

# A Decidable Multi-Modal Logic of Context\*

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## Abstract

We give a logic for formulas  $\phi \multimap \psi$ , with the informal reading "  $\psi$  is true in the context described by  $\phi$ ". These are interpreted as binary modalities, by quantification over an enumerable set of unary modalities  $c \multimap \psi$ , meaning "  $\psi$  is true in context  $c$ ". The logic allows arbitrary nesting of contexts.

A corresponding axiomatic presentation is given, and proven to be decidable, sound, and complete.

Previously, quantificational logic of context restricted the nesting of contexts, and was only known to be decidable in very special cases.

## 1 Introduction

The need for formal systems of reasoning within given contexts, and for migrating between contexts, was pointed out in (McCarthy and Buvač 1994), (McCarthy 1993), (Giunchiglia 1993). See (Buvač 1996), (Buvač et al. 1995), (Gabbay and Nossum 2000), (Serafini and Giunchiglia 2001), (Nossum and Serafini 2002), (Nossum 2002) for some developments of logical systems in this area.

An well-known application of a formal system of context is to localized contexts in the Cyc knowledge base (Reed and Lenat 2002). During the early phases of the Cyc project (Guha 1991) introduced the notation  $\mathbf{ist}(c, \psi)$ , with the reading that "  $\psi$  is true in context  $c$ ".

We use the notation  $c \multimap \psi$  for this, and generalize by allowing formulas in the first coordinate:  $\phi \multimap \psi$ . We give formal semantics corresponding to the informal reading of  $\phi \multimap \psi$  as "  $\psi$  is true in the context described by  $\phi$ ".

This paper is organized as follows: The formula language is defined in the next section, and some examples are given that illustrate some

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\*Accepted for publication in the Journal of Applied Logic

of the issues in interpreting contextualized formulas. Then a model framework is defined, with clauses for interpreting every formula of the language. A deductive system of axiom schemas and rules of inference is given, with proofs of decidability, soundness, and completeness. Finally, some further properties of the logic are given, and some other work in this area is discussed.

## 2 Formula language

The formula language is defined by a Backus-Naur grammar clause. The bases for the definition are a set  $A$  of atomic formulas, and a set  $C$  of context names, both of which are assumed to be enumerably infinite.

**Definition 2.1 (The language  $\Sigma$ )**

$$\Sigma ::= A | \neg \Sigma | \Sigma \rightarrow \Sigma | C \multimap \Sigma | \Sigma \multimap \Sigma$$

Additional connectives  $\leftrightarrow, \wedge, \vee$  etc., and the constants  $\top$  and  $\perp$ , can be added to the language in the usual way.

Our formulas  $c \multimap \phi$  are comparable to  $\mathbf{ist}(c, \phi)$  of (Buvač 1996), but where there is first-order quantification over  $c$  in (Buvač 1996), we have instead formulas  $\psi \multimap \phi$ , the semantics of which will be defined by quantification over a set of modalities  $\{d \multimap \phi\}$ , depending on  $\psi$ .

## 3 Examples

### 3.1 Nested contexts

Recall the story of King Lear, who makes the fateful mistake of disowning his only loving and loyal daughter, entrusting his kingdom to his two other daughters. But they are greedy and selfish, and disaster unfolds. Let  $\psi$  be the formula "he is happy".

$$\text{the play} \multimap (\text{the family} \multimap \neg \psi)$$

The play is usually cast with actors who are quite rational in their personal lives: Imagine such an actor, a devastating Lear on stage, but privately a considerate and sensible father. Therefore,

$$\text{the family} \multimap \psi$$

unless some mishap temporarily interferes:

the family  $\multimap$  ( some mishap  $\multimap \neg\psi$  )

Of course, if a relative visits him at home, and compliments the actor on his performance as King Lear, this makes him happy:

the family  $\multimap$  ( the play  $\multimap \psi$  )

We are concerned that the pattern of surrounding contexts should have bearing on the interpretation of embedded formulas. Some previously proposed logics of context only took account of the innermost context in formulas such as the above (Buvač 1996).

### 3.2 Contexts described by formulas

Context can also be indirectly given by formulas, which then describe the set of contexts where they are true. Consider the example given by (Lewis 1973) in his early formal treatment of counterfactual conditionals: "If kangaroos had no tails, they would topple over."

Writing  $\kappa$  for "kangaroos have no tails", and  $\tau$  for "kangaroos topple over", this becomes

$$\kappa \multimap \tau$$

and our semantical account of it interprets it to be true in circumstances such that all contexts where kangaroos have no tails, are contexts where kangaroos topple over. The models employ a possible-worlds framework to make precise the phrase "circumstances such that".

### 3.3 Incremental context

Context can be perceived as an incremental construction, and reconstructing a coherent context from fragmented information is sometimes necessary. Understanding anaphoric references is a case in point. Consider for example a visitor to one of London's prestigious old university colleges, who is trying his best to find his way to a colleague's office through a confusing maze of underground corridors between various college buildings.

He entered the underground passage, and the lights went out. He swore profusely as he hit his forehead on something. When the lights came back on, he found her office where he had left it two days earlier. The bandaids were underneath the computer manuals.

A benevolent reader will be prone to 'fill in the gaps' in the above paragraph by bringing to bear contextual knowledge and assumptions consistent with the rest of the tale. 'The bandaids' acquire relevance by assimilation of a 'hitting of forehead'-context, although their mention is not preceded by any explicitly defining occurrence. This necessitates a contextual reconstruction based on the preceding narrative, as well as other sources, for understanding. (Kempson et al. 2001) present a technique for building a tree structure of contextual information from text, for use in resolving anaphoric references.

In the semantical framework that follows, there is a function which augments a current state of affairs, say  $w$ , with a contextual component  $c$ , to form a new state of affairs  $w \star c$ . This represents incremental assimilation of contextual information into the current state of affairs.

## 4 Model framework

As indicated above, we are concerned with the ability to interpret formulas in circumstances which are influenced by context, and where current circumstances are augmented when new contextual information arrives. States of affairs change as context is assimilated, they may change again by the addition of more context, and so on.

This leads to a reflection about what it is to be a state of affairs, or a possible world, in our framework: Let us dispense for a moment with any preconception of an original state of affairs from which later states develop, or a final state of affairs towards which previous states are headed, and try to fend exclusively with the sequential assimilation of new bits of contextual information. In that case, all that a state of affairs has to identify it, is the sequence of contexts that has led up to it. If so minded, we may well speak of a possible world  $w$  and how it changes to  $w \star c$  when context  $c$  is assimilated, while remaining ready to identify  $w$  with a sequence  $\bar{c}$  of assimilated contexts. In fact, at several places in the following sections, that is going to be very convenient indeed.

The truth value of a formula will be evaluated at each of a set of points, or possible worlds, in each model. A possible world gives a coherent interpretation of the propositional language fragment, and the worlds are related in a way that determines the interpretation of  $\neg\circ$  formulas.

The interpretation of a formula  $c \neg\circ \phi$  at a point  $w$  depends on the

interpretation of  $\phi$  at another point, which is related to  $w$  and  $c$ .

Each point is related to an enumerable set of other points, one per context. Formulas of the form  $\phi \multimap \psi$  are interpreted at a point by interpreting  $\phi \rightarrow \psi$  at all related points.

Formally, a model in this framework is a triple  $M = \langle W, \star, V \rangle$ , where

- $W \neq \emptyset$  is the set of possible worlds,
- $\star : W \times C \rightarrow W$ , and
- $V : W \times A \rightarrow 2$ .

The  $\star$  component imposes a directed graph on  $W$ , with an edge from  $w$  to  $w\star c$  for each  $w \in W$  and  $c \in C$ . Paths in the graph correspond to sequences of nested contexts.

Whenever  $\bar{c}$  is a finite sequence  $\langle c_1, \dots, c_k \rangle$  of contexts,  $\bar{c} \multimap \phi$  is an abbreviation for  $c_1 \multimap \dots \multimap c_k \multimap \phi$ , and  $w\star\bar{c}$  is an abbreviation for  $w\star c_1 \dots \star c_k$ . When  $\bar{c}$  is empty,  $\bar{c} \multimap \phi$  and  $w\star\bar{c}$  are just  $\phi$  and  $w$ .

**Definition 4.1 (Interpretation)** *The truth value of a formula  $\phi$  at a point  $w \in W$  in a model  $M$  is denoted  $M, w \models \phi$ , and defined by the following clauses, where  $a \in A$ ,  $c \in C$ , and  $\phi, \psi \in \Sigma$ :*

$$M, w \models a \text{ iff } V(w, a)$$

$$M, w \models \neg\psi \text{ iff } M, w \not\models \psi$$

$$M, w \models \phi \rightarrow \psi \text{ iff } M, w \models \phi \text{ implies } M, w \models \psi$$

$$M, w \models c \multimap \psi \text{ iff } M, w\star c \models \psi$$

$$M, w \models \phi \multimap \psi \text{ iff } M, w \models c \multimap (\phi \rightarrow \psi)$$

*for all  $c \in C$*

The symbol  $\multimap$  is doing the work of two here, corresponding to the last two semantical clauses, but it will always be syntactically unambiguous which one is intended.

**Definition 4.2 (Satisfaction)** *We say that a model  $M = \langle W, \star, V \rangle$  satisfies a formula  $\phi$ , denoted  $M \models \phi$ , iff  $M, w \models \phi$  for all  $w \in W$ .*

**Definition 4.3 (Validity)** Truth of a formula  $\phi$  at all points in all models is denoted  $\models \phi$ , and we then say that  $\phi$  is valid.

Every atomic formula  $a \in A$  is interpreted as a proposition which is either true or false, and the connectives  $\neg$  and  $\rightarrow$  are interpreted classically. Therefore, all substitution instances of propositional tautologies are valid in this class of models.

## 5 Deductive system

The deductive system consists of axiom schemas and rules of inference. A finite nonempty sequence  $\bar{\sigma} = \langle \sigma_1, \dots, \sigma_k \rangle$  of formulas is a *derivation* of  $\sigma_k$  iff each element of  $\bar{\sigma}$  either is an axiom or follows from previous elements of  $\bar{\sigma}$  by a rule of inference.

A formula  $\phi$  is a *theorem*, in symbols  $\vdash \phi$ , iff there exists a derivation of  $\phi$ .

These are the axiom schemas and rules of inference:

$\mathcal{A}1$  All instances of propositional tautologies.

$\mathcal{A}2$   $(c \multimap \neg \phi) \leftrightarrow \neg(c \multimap \phi)$

$\mathcal{A}3$   $(c \multimap (\phi \rightarrow \psi)) \rightarrow ((c \multimap \phi) \rightarrow (c \multimap \psi))$

$\mathcal{A}4$   $(\phi \multimap \psi) \rightarrow (c \multimap (\phi \rightarrow \psi))$

$\mathcal{R}1$  
$$\frac{\vdash \phi, \vdash \phi \rightarrow \psi}{\vdash \psi}$$

$\mathcal{R}2$  
$$\frac{\vdash \phi}{\vdash c \multimap \phi}$$

$\mathcal{R}3$  
$$\frac{\vdash \phi \rightarrow (\bar{c} \multimap (c \multimap (\psi \rightarrow \chi)))}{\vdash \phi \rightarrow (\bar{c} \multimap (\psi \multimap \chi))}$$

if  $c$  does not occur in  $\phi$ .

Let us briefly comment on each schema and rule. Schema  $\mathcal{A}1$  and rule  $\mathcal{R}1$  root the system in propositional logic, and  $\mathcal{A}2$  constrains the set of formulas true in each context to be propositionally coherent. Schema  $\mathcal{A}3$  and rule  $\mathcal{R}2$  are reminiscent of K and RN from normal

modal logic, cfr (Chellas 1980). Schema  $\mathcal{A}4$  and rule  $\mathcal{R}3$  are similar in spirit to universal instantiation, resp. universal generalisation, as found in texts on quantified modal logic, e.g. (Hughes and Cresswell 1996).

**Definition 5.1 (Decidability)** *A deductive system is said to be decidable iff there is an effective procedure for deciding membership in its set of theorems.*

When an axiomatic system with finitely many axioms and rules of inference has the *finite model property*, i.e. every formula which is satisfied by all finite models is valid, then theoremhood can be decided by a procedure which alternately enumerates derivations and finite models. Given a formula, eventually either it will pop up as a theorem or its negation will be satisfied in a finite model. This procedure is effective, since satisfaction can be recursively calculated in finite models.

Because our deductive system is finite, we only have to prove the finite model property in the proof of decidability below.

**Definition 5.2 (Soundness)** *An axiom is sound with respect to a model framework iff it is valid, and a rule of inference is sound if it maps valid premises to valid conclusions.*

If all axioms and rules are sound, the whole system is said to be sound, and then

$$\vdash \phi \text{ implies } \models \phi$$

for all formulas  $\phi$ .

**Definition 5.3 (Consistency)** *A formula  $\phi$  is consistent iff  $\nVdash \neg\phi$ .*

A finite set of formulas is said to be consistent iff the conjunction of its members is consistent, and an infinite set is consistent iff all its finite subsets are consistent.

A formula  $\phi$  is *consistent with* a set  $\Gamma$  according to the consistency of  $\Gamma \cup \{\phi\}$ . Clearly, when  $\Gamma$  is a consistent set,  $\phi$  is consistent with  $\Gamma$  iff  $\neg\phi$  is inconsistent with  $\Gamma$ .

**Definition 5.4 (Maximality)** *A consistent set  $\Gamma$  is maximal iff, for all formulas  $\phi$ , consistency of  $\Gamma \cup \{\phi\}$  implies  $\phi \in \Gamma$ .*

For a treatment of maximal consistent sets and their properties, consult e.g. (Chellas 1980).

**Definition 5.5 (Completeness)** *If every valid formula is a theorem;*

$$\models \phi \text{ implies } \vdash \phi$$

*then the system is said to be complete.*

The definitions of interpretation, satisfaction and validity are not extended to sets of formulas, in particular not to infinite sets. Thus the present notion of completeness can be rephrased as "every consistent formula is true at some point in some model", but this does not carry over to infinite sets of formulas.

We now proceed to prove decidability, soundness, and completeness of this axiomatic presentation of the logic.

## 6 Decidability proof

Since we have finitely many schemas and rules, it is sufficient to show that every formula satisfied by all finite models is valid. In the proof it is convenient to rephrase this condition into an equivalent form: Every formula true at some world in a model, is true at some world in a *finite* model.

So, let us take an arbitrary formula  $\phi$  for which there is a model  $M = \langle W_M, \star_M, V_M \rangle$  and a  $w \in W_M$  such that  $M, w \models \phi$ , and construct a model  $F = \langle W_F, \star_F, V_F \rangle$  with finite  $W_F$ , such that  $F, u \models \phi$  for some  $u \in W_F$ .

The possible worlds  $W_F$  will be a subset of  $W_M$ , constructed during a process which scans through all the subformulas of  $\phi$ . At the start,  $W_F$  is empty, and for every subformula, at most one possible world from  $W_M$  is included into  $W_F$ . Thus,  $W_F$  will remain finite throughout the procedure even though  $W_M$  may be infinite.

The world  $u \in W_F$  which will validate  $\phi$  in the constructed finite model  $F$  is  $w$  itself, and  $w$  is included into  $W_F$  at the first step of the procedure.

The order of traversal of  $\phi$  corresponds to the natural order of semantical interpretation, and proceeds analogously to the way in which one would apply the semantical clauses to evaluate  $M, w \models \phi$ .

Each subformula is interpreted at a possible world, or in the case of  $\psi \rightarrow \chi$  formulas, a set of possible worlds. In either case, no more than



a single world from  $W_M$  is incorporated into  $W_F$  for each subformula of  $\phi$ .

During this traversal we also construct part of  $\star_F$ , and build a relation  $\heartsuit$  between elements of  $W_F$  and subformulas of  $\phi$ . This relation keeps track of which subformulas are interpreted at which worlds during traversal.

The traversal procedure can be denoted  $P(w, \phi)$  and defined recursively as follows, letting  $W_F = \{w\}$  and  $\heartsuit = \emptyset$  initially.

$P(w, \phi)$  : First let  $w \heartsuit \phi$ , then proceed by cases:

$\phi \in A$  : Define  $w \star_F c$  to be  $w$  for all  $c \in C$ .

$\phi = \neg\psi$  : Proceed with  $P(w, \psi)$ .

$\phi = \psi \rightarrow \chi$  : Proceed with  $P(w, \psi)$  and  $P(w, \chi)$ .

$\phi = c \multimap \psi$  : Include  $w \star_M c$  into  $W_F$ , and define  $w \star_F c$  to be  $w \star_M c$ . Proceed with  $P(w \star_F c, \psi)$ .

$\phi = \psi \multimap \chi$  : If  $M, w \models \phi$  then choose a  $c \in C$  such that  $M, w \star_M c \models \psi \rightarrow \chi$ , otherwise choose a  $c \in C$  such that  $M, w \star_M c \not\models \psi \rightarrow \chi$ . Include  $w \star_M c$  into  $W_F$ . Define  $w \star_F c$  to be  $w \star_M c$ . Proceed with  $P(w \star_F c, \psi \rightarrow \chi)$ .

Observe that with every invocation of  $P(u, \eta)$  on some world  $u \in W_M$  and some subformula  $\eta$  of  $\phi$ ,  $u$  has already been included in  $W_F$ , and  $u \heartsuit \eta$  is established immediately by  $P$ .

Clearly,  $W_F$  as constructed here is finite, since at most one world is added per subformula of  $\phi$ .

It remains to define  $V_F$ , and to define  $w \star_F c$  for remaining pairs  $w, c$ . The former is simply  $V_M$  restricted to  $W_F \times A$ . For the latter, observe that for all  $w \in W_F$  there is some  $c \in C$  for which  $w \star_F c$  was defined by  $P$ . Given  $w$ , pick one of these  $c$ , and let  $w \star_F d = w \star_F c$  whenever  $w \star_F d$  was left undefined by  $P$ . This completes the construction of  $F$ .

**Lemma 6.1**  $F, u \models \eta$  iff  $M, u \models \eta$ , whenever  $u \heartsuit \eta$ .

**Proof:** By induction on the syntactical structure of  $\eta$ :

$\eta \in A$  : Immediate since  $V_F(u, \eta) = V_M(u, \eta)$ .

$\eta = \neg\psi$  : By the induction hypothesis applied to the syntactically simpler formula  $\psi$ .

$\eta = \psi \rightarrow \chi$  : Ditto for  $\psi$  and  $\chi$ .

$\eta = c \multimap \psi$  : We have  $u \star_F c = u \star_M c$  from the corresponding clause in the definition of  $P$  so  $F, u \models \eta$  iff (by semantics)  $F, u \star_F c \models \psi$  iff (by induction)  $M, u \star_F c \models \psi$  iff (as remarked)  $M, u \star_M c \models \psi$  iff (by semantics)  $M, u \models \eta$ .

$\eta = \psi \multimap \chi$  : If  $M, u \models \eta$ , then for all  $c \in C$  we have  $M, u \star_M c \models \psi \rightarrow \chi$ , in particular for all  $c$  such that  $u \star_M c$  were included in  $W_F$ , and  $F, u \models \eta$  follows. Otherwise some  $c \in C$  was selected so that  $u \star_M c \in W_F$  and  $M, u \star_M c \not\models \psi \rightarrow \chi$ . It follows that  $F, u \star_F c \not\models \psi \rightarrow \chi$ , and hence  $F, u \not\models \eta$  as required.

**Lemma 6.2**  $F, w \models \phi$

**Proof:** Follows from the previous lemma, since  $M, w \models \phi$  and  $w \heartsuit \phi$ .

This completes the proof of decidability.

## 7 Soundness proof

We prove soundness of  $\mathcal{A4}$  and  $\mathcal{R3}$  as examples. The other axiom schemas and rules are no more complicated.

$\mathcal{A4}$  We fix some arbitrary model  $M = \langle W, \star, V \rangle$ , and show that  $\mathcal{A4}$  is true at all  $w \in W$ : assuming the antecedent of the implication true:  $M, w \models \phi \multimap \psi$ , we must prove its consequent true for the same  $M$  and  $w$ :  $M, w \models c \multimap (\phi \rightarrow \psi)$ . But from the assumption we get:  $M, w \models c \multimap (\phi \rightarrow \psi)$  for every  $c \in C$ , which is sufficient.

$\mathcal{R3}$  Taking the premise of the rule as valid in all models:  $M, w \models \phi \rightarrow (\bar{c} \multimap (c \multimap (\psi \rightarrow \chi)))$  for every model  $M = \langle W_M, \star_M, V_M \rangle$  and every  $w \in W_M$ , we prove that the conclusion of the rule is valid in any model  $N = \langle W_N, \star_N, V_N \rangle$ , in other words  $N, u \models \phi \rightarrow (\bar{c} \multimap (\psi \multimap \chi))$  for every  $u \in W_N$ .

Choose an arbitrary  $N$  and a world  $u \in W_N$  such that  $N, u \models \phi$ , to prove  $N, u \models (\bar{c} \multimap (\psi \multimap \chi))$ . Now we take an arbitrary context  $d \in C$ , and prove  $N, t \star_N d \models \psi \rightarrow \chi$  with  $t = u \star_N \bar{c}$ .

To this end, we construct a special model  $M$  from  $N$  as follows: its domain is  $C^*$ , the set of finite sequences of contexts, including the empty sequence  $\epsilon$ . Intuitively, such a sequence points out a world in  $W_N$ , reachable from  $u$  by repeated application of  $\star_N$ . Distinct paths from  $u$  reaching the same world in  $W_N$  count as distinct worlds in  $W_M$ .

The crucial difference between  $N$  and  $M$  is in the interpretation at  $u$ , resp.  $\epsilon$ , of formulas of the form  $\bar{c} \multimap (c \multimap \dots)$  with  $\bar{c}$  and  $c$  as fixed above. Here are the definitions of  $M = \langle W_M, \star_M, V_M \rangle$ :

$$W_M = C^*$$

$$w \star_M e = we \text{ for arbitrary } w \text{ and } e, \text{ except } \bar{c} \star_M c = \bar{c}d$$

$$V_M(w, a) = V_N(u \star_N w, a)$$

We have  $M, \epsilon \models \phi$  since  $N, u \models \phi$  and  $c$  does not occur in  $\phi$ . By the premise of  $\mathcal{R}3$ , i.e.  $M \models \phi \rightarrow (\bar{c} \multimap (c \multimap (\psi \rightarrow \chi)))$ , and application of  $\mathcal{R}1$ , we obtain  $M, \epsilon \models \bar{c} \multimap (c \multimap (\psi \rightarrow \chi))$ . By the model conditions this is equivalent to  $M, \bar{c} \star_M c \models \psi \rightarrow \chi$ , and by the construction of  $M$  it follows that  $N, t \star_N d \models \psi \rightarrow \chi$ , as required.

## 8 Completeness proof

As already remarked, completeness is equivalent to truth of every consistent formula  $\delta$  at some point in some model. The proof uses a special model  $M$  constructed from  $\delta$ , in which the possible worlds are maximal consistent sets of formulas, and where a certain possible world  $w_0$ , which contains  $\delta$ , also validates  $\delta$ . This proof plan is an adaptation of a technique invented by (Henkin 1949) for first-order predicate calculus.

We take an arbitrary consistent formula  $\delta \in \Sigma$ , and construct a model  $M = \langle W, \star, V \rangle$  such that  $M, w \models \delta$  for a particular  $w \in W$ :

- $W = \{w_0\} \cup \{w \star c \mid w \in W, c \in C\}$ , where  $w_0$  is a certain set of formulas, containing  $\delta$  and constructed as described below,
- $w \star c = \{\phi \mid c \multimap \phi \in w\}$ ,
- and  $V(w, a)$  iff  $a \in w$ .

The construction of  $w_0$  proceeds in steps as follows. We start with the set  $\{\delta\}$ , and traverse the whole of  $\Sigma$ , including more formulas as

we go:  $\Sigma$  is clearly enumerable since  $A$  and  $C$  are, so we fix some enumeration  $\Sigma = \langle \sigma_1, \sigma_2, \dots \rangle$ . Now we consider each  $\sigma_i$  in turn, and if  $\sigma_i$  is consistent with  $w_0$ , then we add  $\sigma_i$  to  $w_0$ . If furthermore  $\sigma_i$  is of the form  $(\bar{c} \multimap \neg(\psi \multimap \chi))$ , then we also add  $(\bar{c} \multimap \neg(c' \multimap (\psi \rightarrow \chi)))$  to  $w_0$ , where  $c'$  is chosen as a member of  $C$  that does not occur in any member of  $w_0$ . Since at each stage of the process  $w_0$  is finite, while  $C$  is enumerably infinite, this is always feasible.

**Lemma 8.1**  *$w_0$  is a maximal consistent set.*

**Proof:** The proof is by induction on the number of steps in the construction of  $w_0$ . It is consistent to begin with, and we show that each addition to it preserves consistency. It follows that every finite subset of  $w_0$  will be consistent, therefore  $w_0$  itself will be consistent too. Also it will be maximal, for suppose that  $w_0 \cup \{\sigma_i\}$  is consistent for some  $\sigma_i \in \Sigma$ , then  $\sigma_i \in w_0$ , since it was added in step  $i$  of the process.

Addition of  $\sigma_i$  in the  $i$ 'th step is only done if it preserves consistency, so it remains to show that, after adding  $(\bar{c} \multimap \neg(\psi \multimap \chi))$  consistently, adding  $(\bar{c} \multimap \neg(c' \multimap (\psi \rightarrow \chi)))$  to  $w_0$  also preserves consistency.

To see this, suppose for contradiction that  $(\bar{c} \multimap \neg(c' \multimap (\psi \rightarrow \chi)))$  is inconsistent with  $w_0$ , in other words,

$$\vdash \neg(\phi \wedge (\bar{c} \multimap \neg(c' \multimap (\psi \rightarrow \chi))))$$

where  $\phi$  is the (finite) conjunction of members of  $w_0$  after adding  $\sigma_i$  consistently. Equivalently

$$\vdash \phi \rightarrow \neg(\bar{c} \multimap \neg(c' \multimap (\psi \rightarrow \chi)))$$

or, by repeated application of  $\mathcal{A}2$ ,

$$\vdash \phi \rightarrow (\bar{c} \multimap (c' \multimap (\psi \rightarrow \chi)))$$

But then by  $\mathcal{R}3$ :

$$\vdash \phi \rightarrow (\bar{c} \multimap (\psi \multimap \chi))$$

since  $c'$  is chosen so as to not occur in  $\phi$ . By repeatedly applying  $\mathcal{A}2$  again, we get

$$\vdash \phi \rightarrow \neg(\bar{c} \multimap \neg(\psi \multimap \chi))$$

or equivalently

$$\vdash \neg(\phi \wedge (\bar{c} \multimap \neg(\psi \multimap \chi)))$$

But this contradicts the consistency of  $\sigma_i$  with  $w_0$ , so it follows that the consistency of  $w_0$  is preserved at every step of its construction process.

**Lemma 8.2**  *$w \star c$  is a maximal consistent set whenever  $w$  is.*

**Proof:** The following three parts are sufficient, cfr. (Chellas 1980):

- $w \star c$  contains all theorems: Suppose  $\vdash \phi$ . Then  $\vdash c \multimap \phi$  by  $\mathcal{R}2$ , so  $c \multimap \phi \in w$  since  $w$  is a maximal consistent set. Then it follows that  $\phi \in w \star c$  by the definition of  $\star$ .
- $w \star c$  separates formulas from their negations, i.e.  $\phi \notin w \star c$  iff  $\neg\phi \in w \star c$ :

Expanding the definition of the former we obtain:  $(c \multimap \phi) \notin w$  which is equivalent to  $\neg(c \multimap \phi) \in w$  by the fact that  $w$  is maximal and consistent. By  $\mathcal{A}2$  this is equivalent to  $c \multimap \neg\phi \in w$ , which again by definition of  $\star$  is equivalent to  $\neg\phi \in w \star c$ .

- $w \star c$  is propositionally closed, i.e. if  $\phi \in w \star c$  and  $\phi \rightarrow \psi \in w \star c$  then  $\psi \in w \star c$ :

Suppose  $\phi \in w \star c$ , i.e. by definition  $c \multimap \phi \in w$ , and suppose also  $\phi \rightarrow \psi \in w \star c$ , which develops into  $(c \multimap (\phi \rightarrow \psi)) \in w$ . We must show  $\psi \in w \star c$ , which means  $c \multimap \psi \in w$ . But this follows from  $\mathcal{A}3$  and the fact that  $w$  is a maximal consistent set.

Therefore the set  $w \star c$  is maximal and consistent whenever  $w$  is. By induction on the length of paths in  $W$  starting at  $w_0$ , the previous two lemmas prove that all  $w \in W$  are maximal consistent sets.

**Lemma 8.3**  *$M, w \models \phi$  iff  $\phi \in w$*

**Proof:** The proof is by induction on the syntactical structure of the formulas.

- The atomic case follows from the definition of  $V$ .
- $\neg\phi$ :  $M, w \models \neg\phi$  iff, by definition,  $M, w \not\models \phi$  iff, by induction,  $\phi \notin w$ , iff, since  $w$  is maximal consistent,  $\neg\phi \in w$ .
- $\phi \rightarrow \psi$ :  $M, w \models \phi \rightarrow \psi$  iff, by definition,  $M, w \models \phi$  implies  $M, w \models \psi$ , iff, by induction,  $\phi \in w$  implies  $\psi \in w$ , iff, since  $w$  is maximal consistent,  $\phi \rightarrow \psi \in w$ .
- $c \multimap \chi$ : By definition,  $M, w \models c \multimap \chi$  iff  $M, w \star c \models \chi$ , equivalent by induction to  $\chi \in w \star c$ , which by definition of  $\star$  is equivalent to:  $c \multimap \chi \in w$ .
- $\phi \multimap \psi$ : By definition,  $M, w \models \phi \multimap \psi$  iff for every  $c \in C$ ,  $M, w \models c \multimap (\phi \rightarrow \psi)$ . By induction, this is equivalent to:  $c \multimap (\phi \rightarrow \psi) \in w$  for every  $c \in C$ .

Now suppose that  $\phi \multimap \psi \in w$ . Then, by  $\mathcal{A}4$  and the fact that  $w$  is a maximal consistent set,  $c \multimap (\phi \rightarrow \psi) \in w$  for any  $c$ , which is equivalent to  $M, w \models \phi \multimap \psi$ .

But if, on the contrary,  $(\phi \multimap \psi) \notin w$ , then since  $w$  is a maximal consistent set,  $\neg(\phi \multimap \psi) \in w$ . By construction of the model,

- $w = w_0 \star \bar{c}$  for some sequence  $\bar{c}$  of contexts
- $\bar{c} \multimap \neg(\phi \multimap \psi) \in w_0$
- $\bar{c} \multimap \neg(c' \multimap (\phi \rightarrow \psi)) \in w_0$  for some select  $c'$

It follows that  $\neg(c' \multimap (\phi \rightarrow \psi)) \in w$ , which as remarked above is equivalent to  $M, w \not\models (\phi \multimap \psi)$ .

This completes the proof of the lemma.

**Lemma 8.4**  $M, w_0 \models \delta$

**Proof:** This is now obvious, since  $\delta \in w_0$ .

Since every consistent formula is true somewhere, the logic is complete for our class of models.

## 9 Further properties of the logic

We survey a few additional properties of the logic, in the form of sound axiom schemas and rules of inference.

In view of completeness, the soundness of each of these schemas and rules is sufficient to show that they are derivable in the axiomatic system. Soundness is straightforward to verify every case, so we omit the proofs.

$$\mathcal{A5} \quad (\phi \rightarrow \psi) \rightarrow ((\psi \multimap \chi) \rightarrow (\phi \multimap \chi))$$

$$\mathcal{A6} \quad (\phi \rightarrow \psi) \rightarrow ((x \multimap \phi) \rightarrow (x \multimap \psi)) \text{ for } x \in C \text{ or } x \in \Sigma.$$

We see that  $\multimap$  is antitone in its first coordinate and monotone in its second one. This is intuitively pleasing, and admits a possible reading of the binary modality  $\multimap$  as a kind of conditional, although this was not the main motivation for its definition.

$$\mathcal{A7} \quad (\phi \multimap \phi)$$

$$\mathcal{A8} \quad (x \multimap \phi) \rightarrow ((\phi \multimap \psi) \rightarrow (x \multimap \psi)), \text{ for } x \in C \text{ or } x \in \Sigma.$$

This shows that  $\multimap$  is a partial preorder on  $\Sigma$ , again in keeping with a reading of  $\multimap$  as a conditional symbol.

$$\mathcal{A9} \quad (\top \multimap \phi) \rightarrow (c \multimap \phi)$$

The least specific description of a context is  $\top$ , which describes every context by virtue of being true no matter what.

$$\mathcal{A10} \quad \neg(\top \multimap \neg\phi) \rightarrow ((\phi \multimap \neg\psi) \leftrightarrow \neg(\phi \multimap \psi))$$

This comes close to mimicking  $\mathcal{A2}$  for  $\phi \multimap \psi$ , and can be intuitively understood to say that every non-contradictory formula describes some contexts. Whenever  $\phi$  is not false in every context, then whatever is false in the contexts described by  $\phi$ , is not true there. The restriction on  $\phi$  is to avoid empty quantification.

$$\mathcal{A11} \quad ((c \multimap \phi) \wedge (c \multimap \psi)) \leftrightarrow (c \multimap (\phi \wedge \psi))$$

This and other analogous rigidity principles apply to the classical connectives. Intuitively, they are aspects of the propositional coherence of the set of formulas true in each context.

$$\mathcal{A12} \quad (x \multimap (\phi \rightarrow \psi)) \rightarrow ((x \multimap \phi) \rightarrow (x \multimap \psi)) \text{ for } x \in C \text{ or } x \in \Sigma.$$

When an implication is true in every member of a set of contexts, and the antecedent is true in every member, then the consequent is also true in every member. An easy generalization over  $\mathcal{A3}$ , admitting formulas in the first coordinate of  $\multimap$ .

$$\mathcal{A}13 \quad (\phi \multimap \psi) \leftrightarrow (\top \multimap (\phi \rightarrow \psi))$$

This conversion principle corresponds closely to the semantical clause for  $\phi \multimap \psi$ . Intuitively,  $\top$  is the least specific description, and thus describes every context. Therefore, every subformula  $\top \multimap \dots$  corresponds to a quantifier spanning the whole set  $C$  of contexts.

$$\mathcal{R}4 \quad \frac{\vdash \phi \leftrightarrow \psi}{\vdash (\phi \multimap \chi) \leftrightarrow (\psi \multimap \chi)}$$

$$\mathcal{R}5 \quad \frac{\vdash \wedge \phi_i \rightarrow \psi}{\vdash \wedge (x \multimap \phi_i) \rightarrow (x \multimap \psi)} \text{ for } x \in C \text{ or } x \in \Sigma.$$

These two rules show that, in the terminology of (Chellas 1980), if we look at  $\multimap$  as a binary modality, it is *classical* in its first coordinate and *normal* in its second one. It shares these properties with the class of conditional logics investigated there, and for which the model framework was a class of minimal models. We feel that the present class of models is simpler and more intuitive.

$$\mathcal{R}6 \quad \frac{\vdash \phi}{\vdash x \multimap \phi} \text{ for } x \in C \text{ or } x \in \Sigma.$$

Every theorem is true in every context described by any formula. This is highly intuitive, and is a generalization over  $\mathcal{R}2$ , admitting formulas in the first coordinate of  $\multimap$ .

$$\mathcal{R}7 \quad \frac{\vdash \phi \rightarrow \psi}{\vdash \phi \multimap \psi}$$

Constrains the phrase "described by" in the informal reading of  $\phi \multimap \psi$ . What is implied logically by  $\phi$ , must also be true in the contexts described by  $\phi$ .

## 10 Comparison with other logics of context

The propositional logic of  $\mathbf{ist}(c, \psi)$  is investigated in (Buvač et al. 1995), and augmented with first-order quantification in (Buvač 1996). These are axiomatic systems for reasoning with **ist**-formulas asserted in given contexts. The syntax for asserting  $\phi$  in context  $c$  is  $c : \phi$ , and a central motivation for these logics is the ability to enter a given context, perform some reasoning there according to facts that hold in that context, and exit with the results so obtained. In general, this gives rise



to a stack  $\bar{c}$  of contexts having been entered into and not exited from, and these are the deduction rules governing entry into and exit from contexts:

$$\text{Enter: } \frac{\vdash \bar{c} : \mathbf{ist}(c, \psi)}{\vdash \bar{c}c : \psi} \qquad \text{Exit: } \frac{\vdash \bar{c}c : \psi}{\vdash \bar{c} : \mathbf{ist}(c, \psi)}$$

The Enter rule is not listed in the axiomatic presentation of (Buvač et al. 1995), and it is subsumed by other axioms and rules of the logic. It is included here for symmetry.

The semantics of (Buvač 1996) does not distinguish between  $\mathbf{ist}(c, \mathbf{ist}(d, \phi))$  and  $\mathbf{ist}(d, \phi)$ , which appears anomalous at first blush. The rules for entering and exiting contexts are correspondingly constrained:

$$\text{Enter: } \frac{\vdash x : \mathbf{ist}(c, \psi)}{\vdash c : \psi} \qquad \text{Exit: } \frac{\vdash c : \psi}{\vdash x : \mathbf{ist}(c, \psi)}$$

The logic remembers only the last context that has been entered into. This phenomenon, called *flatness*, is not unavoidable however, and as discussed below, (Nossum 2002) generalizes the propositional logic of  $\mathbf{ist}(c, \phi)$  to an algebraically generated spectrum of context logics where flat contexts are only a special case.

We may compare our

$$\phi \multimap \psi$$

with

$$\forall c : \mathbf{ist}(c, \phi \rightarrow \psi)$$

of (Buvač 1996). The latter formula can be taken as rephrasing our semantical clause for the former. If the two are accepted as variants of each other, then Buvač's system is seen to be strictly more expressive than the one presented here, because it has full first-order quantification, over context variables as well as other variables. But as usual expressivity comes at a price: the system of (Buvač 1996) is not decidable.

To further illuminate the trade-off that afforded us decidability in the present logic, let us point out our axiom  $\mathcal{A}2$ , which may be taken to express that each individual context is a logically coherent and complete entity. That is a stricter assumption than in most other logics of context, including other logics studied by this author (Nossum 2002), (Nossum and Serafini 2002), (Gabbay and Nossum 2000).

van Benthem (van Benthem 1996) finds that the term 'context' denotes a convenient methodological fiction, rather than a well-defined ontological category. His proposal is for an indexing scheme, where each language element can be decorated with an index specifying an intended context for evaluation. This results in a logic where transition between contexts has a natural expression.

Giunchiglia, Serafini et.al. have also developed logical systems of context where transition between contexts is the main concern (Serafini and Giunchiglia 2001). Each context is modelled by a separate natural deductive system, and there are special rules for inter-context deduction. However, there is no direct provision for nesting of contexts, as in  $\mathbf{ist}(c, \mathbf{ist}(d, \phi))$ , although the natural deduction system allows for convenient passage between  $c$  and  $d$ .

Nossum and Serafini have developed natural-deductive systems of context where context combination is catered for through an algebraic component (Nossum and Serafini 2002). Sequential composition of contexts, e.g.  $c, d, e, \dots$ , is represented associatively by algebraic terms, e.g.  $c \oplus d \oplus e \dots$ , and there is provision for algebraic equations on context terms, thus spanning a variety of natural-deductive logics of context including flat contexts, context sets, context multisets, and context sequences.

(Nossum 2002) expands on the idea of algebraic context augmentation in the framework of an axiomatic  $\mathbf{ist}$ -logic in the style of (Buvač et al. 1995). This time, context terms like  $c \oplus d \oplus e$  are introduced into the syntax of the language, as are algebraic equations on ground context terms. The equational varieties within the scope of this approach are the same as in (Nossum and Serafini 2002), including flat contexts, context sets, context multisets, and context sequences. Augmenting a context  $w$  with another one,  $c$ , to form a composite context  $w \oplus c$ , is analogous to going from possible world  $w$  to possible world  $w \star c$  in the logic of the present paper. Ongoing work aims to generalize (Nossum 2002) to wider equational varieties and quasi-varieties, as well as to quantificational logic.

In (Gabbay and Nossum 2000) a quantificational system similar to Buvač's is obtained by self fibring of predicate logics, and decidability is shown in a special case. They also develop a multi-modal logic for  $\mathbf{ist}(\phi, \psi)$ , which is as expressive as the present logic for  $\phi \multimap \psi$ , but they do not prove decidability for it.

## 11 Conclusion

We depart from the notation  $\mathbf{ist}(c, \psi)$  which originates with (Guha 1991), preferring  $c \multimap \psi$  and generalizing to  $\phi \multimap \psi$ . Our system harnesses generalization over contexts in a two-layered multi-modal system, the semantics of one modality quantifying over a set of simpler modalities. This results in a simple, decidable, sound, and complete axiomatic presentation.

## References

- Buvač, Saša. 1996. Quantificational logic of context. In *Proceedings of the Thirteenth National Conference on Artificial Intelligence*.
- Buvač, Saša, Buvač, Vanja, and Mason, Ian A. 1995. Metamathematics of context. *Fundamenta Informaticae*, vol. 23, no. 3.
- Chellas, Brian F. 1980. *Modal Logic: an introduction*. Cambridge University Press.
- Gabbay, Dov and Nossum, Rolf. 2000. Structured contexts with fibred semantics. In P. Bonzon, M. Cavalcanti, and R. Nossum (eds.), *Formal Aspects of Context*. Applied Logic Series, Kluwer.
- Giunchiglia, Fausto. 1993. Contextual reasoning. *Epistemologia*, vol. special issue on I Linguaggi e le Macchine, no. XVI, pp. 345–364. IRST Technical Report 9211-20, IRST, Trento, Italy.
- Guha, Ramanathan V. 1991. *Contexts: A Formalization and Some Applications*. Ph.D. thesis, Stanford University.
- Henkin, L. 1949. The completeness of the first-order functional calculus. *Journal of Symbolic Logic*, vol. 14, pp. 159–166.
- Hughes, G. E. and Cresswell, M. J. 1968. *A New Introduction to Modal Logic*. Routledge.
- Kempson, Ruth, Meyer-Viol, Wilfried, and Gabbay, Dov. 2001. *Dynamic Syntax: The Flow of Language Understanding*. Blackwell.
- Lewis, D. K. 1973. *Counterfactuals*. Basil Blackwell, Oxford.
- McCarthy, John. 1993. Notes on formalizing context. In Wolfgang Wahlster (ed.), *Proceedings of IJCAI'93*.

- McCarthy, John and Buvač, Saša. 1994. Formalizing Context (Expanded Notes). Technical Note STAN-CS-TN-94-13, Stanford University.
- Nossum, Rolf. 2002. Propositional logic for ground semigroups of context. *Logic Journal of the IGPL*, vol. 10, no. 3, pp. 273–297.
- Nossum, Rolf and Serafini, Luciano. 2002. Multicontext logic for semigroups of contexts. In J. Calmet, B. Benhamou, O. Caprotti, L. Henoque, and V. Sorge (eds.), *Artificial Intelligence, Automated Reasoning and Symbolic Computation*, vol. 2385 of *LNAI*, pp. 90–101. Springer-Verlag.
- Reed, Stephen and Lenat, Douglas. 2002. Mapping ontologies into Cyc,. In *AAAI Workshop on Ontologies and the Semantic Web*. <http://www.cyc.com>.
- Serafini, Luciano and Giunchiglia, Fausto. July 2001. ML systems: A proof theory for contexts. *Journal of Logic, Language and Information*.
- van Benthem, Johan. 1996. Changing contexts and shifting assertions. In A. Aliseda-Llera, R. van Glabbeek, and D. Westerståhl (eds.), *Proceedings 4th CSLI Workshop in Logic, Language and Computation*. CSLI Publications.