that all other variables except the interest rate are constant in every period and under each specification. In terms of the real wage, (44) implies that the efficiency wage will be constant over time given that the interest rate is also constant. Thus, an identifying feature of a real efficiency wage set under assumption of an exponential discount function is that it is constant over time according to the linear equation.[30] On the other hand, in the quasi-hyperbolic and hyperbolic cases, since there is heavy discounting of the very near future, in our case, period τ^1, we argue that in order to be effective, the real wage rate during this period should be significantly higher than that set under assumption of an exponential discount function. Our argument is by way of illustration and we make a few assumptions.

Restating the 3 discount functions:

$$Exponential: \quad 1 + \rho + \rho^2 + \rho^3 + ...$$

$$Quasi-hyperbolic: \quad 1 + \beta\rho + \beta\rho^2 + \beta\rho^3 + ...$$

$$Hyperbolic: \quad 1 + \beta_1\rho + \beta_2\beta_1\rho^2 + \beta_3\beta_2\beta_1\rho^3 + ... \;\; (\beta_n > \beta_{n-1} > \beta_{n-2})$$

We note that as we move from the present to the first period, the change in discounting is equal to ρ under exponential discounting, $\beta\rho$ under quasi-hyperbolic discounting and $\beta_1\rho$ under hyperbolic discounting.[31] We first assume that the firm under the belief that employees are exponential discounters determines the employees' pure rate of time preference or r to be 0.1 and goes ahead to set the efficiency wage based on this estimate. This implies that ρ is equal to approximately 0.91 under assumption of perfect capital markets. Recall that from (3), we established the following relationship between the discount rate and the discount factor:

$$\rho = \frac{1}{1+r} \Rightarrow 0.91 = \frac{1}{1+0.1} \tag{45}$$

Thus, the exponential firm having estimated that employees possess a pure rate of time preference of 0.1 will apply a discount factor of 0.91 to each of the 4 periods,

[30] Here, we stress 'real' so as to ignore the effect of external factors that affect the nominal wage rate principally among them, inflation.

[31] It is an elementary process to obtain these results: Since at present and under each discount function we have a value of 1, we look at the rate at which next period's payoffs are discounted and divide through to obtain the change: exponential: $\frac{\rho}{1} = \rho$; quasi-hyperbolic: $\frac{\beta\rho}{1} = \beta\rho$; hyperbolic: $\frac{\beta_1\rho}{1} = \beta_1\rho$.

$\tau^1, \tau^2, \tau^3, \tau^4$. In the case of the quasi-hyperbolic firm, it will have to estimate a level of β which accounts for the extreme impatience employees possess for payoffs that occur in the very near future, in our case, that being τ^1. As noted earlier in the Section 2.2 under quasi-hyperbolic discounting, based on statistical tests by econometricians, Laibson (1997) suggests that β lies in the interval $(0, \frac{2}{3})$. Thus, in our case, we assume that the quasi-hyperbolic firm estimates the value of β to be 0.6 and goes ahead to set an efficiency wage based on this estimate. Having previously established a value for ρ, we have:

$$\beta\rho = 0.6 \cdot 0.91 = 0.55 \; . \tag{46}$$

What we immediately notice is that there is a huge difference between the two factors applicable to period τ^1. The exponential firm will set it's efficiency wage based on a factor of 0.91 whereas the quasi-hyperbolic firm will apply a factor of 0.55. The effective discount rate implied by a factor of 0.55 can be backed out from the general formula[32]

$$r = \frac{1}{\rho} - 1 \Rightarrow r = \frac{1}{0.55} - 1 = 0.82 \; . \tag{47}$$

Thus, taking the two values for r, that is, 0.1 and 0.82 for the exponential firm and quasi-hyperbolic firm respectively, it is clearly evident from these figures that the quasi-hyperbolic firm's discount rate applicable at τ^1 is significantly higher than that of an exponential firm. Similarly, in the case of a hyperbolic firm, r would still be far greater than 0.1 to account for the steep discounting of the very near future. In terms of the efficiency wage that has to be set under the Shapiro & Stiglitz shirking model, we have to examine equation (44) to interpret the impact of having different values for r.[33]

[32]Here, we are not suggesting that ρ is equal to 0.55 since this is the value of $\beta\rho$. What we are trying to establish is the 'effective' discount rate given the additional effect of β.

[33]In terms of how we interpret the differences in the discount rates obtained above, we have to make an assumption regarding a specific wage that at time 0 generates sufficient utility to deter an agent from shirking. If we randomly assign a value of \$10,000 to represent this amount, then we note that at τ^1, the real wage that the exponential firm has to pay to deter cheating is \$11,000 whereas the quasi-hyperbolic firm will have to pay a real wage of \$18,200 to achieve the same purpose. We discount in the usual way to obtain these values: $\$10,000 = \$11,000/(1 + 0.1)$ and $\$10,000 = \$18,200/(1 + 0.82)$.

Proposition 4. *At* $\tau^1, r_h \geq r_{qh} > r_e \ \Rightarrow \hat{w}_h \geq \hat{w}_{qh} > \hat{w}_e$.

Proof. If we denote the quasi-hyperbolic discount rate as r_{qh}, the exponential discount rate as r_e and the hyperbolic discount rate as r_h, comparing the efficiency wages at τ^1 across all 3 cases, we have:

$$r_e V_u + (r_e + b + q)\frac{e}{q} \equiv \hat{w}_e \tag{48}$$

$$r_{qh} V_u + (r_{qh} + b + q)\frac{e}{q} \equiv \hat{w}_{qh} \tag{49}$$

$$r_h V_u + (r_h + b + q)\frac{e}{q} \equiv \hat{w}_h \tag{50}$$

Since $\beta_1\rho \ \leq \ \beta\rho \ < \ \rho \ \Rightarrow r_h \geq r_{qh} > r_e$ at τ^1. From (48), (49) and (50) we conclude:

$$\hat{w}_h \geq \hat{w}_{qh} > \hat{w}_e. \tag{51}$$

Proposition 5. *At* $\tau^2, r_h > r_{qh} = r_e \ \Rightarrow \hat{w}_h > \hat{w}_{qh} = \hat{w}_e$.

Proof. At τ^2, payoffs are discounted at the same factor under both exponential and quasi-hyperbolic discounting, that is, ρ. In the hyperbolic case, the discount factor is $\beta_2\rho$ and the wage applicable to this period is dependent on the firm's estimate of β_2.

$$quasi-hyperbolic: \ \ \frac{\beta\rho^2}{\beta\rho} = \rho, \quad hyperbolic: \ \ \frac{\beta_2\beta_1\rho^2}{\beta_1\rho} = \beta_2\rho$$

Since $\beta_2\rho < \rho \Rightarrow r_h > r_{qh} = r_e$. From (48), (49) and (50), the following holds:

$$\hat{w}_h > \hat{w}_{qh} = \hat{w}_e \tag{52}$$

Proposition 6. *At τ^3 and $\tau^4, r_h < r_{qh} = r_e \ \Rightarrow \hat{w}_h < \hat{w}_{qh} = \hat{w}_e$.*

Proof. The change in discounting in the hyperbolic case from the second to the third and from the third to the fourth periods is equal to $\beta_3\rho$ and $\beta_4\rho$ respectively. In our specification of a hyperbolic function in Section 3.2, we established the following:

$$\beta_n\rho > \beta_{n-1}\rho > \beta_{n-2}\rho \Leftrightarrow r_n < r_{n-1} < r_{n-2}$$

Since $\beta_3\rho > \rho$ and $\beta_4\rho > \rho \ \Rightarrow \ r_h < r_{qh} = r_e$ at τ^3 and τ^4 respectively. From (48), (49) and (50), we conclude:

$$\hat{w}_h < \hat{w}_{qh} = \hat{w}_e \tag{53}$$

To account for falling discount rates over time, the Shapiro & Stiglitz real wage must fall to below the exponential level at τ^3 and τ^4 under the presence of purely hyperbolic discounting employees. Analytically, this is made possible by allowing successive βs to be set at a level above 1. Table 3 shows the real wage profile across all 3 discount functions during the employment contract. The implication is that wages set under the assumption of exponential discounting may not effectively deter shirking in the presence of hyperbolic or quasi-hyperbolic discounting employees.

Time period	Shapiro-Stiglitz real wage
τ^1	$\hat{w}_e < \quad \hat{w}_{qh} \quad \geq \hat{w}_h$
τ^2	$\hat{w}_e = \quad \hat{w}_{qh} \quad < \hat{w}_h$
τ^3	$\hat{w}_e = \quad \hat{w}_{qh} \quad > \hat{w}_h$
τ^4	$\hat{w}_e = \quad \hat{w}_{qh} \quad > \hat{w}_h$

($\hat{w}_e$ - exponential wage, $\hat{w}_{qh}$ - quasi-hyperbolic wage, $\hat{w}_h$ - hyperbolic wage)

Table 3: Real wage comparison under exponential and non-exponential discounting.

4.3 Equilibrium Unemployment under non-exponential discounting

We consider the model of a firm due to Solow (1956) presented in (Blanchard & Fischer 1989, p 455). The firm's production function is given by

$$Y = sF(e(w)L), \tag{54}$$

where s reflects shifts in either technology or the relative price of the firm, L is the number of workers and the other variables as defined previously. We assume that the effort and production functions satisfy

$$e(w) = 0 \;\; for \;\; w = R > 0, \;\; e'(\cdot) \; > 0, \;\; e''(\cdot) < 0 \tag{55}$$

$$F'(\cdot) > 0 \;\; and \;\; F''(\cdot) < 0. \tag{56}$$

The firm thus maximizes profit, $sF(e(w)L) - wL$, over w and L yielding the following first-order conditions:

$$\frac{e'(w^*)w^*}{e(w^*)} = 1, \tag{57}$$

$$e(w^*)sF'(e(w^*)L) = w^*. \tag{58}$$

Under the assumptions above, it follows that wage is independent of s and only determined by (57) which states that the elasticity of effort with respect to wage is equal to one. Therefore (58) gives the level of employment and the equation states that this must be such that the marginal product of an extra worker is equal to the wage.

Reverting back to the Shapiro & Stiglitz shirking model, we introduce an additional variable and assess equilibrium unemployment under exponential and non exponential discounting. Thus, we assume that if a worker is unemployed, the probability of her becoming employed in the next period is a or she remains unemployed with probability $1 - a$. Given

$$V_u = \frac{1}{r}\Big(\frac{a}{a+b+r}\Big)(w-e) \tag{59}$$

We can insert equation (59) into (44) and get

$$w_i = \Big(\frac{a}{a+b+r}\Big)(w-e) + \Big[1 + \frac{r+b}{q}\Big]e \tag{60}$$

where w_i is the wage chosen by the firm and w is the aggregate wage. The firm thus chooses employment such that

$$sF'(L_i) = w_i. \tag{61}$$

4.3.1 General Equilibrium in the exponential case

Assuming perfect information on job options and the existence of homogeneous firms, the firm's wage is equal to the aggregate wage i.e.,

$$w_i = w, \ \forall i. \tag{62}$$

Inserting w for w_i in equation (60), we have

$$w = e + \frac{e(a+b+r)}{q}. \tag{63}$$

We note that for our segmented employment contract, the exponential discount rate r_e is constant over all 4 periods, $\tau^1, \tau^2, \tau^3, \tau^4$ implying that the exponential real wage $\hat{w}_e$ is also constant. In the steady state, the accession rate must be such that the flow into unemployment equals the flow out of unemployment or

$$bL = a(N - L) \Rightarrow a = \frac{bL}{(N - L)} \tag{64}$$

Thus, taking this condition and putting it into (63) yields the equilibrium wage given the NSC.

$$e + \frac{e\{[bL/(N - L)] + b + r\}}{q} \tag{65}$$

In the exponential case, $r = r_e \ \forall \tau$. The aggregate employment is given by the condition:

$$sF'\Big(\frac{L}{M}\Big) = w, \tag{66}$$

where M is the number of firms. We can thus plot an aggregate labor demand curve for a given level of s and the NSC that would prevail in the exponential case at all periods, $\tau^1, \tau^2, \tau^3, \tau^4$ (see Fig. 5). The equilibrium is given by E. Given that the firm rightly believes that employees are exponential discounters and sets wages at the level w^*, then the efficiency wage will serve its purpose. From the firm's point of view, it is uneconomical to pay a wage above w^* since the optimal amount of labor is obtained at this point. On the other hand, paying a lower wage would induce employees to shirk. Shapiro & Stiglitz (1984) call the form of unemployment that would prevail in the labor market given that all firms pay the efficiency wage as "involuntary unemployment". This is contrasted to search unemployment which results from the lack of information on the part of job seekers and which the model accounts for under the assumption of perfect information on job availability. Thus, involuntary unemployment is a necessary condition for an equilibrium since the lack of it implies no cost to losing one's job. In such an instance, an employee would shirk, lose his job and immediately find a new one without any consequence.

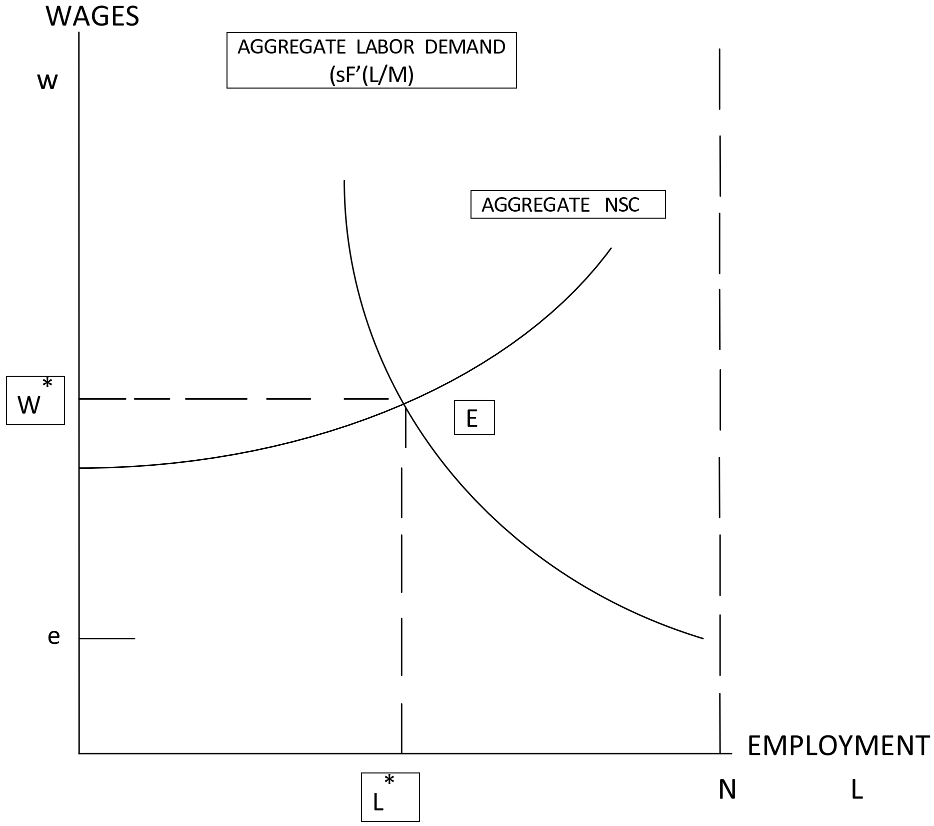


Figure 5: Exponential case: Equilibrium unemployment under efficiency wages (NSC - No Shirking Constraint)

4.3.2 General Equilibrium in the non-exponential case

In order to analyze unemployment equilibrium in the Shapiro & Stiglitz (1984) shirking model under non exponential discounting, we have to segment the employment contract into distinct periods to account for the non-stationarity property of the functions under consideration. Thus, we use the same segmentation previously established. Additionally, we assume that all firms have a homogeneous employment contract of equal duration and employment is initiated at $\tau = 0$ with no further expansion of employment possibilities throughout the duration of the contracts. Thus, any new employment opportunity would thus arise from an employee being dismissed as a result of shirking or some other exogenous event unique to an individual employee. In our analysis, we compare our findings in each case with the baseline case of exponential discounting.

At τ^1, that is, the first period of the contract, both discount rates under quasi-hyperbolic discounting (r_{qh}) and hyperbolic discounting (r_h) exceed the constant exponential discount rate (r_e). This is reflected by the initial steep fall in both discount functions implying extreme impatience for payoffs in the very near future. Therefore, from (65), it follows on that

$$e + \frac{e\{[bL/(N-L)] + b + rqh\}}{q} > e + \frac{e\{[bL/(N-L)] + b + re\}}{q} \quad and, \tag{67}$$

$$e + \frac{e\{[bL/(N-L)] + b + rh\}}{q} > e + \frac{e\{[bL/(N-L)] + b + re\}}{q} \tag{68}$$

Since the aggregate employment is not a function of r, the firm will still aim to maximize its profits by increasing employment until the marginal product of labor (MPL) is equal to the wage. Thus, the condition in (66) still prevails.

Proposition 7. *At τ^1, **the equilibrium under hyperbolic and quasi-hyperbolic discounting is at** $E^{'}$ **implying a higher efficiency wage and a higher unemployment level.***

Proof. The result is simple: From the NSC in (67) and (68), we establish that a higher r will lead to a situation of higher wages and more unemployment. Comparing to the exponential case in Figure 5, we note that a firm that sets its wages under the impression that employees are quasi-hyperbolic or hyperbolic discounters would have to set the efficiency wage at a higher level i.e., $w^{'}$ compared to w in the exponential case at τ^1.

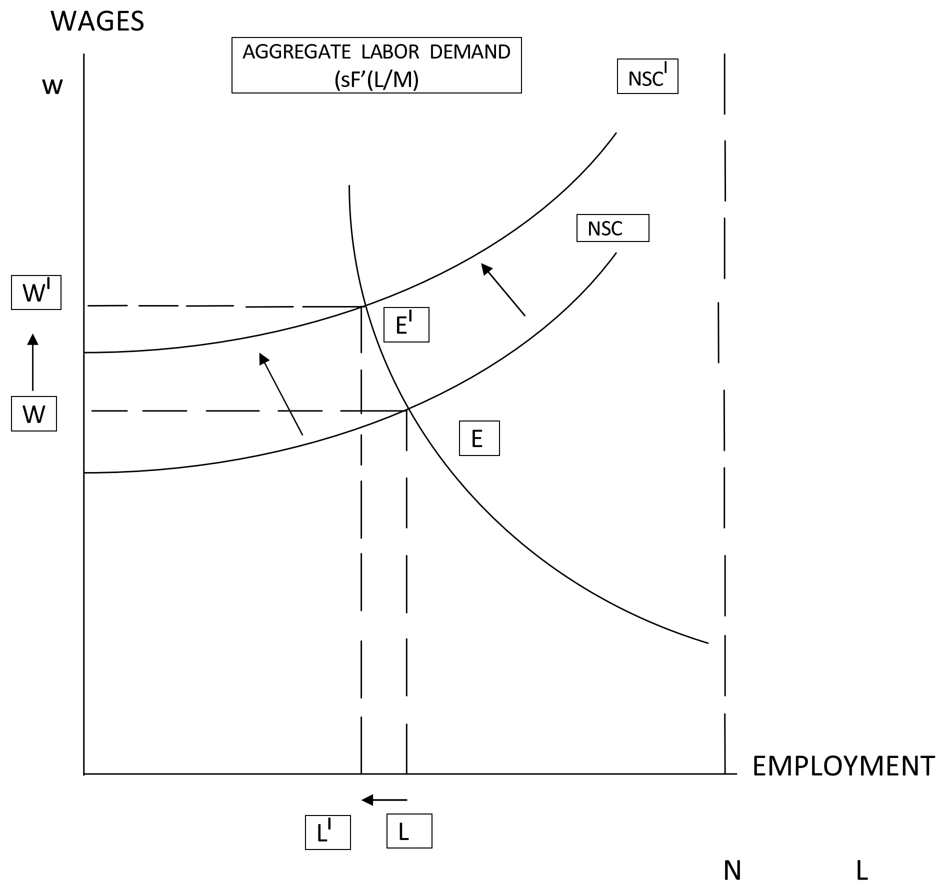


Figure 6: Non-exponential case: Equilibrium unemployment under efficiency wages

Analytically, given (63), we establish a positive relationship between r and w:

$$\frac{d}{dr}\left(e+\frac{e(a+b+r)}{q}\right)=\frac{e}{q}>0. \tag{69}$$

Hence, an increase in r occasions a north-western shift in the NSC curve. There is no shift in the aggregate labor demand curve and thus the equilibrium under non-exponential discounting at τ^1 is at E' translating into a higher efficiency wage and more unemployment in relation to the exponential case.

Proposition 8. *At τ^2, **in the presence of a commitment technology, the equilibrium under quasi-hyperbolic discounting reverts the exponential level** E **and remains at this level for the remainder of the contractual period.***

Proof. At τ^2, the quasi-hyperbolic discount rate falls back to the constant exponential discount rate level.[34] This implies that as seen from today, the real efficiency wage in the quasi-hyperbolic case should drop to the exponential wage level at τ^2. Strictly speaking, it is possible to implement a wage scheme which falls after a certain period. However, from the perspective of the firm, such a drop in the wage rate in the presence of quasi-hyperbolic employees results in a real possibility of shirking once the initial period ends. The dynamic inconsistency inherent in non-exponential discount functions thus pose a problem to the company when setting an effective efficiency wage in the periods following the initial. As a result, we note that the company needs a commitment technology at the initiation of the contract which would limit the options available to an employee in the future. One example would be the company undertaking a specific investment which imparts the employee with skills that are of little or no use outside the firm.[35] The specific assets acquired will bind the employee to the company even with falling real wages until such time some exogenous event results in separation or retirement. We therefore conclude that under the existence of quasi-hyperbolic employees and in the presence of a commitment technology, the equilibrium at τ^2 reverts to E from $E^{'}$ and remains at this level for each of the subsequent contractual periods, τ^3, τ^4 (see Fig. 7). Thus, as we move from τ^1 to τ^2, the fall in r results in a lower wage and less unemployment since there is a south-eastern shift in the NSC curve.

[34] We earlier established that the change in discounting in both the exponential and the quasi-hyperbolic cases in consecutive periods after the first to be ρ. Thus, after the first period, r is also equal in both cases.

[35] Instances of firms undertaking specific human capital investments are not rare. As an example, a company may choose to train its staff on using Microsoft Dynamics NAV software for its enterprize resource planning (ERP) needs knowing that its competitor uses a different software e.g. SAP Business One. Thus, an employee who considers joining the competitor has a disincentive since she needs to invest in learning the new software.

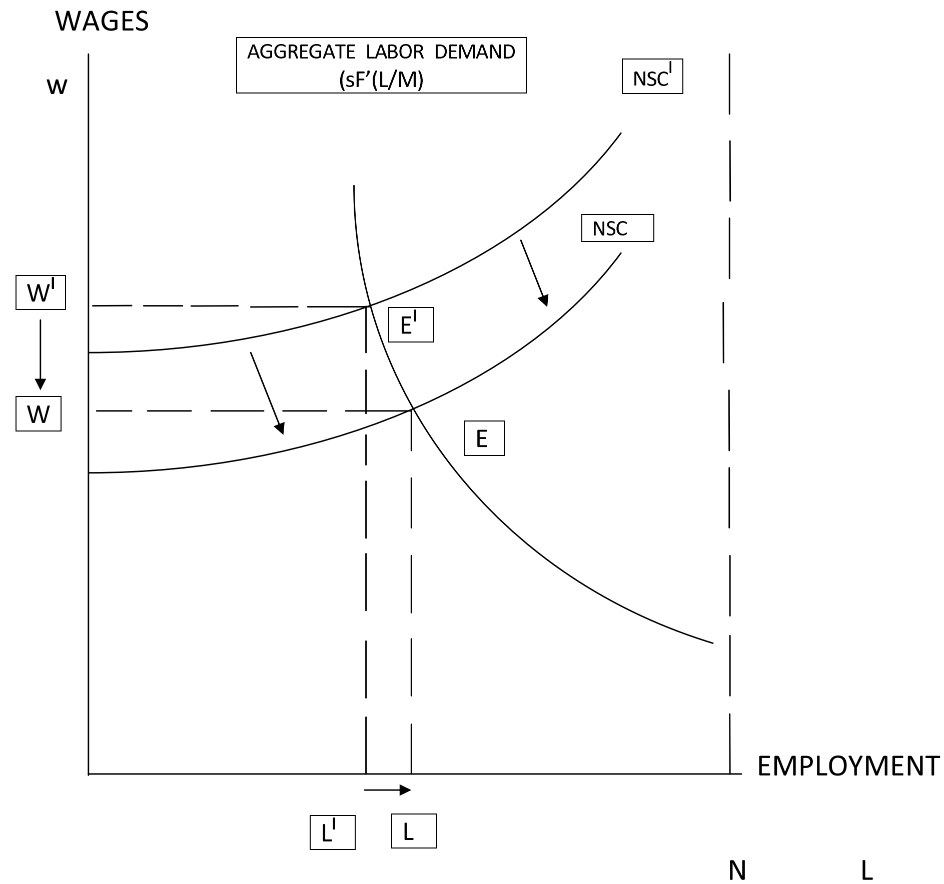


Figure 7: Quasi-hyperbolic case in the presence of a commitment technology (τ^2, τ^3, τ^4): Equilibrium unemployment under efficiency wages

Proposition 9. *At τ^2, **in the presence of a commitment technology, the equilibrium under hyperbolic discounting shifts to E^T leading to a higher efficiency wage and a higher level of unemployment.***

Proof. In the pure hyperbolic case, the firm will have to incorporate falling discount rates over the duration of the contract. At τ^2 and in each of the subsequent periods, the firm has to estimate the rate at which the discount rate falls and in the process the corresponding wage at each period. The wage scheme still requires an accompanying commitment technology to be effective. Assuming that the rate of decline in the discount rate is not steep, at τ^2, we have:

$$e+\frac{e\{[bL/(N-L)]+b+rh\}}{q} > e+\frac{e\{[bL/(N-L)]+b+re\}}{q} = e+\frac{e\{[bL/(N-L)]+b+rqh\}}{q} \tag{70}$$

implying

$$\bar{w}_h > \bar{w}_e = \bar{w}_{qh}. \tag{71}$$

Graphically, as we move from τ^1 to τ^2, there is a south-eastern shift in the NSC curve but the new equilibrium is at a wage rate higher than in the exponential case. We denote this equilibrium E^T (see Fig. 8). Here, wages have declined and there is less unemployment but still more of the labor force is unemployed in comparison to the exponential case ($L^T > L$).

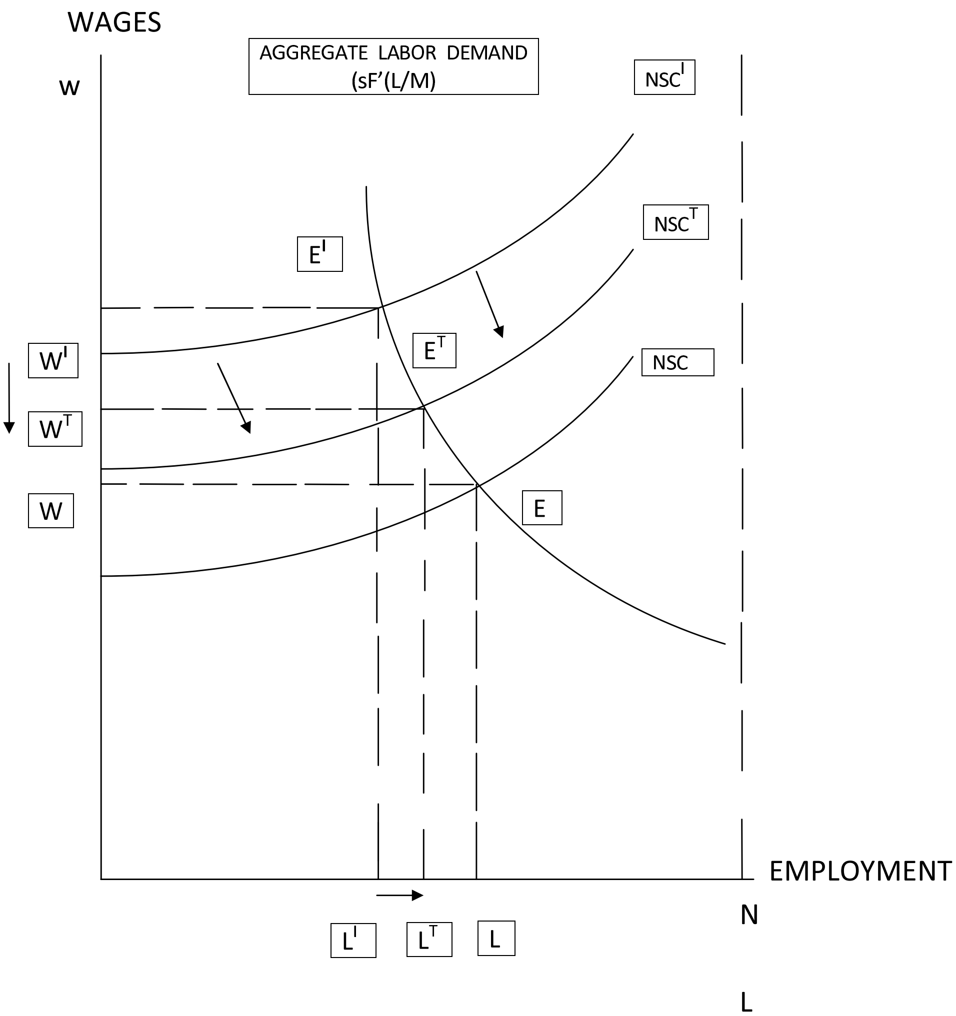


Figure 8: Hyperbolic case in the presence of a commitment technology ($\tau^2, \bar{w}_h > \bar{w}_e = \bar{w}_{qh}$): Equilibrium unemployment under efficiency wages

Proposition 10. *At τ^3 **and** τ^4, **in the presence of a commitment technology, the equilibrium under hyperbolic discounting shifts to** E^3 **and** E^4 **respectively leading to a lower efficiency wage and less unemployment.***

Proof. At τ^3, r drops to below the constant exponential discount rate and drops even further at τ^4, the final period of the contract. Thus, we have:

$$e+\frac{e\{[bL/(N-L)]+b+rh\}}{q} < e+\frac{e\{[bL/(N-L)]+b+re\}}{q} = e+\frac{e\{[bL/(N-L)]+b+rqh\}}{q} \tag{72}$$

implying

$$\bar{w}_h < \bar{w}_e = \bar{w}_{qh}. \tag{73}$$

Thus, in both periods τ^3 and τ^4, we have a south-eastern shift in the NSC. The equilibrium wage falls to E^3 and E^4 respectively leading to a further fall in unemployment (see Fig. 9). Thus, under the existence of pure hyperbolic employees and in the presence of a commitment technology, we conclude that the Shapiro & Stiglitz shirking model is characterized by falling real wages over the course of the employment contract.

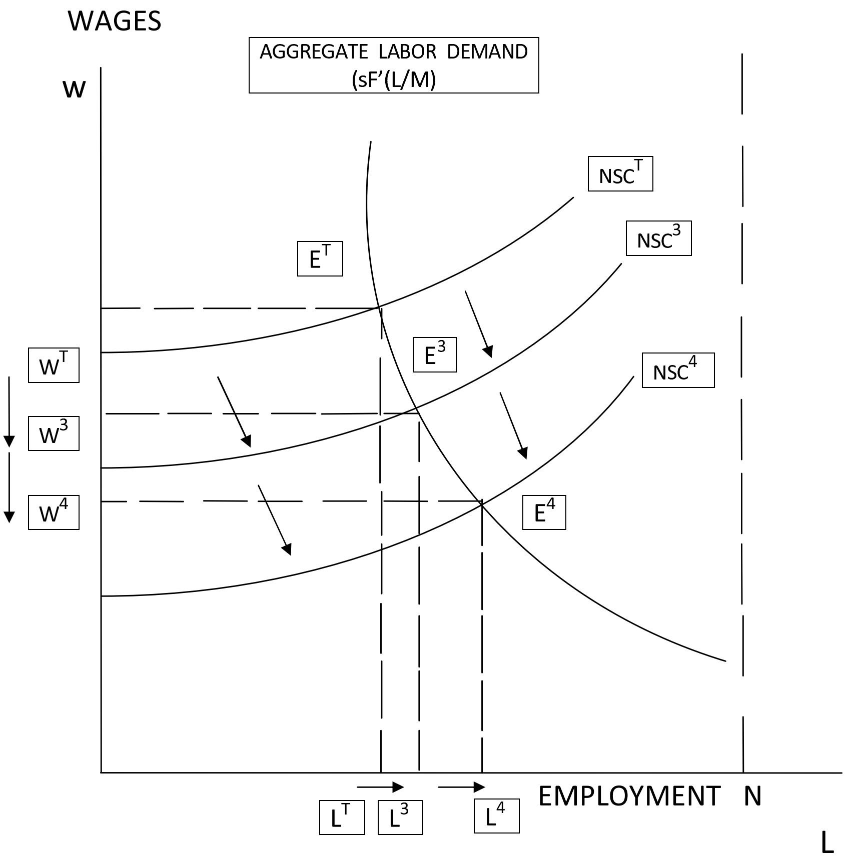


Figure 9: Hyperbolic case in the presence of a commitment technology ($\tau^3, \tau^4, \bar{w}_h < \bar{w}_e = \bar{w}_{qh}$): Equilibrium unemployment under efficiency wages

5 Summary and Conclusion

The growing interest on alternative models of discounting particularly over the last decade points to an awareness among economists and other social scientists on the limitations of Samuelson's (1937) DU model. Despite its lack of empirical validity, the model which leads to an exponential function still remains the dominant model in intertemporal choice analysis. In this paper, we began by revising literature on the motives for discounting. We identified four main motives including pure time discounting, uncertainty and risk, opportunity cost and diminishing marginal utility. We then reviewed assumptions inherent in the DU model including constant discount rate, positive rate of time preference, utility independence, consumption independence, stationary instantaneous utility and independence of discounting from consumption. We examined desirable properties of exponential discounting which may in part explain the prominence of the DU model and these include its stationarity property implying dynamic consistency and its mathematical tractability. Following a review of the literature, we highlighted the major behavioral anomalies in the DU model including declining discount rates, magnitude effect, sign effect, common difference effect, delay-speed up asymmetry and preference over improving sequences of outcomes. In particular, we noted that declining discount rates and the common difference effect contradict the constant discount rate assumption which is the core assumption of the DU model.

We examined alternatives to the DU model that have been suggested in part to account for these anomalies, first among them the quasi hyperbolic model by Phelps & Pollak (1968) and Laibson (1997). We noted that the model partly accounts for the declining discount rates and the common difference effect anomalies. In addition, its similarity to the DU model after the first period implies that it maintains the mathematical tractability of exponential discounting. However, the model falls short of being descriptively accurate since it does not fully account for declining rates into the future and other DU model anomalies. Further, we looked at hyperbolic models that incorporate declining discount rates with the progression of time and identified three functional forms, that is, Ainslie (1975), Herrnstein (1981) and Mazur (1987), and Loewenstein & Prelec (1992). We noted that although these models appear to be descriptively better than both the exponential and quasi-hyperbolic forms, their limitation lies in their lack of convergence thus restricting their application in intertemporal choice analysis.

In the analysis section, we modeled the interaction between two firms in the form of an infinite IPD and analyzed the equilibrium conditions under non exponential discounting necessary to sustain cooperation as an SPE. We found that under quasi-hyperbolic discounting, we require a higher discount factor to induce the cooperative outcome. We analyzed the Shapiro & Stiglitz (1984) Shirking model under non exponential discounting and concluded that wages set under the assumption of exponential discounting may not effectively deter shirking in the presence of hyperbolic or quasi-hyperbolic discounting employees. Finally, we concluded our analysis by examining unemployment equilibrium in the model incorporating Solow (1956). We found that initially under both quasi-hyperbolic and hyperbolic discounting, we have higher wages and more unemployment compared to the exponential case. After the initial period, unemployment and wages in the quasi-hyperbolic case revert to the exponential level but in the hyperbolic case, the Shapiro & Stiglitz shirking model is characterized by falling real wages over the course of the employment contract.

Thus our analysis reveals that divergences from what economic modeling predicts and what is empirically observed can in part be attributed to the discount function employed by the modeler, if in fact what the literature suggests is true. In most cases, this fact is often overlooked and if policies are formulated based on economic models that do not account for this apparent inaccuracy, then they run the risk of not achieving their desired results. In the extreme case, they may lead to potentially harmful effects on welfare. Therefore, at the very least, modelers of social and economic behavior should seriously consider the effect of the discount function employed. The critical evaluation of the assumptions a model is resting on should clearly extend to the aspect of discounting behavior. On the research part, we believe that any further work geared towards developing descriptively adequate models of discounting is a positive contribution to the entire scope of the social sciences and the area of intertemporal choice in particular.

6 Appendix

6.1 Appendix A

A1: Given the infinite series in (10), we prove that the series converges to $\frac{1}{1-\rho}$ below:

$$1 + \rho + \rho^2 + \rho^3 + ... = \sum_{n=0}^{\infty} \rho^n = \frac{1}{1-\rho}$$

We first determine the nth term in the series. Given the beginning of the series and using the notation $s_1, s_2, ..., s_n$ to indicate the 1st, 2nd, ..., nth partial sums respectively, we have:

$$s_1 = 1$$

$$s_2 = 1 + \rho$$

$$s_n = 1 + \rho + \rho^2 + ... + \rho^{n-1} \tag{6.74}$$

Having determined the nth partial sum, i.e., s_n we multiply it by ρ and get:

$$\rho s_n = \rho + \rho^2 + \rho^3 + ... + \rho^n \tag{6.75}$$

Subtracting (6.75) from (6.74) we have:

$$s_n - \rho s_n = 1 - \rho^n.$$

From the above, we can easily solve for s_n:

$$s_n(1-\rho) = 1-\rho^n.$$

$$s_n = \frac{(1-\rho^n)}{1-\rho}, \quad (\rho \neq 1).$$

If $\mid \rho \mid < 1$, $then\, \rho^n \rightarrow\ 0\, as\ n\ \rightarrow\ \infty\, and\, s_n\ \rightarrow\ \frac{1}{1-\rho}$. If $\mid \rho \mid > 1$, $then\ \mid \rho^n \mid \rightarrow\ \infty$ and the series diverges.

However, we know from the DU model that due to the positive rate of time preference assumption, we never have $\mid \rho \mid > 1$. We thus note that (10) converges to $\frac{1}{1-\rho}$.

//

A2: Given the following specification of the DU model in (5)

$$U(c_1, c_2, ..., c_N) = u(c_1) + \frac{u(c_2)}{1+r} + ... + \frac{u(c_N)}{(1+r)^{N-1}}.$$

We show that the discount factor is constant from one period to the next by selecting a representative period m, $m \in \mathbb{Z}_+$.

At period m, the cardinal instantaneous utility function, $u(c_m)$ is discounted at factor:

$$\frac{1}{(1+r)^{m-1}}.$$

In the next period, i.e., $m+1$, $u(c_{m+1})$ is discounted at factor:

$$\frac{1}{(1+r)^m}.$$

The change in discounting over the two periods is given by:

$$\frac{(\frac{1}{1+r})^m}{(\frac{1}{1+r})^{m-1}} \tag{6.76}$$

Solving (6.76) we get:

$$= (\frac{1}{1+r})^m \cdot (1+r)^{m-1} \ = \frac{(1+r)^{m-1}}{(1+r)^m} \ = (1+r)^{m-1-m}$$

$$= (1+r)^{-1} \ = \frac{1}{1+r} \ = \rho$$

//

6.2 Appendix B

B1: Given the IPD model in Section 3, we prove that the game has a unique SPE if played a finite number of times which we denote T, $(1 \leq T < \infty)$, where each firm plays $defect$ in each period :

The proof involves backward induction. First, we determine the NE of the game using firm i's best response function $(i = \{1, 2\}), i \neq j$:

Taking the payoffs for both player i and for player j under each outcome:

$$\pi^i(a^1) = C \qquad \pi^j(a^1) = C,$$

$$\pi^i(a^2) = Z \qquad \pi^j(a^2) = A,$$

$$\pi^i(a^3) = A \qquad \pi^j(a^3) = Z,$$

$$\pi^i(a^4) = S \qquad \pi^j(a^4) = S.$$

Where,

$$a^1 = \{cooperate, cooperate\};$$

$$a^2 = \{cooperate, defect\};$$

$$a^3 = \{defect, cooperate\};$$

$a^4 = \{defect, defect\}.$

and given $A > C > S > Z$, we note player i's best response mapping is:

$$R^i(a^j) = \begin{cases} defect \ \ if \ a^j \ = \ cooperate \\ defect \ \ if \ a^j \ = \ defect \end{cases} \tag{6.77}$$

$\Longrightarrow$ *From* (6.77) ,*we note that the outcome* $(defect, \ defect)$ *constitutes a unique* NE *for this game.*

Thus having established this, we suppose that the game has already been played in $T-1$ periods, and now firm i is ready to play for a final time in period T. In this case, the game is identical to a one shot game and its outcome is identical to that which we have previously established, i.e. $(defect, defect)$. Now, we consider the game played in period $T-1$. Both firms know that after this period, they will have one game to play and the outcome of this game would result in both playing $defect$. Hence, at this period, they both would play their dominant startegy $defect$.

Using backward induction, we note that in each period $T-2, T-3, ..., 1$, $defect$ will be played by both players hence SPE.

//

B2: Given (29) and (30) from section 3.2, we show that deviation is not beneficial for firm 1 if

$$\rho \geq \frac{A-C}{A-S}$$

by solving the inequality:

$$C\Big(\frac{1}{1-\rho}\Big) \geq S\Big(\frac{A}{S} + \frac{1}{1-\rho} - 1\Big)$$

$$\Leftrightarrow \frac{C}{1-\rho} \geq A - S + \frac{S}{1-\rho}$$

$$\Leftrightarrow \frac{C-S}{1-\rho} \geq A - S$$

$$\Leftrightarrow \frac{C-S}{A-S} \geq 1-\rho$$

$$\Leftrightarrow \rho \geq 1-\frac{C-S}{A-S}$$

$$\Leftrightarrow \rho \geq \frac{A-S-(C-S)}{A-S}$$

$$\Leftrightarrow \rho \geq \frac{A-C}{A-S}$$

//

B3: Given (32) and (33) from section 3.2, we show that deviation is not beneficial for firm 1 under quasi-hyperbolic discounting if

$$\rho \geq \frac{A-C}{\beta(C-S)+A-C}.$$

$$\beta C\Big(\frac{1}{\beta}+\frac{1}{1-\rho}-1\Big) \geq A+\beta S\Big(\frac{1}{1-\rho}-1\Big)$$

$$\Leftrightarrow \frac{\beta C}{1-\rho}-\beta C+C \geq \frac{\beta S}{1-\rho}-\beta S+A$$

$$\Leftrightarrow \frac{\beta(C-S)}{1-\rho} \geq \beta(C-S)+A-C$$

$$\Leftrightarrow \frac{\beta(C-S)}{\beta(C-S)+A-C} \geq 1-\rho$$

$$\Leftrightarrow \rho \geq 1-\frac{\beta(C-S)}{\beta(C-S)+A-C}$$

$$\Leftrightarrow \rho \geq \frac{\beta(C-S)+A-C-[\beta(C-S)]}{\beta(C-S)+A-C}$$

$$\Leftrightarrow \rho \geq \frac{A-C}{\beta(C-S)+A-C}$$

//

B4: Given (34) in Section 3.2, we show that the following holds:

$$\frac{d}{d\beta}\,\rho^*(\beta) < 0 \tag{6.78}$$

If we substitute $A-C$ with α and $C-S$ with γ, we can restate $\rho^*(\beta)$ in the following way:

$$\rho^*(\beta) \;=\; \frac{A-C}{\beta(C-S)+(A-C)} \;= \frac{\alpha}{\beta\gamma+\alpha} \;= \alpha \;\cdot\; \frac{1}{\beta\gamma+\alpha}$$

Thus, we derive $\rho^*(\beta)$ with respect to β below:

$$\frac{d\rho^*}{d\beta} \;= \alpha \;\cdot\; \frac{d}{d\beta}\Big(\frac{1}{\beta\gamma+\alpha}\Big) = \alpha \;\cdot\; \frac{0 \;-\; 1 \;\cdot(\beta\gamma+\alpha)'}{(\beta\gamma+\alpha)^2}$$

$$= \;-\alpha \;\cdot\; \frac{\gamma}{(\beta\gamma+\alpha)^2} \;= -\frac{\alpha\gamma}{(\beta\gamma+\alpha)^2}$$

From section 3.1, we established the following relationship: $A > C > S > Z$.

Thus, we note that $\alpha > 0$ and $\gamma > 0$ since:

$$\alpha = \underbrace{(A-C)}_{+} \quad \gamma = \underbrace{(C-S)}_{+} \quad \Rightarrow (A-C)(C-S) \;>\; 0$$

implying:

$$-\frac{\alpha\gamma}{(\beta\gamma+\alpha)^2} \;<\; 0 \;\Leftrightarrow -\frac{(A-C)(C-S)}{(\beta(C-S)+(A-C))^2} \;<\; 0$$

//

B5: Given (34) in Section 3.2, we show that the following holds:

$$\frac{d^2}{d\beta^2}\ \rho^*(\beta) > 0 \tag{6.79}$$

$$\frac{d^2}{d\beta^2}\Big(\frac{A-C}{\beta(C-S)+A-C}\Big) = \frac{d}{d\beta}\Big(-\ \frac{\alpha\gamma}{(\beta\gamma+\alpha)^2}\Big) \ = \ -\alpha\gamma\ \frac{d}{d\beta}\Big(\frac{1}{(\beta\gamma+\alpha)^2}\Big)$$

$$= -\alpha\gamma\ \Big(\frac{0\ -\ 2(\beta\gamma+\alpha)^1\gamma}{(\beta\gamma+\alpha)^4}\Big)$$

$$= -\alpha\gamma\ \cdot\ \frac{-\ 2\gamma(\beta\gamma+\alpha)}{(\beta\gamma+\alpha)^4}$$

$$= 2\alpha\gamma^2\ \cdot\ \frac{\beta\gamma+\alpha}{(\beta\gamma+\alpha)^4}$$

$$= 2\alpha\gamma^2\ \frac{1}{(\beta\gamma+\alpha)^3}\ >\ 0\ \ \Leftrightarrow\ \ \frac{2(A-C)(C-S)^2}{(\beta(C-S)+(A-C))^3}\ >\ 0$$

//

B5: Given (36) from Section 3.2, we show that the following holds:

$$\frac{A-C}{A-S}\ < 1$$

From section 3.1, we established the following relationship: $A > C > S > Z$. Thus,

$$A - C\ <\ A - S \quad |\, -A$$

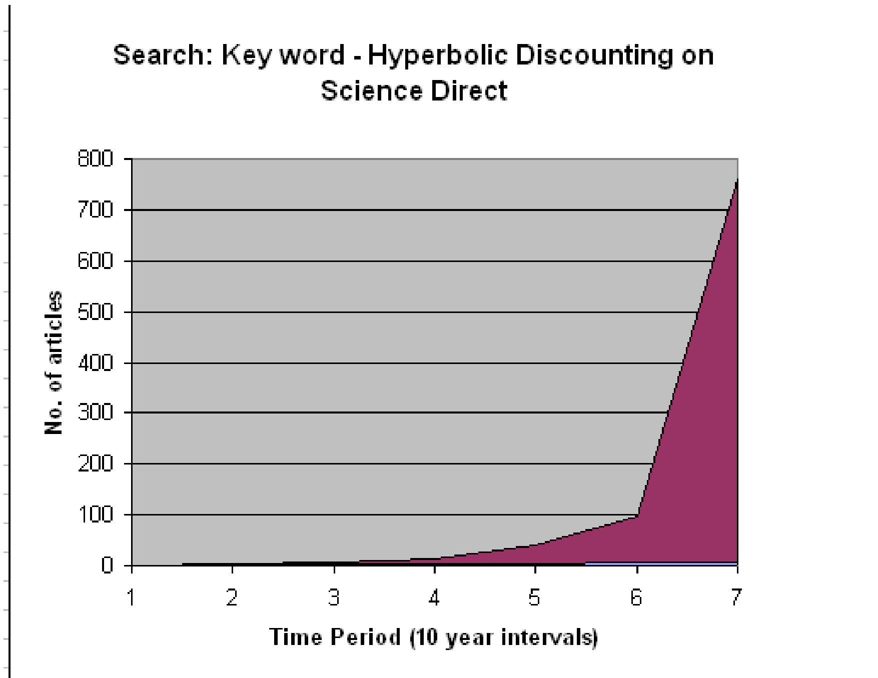
$$\Leftrightarrow \quad -C\ <\ -S \qquad |\, (-)$$

$$\Leftrightarrow \quad C\ >\ S$$

//

6.3 Appendix C

C1: Figure 10 shows the trend over the past century of published articles on Science Direct, one of the world's largest online collections of published scientific research (key word: hyperbolic discounting).



Time Period	Period	No. of articles
1939-1949	1	0
1949-1959	2	0
1959-1969	3	4
1969-1979	4	8
1979-1989	5	35
1989-1999	6	90
1999-2009	7	754

Figure 10: Articles on hyperbolic discounting available on Science Direct (1939-2009)

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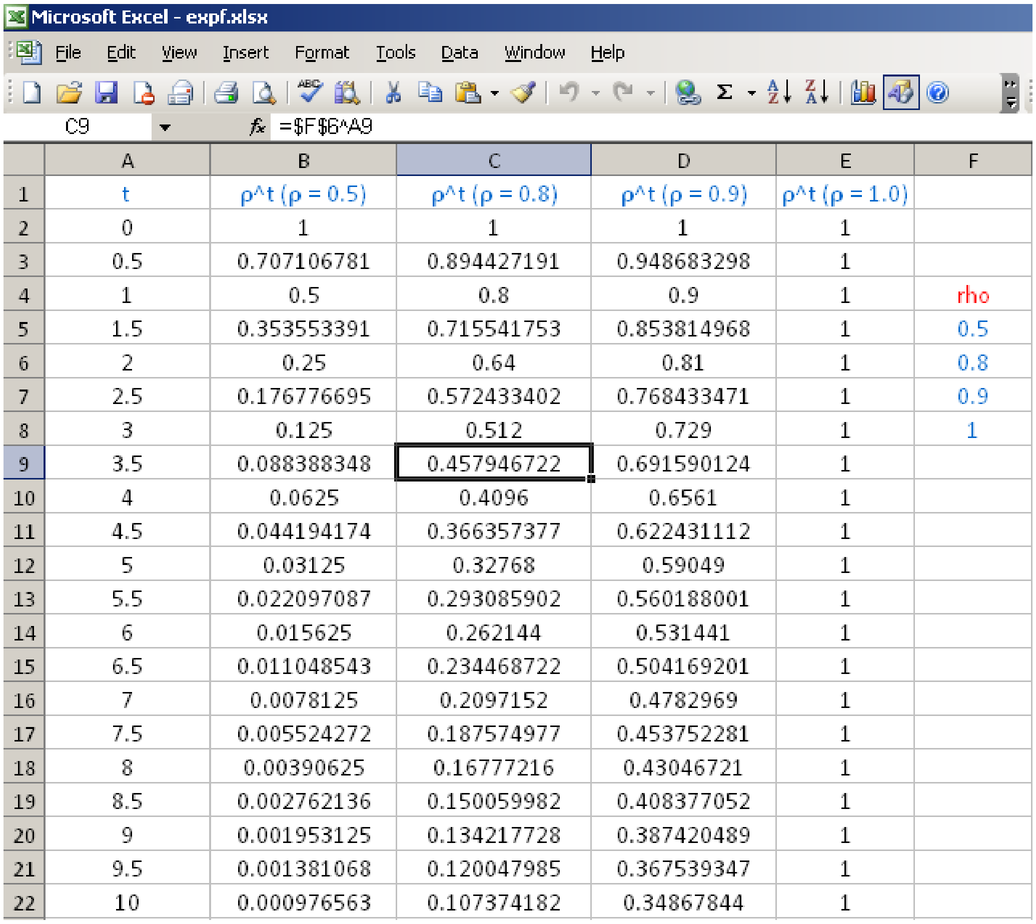
C2: The graphs in Figure 1 were plotted by Stata from data imported from Excel. At time 0, the value of the payoff equals 1 implying no discounting of the present. For each subsequent period, i.e., $t = 1, 2, ...$, the value of the payoff is equal to ρ^t.

The table below gives the output in excel. We set ρ at 0.5, 0.7 and 0.9 although any value in the range (0,1) would suffice. In addition, we give the theoretical limit case wherein $\rho = 1$, implying zero discounting.


Microsoft Excel - expf.xlsx

C9 =$F$6^A9

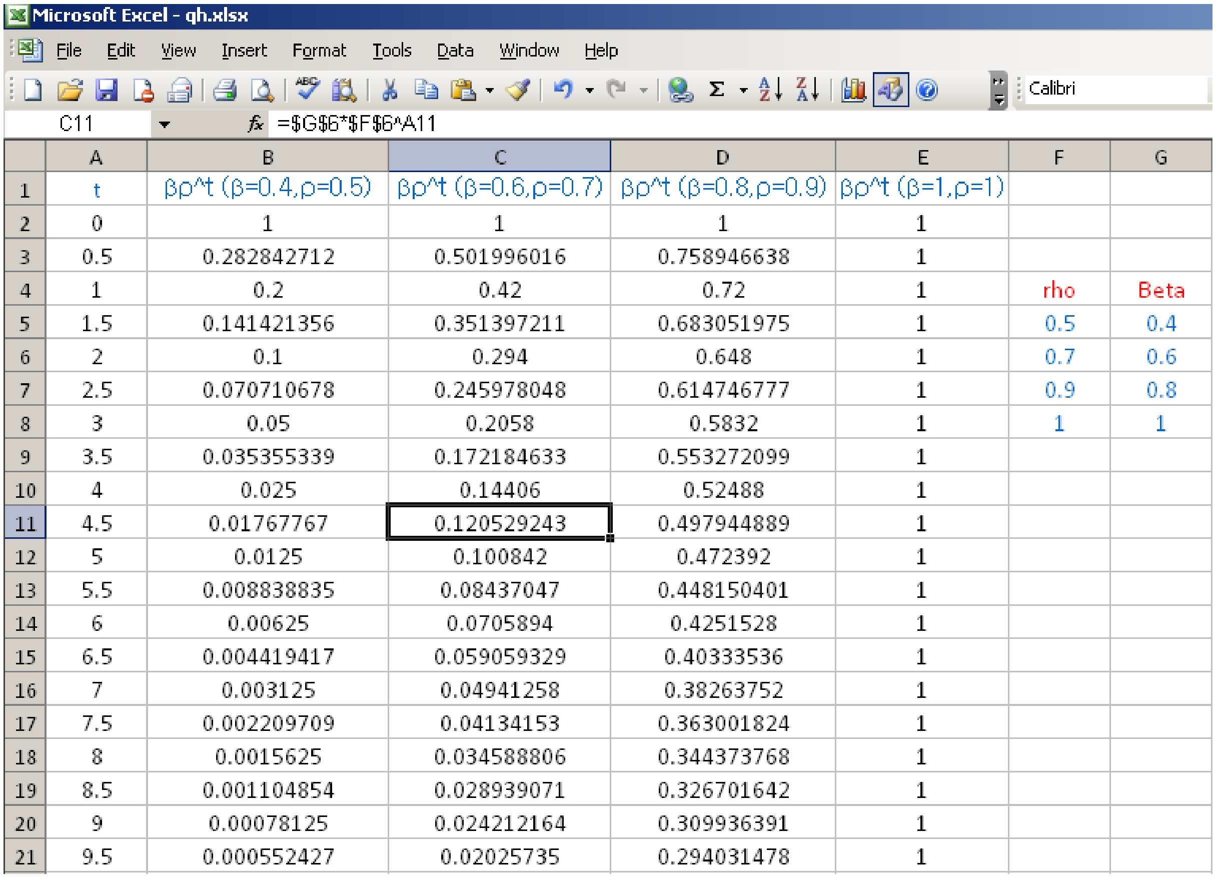
	A	B	C	D	E	F
1	t	ρ^t (ρ = 0.5)	ρ^t (ρ = 0.8)	ρ^t (ρ = 0.9)	ρ^t (ρ = 1.0)	
2	0	1	1	1	1	
3	0.5	0.707106781	0.894427191	0.948683298	1	
4	1	0.5	0.8	0.9	1	rho
5	1.5	0.353553391	0.715541753	0.853814968	1	0.5
6	2	0.25	0.64	0.81	1	0.8
7	2.5	0.176776695	0.572433402	0.768433471	1	0.9
8	3	0.125	0.512	0.729	1	1
9	3.5	0.088388348	0.457946722	0.691590124	1	
10	4	0.0625	0.4096	0.6561	1	
11	4.5	0.044194174	0.366357377	0.622431112	1	
12	5	0.03125	0.32768	0.59049	1	
13	5.5	0.022097087	0.293085902	0.560188001	1	
14	6	0.015625	0.262144	0.531441	1	
15	6.5	0.011048543	0.234468722	0.504169201	1	
16	7	0.0078125	0.2097152	0.4782969	1	
17	7.5	0.005524272	0.187574977	0.453752281	1	
18	8	0.00390625	0.16777216	0.43046721	1	
19	8.5	0.002762136	0.150059982	0.408377052	1	
20	9	0.001953125	0.134217728	0.387420489	1	
21	9.5	0.001381068	0.120047985	0.367539347	1	
22	10	0.000976563	0.107374182	0.34867844	1	

Figure 11: Excel output for Figure 1

//

C3: The graphs in Figure 2 were plotted by Stata from data imported from Excel. At time 0, the value of the payoff equals 1 implying no discounting of the present. For each subsequent period, i.e., $t = 1, 2, ...$, the value of the payoff is equal to $\beta\rho^t$.

The table below gives the output in excel. We set β at 0.4, 0.6 and 0.8 and ρ at 0.5, 0.7 and 0.9 respectively although for both parameters, any value in the range (0,1) would suffice. In addition, we give the theoretical limit case wherein $\beta = 1$ and $\rho = 1$, implying zero discounting.



Microsoft Excel - qh.xlsx

C11 =$G$6*$F$6^A11

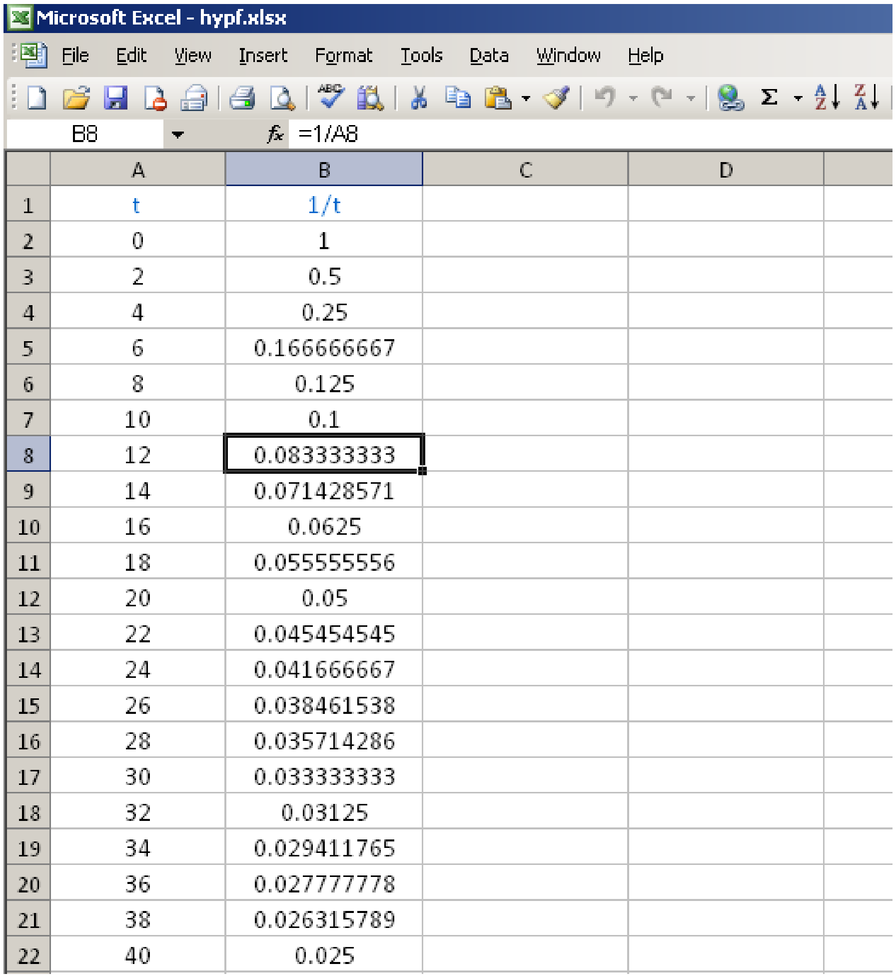
	A	B	C	D	E	F	G
1	t	βρ^t (β=0.4,ρ=0.5)	βρ^t (β=0.6,ρ=0.7)	βρ^t (β=0.8,ρ=0.9)	βρ^t (β=1,ρ=1)		
2	0	1	1	1	1		
3	0.5	0.282842712	0.501996016	0.758946638	1		
4	1	0.2	0.42	0.72	1	rho	Beta
5	1.5	0.141421356	0.351397211	0.683051975	1	0.5	0.4
6	2	0.1	0.294	0.648	1	0.7	0.6
7	2.5	0.070710678	0.245978048	0.614746777	1	0.9	0.8
8	3	0.05	0.2058	0.5832	1	1	1
9	3.5	0.035355339	0.172184633	0.553272099	1		
10	4	0.025	0.14406	0.52488	1		
11	4.5	0.01767767	0.120529243	0.497944889	1		
12	5	0.0125	0.100842	0.472392	1		
13	5.5	0.008838835	0.08437047	0.448150401	1		
14	6	0.00625	0.0705894	0.4251528	1		
15	6.5	0.004419417	0.059059329	0.40333536	1		
16	7	0.003125	0.04941258	0.38263752	1		
17	7.5	0.002209709	0.04134153	0.363001824	1		
18	8	0.0015625	0.034588806	0.344373768	1		
19	8.5	0.001104854	0.028939071	0.326701642	1		
20	9	0.00078125	0.024212164	0.309936391	1		
21	9.5	0.000552427	0.02025735	0.294031478	1		

Figure 12: Excel output for Figure 2

//

The graph in Figure 3 was plotted by Excel. At time 0, the value of the payoff equals 1 implying no discounting of the present. For each subsequent period, i.e., $t = 1, 2, ...$, the value of the payoff is equal to $\frac{1}{t}$. Note that here t represents a unit time delay.

The table below gives the output in excel that generated the graph.



Microsoft Excel - hypf.xlsx

File Edit View Insert Format Tools Data Window Help

B8 f_x =1/A8

	A	B	C	D
1	t	1/t		
2	0	1		
3	2	0.5		
4	4	0.25		
5	6	0.166666667		
6	8	0.125		
7	10	0.1		
8	12	0.083333333		
9	14	0.071428571		
10	16	0.0625		
11	18	0.055555556		
12	20	0.05		
13	22	0.045454545		
14	24	0.041666667		
15	26	0.038461538		
16	28	0.035714286		
17	30	0.033333333		
18	32	0.03125		
19	34	0.029411765		
20	36	0.027777778		
21	38	0.026315789		
22	40	0.025		

Figure 13: Excel output for figure 3

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